Optimal annuity demand in behavioral decision models

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Diese Dissertation beinhaltet die folgenden Teile:

- Introduction
- The effects of hyperbolic discounting and the bequest motive on the retirement portfolio
- Narrow framing, loss aversion and the optimal retirement portfolio with Arrow annuities
- Myopic loss aversion and the demand for life annuities
- Reference-dependent preferences and the demand for annuities

Introduction

What portion of their total wealth should retirees hold in life annuities? Consumption based models, from early examples to more recent models, typically suggest a very high portion or even full annuitization. However empirical evidence shows that actual annuitization degrees are often much lower than that. For example, voluntary annuitization beyond mandatory pension plans seems to occur very rarely. Franco Modigliani, in his Nobel Price acceptation speech, stated that,

It is a well-known fact that annuity contracts, other than in the form of group insurance through pension systems, are extremely rare. Why this should be so is a subject of considerable current interest. It is still ill-understood.

In the face of aging societies and increasing life expectancy this is not merely a scientific curiosity, but carries a high risk for people's financial wellbeing. On the one hand it affects retirees negatively as their pensions sink, because there are less active contributors to the pension funds, on the other hand it burdens the rest of society because states might need to draw on tax money to cover the gaps in subsistence spending for retirees. Because of the magnitude of the potential problems and the ongoing inability of the scientific community to fully and satisfactorily explain the perceived unattractiveness of annuity contracts, a broad variety of research on the topic has been produced in the past two decades. In this context a range of potential explanations have been identified and analyzed. In the realm of rational behavior, prominent examples of this are unfair annuity pricing and the presence of bequest motives. While both of these factors can reduce the demand for life annuities in a rational model, the resulting annuity demand often still exceeds the empirically observed purchasing rates. As a result of this, explanations beyond purely rational factors have been suggested and various other potential obstacles in the annuity market have been identified. These include among others framing, financial illiteracy, underestimated life expectancies and loss aversion. In many cases these suggestions are qualitative in their nature. That means they explain why there is less demand for annuities than predicted, but oftentimes do not produce quantitative estimates on how the resulting annuity demand would look like under their assumptions. This dissertation aims to develop behavioral decision models, that include behavioral within the framework of classical rational models. The goal is to achieve an understanding of how, and to what exact extend, these phenomena affect the demand for life annuities and other life insurance products. A potential justification for this hybrid approach is to assume that the individual retirees at the center of our analysis act rationally in most ways, but are subject to some behavioral bias or fallacy affecting their analysis of investment opportunities, insurance products or their perception of future periods.

The purpose of an annuity and its main selling point is to enable a steady and lifelong stream of consumption for the retiree. When reduced to its characteristics as a mere investment opportunity, the big advantages of annuities regarding consumption are not sufficiently accounted for and as a result, annuities fall behind other investment classes in terms of risk and return characteristics. For this reason, the core of the models that are adopted or proposed in the following papers, is the retiree's utility from consumption over his uncertain remaining life span. However the payoffs from annuities and other forms of insurance only have a direct effect on the insured's wealth on hand and not on his consumption. An evaluation of their effect on the retiree's consumption, and therefore their actual usefulness, can only be conducted when the retiree's optimal consumption policy is known. In addition to that, the wealth levels of the retiree are not only dependent on the payoffs from his insurance contracts, but also from other types of assets the retiree may be invested in. Therefore our analyses rely heavily on optimization and as a result do not only produce the optimal insurance endowments, but also the resulting optimal consumption and investment plans throughout retirement. Besides obtaining optimal insurance and annuity endowments, a minor focus of this dissertation lies on the analysis of these resulting plans and a comparison with the benchmark plans for a fully rational individual.

Altogether this dissertation consists of four separate papers, which cover a total of three behavioral or non-classical phenomena which may affect an individual's financial decision making. The approach outlined above requires that the effects that are incorporated are quantifiable in some way. Therefore we focus on three phenomena that fulfill this requirement. These are hyperbolic discounting, loss aversion in connection with narrow framing of investment decisions, and reference-dependent utility. The way that these phenomena enter our analysis, is through the retiree's preferences described by an objective function. Depending on the phenomenon, their implementation within the framework of a multi-period consumption utility model can be more or less challenging. Furthermore empirical testing of our models is fairly difficult because very detailled financial data for retirees is needed to adequately calibrate the models. We go into more detail on this point later. Because data with the required depth of detail is not readily available, the papers in this dissertation focus solely on the demand side. This means we propose a hypothetical model and obtain the optimal annuity demand within this model. Regarding parameter choices, we resort to the choices most encountered in the literature whenever possible. In some cases, when our models venture into territory, for which there is no established consensus on reasonable parameter values, we conduct our calculations for a variety of potential candidates for these parameters. Our main contributions lie in the proposition of models and a thorough optimization within these models. The central parts of these models are the objective functions, describing the retirees preferences. The aim of this introduction is to give an outline of the similarities and differences between the objective functions in the four papers. In addition to that, we highlight the potential obstacles in their conception and how we overcome them.

Hyperbolic discounting is a special form of time inconsistency. A typical example of this is when an individual, who faces the question whether to take a fixed amount of money or an increased amount of money a week later, prefers a different alternative whether the first amount is due now or in a year's time. If we assume that the individual's preferences remain consistent and that he is able to correctly anticipate his preferences in a year, then he should choose the same alternative regardless of when the first amount is due. In that sense hyperbolic discounting can be seen as a form of irrational behavior. In most classical consumption based life cycle models the typical assumption on time perception is time consistent discounting or exponential discounting. With time consistent discounting, a subsequent period receives β times the weight of the previous period for some $\beta < 1$. This means that the outcomes in t periods from now, or from time $t = 0$, receive the total weight β^t. In total, the classical consumption based multi-period model with uncertain lifetime takes the following form. When $p_{0,t}$ are the probabilities that the individual survives until time t, under the condition that he is alive at time 0, and $u(C_t)$ is the utility from annual consumption, then the individual's objective function for the retirement phase is given by

$$E_0\Big[\sum_{t=0}^{T} p_{0,t}\beta^t u(C_t)\Big].$$

Here we assume that the individual enters retirement at time $t = 0$ and may live up to T more years. The annual consumption level is limited by the individual's wealth on hand, which itself is a result of the individual's previous consumption and investment plans. When the individual invests in risky assets, then the future wealth levels are random variables depending on the outcomes of his investments. Additionally, a bequest motive can be incorporated into this framework by introducing a second utility function v, which describes the retiree's utility from bequest. If we assume that the bequest is transferred immediately upon the death of the retiree, then the retiree receives the additional utility $v(B_t)$ at his time of death. The expanded objective function then takes the form

$$E_0\Big[\sum_{t=0}^{T} p_{0,t}\beta^t (u(C_t) + (1 - p_{t,t+1})\beta v(B_{t+1}))\Big].$$

If we assume hyperbolic instead of exponential discounting, then we further adjust the above objective function by interchanging the exponential discount factors β^t with hyperbolic discount factors DF_t.

On the basis of this objective function, in the first paper we analyze how assuming hyperbolic discounting instead of exponential discounting affects the optimal insurance endowment during retirement. We assume that the retiree has access to a life annuity, which may contain periods with guaranteed annuity payments, and a life insurance policy paying a fixed sum at the time of death of the retiree. The analysis is conducted for various parametrizations of the bequest utility function.

The two subsequent papers propose models that attempt to incorporate the effect of loss aversion on a retiree's demand for annuities. The approach we take is based on the evaluation function from Kahneman and Tversky's prospect theory, which describes an individual's perception of the potential outcomes of a risky gamble or a risky investment. In classical expected utility theory, it is often not the monetary outcomes of individual investments which are evaluated, but the resulting consumption levels or, usually as an approximation for consumption, the agent's total wealth. And it is fluctuations in this consumption levels, or the total wealth levels, that the individual is concerned about, and only indirectly the developments of his assets. In contrast to this, the prospect theory evaluation function is not applied to dimensions that yield immediate utility in the baseline consumption model, such as consumption, but to the immediate monetary outcomes of an investment policy or an individual investment. A characteristic feature is, that the evaluation function from prospect theory distinguishes between gains and losses. This distinction is achieved through the introduction of a reference point. Outcomes that lie above the reference point are counted as gains, outcomes below as losses. The losses are weighted heavier than the gains, which makes them loom larger than potential chances. This uneven assessment of gains and losses motivates the term loss aversion. As a result, an objectively attractive investment opportunity may be perceived as unattractive by a loss averse individual, because he overemphasizes the disadvantageous outcomes. Another important property is that the scope of the investment evaluation is usually not the total outcome of the investment strategy, i.e. the resulting wealth level or the portfolio return, but is often limited to the outcomes of smaller groups of risky assets or even to the outcomes of individual risky assets. This partial evaluation of the investor's portfolio is called narrow framing. Depending on how narrow the framing actually is, such an evaluation may neglect diversification effects with other risky assets. This can make the assets appear more risky than they actually are and therefore even more unattractive to the loss averse investor. Therefore loss aversion may be a potent explanation for the reluctance to invest in risky assets such as stocks or life annuities. The fact that Kahneman and Tversky's cumulative prospect theory builds on a strong empirical foundation underlies its relevance and its potential ability to explain investor behavior.

However cumulative prospect theory was originally conceived as a theory for static decision making under uncertainty. This means, that it is designed to explain preferences regarding risky decisions where the uncertainty about the outcome is resolved immediately, or at least where there are no further timing effects involved, such as accumulated interest or periodic withdrawals. Furthermore for asset classes whose values are constantly fluctuating, such as stock prices, it is not intuitively obvious which changes in value and especially over which time horizon constitute a gain or a loss. Counting every downward movement in the price of a stock as a loss would make the investment appear utterly unattractive. Especially considering that in most continuous time models such as the Black-Scholes-Model, there are infinitely many such downward movements in any time interval. To only assess the outcome of an investment once it has been completely liquidated or reaches maturity seems to be the more reasonable alternative. Unfortunately this approach runs into another obstacle. The structure of the prospect theory evaluation function requires that an investment can be represented in terms of an amount of capital invested and the random rate of return by which this capital growths. But when there are partial withdrawals while the investment is still running, then it is impossible to find a pair consisting of an amount of capital and a rate of return that accurately describes the investment. This means that loss aversion in the strict sense of Kahneman and Tversky's prospect theory is not a suitable tool to describe an individual's evaluation of a life annuity which by definition contains annual or monthly withdrawals. Therefore, we resort to a derivative concept which avoids the problem outlined above, the concept of myopic loss aversion. Myopic loss aversion describes the phenomenon that an investor who makes a long term investment is sensitive to the short term fluctuations of his assets. A typical simplified example is the following. A single gamble with a positive expected outcome may reasonably be declined by an individual, for example due to risk aversion. However, when the individual has the option to repeat the gamble a sufficiently high number of times, then by the law of large numbers the probability of an accumulated loss becomes small enough so that the gamble should eventually be accepted. But when the individual is myopic loss averse, then he may still asses the outcome of each gamble individually and therefore refuse to participate in the repeated gamble. In a similar manner, a stock investment may be viewed as a sequence of gambles in which the price of the stock either goes up or down. When these individual gambles have a positive expected return, then in the long run the probability that the investor faces an accumulated loss should not be prohibitively high. However when the investor is sensitive to short term losses, for example the development of his asset over the course of a month or a single year, then the probability of a loss within this shorter time interval is much higher. As a result, myopic loss averse investors perceive such investments as riskier than they actually are and therefore may abstain from them. For this reason, myopic loss aversion is a popular explanation for low participation rates on equity markets. From a modelling perspective, myopic loss aversion has a particular advantage. When the evaluation horizons and the periods between withdrawals coincide, then myopic loss aversion avoids the problem of assessing investments with intertemporal withdrawals. Furthermore, in models with annual selection of consumption level and investment strategy, it is somewhat reasonable to assume that the evaluation horizon corresponds to the periods between portfolio adjustments[1]. This makes it possible to construct a preference functional that contains annual consumption spending and a loss averse investment evaluation and forms the basis for the model in the second and the third paper.

With the addition of the investment evaluation, the retiree's objective function now contains three sources of utility. The first two, utility from consumption, u_C, and utility from bequest, u_B, are utility functions in the classical sense. The third source of subjective utility is the loss averse investment evaluation function, v. We analyze two different specifications of the scope of the investment evaluation. The first specification, labeled narrow framing, assumes that the investment prospects of each risky asset are assessed individually, the second, labeled broad framing, assumes that the portfolio of all the retiree's risky asset are evaluated at once. For an exogenous reference return r_R, the general form of the objective function for m risky assets with returns $r_{j,t}$ and asset weights $\theta_{j,t}$ takes the form

$$E_0\Big[\sum_{t=0}^{T} p_{0,t}\beta^t(u_C(C_t) + (1 - p_{t,t+1})\beta u_B(B_{t+1})) + \beta^t\kappa\sum_{j=1}^{m} v(\theta_{j,t}r_{j,t}, rf)\Big]$$

in the model specification with narrow framing and

$$E_0\Big[\sum_{t=0}^{T} p_{0,t}\beta^t(u_C(C_t) + (1 - p_{t,t+1})\beta u_B(B_{t+1})) + \beta^t\kappa v(\sum_{j=1}^{m}\theta_{j,t}r_{j,t}, r_R)\Big]$$

in the model specification with broad framing.

Objective functions similar to the two above have been applied in empirical asset pricing models and portfolio optimization models. In these cases the risky assets are usually assumed to be liquid assets which can be traded annually. This is not the case for a life annuity, which is characterized

[1]Even though, as it is the case for the accumulation phase of an individual's life cycle, there may be no withdrawals but further investments in the risky asset.

by payoffs over multiple periods and is usually an illiquid investment which cannot be traded. As a result we cannot apply the above utility functions to a life annuity. To obtain an approximate result on how loss aversion affects the demand for asset classes or types of insurance whose payoff depends on the survival of the retiree, we resort to a simplified version of an annuity in the second paper, the Arrow annuity. The Arrow annuity is a one period financial contract which pays a fixed amount if the investor is still alive in the subsequent period, and nothing if he deceases during the current period. At each time $t = 0, 1, \ldots, T$, i.e. at the beginning of every year, the retiree may purchase an Arrow annuity with an arbitrary payoff as long as its price is within the limits of his budget restrictions. This simplified annuity has all the characteristics of a tradeable one period investment. Therefore the objective functions presented above can be applied within this simplified framework.

In general, loss aversion results in an unfavorable evaluation of the risky assets. That means that in the absence of consumption or bequest utility, the loss averse retiree would refuse to invest in any risky asset because the risky investment prospects result in subjective disutility. Hence, the retiree in our model only invests in a risky asset when the expected gain in classical utility outweighs the disutility associated with risk exposure. Therefore the parameter $\kappa > 0$ plays a crucial role in the objective function. It governs the relative importance of classical and prospect utility. For large values of κ the investor refuses to invest at all in risky assets, for small values the prospect utility only plays a minor role in the investor's assessment and the resulting demand for risky assets stays near the benchmark demand in a model without prospect utility. We conduct the optimization in the second paper for various values of the parameter κ to analyze its effect on the demand for risky assets and the resulting optimal consumption and investment plans.

The analysis in the second paper is a reasonable approximation, because Arrow annuities contain the important characteristics of a life annuity, which are dependency of their payoff on the investor's survival and, when they are priced following actuarial principles, the mortality credit, i.e. the effect, that the survivors receive a return that is higher than the return of the underlying asset, at the cost of the deceased who do not get their investment back. However there are still qualitative and quantitative differences to a real life annuity with payoffs over multiple periods. First, in contrast to the Arrow annuity, a regular life annuity is of a fixed size which cannot be adjusted during the payoff phase. Second, the mortality credit is more pronounced and more nuanced for a regular annuity than for the Arrow annuity. Therefore, to obtain a more accurate analysis of the effect of loss aversion on the demand for annuities, in the third paper we propose an adjustment to the model in the second paper that allows a loss averse investment evaluation of a regular life annuity. Our approach builds on a decomposition of the life annuity into individual Arrow annuities whoose payoffs are of the same size, one for each potential payoff date. Each of the individual Arrow annuities can be regarded as an investment over multiple periods, which either grows annually by a certain factor if the investor survives, or results in a total loss, i.e. grows by the return -1, if the investor deceases. In this sense, we can find an annual return rate representing the development of each individual Arrow annuity. And because there are no intertemporal withdrawals for an individual Arrow annuity, the respective investments can be evaluated by the investment evaluation function in a similiar way as in the second paper. In addition to individual Arrow annuities, we can then regard the portfolio of all Arrow annuities, which is equivalent to the regular life annuity. At any point in time it contains all the Arrow annuities that have not yet been paid out. Therefore its current value or rather the capital which is currently bound in the annuity at time t is the sum of the initial endowments of all the Arrow annuities that are still active multiplied by their respective growth factors for the previous $t - 1$ periods. In this sense we can find an amount of capital for each period that describes the current investment in the whole annuity. The annual portfolio return of the portfolio of Arrow annuities is the sum of the individual annualized returns weighted by the relative sizes of their current endowments. As a result we can find a pair of an amount of capital and an associated random return that describes the annuity investment accurately during every period. This solves the initial problem regarding the applicability of loss averse prospect utility to life annuities and allows us to apply the objective functions from the second paper to a regular life annuity. Again we distinguish between the two model specifications with a narrow scope and a broad scope of framing in the investment evaluation and also conduct the analyses for various values of the crucial parameter κ.

In the fourth paper we analyze the annuitization problem in the context of reference-dependent preferences. More specificly we adapt a model by Köszegi und Rabin to the problem of voluntary annuitization. The resulting preference model is again based on a multi-period model with uncertain lifetime and utility from consumption and bequest, but additionally assumes that the retiree has an initial set of reference levels or beliefs about his future consumption and bequest. We assume that these beliefs are rationally formed, for example on the basis of his annuitization level prior to entering retirement. When the retiree is offered the chance to annuitize further parts of his wealth upon entry to retirement, he measures the benefits of doing so against potential non-beneficial deviations that come with choosing a higher degree of annuitization, for example a lower bequest size if he deceases in an early period. The retiree's objective function then takes the form

$$\begin{aligned} E_0\Big[\sum_{t=0}^{T} p_{0,t}\beta^t((u_C(C_t) + m_C(u_C(C_t) - u_C(C_{R,t}))) \\ + (1 - p_{t,t+1})\beta(u_B(B_{t+1}) + m_B(u_B(B_{t+1}) - u_B(B_{R,t+1})))\Big] \end{aligned}$$

Here $C_{R,t}$ and $B_{R,t}$ are the reference levels regarding consumption and bequest and m_C and m_B are piecewise linear functions which, similar to the concept of loss aversion, have a higher gradient on the negative half axis than on the positive half axis. When the reference value and the actual value are identical, then the reference utility terms cancel out and the preferences reduce to a classical expected utility preference functional. The imbalance between positive deviations and negative deviations results in a preference for the current reference level. As a result, a higher annuitization level may be perceived as unfavorable even though it would lead to a higher classical utility in the absence of the reference utility terms. Therefore the model can explain low annuitization rates by assuming reference dependency and low reference annuitization levels.

The four papers in this dissertation rely heavily on optimization. Because they all contain stochastic optimization problems with multiple periods where the behavior in each period influences the potential scope of action in the subsequent periods, we resort to backwards induction to find the optimal strategies and thus the optimal annuitization degrees in each model. However this procedure requires a large number of computation steps and a variety of numerical methods to make the calculation as time efficient as possible. This is especially true in the fourth paper, because the model there contains the additional dimensions of the reference values, which are typically not constant but random variables themselves. To keep the computation feasible we adopt an optimization method that builds on simulating potential reference levels, and then solving the optimization problem along the simulated trajectories. Furthermore we employ some recently proposed modifications to the regular backwards induction procedure to further decrease the computation time. This dissertation contains a minor methodical contribution in that we modify a method for smooth objective functions to allow an optimization of the non-smooth objective functions occurring in the fourth paper.

In contrast to models for empirical asset pricing, which contain models that are identical, or very similar, to the models we adopt, our analysis only covers the demand side of annuitization. This means we assume that the retiree has access to life annuities or other types of insurance at a certain price and then calculate the optimal demand. However, as it is typical for models for optimal annuity demand, we omit a comparison with empirical data because relevant data is not easily available. To properly calibrate the models to actual data, the data set would have to include very detailed data on retirees' annual consumption, their investment policies, their annuity and insurance endowments and their total wealth. The latter would have to include their whole estate, including possible property and other valuable assets. Without access to such data, models like the ones we propose can only come to hypothetical explanations. However these models may still help to understand how purchases of life insurance and life annuities are motivated and how big the quantitative effects of potential influential factors are. As mentioned at the beginning of this introduction, it is vital for the financial well-being of ageing societies in western countries but also in newly industrialized countries like China, to achieve a high general annuitization degree throughout their whole population. And when, in the face of demographic changes, unfunded pension systems are no longer capable to supply this high annuitization demand, then it is even more important to motivate people to privately annuitize parts of their wealth. But this is only possible when we better understand, how the decision to annuitize is formed.

The effects of hyperbolic discounting and the bequest motive on the retirement portfolio

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Abstract

We study how assuming hyperbolic discounting instead of exponential discounting influences the demand for life annuities and bequest insurance for a life-cycle investor with a bequest motive. We find that hyperbolic discounting leads to an increased annuitization degree and a decrease in the demand for life insurance contracts. Nevertheless hyperbolic discounting leads to higher bequests, mainly at the expense of consumption in the early retirement years. In addition to that we find that bequest insurance is only optimal for strong bequest motives. In this case the direct bequest insurance always dominates guarantee periods in our model.

1 Introduction

This paper explors the combined effect of hyperbolic discounting and a sufficiently strong bequest motive on a life-cycle investor's decision to annuitize his accumulated savings upon entry to retirement. We propose a simple model in which the retiree has a one time only access to a life annuity, which may contain a desired number of periods with a guaranteed payoff, and to a life insurance contract which pays a fixed amount upon the death of the retiree. In our model guarantee periods, as well as life insurance contracts, exclusively serve to ensure a sufficient bequest size. Albeit identical in purpose, both forms of bequest insurance function very differently. The life insurance policy, which can be regarded as an inverse annuity comes with an inverse version of the mortality credit. Because some policy holders may live very long, and thus their life insurance payoff happens at a very late date, policy holders may enjoy a significant discount when ensuring a sufficient bequest size in this way. Annuities with periods with a certain payoff work somewhat differently towards a similiar goal. Typically a lifecycle investor whose wealth is fully annuitised will save out of his annuity to accumulate a certain bequest size over time. In case of an early death however, the amount saved will not be enough to reach the desired bequest size. With a number of periods with guaranteed payoff, this problem may be avoided. Even though the accumulated savings might not be sufficient, the continuing payoff of annuities for a few periods can cover the resulting gap.

Which of both forms of insuring bequests, or wether a combination of both is optimal, may depend partly on the retiree's perception of time. The life policy combined with otherwise full annuitization enables an agent to plan the exact amount that is bequeathed, as well as the exact amount that he himself has at his disposal in each year, independent of his uncertain lifespan. Of course the actual size of his bequest and his annual consumption levels still depend on wether or not he chooses to save out of his annuity income. But if he does not, then he can in theory determine his future financial policy with absolute certainty upon entry to retirement. The effect of annuities with guaranteed payoff periods towards covering the bequest on the other hand is completely dependent on the time of the agent's death. For an agent with a sufficiently strong bequest motive, guarantee periods can only form a part of his strategy to insure his bequest. A complement in the form of a suitable savings and investment policy will always be necessary. The specific form of the retiree's subjective time weighting may influence the relative attractivity of the retiree's various options when it comes to ensuring consumption and bequest. The aim of this paper is to find how the optimal insurance endowment is affected by hyperbolic discounting compared to exponential discounting.

Generally speaking, hyperbolic discounting leads to a strongly increased relative weighting of later periods. This suggests the question, wether or not hyperbolic discounting will affect the demand for longevity insurance, as running out of funds in old age may be more concerning to the investor. On the other hand hyperbolic discounting leads to a lowering of the subjective importance of earlier periods, which may reduce the demand for annuities with periods with certain payoff.

We propose a dynamic model which covers the retirement phase of a life-cycle investor. At the beginning, upon entering retirement, the agent chooses his insurance endowment which cannot be readjusted later. He then subsequently chooses his annual consumption level and his investment policy. At the time of his death, all of his remaining wealth and potential additional payoffs from a life insurance policy, or from guarantee periods in the annuity contract, are transfered to a heir in form of a bequest. In particular, we explicitly model how guaranteed payoffs in future periods affect the bequest utility at the time of death by calculating the certainty equivalent of the outstanding payoffs. We assume that the agent's survival probabilities coincide with the death tables underlying the annuity premium calculation. The agent's preferences follow a time additive power utility specification regarding both consumption and bequest. More specificly we follow De Nardi's [15] specification of bequest utility, which includes the two parameters strength and prevalence in the population of the bequest motive, in addition to the agent's risk aversion. In a first step, we find the optimal insurance endowments for various parametrizations of the bequest utility by means of backwards induction following the principle of dynamic programming. This analysis is conducted twice, once for an exponential discounting agent and once for a hyperbolic discounting agent. In a second step, we simulate forward to obtain consumption and investment paths associated with the optimal insurance endowments.

Our main results are, that hyperbolic discounting increases the annuitization degree, that hyperbolic discounting decreases the demand for direct bequest insurance yet leads to higher bequest sizes and that guarantee periods are never optimal in the cases considered in our analysis. The difference between the optimal annuitization degrees lies between 2.40 and 3.80 percentage points, depending on the parametrization of the bequest utility function. In the benchmark bequest motive parametrization, there is no demand for bequest insurance in both models. However we find that a slight decrease in the demand for the bequest leads to a positive demand for bequest insurance. In the two cases in which there is a demand for bequest insurance, the hyperbolic discounter invests 3.28% and 4.12% less in the life insurance than the exponential discounter, even though we find that hyperbolic discounting leads to an increased sensitivity regarding bequest. Therefore bequest sizes increase for the hyperbolic investor with a simultaneous reduction of their standard deviation. These gains are achieved at the expense of consumption in the early periods. The average consumption level lies slightly above the fully annuitized income for the exponential discounter and, due to the decreased consumption in the early periods, slightly below for the hyperbolic discounter.

1.1 Literature review

A majority of models for annuity demand suggest, that either full or at least fairly high partial annuitization rates are optimal. Yet, as empirically observed, very few people actually annuitize wealth beyond mandatory pension plans. This disparity is known as the annuity puzzle. Even in the presence of a large loading factor in the annuity pricing, annuitization is still very desirable for the expected utility maximizer. Mitchell, Poterba and Warshawsky [40] find that in reasonably calibrated models, a retiree may pay up to a quarter of his wealth just for access to the annuity market, which is far more than the typical loadings on annuities.

In the recent decades, many potential influence factors on the demand for annuitites within the framework of a rational investor have been identified. A good overview is given by Brown [8]. Perceived poor empirical performance of many of these models has lead to a search for answers outside of the domain of a rational investor. See for example Hu [25] or Brown [**?**] for a compendium of behavioral explanations in addition to the time inconsistent discounting that is touched in this paper.

A prevalent argument on the side to recover the classical approach is to assume a bequest motive. This notion goes back to Yaari's [49] early article on consumption under uncertain lifetime. In Yaari's article, and in most quantitative models, bequest motives are given exogenously in form of a utility function over bequest sizes. Some approaches model bequest motives explicitly in the form of their immediate effect on the welfare of the potential heirs. In the latter case, the form of the heirs' utility function for consumption will naturally carry over to an implicit bequest utility function from the point of view of the bequeather. In a way this legitimizes the former approach to model bequest utility directly through a utility function as done in this paper.

In general, bequests can be categorized by the receipient of the inheritage which leads to intramarital, intergenerational and charitable bequest motives. Modigliani [41] provides an overview over these forms of bequest motives. Intergenerational bequests, that is parents bequeathing to children, are sometimes further categorized in the classes incidental/accidental and altruistic/strategic. Accidental intergenerational bequests result from a premature death while saving for longevity. The accidental bequest is not intended and thus should have no effect on the decision to annuitize. A parent with a strategic bequest motive uses his estate as a means to incentivise his children or grandchildren whereas an altruistic agent will give for the sake of giving. Both cases may hinder annuitization. However, the altruistic parent may always just leave his desired bequest size immediately upon entry to retirement and annuitize the rest of his wealth. A similiar argument can be made for the charitable bequest.

There is variety of papers exploring the effects of intergenerational bequest motives on saving and consumption plans of the bequeather and the resulting bequest sizes. For example Gokhale, Kotlikoff, Sefton and Weale [18] analyze the effect of intergenerational bequests on the development of wealth and in particular wealth inequality over many generations. Instead of an exogenous approach on bequest utility, they use a joint preference functional for bequeather and heirs. However they do not actually optimize due to the high number of state variables but apply behavioral heuristics to simulate consumption and saving behaviour. In a more recent paper, Love [37] studies a life-cycle model with random familiy shocks and the possibility to insure sufficient bequest sizes through a life insurance policy. His approach however does not include access to annuity markets. Bernheim, Shleifer and Summers [6] represent an examplary article that studies a non-altruistic bequest motive. They propose a model for a strategic bequest motive in which parents use bequest as an incentive for their heirs.

Furthermore there are many papers concerned with the particular form, and the effect on consumption and saving plans, of intramarital bequest motives. Especially life insurance policies as a means of bequest insurance are usually connected with this form of bequest. Auerbach and Kotlikoff [3] argue, that life insurance policies can play an important role for bequests within a marriage and find that empirically observed life insurance demand is too low for US households. In a paper on the effects of the US social security system Hubener, Maurer and Mitchell [27] find that life insurance for married couples is mainly purchased on the man's lifes. In another paper Hubener, Maurer and Rogalla [28] study the demand for annuities and life insurance for married couples and find that, aside from joint annuities, life insurance may play an important role if there is a significant asymmetry in the annuitization degree of both partners. The latter situation corresponds, in some way, to the strong bequest motive case in our framework.

Vidal-Melia and Lejarraga-Garci [48] study the annuitization decision with bequest motives for a married couple with a joint utility function. Another look on intramarital bequest is given in Brown and Poterba [11], who study the potential benefits of joint life annuities for married couples with the result that utility gains for joint contracts are significantly lower than the utiltiy gains of an annuity for individuals, which can explain the perceived under-annuitization of married couples.

There is also a variety of literature on the general relationship between the demand for life insurance policies and bequest motives. For example Bernheim [5] was among the first to conduct an empirical analysis on the role of life insurance as evidence for a bequest motive and its influence on annuitization. Inkmann and Michaelides [30] follow a similiar approach in a more recent paper.

As it is the case in this paper, many articles study the effect of a general bequest motive on annuitization without concerning themselves with the particular type of bequest. A recent analysis concerning a similiar framework as in this paper is conducted by Lockwood [35]. In accordance with our results, he finds that sufficiently strong bequest motives can significantly lower the demand for annuities. In a more recent paper [36] Lockwood finds that strong bequest motives may additionally increase saving and decrease long-term care insurance purchases. Inkmann, Lopes and Michaelides [29] compare empirical determinants of voluntary annuity demand with their effects in a suitable parameterized life-cycle modell and find that bequest motives can lower the annuity demand and lead to predictions that are comparable to empirically observed values. In accordance with our results they find that a reasonable risk premium in the stock market, in combination with sufficiently strong bequest motives, can lead to less than full annuitization in the optimum. Vidal-Melia and Lejarraga-Garci [47] study the impact of general bequest motives on the annuitization decision under an implicit incorporation of bequest utility into the decumulation phase of a lifecycle model and also find a decrease in annuity demand. Kotlikoff and Spivak [32] find that, without access to annuities, people without a bequest motive may still leave bequests of significant size by accident. This is mainly the result of a premature death while saving for longevity.

Furthermore there is a great amount of literature concerning the general annuitization decision. For example Davidoff, Brown and Diamond [14] find that in the absence of a bequest motive, full annuitization is optimal in a complete market. In a similar perfect market setting with a bequest motive, the agent should choose his desired bequest size upfront and then annuitize the remainder of their wealth. Albeit they do not analyze the effect of access to equity markets, they conclude that any annuitized asset dominates the non annuitized version in the absence of bequest motives.

In contrast to the once, and once only, annuitization paradigm followed here and in most of the literature cited here, there is also a fair amount of literature concerned with timing effects of the annuitization decision. Horneff, Maurer and Stamos [24] study the optimality of gradual annuitzation. Milevsky and Young ([38] and [39]) focus on the optimal timing of the lump sum annuitization decision. The general result is that annuitization should not occur too early. Milevsky and Young find that self annuitization dominates the annuitization products available on the market at the time of their research (2007) before the ages 65 to 70. Annuitization should also not occur too late.

Peijnenburg, Nijman and Werker [42] conduct a similiar analysis of the retirement phase of a life-cycle with a bequest motive both absent and present. Their findings suggest, that even in the presence of equity markets, full annuitization may still be optimal. In contrast to the Black-Scholes equity prices underlying equity returns in our paper, they assume a more unfavorable return specification, which leads to higher annuity demand in their model.

The potential gains from annuitized equity investments, in the form of variable annuities, have also been the subject of various recent studies. See for example the recent articles by Horneff, Maurer, Mitchell and Stamos [21] and Horneff, Maurer, Mitchell and Rogalla [20]).

A comparison of the benefits of self-annuitization via equity markets and regular annuitization is conducted by Albrecht and Maurer [2]. They find that self annuitization carries a high risk of consumption shortfall, thus supporting the results in this paper, that while equity investments lead to less than full annuitization in the optimum, they are far from fully replacing life annuities as a protection against longevity risk.

This paper introduces the notion of hyperbolic discounting into to the general problem of the optimal annuity and insurance endowment in the retirement phase of the life cycle. There are already various studies of the effects of hyperbolic discounting in the lifecycle on consumption and asset allocation. Laibson [33] finds that hyperbolic discounting can explain general overspending and in particular undersaving for retirement. Diamond [16] studies the effect of quasi-hyperbolic discounting on consumption paths and saving for retirement. Schreiber and Weber [**?**] analyze the effect that hyperbolic discounting has on the decision wether to take a lump sum or to annuitize accumulated saving upon entry to retirement, suggesting that time inconsistent discounting may explain low annuitization rates.

The benchmark household in our paper is wealthy enough that claiming social security benefits can be avoided altogether. However in a further analysis, we also study optimal insurance and annuity endowments for poorer households. In that case there is strong interaction between voluntary annuitization and government subsidies, which also perform the role of a longevity insurance. The potential effects of such subsidies are subject to a variety of articles. Hubbard [26] for example, finds a significant impact of anticipated social security in form of a government subsidy on the household portfolio allocation. Rust and Phelan [44] study the retirement behavior of poorer households and find that their lack of access to fairly priced annuitites, as well as incentives arising from social security, can explain under-annuitization among less wealthy households. In a life-cycle analysis, Benitez-Silva [4] finds that the presence of social security can have a crucial impact on private annuitization rates. Caliendo, Guo and Hosseini [12] study the question wether social security is a substitute for anuity markets and find that social security may crowd out bequests.

2 The Model

2.1 The agent's decision problem

We assume an exemplary agent who, at age 65 (time $t = 0$) enters retirement with savings of size $W > 0$ at his disposal. The agent may reach a maximum age of 100 years (time $T = 35$) but may also deceases at a prior time. Without a stream of labor income, the retiree depends entirely on his accumulated savings W or returns generated from investing parts of it to finance his future spendings. Aside from consumption by himself, the agent may also transfer parts of his wealth to a heir in form of a bequest at the time of his death.

We assume that there are two general classes of assets available to finance his future consumption. Initially, i.e. at time $t = 0$, the agent has access to the following types of insurance products:

- An actuarially fair priced life annuity that pays a fixed amount A at the beginning of every year under the condition that the agent is still alive at that time. Furthermore, the agent has the option to fix a number N_A of periods in which the annuity will be paid out regardless of the survival state of the agent. Here we formally assume that the certainty equivalent of the outstanding annuities with certain payoff will be added to the bequest in case of a premature death of the agent.

- An actuarially fair priced life insurance policy paying a fixed amount a at the time of death of the agent. This amount will be added to the bequest left to the heir.

Both types of insurance are assumed to be illiquid[1] and can only be purchased once at the beginning of the planning horizon. We represent the agent's insurance endowment by the triple $\mathbb{A} = (A, N_A, a)$. In a later section we relax the assumption that prices are fair and study the demand for annuities and life insurance in the presence of a loading on the prices. For the remainder of this section, we assume that a fixed endowment $\mathbb{A}$ has been chosen and outline the dynamic consumption/investment problem the agent faces throughout his retirement under this endowment. We then deduce the agent's optimal behavior given $\mathbb{A}$ and thereby the value of the endowment according to the preferences we specify below. Comparing the values for different endowments then gives us the agent's optimal longevity insurance investments. To simplify the notation we omit indicating the fixed endowment $\mathbb{A}$ in the following.

In $t = 0$, after the insurance endowment has been selected and then subsequently, at the beginning of every year (time $t = 1, 2 \dots, T$), the investor chooses his annual consumption level C_t and allocates the rest of his cash on hand between a riskless bond account, paying a fixed interest rate R_f, and a stylized stock investment, paying a risky return R_t. We let $0 \leq c_t \leq 1$ denote the fraction of cash on hand that is consumed at time t and $0 \leq \theta_t \leq 1$ the fraction of the remaining cash that is invested in the stock account, implying that $(1 - \theta_t)$ of the remaining cash will be invested in the bond account. At no other point in time may the agent interfere with either consumption or investment policy. Therefore his decision process, with the exception of the initial insurance endowment $\mathbb{A}$, is described by the pair $\gamma_0 = (c_t, \theta_t)_{t \in [0,T]}$. The restriction to the unit intervals for both policy parameters ensure firstly, that consumption may not exceed current wealth, i.e. that $C_t \leq W_t$ holds, and secondly implies a no short selling as well as a no borrowing constraint. We further assume that there is a base level of consumption $\underline{C} \leq C_t$ that the agent needs to maintain at any year. If his wealth on hand, and thus his consumption level, falls below this threshold, the agent receives a government subsidy to bring his consumption up to the minimum level $\underline{C}$. When this happens the agent is forced to consume his whole wealth at once and is therefore reliant on the subsidy for the rest of his life. To make an agent contribute to his cost of living to as much an extent as possible, long term government subsidies usually demand liquidation of any of the agent's assets, even at very low prices. We thus assume that any pending life insurance payoff, as well as any pending annuities with outstanding periods with certain payoff, have to be sold back to the insurer when receiving the subsidy. While this may violate the nonliquidity of these assets, we further assume that the general costs, and or legal regulations, prevent the agent from selling off insurance contracts at any other times[2].

The agent's budget restriction, as implied by the above considerations, is therefore

$$W_{t+1}^L = \max\{W_t(1 - c_t)(\theta_t(1 + R_{t+1}) + (1 - \theta_t)(1 + R_f)) + A, \underline{C}\} \tag{1}$$

if the agent is alive at time $t + 1$ and

$$W_{t+1}^D = W_t(1 - c_t)(\theta_t(1 + R_{t+1}) + (1 - \theta_t)(1 + R_f)) + a \tag{2}$$

if the agent deceases during period $t+1$, has not yet received a government subsidy in the previous periods and there are no outstanding annuities from periods with a certain payoff. If there are

[1] Generally any insurance policy with a certain payoff, such as the life insurance above, may be subject to repurchase. However this usually involves significant costs. We later weaken the illiquidity assumption under specific circumstances, yet the agent will never be allowed to actively terminate the contract by himself.

[2] It should be noted that the effects of a government subsidy are only significant for an agent with a fairly low initial wealth W. The agent in our benchmark case optimally chooses an annuity that is well above subsidy level. This makes receiving a subsidy impossible for the agent as long as bankruptcy of the insurer is excluded. It does however have significant effects on the annuitization decision of poorer households as we show later.

such outstanding annuities, then his bequest size cannot be calculated as the wealth on hand at time $t+1$, since future cash flows and possible returns from investing parts of these chash flows between the time of death and the last annuity payment have to be taken into account. Therefore in that case we set

$$W_{t+1}^D = \mathrm{CE}_A^{N_A - t} \tag{3}$$

where CE_A^s denotes the certainty equivalent of the s outstanding annuities of size A in addition to the current wealth at the time of death as given by equation (2). We give an exact specification of CE_A^s after the utility functions for consumption and bequest have been specified.

If the agent has already received a government subsidy at time t, his total wealth will be zero and any pending life insurance policy contracts cancelled. Thus there are no remaining assets to bequeath, i.e.

$$W_{t+1}^D = 0. \tag{4}$$

The inital value of the agent's wealth process, $W_0^L = W_0$, is determined by the agent's wealth after purchasing the insurance endowment $\mathbb{A}$ in $t = 0$. Specificly we have

$$W_0 = W - P_{A,N_A} - P_a + A \tag{5}$$

where P_{A,N_A} is the price of an annuity of size A with N_A periods with certain payoff and P_a is the price of a life insurance policy paying the amount a at the time of death of the agent.

Before we introduce the notion of hyperbolic discounting, we specify the benchmark model with exponential discounting. In the most general form, time discount factors are written as

$$DF_t = (1+i)^{-\alpha(t)}. \tag{6}$$

where the function α describes the agent's perception of time. Applying the identity function $\alpha(t) = t$ results in the exponential discount factors $DF_t = (1+i)^{-t} = \beta^t$ where $\beta = \frac{1}{1+i}$. In this case the discount factors $DF_t = \beta^t$ are time consistent, i.e. the multiplicative property

$$DF_{s+t} = DF_s \cdot DF_t \tag{7}$$

holds for all $s < t$[3].

Aside from the incorporation of hyperbolic discounting, which will be discussed in detail below, our preference framework consists of a time-additive expected utility model. At any time $t = 0, 1, 2, \ldots, T$ the retiree, provided that he is still alive at that time, receives utility from consumption, $u(C_t) = u(c_t W_t^L)$. At the time of his death he receives a final utility from bequest, $v(W_t^D)$. For an agent, who is alive at time t, we let p_t denote the probability of surviviving the period $t+1$. Furthermore we let $p_{0,t} = \prod_{s=0}^t p_s$ denote the unconditional[4] probability that the investor is alive at time t. Under the above assumptions the agent's preferences in $t = 0$, under the fixed annuity and insurance endowment $\mathbb{A}$, are described by the functional $\Phi_0 \colon [0,\infty) \times \Gamma_0 \mapsto \mathbb{R}$ with

$$\Phi_0(W, \gamma_0) = E_0 \left[\sum_{t=0}^{T} p_{0,t-1} \beta^t \left(p_t u(c_t W_t^L) + (1-p_t) v(W_t^D) \right) \right]. \tag{8}$$

Analogously, we can consider the investor's preferences (re-)started at any later time $t > 0$ for some wealth level W_t. To this end we let $\gamma_t = (c_s, \theta_s)_{s=t,t+1,\ldots,T}$ denote the agent's decision process started at t, with c_s and θ_s following the constraints formulated above for all $s = t, t+1, \ldots, T$.

[3] Note that the functional equation $f(t+s) = f(t) \cdot f(s)$ with $f \neq 0$ is uniquely solved by the family of exponential functions $f(t) = e^{u \cdot t}$.

[4] The probability is unconditional with respect to our model. In reality, $p_{0,t}$ is the probabilty that a male individual who has reached age 65 will reache age $65 + t$.

The agent's preference functional for the remaining time optimization problem at time t is then given by[5]

$$\Phi_t(W_t, \gamma_t) = E_t\left[\sum_{s=t}^{T} p_{0,s-1}\beta^{s-t}\left(p_s u(c_s W_s^L) + (1-p_s)v(W_s^D)\right)\right]. \tag{9}$$

An agent whose wealth once falls below the subsistence threshold $W_t < \underline{C}$, is forced by the above constraint to consume the base level subsidy consumption in each period, i.e. $C_t = \underline{C}$ and thus $c_t = 1$. In turn $(1-c_t)\theta_t = 0$, which means the retiree looses control over both his consumption and his investment policy[6]. As a consequence the retiree's preferences are constant on $W_t < \underline{C}$ which imposes the lower bound

$$\Phi_t(W_t, \gamma_t) \geq \Phi_t(\underline{C}) \tag{10}$$

for all $t = 0, 1, \ldots, T$ and $W_t \geq 0$ on the agent's time t preferences.

Let U_t denote the set of time t decision policies γ_t, which are admissible according to the previously established constraints. The investor's value function is defined as

$$V_t(W_t) = \sup_{\gamma_t \in U_t} \Phi_t(W_t, \gamma_t) \tag{11}$$

For $W_t > \underline{C}$, by lemma A.2 ii), the value function satisfies the Bellmann equation in the following form

$$V_t(W_t) = \sup_{\substack{0 \leq c, \theta \leq 1 \\ cW_t \geq \underline{C}}} \left\{ u(cW_t) + p_t\beta E_t\left[V_{t+1}(W_{t+1}^L)\right] + (1-p_t)\beta E_t\left[v(W_{t+1}^D)\right]\right\}. \tag{12}$$

In the subsidy case $W_t \leq \underline{C}$, the value function is equal to the lower bound from (10)

$$V_t(W_t) = \Phi_t(\underline{C}). \tag{13}$$

The time $t = T$, due to $p_T = 0$, marks the endpoint in the planning horizon and simultaneously provides a terminal condition for the difference equation imposed in (12)

$$V_T(W_T) = \sup_{\substack{0 \leq c, \theta \leq 1 \\ cW_T \geq \underline{C}}} \left\{ u(cW_T) + \beta E_t\left[v(W_{T+1}^D)\right]\right\}. \tag{14}$$

With the benchmark model established we now turn to the specification with hyperbolic discounting. The latter is a form of inconsistent subjective time evaluation. A consistent time evaluation implies that the relative importance that is assigned to incidents at two points in time only depends on the distance between those two points. Considering trade-offs between consumption and saving between this year and the next, would result in the same preference as the trade-off between consumption and saving twenty years and twenty-one years in the future. This is precisely what equation (7) entails. As evidence by Thaler [46] suggests, some economic agent's subjective perception of the value of future payoffs may violate this principle. A famous example of such a violation are probant's answers to the hypothetical questions

- Choose between one apple today (A1) and two apples tomorrow (A2)
- Choose bewteen one apple in a year (B1) and two apples in a year and one day (B2).

[5]From a technical standpoint, the processes $(W_s^L)_{s\geq t}$ and $(W_s^D)_{s\geq t}$ occuring on the right side of equation (35) are stochastic processes, that follow the dynamics described by the budget constraints (1), (2) and (4), started at time t with the initial value W_t and controlled by the policy process γ_t. In the literature on control theory, a superscript in the form of $(W_s^{(t,W_t,\gamma_t)})_{s\geq t}$ is often added as an indication. For simplicity we omit any such notation throughout this paper.

[6]More precisely we then have $U_t = \{(1,0)\}$, i.e. the set of feasible decision parameters collapses to a point.

Here A1 is frequently preferred over A2, yet simultaneously B2 is preferred over B1. Since the relative importance of future periods is a crucial factor in multi-period decision problems, the question arises, if and to what extend, time inconsistent discounting influences the demand for longevity and bequest insurance.

In this paper we adopt Harvey's[7] approach [19] on time inconsistent, or more specificly hyperbolic discounting, where

$$\alpha(t) = \frac{r}{\ln(1+i)} \ln(1+t) \tag{15}$$

which results in the discount factors

$$DF_t = (1+t)^{-r}. \tag{16}$$

A comparison of the weight assigned to the individual periods with exponential and hyperbolic discounting is displayed in figure 1. The two most striking effects of hyperbolic discounting are first, that earlier periods receive a lower weight while later periods, here starting at age 81, receive a higher weight than in the exponential discounting counterpart. The second effect is the time inconsistency itself. Whereas the relative importance of two succeeding periods always equals $\beta = .96$ for the exponential discounter, the hyperbolic discounter's relative weight of a succeeding period compared to the previous period, from the point of view of $t = 0$, increases over time. In our parametrization of hyperbolic discounting, the second period receives a relative importance of .8581 compared to the first period, whereas the last period receives a relative importance of .9940 compared to the second to last period.

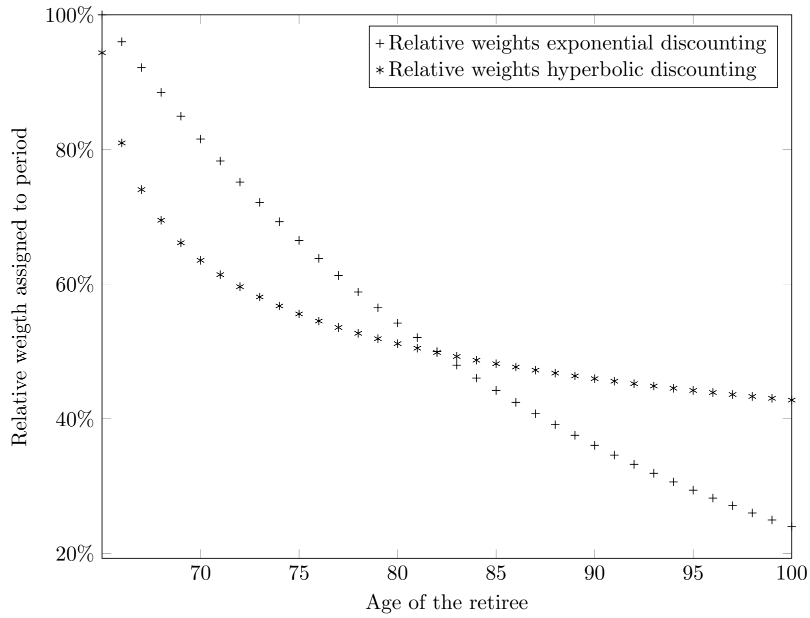

Figure 1: *Exponential versus Hyperbolic Discount Factors - agent's age plotted versus subjective relative time weighting of that period. For true comparability the values contained in the figure are not the discount factors* DF_t *applied in the model but the normalized versions* $DF_t(\sum_{t=0}^{T+1} DF_t)^{-1}$. *100% on the y-axis represents the value of the normalized weight of the first period for the exponential discounter.*

To derive the Bellmann equation for the hyperbolic discounter, the retiree's preference functional $\Phi_0(\cdot,\cdot)$ must be decomposable in an analoguos manner as above. This requires a multiplica-

[7] It should be noted that there are various approaches to hyperbolic or more general forms of discounting. However Abdellaoui, Attema and Bleichrodt [1] find that Harvey's model yields the best fit to their data.

tive property in the form of (7) to hold. Therefore we decompose the hyperbolic discount factors DF_t in the following way for $s \geq t$

$$DF_s = (1+s)^{-r} = (1+t)^{-r} \cdot \left(\frac{(1+s)^{-r}}{(1+t)^{-r}}\right) = (1+t)^{-r} \cdot \left(\frac{1+s}{1+t}\right)^{-r} = DF_t \cdot \left(\frac{1+s}{1+t}\right)^{-r}. \quad (17)$$

This leads us to the artificial intertemporal discount factors[8] from time t to time $s \geq t$

$$DF_{t,s} = \left(\frac{1+s}{1+t}\right)^{-r} \quad (18)$$

which satisfy

$$DF_s = DF_t \cdot DF_{t,s}. \quad (19)$$

By Lemma A.2 i), as with the exponential discounting agent, we can now construct a difference equation for the family Φ_t. The problem's Bellmann equation for wealth levels $W_t \geq \underline{C}$ then takes the form

$$V_t(W_t) = \sup_{\substack{0 \leq c,\theta \leq 1 \\ cW_t \geq \underline{C}}} \left\{ u(cW_t) + p_t DF_{t,t+1} E_t \left[V_{t+1}(W_{t+1}^L) \right] + (1-p_t) DF_{t,t+1} E_t \left[v(W_{t+1}^D) \right] \right\} \quad (20)$$

with the terminal condition

$$V_T(W_T) = \sup_{0 \leq c,\theta \leq 1} \left\{ u(cW_T) + DF_{T,T+1} E_t \left[v(W_{T+1}^D) \right] \right\}. \quad (21)$$

For $W_t \leq \underline{C}$, the problem once again reduces to

$$V_t(W_t) = \Phi_t(\underline{C}). \quad (22)$$

2.2 Utility specifications

There are various utility specifications that could be applied to the general model presented above. In this paper we follow the choices most encountered in the relevant literature.

Regarding the agent's utility from consumption, we assume that it is given by the power utility specification

$$u(x) = \frac{1}{1-\gamma} x^{1-\gamma}. \quad (23)$$

To model the agent's utility from bequest we follow De Nardi's approach [15] where a bequest of size b yields the utility

$$v(b) = \frac{\omega}{1-\gamma} \left(\psi + \frac{b}{\omega}\right)^{1-\gamma}. \quad (24)$$

Here ω represents the individual's desire to leave a bequest, the strength of the bequest motive, and ψ can be referred to as the degree to which bequests are a luxury good[9]. For $\psi > 0$ the retiree will only leave a bequest if his wealth allows a bequest of sufficient size in addition to his own consumption. This has the consequence that a retiree in low wealth states shifts his priority away

[8]Note that these artificial factors are not the discount factors the investor would apply in time $t \geq 0$.

[9]To illustrate the function of the two parameters in v it is best to regard the simpler problem with no time weighting, a fixed wealth level W, a fixed lifetime of T years and no government subsidy. The optimal consumption resulting from first order conditions is then $C = (W + \omega\psi)/(\omega + T)$ and the optimal bequest is $B = \omega(C - \psi)^+$, i.e. the bequest covers ω periods of spending the amount $C - \psi$ or the amount the agent's own consumption exceeds ψ. If the agent cannot bequeathe an amount which exceeds ψ for ω years then the optimal bequest is zero.

from bequest to his own consumption and is thus less risk-averse over bequests than consumption.

Because an agent who receives a subsidy is forced to consume all of his available wealth in every period, welfare recipients invariably leave a bequest of size zero. Therefore they receive the minimum bequest utility

$$v(0) = \frac{\omega}{1-\gamma}\psi^{1-\gamma}. \tag{25}$$

If there are outstanding annuities with certain payoff, we set the bequest B to equal the certainty equivalent of these outstanding payments in addition to the current wealth of the retiree at the time of his death. Taking into account possible investment returns until the last annuity is paid out, we define the certainty equivalent $CE_A^{t,s}$ of s outstanding annuities of size A at time t to be[10]

$$CE_A^{t,s} = v^{-1}\left(DF_{t,t+s-1} \sup_{\gamma'_{s-1}\in U'_{s-1}} E\Big[v(W_{s-1})\Big]\right) \tag{26}$$

where W_{s-1} denotes the resulting wealth from optimally investing the initial wealth according to (2) at the time of death of the agent and then subsequently receiving and investing the pending annuities. Here U'_s denotes the set of all admissible s-periods investment strategies γ'_s following the same constraints as the agent's investment policies. In the case of only one outstanding annuity payable at time t the formula reduces to

$$CE_A^{t,1} = W_t^D + A \tag{27}$$

where W^D is the bequest size in the absence of outstanding periods given by equation (2). We refer to appendix A.2 for a detailled treatment of the optimization problem contained in (26).

The lower bound on the agent's preferences, given by equation (10) together with the specific utility specification presented above, allows us to analytically compute a lower bound for the agent's value function. This proves useful in the numeric solution algorithm. For the problem with exponential discounting it holds according to lemma A.1 that

$$V_t(W) \geq V_t(\underline{C}) = \sum_{s=t}^{T} p_{t,s}(1-p_s)\left(\frac{\beta^{t-s+1}-1}{\beta-1}u(\underline{C}) + \beta^{t-s+1}v(0)\right) \tag{28}$$

for all t and $W \geq 0$. A similiar bound without explicit solution exists for the problem with hyperbolic discounting.

2.3 Insurance Pricing

We assume that the annuity as well as the life insurance policy are both actuarilly fair priced and we omit a loading in the premium calculation, i.e. the prices are given by the expected discounted payoffs of both financial contracts. Hence we have

$$P_{A,N_A} = A\left(\sum_{t=0}^{N_A-1}(1+R_A)^{-t} + \sum_{t=N_A}^{T} p_{0,t}(1+R_A)^{-t}\right) \tag{29}$$

for an annuity of size A with N_A periods with certain payoff and

$$P_a = a\sum_{t=1}^{T+1} p_{0,t}(1-p_{t-1})(1+R_A)^{-t} \tag{30}$$

for a life insurance with payoff of size a at the time of death of the agent. We assume the calculatory interest rate R_A to be slightly higher than the riskfree interest rate R_F. Throughout this paper

[10] v is strictly concave and thus invertible on $[0,\infty)$ with $v^{-1}(x) = \omega\left\{\left(x\cdot\frac{1-\gamma}{\omega}\right)^{\frac{1}{1-\gamma}} - \psi\right\}$.

we let the agent's individual survival probabilities coincide with those in the agent's cohort, which form the basis of the above pricing formulas. The issue of adverse selection in the annuity market is therefore not captured in our model.

2.4 Return specifications and Parameter Choice

The parameter choices in this paper regarding the life cycle model underlying our approach reflect the choices most encountered in the literature. We refer for example to Cocco and Gomes [13] for a standard life cycle model. We assume that the returns R_t of the risky stock investment are independently and identically lognormally distributed with parameters μ and σ such that the expected return is $E[R_t] = 8\%$ with a standard deviation of $\sigma(R_t) = 20\%$. This corresponds to a standard Black-Scholes economy where asset prices are driven by the dynamics

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{31}$$

with the initial value $S_0 = 1$. The riskfree interest rate R_f is set to 2%, implying a risk premium of 6%.

As in Brown and Poterba [11] the agent enters retirement at age 65. The maximum age is set to 100 years. The agent's relative risk aversion parameter γ is set to 5. Our parametrization of the two additional parameters in the bequest utility function is based on De Nardi [15][11]. We follow Peijnenburg, Nijman and Werker [42] and set the luxury parameter in relation to the retiree's fully annuitized income, the FAI, specified below. The resulting parameters are $\omega = 7.81$ and $\psi = 0.67 * FAI$. The constant time discounting parameter in the exponential discounting is set to $\beta = 0.96$. The hyperbolic discounting parameter is calibrated such that period 15 receives the same weight as in the specification with time consistent discounting, following Schreiber and Weber's parametrization [45].

The survival probabilities used in the pricing of the insurance products as well as the agent's individual survival probabilities are both taken from german death tables[12] and we assume a male policyholder. The agent's initial wealth is choosen such that full annuitization, i.e. investing the complete initial wealth W in the annuity with zero periods with certain payoff and no life insurance, results in a yearly annuity in advance of $FAI = 25000$.

3 Numerical Solution Technique

Given a fixed insurance endowment (A, N, a) we solve the agent's optimal consumption and investment problem via backwards induction by means of the bellmann equations (12) and (20). The terminal conditions in equations (44) and (21) allow us to compute V_T. Given the value function V_t at any time t, the Bellmann equations allow us to compute V_{t-1} and subsequently we obtain V_0, as well as the optimal consumption and investment policies (c_t, θ_t) along the time horizon $t = 0, \ldots, T$.

To calculate V_t in any time step we proceed as follows. We define a dynamic time dependant grid $\{w_l\}_{l=0,\ldots,L_t}$ for the endogenous variable W_t^L. Starting at time T we compute $V_T(w_l)$, $c_T(w_l)$ and $\theta_T(w_l)$ for all $l = 1, \ldots, L_T$ by solving the terminal conditions (44) or (21). The affine nature of the budget constraint and the bequest utility prevent the typical separation of consumption and investment problem and we resort to a numerical solution technique to obtain the optimal policies. In the subsequent steps $t = T-1, ..., 0$ we use cubic spline interpolation on the grid $(w_l, V_{t+1}(w_l))$

[11] Note that De Nardi's original formulation reads $v(b) = \psi_1 \left(1 + \frac{b}{\psi_2}\right)^{1-\sigma}$ with $\psi_1 = -9.5$ and $\psi_2 = 11.6$. In contrast to De Nardi where the relative risk aversion is set to $\sigma = 1.5$ we set $\gamma = 5$ to keep the risk aversion parameter consistent with the utility from consumption.

[12] Source: Sterbetafel 2009/11 Deutschland männlich, Periodensterbetafeln für Deutschland 2009/2011, Statistisches Bundesamt, Wiesbaden 2012.

to calculate the value of $V_{t+1}(w)$ for arbitrary w. A slight modification is necessary to capture the piecewise constant nature of V_{t+1} for wealth levels below the subsidiary threshold. Inherited from the curvature of the utility functions u and v the value function V_t is, with an exception at the right side of the interval $[0, \underline{C}]$ where it is constant, strictly concave with diminishing curvature for higher wealth levels. As the approximation error for spline interpolation depends on the curvature between sample points we choose an exponentially growing base wealth grid $\{w_l\}_{l=0,\ldots,L_0}$ with $w_0 = \underline{C}$ and $L_0 = 30$. In addition to that we let the wealth grid grow linearly in each time step by adding a new grid point $w_{L_{t+1}} = w_{L_t} + \Delta_w$ in every time step to prevent extrapolation for values exceeding w_{L_t} while calculating V_{t+1}[13].

The conditional expectations, ocurring in each time step's optimization problem, are calculated using Gauss-Hermite-Quadrature (GH-Quadrature) with $n = 32$ sample points. To achieve an optimal efficency we follow Liu and Pierce [34] and apply a transformation of the standard weights and sample points in GH-Quadrature which accounts for the specific parametrization (μ, σ) of the lognormal returns in our model.

We conduct the above steps on a coarse grid of insurance endowments to obtain the location of the optimal endowment. A search algorithm gives us the optimal value on the coarse grid and thus a first estimator for the solution. In a second step we refine the insurance grid to the desired precision around the estimator obtained in the first step. A second search within the solutions on the fine grid gives us the optimal insurance endowment. Because the insurance endowment is three dimensional, choosing a fine grid in the first step results in very high number of optimization runs. The two step procedure avoids this curse of dimensionality.

The computational method applied here, to obtain the value function on the coarse grid, is a typical numerical procedure in dynamic portfolio optimization. Further refinement of the number of grid points in the various approximations does not change our results at the reported precision. Similiar methods are applied by Horneff, Maurer and Stamos ([23] and [22]).

4 Results

4.1 Optimal insurance endowments

We present the optimal insurance endowments in table 1. The table contains the results for the model specifications with exponential discounting, as well as hyperbolic discounting. Besides the results in the benchmark bequest utiltiy specification, the table also contains the results for the models in which the strength of the bequest motive and the luxury parameter are increased and decreased by 50%, as well as a no bequest motive case.

The two main effects of hyperbolic discounting are: a) it increases the annuitization degree. In the parameter constellations considered in our analysis the hyperbolic discounter's annuitization degree increases by between 2.40 and 3.80 percentage points. b) Hyperbolic discounting decreases the demand for bequest insurance. In a constellation with the luxury parameter lowered by 50%, we find that the hyperbolic discounter invests 3.28% less in direct bequest insurance. When the strength parameter is increased by 50%, in addition to the decrease in the luxury parameter, the hyperbolic discounter invests 4.12% less in the life insurance policy. All other parameter constellations, especially the benchmark bequest motive specification, do not result in positive demand for bequest insurance. Contrary to their popularity among retirees, guarantee periods are never optimal in the cases considered here.

[13] Possible extrapolation on the left side of the wealth grid is not an issue since the value function is constant and equal to $\Phi_{t+1}(\underline{C})$ for all $W_{t+1} < \underline{C}$.

	Optimal Insurance Endowment			
	Annuity Size	Guarantee Periods	Life Pay-off	Total fraction of wealth in insurance
Exponential Discounting				
$\omega = 7,81, \psi = 0,67$	19800 (.7920)	1	0 (.0000)	.7920
$\omega = 3,91, \psi = 0,67$	21075 (.8430)	1	0 (.0000)	.8430
$\omega = 11,7, \psi = 0,67$	18900 (.7560)	1	0 (.0000)	.7560
$\omega = 7,81, \psi = 0,34$	18575 (.7430)	1	47235 (.0610)	.8040
$\omega = 7,81, \psi = 1.01$	21000 (.8400)	1	0 (.0000)	.8400
$\omega = 11,7, \psi = 0,34$	16800 (.6720)	1	75111 (.0970)	.7690
No bequest motive	22750 (.9100)	1	0 (.0000)	.9100
Hyperbolic Discounting				
$\omega = 7,81, \psi = 0,67$	20750 (.8300)	1	0 (.0000)	.8300
$\omega = 3,91, \psi = 0,67$	21750 (.8700)	1	0 (.0000)	.8700
$\omega = 11,71, \psi = 0,67$	19750 (.7900)	1	0 (.0000)	.7900
$\omega = 7,81, \psi = 0,34$	19250 (.7700)	1	45686 (.0590)	.8290
$\omega = 7,81, \psi = 1.01$	21550 (.8620)	1	0 (.0000)	.8620
$\omega = 11,7, \psi = 0,34$	17725 (.7090)	1	72013 (.0930)	.8020
No bequest motive	23350 (.9340)	1	0 (.0000)	.9340

Table 1: *Optimal insurance endowments for varying bequest motive parameterizations. The numbers in brackets indicate the fractions of the initial wealth invested in the annuity and the life insurance. The fraction in the Annuitized Wealth Column does include the price adjustment for guarantee periods. The poorer household starts with only* 50% *of the wealthier household's initial wealth* W_0.

The strength parameter ω controls the relative importance of bequest with respect to annual consumption. In the case of $\psi = 0$, and in the absence of time discounting, saving effects and uncertainty, a retiree would bequeathe ω times his annual consumption. In the benchmark bequest motive specification, the difference in the optimal annuitization degrees for exponential and hyperbolic discounting is 3.80%. This is the largest discrepancy between annuitization rates in the cases considered in our analysis. When the strength of the bequest motive is reduced by 50%, the annuitization rates increase to 84.30% with exponential discounting, and to 87.00% with hyperbolic discounting. This means the difference between annuitization rates decreases to 2.70%. In the complete absence of a bequest motive, i.e. $\omega = 0$ in our model, the annuity demand in the optimum further increases to 91.00% with exponential discounting, and to 93.40% with hyperbolic discounting. This is a further decrease in the difference between the annuitization rates to 2.40%. With neither a bequest motive nor other potential events that cause demand for liquid wealth, such as medical expenses or other forms of background risk, the only factor that reduces the demand for annuities is the relatively high equity risk premium and the lack of an annuitzed equity investment in our model. This lack in the insurance menu in our model may be realistic as it is in accordance with the finding in Milevsky and Young [38], that the little variety of annuity forms that are actually available on the market can be a limiting factor in annuity demand. Even in the presence of a bequest motive, the access to equity markets holds an incentive to annuitize less. Figure 5 contains a comparison that shows the utility gains from annuitization for different risk premia specifications in the benchmark parametrization with exponential discounting. Even for the lower risk premium of 4%, which exceeds the calculatory interest rate r_A by only 2%, the optimal annuitization is less than 90%.

The luxury parameter ψ has a non-linear effect with respect to the retiree's bequest utility. In a simplified model, the retiree would bequeathe ω times the amount that his own annual consumption exceeds the threshold ψ, if he can afford it, and nothing otherwise. A higher ψ implies that the degree to which bequest is a luxury good increases. When ψ is lowered by 50% the degree to which bequest is a luxury good decreases. This implies that the demand for bequests increases when the retiree's wealth level remains constant. Furthermore the retiree's risk aversion[14] over bequest increases which in turn results in a demand for direct bequest insurance. The exponential discounter invests 6.10% of his initial wealth and the hyperbolic discounter 5.90% of his initial wealth in the life insurance. The purchases of bequest insurance are offset by reduced degrees of annuitization. As a result the overall insurance endowment sizes do not differ much between the two time discounting specifications.

4.2 Optimal wealth, consumption, bequest and equity exposure

Figure 2 compares averaged optimal trajectories of the state and control variables wealth on hand, consumption, bequest and equity exposure for the models with exponential and hyperbolic discounting in the benchmark bequest motive parametrization. It shows that intertemporal variation in consumption increases for the hyperbolic discounter. More precisely, consumption is lower in the early periods than in the exponential discounting case and higher in the later periods. The same relationships holds for bequests. An early death leads to smaller bequest whereas bequests left in the late retirement years are bigger. This is a result of the relative underweighting of early periods and relative overweighting of late periods[15]. In the presence of a sufficiently strong bequest motive there is an incentive to preserve and even increase wealth on hand over parts of the retirement phase. Togehter with the consumption behaviour this leads to a symmetrical hump shape in the wealth trajectory of the exponential discounter. For the hyperbolic discounter this trajectory is tilted to the left and negatively skewed. This relation exists with all the displayed processes.

The remainder of this section contains a brief outline of the main findings regarding the retiree's optimal behaviour followed by an in-depth analysis of the individual paths. Alongside the benchmark bequest motive parametrization ($\omega = 7,81, \psi = 0,67$) we also analzye the results for the specification with a stronger bequest motive in form of a decreased luxury parameter ($\omega = 7,81, \psi = 0,34$) and for the no bequest motive case ($\omega = 0$).

Our main finding is that, in the two cases with a bequest motive, hyperbolic discounting leads to higher bequests at the expense of consumption, especially in the early retirement years. Even though late life consumption is higher for the hyperbolic discounter, the disproportional decrease in the early retirement periods leads to a decrease in total. This behaviour can be attributed to the fact that, at the time of retirement the hyperbolic discounter assigns a relatively low weight to the early periods, which results in lower actual consumption in those years. On the other hand higher savings and disproportionally higher equity exposure in the later years lead to higher bequest sizes for the hyperbolic discounter. The trajectories in the absense of a bequest motive differ significantly from the other two specifications. In this case consumption is generally higher at the expenses of bequests. The average consumption paths remain stable on a high level compared with the bequest motive cases in the first half of retirement, and then slowly decline in the second half as the retirees' funds run out. In contrast to the bequest motive cases, on average the hyperbolic discounter has a lower intertemporal variation in consumption than the exponential discounter.

We turn back to the benchmark bequest motive and analyze the case of the exponential discounter first. His average wealth on hand trajectory forms a hump shape. The maximum average wealth level $(10, 639)$ is reached at age 81. Disregarding the bequest motive, the annuity acts as a riskless bond investment from the perspective of portfolio optimization. The relatively high annuitization degree means that the agent's total asset allocation, that is wealth on hand and capital bound in the annuity, is heavily focused on a riskless asset class. Wealth on hand is therefore

[14]The relative risk aversion over bequests is $\frac{-bv''(b)}{v'(b)} = \frac{\gamma}{\omega} \frac{b}{(\psi + \frac{b}{\omega})}$.

[15]Compare figure 1 which contains a comparison of the discount factors in both models.

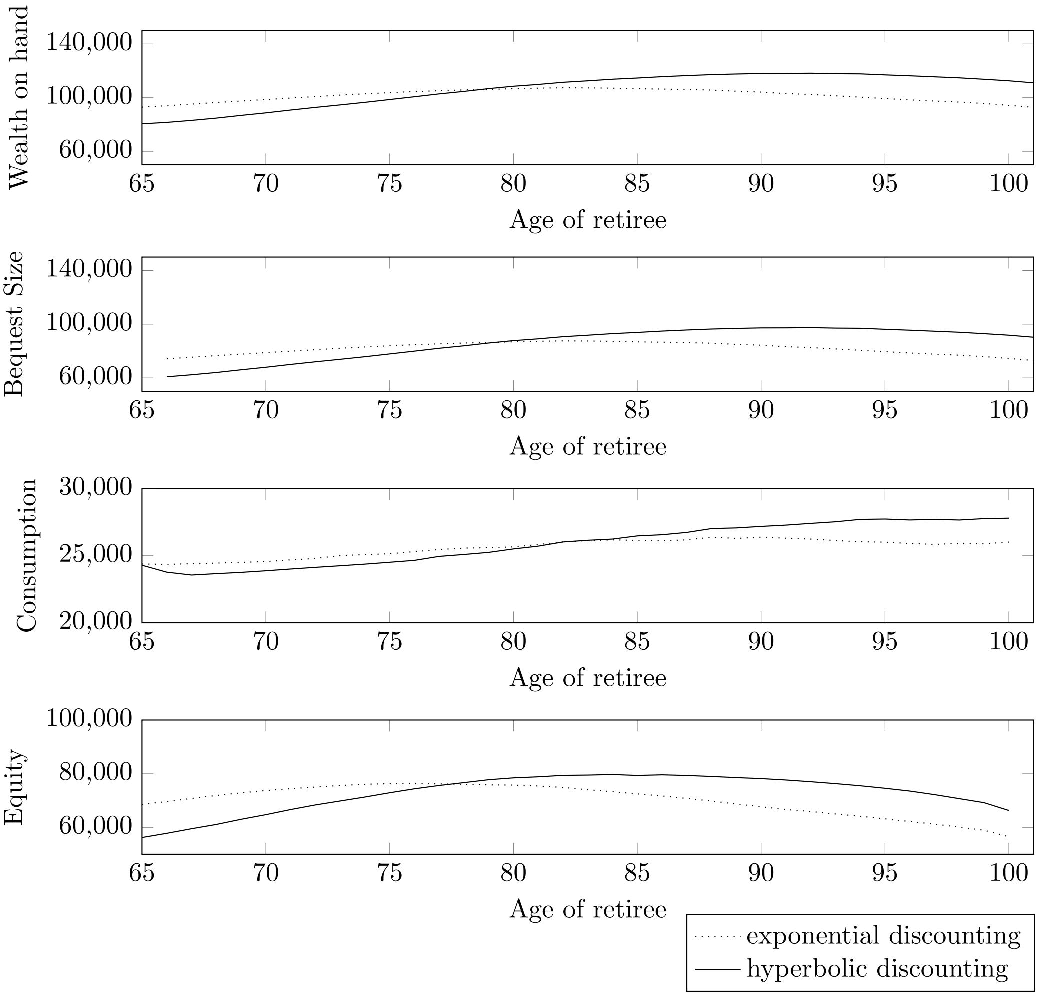

Figure 2: *Averaged trajectories for wealth, bequest, consumption and stock investment calculated from* $N = 10000$ *forward simulations in the benchmark bequest parametrization for both time discounting specifications. The optimal annuity endowment is* 79.20% *of the initial wealth for the exponential discounter and* 83.00% *for the hyperbolic discounter. Both agents abstain from both forms of bequest insurance.*

unusually reliant on risky equity investments to compensate for this deviation from the optimal portfolio composition. On average, the retiree invests 69.05% of his liquid wealth in equity markets. This leads to high standard deviations for the wealth on hand. Disregarding the first period where wealth levels are fixed, the mean standard deviation in any later period is equal to 49.08% of the average wealth level at that period. However due to the high degree of annuitization, the consumption paths are less volatile. Ignoring premature death and again excluding the first period, the mean standard deviation of annual consumption amounts to only 17.51% of the average consumption level $\bar{C}$. The latter is equal to $\bar{C} = 25,535$ which exceeds the fully annuitized income by $2,14\%$. Average consumption is slightly lower than the fully annuitized income in the first 9 years of retirement, however these deviations do not execeed a shortfall of 3%. Furthermore the average consumption level that is actually experienced by the retiree, i.e.

$$\sum_{\tau=0}^{T} p_{0,t}(1 - p_{t+1}) E\left[\frac{1}{\tau}\sum_{t=0}^{\tau} C_t\right] \tag{32}$$

is equal to $25,113$ and thus below $\bar{C}$ but still above the fully annuitized income. The consumption smoothing effect of the annuity is noticeable in the low intertemporal variation in average consumption. The standard deviation within the average consumption levels in all periods is only 2.61% (665.29).

On average the retiree dies at age 85 and leaves a bequest which amounts to 3.31 times the average experienced consumption, or 83,102 in total. In the simulation bequest sizes range from 11.52% (8,310) to 593.32% (493,060) of the average bequest size. The sample standard deviation is 68,18% (56,660) of the average bequest size.

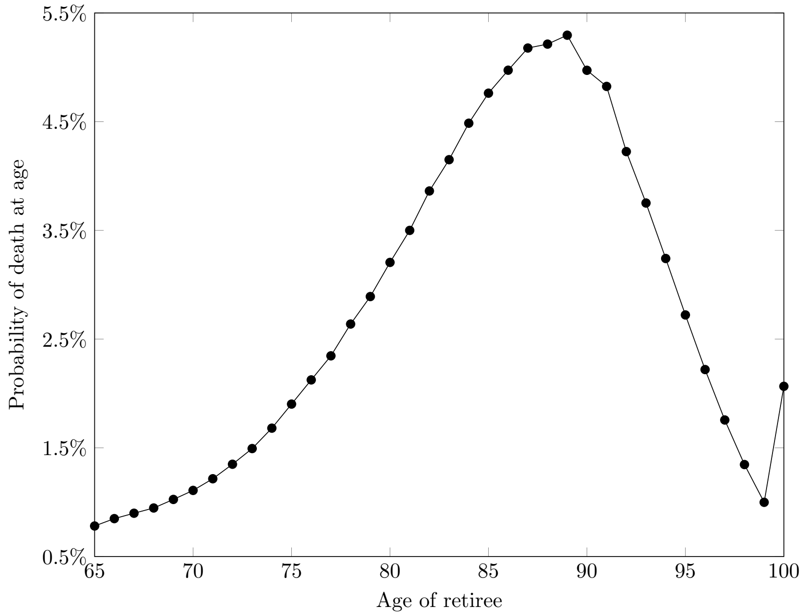

Figure 3: *Probability mass function (pmf) of time of death in our model. The spike at time T results from cutting of the tail of the actual distribution.*

The hyperbolic discounter chooses a slightly lower average consumption level in the first 16 periods. Noticeable is the immediate drop in the optimal consumption levels after the first period. This is a result of the sharp drop in the relative importance the hyperbolic discounter assigns to early periods, which is steepest in the first period. The average consumption level assuming no premature death, $\bar{C}$, is 25,883 which is an increase of 1.01% compared with the exponential discounter. This increase stems from a disproportional increase in consumption in the late retirement years. But because the retiree's survival until those late years is uncertain, the actually experienced average consumption is only 24,859 and thus lower than the exponential discounter's and the fully annuitized income. The intertemporal variation in consumption is 5.83% or 2.23 times that of the exponential discounter. A similiar imbalance is observable in the exposure to equity markets. Again average wealth on hand along to whole timeline is higher by 4.88%, even though the initial wealth on hand is lower. In turn, the average bequest size along to whole timeline also exceeds that of the exponential discounter by 5.93%. However in this case, the resulting average bequest size surpasses that of the exponential discounter by 9.43%. On average, the hyperbolic discounter leaves a bequest of 3.66 times his average experienced consumption. This is a relative increase of 10.57% compared to the model with time-consistent discounting. The standard deviation in total bequest size is very close to that of the exponential discounter. In relative terms the standard deviation decreases by 4.52 percentage points to 63.66% of the average bequest size.

The relationship between the two models remains qualitatively similiar when the luxury parameter is lowered and thus the demand for bequest is increased. In an exemplary parameter specification where the parameter ψ is lowered by 50%, both retirees decrease their annuitization degree and purchase direct bequest insurance[16]. In both cases the decrease in capital invested in the annuity offsets the capital invested in bequest insurance so that the total insurance endowment is very close to the benchmark endowment. There is no noticeable increase in equity exposure and cash on hand compared with the benchmark model. Because annuity payments are lower, the average consumption along the whole timeline is thus lower with $\bar{C} = 23,910$ for the exponential discounter and $\bar{C} = 24,330$ for the hyperbolic discounter, which results in an actually experienced average consumption of 23,408 in the former and 23,270 in the latter case. The relative difference between the two models in experienced average consumption is almost identical to the benchmark bequest motive. The intertemporal variation in consumption increases absolutely and relatively in both cases to 3.03% for the exponential discounter and 6.27% for the hyperbolic discounter. The second consequence of the higher demand for bequest is an increase in bequest sizes. The exponential discounter now bequeathes on average 5.67 times his average experienced consumption, in total 132,780. Again the hyperbolic discounter leaves larger bequests on a proportional and absolute basis with an average bequest size of 6.12 times the average experienced consumption or 142,070 in total. The gains in average bequest sizes compared with the benchmark bequest motive are almost entirely covered by the bequest insurance. As noted above there is no significant change in saving and investing behaviour so that bequest insurance is mainly financed by reducing consumption. On average the retirees now leave their benchmark bequest plus the amount covered by bequest insurance. Because the latter is fixed the relative standard deviation of bequest sizes reduces to 40.41% for the exponential and 41.00% for the hyperbolic discounter.

The absence of a bequest motive has a strong impact on the retirees' optimal behaviour. The optimal annuitization degrees are 91.00% for the exponential discounter and 93.40% for the hyperbolic discounter, constituting an increase of 11.8 percentage points in the former and 10.4 percentage points in the latter case compared with the results for the benchmark bequest motive. Now consumption is the only source of utility in the model. Under a sufficient degree of annuitization there is thus no need to keep wealth on hand above certain levels. On the contrary the retirees aim to consume all their wealth during their retirement. On average, this leads to higher consumption in the early periods. Therefore, in contrast to the two previous cases with a bequest motive present, late life consumption is decreasing. In parts this is also due to the fact that equity exposure decreases with decreasing wealth and thus the retirees miss out on the equity returns that retirees with a bequest motive receive as a result of their savings behaviour. Despite these factors, overall consumption is obviously higher in the no bequest motive case. The experienced average consumption level is 25,343 for the exponential discounter and 25,303 for the hyperbolic discounter. These are increases of 0.91% and 1.70% compared with the benchmark model. In comparison with the bequest motive cases the intertemporal variation in consumption increases absolutely and relatively to 4.75% for the exponential discounter. In contrast to this the hyperbolic discounter experiences the lowest intertemporal variation in consumption of all considered cases, with a standard deviation of 3.53%. Unlike in the previous cases this value is smaller than the exponential discounter' counterpart.

Any bequests left in the no bequest motive case are purely accidental and do not yield utility. Bequest sizes are significantly lowered compared with the previous case. On average, bequests amount to 9,757 for the exponential discounter and 10,615 for the hyperbolic discounter, in other terms 38.50% and 41.95% of the average experienced consumption. The standard variations in bequest sizes are 1.44 and 1.24 times the average bequest sizes, also considerably lower than in the previous cases.

[16]We refer to appendix 4, figure 6 for an illustration of the average trajectories in this case.

4.3 Who needs bequest insurance?

The figure 4 displays the minimum value for the strength of the bequest motive, the parameter ω, for which at least 1% of the initial wealth is invested in the life insurance policy for a range of values for the luxury parameter ψ. For $\psi > 0.6 \cdot FAI$, none of the ω-values considered in our analysis, $0 \leq \omega \leq 10$, generate positive demand for bequest insurance. We note that the benchmark bequest motive specification, $\omega = 7,81, \psi = 0,67$, lies just outside the critical parameter range. This implies that even small deviations towards a higher demand for bequest, e.g. a lowering of ψ alone or coupled with an increase in ω, may result in positive demand for life insurance.

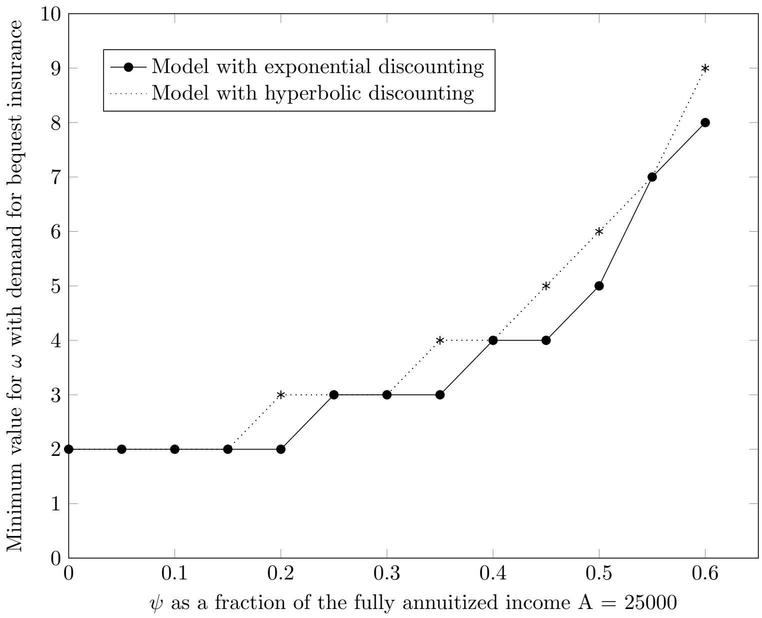

Figure 4: *Minimum strength of bequest motive parameter value ω which results in a postive demand for bequest insurance plotted versus degree of luxury good parameter ψ. Missing values imply that there is no demand for bequest insurance up to the highest value for ω in our model. All parameter combinations above the graph specify a model that generates a positive life insurance demand.*

The figure 4 shows that hyperbolic discounting slightly reduces the demand for bequest insurance. This may seem counterintuitive because the periods in which the agent's probability of death is comparatively high fall within the range of periods which are overweighted by the hyperbolic discounter compared to the exponential discounter[17]. However as discussed in the two previous sections the hyperbolic discounter chooses a higher degree of annuitization combined with higher savings which directly translate into higher wealth levels and therefore higher bequest sizes in the later periods. Thus bequest sizes are, at least in the parameter constellations explicitly considered in the two last sections, the smallest in the early periods. Because these periods are underweighted by the hyperbolic discounter compared with the exponential discounter, hyperbolic discounting reduces the demand for bequest insurance. Nevertheless as we analyzed in the previous section, the hyperbolic discounter's higher savings in the second half of retirement lead to bigger bequest sizes.

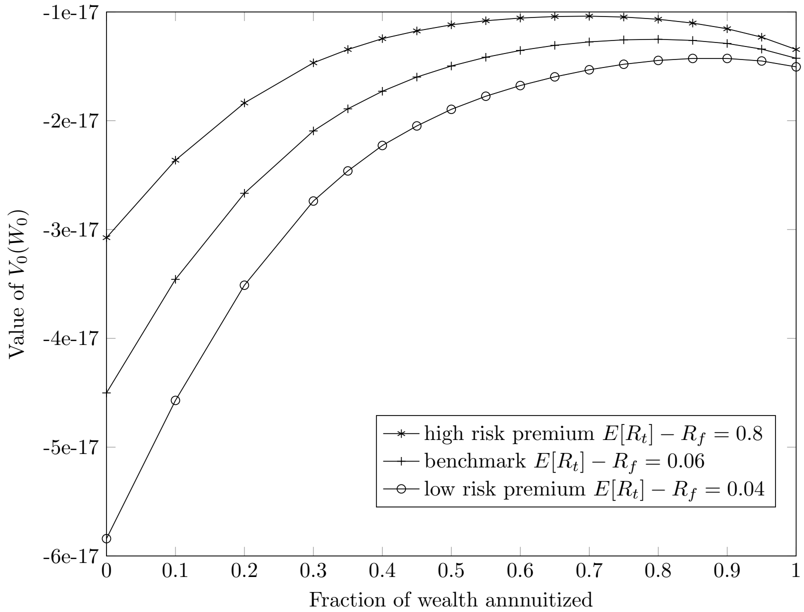

Figure 5: *Utility gains from annuitization with varying risk premia for the hyperbolic discounter in the benchmark bequest motive parametrization.*

4.4 The equity risk premium and the annuitization decision

The decision to purchase a financial asset cannot be made without the consideration of alternative assets. In a model without a bequest motive and in which all potential assets are also available in an annuitized form, the annuitized assets will dominate their non-annuitized counterparts under a broad set of assumptions including unfair pricing (Davidoff, [14]). However Milevsky and Young ([38]) point out that the assumption that an equivalent annuitized version of any potential asset exists is far from realistic. In their model self-annuitization via equity markets dominates the actually available annuity products on the market upto retirement ages between 65 and 70. In the presence of a bequest motive equity markets play a second role besides increasing consumption, which is ensuring a sufficient bequest size. The optimal degree of annuitization therefore depends on the trade-off between the longevity insurance provided by the annuity and the attractive return potential of equity investments. This holds especially true when annuitized equity is not available.

Figure 5 shows the utility gains from various degrees of annuitization for three different risk premia specifications. In any case full annuitization is not optimal. In the Black-Scholes framework assumed in this paper, even the small risk premium of 4% (2% compared to the calculatory interest rate), leads to annuitization rates that are below 90%. It should be noted that these results not only depend on the size of the risk premium but also the general return specification. In a very similiar setting which assumes normal returns, Peijnenburg [42] finds full annuitization to be optimal for all the bequest utility parametrizations discussed in this paper at an identical risk premium.

4.5 Initial savings and crowding out effect of government subsidy

In the parameter constellations considered so far, the access to a subsidy has no effect on the retiree's optimal strategy. With lower initial wealth levels this is no longer true. When the luxury parameter ψ is proportionally adjusted, i.e. calibrated against the adjusted fully annuitized

[17] Compare figures 1 and 3.

income FAI, than it holds that $V_0(pW|A) = p^{1-\gamma}V_0(W|A)$ for $A \geq \underline{C}$ for both time weighting specifications. This is the case because both types of utility functions are positive homogeneous of degree $1-\gamma$[18] and the sufficient annuity size prevents the retiree from ever being eligible for a subsidy. This means that optimal insurance endowments and behaviour variables such as consumption and equity exposure can be calculated from the benchmark intial wealth case by scaling them accordingly. However for sufficiently small p the access to a subsidy does have an effect on the retiree. When the initial wealth is not high enough to ensure a steady consumption $C_t \geq \underline{C}$ and an eventual bequest through partial annuitization with a sufficiently high probability, then the retiree may be tempted to either overconsume in the early retirement years and then resort to the subsidy as his funds run out or try to increase his wealth through an aggressive equity investment strategy. Both cases lead to a severe decrease in annuitization levels.

	Optimal Insurance Endowment		
	$100\%W_0$	$20\%W_0$	$19\%W_0$
Exponential Discounting			
$\omega = 7,81, \psi = 0,67$	80%/0%	6%/0%	0%/0%
Hyperbolic Discounting			
$\omega = 7,81, \psi = 0,67$	83%/0%	9%/0%	0%/0%

Table 2: *The effect of low initial wealth levels on the optimal insurance endowments in the benchmark bequest motive parameterization. The first number indicates the fraction of the initial wealth that is annuitized, the second number indicates the fraction of the initial wealth that is invested in the life insurance. In all cases, the optimal endowments do not contain additional guarantee periods.*

Table 2 contains the optimal insurance endowments for two households with low initial wealth levels in addition to the optimal endowment in the benchmark initial wealth for the benchmark bequest motive specification. When the household has only 20% of the benchmark initial wealth available, which means that the full annuitized income is equal to government subsidy $FAI = \overline{C} = 5000$, then there is still a little demand for annuities in our model. However for lower initial wealth levels such as 19% this demand vanishes and both retirees abstain from annuitization.

[18] Let AF denote the annuity factor according to equation 29 with zero guarantee periods. The fully annuitized income FAI must satisfy $FAI = W/AF$ in the benchmark case and $FAI_p = Wp/AF$ for a household with only $100p\%$ of the benchmark inital wealth W. Let $\psi = FAI \cdot \bar{\psi}$ and $\psi_p = FAI_p \cdot \bar{\psi}$ denote the respective luxury parameters. Then we have for the bequest utility v_p with luxury parameter ψ_p that

$$\begin{aligned}
v_p(pB) &= \frac{\omega}{1-\gamma}\left(\psi_p + \frac{pB}{\omega}\right)^{1-\gamma} \\
&= \frac{\omega}{1-\gamma}\left(FAI_p\bar{\psi} + \frac{pB}{\omega}\right)^{1-\gamma} \\
&= \frac{\omega}{1-\gamma}\left(\frac{Wp}{AF}\bar{\psi} + \frac{pB}{\omega}\right)^{1-\gamma} \\
&= p^{1-\gamma}\frac{\omega}{1-\gamma}\left(\frac{W}{AF}\bar{\psi} + \frac{B}{\omega}\right)^{1-\gamma} \\
&= p^{1-\gamma}\frac{\omega}{1-\gamma}\left(\psi + \frac{B}{\omega}\right)^{1-\gamma} \\
&= p^{1-\gamma}v(B)
\end{aligned}$$

.

4.6 Pricing effects

In order to compensate the insurer for administration costs and general expenses, the actual price P offered for an annuity is typically higher than the actuarial fair price. This section analyzes the effects of such deviations from the fair price on the demand for annuities and life insurance. A way to measure the degree to which an annuity or a life insurance is priced unfairly is the Money's Worth of an annuity. It is defined as the quotient of the expected net present value of the annuity payments, i.e. the actuarial fair price, and the actual price offered for the annuity. Empirical studies have found that the Money's Worth of an annuity depends on several factors. There are differences between different countries and wether or not the annuitiy payments are inflation protected or nominal. Furthermore the Money's Worth can be different for compulsatory annuity markets and for voluntary annuity markets. Brown, Mitchell and Poterba [10] report a Money's Worth of 85% for real annuities on the compulsatory market in the UK and James and Vittas [31] measure a Money's Worth of 80% for real annuities on the vountary market in the UK. For nominal annuities in the UK, Finkelstein and Poterba [17] report a Money's Worth of 87% for voluntary annuities and 90% for compulsatory annuities. In the US market, Poterba and Warshawski [43] measure a Money's Worth of 85% for nominal and 70% for real annuities.

In our analysis we consider the exemplary cases of a Money's Worth of 85% and 70% for a variety of bequest motive specifications in the models with exponential discounting and hyperbolic discounting. We keep the assumption that the agent's initial wealth is such that full annuitization assuming fair pricing results in an annuity of size 25000. However due to unfair pricing, the full annuitized income FAI is lower in the cases considered in this section. To keep the results comparable to the previous results we assume that the degree to which bequests are luxury goods, captured in the parameter ψ which is calibrated relative to the FAI, is still calibrated relative to the full annuitized income in the case of fair pricing. The resulting optimal insurance endowments are displayed in table 3. As it is the case with actuarilly fair priced annuities, guarantee periods are never optimal in the cases considered here.

	Optimal Insurance Endowment		
	Actuarilly Fair	MW 85%	MW 70%
Exponential Discounting			
$\omega = 7,81, \psi = 0,67$	.7920/.0000	.7080/.0000	.6420/.0000
$\omega = 3,91, \psi = 0,67$	.8430/.0000	.7540/.0000	.6740/.0000
$\omega = 7,81, \psi = 0,67$	.7430/.0610	.6650/.0320	.4950/.0000
No bequest motive	.9100/.0000	.8170/.0000	.6890/.0000
Hyperbolic Discounting			
$\omega = 7,81, \psi = 0,67$	.8300/.0000	.7390/.0000	.6620/.0000
$\omega = 3,91, \psi = 0,67$	.8700/.0000	.7810/.0000	.6790/.0000
$\omega = 7,81, \psi = 0,67$	.7700/.0590	.6780/.0200	.5520/.0000
No bequest motive	.9340/.0000	.8330/.0000	.7240/.0000

Table 3: *Pricing effects on the optimal insurance endowments for varying bequest motive parameterizations. The first number indicates the fraction of the initial wealth that is annuitized, the second number indicates the fraction of the initial wealth that is invested in the life insurance. In all cases, the optimal endowments do not contain additional guarantee periods.*

In all model and bequest motive specififcations considered in this section, unfair pricing has a strong effect on the optimal insurance endowment. One of the reasons for this is the access to the equity market. When the return of the annuity investment decreases due to higher prices the relative attractiveness of equity investments increases. However even in the case of a Money's Worth of only 70% the retiree still invests a substantial amount in annuities. With the exception of the model specification with the strongest demand for bequest, the model with the lowered luxury parameter, the optimal annuitization degree is always above 60%. In the benchmark bequest motive specification, unfair pricing has a stronger effect on the hyperbolic discounter than on the exponential discounter. For the former, annuity demand reduces by 10.96% when the Money's Worth is 85% and by 20.24% when the Money's Worth is 70%. For the latter annuity demand reduces by 10.61% and 18.94% respectively. In turn, unfair pricing reduces the differences between the exponential and the hyperbolic discounter. The difference between the optimal annuity demand reduces from 4.80% in the model with fair pricing to 3.12% in the model with a Money's Worth of 70%. The situation is fairly similiar for the bequest motive parameterization with a lowered strength parameter. In the model parameterization with a lowered luxury parameter the demand for annuities is affected in a similiar magnitude than in the two previous cases when the Money's Worth is lowered to 85%. The demand for bequest insurance is affected more drastically. In the model with exponential discounting the retiree invests 47.54% less in the life insurance than he would with fair pricing. In the model with hyperbolic discounting he invests 66.10% less. When the Money's Worth is only 70%, the demand for bequest insurance vanishes completely in both model specifications. Furthermore the retirees now decrease their annuitization degrees much stronger than in the previous cases. The exponential discounter by 33.38% and the hyperbolic discounter by 28.31% compared to the model with fair pricing. To finance their demand for bequests the retirees now rely much heavier on equity investments than on life insurance products. Finally, in the model without a bequest motive, the changes in annuity demand are again quantitatively similiar to the first two cases. The exponential discounter reduces his annuity investment by 10.22% when the Money's Worth reduces to 85% and by 24.29% when the Money's Worth reduces to 70%. The hyperbolic discounter reduces his annuity demand by 10.81% and 22.48%.

5 Conclusion

We find that hyperbolic discounting increases the demand for life annuities because it shifts the retiree's priorities towards the later periods of the retirement phase. Furthermore the hyperbolic discounter invests less in life insurance contracts. Because death, and therefore the timing of the bequest, is more likely to occur in the second half of the retirement phase, and because these periods receive a higher subjective weight by the hyperbolic discounter, it is bequest sizes in these periods that he is particularly concerned about. His strategy to achieve high bequest sizes in these periods is to consume less while receiving higher annuity payments than the exponential discounter and so to save for bequest out of his income. Furthermore we find that guarantee periods are never optimal in the cases considered in this paper.

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Appendix

A1

Lemma 1 (A.1)**.** *Assuming a subsidy consumption level of size* $\underline{C}$ *and a bequest utility of the form* (24)*, the value function for the exponential discounting agent satisfies*

$$V_t(w) \geq \sum_{s=t}^{T} p_{t,s}(1-p_s)\left(\frac{\beta^{t-s+1}-1}{\beta-1}u(\underline{C}) + \beta^{t-s+1}v(0)\right) \tag{33}$$

for all $t \in [0, T]$*.*

Proof. Using that $\sum_{k=0}^{s-1}\beta^k = \frac{\beta^s-1}{\beta-1}$, a quick calculation yields

$$\sum_{k=t}^{s}\beta^{k-t} = \frac{\beta^{s+1-t}-1}{\beta-1}. \tag{34}$$

Let $p_{t,s*} = p_{t,s}(1-p_s)$ denote the probability that the individual deceases in period $t+1$. Because the subsistence consumption level must be maintained at all times, we must have $C_t = c_t W_t \geq \underline{C}$ for all $t \in [0, T]$. Using this and the definition of the value function V_t (11), the definition of the preference functional Φ (35), the Law of Total Expectation and (34) we find that for all wealth levels $w \geq 0$ and strategies $\gamma_t \in \Gamma_t$

$$\begin{aligned}
V_t(w) &\geq \Phi_t(w, \gamma_t) \\
&= E_t\left[\sum_{s=t}^{T} p_{0,s-1}\beta^{s-t}\left(p_s u(c_s W_s^L) + (1-p_s)v(W_s^D)\right)\right] \\
&\geq E_t\left[\sum_{s=t}^{T} p_{0,s-1}\beta^{s-t}\left(p_s u(\underline{C}) + (1-p_s)v(0)\right)\right] \\
&= \sum_{s=t}^{T} p_{t,s*}\left[\left(\sum_{k=t}^{s}\beta^{k-t}u(\underline{C})\right) + \beta^{s+1-t}v(0)\right] \\
&= \sum_{s=t}^{T} p_{t,s}(1-p_s)\left(\frac{\beta^{t-s+1}-1}{\beta-1}u(\underline{C}) + \beta^{t-s+1}v(0)\right)
\end{aligned}$$

□

Lemma 2 (A.2 Bellmann Equation)**.** *i) The value function for the hyperbolic discounting agent satisfies the Bellmann equation* (20) *with the terminal condition* (21)*.*

ii) The value function for the exponential discounting agent satisfies the Bellmann equation (12) *with the terminal condition* (44)*.*

Proof. i)

Given the current wealth level W_t and a feasable (remaining time) strategy γ_t we can now formulate the investor's preferences assuming an initial annuity endowment (A, N_A, a) when starting at a fixed time t. Here we assume that the investor is still alive at time t and hence set $p_{t,r} = 1$ for all $r \leq t$ and $p_t = 1$. Using the multiplicative properties of the cumulative survial probabilities, the decomposed discount factors and the law of iterated expectations we obtain

$$
\begin{aligned}
\Phi_0(W_0,\gamma_0) =& E_0\left[\sum_{s=0}^{T} p_{0,s-1} DF_s \left(p_s u(c_s W_s^L) + (1-p_s)v(W_s^D)\right)\right]\\
=& E_0\left[\sum_{s=0}^{t-1} p_{0,s-1} DF_s \left(p_s u(c_s W_s^L) + (1-p_s)v(W_s^D)\right)\right]\\
&+ E_0\left[\sum_{s=t}^{T} p_{0,s-1} DF_s \left(p_s u(c_s W_s^L) + (1-p_s)v(W_s^D)\right)\right]\\
=& E_0\left[\sum_{s=0}^{t-1} p_{0,s-1} DF_s \left(p_s u(c_s W_s^L) + (1-p_s)v(W_s^D)\right)\right]\\
&+ DF_t \cdot p_{0,t} \cdot E_0\left[E_t\left[\sum_{s=t}^{T} p_{t,s-1} DF_{t,s} \left(p_s u(c_s W_s^L) + (1-p_s)v(W_s^D)\right)\right]\right]\\
=& E_0\left[\sum_{s=0}^{t-1} p_{0,s-1} DF_s \left(p_s u(c_s W_s^L) + (1-p_s)v(W_s^D)\right)\right]\\
&+ DF_t \cdot p_{0,t} \cdot E_0\left[\Phi_t(W_t,\gamma_t)\right]
\end{aligned}
$$

where

$$
\Phi_t(W_t,\gamma_t) = E_t\left[\sum_{s=t}^{T} p_{t,s-1} DF_{t,s}\left(p_s u(c_s W_s^L) + (1-p_s)v(W_s^D)\right)\right] \tag{35}
$$

denotes the investor's time t preference functional.

From the definition of Φ_t we can obtain the following difference equation for the family $(\Phi_t)_{t\in[0,T]}$ by again invoking the tower property of the conditional expectation and the multiplicative properties of the survival probabilities and the intertemporal discount factors

$$
\Phi_t(W_t,\gamma_t) = E_t\left[\sum_{s=t}^{T} p_{t,s-1} DF_{t,s}\left(p_{s-1} u(c_s W_s) + (1-p_{s-1})v(W_s^D)\right)\right] \tag{36}
$$

$$
= DF_{t,t} u(c_t W_t) + E_t\left[\sum_{s=t+1}^{T} p_{t,s-1} DF_{t,s}\left(p_{s-1} u(c_s W_s) + (1-p_{s-1})v(W_s^D)\right)\right] \tag{37}
$$

$$
= u(c_t W_t) + E_t\left[\sum_{s=t+1}^{T} p_{t,s-1} DF_{t,s}\left(p_{s-1} u(c_s W_s^L) + (1-p_{s-1})v(W_s^D)\right)\right] \tag{38}
$$

$$
= u(c_t W_t) + DF_{t,t+1} p_t E_t\left[E_{t+1}\left[\sum_{s=t+1}^{T} p_{t+1,s-1} DF_{t+1,s}\left(p_{s-1} u(c_s W_s^L) + (1-p_{s-1})u(W_s^L - A)\right)\right]\right] \tag{39}
$$

$$
+ DF_{t,t+1}(1-p_t) E_t\left[v(W_t^D)\right] \tag{40}
$$

$$
= u(c_t W_t) + DF_{t,t+1} E_t\left[p_t \Phi_{t+1}(W_{t+1}^L, \gamma_{t+1}) + (1-p_t)u(W_t^D)\right]. \tag{41}
$$

The preference functional Φ_t, the wealth dynamics described in (1),(2) and (4), the control set U_t and the underlying probability space containing the stock returns satisfy the general conditions as given in Bertsekas [7] to allow for a dynamic programming approach[19]. Thus by taking the supremum at both sides of

$$\Phi_t(W_t, \gamma_t) = u(c_t W_t) + DF_{t,t+1} p_t E_t \left[\Phi_{t+1}(W_{t+1}^L, \gamma_{t+1}) + (1 - p_t) v(W_t^D)) \right] \tag{42}$$

we obtain the Bellmann equation for the investor's optimization problem

$$V_t(W) = \sup_{\substack{0 \le c, \theta \le 1 \\ cW_T \ge \underline{C}}} \left\{ u(cW) + DF_{t,t+1} p_t E_t \left[V_{t+1}(W_{t+1}^L) \right] + DF_{t,t+1}(1 - p_t) E_t \left[v(W_t^D) \right] \right\} \tag{43}$$

with the terminal condition

$$V_T(W) = \sup_{\substack{0 \le c, \theta \le 1 \\ cW_T \ge \underline{C}}} \left\{ u(cW_T) + DF_{T,T+1} E_T \left[v(W_{T+1}^D) \right] \right\}. \tag{44}$$

ii) Follows from i) by setting $DF_{t,t+1} = \beta$.

□

[19] These are mainly measureability conditions regarding the functions contained in the dynamics of the budget constraints $W_{t+1}^L = f_1(W_t^L, c_t, \theta_t, R_{t+1})$ and $W_{t+1}^D = f_2(W_t^L, c_t, \theta_t, R_{t+1})$ as well as in the objective function $\Phi_t(W_t, \gamma_t) = E_t \left[\sum_{s=t}^T f_3(s, W_s^L, W_s^D, c_s, \theta_s) \right]$ and the stochastic process $(R_t)_{t=1,T+1}$. The measurability of f_1, f_2 and f_3 follows from their continuity. The random returns are adapted to the filtration by definition because $F_t = \sigma(R_t)$.

A2 - Certainty equivalent of outstanding annuities

The dynamic portfolio optimization problem contained in the definition of the certainty equivalent $CE_A^{t,s}$ in (26)

$$CE_A^{t,s} = v^{-1}\left(DF_{t,t+s-1} \sup_{\gamma'_{s-1} \in U'_{s-1}} E\Big[v(W_{s-1})\Big]\right)$$

can be formulated as follows:

$$\begin{aligned} \underset{\gamma'_{s-1}=(\theta_t,\ldots,\theta_{t+s-2})}{\text{maximize}} \quad & E\Big[v(W_{s-1})\Big] \\ \text{subject to} \quad & W_{r+1} = W_r(\theta_t(1+R_{r+1}) + (1-\theta_r)(1+R_f)) + A,\ r = t,\ldots,t+s-2, \\ & W_t = W_t^D + A, \\ & 0 \le \theta_r \le 1,\ r = t,\ldots,t+s-2, \end{aligned} \tag{45}$$

where W_t^D is time t wealth in the death case defined by (2). Once again we resort to dynamic programming to solve this problem. The Bellmann equation for this problem is

$$V_r(w) = \sup_{\theta\in[0,1]} E\left[V_{r+1}\left(w(\theta(1+R_{r+1}) + (1-\theta)(1+R_f)) + A\right)\right] \tag{46}$$

with the terminal condition

$$V_{s-2}(w) = \sup_{\theta\in[0,1]} E\left[v\left(w(\theta(1+R_{t+s-1}) + (1-\theta)(1+R_f)) + A\right)\right]. \tag{47}$$

In both cases we optimize a contiuous function over the compact domain $[0,1]$ which implies that the suprema are attained and thus the existence of the maximum in the optimization problem (45). We refer to Remark 1 in Appendix A1 for details about the extistence of the integerals in (46) and (47).

We solve the problem via backwards induction. We define an exponentially placed and dynamically growing grid $\{w_l\}_{l=1,\ldots,L_t}$ with $L_0 = 30$ and $w_0 = A$ for the wealth process $W_t,\ldots,W_{t+r-2}$ and interpolate the value function from the pairs $(w_l, V_t(w_l))$ by Piecewise Cubic Hermite Interpolation. Conditional expectations are calculated via Gauss-Hermite-Quadrature with $n = 32$ sample points transformed to best suit the lognormal return specification of our model.

A3 - Wealth, consumption, bequest and equity exposure trajectories

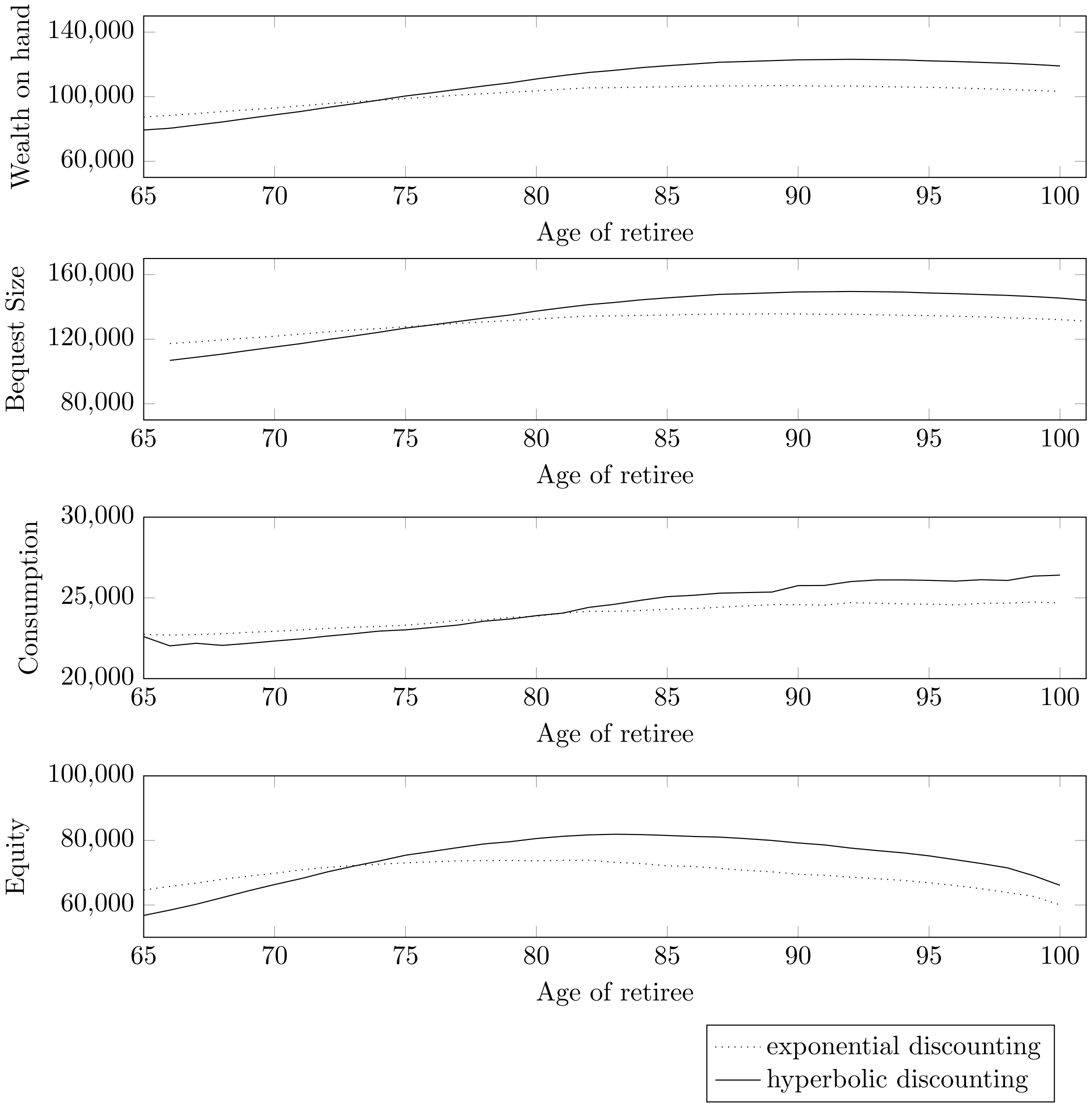

Figure 6: *Averaged trajectories for wealth, bequest, consumption and equity exposure calculated from* $N = 10000$ *forward simulations in the bequest motive parametrization with low luxury parameter* $\psi = 0.34$ *for both time discounting specifications. The optimal annuity endowments are* $A = 18,575$ *(*74.30%*) and* $a = 47,235$ *(*6.10%*) [*80.40% *total] for the exponential discounter and* $A = 19,250$ *(*77.00%*) and* $a = 45,686$ *(*5.90%*) [*82.90% *total] for the hyperbolic discounter. The numbers in brackets indicate the fraction of the initial wealth that is invested in the respective insurance class.*

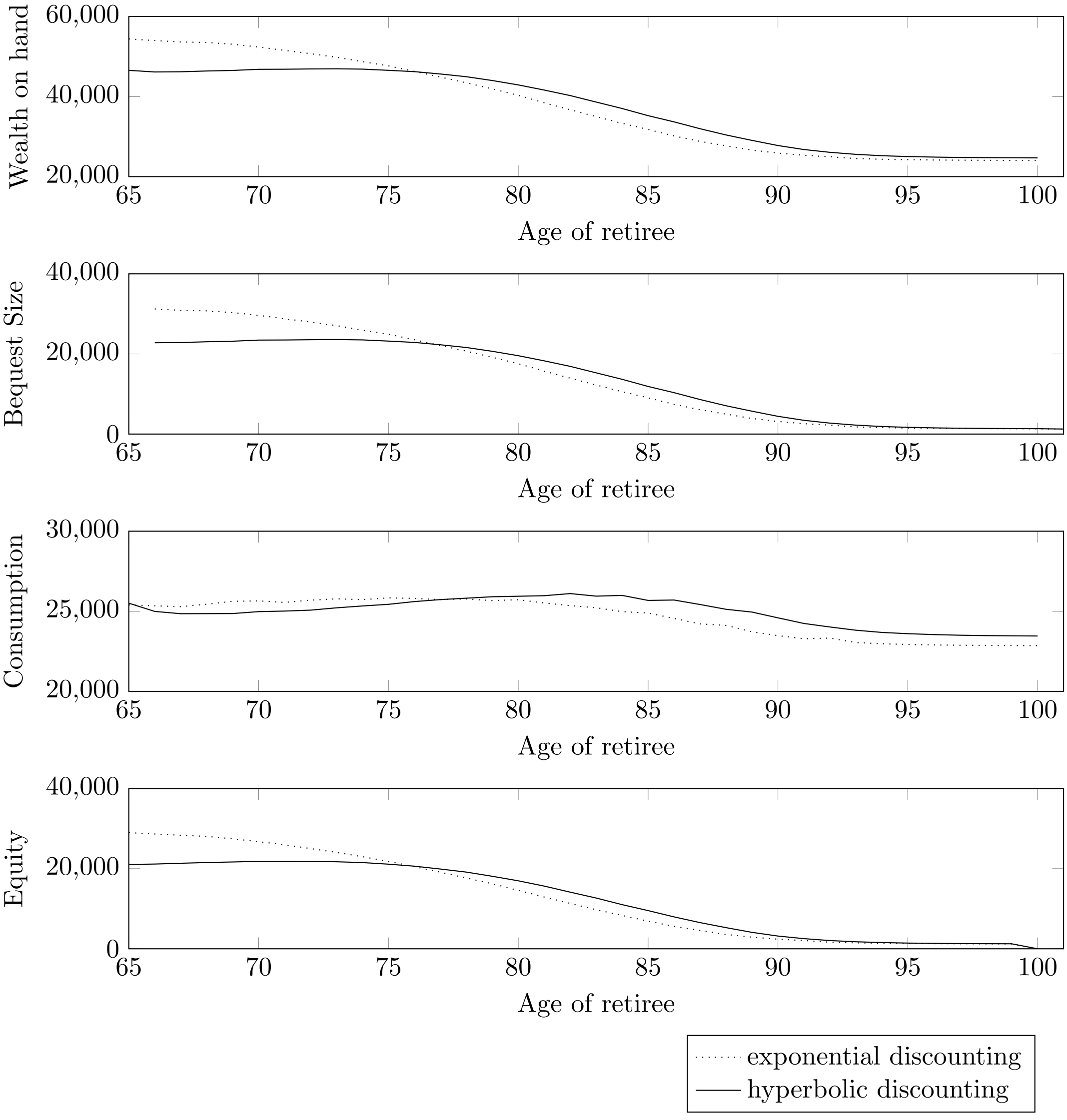

Figure 7: *Averaged trajectories for wealth, bequest, consumption and equity exposure calculated from* $N = 10000$ *forward simulations in the no bequest motive parametrization* ($\omega = 0$) *for both time discounting specifications. The optimal annuity endowments are* $A = 22,750$ *(91.00%) for the exponential discounter and* 23350 *(93.40%) for the hyperbolic discounter. The numbers in brackets indicate the fraction of the initial wealth that is invested in the respective insurance class.*

Narrow framing, loss aversion and the optimal retirement portfolio with Arrow annuities

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Abstract

We study the demand for Arrow annuities, a stylized form of one-period annuities, in a dynamic preference model that combines expected utility theory and loss averse investment evaluation. A particular focus of this paper lies on the parametrization of the relative importance of the investment evaluation. We find that the optimal demand for Arrow annuities reacts less sensitive to an increase of the relative importance of the investment evaluation than the demand for stocks. While loss aversion always reduces the total exposure to risky assets, we find that it can, in some cases, increase the demand for Arrow annuities compared to a benchmark model without loss aversion.

1 Introduction

A recurring finding in the recent literature on annuity demand, is that the empirically observed lack of voluntary annuitization may be attributed to the particular form the annuitization decision is displayed to the decision maker. This form may or may not conincide with the economic models aiming to derive the optimal demand for annuities. The typical life-cycle models only focus on the retiree's utility from consumption. And when these models also include bequest motives in the form of utility from bequest, it is usually an approximation for the heirs' additional utility from consumption that the bequest grants them. Therefore these models fall into the category of a consumption frame for the annuitization decision. But experimental evidence, for example by Brown, Kling, Mullainathan and Wrobel [12] suggests, that indivduals actually choose annuity levels as predicted by those models, when their optimization problem is framed in terms of consumption smoothing and longevity insurance. It is only when the decision is approached from an investment perspective, which means comparing the risk and return profile of an annuity with other asset classes, that individuals refrain from voluntary annuitization. The literature concerning the annuity puzzle, the forementioned gap between suggested and observed voluntary annuitization, has so far, at least to the best of our knowledge, not produced a dynamic model for the annuitization decision that adds an investment evaluation to the classical consumption frame. There are dynamic models, such as Hu and Scott [23], that can be classified as investment frames for the annuitization decision. However these models do not incorporate utility from consumption at all, and thus neglect the huge potential of annuities as an insurance against longevity risk. The aim of this paper is to propose a simple dynamic model that incorporates the effects that are present in both frames, the benefits of life annuities when it comes to insuring a stable consumption throughout the whole retirement phase and the possibly disadvantageous evaluation of the return characteristics of an annuity. A general problem that occurs when comparing annuity returns with the returns of other asset classes is their unknown number of payment periods. Any liquid asset, such as a stock investment, can be represented by its annual return characteristics. From the return characteristics and the amount of cash invested in the stock we can calculate an annual investment evaluation in the sense of Benartzi and Thaler [7]. They introduce the concept of myopic loss aversion which applies a loss averse investment evaluation to the development of a short term stock investment. However this concept cannot be easily applied to life annuities because they contain annual payments over multiple periods. Because there is no natural way to break down an annuity investment into individual one period returns and an associated amount of capital invested per period, there is no straightforward way to implement loss aversion over annuities in a dynamic model with annual consumption and investments in other asset classes. To avoid this problem we reduce our analysis to a stylized form of a life annuity. Instead of annuities with payoffs over multiple periods for the rest of the purchaser's lifetime, we focus on one-period annuities whoose payoff is conditional on the purchaser surviving the subsequent period and which may be purchased annually, starting upon entry to retirement.

We assume our agent has preferences similiar to the models proposed by Barberis and Huang ([3] and [4]). Their approach combines consumption evaluation according to expected utility theory and a loss averse investment evaluation based on Kahnemann and Tversky's Cumulative Prospect Theory [30]. Our model focuses on the retirement phase of the life-cycle with an uncertain life time. The agent's task is to find the optimal consumption and investment plan for his accumulated savings. To finance his future consumption and a potential bequest, the agent may divivde his wealth on hand among three different asset classes. One of which are Arrow annuities, which are a form of Arrow-Debreu-securities that pay a fixed amount if the investor is still alive in the subsequent period and nothing if the investor deceases. We start our analysis with a simple model in which it is possible to partially derive an analytical solution. In a second step we expand our model to allow for a more sophisticated and realistic treatment of the retiree's bequest motive following De Nardi's [18] specification of bequest utility. Within this framework we derive a benchmark consumption and investment strategy using a numerical dynamic programming approach. In the third and final step we incorporate the concept of narrow framing into the investor's preferences. This concept assumes that an investor may evaluate the outcome of an investment independent from the overall effect on his investment goal. A simple example is that an investor would be subject to a form of negative utility if he faced a loosing stock in an otherwise winning portfolio. In our model, the evaluation of the individual assets, which may be applied to bundles of assets or to the single assets individually, follows the s-shaped evaluation function from Kahneman and Tversky's cumulative prospect theory (CPT), which distinguishes between losses and gains by means of a reference point. A central aspect of CPT is loss aversion. This means that losses will be weighted more heavily than potential gains.

The preferences in our model are time-additive and additive with respect to the different sources of utility. The latter means that the agent annually receives the sum of the utility from consumtpion or bequest depending on his survival status and the subjective utility from the loss averse investment evaluation. Combining these utility functions with different curvatures runs into the problem of finding a normalization parameter that harmonizes the effects of both utility function. Therefore we pay particular attention to the choice of the relative weighting parameter of standard utility versus CPT-utility in our model. It is this parameter that determines the degree to which our preference functional represents more of an investment frame or more of a consumption frame.

We find that, depending on its relative importance within the preference functional, the loss averse investment evaluation can lead to a broad range of effects, ranging from only minor deviations from the optimal strategy in a benchmark model without loss aversion, to complete abstinence from annuity and equity markets. We further find that whether risky assets are framed individually or as a group does have a strong effect on the optimal investment policies. When all risky assets are framed and evaluated as a whole, the agent can form portfolios that are particulary well-suited to the loss averse evaluation in the sense that they cause significantly less losses than the individual assets. This effect goes beyond classical diversification effects, due to the non-smoothness of the CPT evaluation function and the heavier weighting of losses. Therefore, exposure to risky assets does increase noticeably compared to the situation in which assets are framed individually. In one parametrization in our analysis, the broader framing even results in a higher annuitization degree than the benchmark model without loss aversion. Although the total exposure to risky assets remains lower than in the benchmark model, this shows that a loss averse investment evaluation in the sense of our approach does not necessarily decrease the demand for annuities.

Literature review

The notion that voluntary annuitzation rates are too low compared to their potential benefits for a retiree dates back to Yaaris's influential article on portfolio choice facing an uncertain lifetime [32]. This is the core of the term annuity puzzle. A detailed discussion of this puzzle is given by Benartzi, Previtero and Thaler [6].

Throughout the last decades, various rational explanations have been sought out to explain the observed under-annuitization. Some see under-annuitization as evidence for a bequest motive. Precautionary savings and unfair annuity pricing are other popular explanations. Recently other factors have been examined, that arise from non-rational or at least only partly rational behaviour of the individual retiree. See for example Brown [11] for a thorough compendium on potential rational and psychological determinants of annuity demand. An analysis of how behavioural approaches may effect more general types of insurance is given by Richter, Schiller and Harris [28]. Empirical testing of several of the rational, as well as the behvioural hypotheses, is conducted by Goedde-Menke, Lehmensiek-Starke and Nolte [21], who find framing, distrust, bequest, and self-selection to be the strongest obstacles in the annuity market. A further psychological look on how annuities are evaluated is given by Duxbury, Summers, Hudson and Keasey [20]. General empirical determinants of annuity demand are found to be wealth, gender, financial literacy and framing for example by Cappelletti, Guazzarotti and Tommasino [14] or Agnew and Szykman [2].

The framing of the annuitization decision, which is also a main part of this paper, is also the focus of other literature on the demand for annuities. The basic notion is that while annuities may seem unbeatable when it comes to ensuring late life consumption, the may not seem so attractiv on other scales and thus compare unfavourably to other investment opportunities. Indeed, annuities are a bad investment conditional on an early death of the retiree and generally include a high risk without the associated high returns. Brown et al. [12] conduct an experiment in which the annuitization decision is formulated to probants in two different ways. One way focuses on lifelong consumption smoothing and one way focuses on the risk return characteristics of the annuity. They find that many prefer the annuity when presented with the "consumption frame" but that the preference rate for the annuity decreases from 72% to 21% under the "investment frame". Agnew, Anderson, Gerlach and Szykman [1] take a look at how framing in terms of marketing annuities may effect customers and in turn their demand for annuities.

We propose a preference model which takes into acount both utility from consumption as well as subjective utility from the investment decision. The latter relies on a variant of the evaluation function in Kahnemann and Tverky's [30] Cumulative Prospect Theory (CPT). Building on the CPT framework, Benartzi and Thaler [7] introduce the concept of myopic loss aversion, that is a loss averse investor who evaluates the development of his portfolio in short intervals even though he may have long-term investment goals. Their aim is to provide a possible explanation for a problem which is possibly related to the annuity puzzle, the equity premium puzzle. Shefrin and Statman's paper [29] represents another early approach that incorporates aspects of CPT into the problem of optimal portfolio choice.

We follow Barberis and Huang in our model proposition who study general optimal portfolio and asset pricing problems with a particular focus on the equity premium puzzle (see for example [3],[5] and [4]). Their results fortify the importance of narrow framing in the investment decision, that is the separated evaluation of individual asset types independent of the total outcome of the portfolio. Several degrees are possible, in which investment outcomes may be bundled by assets or actually evaluated individually. In a similiar way we implement a form of narrow framing which combines equity and annuity investments and one with a separate evaluation. Another similar behavioural optimal portfolio approach is conducted by De Giorgi and Legg [17]. In contrast to the previous papers they also include CPT's aspect of probability weighting into their preference fucntional.

Hand in hand with investment evalution under narrow framing typically goes the concept of loss aversion, i.e. a heavier weighting of adverse market outcomes. Other optimal portfolio studies that follow these principles are conducted by Berkelaar, Kouwenberg and Post [8] and Magi [26]. A special implementation of loss aversion in a household portfolio situation can be found in Dimmock and Kouwenberg [19]. A continous-time behavioural portfolio problem is solved by Rsonyi and Rodrigues [27]. Van Bilsen, Laeven and Nijman [31] study a dynamic investment and consumption problem with endogenous updating of the investor's reference point, a key concept in loss aversion that distinguishes investment gains from losses. Blake, Wright and Zhang [10] study the optimal investment plans under loss aversion in the accumulation phase of a life-cycle. As is typical for loss averse investors, they find that the optimal policy characteristic is dependant on the current wealth level of the agent. More specificly, the agent is more risk seeking for low wealth levels and switches to portfolio insurance once he reaches sufficiently high wealth levels.

In a setting similiar to this paper, Gottlieb and Mitchell [22] analyze the effect that combining standard expected utility and CPT evaluation has on the demand for long-term care insurance. In contrast to a combined preference functional, Chen, Hentschel and Klein [15] study the effects of guarantees in life insurance on expected utility and CPT investors.

In a non-dynamic framework, Hu and Scott [23] model the annuitization decision for a loss averse investor using the CPT evaluation function. They find that loss aversion leads to low annuitization rates and furthermore increases the demand for guarantee periods for the few people that purchase annuities. In contrast to our paper, their model does not include the beneficial effects of an annuity on the retiree's lifelong consumption and does therefore classify as a pure investment frame.

2 The model

We regard an exemplary agent who enters retirement at age 65 (time $t = 0$) with the accumulated savings $W > 0$ at his disposal. We assume that the agent has no further assets such as pre-annuitized wealth or future labour income. To finance future consumption and a possible bequest, the agent has access to a menu of three investment types. At the beginning of every year, the agent chooses his annual consumption level C_t, starting in $t = 0$ and then allocates the rest of his wealth on hand $W_t - C_t$ among the three asset classes. We assume that the agent may survive every year with a positive probability until he reaches a maximum age of 100 years (time $T = 35$). At $t = T$, he chooses his final consumption level and investment strategy, the result of which becomes his bequest at time $T + 1$. When the agent deceases during some earlier period t, the remaining assets, if their payoff is not conditional on the agent's survival, will also be transfered to an heir in form of a bequest at time t.

The first asset available to the agent is a riskfree bond, paying a fixed interest rate R_f. The second asset is a stylized stock investment whose underlying price process follows a geometric Brownian Motion, i.e. pays a lognormally distributed return $1 + R_t$[1]. The third asset is a one period Arrow annuity, a financial contract that pays a fixed amount $A > 0$ if the investor survives the subsequent period and nothing if the investor deceases. Let p denote the probability that an

[1]Throughout this paper we will assume that all probabilities and conditional and unconditional expectations are taken with respect to the canonical probability space generated by the sequence of returns $(R_t)_{t\in[1,T+1]}$

investor survives the subsequent period. The expected present value of the Arrow annuity payout is then given by

$$P_A = \frac{1}{1+R_f}\Big(p \cdot A + (1-p)\cdot 0\Big) = \frac{pA}{1+R_f}. \tag{1}$$

Assuming that the calculatory interest rate is equal to R_f and a loading of size zero, P_A is the actuarially fair price of an Arrow annuity. Under this assumptions the investor's return in case of survival is

$$R_A = \frac{A}{P_A} - 1 = A \cdot \frac{1+R_f}{pA} - 1 = \frac{1+R_f}{p} - 1 \tag{2}$$

and a total loss $R_A = -1$ otherwise. As survival probabilities vary with the investor's age we let

$$R_{A,t} = \frac{1+R_f}{p_t} - 1 \tag{3}$$

denote the respective annuity return in period t and $Q_{A,t} = 1 + R_{A,t}$ the respective growth factor. We assume that all assets can be purchased without any transaction costs.

For all $t = 0, 1, 2, \ldots, T$ we let $0 \leq c_t \leq 1$ denote the fraction of wealth on hand W_t that is consumed in the following period and $\theta_t^S \geq 0$ and $\theta_t^A \geq 0$ define the respective fractions of the investor's wealth after consumption which are invested in the risky asset and the Arrow annuity. To allow neither borrowing nor short selling either of the asset types, we require that $0 \leq \theta_t^S, \theta_t^A \leq 1$ and $\theta_t^S + \theta_t^A \leq 1$ for all $t = 0, 1, \ldots, T$. The remaining fraction $1 - \theta_t^S - \theta_t^A$ is assumed to be invested in the riskless asset. In the following we let U denote the set of feasible decision policies $\gamma_0 = (c_t, \theta_t^S, \theta_t^A)_{t\in[0,T]}$ abiding the constraints above. Since the payoff of an Arrow annuity is conditional on the agent's survival, the agent's wealth process depends on his survival state. If the agent is alive at time $t+1$, his wealth on hand is given by

$$W_{t+1}^L = W_t(1-c_t)(\theta_t^S(1+R_{t+1}) + \theta_t^A(1+R_{A,t+1}) + (1-\theta_t^S-\theta_t^A)(1+R_f)). \tag{4}$$

If the investor deceases during period $t+1$, his wealth on hand at time $t+1$, and thus the size of a possible bequest is

$$W_{t+1}^D = W_t(1-c_t)(\theta_t^S(1+R_{t+1}) + (1-\theta_t^S-\theta_t^A)(1+R_f)). \tag{5}$$

The agent's wealth at all subsequent times after his death is zero.

An economic agent in a developed country is usually protected from falling beneath some subsistence level in terms of his consumption. Typically this form of protection from extreme poverty comes in form of social security, which is basically a conditional annuity that the agent may draw on. This implies that independent of the agent's initial wealth and economic decision, he will still be able to maintain a certain base level of consumption $\underline{C} \geq 0$ through a form of government subsidy. Obviously, such a subsidy serves as a form of longevity insurance and may thus lead to a crowding out effect regarding private longevity insurance in form of voluntary annuitization. Access to a social security subsidy leads to the modified budget equation in case of survival

$$W_{t+1}^L = \max\left(W_t(1-c_t)(\theta_t^S(1+R_{t+1}) + \theta_t^A(1+R_{A,t+1}) + (1-\theta_t^S-\theta_t^A)(1+R_f)), \underline{C}\right). \tag{6}$$

To simplify matters on a numerical level[2] we constrain the agent's decision set U_t to force him to maintain the base level consumption $\underline{C}$ in all periods by requiring the additional condition $C_t \geq \underline{C}$ at all times $t = 0, \ldots, T$. This implies that an agent who has already received a subsidy in previous periods, and thus by the above constraint has consumed his whole wealth in those

[2] Since our agent's optimal consumption will typically lie way above $\underline{C}$, forcing a minimum consumptin does not have a significant impact on the agent's policy.

previous periods, will leave a bequest of size 0 at the time of his death.

We assume that the investor has preferences similiar to the time-additive model by Barberis and Huang [3] which aggregates expected utility from consumption and prospect theory investment evaluation. Because we also include utility from bequest into our model our investor is, in total, subject to three sources of utility. At the beginning of every year $t = 0, 1, \ldots, T$, if the agent is still alive, he receives utility from consumption $u(C_t) = u(c_t W_t^L)$. At the time of his death he receives an additional utility from bequest $v(W_t^D)$, depending on the strength of his bequest motive $\omega \geq 0$. The concrete specifications for u and v are discussed below. In addition to the first two classical sources of utility, the agent also experiences a subjective utility from the potential prospects of his risky investments. We borrow from Kahneman and Tversky's Cumulative Prospect theory (CPT) for the specification of this third form of utility. They originally proposed the evaluation function[3]

$$u_{CPT}(x) = \begin{cases} x^\alpha & \text{for } x \geq 0 \\ -\lambda(-x)^\alpha & \text{for } x < 0 \end{cases} \tag{7}$$

for parameters $\alpha \in (0, 1)$ and $\lambda > 1$ [4]. The parameter restriction for λ implies that the investor is loss averse in the assessment of his investment opportunities. This means that he will assign a higher weight to the subjective disutility derived from adverse developments compared to the positive subjective utility derived from beneficial developments. Contrary to expected utility theory this function is not applied to a consumption or a wealth level, but to gains and losses. This differentiation is achieved through the introduction of a reference rate of return r_0. The investor assesses the success of an investment of size W by comparing its return Wr to the benchmark return Wr_0. If $Wr - Wr_0 \geq 0$, it is considered a gain and a loss otherwise. We set r_0 to be equal to the riskless rate of return for the remainder this paper. In analogy to expected utility theory we assume that the subjective utility associated with a risky investment is given by the expected value of the evaluation function u_{CPT}. Probability distortion, another main feauture of Kahnemann and Tversky's CPT, is not included in our model. In total the CPT evaluation functional for an investment of size W with the random return r is thus

$$E\left[u_{CPT}(Wr - Wr_0)\right] = W^\alpha E\left[u_{CPT}(r - r_0)\right]. \tag{8}$$

Since the term $E\left[u_{CPT}(r - r_0)\right]$ in the above equation (10) is independent of the fraction of wealth invested, we define

$$K(r, r_F) := E\Big[u_{CPT}(r - r_f)\Big] \tag{9}$$

and the investment evaluation function for an individual asset with return r reduces to the power function

$$E\left[u_{CPT}(Wr - Wr_0)\right] = K(r, r_F) W^\alpha. \tag{10}$$

In most applications, the concept of risk aversion is typically applied to outcomes that are related to the resulting total wealth level of a risky course of action. For example the annual consumption level, which, according to the life-cycle hypothesis, is choosen in dependance of the current wealth level of an agent. The current wealth level in turn results from the overall investment and saving strategy of the agent. In contrast to this, the concept of loss aversion is usually understood as an aversion towards adverse outcomes of specific parts of an agent's investment strategy. Therefore the subjective evaluation function (10) in our model is not applied to the total portfolio outcome but to either individual risky assets or groups of risky assets. In this paper we differentiate between two model specifications regarding the scope of the investment evaluation. This differentiation follows Barberis and Huang's [3] approach. The first form is narrow framing (NF), which assumes that the prospects of each risky asset are assessed individually. This increases

[3]We use the term evaluation function to distinguish Kahnemann and Tversky's approach from a classical Bernoulli utility function

[4]The original evaluation function proposed by Kahnemann and Tversky allows for a different curvature parameter $\alpha \in (0, 1)$ for gains and losses. However their proposed parameterisation results in identical values of $\alpha = 0.88$

the subjective riskiness of the individual asstes because diversification effects are ignored. The second form is broad framing (BF), which assumes that the assessed outcome variable is the return of the portfolio of both risky assets, the stylized stock and the Arrow annuity. The remainder of this section introduces the resulting preference functionals in the two model specifications and their associated value functions.

We let p_t denote the conditional probability that an agent who is alive at time t survives until time $t+1$. Then the product $p_{0,t} = \prod_{s=0}^{t} p_s$ gives the unconditional probability that an agent who has reached age 65 survives until time t. Combining the expected utility framework with the subjective evaluation function in an additive way yields the time-additiv preference functional $\Phi\colon (0,\infty) \times U \mapsto \mathbb{R}$ describing the agent's optimization problem at time $t = 0$ given by

$$\Phi_0(W, \gamma_0) = E_0\Bigg[\sum_{t=0}^{T} p_{0,t-1}\beta^t \Big(p_t u(c_t W_t^L) + (1-p_t)\omega v(W_t^D) \tag{11}$$

$$+ \kappa \sum_{i\in\{S,A\}} (W_t^L \theta_t^i)^\alpha E[u_{CPT}(R_{t+1}^i - R_0)]\Big)\Bigg]. \tag{12}$$

for the narrow framing agent and

$$\Phi_0(W, \gamma_0) = E_0\Bigg[\sum_{t=0}^{T} p_{0,t-1}\beta^t \Big(p_t u(c_t W_t^L) + (1-p_t)\omega v(W_t^D) \tag{13}$$

$$+ \kappa\Big(W_t^L\Big(\sum_{i\in\{S,A\}} \theta_t^i\Big)\Big)^\alpha E\Big[u_{CPT}\Big(\Big(R_{P,t+1}\Big) - R_0\Big)\Big]\Big)\Bigg] \tag{14}$$

for the broad framing agent. Here

$$R_{P,t+1} = \sum_{i\in\{S,A\}} \frac{\theta_t^i}{\theta_t^S + \theta_t^A} R_{t+1}^i$$

denotes the return of the portfolio of the framed assets and κ describes the relative weighting of subjective investment evaluation and utility from annual consumption and bequest. In other words the degree to which the agent's optimization problem can be characterized as an investment frame or a consumption/ bequest frame. In both specifications, with narrow framing and broad framing, the utilities from the subjective evaluation functions are assumed to effect the agent at the time a risky investment is placed and not when it is resolved. Because in the subjective evaluation, the agent assesses the investment solely on its individual characteristics and independent of its effect on his future financial well-being, the subjective evaluation utility is not discounted by the agent's survival probability and the time-weighting factor β.

In addition to the time $t = 0$ preferences above, we can define the investor's preference functional Φ_t, describing the problem (re-)started at a later time $t = 1, \ldots, T$. The problem's value function V_t is then defined as the optimized remaining time preference functional formulated as a function of the endogenous state varibale wealth at time t, i.e.

$$V_t(W_t) = \sup_{\gamma_t \in U_t} \Phi_t(W_t, \gamma_t). \tag{15}$$

In both problem formulations, the value function satisfies the Bellmann equation[5]. In the model

[5] We refer to Bertsekas and Shreve's compendium "Stochastic optimal control: The discrete time case" [9] for a thorough discussion of technical conditions under which the Bellmann equation is satisfied.

with narrow framing the problem's Bellmann equation is

$$V_t\big(W_t\big) = \sup_{\substack{(c,\theta^S,\theta^A)\in[0,1]^3\\ \theta^S+\theta^A\leq 1}} \Bigg\{ u(cW_t) + \beta p_t E_t\Bigg[V_{t+1}\Bigg(W^L_{t+1})\Bigg)\Bigg] + \beta(1-p_t)\omega E_t\Bigg[v(W^D_{t+1}\Bigg]$$
$$+ \kappa(W_t\theta^S)^\alpha K(R_{S,t+1},R_f) + \kappa(W_t\theta^A)^\alpha K(R_{A,t+1},R_f)\Bigg\}. \tag{16}$$

with the terminal condition

$$V_T\big(W_T\big) = \sup_{(c,\theta^S)\in[0,1]^2} \Bigg\{ u(cW_T) + \beta\omega E_T\Bigg[v(W^D_{T+1})\Bigg]$$
$$+ \kappa(W_T\theta^S)^\alpha K(R_{S,T+1},R_f) + \kappa(W_T\theta^A)^\alpha K(R_{A,T+1},R_f)\Bigg\}. \tag{17}$$

In the model with broad framing the Bellmann equation is

$$V_t\big(W_t\big) = \sup_{\substack{(c,\theta^S,\theta^A)\in[0,1]^3\\ \theta^S+\theta^A\leq 1}} \Bigg\{ u(cW_t) + \beta p_t E_t\Bigg[V_{t+1}\Bigg(W^L_{t+1})\Bigg)\Bigg] + \beta(1-p_t)\omega E_t\Bigg[v(W^D_{t+1})\Bigg]$$
$$+ \kappa\Big(W_t(\theta^S+\theta^A\Big)^\alpha K(R_{P,t+1},R_f)\Bigg\}. \tag{18}$$

with the respective terminal condition

$$V_T\big(W_T\big) = \sup_{(c,\theta^S)\in[0,1]^2} \Bigg\{ u(cW_T) + \beta\omega E_T\Bigg[v(W^D_{T+1})\Bigg]$$
$$+ \kappa\Big(W_T(\theta^S+\theta^A)\Big)^\alpha K(R_{P,T+1},R_f)\Bigg\} \tag{19}$$

where $R_{P,t+1}$ again denotes the return of the framed component of the portfolio.

The general model above allows for a variety of specifications regarding the choice of the utility functions. The specific choices in this paper are made under consideration of analytical tractability and prevalence in the literature. Throughout this paper we assume that utility from consumption follows a power utility specification where

$$u(x) = \frac{1}{1-\gamma}x^{1-\gamma} \tag{20}$$

with a constant coefficient of relative risk averison $\gamma > 1$. This specification reflects the choice most encountered in the literature.

A simple specification for the bequest utility results from setting $v = u$. This assumption, while simplified, is not entirely unjustified. Assuming altruistic bequest motives, the retiree is concerned about the utility from consumption of his potential heir and assuming that the heir also receives power utility $u(C)$ from his consumption, the form of the utility function for consumption carries over, to some extend, to the form of the bequest utility. While setting $\overline{C} = 0$, $\kappa = 0$ and $v = u$ throws out many crucial features of the general model above, it is still a useful benchmark specification because it allows a degree of analytical tractability that is lost in more sophisticated models. In the following we refer to this specification as the simple model.

The simple model however has some obvious shortcomings which can lead to unrealistic predictions and unwise normative implications. Consider for example an agent with a low initial wealth. To eliminate the risk of running out of funds in late years, the longevity risk, the agent will reduce his consumption to fairly low levels right from the start in $t = 0$. Furthermore the specification of the agent's bequest utility implies that the agent will always save up a proportional part of his wealth to retain a bequest of reasonable size in the next period, even if it means reducing his own consumption to unsustainably low levels, at least if $\omega > 0$. These two effects can lead to an unrealistically low consumption under certain circumstances.

A more sophisticated take on bequest utility is given by De Nardi's [18] approach. While the parameter ω in the model above is able to control the relative importance of bequest with respect to annual consumption, there is a lack of flexibility regarding this ratio when it comes to different wealth states of the investor. As hinted at above, an investor who is low on funds may shift his priorities away from bequest to his own subsistence. This shortcoming is resolved in De Nardi's specification where a bequest of size B yields the utility

$$v(B) = \frac{1}{1-\gamma}\left(\psi + \frac{B}{\omega}\right)^{1-\gamma}. \tag{21}$$

The additional parameter ψ contained in this formulation can be interpreted as the prevalence of a bequest motive in the population or, more in line with the above reasoning, the degree to which bequests are a luxury good[6]. Among other effects, this ensures that the agent's relative importance assigned to bequest is in accordance with his ability to bequeathe a part of his wealth while still maintaining reasonable consumption levels himself.

The advanced bequest specification and the assumption of a non-zero government subsidy have some implications for the numerical properties of the problem. An agent with access to a subsidy can rely on receiving at least the base consumption utility $u(\underline{C})$ in every year. Hence the agent's time t value function is bounded from below by

$$V_t(W_t) \geq \Phi_t(\underline{C}) \tag{22}$$

for all $W_t \geq 0$ and $t = 0, \ldots, T$. Furthermore the requirement $C_t \geq \underline{C}$ implies that $c_t = 1$ for $W_t \leq \underline{C}$, i.e. the agent's decision set collapses to the point $(1, 0)$. In turn this implies that the agent's value function equals

$$V_t(w) = \underline{V}_t := \Phi_t(\underline{C}) \tag{23}$$

for $w \leq \underline{C}$. Lemma 2 in the appendix provides the exact expression for this boundary

$$\underline{V}_t = \sum_{s=t}^{T} p_{t,s}(1-p_s)\left(\frac{\beta^{t-s+1}-1}{\beta-1}u(\underline{C}) + \beta^{t-s+1}v(0)\right). \tag{24}$$

For all $W_t \geq \underline{C}$ the value function still satisfies the respective Bellman equations given by equations (16) and (18).

3 Parameter choice

A summary of the model calibration is given by table 1. We assume that the underlying price process of the stylized stock investment follows a geometric Brownian Motion with the dynamics

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{25}$$

[6] To achieve a better understanding of the parameters in v it helps to regard the simpler problem with no time weighting, a fixed wealth level W, a fixed lifetime of T years, no investment opportunities and no government subsidy. The optimal consumption resulting from first order conditions is then $C = (W + \omega\psi)/(\omega + T)$ and the optimal bequest is $B = \omega(C - \psi)$, i.e. the bequest covers ω periods of spending the amount $C - \psi$, i.e. the amount the agent's own consumption exceeds ψ. If the agent cannot bequeathe an amount which exceeds ψ for ω years then the optimal bequest is zero.

which implies that equity returns are i.i.d lognormally distributed. The parameters μ and σ are chosen such that the expected return in the benchmark case is $E[R_t] = 6\%$ with a standard deviation of $\sigma(R_t) = 20\%$. The risk free rate is set to $R_f = 2\%$ which implies a risk premium $E[R_t] - R_f$ of 4%. This is a standard calibration in life cycle models as for example in Cocco and Gomes [16]. The calculatory interest rate R_A is set to equal the risk free rate R_F.

Model parameters	
Expected utility framework	
W_0	400000
T	35
age in $t = 0$	65
age in $t = T$	100
γ	5
r_f	2%
r_A	2%
$\mathrm{E}[r_t]$	6%
$\sigma[r_t]$	20%
β	0.96
Bequest motive	
ω Simple Model w/o Bequest Motive	0
ω Simple Model w/ Bequest Motive	131.61
ω Advanced Model	7.81
ψ	$0.67 \cdot FAAC$
$FAAC$	20513
Loss averison	
λ	2
α	0.88

Table 1: Model parameters

The time weighting parameter and the risk aversion parameter follow standard life cycle assumption and are set to $\beta = 0.96$ and $\gamma = 5$ which are also standard assumptions as in Cocco and Gomes [16]. Entry to retirement is assumed to be at age 65 as for example in [13] and we allow a maximum age of 100 years which implies a time horizon of $T = 35$ years.

In the non-simple bequest utility specification the benchmark parameters are set to $\omega = 7.81$ and $\psi = 0.67 * FAAC$. Here $FAAC$, the fully annuitized average consumption, denotes the average experienced optimized consumption level $FAAC = E[\sum_{t=0}^{\tau} C_t] = 14674$ under full annuitization[7], i.e. $\theta_t^A = 1$ for all $t \in [0, T]$. $\tau \in [0, T]$ denotes the time of death of the retiree, i.e. the last time the agent is alive. This is in line with the original parametrizaion in De Nardi [18], with the exception that the risk aversion is adjusted to be identical to the risk aversion concerning consumption in our model.

The strength of the bequest motive in the simple model is choosen such that the average

[7]In order to obtain the $FAAC$, we solve the agent's original optimization problem under the assumption that $\omega = 0$, i.e. in the absence of a bequest motive, and $\theta_t^A = 1$ for all $t \in [0, T]$. Aside from the two modifications, the optimization technique is identical to the one applied for the original problem. The $FAAC$ itself is computed from the optimal consumption values by forward simulation.

bequest size in the benchmark model yields the same bequest utility as in the advanced model. Hence

$$\omega_{simple} = \omega \left(\frac{\psi + \frac{\bar{B}}{\omega}}{\bar{B}} \right)^{1-\gamma}, \tag{26}$$

where $\bar{B}$ is defined as the average bequest size in the benchmark model with $\kappa = 0$. In either bequest utility specification, the no bequest motive case is represented by the parameter choice $\omega = 0$.

The survival probabilities used in the pricing of the insurance products as well as the agent's individual survival probabilities are both taken from german death tables[8] and we assume a male policyholder in both cases. Furthermore we omit a loading on the premium calculation.

The calibration of the subjective evaluation function for the loss averse investor follows Kahnemann und Tversky's specification [30] whith $\alpha = 0.88$ and $\lambda = 2$.

4 Solution technique

4.1 Optimal consumption and investment policy in the simple model

Before we solve the general problem we focus on the simple model for which, at leats in parts, an analytical solution can be obtained. The starting point is the problem's Bellmann equation[9] which allows us to to solve the optimization problem by backwards induction. By lemma 1 in the appendix, V_t in this simple case is positively homogeneous with degree $1-\gamma$, i.e.

$$V_t(w) = w^{1-\gamma} V_t(1) \tag{27}$$

for all $t = 0, 1, \ldots, T$. An important consequence of this result is that the optimisation can be conducted independent of the current wealth level. To simplify notation we define

$$\Gamma_t^{(1)} = E_t \left[(\theta Q_t + \alpha Q_{A,t} + (1-\theta-\alpha) Q_f)^{1-\gamma} \right] \tag{28}$$

and

$$\Gamma_t^{(2)} = E_t \left[\left(\theta Q_t + (1-\theta-\alpha) Q_f \right)^{1-\gamma} \right]. \tag{29}$$

Invoking (27), (28) and (29) we can rewrite the Bellmann equation for $t = 0, 1, \ldots, T-1$ as

$$\begin{aligned}
&V_t(w) \\
&= w^{1-\gamma} \cdot \sup_{(c,\theta,\alpha)\in[0,1]^3} \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \beta p_t (1-c)^{1-\gamma} V_{t+1}(1) \Gamma_t^{(1)} + \beta(1-p_t)\omega \frac{(1-c)^{1-\gamma}}{1-\gamma} \Gamma_t^{(2)} \right\} && (30) \\
&= w^{1-\gamma} \cdot \sup_{(c,\theta,\alpha)\in[0,1]^3} \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \frac{(1-c)^{1-\gamma}}{1-\gamma} \beta \Big(p_t V_{t+1}(1) \Gamma_t^{(1)} + (1-p_t)\omega \Gamma_t^{(2)} \Big) \right\} && (31) \\
&= w^{1-\gamma} \cdot \max_{c\in[0,1]} \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \frac{(1-c)^{1-\gamma}}{1-\gamma} \cdot \min_{(\theta,\alpha)\in[0,1]^2} \beta \Big(p_t (1-\gamma) V_{t+1}(1) \Gamma_t^{(1)} + (1-p_t)\omega \Gamma_t^{(2)} \Big) \right\} && (32)
\end{aligned}$$

[8]Source: Sterbetafel 2009/11 Deutschland männlich, Periodensterbetafeln für Deutschland 2009/2011, Statistisches Bundesamt, Wiesbaden 2012.

[9]In the simple model the Bellmann equation is defined by equations (16) and (17) with $\kappa = 0$.

where separation of the problems concerning consumption and asset allocation is justified by the fact that for any fixed $c \in [0,1]$ the function

$$g\colon [0,+\infty) \to (-\infty,0), \quad g(x|c) = \frac{c^{1-\gamma}}{1-\gamma} + \frac{(1-c)^{1-\gamma}}{1-\gamma} \cdot x$$

is strictly decreasing in $x > 0$. Furthermore a realistically ($\gamma > 1$) calibrated CRRA-utility function is negative and strictly increasing. This implies that $V_t < 0$ for all t and therefore $(1-\gamma)\cdot V_t(1) > 0$. Because $\Gamma_t^{(1)}, \Gamma_t^{(2)} > 0$ holds for all t by definition we conclude that

$$\Gamma_t := \beta\Big(p_t(1-\gamma)V_{t+1}(1)\Gamma_t^{(1)} + (1-p_t)\omega\Gamma_t^{(2)}\Big) > 0 \tag{33}$$

for all t. Independent of the choice of c, a smaller value of x will yield a higher function value $g(x|c)$. Therefore a smaller value of Γ_t will yield a higher function value $g(\Gamma_t|c)$ and thus separation of the optimization problems is justified. Both suprema in equation (32) are attained because they are taken over the compact sets $[0,1]$ and $[0,1]^2$.

At each time $t = 0, 1, \ldots, T-1$ we can solve (32) given $V_{t+1}(1)$ in two steps:

- **Step 1:** We find the optimal asset allocation by calculating

$$\hat{\Gamma}_t = \min_{(\theta,\alpha)\in[0,1]^2} \beta\Big(p_t(1-\gamma)V_{t+1}(1)\Gamma_t^{(1)} + (1-p_t)\omega\Gamma_t^{(2)}\Big).$$

- **Step 2:** Given $\hat{\Gamma}_t$ we can calculate $V_t(1) = \max_{c\in[0,1]} g(\hat{\Gamma}_t|c)$.

It is not possible to derive an analytic expression for the optimum in step 1. Hence numerical calculation of the optimal asset allocation is neccesary. However given $\hat{\Gamma}_t$, the maximum in step 2 can be calculated from first order conditions. For any $x > 0$ we have

$$\frac{d}{dc}\, g(x|c) = c^{-\gamma} - (1-\gamma)(1-c)^{-\gamma}x = 0 \tag{34}$$

which is solved by

$$\hat{c} = \frac{x^{\frac{1}{-\gamma}}}{1 + x^{\frac{1}{-\gamma}}} \in (0,1). \tag{35}$$

The strict concavity of g further implies that $\hat{c}$ is the unique maximum $\hat{c} = \mathrm{argmax}_{c\in[0,1]} g(\hat{\Gamma}_t|c)$.

On the basis of these results we can now set out to compute the value function V_t and the associated optimal consumption and asset allocation strategies $(\hat{c}_t, \hat{\theta}_t^S, \hat{\theta}_t^A)$ by means of backward induction starting in time T where

$$V_T(w) = w^{1-\gamma} \cdot \sup_{(c,\theta)\in[0,1]^2} \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \frac{(1-c)^{1-\gamma}}{1-\gamma}\beta\Gamma_T^{(2)} \right\}. \tag{36}$$

After the optimized value $\hat{\Gamma}_T = \min_{\theta\in(0,1)} \Gamma_T$ has been determined, we can invoke (35) to find the optimal consumption parameter $\hat{c}_T$ given by

$$\hat{c}_T = \frac{\hat{\Gamma}_T^{\frac{1}{-\gamma}}}{1 + \hat{\Gamma}_T^{\frac{1}{-\gamma}}}. \tag{37}$$

Using this in (36) yields

$$V_T(w) = \frac{w^{1-\gamma}}{1-\gamma} \left[\left(\frac{\hat{\Gamma}_T^{\frac{1}{-\gamma}}}{1 + \hat{\Gamma}_T^{\frac{1}{-\gamma}}} \right)^{1-\gamma} + \beta\hat{\Gamma}_T \left(\frac{1}{1 + \hat{\Gamma}_T^{\frac{1}{-\gamma}}} \right)^{1-\gamma} \right] \tag{38}$$

and especially

$$V_T(1) = \frac{1}{1-\gamma}\left[\left(\frac{\hat{\Gamma}_T^{\frac{1}{-\gamma}}}{1+\hat{\Gamma}_T^{\frac{1}{-\gamma}}}\right)^{1-\gamma} + \beta\hat{\Gamma}_T\left(\frac{1}{1+\hat{\Gamma}_T^{\frac{1}{-\gamma}}}\right)^{1-\gamma}\right] \tag{39}$$

Plugging this into (33) with $t = T-1$ now allows us to find the optimal time $T-1$ asset allocation $(\theta_{T-1}, \alpha_{T-1})$ together with Γ_{T-1}. As in $t = T$ this yields the optimal consumption using (35) and the time $T-1$ value function V_{T-1} as in (38). Repeating this procedure until $t = 0$ produces the optimal strategies $(\hat{c}, \hat{\theta}, \hat{\alpha})$ where

$$\hat{c}_t = \frac{\hat{\Gamma}_t^{\frac{1}{-\gamma}}}{1+\hat{\Gamma}_t^{\frac{1}{-\gamma}}}. \tag{40}$$

and the value function V with

$$V_t(w) = \frac{w^{1-\gamma}}{1-\gamma}\left[\left(\frac{\hat{\Gamma}_t^{\frac{1}{-\gamma}}}{1+\hat{\Gamma}_t^{\frac{1}{-\gamma}}}\right)^{1-\gamma} + \beta\hat{\Gamma}_t\left(\frac{1}{1+\hat{\Gamma}_t^{\frac{1}{-\gamma}}}\right)^{1-\gamma}\right]. \tag{41}$$

4.2 Solution technique in the advanced model

In principle, the advanced model can also be solved by backwards induction following the same logic as above. However the introduction of a subjective investment evaluation, as well as social security and the affine nature of the bequest utility given by equation (21), result in optimal strategies that are no longer independent of the investor's current wealth level. Separation of the consumption and the investment problem is then no longer possible and we fully resort to numerical optimization methods in each step of the backwards induction.

The starting point is again the terminal condition for V_t imposed by the time T Bellmann equation. For any fixed wealth levels w_l we can find $V_T(w_l)$ and the respective optimal strategies $(c_T(w_l), \theta_T^S(w_l), \theta_t^A(w_l))$ by numerically solving the static optimization problem[10]

$$V_T(w_l) = \max_{\substack{0\le c,\theta,\alpha\le 1\\ \theta+\alpha\le 1}} \left\{ u(cw_l) + \beta E_T\left[v((w_l(1-c)(\theta(1+R_T) + (1-\theta-\alpha)(1+R_f)) \right] \right\}. \tag{42}$$

We repeat this procedure to compute pairs $(w_l, V_T(w_l))$ on a grid of wealth levels w_l. The smoothness of V_T allows us to construct an interpolant $\hat{V}_T$ for V_T from these pairs. Given $\hat{V}_T$, the time $T-1$ optimization problem given by (16) or (18) depending on the scope of the framing reduces to a static optimization problem for any fixed wealth level w_l and we can compute V_{T-1} for a grid of wealth levels as well as the optimal policies $(c_{T-1}(w_l), \theta_{T-1}(w_l), \alpha_{T-1}(w_l))$. We repeat this step until we obtain the time $t = 0$ value function V_0.

In the procedure described above, we use a dynamic time dependant grid $\{w_l\}_{l=0,\ldots,L_t}$ where we add an additional grid point $w_{L_{t+1}} = w_{L_t} + \Delta_w$ in each time step to prevent extrapolation when calculating the value function[11]. The $L_0 = 30$ grid points in the base grid are exponentially placed to increase the efficieny in the interpolation of V_t, which has a much higher curvature for lower wealth levels and is asymptotically linear for high wealth levels. The base grid spans from $w_0 = \underline{C}$ to $w_{30} = W_0$ where W_0 denotes the initial wealth of the agent.

[10]The equation below gives the time T optimization problem in the advanced model without loss averse investment evaluation. In the model with loss averse investment evaluation appears the additional term $\kappa(w_l(\theta^S + \theta^A)))^\alpha K(R_{P,T+1}(\theta^S, \theta^A), R_f)$ in the model with broad framing and $\kappa(w_l\theta^S))^\alpha K(R_{S,T+1}, R_f) + \kappa(w_l(\theta^A)))^\alpha K(R_{A,T+1}, R_f)$ in the model with narrow framing on the right hand side of the equation.

[11]Extrapolation on the left side of the wealth grid is not an issue because the value function is constant and equal to $\Phi_{t+1}(\underline{C})$ for all $W_{t+1} < \underline{C}$.

To interpolate V in each time step we use cubic spline interpolation with a modification to be constant to $\Phi_{t+1}(\underline{C})$ on the left side of the wealth grid. The conditional expectations ocurring in each time step's optimization problem are calculated using Gauss-Hermite-Quadrature (GH-Quadrature) with $n = 32$ sample points. To achieve an optimal approximation, we follow Liu and Pierce [25] and apply a transformation of the standard weights and sample points in GH-Quadrature to account for the specific parametrization (μ, σ) of the lognormal returns in our model.

5 Results

5.1 Optimal Arrow annuity demand

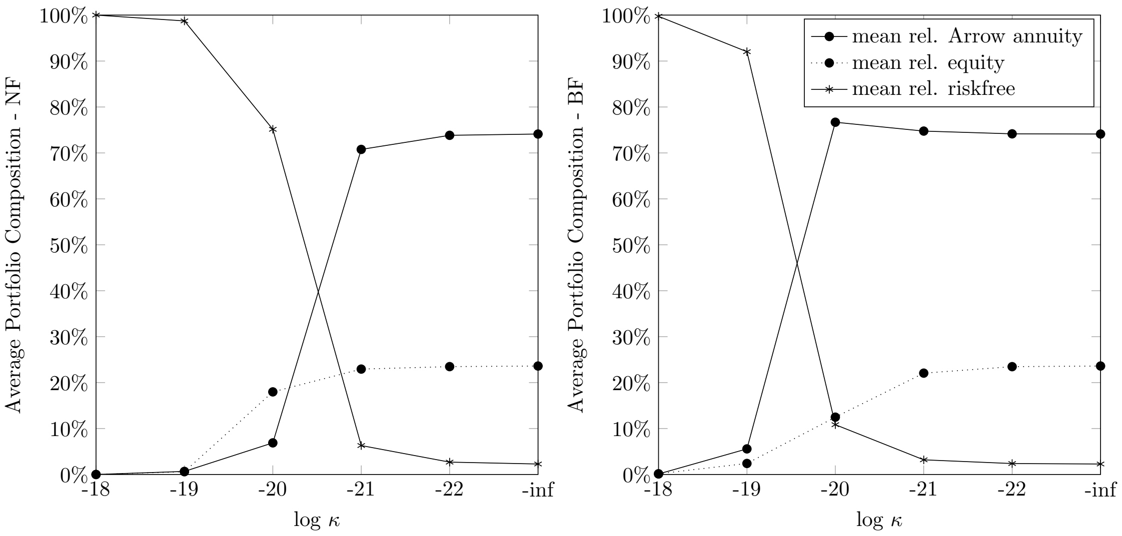

Figure 1: *Average portfolio composition for the narrow framing agennt (NF) and the broad framing agent (BF) plotted versus* $\log(\kappa)$. *Results are calculated as the sample mean of* $N = 10000$ *simulated values obtained by forward simulation for the relative annual annuitization level* $\frac{1}{\tau}\sum_{t=1}^{\tau} a_t$, *relative equity level* $\frac{1}{\tau}\sum_{t=1}^{\tau} \theta_t^S$ *and relative risk-free level* $\frac{1}{\tau}\sum_{t=1}^{\tau}(1 - a_t - \theta_t^S)$ *where* τ *denotes the time of death of the retiree.*

We examine the results in the advanced model specification first, with a focus on the effect of loss aversion. Afterwards we analyze the validity of the simple model in the case without investment evaluation. In the former case, the optimal annual annuitization level depends on the age as well as the current wealth of the investor. A visualization of the exemplary effect of loss aversion for the narrow framing investor on the optimal portfolio composition is given in figure 1. It contains the averaged shares of the investor's portfolio held in Arrow annuities, equity and riskfree bonds. The values are calculated by averaging the mean portfolio shares, calculated as $\frac{1}{\tau}\sum_{t=1}^{\tau}(1 - c_t)a_t$ for Arrow annuities, over $N = 10000$ simulated trajectories. The mean death age is 86.09 with a standard deviation of 7.88 years. A more detailed overview of the results for both model specifications (NF and BF) is given in table 2. We refer to the table caption for a detailed description on how the individual values are calculated.

Effective parametrizations for κ lie between 10^{-23}, for which the optimal portfolio composition is very close to the benchmark consumption frame, with an average annuity level of 67%, and 10^{-18}, which results in complete abstinence from annuity and equity markets. This parameter range is close to the parametrization in the similiar model by Barberis and Huang [3]. In an equilibrium model with a time-additive expected utility specification they estimate the relative strength of the investment evaluation to be $0.45\bar{C}^{1-\gamma}$ where $\bar{C}$ denotes the part of the agent's income which is not narrowly framed, e.g. labour income. Assuming an annual labour income of 50000 and no dividends this results in a relative strength of the investment frame of 10^{-20}. In a similiar time-additive model specification, where health insurance is the only framed asset, Hwang [24] finds that for values between 10^{-9} and 10^{-13}, depending on the agent's risk aversion, purchase of actuarilly fair health insurance is rejected.

Abstinence from annuity markets means that the investor's total estate is subject to bequest. Therefore the relative strength of the investment evaluation captured in the parameter κ has a strong impact on the bequest size. Because the overly large bequest sizes are not motivated by the desire to leave a bequest but by the reluctance to annuitize, loss averse investors involuntarily leave too large bequests. Such effects are to be expected when loss aversion is implemented in broader problems like retirement portfolios because unlike risk aversion, the impact of loss aversion is independent from the intended goals of the investment strategy.

	Mean rel. annuitiza-tion	Intertemp. variation annuitiza-tion	Mean rel. equity exposure	Intertemp. variation equity exposure	Mean bequest size	Std. bequest size
Consumption Frame						
$\kappa = 0$	.7411	.1353	.2361	.1208	.0940	.7140
Investment Frame (NF)						
$\log \kappa = -23$	.7383	.1354	.2348	.1273	.0945	.7179
$\log \kappa = -22$	.7076	.1378	.2294	.1171	.1056	.7243
$\log \kappa = -21$	.0688	.2878	.1798	.1503	.2950	.8759
$\log \kappa = -20$	.0067	.2866	.0060	.0868	.3547	.6781
$\log \kappa = -19$	.0000	.0000	.0000	.0000	.3768	.6142
Investment Frame (BF)						
$\log \kappa = -23$	.7415	.1355	.2347	.1190	.0938	.7154
$\log \kappa = -22$	.7475	.1391	.2206	.1215	.0911	.7173
$\log \kappa = -21$	.7668	.1599	.1248	.1668	.0788	.7690
$\log \kappa = -20$	.0557	.3700	.0240	.1227	.2949	.8686
$\log \kappa = -19$	.0011	.1100	.0016	.1138	.3764	.6164

Table 2: *Mean and standard deviations for the portfolio composition and bequest size for various model specifications. The values are calculated from $N = 10000$ individual values derived by forward simulation. The respective formulae for the individual values from which the means are calculated are $\frac{1}{\tau}\sum_{t=1}^{\tau}(1-c_t)a_t$ for the mean relative annuitization and $\frac{1}{\tau}\sum_{t=1}^{\tau}(1-c_t)\theta_t^S$ for the mean relative equity. Here τ denotes the (random) time of death of the retiree. The intertemporal variations are calculated as the standard deviations of the average per period portfolio shares over the whole time horizon. The values regarding bequest sizes are the sample mean and sample standard deviation over all N bequest sizes.*

A comparison of the average portfolio compositions in both model specifications shows that the scope of the framing can have a strong effect on the agent's investment strategy. Because diversification effects are ignored in the narrow framing agent's investment evaluation, the risky assets are perceived as riskier than in the broad framing agent's evaluation. Furthermore the broad framing agent has the ability to exercise some control over the return distribution that is evaluated compared to the narrow framing agent. Therefore the narrow framing agent's risk exposure, in total, is smaller than that of the broad framing agent in all cases considered here. This is not true for the individual asset classes. For $\log \kappa = -21$, the narrow framing agent's equity exposure exceeds that of the broad framing agent. However in the same model specification, the broad framing agent invests a much bigger share of his available wealth in Arrow annuities. In fact, this share is slightly larger than the benchmark agent's mean annuitization degree. Because of the reduced equity exposure, the total risk exposure is still below the benchmark agent. Therefore in the case $\log \kappa = -21$, loss aversion leads to a reduced risk exposure for the broad framing agent, but to a higher annuitization degree. This shows that in the broad framing model, loss aversion cannot explain the annuity puzzle in all cases but may even provide further evidence for it. Nevertheless, loss aversion can explain lower annuity and equity demand in most model specifications considered here. In the model with narrow framing, loss aversion leads to a reduction in annuity and equity demand in all the parameterizations considered here.

The respective values regarding the mean relative annuitization and the mean relative equity exposure are given further below where we discuss the adequacy of the simple model when it comes to being an approximation of the advanced model.

5.2 Optimal wealth, consumption, bequest and equity exposure

This section contains an analysis of the optimal wealth, consumption and asset allocation trajectories for selected model specifications. As in the previous section, all the values below are calculated by forward simulation ($N = 10000$) using the respective optimal strategies and then averaging the resulting trajectories for the state and control variables. At first we analyze the advanced model without loss aversion to obtain a benchmark to compare the models with loss aversion to. In the way it is applied in this paper, loss aversion acts mainly as a restriction on the agent's willingness to invest in risky assets. This leads to negative wealth effects which translate mainly in a decline in consumption. Even though wealth levels are lower in most cases for the loss averse investor, abstinence from annuity markets can often lead to larger bequest sizes. See figure 2 for an overview of the average portfolio composition through time in the cases considered in this section.

We begin our analysis with the benchmark model with no prospect utility, i.e. $\kappa = 0$. Because of the lack of an investment evaluation, the agent's investment strategy does not directly enter his preferences but only indirectly via the resulting distribution of consumption and bequests. In contrast to bequest sizes, which can strongly depend on the age of death, the consumption levels can be controlled by the agent to a much higher degree. For this reason we begin our analysis with the optimal consumption paths. The average consumption level on the whole time horizon is 17450, which is 4.36% of the agent's initial wealth. The intertemporal variation of consumption, measured by the standard deviation of the mean per period consumption, is 6236 or 35.74% of the average consumption level on the whole time horizon. Because the probability that the agent actually experiences the planned consumption levels decreases over time the agent has reason to prioritize consumption in the early periods. As a result of this, his consumption is highest (24059 or 6.01% of the agent's initial wealth) in the first period and lowest (5234 or 1.30% of the agent's initial wealth) in the last period. Furthermore the average consumption level that is actually experienced by the agent[12] is given by 21459 (5.36% of his initial wealth) which is a 22.97% increase compared to the average consumption level along the whole time horizon.

Besides consumption, the agent also exercises immediate control over his investment strategy. As mentioned in the previous section the agent holds on average 74.11% of his wealth after consumption in Arrow annuities, 23.61% in equity and 2.29% in riskless bonds. As in the previous sections these values give the actually experienced asset allocation and not the averages along the whole time horizon. For individual periods, the optimal asset allocation depends on the current

[12] The average experienced consumption level is given by

$$\sum_{\tau=0}^{T} p_{0,t}(1 - p_{t+1}) E\left[\frac{1}{\tau}\sum_{t=0}^{\tau} C_t\right]. \tag{43}$$

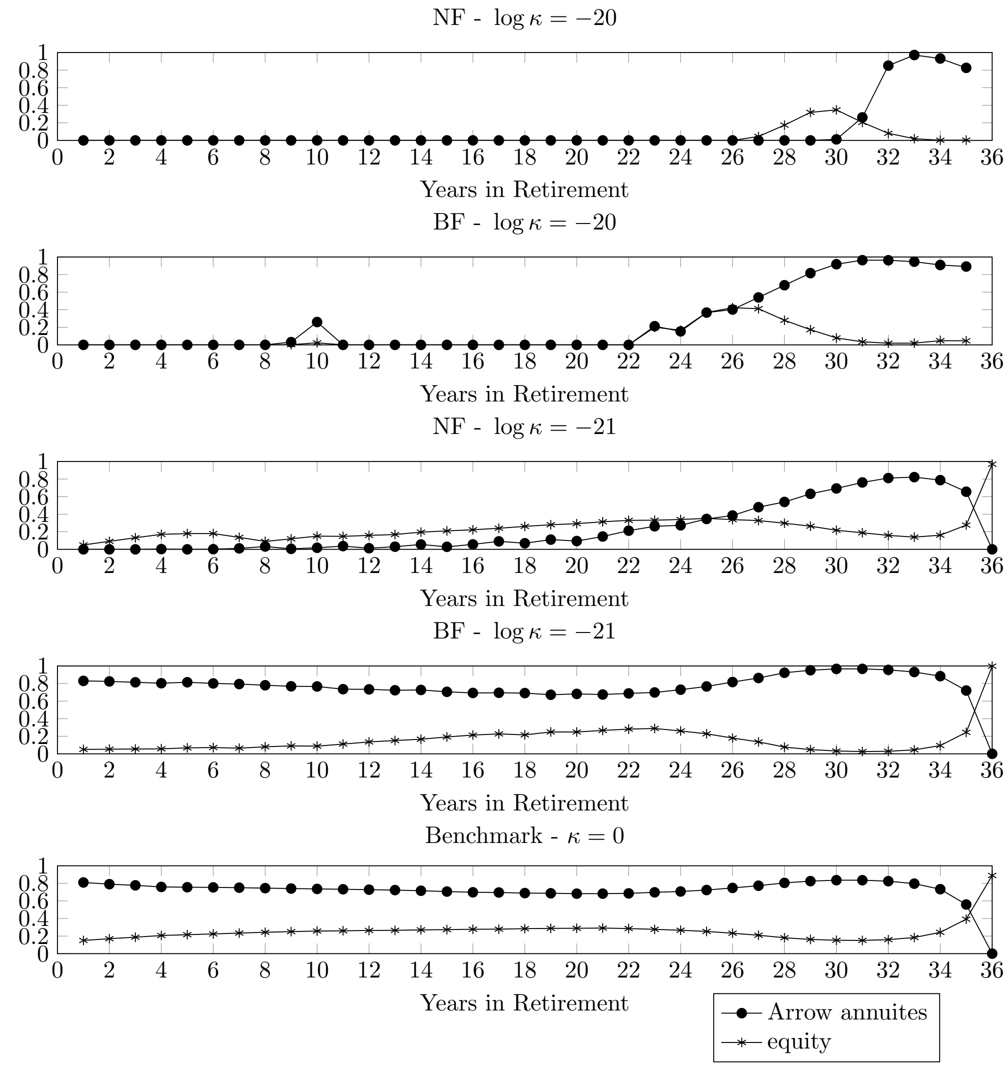

Figure 2: *Average Portfolio Compositions in each year of retirement for the model specifications considered in this section.*

wealth level which is generally decreasing over time and the survival probability of the agent. Therefore there is a high intertemporal variation within the investment trajectories.

The agent's average portfolio share of Arrow annuity is 81% in the first period and then continouosly decreases to 68.27% in period 20. In the remaining retirement years the average portfolio share of Arrow annuities increases again to 83.57% in period 31 after which it decreases to 55.91% in the second to last period. The agent's average equity exposure forms a mirrored image of his annuity exposure. In the first period, his portfolio share of equity is 15%. Afterwards his average portfolio share of equity increases continuously to 29.10% in period 21, followed by a steady decrease to 15.04% in period 31, followed by an increase to 39.21% in the second to last period. Excluding the last period, the agent holds at the most 4.87% of his wealth after consumption in bonds, at the least riskless bonds make up 0.66% of the agent's portfolio. The last period investment strategy differs from the previous periods since the agent faces certain death and is thus no longer concerned with consumption in the future. This leads to riskier investment strategies again because the agent is less risk averse over bequests than over consumption in the advanced model.

On average, his final period portfolio consists of 88.97% equity and 11.03% riskless bonds.
The development of wealth on hand over time is a result of the agent's consumption and investment strategy. In the absence of a bequest motive the agent would aim to consume all of his wealth within his remaining lifespan. Although the presence of a bequest motive introduces an incentive to keep wealth levels on a certain level throughout the whole time horizon, the agent's optimal wealth trajectories still result in wealth levels which are close to zero in the final period. Because survival until the late periods is not that likely the agent prioritizes consumption in the early and middle periods over high bequests in the late periods. Furthermore in contrast to the riskless bond, which is always dominated by the Arrow annuity for an agent without a bequest motive, wealth on hand may be invested in equity mainly because of the attractive risk premium and not to preserve wealth for bequests in the case of death. for this reason, funds that are invested in equity also lead to larger bequests even though that is not the main intention of the investment. The bequests that result from these investments are accidental bequests. Because the agent's wealth and the absolute size of the agent's equity investments decrease over time, the size of accidental bequests also decreases. In total, these effects lead to a high degree of assymetry between early bequests and late bequests. While death within the first 20 periods results in a bequest size between 10.45% of the agent's initial wealth and 21.86% of the agent's initial wealth, death in the later years of retirement leads to bequest sizes between 9.18% of the agent's initial wealth and .06% of the agent's initial wealth. The average realized bequest size 37590 or 9.40% of the agent's initial wealth. The standard deviation of bequest sizes is 71.40% of the average bequest size.

In the following we turn to the advanced model with prospect utility. We analyze the cases $\log \kappa = -20$ and $\log \kappa = -21$ in the model specification with narrow framing and broad framing. The introduction of the loss inverse investment evaluation means that the agent now has to balance his desire for sufficient consumption and bequest levels with his reluctance to make risky investments. For higher values of κ, the agent becomes more reluctant to accept possible losses, which eventually leads to complete abstinence from annuities and equity investments. From the point of view of a classical expected utility optimizer, this behaviour is irrational as it leads to lower expected utility from consumption and bequest and to some extend to generally lower consumption and bequest levels. After a discussion of the optimal trajectories for the loss averse agent, we analyze the size of these decreases in utility which are the price for abstaining from possible losses.

At first we look at the parameter choice $\log \kappa = -20$. In the model specification with narrow framing, where diversification effects are ignored when the investment prospects of the two risky asset classes are evaluated, the introduction of the loss averse investment evaluation has a crucial effect on the agent's investment behaviour. On average along the whole timeline, the narrow framing agent only holds 10.70% of his wealth after consumption in Arrow annuities. Because of the relatively high survival probabilities in the earlier periods, the mortality credit of the Arrow annuities in those periods is fairly low. Especially in earlier periods this makes Arrow annuities a low-return/high-risk investment because they result in a total loss in the case of death. For the expected utility from consumption, the total loss in the death case is negligible because there is no more demand for consumption in the death case. In contrast to this, the loss averse prospect utility framework evaluates the investment in Arrow annuities solely by its inherent return characteristics and is thus sensitive to the total loss in the death case. Furthermore the relative importance of prospect utility within the preference functional depends on the size of an investment. Since the agent's wealth process is typically decreasing over time, the relative strength of the prospect utility is higher in the early periods when the agent has a lot of cash to invest. For this reason Arrow annuities appear particularly unattractive in earlier periods. In this case this leads to a complete reluctance to invest in Arrow annuities in the first 29 years of retirement[13] In the re-

[13]It should be noted that the agent could theoretically adjust the size of his investments until the negative effects of loss aversion on the willingness to invest are balanced with the positive effects on future consumption and bequest

maining years of the retirement phase the decreasing wealth levels result in a weakening of the effect of loss aversion relative to consumption and bequest utility and therefore the agent becomes more willing to invest in Arrow annuities. Between period 30 and 35 the agent's average portfolio share of Arrow annuities is 64.23%. In total, because Arrow annuity investments are only attractive in the final years of retirement, the actually experienced average portfolio share of Arrow annuitites is only 0.67% and therefore significantly lower than the average along the whole timeline.

The effect of loss aversion on the agent's willingness to invest in equity is similiar to the effect on Arrow annuities. Again, in the early periods, the relatively high wealth on hand leads to a strong effect of loss aversion which results in a reluctance to invest. For $\log \kappa = -20$, the agent abstains from equity markets until period 26 where he has on average 16.51% of his initial wealth remaining. In the remaining years of retirement, the agent first increases his equity portfolio share to an average maximum of 34.74% in period 30 and then reduces it to an average of 0.32% in the second to last period. In this case the agent actually consumes all of his wealth in his final period. The average portfolio share of equity is 0.60%. In this case the agent actually consumes all of his wealth in his final year of retirement. Therefore there is no investment strategy for the final period.

The overall reluctance to invest in risky assets comes with a price regarding the agent's consumption. Abstinence from equity and annuity markets results in lower investment returns and therefore lower wealth levels. Now the average consumption level along the whole timeline is 14895 and the average consumption level that is actually experienced is 17352. This is a 14.64% reduction of the average along the whole timeline and a 19.14% reduction of the actually experienced consumption compared with the benchmark agent who is not loss averse. From the perspective of bequests, loss aversion has two contrary effects. Abstinence from equity markets means the agent misses out on the high risk premia which leads to lower wealth levels in most cases and therefore to lower bequests. Abstinence from annuity markets means the agent's whole estate is subject to bequest which leads to higher bequest levels. For the parameter choice $\log \kappa = -20$, the second effect dominates strongly. Average bequest levels are 142948 which is a 280.28% increase compared to the benchmark model. The standard deviation of the bequest size is 67.80% of the mean bequest size. This is the case because the agent now leaves very high accidental bequests if he dies in the early retirement periods, because of his abstinence from annuity markets, followed by relatively low bequests in the late periods of his retirement because of his lower wealth levels compared to the benchmark agent.

The reluctance to invest in risky assets is not as strong for the broad framing agent. In addition to the consideration of diversification effects in the investment evaluation, the broad framing agent also has some control over the return distribution that is evaluated. The narrow framing agent always evaluates the return characteristics of the individual assets independent of the actual portfolio composition. It is only the weighting of the resulting return evaluations that is affected by the agent's portfolio choice. In contrast to this, the broad framing agent can design a portfolio whose return distribution is particularly well adjusted to the form of the loss averse investment evaluation. As a result, the average portfolio share of Arrow annuities, as well as equity, increases compared to the narrow framing agent. The former to 5.57% (compared to 0.67%) and the latter to 2.40% (compared to 0.60%). Even though the broad framing agent does also abstain from annuity and equity markets in most of the early periods, he does not abstain in all periods and furthermore starts to invest larger amounts much earlier than the narrow framing agent. Between period 22 and the second to last period, the agent's average portfolio share of Arrow annuities is 67.41%. His average portfolio share of equity during that time span is 17.45%. The resulting wealth effects of the more balanced portfolio translate into a slight increase in consumption. The average consumption that is actually experienced is 18082 which is a 4.04% increase compared to the narrow framing agent but still a 15.74% decrease compared to the benchmark agent. The

utility. However for technical reason we employ the not unplausible restriction that the minimal portfolio share must be at least 1% of the agent's wealth on hand.

average bequest size reduces by 17.48% compared to the narrow framing agent to 117961. In relative terms, bequests are more volatile for the broad framer with a standard deviation of 86.86% of the mean bequest size.

In the case $\log \kappa = -21$, both the narrow framing and the broad framing agent increase their risk exposure significantly but still remain below the benchmark levels in terms of total risk exposure. However there are now distinct differences between the optimal portfolio structure of the narrow framing agent and the broad framing agent. We begin our analysis with the narrow framing agent. Because diversification effects are ignored in the subjective investment evaluation, a decrease in κ leads to a decrease in the reluctance to invest in the individual risky assets and thus the portfolio shares of both risky assets increase. Arrow annuities are still perceived as less attractive from a pure investment point of view because of the total loss in the death case. For this reason the increase in the equity portfolio share is larger in size and the resulting portfolio share much closer to the benchmark portfolio share than the Arrow annuity portfolio share. The mean Arrow annuity portfolio share is now 6.88%, which is only 9.28% of the benchmark annuitization degree, and the mean equity portfolio share is now 17.98%, which is 76.15% of the benchmark equity exposure. The higher risk exposure, compared to the previous case, results in higher wealth levels which lead to an increase in consumption. In this case the narrow framing agent experiences an average consumption level of 19325. The average along the whole time horizon is 16142. These are increases of 11.37% and 8.37% compared to the case $\log \kappa = -20$ but still 9.94% and 7.50% below the values for the benchmark agent who is not loss averse. Average bequest sizes decrease for the narrow framing agent compared to the previous case. Albeit the higher risk exposure leads to higher wealth levels in many cases, the relatively strong increase in the degree of annuitization means that a smaller part of the agent's wealth is subject to bequest. In this case, the average bequest size decreases by 16.83% compared to the narrow framing agent in the previous case. Compared to the benchmark agent this is still an increase of 213.83%. In relative terms the standard deviation of bequest sizes increases to 87.59% of the mean bequest size compared to 67.81% of the mean bequest size for the narrow framing agent in the previous case and 71.40% of the mean bequest size in the benchmark case.

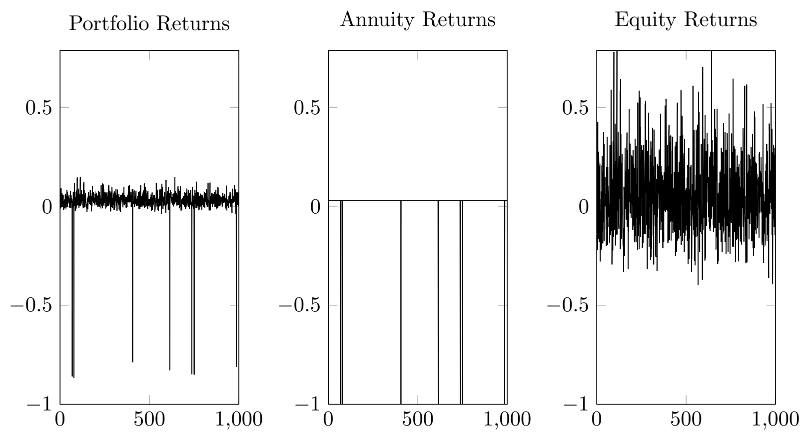

Figure 3: *Comparison of* $N = 1000$ *simulated returns of the risky component of the optimal portfolio and of the individual assets in the first period. The risky component of the agent's portfolio contains* 94.32% *Arrow annuities and* 5.68% *equity.*

The broad framing agent's investment strategy differs drastically from the narrow framing agent's portfolio choice. The lower relative weight of the loss averse investment evaluation compared to the case $\log \kappa = -20$ allow the broad framing agent to construct a portfolio with a relatively high risk exposure for which the negative effect of the loss averse investment evaluation does not dominate the beneficial effects of increased consumption and bequest utility. See figure 3 for a comparison of the individual asset returns and the return of the optimal portfolio in the first period. This leads to a strong increase in the agent's annuitization degree to an average of 76.68%, which slightly surpasses that of the benchmark agent. However, equity exposure is reduced compared to the narrow framing agent with an average of 12.48%. In total the average portfolio share of the risky assets is 89.16% which is 8.56 percentage points below that of the benchmark agent. Compared to the narrow framing agent, the average experienced consumption level of the broad framing agent increases by 7.08% to 20694 which is 3.56% below the level of the benchmark agent. Due to high annuitzation degree the mean bequest sizes decreases drastically compared with the narrow framing agent. Due to the relatively low equity exposure compared to the benchmark agent the broad framing agent reaches lower wealth levels in most cases and thus leaves smaller bequests than the benchmark agent. The mean bequest size is 31526 which is 16.17% below the benchmark level. The standard deviation of bequest sizes is 76.90% of the mean bequest size.

5.3 Comparison of the simple and the advanced model

	Mean rel. annuitiza-tion	Intertemp. variation annuitiza-tion	Mean rel. equity exposure	Intertemp. variation equity exposure	Mean bequest size	Std. bequest size
Advanced Model	.7411	.1353	.2361	.1208	.0940	.7140
Simple Model	.7405	.1967	.2018	.0061	.1100	.3810

Table 3: *Comparison of the mean and the standard deviations of the portfolio compositions and bequest sizes for the benchmark investor in the simple model and the advanced model without loss aversion. The values for the advanced model are calculated from $N = 10000$ individual values derived by forward simulation. The respective formulae for the individual values from which the means are calculated are $\frac{1}{\tau}\sum_{t=1}^{\tau}(1-c_t)a_t$ for the mean relative annuitization and $\frac{1}{\tau}\sum_{t=1}^{\tau}(1-c_t)\theta_t^S$ for the mean relative equity. Here τ denotes the (random) time of death of the retiree. The intertemporal variations are calculated as the standard deviations of the average per period portfolio shares over the whole time horizon. The values regarding bequest sizes are the sample mean and sample standard deviation over all N bequest sizes.*

This section analyzes the differences between the agent's optimal behaviour and its effects on consumption paths and bequest sizes. A comparison of some key results in both models is given in table 3. A visualization of the agent's optimal portfolio composition through time is given in figure 4. We note that in the simple model, the optimal portfolio composition is independent of agent's wealth level and only dependent on the time. Thus the values given in figure 4 are exact values and not averages. The results in the simple model are comparatively similiar to the results in the advanced model without loss aversion. Excluding the loss averse investment evaluation, the differences between the two models are the different bequest utility specifications and the presence of a governement subsidy. In the simple model, utility from bequest is calculated using the same utility function as for consumption, albeit weighted differently than annual consumption. This means that the agent is as risk averse about bequests as he is about consumption. This and the fact that bequest utility typically receives a higher relative weighting than consumption utility means that agents with low wealth levels sometimes choose unrealistically low consumption levels in order to have enough chash left for sufficient bequest sizes. The affine formulation of bequest utility in the advanced bequest motive specification prevents this behaviour. Because in this formulation, bequests are essentially luxury goods who are only demanded when the agent can afford a sufficiently high bequest size relative to his own consumption. This difference between the two models does have observable effects on the agent's behaviour especially in the second half of the retirement phase. To ensure that his available funds always allow a bequest of sufficient size, the agent in the simple model reduces his portfolio fraction of Arrow annuities much earlier than the agent in the advanced model who is less risk averse about bequests, especially about low wealth level bequests. In earlier periods however, when the agent's wealth on hand is relatively high and thus the demand for bequest in the advanced model is stronger, the agent in the advanced model invests a smaller fraction in Arrow annuitites than the agent in the simple model. Because the agent is more likely to be alive in earlier periods than in later periods, the average portfolio share of Arrow annuities in the simple model is almost equal to that in the advanced model. In the simple model the agent holds on average 74.05% of his wealth after consumption in Arrow annuities, in the advanced model without loss aversion the agent holds on average 74.11% of his wealth after consumption in Arrow annuities. This behaviour has an effect on the agent's consumption paths. The high

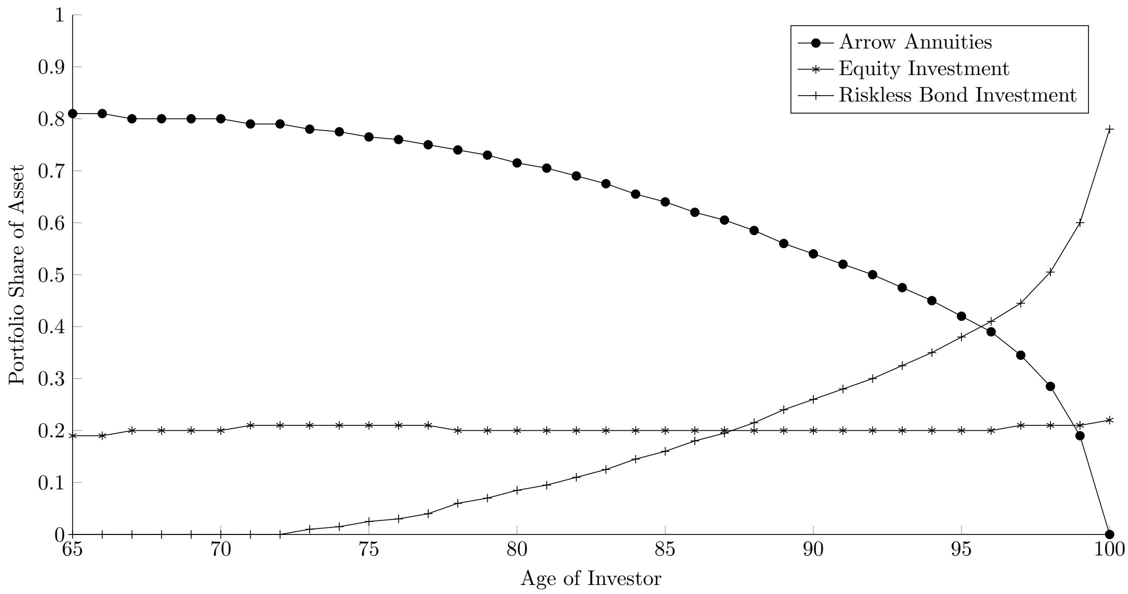

Figure 4: *Optimal Portfolio composition in the simple model. The values above are the exact portfolio weights which are only dependent on the time and not on the wealth level of the agent.*

annuitization degree in the earlier periods leads to high wealth levels in the case of survival, which in turn leads to high annual consumption in these periods. In contrast to the simple model, the agent in the advanced model does invest in riskless bonds in the early periods which means that he misses out on the mortality credit in the case of survival. This results in lower wealth levels and thus lower consumption levels in the early periods in the advanced model. However this effect diminishes over time. With the passing of time the agent's survival probabilities decrease and thus death becomes more likely compared to the earlier periods. Therefore the agents are more concerned about a sufficient bequest size in the later periods. In the simple model this leads to a quick reduction of the share of Arrow annuities in the agent's portfolio. This effect is not as strong in the advanced model because the agent is generally not as risk averse about bequests and especially not as concerned with very low bequest sizes. Therefore the portfolio share of Arrow annuities remains more stable than in the simple model. The standard deviation of the average per period portfolio share is 13.53% in the advanced model and 19.67% in the simple model. Together with a lower relative demand for bequest in the second half of retirement, this leads to slightly higher consumption levels for the agent in the advanced model in the second half of the retirement phase. Conversely, wealth after consumption, and therefore bequest sizes, are higher in the simple model in the second half of the retirement phase. In the first half of the retirement phase, the average wealth after consumption levels of both agents are very similiar. Overall this leads to higher bequest sizes in the simple model. The average bequest size in the simple model exceeds the average bequest size in the advanced model by 17.02%. Furthermore bequests are far less volatile in the simple model with a standard deviation of 38.10% compared to 71.40% in the advanced model.

As already mentioned above, the higher relative concern for bequest in the simple model leads to generally smaller average consumption levels in the simple model. In all but five periods, which are all within the first 12 years of retirement, the average consumption level is higher in the advanced model. In the final 14 years of the retirement phase the average consumption level in the advanced model is between 10% and 22% higher than in the simple model. In total the average experienced annual consumption level is 21026, which is a 2.02% decrease compared to the advanced model.

The second difference between the two models, the presence of a government subsidy, does not substantially effect the results. This is the case because we assume an initial wealth of sufficient size, that leads to an optimal behaviour in which the agent is very rarely forced to actually access the subsidy. The situation would change however for lower initial wealth levels.

To summarize, the simple model slightly shifts priorities to bequest sizes at the expense of annual consumption in the second half of the retirement phase. This leads to slightly higher average bequest sizes and a slightly lower average annual consumption. Assuming that De Nardi's bequest motive specification used in the advanced model is the more accurate representation of an individual's actual desire to leave a bequest, it can be concluded that the simple model overemphasizes bequests and that normative conclusions should rather be drawn on the basis of the advanced model. Furthermore as long as the agent's initial wealth level is sufficiently high, the access to a government subisdy does not lead to significant changes in the results.

6 Conclusion

This paper proposes a way to introduce loss averse investment evaluation into a dynamic framework which also includes utility from consumption and bequest. To overcome the difficulties in representing an annuity investment in terms of its return characteristics we resort to a stylized form of an annuity. Within this framework we find that a sufficient degree of loss aversion can explain the empirically observed under-annuitization. It remains to be shown in future research, wether or not a similiar concept can be applied to actual multi-period life annuities. Furthermore, an important factor in our model is the relative strength of the loss averse investment evaluation with respect to the classical utility. Because this factor heavily depends on the size of the investment and the annual consumption levels, a constant factor as proposed in our model cannot be adequat in all cases. Therefore further analysis can concern itself with a suitable dynamic factor that results in a harmonic and stable interaction of loss averse investment evaluation and classical utility.

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Appendix I

Lemma 1. *The value function defined by equations* (16) *and* (18) *is positively homogeneuous with degree* $1-\gamma$.

Proof.

$$V_T(w) = \sup_{(c,\theta)\in[0,1]^2} \left\{ \frac{(wc)^{1-\gamma}}{1-\gamma} + \beta E_T\left[\frac{\big((1-c)w(\theta Q_T + (1-\theta)Q_f)\big)^{1-\gamma}}{1-\gamma} \right] \right\} \tag{44}$$

$$= w^{1-\gamma} \cdot \sup_{(c,\theta)\in[0,1]^2} \left\{ \frac{(c)^{1-\gamma}}{1-\gamma} + \beta E_T\left[\frac{\big((1-c)(\theta Q_T + (1-\theta)Q_f)\big)^{1-\gamma}}{1-\gamma} \right] \right\} \tag{45}$$

$$= w^{1-\gamma} \cdot V_T(1) \tag{46}$$

Using this we obtain for V_{T-1} that

$$\begin{aligned}
V_{T-1}(w) = \sup_{(c,\theta,\alpha)\in[0,1]^3} & \left\{ \frac{(wc)^{1-\gamma}}{1-\gamma} \right. \\
& + \beta p_{T-1} E_{T-1}\left[V_T\Big((1-c)w(\theta Q_{T-1} + \alpha Q_{A,T-1} + (1-\theta-\alpha)Q_f) \Big) \right] \qquad (47) \\
& \left. + \beta(1-p_{T-1}) E_{T-1}\left[\frac{\big((1-c)w(\theta Q_{T-1} + (1-\theta-\alpha)Q_f)\big)^{1-\gamma}}{1-\gamma} \right] \right\} \\
= \sup_{(c,\theta,\alpha)\in[0,1]^3} & \left\{ w^{1-\gamma} \frac{(c)^{1-\gamma}}{1-\gamma} \right. \\
& + \beta p_{T-1}(1-c)^{1-\gamma} w^{1-\gamma} V_T(1) E_{T-1}\left[(\theta Q_{T-1} + \alpha Q_{A,T-1} + (1-\theta-\alpha)Q_f)^{1-\gamma} \right] \qquad (48) \\
& \left. + \beta(1-p_{T-1})(1-c)^{1-\gamma} w^{1-\gamma} E_{T-1}\left[\frac{\big(\theta Q_{T-1} + (1-\theta-\alpha)Q_f\big)^{1-\gamma}}{1-\gamma} \right] \right\} \\
= w^{1-\gamma} \cdot \sup_{(c,\theta,\alpha)\in[0,1]^3} & \left\{ \frac{(c)^{1-\gamma}}{1-\gamma} \right. \\
& + \beta p_{T-1}(1-c)^{1-\gamma} V_T(1) E_{T-1}\left[(\theta Q_{T-1} + \alpha Q_{A,T-1} + (1-\theta-\alpha)Q_f)^{1-\gamma} \right] \qquad (49) \\
& \left. + \beta(1-p_{T-1})(1-c)^{1-\gamma} E_{T-1}\left[\frac{\big(\theta Q_{T-1} + (1-\theta-\alpha)Q_f\big)^{1-\gamma}}{1-\gamma} \right] \right\} \\
= w^{1-\gamma} \cdot V_{T-1}(1). & \qquad (50)
\end{aligned}$$

It then follows by induction that

$$V_t(w) = w^{1-\gamma} \cdot V_t(1) \tag{51}$$

for all $t \in [0, T]$.

□

Lemma 2. *Assuming a subsidy consumption level of size* $\underline{C}$ *and a bequest utility function of the form* (21), *the value function defined by equations* (16) *and* (18) *satisfies*

$$V_t(w) \geq \sum_{s=t}^{T} p_{t,s}(1-p_s) \left(\frac{\beta^{t-s+1}-1}{\beta-1} u(\underline{C}) + \beta^{t-s+1} v(0) \right) \tag{52}$$

for all $t \in [0, T]$.

Proof. Using that $\sum_{k=0}^{s-1} \beta^k = \frac{\beta^s - 1}{\beta - 1}$, a quick calculation yields

$$\sum_{k=t}^{s} \beta^{k-t} = \frac{\beta^{s+1-t} - 1}{\beta - 1}. \tag{53}$$

Let $p_{t,s*} = p_{t,s}(1 - p_s)$ denote the probability that the individual deceases in period $t+1$. Because we enforce a minimum subsistence consumption level we have $C_t = c_t W_t \geq \underline{C}$ for all $t \in [0, T]$. Using this and the definition of the value function V_t (15), the definiton of the preference functional Φ (11) and (13), the Law of Total Expectation and (53) we find that for all wealth levels $w \geq 0$ and strategies $\gamma_t \in \Gamma_t$

$$\begin{aligned}
V_t(w) &\geq \Phi_t(w, \gamma_t) \\
&= E_t \left[\sum_{s=t}^{T} p_{0,s-1} \beta^{s-t} \left(p_s u(c_s W_s^L) + (1 - p_s) v(W_s^D) \right) \right] \\
&\geq E_t \left[\sum_{s=t}^{T} p_{0,s-1} \beta^{s-t} \left(p_s u(\underline{C}) + (1 - p_s) v(0) \right) \right] \\
&= \sum_{s=t}^{T} p_{t,s*} \left[\left(\sum_{k=t}^{s} \beta^{k-t} u(\underline{C}) \right) + \beta^{s+1-t} v(0) \right] \\
&= \sum_{s=t}^{T} p_{t,s}(1 - p_s) \left(\frac{\beta^{t-s+1} - 1}{\beta - 1} u(\underline{C}) + \beta^{t-s+1} v(0) \right)
\end{aligned}$$

□

Myopic loss aversion and the demand for life annuities

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Abstract

We propose a model that allows a loss averse evaluation of a life annuity to enter into the retirement phase of a life-cycle model. Our approach consists of a workaround using the concept of myopic loss aversion and a representation for an annuity investment consisting of annual returns and the capital bound in the annuity. We explore the portfolio effects of our model in two cases, distinguished by the degree to which the investor frames his asset developments narrowly. Optimization within the two models is conducted for various parametrizations of the relative strength of the investment evaluation.

1 Introduction

In theory life annuities are the prime tool to ensure a steady consumption throughout retirement and a powerful insurance against longevity risk. However, observed voluntary annuitization rates beyond the mandatory rates are fairly low. Using data from a survey on 401(k)plans in the United States, Schaus [29] reports that only 6% opted for annuitization when it was available. This discrepancy, known as the *annuity puzzle*, has been at the center of a large variety of research in the recent decades. Attemps to explain this phenomen range from minor adjustments to the classical theory to behavioral models assuming either bounded rationality or a complete lack of rationality on the side of the investor. A popular explanation that falls somewhere in the middle of these categories is the concept of loss aversion. Originally made famous as a central characteristic of Kahneman and Tversky's prospect theory, loss aversion has been applied to a broad range of financial decision problems. Unlike risk aversion, which usually describes the sensitivity to fluctuations in the investor's consumption or total wealth level, loss aversion represents a sensitivity to adverse outcomes of his risky investments. Thus the notion of loss aversion generally implies that some form of *narrow framing* occurs. This means that the investor is sensitve to fluctuations of individual assets or asset classes isolated from the fluctuations of his whole portfolio or other sources of income. When the concept of loss aversion is applied to a dynamic framework, it is not immediately clear which fluctuations in assets constitute fluctuations the investor should be sensitive about. Losses or gains in an asset could be registered by the investor at their time of maturity or when the assets are sold. On the other hand the investor could also care about intertemporal fluctuations of the value of his assets. Assuming the latter is the general assumption of myopic loss aversion, a concept first introduced by Benartzi and Thaler [5]. Models assuming myopic loss aversion have been applied to a problem similiar to the annuity puzzle, the equity premium puzzle. For volatile assets such as stocks, a short-term loss averse evaluation of the investment development can make a long-term investment seem much less attractive than it may seem from a classical expected utility framework. Equilibrium models including myopic loss aversion therefore generate risk premia closer to the observed values than classical models. Empirical evidence, as for example given in Benartzi and Thaler [6], and theoretical arguments aside, myopic loss aversion has the additional benefit from a modelling perspective that it allows a very canonical implementation in dynamic frameworks, such as multi-period consumption/investment problems. In such applications of myopic loss aversion, the evaluation horizon for narrowly framed assets usually coincides with the timing of the consumption and investment decision. In such frameworks, loss aversion can be incorporated in the investor's preferences as an additional source of utility or rather, because loss aversion usually results in an unfavourable evaluation from an investment perspective, disutility.

While loss aversion itself has been included in models of annuity demand before, these models are generally static and therefore necessarily separate the consumption and the investment problem. This is the approach taken for example by Hu and Scott [22]. The downside to these approaches is, that they ignore the huge potential of annuities when it comes to consumption smoothing and ensuring late life consumption. But to include loss aversion in a reasonable dynamic model, the loss averse investment evaluation of the annuity and other risky assets has to be embedded within a multi-period consumption/investment problem. This however, is not without difficulties. There is no straight forward generalization of models including loss aversion about stocks to include annuites. In this paper, we propose a workaround to this problem building on myopic loss aversion applied to all risky assets, including the life annuity. The general assumption is that the investor narrowly frames the annual development of the capital bound in the annuity and therefore displays sensitivity to the short term development of this asset. Because the life annuity payments are conditional on the investor's survival, the capital bound in the annuity is constantly exposed to the risk of a total loss. For an investor who frequently assesses his investment prospects, this looming risk may render the annuity unattractive. Since the annuity is an illiquid investment, hence cannot be traded after the purchase, our model allows two interpretations when myopic loss aversion enters the investor's preferences. The investor may either actually conduct an annual loss averse evalution of the risky component of his portfolio, in our case stocks and the annuity, as is the case with stocks in the myopic loss averse models by Baberis and Huang ([2] or [4]). Or another interpretation is that the whole evaluation of the annuity investment is conducted at the time of its purchase. That is the investor anticpiates the development prospects for all future periods and evaluates them at once. Because it is not when uncertainty is resolved that loss aversion effects the investor but the expected value[1] of this effect, that drives the investor, the actual timing is irrelevant to the results. However in both cases myopic loss aversion demands that assets are represented in terms of bounded capital and annual returns, which is typically not the form a life annuity is presented. We offer a solution to this problem and thus a way to implement myopic loss aversion over a life annuity. We call our approach a workaround, because wether or not the investor is actually myopic concerning the annuity is not central to our model. We can always interpret the sum of all future periodwise investment evaluations as the loss averse investment evaluation at the time the annuity is purchased. To counter potential criticism regarding the complexity of our approach, a workaround like this is needed because, as we will explicate below, a static loss averse evaluation of an annuity is not possible while still properly representing the properties of the annuity.

Our approach builds on a decomposition of the life annuity into individual Arrow-Debreu-Securities, in the following refered to as Arrow annuities, representing the individual annual payoffs. The price of the whole annuity naturally decomposes into the sum of the prices of the individual Arrow annuities. Given the price of an individual Arrow annuity and its payoff, we can caluculate the total and the annualized return on investment in the case that the agent survives until maturity. For a single Arrow-Annuity this means, broken down into a single period, that the endowment invested into this particular annuity will grow by its respective annual return with the probability that the agent survives the current period, or will be lost completely with the probability that the agent deceases during the current period. Applied to the original life annuity, we can form the portfolio of all the Arrow annuities representing the outstanding annuity payoffs

[1] It should be noted here that general applications of loss aversion do not neccessarily contain expected values but assume probability distortion and thus result in a more general evaluation functional. However wether these evaluations affect the investor annually or through anticipation all at once is not affected by the particular choice of the functional.

and calculate their "portfolio" return as a weighted sum of the individual annual returns. Again, broken down into a single period, this means that the current annuity endowment will grow by this average annual return with the probaility that the agent survives the current period and by the factor zero with the probability that the agent deceases during the current period. This line of reasoning yields a framework for myopic loss aversion to apply. Given an appropriate reference rate of return, the myopic loss averse agent annually receives prospect utility regarding the capital bound in the annuity investment, by comparing the prospects "growth with the annual average return" and "total loss in case of death" with the reference return.

Our general model combines a classical expected utility framework with investment evaluation charaterized by myopic loss aversion. The model spans the entire retirement phase of a life-cycle agent. We assume that the agent has a one time access to the annuity market upon entry to retirement. He may further annually invest his remaining wealth after consumption in two liquid asset classes, representing a riskfree bond investment and a stylized stock investment. At the beginning of each year he receives utility from consumption. In addition to that he may receive utility from bequest at the time of his death. Following the concept of myopic loss aversion, the agent also evalutes the short-term development prospects of his investments on an annual basis. This evaluation results in a third form of utility, which we refer to as prospective utility, which is assumed to be characterized by loss aversion. In this context, loss averison entails an uneven assessment of losses and gains, i.e. adverse or advantageous deviations from a given reference point, in the sense of Kahneman and Tversky's prospect theory. Within our model we further differentiate between narrow and broad framing. Narrow framing describes the notion that an asset's prospects are evaluated individually and independently of the performance of the whole portfolio. Broad framing describes an agent whose investment evaluation relies on the aggregated return of the risky component of his total portfolio and not on the individual assets. For a typical investment, a higher degree of narrow framing leads to a less favourable subjective investment evaluation because diversification effects vanish and individual assets are perceived as far riskier than they actually are.

Obviously this paper comes with that caveat that our results are entirely hypothetical. In contrast to applications of myopic loss aversion in the context of tradeable assets, there is no intuitive reason that an investor would perform an analysis such as ours when deciding wether to annuitize or not. However as myopic loss aversion has become a popular tool in descriptive analysis of household and professional portfolio decisions, the question arises what portfolio effects we can expect when the concept is applied to illiquid assets. As mentioned above the only difference between an annuity and an equity investment from a mere investment perspective is the liquidity of the annuity. Hence there is no a priori reason why myopic loss aversion should not apply to life annuities.

As it is to be expected, we find that a sufficiently strong relative weighting of the loss averse investment evaluation leads to a reluctance to invest in risky assets. Therefore, a gradual increase of the strength of the relative weighting results in a decrease in both annuitization degree and average equity exposure, which eventually leads to complete abstinence from both annuity and equity markets. This decrease in exposure to risky assets depends in parts on the scope of the framing the investor exhibits. When the two risky assets in our model are evaluated together, the broad framing case, the reluctance to make risky investments is not as strong as when they are evaluated individually, the narrow framing case. However this effect is either non-existing or fairly small for the annuity as a risky asset but very noticeable for the investor's equity exposure. As we discuss in detail below this is a result of the broad framing investor's ability to construct certain portfolios that result in a much less unfavourable evaluation than the individual assets. Because the evaluation function is not a smooth function due to the property of loss aversion, this effect goes beyond typical diversification effects.

Literature review

The concept of loss aversion, or in our context more precisely the evaluation of risky investments according to the principles of Kahneman and Tversky's cumulative prospect theory (CPT) [33], has been applied to problems involving optimal choice in numerous ways in the past decades. An early approach which is more related to classical static portfolio choice problems than to the dynamic framework analyzed in this paper is given by Shefrin and Statman's behavioral portfolio theory [31].

Dynamic models assuming loss aversion usually resort to the concept of myopic loss aversion for the reasons discussed above. The concept of myopic loss aversion was first introduced by Benartzi and Thaler [5] as a way to explain the equity premium puzzle. Since then various dynamic optimal portfolio or consumption/investment models have been proposed that build on the idea of a loss averse evaluation of the short term development of a risky asset. Recent examples include among others Dimmock and Kouwenberg [18] who study the effect of loss aversion in a classical household portfolio framework, De Giorgi and Legg [16] who include probability weighting in their analysis, a feature of CPT that is often ignored in other models, Rsonyi and Rodrigues [27] and Jin and Zhou [23] who solve optimal portfolio selection problems under loss aversion in continous time models, Van Bilsen, Laeven and Nijman [34] who study a dynamic investment and consumption problem in which the investor's reference point, the threshold that distinguishes investment losses from investment gains, is not constant but dependant on the investor's current state and his past investments. Other portfolio selection models that include the concepts outlined above are studied by Berkelaar, Kouwenberg, and Post [7] and Magi [25]. Our approach on myopic loss aversion follows the models by Barberis and Huang (see for example [2],[4] and [3]) who in particular, as it is the case in this paper, place emphasis on the scope of the investor's framing.

In a setting related to this paper, Blake, Wright and Zhang [8] conduct a loss averse evaluation of investment plans in the accumulation phase of a life-cycle. They find that the optimal investment depends on the wealth level of the investor. For low wealth levels the investor is more risk seeking and switches to portfolio insurance strategies once he reaches higher wealth levels. In our paper which focuses on the decumulation phase of the life cycle we find that this effect reverses itself, as the retiree's wealth level is decreasing over time. In the beginning of the retirement phase, when wealth levels are the highest, the investor is reluctant to invest in risky assets but gradually increases equity exposure as his total funds decrease.

Empirical evidence for myopic loss aversion is given for example by Thaler, Tversky, Kahneman and Schwartz [32]. Haigh and List [21] provide experimental evidence that loss aversion is not only displayed by individual investors who may lack financial literacy but also by professional traders. Contrary to intuition they report stronger degrees of myopic loss aversion for the professionals than for students in their experiment.

Besides the subject of optimal portfolio choice under loss aversion, this paper contributes to the theory on the demand for life annuities and more specificly on the annuity puzzle. A branch of research that goes back to an article by Yaari on portfolio choice facing an uncertain lifetime [35]. Yaari finds that under certain conditions full annuitization is optimal. A more recent study which represents a more sophisticated, yet conceptually similiar approach, is conducted by Davidoff, Brown and Diamond [15]. They find that depending on the circumstances, full or at least high annuitization degrees are optimal. The inability of the classical rational models to explain the low observed voluntary annuitization rates has jolted many approaches that seek an explanation for low annuity demand outside of the realm of rational models. A broad compendium containing rational and psychological factors of annuity demand is given by Brown [9]. Richter, Schiller and Schlesinger [28] study how behavioral obstacles may effect demand for general insurance products. Hu and Scott [22] put a particular focus on how behavioral effects may hinder the purchase of annuities. They also propose a model which applies loss aversion to a life annuity. However they only focus on the investment evaluation without considering the beneficial effects of a life annuity regarding longevity risk. Furthermore there is a variety of literature that is concerned with empirical testing of the various rational and psychological effects proposed in the aforementioned articles and beyond. Among others, examples include Goedde-Menke, Lehmensiek-Starke and Nolte [19], Cappelletti, Guazzarotti and Tommasino [12] or Agnew and Szykman [1]. General findings are that wealth, gender, financial literacy, framing, distrust, bequest and self-selection are important determinants of annuity demand.

Especially the concept of framing is an important part in this paper. The dynamic models proposed by Yaari [35] and Davidoff, Brown and Diamond [15] typically only consider utility from consumption and sometimes utiltiy from bequest. Therefore they fall into the category of a consumption frame for the annuitization decision. However Brown, Kling, Mullainathan and Wrobel [10] find experimental evidence that individuals actually choose annuitization levels as proposed by those models when the annuitization decision is framed in terms of consumption and longevity risk. But when the annuity investment is instead portrayed in terms of its return characteristics, then the preference rates for annuities fall from 72% to 21%. The myopic loss averse models listed above can be considered a hybrid frame which combines consumption and investment considerations. This is a suitable approach for life annuities because mere investment frame models ignore the huge potential of annuities when it comes to ensuring a stable consumption throughout retirement and thus result in a too unfavourable evaluation of the life annuity.

The idea to use behavioral effects to explain the demand for insurance is adopted in other recent papers. For example Gottlieb and Mitchell [20] analyze the effect a combined utility and investment evaluation has on the demand for long-term care insurance. Chen, Hentschel and Klein [13] analyze how guarantees in life insurance products are evaluated by CPT investors in contrast to expected utility investors. An analysis of the demand for annuities under loss aversion using a stylized form of one period annuities, which avoids most of the difficulties in applying loss aversion to annuities, is conducted by Schneider [30].

2 The model

Our model spans the retirement phase of the life-cycle. At age 65, at time $t = 0$, our exemplary agent enters retirement with the accumulated savings $W > 0$. He can live up to maximum age of 100 years (time $T = 35$), but may also decease during any prior year. We assume that the agent has no additional assets such as pre-annuitized wealth or future labour income. To finance his future consumption and a potential bequest the agent has access to a menu of three investment classes. Two of them are liquid assets who can be rebalanced on an annual basis. The third is an illiquid life annuity, which cannot be traded after the purchase. At the beginning of every year, starting in $t = 0$, the agent chooses his annual consumption level C_t and then allocates the rest of his wealth on hand $W_t - C_t$ among the two liquid asset classes.

The two liquid assets available to the agent consist of a riskfree bond paying a fixed interest rate R_f and a stylized stock investment whose underlying price process follows a geometric Brownian Motion, i.e. pays a lognormally distributed return $1+R_t$[2]. We assume that there are no transaction costs and furthermore that the annuity is priced fairly. That means the price of an annuity in advance of size A is given by

$$P_A = A \sum_{t=0}^{T} p_{0,t}(1 + R_A)^{-t} \tag{1}$$

[2]Throughout this paper we will assume that all probabilities and conditional and unconditional expectations are taken with respect to the canonical probability space generated by the sequence of returns $(R_t)_{t\in[1,T+1]}$

where $p_{0,t}$ are the assumed survival probabilities from the agent's cohort. Throughout this paper we assume that these probabilities coincide with the agent's individual survival probabilities. $R_A \geq R_f$ is the assumed interest rate.

For all $t = 0, 1, 2, \ldots, T$ we let $0 \leq c_t \leq 1$ denote the fraction of wealth on hand W_t that is consumed in the subsequent period and $\theta_t \geq 0$ denote the fraction of wealth after consumption which is invested in the stock market. The remaining fraction $1-\theta_t$ is invested in the riskless asset. To allow neither borrowing nor short selling either of the asset types we require that $0 \leq \theta_t \leq 1$ for all $t = 0, 1, \ldots, T$. Since the payoff of the annuity is conditional on the agent's survival, the agent's wealth process (or more formally - the agent's budget constraint) depends on his survival state. There are three possible states. If the agent is alive at time $t+1$ his wealth on hand is given by

$$W_{t+1}^L = W_t(1 - c_t)(\theta_t(1 + R_t) + (1 - \theta_t)(1 + R_f)) + A. \tag{2}$$

If the investor deceases during period $t+1$, his wealth at time $t+1$ is subject to bequest. The time $t+1$ annuity however, will not be paid out. The wealth on hand at that time $t+1$, and thus the size of his bequest is therefore

$$W_{t+1}^D = W_t(1 - c_t)(\theta_t(1 + R_t) + (1 - \theta_t)(1 + R_f)). \tag{3}$$

The agent's wealth on hand at all later times after his bequest is zero.

The concept of myopic loss aversion suggests that the agent narrowly frames a part of the risky component of his wealth process. In a typical model this would concern the stock investment $W_t(1 - c_t)\theta_t$. Annual fluctuations in this asset class would then, in addition to their intermediate effect on consumption and bequest, immediately effect the agent via some form of "gain-loss-utility". This investment evaluation is usually characterized by loss aversion, which means that the agent is more sensitive to potential losses than to potential gains. Wether or not an annual return is considered a gain or a loss is determined by comparison with some reference return, such as zero or the risk-free rate. Yet in our framework, stocks are not the only risky asset the agent holds. There is also the annuity. Although the annual payoff A itself is fixed, the actual return on investment of the annuity may vary to a great extend (see figure 1). So albeit not tradeable, the annuity contains all the features a myopic loss averse investor may be sensitive about. But the general model framework sketched above demands that the annuity investment must be represented in the form of an annual variable rate of return to allow a myopic loss averse evaluation.

To obtain such a representation we decompose the whole annuity into individual Arrow-Debreu-Securities A_t, $t = 0, \ldots, T$. Each A_t is a finacial contract made at time $t = 0$, paying the amount A to the agent if he is alive at time t and 0 if he is not. The price P_t of A_t is

$$P_t = Ap_{0,t}(1 + R_A)^{-t}. \tag{4}$$

Obviously the purchase of all $T+1$ Arrow annuities is identical to the purchase of the life annuity and the following identity holds for the respective prices

$$P = \sum_{t=0}^{T} P_t. \tag{5}$$

Each Arrow-Debreu-Security A_t, which will be denoted Arrow-Annuity in the following, has a total return of

$$1 + R_{A,0,t} = 1 + R_{A,0,t}^L = \frac{A}{P_t} = \frac{(1 + R_A)^t}{p_{0,t}} \tag{6}$$

if the agent survives until time t and a return $R_{A,0,t} = R_{A,0,t}^D = -1$ if the agent dies prior to its maturity. The respective annualized returns are

$$1 + R_{A,t} = (1 + R_{A,0,t}^L)^{1/t} = \frac{(1 + R_A)}{p_{0,t}^{1/t}} \tag{7}$$

in case of survival and $R_{A,t} = -1$ otherwise. Suppose the agent holds one arbitrary Arrow-Annuity A_t at time $s < t$. The current "endowment"[3] of this asset is $P_t(1+R_{A,t})^s$. If the investor survives one more period until time $s+1$ this value will grow by the factor $1+R_{A,t}$. If the investor deceases during the current period, the invested capital will be lost, i.e. grow by the factor 0. We let

$$E_s^t = \begin{cases} P_t(1+R_{A,t})^s, & t > s \\ 0, & t \leq s \end{cases} \tag{8}$$

denote the current endowment in the single Arrow-Annuity A_t at time s. Now suppose the agent holds all $T+1$ Arrow annuities of the same size A, i.e. the agent holds the life annuity. At time s, the current annuity endowment is the sum of the endowments of the individual Arrow annuities which have not yet matured

$$E_s := \sum_{t>s}^{T} E_s^t = \sum_{t>s}^{T} P_t(1+R_{A,t})^s. \tag{9}$$

If the agent survives the following period, the endowments will grow by their respective annual rates $1 + R_{A,t}$, and thus the whole portfolio of the Arrow annuities will grow by the "portfolio" return

$$1 + R_{PA,s} = \sum_{t>s}^{T} \frac{E_s^t}{E_s}(1 + R_{A,t}) \tag{10}$$

if the agent survives the current period and $R_{PA,s} = -1$ if the agent deceases in the current period.

This investment representation of the life annuity permits comparison with annual investments such as stock holdings or general annual reference returns. Each year, the annuity exposure E_t, which can be thought of as the capital invested in the annuity at time t, grows by the risky return $R_{A,t}$. In a similiar manner as the stock exposure $W_t(1-c_t)\theta_t$ grows by the risky return R_t. The remaining difference between the two assets is their liquidity. To account for this, the appropriate reference return has to be set higher than the reference return for liquid investments. Furthermore our investment representation is consistent with the actual payoff of the annuity and only regards funds that are actually invested in the annuity and thus avoids potentially problematic reinvestment assumptions.

We assume a combined preference functional similiar to the models proposed by Barberis and Huang ([2],[4] or [3]), who also build on the assumption of myopic loss aversion. These models typically contain two types of sources of utility. In our framework these are the classical forms of utility from consumption and bequest and a prospective utility from myopic investment evaluation. We further assume, as in [2], that utility is additive in time as well as in type[4]. The specific timing when the utilities effect the agent, i.e. when they are felt by the agent, is as follows. At the beginning of every year $t = 0, 1, \ldots, T$, if the agent is still alive, he receives utility from consumption $u(C_t) = u(c_t W_t^L)$. If he deceases during period t he receives utility from bequest $v(W_t^D)$ at time t, weighted by the strength of his bequest motive $\omega \geq 0$. Subjective investment evaluation occurs annually beginning at time $t = 0$. Since we assume a myopic investor, only fluctuations within the short time horizon of one year are considered. Thus at time t the investor experiences loss averse prospect utility m over the potential investment outcomes within period

[3]This does not represent an objective evaluation of the asset itself. Rather it describes the growth of the investment $K_s = P_t(1+R_{A,t})^s$ from the initial endowment P_t to the final, but conditional, payoff A.

[4]The more recent models proposed in [4] and [3] assume recursive instead of additive utility. The recursive specification has the advantage that it allows tractability in the equilibrium and thus to derive statements about the equity risk premium and stock demand in the optimum. Because for now our model concerns only with the effects on the demand side, i.e. the relative attractiveness of annuities, we stick to the additive formulation which allows a greater degree of intuitive traceability concerning the preferences themselves. Another noteworthy advantage of recursive utility models is that they avoid that the rate of intertemporal substitution is directly linked to the risk aversion parameter.

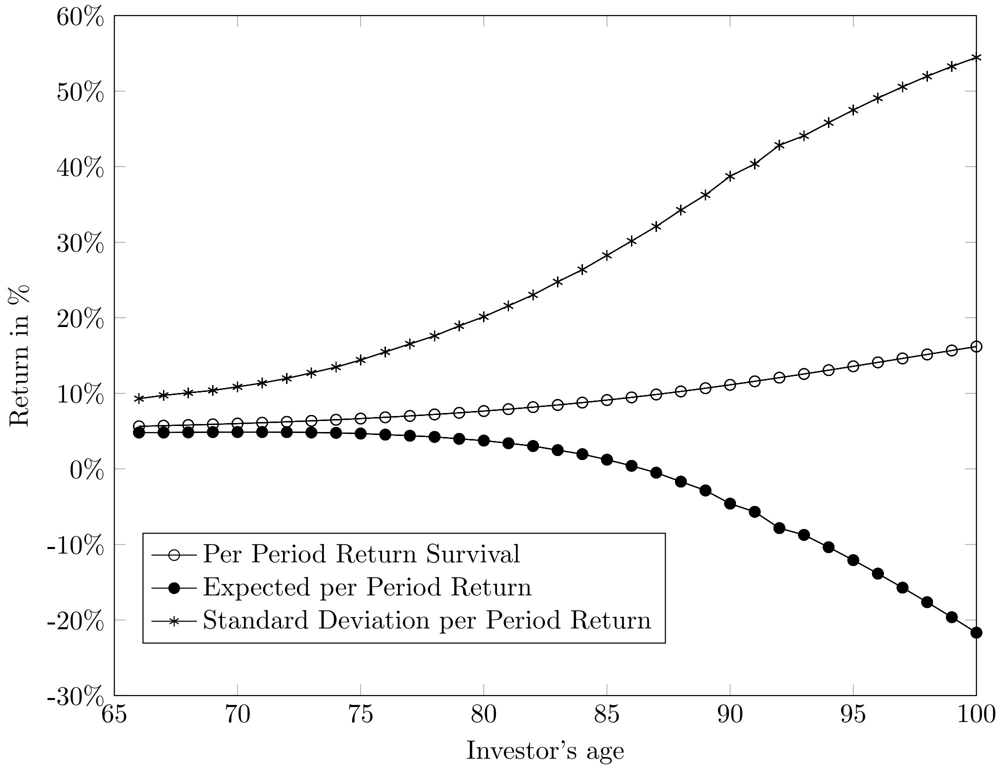

Figure 1: *The figure displays the annual per period returns $R_{PA,s}$ if the investor survives and their expected values and standard deviations. After* 13 *periods the risk premia of the annuity $E[R_{PA,s}] - R_A$ become negative.*

$t+1$, i.e. over the risky returns R_t and $R_{PA,t}$. Contrary to classical expected utility theory this function is not applied to a consumption or a wealth level but to gains and losses. Differentiation between gains and losses is achieved by comparison with a reference return R_r. The investor assesses the success of an investment of size W by comparing its return WR to the benchmark return WR_r. If $WR - WR_R \geq 0$, it is considered a gain and a loss otherwise. Our specification of the function m, which is discussed below, is positively homogeneous with degree $\alpha > 0$. The prospective investment evaluation, defined as the expected prospective utlity, can therefore be written in the form

$$E[m(WR - WR_r)] = W^\alpha E[m(R - R_r)].$$

The appropriate reference return depends on the type of asset or the composition of the portfolio that is narrowly framed. In the following, we will assume that the agent's reference return is the risk-free rate R_f for liquid investments and the assumed interest rate R_A for the illiquid annuity investment. We distinguish between two model specifications regarding the scope of the agent's framing. The narrow framing (NF) specification assumes that the agent frames and thus evaluates both risky assets, stock and annuity, individually. The broad framing (BF) specification assumes that the whole risky component of the agent's portfolio is framed and evaluated at once. In the latter case the total size of the risky component is

$$E_t^{BF} = E_t + W_t(1 - c_t)\theta_t. \tag{11}$$

The respective portfolio return $R_{P,t}$ is defined as the sum of the individual returns R_t and $R_{PA,t}$

weighted with the relative size of their current endowment, i.e.

$$R_{P,t} = \frac{W(1-c_t)\theta_t}{E_t + W_t(1-c_t)\theta_t} R_t + \frac{E_t}{E_t + W(1-c_t)\theta_t} R_{PA,t}. \tag{12}$$

Because the portfolio is compounded of liquid and illiquid assets the appropriate reference return is the weighted sum of the two respective reference returns, i.e

$$R_{R,t} = \frac{W_t(1-c_t)\theta_t}{E_t + W_t(1-c_t)\theta_t} R_f + \frac{E_t}{E_t + W_t(1-c_t)\theta_t} R_A. \tag{13}$$

In the following we use the short notation $p_t = p_{t,t+1}$ to denote the conditional periodwise survival probabilities. For any feasible consumption/ investment plan γ_0, the time $t = 0$ preference specification (NF) with individual framing of the risky assets takes the form

$$\begin{aligned}\Phi_0^{NF}(W,\gamma_0) =& E_0\Bigg[\sum_{t=0}^{T} p_{0,t-1}\beta^t \Bigg(p_t u\big(c_t W_t^L\big) + (1-p_t)\omega v\big(W_t^D\big) \\ &+ \kappa(E_t)^\alpha m\big(R_{PA,t} - R_A\big) \\ &+ \kappa(W_t(1-c_t)\theta_t)^\alpha m\big(R_{t+1} - R_f\big)\Bigg)\Bigg].\end{aligned} \tag{14}$$

The preference specification (BF) with broad framing is given by

$$\begin{aligned}\Phi_0^{BF}(W,\gamma_0) =& E_0\Bigg[\sum_{t=0}^{T} p_{0,t-1}\beta^t \Bigg(p_t u\big(c_t W_t^L\big) + (1-p_t)\omega v\big(W_t^D\big) \\ &+ \kappa\big(E_t + W_t(1-c_t)\theta_t\big)^\alpha m\big(R_{P,t+1} - R_{R,P}\big)\Bigg)\Bigg].\end{aligned} \tag{15}$$

The model parameter κ describes the relative impact of prospective utility with respect to classical utility from annual consumption and bequest.

Let $U := U_0$, resp. U_t, denote the set of feasible investment/consumption plans starting at time 0, resp. time t, that abide the constraints formulated above. Analogous to the time $t = 0$ preferences we can formulate the respective preferences $\Phi_t^{NF/BF}(W,\gamma_t)$ for the problem (re-)started at any later time $t = 1, 2, \dots, T$. The problem's value function $V_t^{NF/BF}$ is defined as the optimized preference functional $\Phi_t^{NF/BF}$

$$V_t^{NF/BF}(W_t) = \sup_{\gamma_t \in U_t} \Phi_t^{NF/BF}\Big(W_t, \gamma_t\Big). \tag{16}$$

In both model specifications the value functions satisfy the Bellmann equation, which states that

$$\begin{aligned}V_t^{NF}\big(W_t\big) = \sup_{(c,\theta)\in[0,1]^2} \Bigg\{ & u\big(cW_t\big) + \beta p_t E_t\Bigg[V_{t+1}\big(W_{t+1}^L\big)\big)\Bigg] + \beta(1-p_t)\omega E_t\Bigg[v\big(W_{t+1}^D\big)\Bigg] \\ &+ \kappa(E_t)^\alpha E_t\Bigg[m\big(R_{PA,t} - R_A\big)\Bigg] + (W_t(1-c)\theta)^\alpha E_t\Bigg[m\big(R_t - R_f\big)\Bigg]\Bigg\}.\end{aligned} \tag{17}$$

holds with with the terminal condition

$$V_T^{NF}\big(W_T\big) = \sup_{(c,\theta)\in[0,1]^2} \Bigg\{ u\big(cW_T\big) + \beta\omega E_T\Bigg[v\big(W_{T+1}^D\big)\Bigg] + (W_T(1-c)\theta)^\alpha E_T\Bigg[m(R_T - R_f)\Bigg]\Bigg\}. \tag{18}$$

and that

$$V_t^{BF}(W_t) = \sup_{(c,\theta)\in[0,1]^2} \Bigg\{ u(cW_t) + \beta p_t E_t\Big[V_{t+1}(W_{t+1}^L))\Big] + \beta(1-p_t)\omega E_t\Big[v(W_{t+1}^D)\Big] + \kappa(E_t + W_t(1-c)\theta)^\alpha E_t\Big[m(R_{P,t} - R_{R,t})\Big]\Bigg\}. \tag{19}$$

holds with the terminal condition

$$V_T^{BF}(W_T) = \sup_{(c,\theta^S)\in[0,1]^2} \Bigg\{ u(cW_T) + \beta\omega E_T\Big[v(W_{T+1}^D)\Big] + \kappa(W_T(1-c)\theta)^\alpha E_T\Big[m(R_T - R_f)\Big]\Bigg\}. \tag{20}$$

Because there are no outstanding annuities in the final period, the terminal conditions for both model specifications are identical.

Loss aversion impacts the investor as a force on behavior caused by the prospect of a potential loss of invested funds. In the model specification with broad framing (BF), the impact at time t depends not only on the annuity exposure but also on the current wealth level W_t and the investment strategy θ_t of the investor. The magnitude of the impact is therefore a random variable which is not known prior to time t. On the contrary the (NF)-investor, who frames his individual assets narrowly, knows his annuity exposure at all later times at the moment the annuity is purchased. Therefore the (NF)-investor can anticipate the total impact of all future investment evaluations regarding the annuity. This allows an alternative interpretation of the model in which loss aversion regarding the annuity does not cause a recurring annual impact but a one time only evaluation at the time of purchase. Formally we can rewrite the preference functional (NF) in the form

$$\Phi_0^{NF}(W,\gamma_0) = E_0\Bigg[\sum_{t=0}^T p_{0,t-1}\beta^t \Bigg(p_t u(c_t W_t^L) + (1-p_t)\omega v(W_t^D) + \kappa(W_t(1-c_t)\theta_t)^\alpha E\Big[m(R_{t+1} - R_f)\Big]\Bigg)\Bigg] + \kappa\mu(A) \tag{21}$$

where $\mu(A)$ denotes the total sum of investment evaluations regarding the annuity, i.e.

$$\mu(A) = \sum_{t=0}^T \beta^t p_{0,t}(E_t)^\alpha E\Big[m(R_{PA,t} - R_A)\Big]. \tag{22}$$

This component $\mu(A)$ can be interpreted as the investment evaluation of the annuity at the time of purchase in the classical sense of a static loss averse investment evaluation. However the workaround using myopic loss aversion to obtain this component is still necessary, because it makes sure the bounded capital and the return on that capital are measured exactly. Without the breakdown into individual periods, there is no proper way to calculate an a priori return/bounded capital pair for any potential remaining life time because the annuity is characterized by intertemporal payoffs. In particular there is no random variable R' with a suitable reference return R for which

$$\mu'(A) = P_A^\alpha E_0[m(R' - R)] \tag{23}$$

constitutes an appropriate representation of the annuity investment. This underlines the usefulness of the myopic loss aversion approach on annuities.

In the remainder of this section we specify the three utility functions. We note that the model generally allows for a broad class of specifications for all three sources of utility. Our selection reflects the choices most encountered in the relevant literature. This choice is not as clear with the prospect utility specification as there seems to be much less consensus on the function itself and wether or not probability weighting should be applied.

Throughout this paper we assume that the utility derived from consumption is given by the power utility where

$$u(x) = \frac{1}{1-\gamma} x^{1-\gamma} \tag{24}$$

with the constant coefficient of relative risk averison $\gamma > 1$.

Regarding bequest utility we follow De Nardi's ([17]) approach, where a bequest of size B yields the utility

$$v(B) = \frac{1}{1-\gamma} \left(\psi + \frac{B}{\omega} \right)^{1-\gamma}. \tag{25}$$

Here the parameter ω controls the relative strength of the bequest motive with respect to other forms of utility. The second parameter ψ can be interpreted as the prevalence of a bequest motive in the population or the degree to which bequests are a luxury good. The affine formulation ensures that, in the optimum, the agent is only willing to leave a bequest at the expense of his own consumption, if his wealth allows for a sufficiently high bequest size without unreasonably reducing his own annual consumption[5].

We borrow the prospective investment utility function from Kahneman and Tversky's prospect theory [33]. They originally suggested the evaluation function

$$m(x) = \begin{cases} x^{\alpha} & \text{for } x \geq 0 \\ -\lambda(-x)^{\alpha} & \text{for } x < 0 \end{cases} \tag{26}$$

with a Loss Aversion parameter $\lambda > 1$ and a curvature parameter $\alpha \in (0, 1)$ [6].

3 Parameter choice

A summary of the parameter choices in this paper is displayed in table 1. Regarding the expected utility framework the parameter values reflect the choices most encountered in the literature. As in Cocco and Gomes [14], the time disocunt factor is set to $\beta = 0.96$, the risk aversion parameter of the CRRA utiltiy function is set to $\gamma = 5$ and the risk premium is set to $E[R_t] - R_f = 4\%$. We assume a standard Black-Scholes economy where asset prices follow a geometric Brownian Motion which implies that the periodwise equity returns are i.i.d lognormally distributed. The parameters of the geomteric Brownian Motion are chosen such that the expected return is $E[R_t] = 6\%$ and the standard deviation of equity returns is $\sigma(R_t) = 20\%$. The interest rate on riskfree bond investments is set to $R_f = 2\%$. The calculatory interest rate in the pricing of the annuity is set to $R_A = 4\%$.

The parametrization of the bequest utility function is based on De Nardis [17] original specification. We follow Peijnenburg, Nijman and Werker [26] and adjust the luxury parameter according to the retiree's absoulte wealth level by multiplying it with the fully annuitized income, the FAI. This results in the parametrization $\omega = 7.81$ and $\psi = 0.67 * FAI$. In contrast to De Nardi we set

[5]To achieve a better understanding of the parameters in v it helps to regard the simplified problem with no time weighting, a fixed wealth level W, a fixed lifetime of T years and no government subsidy. The optimal consumption resulting from first order conditions is then $C = (W + \omega\psi)/(\omega + T)$ and the optimal bequest is $B = \omega(C - \psi)$, i.e. the bequest covers ω periods of spending the amount $C - \psi$, i.e. the amount the agent's own consumption exceeds ψ. If the agent cannot bequeathe an amount which exceeds ψ for ω years then the optimal bequest is zero.

[6]Although the original evaluation function proposed by Kahnemann and Tversky allows different curvature parameters for gains and losses, their proposed parameterisation results in an identical value $\alpha = 0.88$

Model parameters	
Expected utility framework	
T	35
age in $t = 0$	65
age in $t = T$	100
γ	5
r_f	2%
r_A	4%
$\mathrm{E}[r_t]$	6%
$\sigma[r_t]$	20%
β	0.96
Annuity pricing	
FAI	25000
W	$FAI * P_A(1)^{-1}$
Bequest motive	
ω w/ Bequest Motive	7.81
ψ	$0.67 \cdot FAI$
Loss averison	
λ	2
α	0.88

Table 1: Model parameters

the risk aversion parameter of bequest utiltiy to $\gamma = 5$ to be in accordance with the consumption utility specification in our paper. The loss averse evaluation function is calibrated in accordance with Kahnemann und Tversky [33], who set $\alpha = 0.88$ and $\lambda = 2$.

The survival probabilities occuring in the retiree's preferences and the annuity pricing calculation are taken from german death tables[7] We assume that annuity pricing is actuarilly fair and we chose an inital wealth that results in a fully annuitized income of $FAI = 25000$. As in Brown and Poterba [11] we assume that the retiree enters retirement at age 65 and may reach a maximum age of 100 years.

4 Solution Technique

In contrast to standard consumption and investment problems a separation of the consumption and the investment problem and thus an analytical solution is not possible in our setting. This is due to the affine form of the bequest utility function and the investor's wealth process. Therefore we resort to a numerical method based on a backwards recursion. We fix an annuity level A and regard the respective Bellmann equations given by (17) and (19). To indicate the current annuity size we write $V_t(|A)$ for the value function at time t, assuming an annuity of size A. The recursion starts at the terminal conditions (18) and (20). For any wealth level w_l these terminal conditions give a static one period optimization problem which can be solved numerically to obtain the value of the value function itself $V_T(w_l|A)$ and the associated optimal control values $(c_T(w_l), \theta_T(w_l))$.

[7] Source: Sterbetafel 2009/11 Deutschland männlich, Periodensterbetafeln für Deutschland 2009/2011, Statistisches Bundesamt, Wiesbaden 2012. We assume a male policy holder.

In the model with narrow framing the respective optimization problem is given by

$$V_T^{NF}(w_l|A) = \max_{\substack{0\le c,\theta,\alpha\le 1\\ \theta+\alpha\le 1}} \Bigg\{ u(cw_l) + \beta E_T\Big[v((w_l(1-c)(\theta(1+R_T)+(1-\theta)(1+R_f))\Big] + \kappa(w_l(1-c)\theta)^{\alpha} E_T\Big[m\big(R_T-R_f\big)\Big]\Bigg\}. \tag{27}$$

Because the domain of the consumption and the investment control parameters are compact sets, there is always a solution for which the supremum and therefore the maximum is attained. This optimization step is repeated for a fixed grid of wealth levels w_l which produces the pairs $(w_l, V_T(w_l|A))$. Because the problem's value function is a smooth function, we can use cubic spline interpolation to obtain the values of $V_T(w|A)$ for arbitrary values of w within the boundarys of the wealth grid. This allows us to formulate the time $T-1$ optimization problem as one period optimization problem using the time $T-1$ Bellmann equation. In the exemplary case of the model with narrow framing, the respective optimization problem assuming a fixed wealth level w_l is given by

$$V_{T-1}^{NF}(w_l|A) = \max_{\substack{0\le c,\theta,\alpha\le 1\\ \theta+\alpha\le 1}} \Bigg\{ u\big(cw_l\big) + \beta p_{T-1} E_{T-1}\Big[V_T(W_T^L|A)\Big] + \beta(1-p_{T-1})\omega E_t\Big[v\big(W_T^D\big)\Big] + \kappa(E_{T-1})^{\alpha} E_{T-1}\Big[m\big(R_{PA,T-1}-R_A\big)\Big] + (w_l(1-c)\theta)^{\alpha} E_{T-1}\Big[m\big(R_{T-1}-R_f\big)\Big]\Bigg\} \tag{28}$$

where $V_{t+1}(W_{t+1}^L)|A)$ is computed by cubic spline interpolation as discussed above. We then repeat the steps above until we reach the time $t=0$ value function. This procedure is conducted for annuitization levels ranging from 0% to 100% in steps of 10%. The annuitization degree for which the value function at time $t=0$, evaluated at the appropriate initial value, i.e. $V_0(W_0-P_A+A|A)$, has the highest value is the optimal annuitization degree.

We use a time dependant wealth grid with $\{w_l\}_{l=0,\dots,L_t}$ with a base grid at time $t=0$ with $L_0 = 30$ grid points spanning from $w_0 = epsilon$ to $w_{30} = W$ where $\epsilon < 1$ is a sufficiently small level that is never reached in the optimal strategies. Because it is never reached we assume that the value function is constant for wealth levels below ϵ. Because wealth levels may increase during any period which could make extrapolation necessary we add an additional grid point $w_{L_{t+1}} = w_{L_t} + \Delta_w$ in each time step. Furthermore because the value function has a higher curvature for low wealth levels we use exponentially placed grid points in the base grid to increase the efficiency of our algorithm.

All conditional expectations, which are essentially expected values of a function of a lognormally distriuted random variable, are calculated using Gauss-Hermite-Quadrature with $n=32$ sample points. We apply Liu and Pierce's [24] correction formula for the sample points and the weights in the quadrature which increases the accuracy of the approximation. Changing the number of sample points in the quadrature or the number of points in the wealth grid does not affect our results at the reported precision.

5 Results

5.1 Main Results - Annuity Demand

The main focus of this paper lies on the effect of a myopic loss averse evaluation of an annuity investment on the demand for life annuities. The strength of the effect of loss aversion on the investor is primarily driven by the parameter κ, which controls the relative weighting of loss averse investment evaluation with respect to the utility from consumption and bequest. We find that effective parametrizations of our model lie between $\kappa = 10^{-19}$ and $\kappa = 10^{-22}$ for both model specifications. For values on the lower bound of theses ranges, loss averse investment evaluation has no effect on the annuity demand, i.e. the demand is identical to the benchmark model with $\kappa = 0$. For values on the upper bound of that range, loss aversion leads to complete abstinence from annuity and equity markets. The extremely low numerical values of the parameter κ result from the different curvature of the CRRA utilty function and Kahneman and Tversky's evaluation function. For representational purposes it can make sense to rescale the parameter κ to achieve a better understanding of the relative importance of both effects on the retiree. To this end we define the adjusted versions of the parameter $\kappa_{rel} = \kappa \cdot A^{-(1-\gamma)}$, where $A = 25000$ is the full annuitized income (FAI)[8]. After rescaling, the effective parametrizations lie between $\kappa_{rel} = 0.000391$ and $\kappa_{rel} = 0.0391$.

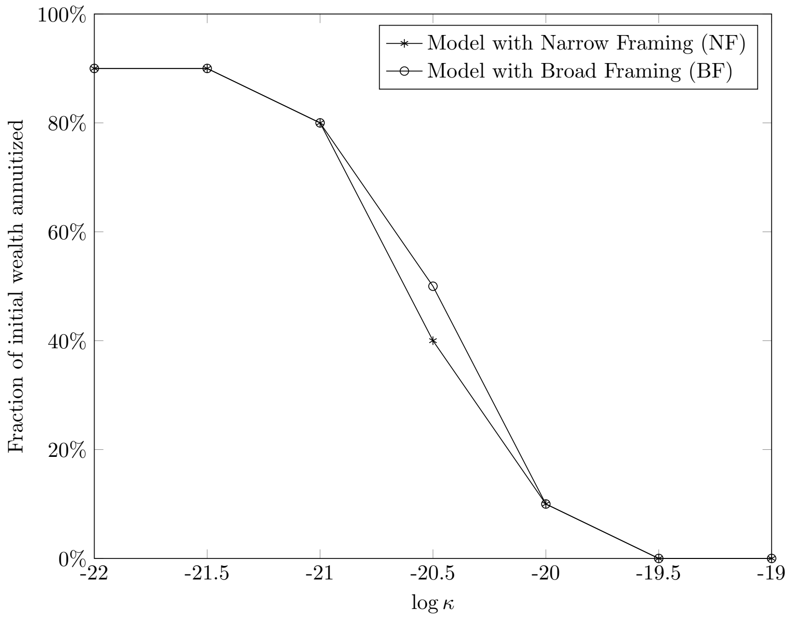

Figure 2: *Optimal annuity demand as a fraction of the retiree's initial wealth.*

Figure 2 displays the annuity demand in the optimum for both model specification for various values of the parameter κ. In the benchmark model without investment evaluation, i.e. $\kappa = 0$, and actuarilly fair annuity pricing, the retiree has 90% of his wealth annuitized upon entry to retirement. In the presence of loss aversion, an increase of the parameter κ results in a decline of the perceived attractiveness of life annuities and therefore a decline in the optimal annuitization degree. This effect is of similar strength in both model specifications, narrow framing and broad framing. For $\log \kappa = -21.5$, or equivalently $\kappa_{rel} = 0.0001235$, or values lower than that, the negative effect of loss aversion is not strong enough to result in a different annuity demand than the benchmark model. When the parameter κ increases the annuity demand gradually decreases. For $\log \kappa = -19.5$ or equivalently $\kappa_{rel} = 0.01235$, or values higher than that, loss aversion completely prevents the retiree from purchasing life annuities. For all but one value of κ considered in our analysis, the optimal annuity demand is equal in both model specifications. For $\log \kappa = -20.5$

[8]In general the scaling factor should not be a constant, because the relative impact of loss aversion depends heavily on the size of the annual consumption in relation to the invested capital. This problem is neglectable here because we regard a very specific situation in which the size of these factors is within a constant range. In more general models a dynamic scaling factor such as non-framed income is used.

or equivalently $\kappa_{rel} = 0.00123$, the model with broad framing results in an optimal annuitzation degree of 50% while the model narrow framing results in an optimal annuitization degree of 40%.

In the model with narrow framing, each asset is evaluated individually. This means that diversification effects between the two risky asset classes, which are uncorrelated, are ignored by the retiree. In other words, a loss in one asset that is offset by a gain in the other asset would still be interpreted as a loss. This is no longer true for the broad framing investor, who evaluates the risky component of his portfolio as a whole. Therefore broad framing generally leads to a more favourable evaluation of risky assets and thus to a higher willingness to invest in risky assets. This effect is considerably more pronounced for the investor's willingness to invest in equity markets than for his willingness to annuitize. Figure 3 displays the average portfolio share of equity $E[\tau^{-1)} \sum_{t=0}^{\tau} \theta_t]$ for the different parametrizations of κ for both model specifications.

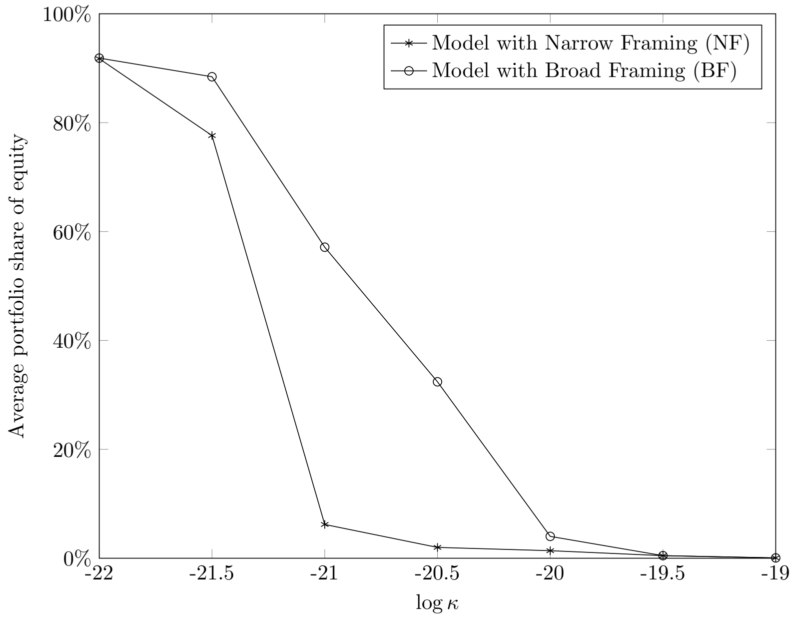

Figure 3: *Average portfolio share of equity for different values of the parameter κ. The values are calculated as $N^{-1} \sum_{i=1}^{N} \left(\sum_{t=1}^{T} P(\tau = t) \left(t^{-1} \sum_{s=0}^{t-1} \theta_{t,i} \right) \right)$ where τ denotes the time of death and the $\theta_{t,i},\ i = 1, \ldots, N$ are obtained by forward simulation using the optimal strategies according to the model specifications with* $N = 10000$.

In contrast to the optimal annuity demand the average portfolio share of equity does differ between the two model specifications for all values of κ considered in our analysis, except for the boundary values $\log \kappa = -22$ and $\log \kappa = -19$. In these cases the average portfolio share of equity coincides with the benchmark model in the former case and is zero in the latter case. For values in between, the narrow framing investor consistently invests a smaller fraction of his available wealth in equity. This is not only true on average across the whole time horizon, as the figure shows, but also in each period as we see in the next section. Overall loss aversion with narrow framing effects the willingness to invest in equity much stronger than broad framing does. This leads to especially high differences in the average portfolio share of equity for the parameter values $\log \kappa = -21$ and $\log \kappa = -20.5$. For $\log \kappa = -21$ the broad framing investor's portfolio share of equity is almost ten times as big as that of the broad framing investor. We note that in this case the annuitization degree is equal for both investors, so that wealth on hand is of comparable size. As we discuss below, consumption behavior is fairly similiar in both model specifications. Therefore, due to the higher portfolio share of equity the broad framing investor ends up with higher wealth levels in many cases which means that there is often an even higher discrepancy between the absolute equity exposure of both investors. For $\log\kappa = -20.5$ the absolute difference in the average portfolio shares of equity between both models is smaller with 30.41 percentage points, but larger in relative terms. In this case the broad framing investor's average portfolio share of equity is over 16 times as large as that of the narrow framing investor. Even though in this case the annuitization degree is higher for the broad framer, which means that he starts off with less wealth on hand, the higher income from the more effective investment strategy eventually results in higher wealth levels than the narrow framing investor. The difference in the average absolute equity exposures is therefore even higher in relative terms than the difference in average portfolio shares. In this case the broad framer has an average equity exposure which is slightly below 20000 which is more than 160 times the average equity exposure of the narrow framing investor. For the previous value $\log\kappa = -21$ the average equity exposure of the broad framer is merely slightly more than 14 times bigger for the broad framing investor than for the narrow framing investor. For higher values of κ both investors have a high reluctance to invest in equity and only have average portfolio shares of equity below 5% and less.

As we discuss below in more detail, the broad framing investor has the advantage that he himself can choose the portfolio that is subject to the loss averse investment evaluation. The narrow framing investor for example always evaluates the equity investment in the same way and it is only the size of his investment that he can control. The broad framing investor on the other hand can build portfolios that are evaluated in a more favourable way than the individual assets. For example if he adds a very small share of equity to a portfolio that otherwise only consists of annuities, then losses in equity themselves very rarely result in the total return being below the reference point when there is no loss in the annuity investment. Therefore the broad framing investor can often add a little equity to his portfolio without increasing the probability of a perceived loss and therefore benefit from the equity risk premium for "free". So in addition to ignoring diversification effects, the narrow framing investor has another disadvantage compared to the broad framing investor which results in even less participation in equity markets.

Naturally, the more cautious investment strategies of the loss averse investors come at the price of missing out on welfare gains in many cases. On average this leads to less income over the whole time horizon which results in lower consumption levels. The expected average consumption levels $E[\tau^{-1)}\sum_{t=0}^{\tau} C_t]$ for the different parametrizations of κ for both model specifications are displayed in figure 4. With an increasing strength of the effect of loss aversion, the gradual withdrawal from annuity and equity markets leads to a severe decline of the average annual consumption levels. Eventually, for large enough κ, the total reluctance to invest in risky assets leads to a 38.54% reduction in the average annual consumption level compared with the benchmark investor who is not loss averse. Due to his more effective investment strategies, the broad framer has slightly higher annual consumption levels on average. However as the differences in the investment strategies between the two models are mostly limited to equity investments, the differences in income are moderate in size and therefore the differences in annual consumption are also not that big. The difference between both model specifications is highest in our analysis for $\log\kappa = -20.5$. In this case the narrow framing investor consumes on average 5.12% less than the broad framing investor.

In the final part of this section we analyze the effect of myopic loss aversion on bequest sizes. Figure 5 contains the means and the standard deviations of bequest sizes in the two model specifications for different values of the parameter κ. In the benchmark case without myopic loss aversion the retiree's annuitization degree of 90% is almost high enough to finance the retiree's total annual consumption. All of the retiree's liquid wealth that is not consumed is invested in equity, at least in the early years of retirement. However not all of the annnual income from this investment is needed to finance the remaining part of the retiree's annual consumption so that the wealth on hand is actually increasing throughout the retirement phase. This means that

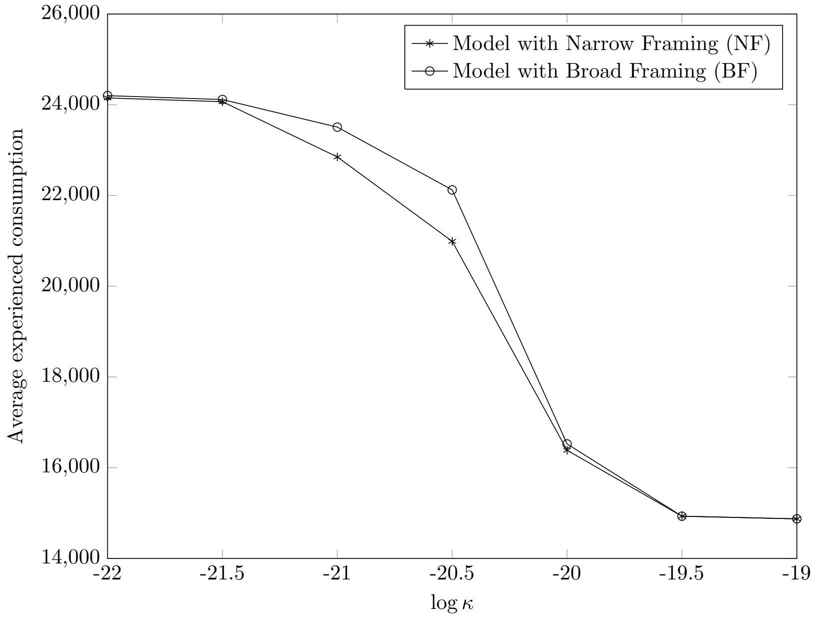

Figure 4: *Average experienced consumption levels for different values of the parameter κ. The values are calculated as* $N^{-1}\sum_{i=1}^{N}\left(\sum_{t=1}^{T}P(\tau=t)\left(t^{-1}\sum_{s=0}^{t-1}C_{t,i}\right)\right)$ *where τ denotes the time of death and the $C_{t,i} = c_{t,i}W_{t,i}^{L},\ i = 1,\ldots,N$ are annual consumtpion levels obtained by forward simulation using the optimal strategies according to the model specifications with $N = 10000$.*

benchmark retiree's bequest is larger when he deceases during a later period. This is not the case for the myopic loss averse investor. In both model specifications and for all the parameter values of κ which are sufficiently high to have a noticeable effect on the investor, the wealth on hand and therefore the bequest size is decreasing throughout the retirement phase. Especially for high values of κ the reluctance to annuitize leads to very high initial wealth levels. However due to the little income from annuities, or even the complete lack thereof, and furthermore the reluctance to invest the liquid wealth in equity, the total income of the retiree is very low. Because the drain of funds through annual consumption is not compensated by a sufficient stream of income, the retiree has to choose significantly lower consumption levels to prevent himself from running out of funds during his remaining lifespan. This means that his wealth on hand is decreasing slowly. Only in the last ten years of the retirement phase does his wealth on hand fall below the wealth level in the benchmark model. Hence bequest are bigger in the models with myopic loss aversion throughout most of the retirement phase. Especially a death in the early retirement years leads to very high bequests for the loss averse investors with high κ values because almost their whole initial wealth is subject to bequest. However these high bequests are not intentional but merely a result from the lack of annuitization. Such bequests, which are due to a premature death and do not represent the desired bequest size, are often called accidental bequests in the literature. In the benchmark model there are no accidental bequests. Roughly put, the retiree chooses how much he wants to bequeathe and annuitizes the remainder of his wealth. In contrast to this, the loss averse investors with high κ values either leave too large bequests, or accidental bequests, when they die early and too small bequests when they die very late. This means that the standard deviations of the bequest size are very low for the benchmark model and models with lower κ parameters and very high for very loss averse investors.

The investors which are not too loss averse to participate in equity and annuity markets leave bequests with sizes between 10% and 14% of their initial wealth. As mentioned, there is very little

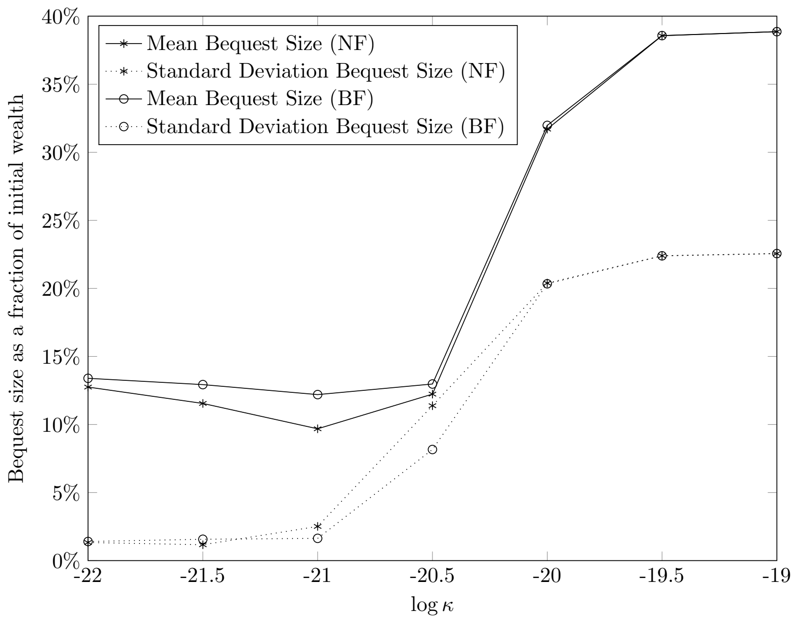

Figure 5: *Mean bequest sizes and standard deviations as a fraction of the retiree's initial wealth for different values of the parameter κ. The values are the sample means and sample standard deviations calculated from $N = 10000$ sample paths obtained by forward simulation using the optimal strategies according to the model specifications.*

variation in bequest sizes in the models with lower κ parameters. In the models with higher κ values, mean bequest sizes range between 30% and 40% of the retiree's initial wealth with standard deviations between 20% and 25% of the initial wealth level. For the parameter values $\log \kappa = -21.5$ and $\log \kappa = -21$, the mean bequest sizes in the model with broad framing are slightly bigger than in the model with narrow framing. In all other cases there are no big differences between the average bequest sizes and their standard deviations for the two model specifications.

5.2 Optimal Wealth, Consumption and Investment Paths

In this section we analyze the average trajectories of the various state and control variables of the two model specifications under various parametrizations for κ. Figure 6 displays the average wealth, consumption and portfolio share of equity paths for the parameter values $\log \kappa = -21$, $\log \kappa = -20$, $\log \kappa = -19$ in the model with narrow framing and the benchmark model without loss aversion. The respective paths for the model with broad framing are displayed in figure 7. The values for wealth on hand are the available funds before consumption. Subtracting the size of the annuity from wealth on hand gives the bequest if the retiree dies in the previous period. This illustrates the strong effect of loss aversion on the bequest size that is discussed in the previous section.

The two retirees contained in both figures who are most hindered by loss aversion, i.e. the models with $\log \kappa = -20$ and $\log \kappa = -19$, have annuitization degrees of 10% and 0%. This means that they end up with a very high initial wealth levels. On average they choose their consumption levels such that their wealth decreases at a constant speed throughout the whole retirement phase. Their income, which stems mostly from risk free bond investments, decreases when the size of the funds invested decreases, i.e. their wealth on hand. This means the annual consumption levels have to be lowered to prevent the wealth levels from falling at an ever faster speed. As a result the annual consumption levels, which are already around 20% lower than in the benchmark model, are steadily decreasing even further. In the model with narrow framing and

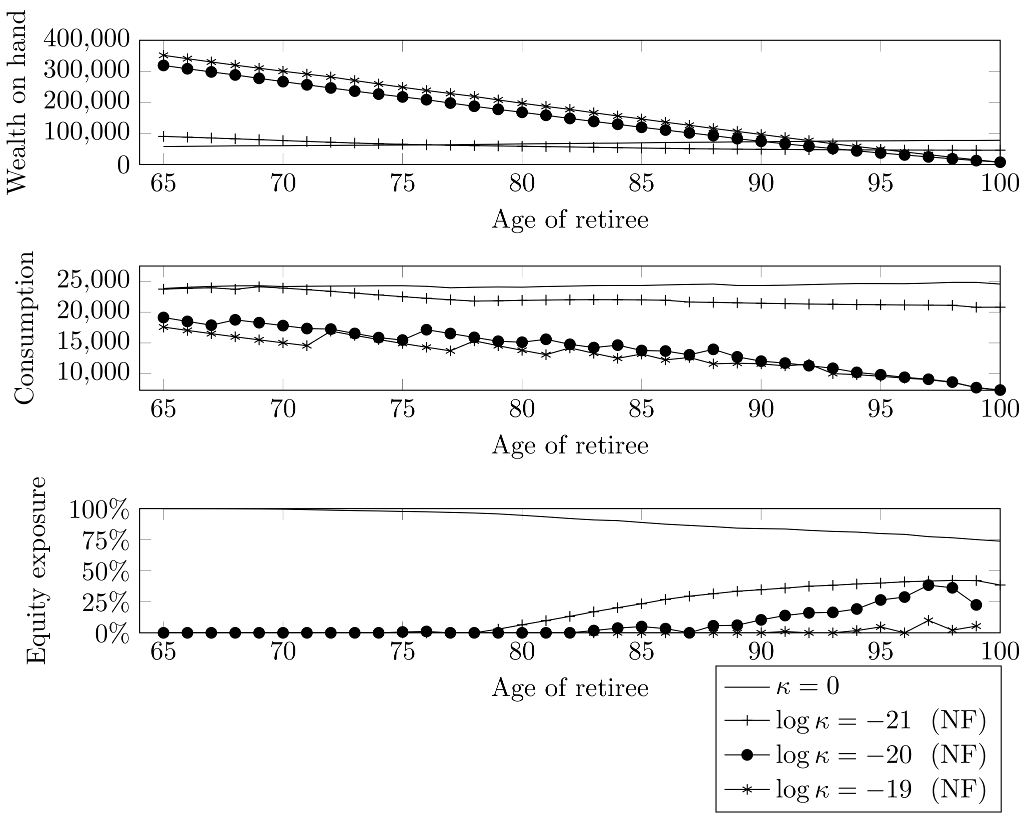

Figure 6: *Average trajectories in the model with narrow framing (NF). The values displayed are averages* $\bar{X}_t = \sum_{i=1}^{N} \frac{X_{t,i}}{N}$ *over* $N = 10000$ *sample paths* $X_{t,i}$ *obtained through forward simulation using the optimal strategy according to the model specification. Wealth on hand denotes the retiree's wealth before consumption. The values at each point in time are conditional on the investor's survival until that period.*

$\log \kappa = -19$, the annual consumption decreases by almost 300 per year on average. If the retiree survives until the final years of the time horizon in the model, this steady decline results in annual consumption levels that have dropped by almost 60% compared to consumption in the first period.

A similar yet drastically less pronounced effect is observable in the models with $\log \kappa = -21$. In contrast to the previous two cases, the retirees in both model specifications choose a fairly high annuitization degree of 80%. On the one hand this means that they start off with initial wealth levels of similiar low size as the benchmark investor. On the other hand this also means that they have annual income through the annuity which finances a great part of their annual consumption. However there is a higher financing gap than in the benchmark model due to the slightly lower annuitization degree. Furthermore the investors are not as willing to invest in equity as the benchmark investor, even though they do invest in equity, but especially in the early years much more cautiously. This results in a further reduction of their annual income which is overall not enough to completely finance their annual consumption. Hence their wealth on hand levels are decreasing in contrast to the wealth of the benchmark investor. As in the previous two cases their wealth on hand decreases somewhat steadily. Because their income also decreases due to their wealth decreasing, they too have to adjust their annual consumption over time. But because they have a steady non-decreasing source of income, the annuity, they only need to slightly lower their consumption each year. In total, if they survive until the end of the time horizon in our model, their annual consumption drops by 12.35% compared to the first period in the model with

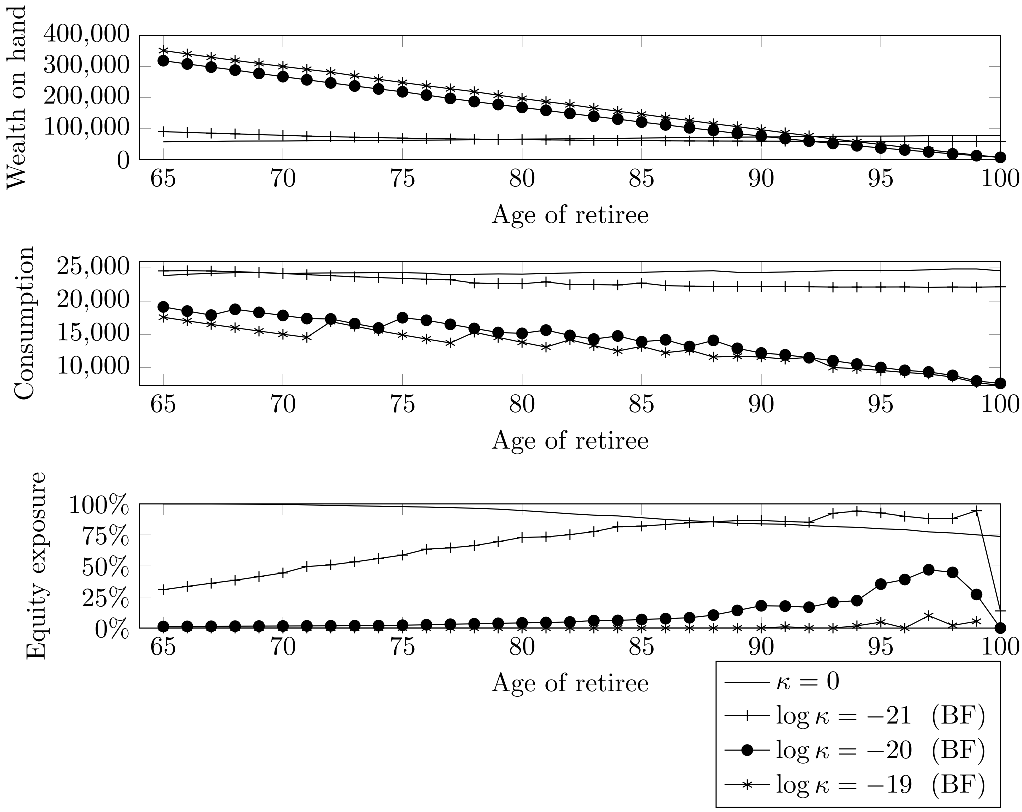

Figure 7: *Average trajectories in the model with broad framing (BF). The values displayed are averages* $\bar{X}_t = \sum_{i=1}^{N} \frac{X_{t,i}}{N}$ *over* $N = 10000$ *sample paths* $X_{t,i}$ *obtained through forward simulation using the optimal strategy according to the model specification. The values at each point in time are conditional on the investor's survival until that period.*

narrow framing and by 9.75% compared to the first period in the model with broad framing. In addition to reducing his consumption more slowly than the narrow framer, the broad framer also starts with 3.28% higher consumption in the first period. As seen in the previous section, the broad framer also leaves larger and less risky bequests than the narrow framer for the parameter $\log \kappa = -21$. The overall higher income levels of the broad framer, which manifest themselves in noticeable differences in consumption levels and bequest sizes, are a result of the significantly higher equity exposure of the broad framing investor which can be seen by comparing figures 6 and 7 and also in figure 3. While the broad framer's investment strategy is still more cautious than the benchmark investor's investment strategy, he has on average 60% of his wealth on hand invested in equity. Because the capital bound in the annuity decreases over time, the negative effect of loss aversion on the willingness to invest, which depends on the size of the investment, decreases over time. Therefore the broad framing investor can increase his relative equity exposure throughout the retirement years. In the last 15 years of retirement, he even invests higher fractions of his wealth in equity than the benchmark investor. As the decrease in capital bound in the annuity has no effect on the narrow framer's investment evaluation of the equity investment, this effect is not present in the model with narrow framing. However the decrease in wealth on hand over time also leads to smaller investment sizes and therefore the narrow framer too can increase the relative size of his equity exposure over time.

5.3 Optimal Portfolios under broad framing

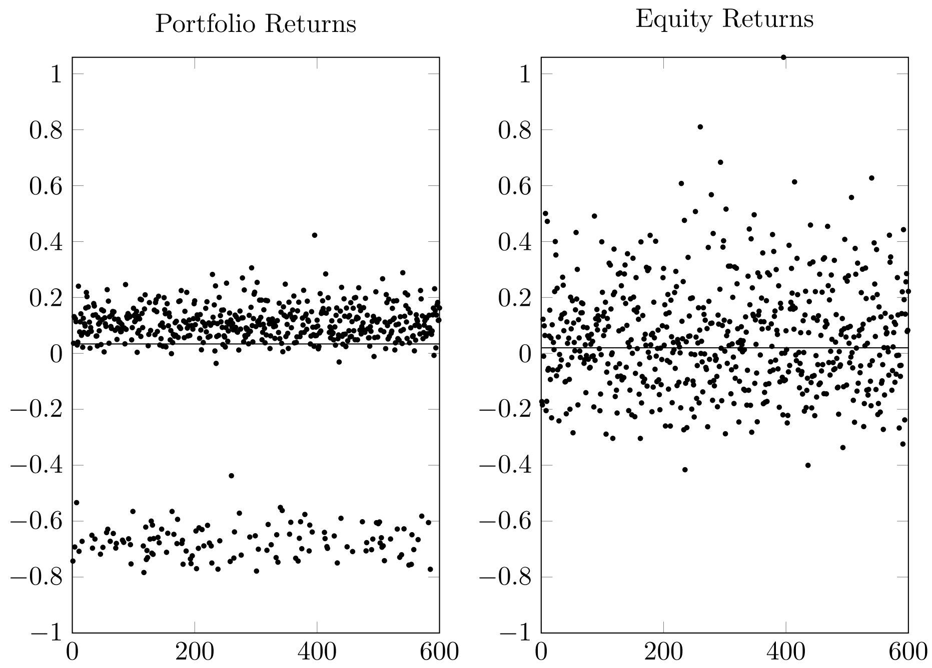

Figure 8: *$N = 600$ simulated returns for the risky component of an exemplary optimal portfolio and of the stationary equity investment. The portfolio represents the average equity exposure of the broad framing investor with $\log \kappa = -21$ in period 30. The portfolio consists of 69% annuities and 31% equity. The horizontal lines represent the respective reference points.*

As already mentioned above, there is another difference in the investment evaluation between the models with broad framing and narrow framing that goes beyond the classical notion of diversification. In the model with narrow framing, equity investments of equal size result in the same evaluation in each period and independant of the annuity investment. In other words the probability of a loss in the equity investment is always the same due to the assumed stationarity of the equity return distribution. This is not the case for the model with broad framing. Here the probability of a loss depends on the composition of the portfolio. Figure 8 displays $N = 600$ simulated returns of the risky component of an exemplary portfolio of the broad framing investor and of the equity investment alone. The risky component consists of 69% annuities and 31% equity. Due to the individual return characteristics of both assets, a loss in the annuity almost always results in a loss in the portfolio with respect to the appropriate reference point formed according to equation 13. A loss in the equity investment on the other hand almost never results in a loss in the portfolio with respect to the appropriate reference point, when there is not simultaneously a loss in the annuity. This can be seen from the fact that there are very few losses of the portfolio which are only slightly below the reference point. These are the losses that are due to equity losses even though the annuity investment did not result in a loss. In the particular example in the figure 46% of the equity investments and 21% of the annuity investments result in a loss. The return of the particular portfolio in the example is counted as a loss in 27% of the cases. So when the narrow framer evaluates an equity investment he has to accept an individually evaluated asset with a 46% chance of a loss. When the broad framer evaluates the same investment he only has to accept that the chance of a loss in his portfolio increases from 21% to 27%. This shows that the broad framer can invest in a substantial share of equity, such as 31% of his total investment in the example, and benefit from the equity risk premium, while still perceiving the possibility of a portfolio loss much less likely than if the equity investment is evaluated individually. Because losses are weighted more heavily than gains in the investment evaluation, a lower chance of a loss leads to a more favourable evaluation of an investment and therefore to a higher willingness to invest. Besides the regular benefits of diversification, this effect is a reason why the broad framing investor has so much higher equity portfolio shares than the narrow framing investor for some values of κ.

6 Conclusion

We propose a model that applies the concept of myopic loss aversion to an annuity investment within the framework of a life cycle model. For a sufficiently high strength of the loss averse investment evaluation, we find that loss aversion can explain low annuitization rates up to complete abstinence from annuity markets. We further find that the the scope of the framing, that is wether or not the investor evaluates individual assets alone or together, can have a strong effect on the equity exposure but has none or only small effects on the investor's annuitization degree. When an annuity is represented in the form that we propose in this paper, the only difference from an investment point of view between an annuity investment and an equity investment is the liquidity. This begs the empirical question wether individuals also exhibit myopic loss aversion over illiquid investments or not. It remains to be seen wether loss aversion can be applied to a life annuity in a hybrid consumption/investment frame in a more natural form as in our model. As we explained above, a straight forward implication of loss aversion with respect to an annuity is not possible, because there is no pair of an amount of capital and a random return that adequatly represents an annuity investment.

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Reference-dependent preferences and the demand for annuities

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Abstract

We apply a dynamic reference-dependent preference model to the problem of voluntary annuitization. We find that in most cases, a retiree with preexisting beliefs about how much he wants to annuitize will not choose higher annuitization levels when offered the possibility to do so. In particular a retiree who only has his mandatory annuitization rate, will not voluntarily annuitize further parts of his wealth if he had not previously planned to do so. Therefore reference-dependence in the preferences and low reference annuitization rates are a possible explanation for the low voluntary annuitization rates.

1 Introduction

We propose a model that applies Köszegi und Rabin's [19][1][2] dynamic preference functional with reference-dependency to the problem of voluntary annuitization upon entry to retirement. Their model consists in parts of a standard dynamic expected utility framework. An extra input factor in addition to the investment and consumption plan in their model is a set of beliefs or reference points about future consumption. In addition to absolute utility from consumption, and in our case also bequest, the agent also receives relative utility arising from potential deviations from his reference points. The relative utility is characterized by loss aversion, which means that adverse deviations from the reference points have a stronger effect on the agent than advantageous deviations from the reference points. Köszegi und Rabin propose a weak and a strong solution concept to their problem. In the weak form, a plan is optimal if, under the current set of beliefs (or the reference points generated by the plan), any deviation from it results in a decrease in total utility. A plan is a solution in the strong sense, if it is a solution in the weak sense, and its total utility is greater than or equal to the total utility of all other plans that are a solution in the weak sense. A strong solution is called a *preferred personal equilibrium (PPE)*. Köszegi und Rabin's model generally allows for a variety of ways in which beliefs or reference points are formed. Our approach is outlined in the following. In this paper we assume that the crucial decision variable is the degree of annuitization. Given a fixed degree of annuitization, we assume that the reference points are the optimal and possibly random consumption plans and associated bequest sizes that arise from standard expected utility optimization under the fixed annuitization degree. This is consistent with the theory because following the plan means that there are no deviations from the original plan, and thus for this strategy, Köszegi und Rabin's model actually reduces to an expected utility optimization. When the agent considers other degrees of annuitization, then his consumption plan may not coincide with the reference points and thus in addition to maximizing utility from consumption, the agent will also try to minimize deviations from reference levels. So in this case the optimization is not identical to expected utility optimization. Depending on the

[1]Especially relevant to our model is section IV. Wealth and Consumption in intertemoporal choice, p. 924-929.
[2]See also Köszegi und Rabin [17], which contains a non dynamic version of the model.

solution concept, this approach allows us to approach the problem of how much to annuitize upon entry to retirement from two angles:

1.) Does an investor who had previously not planned to annuitize his wealth, and thus have respective reference points associated with non-annuitization, change his mind and annuitize, or is non-annuitization a weak solution in the above sense?

This question may be adjusted to include situations in which the investor's or the retiree's wealth is already annuitized to some degree, due to mandatory annuitization, and he is offered the possibility to voluntarily annuitize further parts of his wealth. Then his reference points are based on his previous annuitization degree and the question becomes wether or not does the investor voluntarily annuitize additional parts of his wealth. When low levels of preannuitized wealth are weak solutions, then reference-dependent models could explain the low observed voluntary annuitization levels of retirees. The second question regards the strong solution concept.

2.) Which annuitization degree is a preferred personal equilibrium?

When there are weak solutions which are not a strong solution, then this means that there may be retirees in our model who choose a plan that they would otherwise reject in a classical expected utility framework. This means that they follow a plan that yields less classical utility from consumption and bequest than the optimal plan in the classical sense. Therefore from a rational point of view, it is best to choose the plan as a reference level that is the preferred personal equilibrium. In our case this plan coincides with the optimal plan in the classical expected utility framework. This means that picking the optimal reference annuitization degree in a rational way, i.e. by expected utility maximization, yields the highest total utility in our model. The more interesting case in our analysis is therefore which annuitization levels, which are not optimal in the classical sense, are still weak solutions to the reference-dependent consumption and investment problem.

We find that in our application of Köszegi und Rabin's model, all annuitization rates besides full annuitization and no annuitization are weak solutions. This means that most retirees would not opt for voluntary annuitization beyond their reference annuitization rate in our model. Assuming that many retirees do not plan to voluntarily annuitize beyond mandatory rates in the first place, reference-dependent preferences may be a possible explanation for the low empirically observed actual voluntary annuitization rates.

In addition to a contribution to the literature on voluntary annuitization, this paper contains a minor methodical contribution by extending the numerical solution method based on a semi-simulation approach applied by Koijen, Nijman and Werker [16] to a non-smooth objective function.

2 Literature review

We build in parts on the literature on reference-dependent preferences. Mainly we adopt the general dynamic model of Köszegi und Rabin's [19] which builds on models such as Matthey [20] and Koszegi and Rabin ([18],[17]). We apply a special case of their model to the problem of voluntary annuitization.

There is a large variety of research on what optimal annuitization rates should be and why observed annuitization rates are often much lower than that. For example Schaus [22] reports from a survey on 401(k) plans, that only 6% of the retirees who were able to choose voluntary annuitization did actually do so. This reluctance to annuitize leads to much lower annuitization rates than are found to be optimal from a rational point of view. For example Davidoff, Brown and Diamond [10] find that high annuitization rates or even full annuitization is optimal under a broad set of circumstances including unfair annuity pricing. Peijnenburg, Nijman and Werker [21] obatin similar high annuitization levels. To explain the differences between the theoretical results and the empirical observations, a variety of explanations beyond rational models has been sought out. A good overview over a broad set of potential determinants of annuity demand, both rational and irrational, is given by [5]. Empirical testing of some of those hypothesisis is conducted in various publications, for example Goedde-Menke, Lehmensiek-Starke and Nolte [13] and Cappelletti, Guazzarotti and Tommasino [6]. They find that among others wealth, finacial literacy, bequest motives and framing are determining factors of annuity demand.

The reference-dependent preference model by Köszegi und Rabin builds in parts on a loss averse evaluation of gains and losses in classical utility. Their loss averse evaluation function is based on Kahneman and Tversky's cumulative prospect theory [26]. There are various examples in the literature that base models for annuity and equity demand on this evaluation function. How a loss averse investment evaluation can affect the optimal demand for annuities in a theoretical setting is analyzed for example by Hu and Scott [15]. Examples of how loss aversion can be included in life cycle models to explain equity demand are given by Benartzi and Thaler [3], De Giorgi and Legg [11] and Barberis and Huang ([1],[2]). Applications which are similiar in nature, yet applied to the demand for general insurance products, can be found in Gottlieb and Mitchell [14] and Chen, Hentschel and Klein [8]. Specific applications of loss aversion within a life cycle model to explain annuity demand can be found in Schneider ([23], [25]).

3 The model

Central to our analysis is a retiree's decision, how much of his total wealth W to annuitze upon entry to retirement ($t = 0$). The retiree may already have a fraction of his initial wealth $W_{pa} \leq W$ preannuitized, providing him with an annuity in advance of size A_{pa}. We further assume that the retiree has a one time only access to the annuity market at the time he enters retirement. We let $0 \leq p_R \leq 1$ denote the total percentage of his initial wealth W the retiree plans to annuitize acording to his preexisting beliefs, i.e. p_R is the reference annuitization degree. A key component of our model is that even though the retiree has preexisting beliefs about voluntary annuitization, he may still consider choosing a differing annuitization degree[3]. We let $W_{va} \leq W - W_{pa}$ denote the amount of wealth the retiree voluntarily annuitizes in addition to the preanuitized wealth in the new plan and A_{va} the resulting annuity in advance. The resulting total annuitization degree with respect to the initial wealth on hand W in the new plan is denoted by $p_A = (W_{va} + W_{pa})/W$. In a similiar manner as above we let $W^R_{va} = p_R W - W_{pa}$ denote the total amount of wealth the retiree had planned to voluntarily annuitize according to his reference plan and A^R_{va} the corresponding annuity in advance. Furthermore we let $A = A_{pa} + A_{va}$ denote the total annuity the retiree receives in the new plan and $A^R = A_{pa} + A^R_{va}$ the total annuity according to the reference plan.

The retiree may live up to another T years or decease during any prior period. For $s \leq t$ we let $p_{s,t}$ denote the probability of surviving until time t, conditional on being alive at time s. For two successive years we use the short notation $p_t = p_{t,t+1}$. We assume that annuities are priced fairly and that the survival probabilities in the pricing calculation coincide with the retiree's actual survival probabilities. This means that voluntarily annuitizing the amount W_{va} yields the additional voluntary annuity in advance of size

$$A_{va} = W_{va}\Big(\sum_{t=0}^{T} p_{0,t}(1 + R_A)^{-t}\Big)^{-1}.$$

[3]Assuming that a retiree has preannuitized wealth from his employer's pension plan, this may correspond to the situation that the retiree had never considered the possibility of voluntary annuitization prior to his retirement, but is informed about this possibility by an investment consultant or receives an offer by an insurance company upon entry to retirement.

We let W_t^L denote the retiree's wealth on hand at time t under the condition that he is still alive and immediately after receiving the annuity A. At the beginning of each period, starting at $t = 0$, the retiree chooses his annual consumption level $C_t \leq W_t^L$. The remainder of his wealth $W_t - C_t$ may be invested at a fixed risk-free interest rate R_f or at a risky return R_t, where $R_1, \ldots, R_T$ is a process of i.i.d. random variables. We let $0 \leq \theta_t \leq 1$ denote the fraction of wealth after consumption that is invested in the risky asset. The remaining fraction of the retiree's liquid wealth $1 - \theta_t$ is invested at the riskless rate. This leads to the budget equation

$$W_{t+1}^L = (W_t - C_t)(\theta_t R_t + (1 - \theta_t) R_f) + A \tag{1}$$

which describes the retiree's wealth on hand if he survives until time t and

$$W_{t+1}^D = W_{t+1}^L - A \tag{2}$$

if he deceases during period $t + 1$. Initially the retiree starts with wealth on hand

$$W_0^L = W - W_{va} + A. \tag{3}$$

When the retiree dies during period $t+1$ the amount W_{t+1}^D is transfered to an heir in the form of a bequest $B_{t+1} = W_{t+1}^D$. The agent's wealth in all later periods after his death is zero.

We let $\boldsymbol{U(p_A)}$ denote the set of pairs of (random) consumption and investment plans $(\boldsymbol{C_0}, \boldsymbol{\theta_0})$ with $\boldsymbol{C_0} = (C_0, C_1, \ldots, C_T)$ and $\boldsymbol{\theta_0} = (\theta_0, \theta_1, \ldots, \theta_T)$ which are feasible in the sense that $0 \leq \theta_t \leq 1$ for all $t = 0, 1, \ldots, T$ and that $C_t \leq W_t^L$ for all $t = 0, 1, \ldots, T$ where W_t^L is the time t wealth level resulting from the dynamics described by equations (1) and (3), by applying the consumption plan $\boldsymbol{C_0}$ and the investment plan $\boldsymbol{\theta_0}$ under the voluntary annuitization degree p_A. Furthermore we let $\boldsymbol{B_0(C_0, \theta_0)}$ denote the associated process of bequest sizes which are determined by equation (2).

Derived from the preexisting belief about his voluntary annuitization degree the retiree starts with a set of beliefs or reference points

$$\boldsymbol{F_0} = \{(C_{t,R}, B_{t,R})\}_{s=0,\ldots,T} \tag{4}$$

about present and future consumption and bequest. Because the retiree's wealth process is non-deterministic due to the risky asset, the reference levels $C_{t,R}$ and $B_{t,R}$, denoting the planned consumption level $C_{t,R}$ at time t, assuming survival until time t, and the corresponding bequest size assuming death during period t, are random variables, depending on the wealth on hand at time t and thus the outcome of the previous asset returns. We assume that the reference plans are feasible, i.e. that $\boldsymbol{C_{0,R}} = (C_{0,R}, \ldots, C_{T,R}) \in \boldsymbol{C(p_R)}$ and $\boldsymbol{B_{0,R}} = \boldsymbol{B_0(C_{0,R}, \theta_{0,R})}$ for a feasible investment plan $\boldsymbol{\theta_0}$[4].

In total, the retiree is subject to four sources of utility. Two of which are "classical" sources of utility and two are "gain-loss utilities" derived from deviations in classical utility from prior beliefs. Specificly we have

1. utility from consumption $m_C(C_t)$, if the retiree is alive at time t,
2. utility from bequest $m_B(B_t)$, if the retiree has deceased during period t,
3. gain-loss utility $v(m_C(C_t) - m_C(C_{t,R}))$ regarding consumption, if the retiree is alive at time t and
4. gain-loss utility $v(m_B(B_t) - m_B(B_{t,R}))$ regarding bequest, if the retiree has deceased during period t.

[4]In the same manner as a pair of a feasible consumption and investment plan defines a unique process of bequest sizes, a feasible combination of a reference consumption plan and a process of reference bequest sizes defines a unique reference investment plan $\boldsymbol{\theta_{0,R}}$.

Here m_C and m_B are standard utility specification for consumption and bequest and v is an evaluation function which is characterized by some degree of loss aversion. The specific choices for the utility functions m_C, m_B and v are discussed below.

To shorten notation we let

$$\tilde{m}_C(C_t, C_{t,R}) = m_C(C_t) + v(m_C(C_t) - m_C(C_{T,R})) \tag{5}$$

and

$$\tilde{m}_B(B_t, B_{t,R}) = m_B(B_t) + v(m_B(B_t) - m_B(B_{T,R})) \tag{6}$$

denote the combined utiltiy functions for consumption and bequest.

The retiree's objective function Φ_0 at time $t = 0$ is the sum of all sources of utility whereas utilty from future periods is discounted by the time discount factor β. For any feasible consumption and investment plan, $\boldsymbol{C_0}$ and $\boldsymbol{\theta_0}$, and a set of reference levels $\boldsymbol{F_0}$, the retiree's time $t = 0$ objective function is given by[5]

$$\Phi_0(\boldsymbol{C_0}, \boldsymbol{B_0}(\boldsymbol{C_0}, \boldsymbol{\theta_0}), \boldsymbol{F_0}) = E_0\left[\sum_{t=0}^{T} p_{0,t}\beta^t \Big(\tilde{m}_C(C_t, C_{t,R}) + (1 - p_t)\tilde{m}_B(B_t, B_{t,R})\Big)\right] \tag{7}$$

where $p_T = 0$. In the trivial case that $\boldsymbol{C_0} = \boldsymbol{C_{0,R}}$ and $\boldsymbol{\theta_0} = \boldsymbol{\theta_{0,R}}$, the gain-loss terms cancel out and the objective function reduces to a classical expected utility framework.

As in Köszegi und Rabin's model [19], we propose a weak and a strong solution concept for the retiree's decision problem. The weak solution concept answers the question, wether a retiree with an existing reference plan keeps or abolishes that plan, if he makes his decision according to the preferences defined by equation (7). It requires that the reference annuitization degree p_R allows a feasible consumption and investment plan that yields a higher objective value with its own consumption and bequest levels as reference points, than all feasible plans for all other annuitization degrees under the same reference points.

Definition 1. *An annuitization degree p_R is a personal equilibrium (PE), if there exists a feasible reference consumption and investment plan $(\boldsymbol{C_{0,R}}, \boldsymbol{\theta_{0,R}}) \in \boldsymbol{U}(\boldsymbol{p_R})$ with reference points $\boldsymbol{F_0} = (\boldsymbol{C_{0,R}}, \boldsymbol{B_0}(\boldsymbol{C_{0,R}}, \boldsymbol{\theta_{0,R}}))$, such that for all annuitization levels $p \in [0, 1]$*

$$\Phi_0(\boldsymbol{C_{0,R}}, \boldsymbol{B_0}(\boldsymbol{C_{0,R}}, \boldsymbol{\theta_{0,R}}), \boldsymbol{F_0}) \geq \sup_{(\boldsymbol{C'_0}, \boldsymbol{\theta'_0}) \in \boldsymbol{C}(\boldsymbol{p})} \Phi_0(\boldsymbol{C'_0}, \boldsymbol{B_0}(\boldsymbol{C'_0}, \boldsymbol{\theta'_0}), \boldsymbol{F_0}). \tag{8}$$

If the retiree's annuitization degree p_R is a PE, then a retiree who makes his decision according to the preference functional (7) sticks with his planned annuitization level. If this is not the case then he chooses the annuitization level that allows the plan with the highest objective value under the old reference points. However there might be multiple PE. The strong solution concept requires that the reference level p_R is the PE that results in the highest objective value under its best plan among all PEs.

Definition 2. *An annuitization degree p_R is a preferred personal equilibrium (PPE), if it is a PE and there exists a feasible reference consumption and investment plan $(\boldsymbol{C_{0,R}}, \boldsymbol{\theta_{0,R}}) \in \boldsymbol{U}(\boldsymbol{p_R})$*

[5] As mentioned earlier this is a special application of Köszegi und Rabin's [19] preferences which contains bequest utility in addition to utility from consumption depending on the survival state of the retiree. In our model the only source of uncertainty is the development of the risky asset. Whenever uncertainty is resolved, i.e. after any investment period, the retiree knows his current wealth level and therefore his potential bequest and consumption levels. These levels are compared to the retiree's reference levels which results in gain-loss utility in the current period. In Köszegi und Rabin's general model it is possible that resolved uncertainty also yields expected gain-loss utility about consumption in all future periods. This multiple counting of the effect of one investment can have implications on the preferences for the timing of information. More specificly in their model gain-loss utiltiy from period s in period t is assigned the weight $\gamma_{t,s} \geq 0$ with $\gamma_{t,t} = 1$. Out model corresponds to case $\gamma_{t,t} = 1$ and $\gamma_{t,s} = 0$ for all $s > t$, which omits future period gain-loss utility entering the model.

with reference points $\boldsymbol{F_0} = (\boldsymbol{C_{0,R}}, \boldsymbol{B_0}(\boldsymbol{C_{0,R}}, \boldsymbol{\theta_{0,R}}))$, *such that for all other PE* $p_{R'}$ *with consumption and investment plans* $(\boldsymbol{C'_{0,R'}}, \boldsymbol{\theta_{0,R'}}) \in \boldsymbol{U}(\boldsymbol{p'_R})$ *and associated reference points* $\boldsymbol{F'_0} = (\boldsymbol{C'_{0,R}}, \boldsymbol{B_0}(\boldsymbol{C'_{0,R'}}))$

$$U_0(\boldsymbol{C_{0,R}}, \boldsymbol{B_0}(\boldsymbol{C_{0,R}}, \boldsymbol{\theta_{0,R}}), \boldsymbol{F_0}) \geq U_0(\boldsymbol{C'_{0,R'}}, \boldsymbol{B_0}(\boldsymbol{C'_{0,R'}}), \boldsymbol{F'_0}). \tag{9}$$

4 Utility specifications

There are various utility specifications that could be applied to the general model presented above. In this paper we follow the choices most encountered in the relevant literature.

We assume that the retiree receives CRRA-utility from his annual consumption, i.e. a consumption level of size C yields the utility

$$m_C(C) = \frac{1}{1-\gamma} C^{1-\gamma} \tag{10}$$

with constant relative risk aversion γ.

We further assume that the retiree receives utility from bequest according to De Nardi's [12] definition of bequest utilty, i.e. the utility of a bequest of size B is defined as

$$m_B(B) = \frac{\omega}{1-\gamma} \left(\psi + \frac{B}{\omega} \right)^{1-\gamma}. \tag{11}$$

The two additional parameters ω and ψ describe the strength of the bequest motive and the degree to which bequest is a luxury good. The affine formulation ensures that the retiree only leaves a bequest if his funds simultaneously allow for sufficiently high consumption levels for himself and a sufficiently high bequest size[6]. If this is not the case then the retiree does plan to leave an intentional bequest of size zero. However such a retiree typically still leaves an accidental bequest, if not all of his wealth is annuitized and he does not survive until the end of the time horizon of our model.

The gain-loss utility function is a special case of the loss-averse evaluation function proposed in Kahnemann and Tversky's prospect theory [26].

$$v(x) = \begin{cases} x & \text{for } x \geq 0 \\ \lambda x & \text{for } x < 0 \end{cases} \tag{12}$$

with a Loss Aversion parameter $\lambda > 1$[7].

5 Parametrization

We adopt some often encountered settings in life cycle models with a time discount factor of $\beta = 0.96$ and a parameter of risk aversion $\gamma = 5$ as for example in Cocco and Gomes [9]. The bequest utility parameters used in our model are based on De Nardi's original specification [12]. As in Peijnenburg, Nijman and Werker [21] we adjust the luxury parameter to the retiree's fully annuitized income FAI which results in the parameters $\omega = 7.81$ and $\psi = 0.67 * FAI$. The loss

[6] A simple example helps to illustrate the effect of De Nardi's bequest utility specification. In a model with deterministic age of death after T years and no investment opportunities and no time discounting the retiree splits his initial wealth W in T annual consumption levels of size $(W + \omega\psi)/(\omega + T)$ and a bequest of size $B = \omega(C - \psi)^+$. That means the retiree bequeathes ω times the amount his annual consumption surpasses the luxury threshold ψ if his available funds are of sufficent size and nothing otherwise.

[7] As in Köszegi und Rabin [19] we assume a piecewise linear function instead of Kahnemann and Tversky's original formulation.

aversion parameter in the gain-loss utiltiy is set to $\lambda = 2$.

Furthermore we assume a Black Scholes economy in which equity returns are lognormally distributed with an expected return of $E[R_t] = 8\%$ and standard deviation of $\sigma(R_t) = 20\%$. We further assume that the risk free rate is $R_f = 2\%$ and the calculatory interest rate used in the pricing of the annuity is $R_A = 4\%$. In addition to that we assume that the annuity is priced fairly.

The retiree in our model enters retirement at age 65 and the retirement phase may span up to $T = 35$ periods. The survival probabilities in the model and in the annuity pricing calculation are the male survival probabilities taken from german death tables[8]. The retiree's initial wealth is choosen such that full annuitization results in the fully annuitized income $FAI = 25000$.

6 Solution method

Before we conduct the main optimization, we need to optimize the classical preference functional without gain-loss terms under the reference annuitization degree. This gives us the optimal reference consumption and investment plan on a fixed grid of wealth levels at each point in time. The optimization algorithm used to obtain these reference values is a simplified version of the algorithm used in the main optimization which is described in detail below.

We then simulate forward using these optimal reference consumption and investment plans to obtain $N = 1000$ trajectories for consumption and asset allocation. For wealth levels outside of the wealth grid we use linear interpolation to determine the optimal consumption level and the optimal fraction of the retiree's wealth on hand that is invested in the risky asset. This gives us N trajectories of reference values $X_t(i) = (C_{t,R}(i), \Theta_{t,R}(i), \Theta^C_{t,R}(i))$, where $C_{t,R}(i)$ denotes the realized consumption level and $\Theta_{t,R}(i)$ and $\Theta^C_{t,R}(i)$ denote the funds allocated to the risky asset and the funds allocated to the riskless asset at time t on trajectory $i = 1, \ldots, N$.

In the next step, we calculate the value function at time $t = 0$ for a grid of annuitization levels $(A_1, \ldots, A_K)$ ranging from no annuitization to full annuitization with a stepsize of 10% of the agent's initial wealth. For each annuitization level A_i we find the value function using the the backwards recursion outlined below. In the final step we compare the highest total utility represented by the time $t = 0$ value functions evaluated at the respective inital wealth level after the annuity purchase, i.e.

$$\operatorname*{argmax}_{A_1,\ldots,A_K} V_0(W - W_{va} + A|A) \tag{13}$$

to the total utility in the reference case.

For each annuitization level, we apply a backwards recursion that builds on the solution method applied by Koijen, Nijman and Werker [16], which consists of a combination of methods. These are the simulation approach applied by Brandt, Goyal, Santa-Clara and Stroud [4], which enables optimization in the presence of a high number of state varibles, and Carrol's [7] method of constructing the grid for the endogenous state variable wealth on hand within the recursion from a post-consumption grid. The latter simplifies the optimization in each time step and in particular makes it possible to find the optimal consumption analytically.

However the approach in Koijen, Nijman and Werker [16] demands that all the functions arising in the optimization problem are continuously differentiable. Because of the kink in the gain-loss utility function this is not the case in our problem. We overcome this problem by applying the following procedure. At first we approximate the gain-loss utility function in a sufficently small

[8]Source: Sterbetafel 2009/11 Deutschland männlich, Periodensterbetafeln für Deutschland 2009/2011, Statistisches Bundesamt, Wiesbaden 2012.

enviroment around the kink, with a continuously differentiable function in such a way, that the resulting approximated gain-loss utility function is continuously differentiable everywhere. This allows us to formulate an Euler equation for optimal consumption. Then we regard three separate cases. At first we look for a local maximum on the two outside intervals, left and right of the approximated section, which do not contain the kink. Because of the strict concavity on these intervals, any local maximum on either of the intervals must be a global maximum on its respective interval. Furthermore from the specific form of the solution to the first order conditions on both intervals, it follows that there can not be a local maximum on both intervals simultaneously. Furthermore if we find a local maximum on one such interval, then there can also be no further local maximum on the boundaries of the interval. This is again the case, because of the strict concavity on both intervals. This means that any local maximum on either interval is the global solution to the Euler equation. If there is no local maximum on either interval, then there must be a local maximum in the approximated interval. This is the case because, as we explicate below in more detail, the optimized function cannot be asymptotically increasing. The intuitive reason for this is that, for very low consumption levels, the total utility must be increasing at some point, and for very high consumption values it must be decreasing again at some other point. Therefore the preference functional must have a local maximum at some point. Thus, if there is no local maximum in either outside interval, then there must be a local maximum in the approximated interval which contains the kink. In this case we set the optimal consumption to be equal to the value at which the kink the lies, which is the reference consumption level. Because we can always approximate the preference functional in this way with arbitrary precision, and especially make the approximated interval arbitrary small, this means that the actual optimal consumption value must actually lie on the kink. Therefore the optimal consumption derived in the above way is not an approximation, but the true maximum of the original function.

However there is also another problem in implementing the method of Koijen, Nijman and Werker [16]. The kink in the gain-loss utility function, wether smoothened by approximation or not, leads to Euler equations that can no longer be approximated using a regression on the state variables. Therefore we can no longer find the optimal asset allocation by computing the roots of a fitted polynom, as in the method of Koijen et al. [16], but have to resort to a grid search instead. However this is not too problematic, because for our purposes, restriction to a coarse grid of potential asset allocation parameters does not change the optimal annuitization degrees. Furthermore we have to perform a case-by-case analysis when we compute the optimal consumption, because in our problem the Euler equation takes different forms for values above and below the reference consumption level.

The remainder of this section contains a detailled desrciption and validation of our optimization procedure.

In order to apply variational calculus we smoothen the gain-loss utility function v in a sufficiently small enviroment around zero, i.e. define v_ϵ by

$$v_\epsilon(x) = \begin{cases} v(x) & \text{for } |x| > \epsilon \\ h_\epsilon(x) & \text{for } |x| \leq \epsilon \end{cases} \tag{14}$$

where h is a differentiable and strictly increasing function with $h(-\epsilon) = -2\epsilon$ and $h(\epsilon) = \epsilon$ as well as $h'(-\epsilon) = 2$ and $h'(\epsilon) = 1$[9]. The resulting function v_ϵ is differentiable everywhere. We let

$$\tilde{m}_{C,\epsilon}(C_t, C_{t,R}) = m_C(C_t) + v_\epsilon(m_C(C_t) - m_C(C_{t,R})) \tag{15}$$

and

$$\tilde{m}_{B,\epsilon}(B_t, B_{t,R}) = m_B(B_t) + v_\epsilon(m_B(B_t) - m_B(B_{t,R})) \tag{16}$$

[9] Such a function can be constructed as the limit of a sequence of piecewise defined increasing affine functions with the break points $x_{i,n}$, $i = 1, \ldots, n$ and $h_n(x) = 2x$ for $-\epsilon \geq x \geq x_{1,n}$ and $h_n(x) = x + h_n(x_{n,n})$ for $x_{n,n} < x \geq \epsilon$.

denote the resulting modified composite utility functions and $V_\epsilon(W_T, X_T, T)$ denote the value function of the optimization problem with the modified utility functions. Because the approximation error in the individual utility functions is bounded above by 3ϵ, the total approximation error of the value function at any time point is bounded above by $6(T+1)\epsilon$. Hence we can always find a sufficiently small ϵ that results in an arbitrary small total approximation error. To simplify notation we omit the ϵ subscript in the following. If not otherwise stated, any references to the value function and the utility functions refer to their modified version.

We solve the optimization problem using backward recursion on each of the N trajectories. At each point in time we start by finding the optimal asset allocation on a grid of $M = 30$ exponentially placed post-consumption wealth levels $a_t = W_t - C_t$. Using the modified value function then allows us to solve for the optimal consumption analytically. In the following we outline the general steps of the backwards recursion.

Time T: The time T value function is given by

$$V(W_T, X_T, T) = \max_{\substack{0 \le C_T \le W_T \\ 0 \le \theta_T \le 1}} \Big\{ \tilde{m}_C(C_T, C_{T,R}) + E_T\Big[\beta \tilde{m}_B(B_{T+1}, B_{T+1,R})\Big]\Big\}. \tag{17}$$

We start by optimizing over the asset allocation for each post-consumption grid level $a_T(j), j = 1, \ldots, N$ and each trajectory $X_T(i), i = 1, \ldots, M$,

$$\hat{\theta}_T = \operatorname*{argmax}_{\theta_T} E_T\Big[\beta \tilde{m}_B(B_{T+1}, B_{T+1,R}) | a_T = a_T(j), X_T = X_T(i)\Big]. \tag{18}$$

To this end, we calculate the conditional expectations on the right hand side of the above equation on a grid of potential portfolio shares of equity from 0% to 100% in step sizes of 5%, and apply grid search to find the optimal value. Choosing a finer grid results in significantly longer computation time without changing the optimal annuity endowment at the reported precision.

In a second step, due to the smoothness of the modified value function, the optimal consumption can be derived using the first order condition

$$\frac{\partial}{\partial C_T} \tilde{m}_C(C_T, C_{T,R}) = E_T\Big[\beta \frac{\partial}{\partial B_{T+1}} \tilde{m}_B(B_{T+1}, B_{T+1,R}) R_T^P | a_T = a_T(j), X_T = X_T(i)\Big] \tag{19}$$

where R_T^P deontes the return of the optimal portfolio. To calculate the optimal consumption from this equation, we approximate the derivatives of the modified functions $\tilde{m}_B$ and $\tilde{m}_C$ with the left derivatives of the original functions, i.e. by setting the derivative of the gain-loss utility function to be

$$\frac{\partial}{\partial x} v(x, y) = \begin{cases} 1 & \text{for } x \ge y \\ 2 & \text{for } x < y \end{cases} \tag{20}$$

and thus

$$\frac{\partial}{\partial x} m_C(x, y) = \begin{cases} 2\frac{d}{dx} m(x, y) & \text{for } x \ge y \\ 3\frac{d}{dx} m(x, y) & \text{for } x < y \end{cases}. \tag{21}$$

The modified function $\frac{\partial}{\partial x} m_{C,\epsilon}(x, y)$ is continuous with $\lim_{x \to 0} \frac{\partial}{\partial x} m_{C,\epsilon}(x, y) = \infty$ and $\lim_{x \to \infty} \frac{\partial}{\partial x} m_{C,\epsilon}(x, y) = 0$. Because $\frac{\partial}{\partial x} \tilde{m}_B(x, y) > 0$ for all $x, y \ge 0$ we also have

$$E_T\Big[\beta \frac{\partial}{\partial B_{T+1}} \tilde{m}_B(B_{T+1}, B_{T+1,R}) R_T^P | a_T = a_T(j), X_T = X_T(i)\Big] > 0. \tag{22}$$

Therefore equation (19) must have at least one solution. Hence we first check if there is a solution $\hat{C}_T(i, j)$ to (19) on the intervals $(0, C_{T,R})$ and $(C_{T,R}, \infty)$ by approximating the derivative of $m_{C,\epsilon}$ as described above. If this is the case, then $\hat{C}_T(i, j)$ is a local maximum on its respective interval because m_C is strictly concave on each of the two intervals. But because m_C is continouos and strictly concave on both intervals, any local maximum in the open intervals must also be a global maximum in the semi-closed intervals $(0, C_{T,R}]$ or $[C_{T,R}, \infty)$. If there is no local optimum in one of the intervals, then the supremum must lie at the border of that interval. Furthermore because both semi-closed intervals share the common point $C_{T,R}$, and the optimal consumption can neither be zero nor unbounded[10], any local maximum in either interval must also be the global maximum. If however there is no solution in either open interval, then the solution to (19) must lie in the interval $[C_{T,R} - \epsilon, C_{T,R} + \epsilon]$, in which case we set $\hat{C}_T(i,j) = C_{T,R}$.

After the optimal consumption level and asset allocation is determined, we construct the pre-consumption wealth grid on each trajectory by setting

$$w_T(i,j) = \hat{C}_T(i,j) + a(j). \tag{23}$$

We then calculate the value function for all points on the wealth grid and all trajectories by plugging the optimal strategies into (17). All conditional expectations in the optimization are calculated using Gauss-Hermite quadrature.

Time t: The time t value function is given by

$$\begin{aligned} V(W_t, X_t, t) = \max_{\substack{0 \le C_t \le W_t \\ 0 \le \theta_t \le 1}} \Big\{ & \tilde{m}_C(C_t, C_{t,R}) + p_t E_t \Big[\beta V(W_{t+1}, X_{t+1}, t+1) \Big] \\ & + (1 - p_t) E_t \Big[\beta \tilde{m}_B(B_{t+1}, B_{t+1,R}) \Big] \Big\}. \end{aligned} \tag{24}$$

Again we start by optimizing over the asset allocation for each point on the pre-consumption grid $a(j)$ and each trajectory $X_t(i)$

$$\hat{\theta}_t = \underset{\theta_t}{\operatorname{argmax}}\, E_t \Big[p_t \beta V(W_{t+1}, X_{t+1}, t+1) + (1 - p_t) \beta \tilde{m}_B(B_{t+1}, B_{t+1,R}) | a_t = a_t(j), X_t = X_t(i) \Big] \tag{25}$$

Again we compute the conditional expectations in (25) by using the Gauss-Hermite quadrature formula. To compute the time $t+1$ value function in the expression above, we employ a two step interpolation. For any sample return r_i from the quadrature method, we can calculate the resulting $t+1$ reference wealth level $W_{t+1,R}(r_i)$. We then employ nearest neighbour interpolation and find the trajectory i' whose time $t+1$ wealth level lies closest to $W_{t+1,R}(r_i)$. In the previous time step we calculated the values $V(w_{t+1}(i', j), X_{t+1}(i'), t+1)$ for all grid points $w_{t+1}(i', j)$, $j = 1, \ldots, M$ on trajectory i'. Using these points we employ cubic spline interpolation to find the value of V for wealth levels outside of the grid[11].

The first order condition for optimal consumption $\hat{C}_t$ is then given by

$$\begin{aligned} \frac{\partial}{\partial C_t} \tilde{m}_C(\hat{C}_t, C_{t,R}) = {} & p_t E_t \Big[\beta \frac{\partial}{\partial W_{t+1}} V(W_{t+1}, X_{t+1}, t+1) R_t^P | a_t = a_t(j), X_t = X_t(i) \Big] \\ & + (1 - p_t) E_t \Big[\beta \frac{\partial}{\partial B_{t+1}} \tilde{m}_B(B_{t+1}, B_{t+1,R}) R_t^P | a_t = a_t(j), X_t = X_t(i) \Big] \end{aligned} \tag{26}$$

[10]The first statement stems from the fact that $\lim_{c \to 0} m_C(0) = -\infty$. The second statement results from the diminishing marginal utility of consumption. There must always be some point c_0 at which increasing the size of funds invested in the risky asset by one unit results in a higher expected utility from bequest than increasing consumption by one unit. Therefore if the size of funds invested in the risky asset is bounded ($a_T(j) = a < \infty$) then the optimal consumption must also be bounded.

[11]It would be possible to perform an actual two dimensional interpolation to better approximate the value function. However we find that this is not necessary for sufficiently high N as switching to nearest neigbour interpolation does not change our results at the reported precision.

where R_t^P again denotes the return of the optimal portfolio. Taking the total derivative of (24) with respect to W_t yields

$$\frac{\partial}{\partial W_t}V(W_t,X_t,t) = p_t E_t\Big[\beta\frac{\partial}{\partial W_{t+1}}V(W_{t+1},X_{t+1},t+1)R_t^P|a_t = a_t(j), X_t = X_t(i)\Big]$$
$$+(1-p_t)E_t\Big[\beta\frac{\partial}{\partial B_{t+1}}\tilde{m}_B(B_{t+1},B_{t+1,R})R_t^P|a_t = a_{T-1}(j), X_t = X_t(i)\Big]. \tag{27}$$

From (26) and (27) we conclude that

$$\frac{\partial}{\partial C_t}\tilde{m}_C(\hat{C}_t, C_{t,R}) = \frac{\partial}{\partial W_t}V(W_t,X_t,t). \tag{28}$$

Therefore the first order condition (26) can be written in the form

$$\frac{\partial}{\partial C_t}\tilde{m}_C(\hat{C}_t, C_{t,R}) = p_t E_t\Big[\beta\frac{\partial}{\partial C_{t+1}}\tilde{m}_C(\hat{C}_{t+1},C_{t+1,R})R_t^P|a_t = a_t(j), X_t = X_t(i)\Big]$$
$$+(1-p_t)E_t\Big[\beta\frac{\partial}{\partial B_{t+1}}\tilde{m}_B(B_{t+1},B_{t+1,R})R_t^P|a_t = a_t(j), X_t = X_t(i)\Big] \tag{29}$$

This allows us to compute the expression on the right hand side of (29) using the known values for the optimal time $t+1$ consumption without having to resort to a numerical evaluation of the value function's partial derivative. Specificly, to calculate $\hat{C}_{t+1}$ for wealth levels outside of the grid, we again resort to a two-step interpolation procedure. For any sample return r in the quadrature formula, we can calculate the resulting $t+1$ wealth level in the current and the reference case. Interpolating from the optimal reference consumption grid then gives us the reference consumption $C'_{t+1,R}$ for the return r. It is not as easy to obtain the associated optimal $t+1$ consumption level, because it depends on the reference consumption level. Furthermore we only have grids of optimal consumption levels for a limited number of reference consumption levels. Therefore, in order to obtain the optimal $t+1$ consumption level by interpolation, we first have to construct a grid of wealth levels and associated optimal consumption values for each reference consumption value that we calculate in the first step. To this end, we find the trajectories i' and i'', containing the closest reference consumption values that are larger and smaller than $C'_{t+1,R}$. For these trajectories, we linearly interpolate the $t+1$ post-consumption wealth grids and the $+1$ optimal consumption values to obtain a grid of optimal consumption values for the particular reference consumption level $C'_{t+1,R}$. From this grid, we can then compute the optimal time $t+1$ consumption by using linear interpolation. While this is a slightly more involved calculation than for the optimal asset allocation, we note that these calculations only have to be performed once for each trajectory and post-consumption grid point. Once the conditional expectations are calculated we solve equation (29) to obtain $\hat{C}_t(i,j)$ in the same way as at time T.

In the last step we finally construct the pre-consumption wealth grid

$$w_t(i,j) = \hat{C}_t(i,j) + a(j) \tag{30}$$

and calculate the time t value function according to (24) using the optimal strategies and nearest neighbour interpolation of the time $t+1$ value function.

We repeat the above step until we reach time $t = 0$. This gives us the time $t = 0$ value function for the fixed annuitization level A. We then follow the principles outlined above to find the optimal annuitization level and the check wether this is higher than the reference case or not.

We conducted a variety of robustness tests for the above algorithm regarding the number of grid points and trajectories. Both numbers are similiar to the choices in Koijen et al. [16]. We find that increasing the number of grid points and simulated trajectories in the optimization procedure does not change the optimal annuity endowments at the reported precision.

7 Results

We find that all annuitization levels in our analysis, except no annuitization and full annuitization, are personal equilibriums or weak solutions of the retiree's optimization problem. That means that any retiree with a reference annuitization level between 10% and 90% of his initial wealth will not change and especially not increase his annuitization degree if he makes his decision according to the reference-dependent preference functional given by equation (7). This result is independent of how much wealth the retiree has preannuitized, it is only his reference annuity level that affects the result. However this means that when reference annuity levels are based on the preannuitized wealth, in other words the retiree does not intend to or did never consider to voluntarily annuitize, then they will not do so in our model.

As we discuss below, the two cases in which the retiree does deviate from his original reference plan include rather extreme strategies to beat the reference values. In the case of no annuitization as the reference point, the retiree switches to an annuitization degree of 40%. A retiree with full annuitization as the reference point, will decrease his annuitization degree to 90%. If we assume that people's preexisting annuitization degrees are based on mandatory annuitization and that these annuitization degrees form their reference values, then people will not opt for voluntary annuitization in our model. In this way our model can explain the low levels of observed voluntary annuitization rates.

The strong solution in our model, or the preferred personal equilibrium, is an annuitization degree of 80%. This is also the optimal annuitization degree when the retiree chooses his annuitization level according to standard expected utility theory. Naturally this must be the case in our model, because the preference functional from equation (7) reduces to a classical expected utility objective function when the strategy that is evaluated coincides with the reference level. As this is true for all weak solutions, the optimal solution in the classical problem must be the strong solution in our problem, if and only if it is a weak solution. The retiree's behaviour in the PPE consists of high consumption levels mixed with an aggressive equity investment strategy. His consumption is increasing on average and slightly below the FAI in the early periods and slightly above in the second half of the retirement. His wealth process, and therefore his process of bequest sizes, forms a hump shape. For a thorough analysis of the retiree's optimal behaviour in the PPE, we refer to Schneider [24].

In the remainder of this section we focus on the wealth and consumption trajectories in the cases when the retiree switches annuitization levels. i.e. in the cases with no annuitization and full annuitization. As we will see, this exposes some unrealistic properties of our model. Because adverse deviations in the consumption and bequest levels are punished more heavily than gains of similiar size, a main concern of the retirees in our model is to reduce the number of adverse deviations. The optimal strategy to achieve this is to accept a few deviations in the early periods by consuming less than the reference level and thus reach higher wealth levels than in the reference plan. Once wealth levels are sufficiently high, the retiree can enjoy higher consumption and bequest levels in all the remaining periods and therefore receive the additional positive utility from the gains in consumption and bequest utility. However this leads to a high intertemporal variation in consumption with average consumption ranging between 3500 in the first period and 60000 in the final period in the exemplary case of full annuitization as the reference level. In contrast to this the reference consumption plan in this case, is fairly stabble with consumption levels ranging between 20000 and 30000. A visualization of these average trajectories is contained in figure 1. The situation is similar under no annuitization as the reference point. With this in mind, the extreme cases in which the retiree actually switches annutization degrees may be regarded more as numerical oddities than realistic consumption and investment plans. This is especially true because they violate a general principle of the life-cycle hypothesis which underlies our model, that economic agents seek to smoothen their consumption across their whole lifetime. The opposite is the case in the two outlier cases no annuitization and full annuitization.

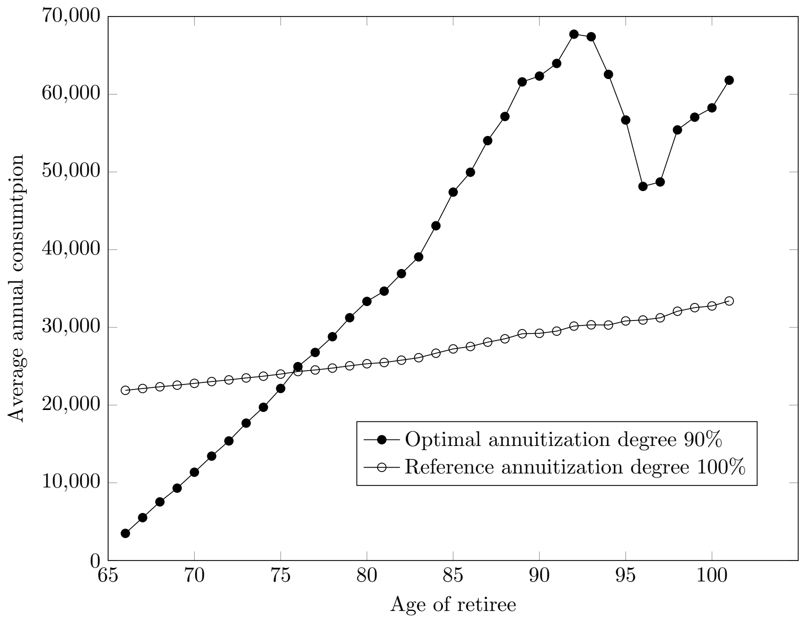

Figure 1: *Average annual consumption in the reference case with full annuitization and the optimal annuitization degree under that reference level. The values are calculated as the periodwise average values from $N = 1000$ trajectories obtained by forward simulation using the optimal strategy.*

8 Conclusion

We find that all annuitization degrees besides no and full annuitization are weak solutions to the retiree's optimization problem. This especially implies that retirees who had previously not planned to voluntarily annuitize parts of their wealth do not do so when offered the possibility, when they actually evaluate the outcome of the decision according to the preference functional in this paper. Therefore, reference-dependence may be a possible explanation for the almost non-existent voluntary annuitization rates. However we also find that reference-dependent preferences may also result in extreme behaviour when gain-loss utility is weighted as heavily as it is in this paper. Better results may be achieved by either reducing the weight of the effect of gain-loss utility in the preference functional or using a different form for the gain-loss utility function. For example the original prospect theory evaluation function proposed by Kahneman and Tversky [26] would lead to lower additional positive utility for large positive deviations from the reference values. This may prevent the extreme strategies we obtain in the cases with no annuitization and full annuitization from being optimal and may lead to more reasonable and realistic behavior.

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