Stokes Matrices of Isolated Hypersurface Singularities

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Sven Balnojan
aus Henstedt-Ulzburg

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Dekan: Dr. Bernd Lübcke, Universität Mannheim
Referent: Professor Dr. Claus Hertling, Universität Mannheim
Korreferent: Professor Dr. Duco van Straten, Universität Mainz

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Abstract

This thesis provides a classification of Stokes matrices of isolated hypersurface singularities (*ihs*), based on their spectrum in the case that the intersection form on the middle homology is positive (semi-) definite. We also provide results for the case of an indefinite intersection form, namely we show that certain matrices, so-called HOR matrices, carry a form of a polarized mixed Hodge structure which enables the definition of a spectrum for them.

The first part of thesis deals with the positive (semi-) definite case. To investigate the root lattices that arise from any upper triangular matrix with ones on the diagonal, we introduce the language of *p.n. root lattices*. Extending old results, we establish their classification and compute important numbers for (lattice, sublattice) pairs. We prove an interesting theorem which is that any root generating system of a root lattice as $\mathbb{Z}$-module also contains a $\mathbb{Z}$-basis. Using those results we are able to prove that an upper triangular matrix S, where $S + S^t$ is positive (semi-) definite, equipped with an ad-hoc spectrum in this case, can be classified as belonging to an *ihs* if the spectrum of their variance fulfills the Hertling variance inequality, their Coxeter Dynkin diagram is connected and a trace is correct.

The second part of the thesis deals with the indefinite case. We recall the classification of isometric triples and thereby of real Seifert pairs, based on works by Milnor and Némethi. We use those abstract results to provide a Thom-Sebastiani formula for $TEZP$ structures. Using the classification, we are able to prove that any so-called HOR matrix carries a Steenbrink polarized mixed Hodge structure. This in turn enables us to define a way of assigning a spectrum, only based on the HOR matrix and the monodromy eigenvalues, which agrees with the Steenbrink spectrum, in the case the matrix represents a distinguished basis of an *ihs*. This work is based the physicists Cecotti & Vafa and conjectures they made. Building upon this spectrum, we are able to prove some results for chain type singularities, which serve as ample evidence, that their conjecture is true, and a natural way of assigning spectral numbers should exist.

Zusammenfassung

Diese Arbeit klassifiziert Stokesmatrizen von isolierten Hyperflächensingularitäten (*ihs*) anhand ihres Spektrums, falls die Schnittform auf der mittleren Homologie positiv (semi-) definit ist. Wir bieten ebenfalls mehrere Resultate für den Fall einer indefiniten Schnittform. Insbesondere zeigen wir, dass die sogenannten HOR Matrizen eine Form von einer polarisierten gemischten Hodge Struktur tragen. Diese ermöglicht uns die Definition eines Spektrums für solche Matrizen.

Der erste Teil der Arbeit behandelt den positiv (semi-) definiten Fall. Jede obere Dreiecksmatrix mit Einsen auf der Diagonalen gibt uns ein Wurzelgitter. Um diese zu untersuchen definieren wir sogenannte *p.n. root lattices*. Aufbauend auf alten Resultaten erstellen wir deren Klassifizierung und berechnen wichtige Zahlen für Paare von (Gitter, Untergittern). Wir beweisen einen interessanten Satz, dass jedes Erzeugendensystem bestehend aus Wurzeln, eines Wurzelgitters als $\mathbb{Z}$-Modul auch eine $\mathbb{Z}$-Basis enthält. Mit diesem Resultat sind wir in der Lage, Folgendes zu beweisen: Eine ganzzahlige obere Dreiecksmatrix S, gehören zu einer *ihs* dann, wenn $S + S^t$ positiv (semi-) definit ist, ihre Spektralzahlen die Hertling Varianzungleichung erfüllen, ihr Coxeter Dynkin Diagram zusammenhängend ist, und ihre Spur korrekt ist.

Der zweite Teil der Arbeit behandelt den indefiniten Fall. Wir erinnern an die Klassifikation von *isometric triples* und erstellen darüber die Klassifikation von *real Seifert pairs*, basierend auf Arbeiten von Milnor und Némethi. Diese abstrakten Resultate nutzen wir, um eine Thom-Sebastiani Formel für $TEZP$-Strukturen anzugeben. Mit der Klassifikation können wir die sogenannten HOR Matrizen mit einer Steenbrink gemischten Hodge Struktur ausrüsten. Diese wiederum lässt uns ein Spektrum für HOR Matrizen definieren. Dieses Spektrum stimmt mit dem Steenbrink Spektrum überein, sollte eine Vermutung von Orlik & Randell über ausgezeichnete Basen von Kettentyp Singularitäten stimmen. Die Arbeit über das Spektrum basiert auf Ideen und Vermutungen der Physiker Cecotti & Vafa. Aufbauend auf diesem Spektrum sind wir in der Lage, Resultate für Kettentyp Singularitäten zu beweisen.

A special thank you to my amazing wife and my gorgeous children for all their support and patience.

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Contents

1 Introduction

The essence of this thesis is contained in two major conjectures. Steenbrink equipped isolated hypersurface singularities (short *ihs*) with a mixed Hodge structure and a spectrum. In 2002 Hertling formulated a surprising conjecture, that the so-called *variance* of the *spectrum* of an *ihs* is always less than or equal to its *spectral width* divided by 12. In 1993 the physicists Cecotti and Vafa wrote about the assignment of spectra to *matrices*. Specifically, they formulated an idea.

Conjecture. ***(CV) There should exist a very natural way to assign a* spectrum *to* any real upper triangular matrix S *with ones on the diagonal and such that* $S^{-1}S^t$ *has unitary eigenvalues. The spectrum should coincide with the Steenbrink spectrum (up to a shift), in case* S *belongs to a distinguished basis of an* ihs.**

We call those matrices *Stokes matrices* and denote their space by $T(n,\mathbb{R})$ when S is a $n \times n$ matrix. This thesis makes progress on this conjecture. The second conjecture we work on is inspired by the Hertling conjecture. We use the idea that the variance is an important datum for singularities. We formulate the following conjecture in the "opposite direction" of the Hertling conjecture.

Conjecture. ***(Classification) We can identify the* Stokes matrices $S \in T(n,\mathbb{R})$ *that "belong" to an* ihs *based on their second Bernoulli moment which is the variance of its spectrum (provided by the CV conjecture).***[1]

If the matrix induces a positive (semi-) definite intersection form, we can give a positive answer, yes, in the chapters 3-4. If the matrix induces an indefinite intersection form, we

[1] We mention "*Bernoulli moments*" for a specific reason; The proofs, in generic form, in this thesis work only with the variance, the 2nd Bernoulli moment. But in chapter 7, we will highlight why we believe, that the higher Bernoulli moments are part of the final picture.

can provide ample evidence for the CV conjecture in the form of multiple hard theorems, equipping the subclasses of HOR matrices with abstract forms of (Steenbrink) polarized mixed Hodge structures, in the chapters 5-6.

We will now provide a sketch of what we mentioned, just enough to highlight the structure of the thesis. The detailed description, notations, and all important definitions are set in chapter 2.

1.1 Overall structure of the thesis

Let us denote the spectrum by $\alpha_1 \leq ... \leq \alpha_n$. The Hertling conjecture then is:

Conjecture. *([He02, Conjecture 14.8]) The variance of an* ihs $f(x_0, ..., x_m)$ *with Milnor number* n *satisfies the inequality:*

$$\mathrm{Var}(Sp(f)) := \frac{1}{n}\sum_{j=1}^{n}\left(\alpha_j - \frac{m-1}{2}\right)^2 \leq \frac{\alpha_n - \alpha_1}{12}. \tag{1.1}$$

Here $\mathrm{Var}(Sp(f))$ can be considered as the variance of the spectrum. Hertling also conjectured a whole series of this type of inequalities generalizing the variance into *higher moments,* and this inequality into inequalities on so-called *Bernoulli moments*. The conjecture came with a proof of equality in the case of quasihomogeneous singularities. For brevity let us denote

- $w(f) := \alpha_n - \alpha_1$ the *spectral width* of an *ihs* f
- $w(S) := \alpha_n - \alpha_1$ the *spectral width* of an upper triangular matrix S. We disregard for a moment that this is not even defined. We also assume the tuple $\alpha_1, ..., \alpha_n$ to be ordered by size, which might not be the case as we see later.

The notation is taken from [St96]. There Steenbrink notices, that $w(f)$ is useful in determining whether an ICIS is simple, unimodal or of higher-modality.

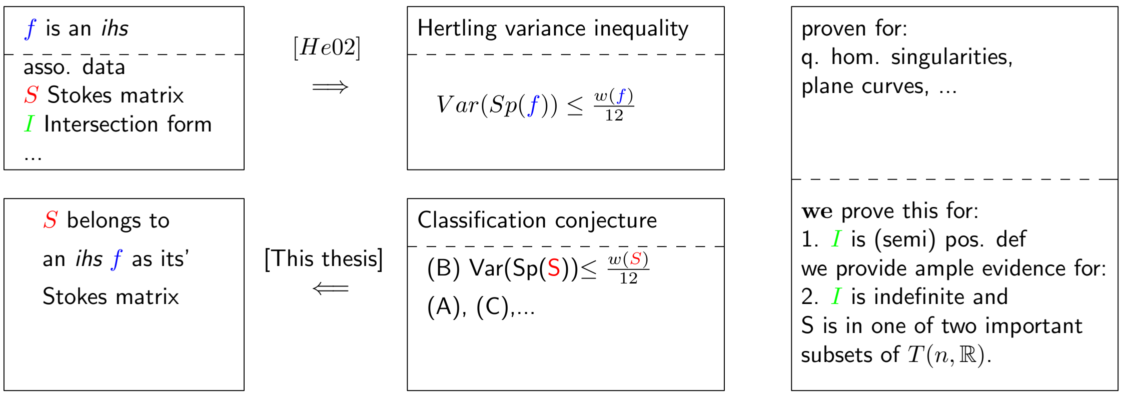
An *ihs* naturally comes equipped with quite a few geometric pieces of data, namely its middle homology groups, an intersection form I on those, the Stokes matrix codifying a distinguished basis' intersection numbers and so on. The *ihs* gets equipped by Steenbrink with a mixed Hodge structure, and thus a spectrum, and the conjecture from [He02] is able to characterize the variance of this spectrum, in the cases, where this conjecture is proven.

On the other hand, take any Stokes matrix $S \in T(n, \mathbb{R})$ with integer entries. Singularity theorists already know a few conditions S must fulfill to be the Stokes matrix belonging to a distinguished basis of an *ihs*.

Those conditions are for instance

- (A) S represents a Coxeter Dynkin diagram that is connected.
- (C) Set $M = S^{-1}S^t$. Then $\mathrm{tr}(M) = 1$.

But we also know those conditions cannot be enough, as there are operations taking a distinguished basis into a weakly distinguished basis while preserving those conditions. Our goal is to work on a condition (B), that includes the Bernoulli moments in a characterization. With those notations and thoughts, we may sketch the core part of this thesis:



The classification conjecture splits into cases.

For **Case 1**, $S + S^t$ is positive (semi-) definite, there is a simple "ad-hoc" way to assign spectral numbers. And in such a way, that if S is indeed the Stokes matrix of an *ihs*, then the spectral numbers are the same as the Steenbrink spectral numbers. The main difficulty, in this case, is to "quotient out" the possible radical of I, which leads us to the machinery of (p.n.) root lattices.

For **Case 2**, $S+S^t$ is indefinite and S is in $T_{\mathrm{HOR1}}(n, \mathbb{R}) \cup T_{\mathrm{HOR2}}(n, \mathbb{R})$, the two important subsets of $T(n, \mathbb{R})$, the problem of assigning spectral numbers, the CV conjecture, is quite hard. Here we need a very abstract treatment of the associated data, so-called *Seifert forms* and *Steenbrink polarized mixed Hodge structures*. Those, in turn, let us define a spectrum and thus provide a very precise idea for the Cecotti & Vafa conjecture.

So in both cases, we have quite a bit of preparation to do. We structure it in the following way.

1. **Chapter 3** (on *p.n. root lattices*) prepares **chapter 4** (proof of positive (semi-) definite case).

2. **Chapter 5** (on *Seifert forms*) prepares **chapter 6** (proof of important results for the indefinite case, related to *chain type singularities* and the CV conjecture).

The final **chapter 7** is separate. The Hertling variance inequality used in the classification of case 1 only includes the *second Bernoulli moment*. We, however, are convinced, that the essence of Stokes matrices of *ihs*, compared to Stokes matrices of *ICIS* for instance, or any unassociated matrix $S \in T(n, \mathbb{R})$ is captured in the full series of *Bernoulli moments*, not just the first one. With that in mind, chapter 7 serves two purposes.

1. To provide a bigger picture and a very rough idea of how a completely general, that means in particular including non quasihomogeneous singularities, classification could work.

2. To list all the missing pieces, i.e. all the unproven conjectures.

1.2 Detailed structure of the chapters

The thesis is structured into chapters 1-7, each of which contains sections and subsections. The numbering of all theorems, definitions, remarks and the likes is in the style *"chapter.consecutive-number"*. So the reference "theorem 2.18" refers to a theorem in chapter 2, which is the 18th numbered piece in that chapter. This is important as we refer back and forth between different chapters. We use the notation "Sp" for the ad-hoc spectrum in chapter 4, "Sp" for the spectrum in chapter 6 and "Spp" for the spectral pairs in chapter 6.

Some of the material was contained in the authors' master thesis at the University of Mannheim ([Ba15]). The material in chapter 3 contains a good part of the material in the arXiv preprint [BH16]. The material in chapter 5 contains a good part of the material in the arXiv preprint [BH17a] and the chapter 6 contains a good part of the material in the arXiv preprint [BH17b].

Chapter 2 contains two sections. The first recalls the important facts and definitions of singularity theory that are needed in this thesis. The second section summarizes the premise, main conjecture and main results of this thesis. We define the space $T(n, \mathbb{R})$ of all *Stokes matrices*, Coxeter Dynkin diagrams and state the classification conjecture.

Chapter 3 is a general treatment of so-called *p.n. root lattices*. The application to singularity theory is based on the first four sections and is mostly contained in sections 3.5 and 3.6, where we carry out a quotient construction and thereby get a *p.n. root lattice*. The first four sections recall definitions of *p.n. root lattices*, basic facts about their classification based on work by [Dy57] and extend it in a specific way. In particular take isomorphism pairs (lattice, subroot lattice), denote them by (L, L_1). We calculate L/L_1, as well as multiple minimal ways of reducing L to L_1 or extending L_1 to L. They are encapsulated in numbers $k_1, ..., k_5$. The goal in section 3.6 is to calculate the final number k_5 which is based on

$k_1, ..., k_4$. The number k_5 is essential in the next section as it controls the existence of quasi Coxeter elements. An interesting result we prove in slightly more generality than needed is the following. “Any generating set of roots of a root lattice as a $\mathbb{Z}$-module contains a $\mathbb{Z}$-basis of it.” Another new notion in this chapter is the notion of strict quasi Coxeter elements for inhomogeneous root lattices.

Chapter 4 contains the proof of conjecture 2.12, the positive (semi-) definite case. First, an ad-hoc definition of the spectrum for this case is given. Then calculating rules are deducted. With that in hand, the proof is carried out in two steps. In the positive definite case, a basic lattice bundle is constructed, the monodromy is identified as a quasi Coxeter element. Calculations exclude all but the Coxeter elements and using transitivity results from Deligne on the braid group orbit, we are able to finish the proof. In the case of a non-zero dimensional radical, a quotient construction is carried out, building on the material in chapter 3, especially section 3.6. Using a transitivity result from Kluitmann, the proof is finished.

Chapter 5 consists of four steps. Step 1 is the classification of Seifert forms, which is done via the classification of isometric triples. Step 2 is the connection of this data to the then defined Steenbrink PMHS. Step 3 connects this data to three equivalent pieces of data, namely sums of two isometric triples, Seifert form pairs, and holomorphic bundles on $\mathbb{C}^*$ with a flat holomorphic connection and a flat real subbundle and a certain flat pairing P between the fibers at $z \in \mathbb{C}^*$ and $-z$. In Step 4 we apply this machinery to the case of *ihs*. We provide a Thom-Sebastiani formula for *TEZP* structures.

Chapter 6 introduces the subspaces T_{HOR1} and T_{HOR2}. It establishes Steenbrink PMHS for them and equips them, based on the results of chapter 5, with a spectrum. It is shown, that in the case of chain type singularities, this spectrum coincides with the Steenbrink spectrum up to a shift. The whole spaces $T(n, \mathbb{R})$ are stratified into their eigenvalue and Seifert form strata for $n = 2, 3$. Facts and results for M-tame functions will be proven to relate to the conjectures made by Cecotti & Vafa.

Chapter 7 consists of two sections. The second section simply reviews all the unresolved conjectures related to, and made in this thesis. The first part explains the relevance of the higher Bernoulli moments in terms of classifying Stokes matrices. In particular, we prove a result up to Milnor number $\mu = 30$, which, providing a conjecture made by Orlik & Randell is true, classifies a subset of the space of all Stokes structures as belonging to chain type singularities. Here additional conditions on the higher Bernoulli moments are necessary. The Hertling variance inequality is not sufficient.

2 Classifying *ihs*

This chapter is divided into two sections. The first can be skipped by the reader with experience in singularity theory. We cover in it

- some essentials of singularity theory, especially isolated hypersurface singularities (*ihs*).
- The *lattices* that emerge on the middle homology.
- Essential facts about *root lattices* in general.

The second section contains important definitions, conjectures, and recipes which we work with in the remaining part of the thesis. Those include

- *Stokes matrices*, and the space of Stokes matrices $T(n, \mathbb{R})$.
- The notion of a *CDD* (*Coxeter Dynkin diagram*).
- The idea of a *spectral recipe*, a way of assigning spectral numbers to any CDD, based on [CV93].
- The *classification conjecture* and the more refined subspaces $T_{\text{HOR1}}(n, \mathbb{R}), T_{\text{HOR2}}(n, \mathbb{R}) \subset T(n, \mathbb{R})$, the so-called HOR spaces.

2.1 *ihs* and their lattices

First, we recall basics of *ihs*, in particular, their classification based on the modality and Milnor number, facts about the monodromy, the spectrum and the Hertling variance inequality. Then, in the second subsection, we introduce classic *root lattices* as they arise from *ihs*. In the next chapter, we will go beyond classical root lattices to *p.n. root lattices*, because they appear as quotients for *ihs*.

2.1.1 Singularity theory in a nutshell

References for most of the results and for the theory of isolated hypersurface singularities in general are [AGV85], [AGV88] and [Eb01]. With the following changes the notations and conventions here, including the definitions of all the pairings, and in [AGV88] and [Eb01] are compatible. Cf. also to the subsection 2.2.3 for the geometry of *basic lattice bundles* and *distinguished bases.* We use some of the notation from that subsection already here.

here	[AGV88]	[Eb01]
n	μ	μ
m	$n-1$	n

Let $f:(\mathbb{C}^{m+1},0)\to(\mathbb{C},0)$ be a holomorphic function germ with an isolated singularity at 0, that means 0 is an isolated zero of the Jacobi ideal $(\partial f/\partial x_0,...,\partial f/\partial x_m)$. Let

$$n := \dim \mathcal{O}_{\mathbb{C}^{m+1}}/(\partial f/\partial x_0,...\partial f/\partial x_m) < \infty$$

be the *Milnor number* of f. For the following notions and facts compare [AGV85], [AGV88] or [Eb01]. One can choose a *universal unfolding* of f, a *good representative* F of it with *base space* $M\subset\mathbb{C}^n$, and a generic parameter $t\in M$. Then $F_t:X\to\Delta$ is a holomorphic map with $X\subset\mathbb{C}^{n+1}$ open, $\Delta\subset\mathbb{C}$ a disk, such that the one singularity of f splits into n A_1-singularities of F_t with pairwise different critical values $u_1,...,u_n$ (the numbering of the critical values is chosen once and for all) and such that the situation in 2.14 (a) is satisfied. Define $U:=\{u_1,...,u_n\}$ and $X':=X\setminus F_t^{-1}(U)$. Then $F_t:X'\to\Delta\setminus U$ is a locally trivial C^∞ fiber bundle, the fibers are called *Milnor fibers*, they are homotopy equivalent to bouquets of n m-dimensional spheres. Define

$$H_{\mathbb{Z}} := \bigcup_{z\in\Delta\setminus U} H_m(F_t^{-1}(z),\mathbb{Z}).$$

The homology groups of the fibers are free $\mathbb{Z}$-modules of rank n.

In addition, one can choose a good representative of f as $f:X\to\Delta$ with $X=\{x\in\mathbb{C}^{m+1}\ |x|<\varepsilon\}\cap f^{-1}(\Delta)$ for a sufficiently small $\varepsilon>0$ and $\Delta=\{\tau\in\mathbb{C}\,|\,|\tau|<\delta\}$ a small disk around 0 (first choose ε, then δ). Then $f:X'\to\Delta'$ with $X'=X-f^{-1}(0)$ and $\Delta'=\Delta-\{0\}$ is a locally trivial C^∞-fibration, the Milnor fibration.

Arnold started a classification of the zoo of all isolated hypersurface singularities, using the *modality* of a singularity [AGV85]. The modality is defined slightly different, but turns out to be the dimension of the μ-constant stratum within the base space M of a universal unfolding of a singularity [Ga74]. The *ihs* with modality 0 are the *simple singularities*

$$A_n \ (n \geq 1, m \geq 0), \quad D_n \ (n \geq 4, m \geq 1), \quad E_6, E_7, E_8 \ (m \geq 1),$$

the lower index is the Milnor number. The *ihs* with modality 1 split into three quite different types:

(α) The *simple elliptic singularities.* There are three 1-parameter families of singularities denoted by $\widetilde{E}_6 \ (m \geq 2), \widetilde{E}_7 \ (m \geq 1), \widetilde{E}_8 \ (m \geq 1)$.

(β) The *hyperbolic singularities.* For each triple $(p, q, r) \in \mathbb{N}$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ there is a 1-parameter family denoted by T_{pqr} ($m \geq 1$ if $2 \in \{p, q, r\}$, $m \geq 2$ if $2 \notin \{p, q, r\}$).

(γ) The *exceptional unimodal singularities.* There are 14 1-parameter families, 6 with $m \geq 1$ and 8 with $m \geq 2$.

The following theorem collects the results which hold for all *ihs* and establish the geometric data of a basic lattice bundle (defined and studied in detail below in subsection 2.2.3) in the underlying case of a *ihs*.

Theorem 2.1. *[Eb01] (a) There exists an up to the signs of the elements unique tuple $\underline{\delta}^0 = (\delta_1^0, ..., \delta_n^0)$ and a unique flat pairing I on $H_\mathbb{Z}$ such that the tuple $(H_\mathbb{Z} \to \Delta \setminus U, \underline{\delta}^0, m, I)$ is a basic lattice bundle with pairing.*

(b) ([Eb01, Theorem 5.9], [Eb87]) Suppose that m is even and that the ihs *f is not one of the hyperbolic singularities of type T_{pqr} with $(p, q, r) \notin \{(3, 3, 4), (2, 4, 5), (2, 3, 7)\}$. Then*

$$\Lambda_{van} = \{\delta \in H_{\mathbb{Z},r} \,|\, I(\delta, \delta) = 2 \text{ and } I(\delta, H_{\mathbb{Z},r}) = \mathbb{Z}\}. \tag{2.1}$$

(c) ([AGV88, Eb01]) The monodromy M is quasiunipotent, i.e. the eigenvalues are unit roots.

(d) ([AC75]) The trace of the monodromy M is $trM = (-1)^{m+1}$. ([Eb01, Korollar 5.8]) And the CDD of any weakly distinguished basis (definition 2.17 (b)) is connected.

Remark 2.2. *ihs* in one μ-homotopy class have the same basic lattice bundle with pairing (up to homotopy of the basis $\Delta \setminus U$).

Theorem 2.3. *([AGV85], Arnold) Suppose that $m \equiv 0 \mod 4$. Then the only* ihs *where I on $H_{\mathbb{Z},r}$ is positive definite, are the simple singularities. The only* ihs *where I on $H_{\mathbb{Z},r}$ is positive semidefinite, but not positive definite, are the simple elliptic singularities. For all other* ihs*, I on $H_{\mathbb{Z},r}$ is indefinite.*

Steenbrink ([AGV88], or [He02]) associated to any *ihs* a *mixed Hodge structure* and a *spectrum* $Sp(f) = (\alpha_1, ..., \alpha_n)$ of numbers $\alpha_1, ..., \alpha_n \in \mathbb{Q} \cap (-1, m)$ with $\alpha_1 \leq \alpha_2 \leq ... \leq \alpha_n$, with the symmetry

$$\alpha_j + \alpha_{n+1-j} = m - 1, \tag{2.2}$$

and such that $e^{-2\pi i \alpha_1}, ..., e^{-2\pi i \alpha_n}$ are the eigenvalues of the monodromy. The choice of the numbers $\alpha_1, ..., \alpha_n$ within all numbers β such that $e^{-2\pi i \beta}$ is an eigenvalue of the monodromy comes from the mixed Hodge structure.

Conjecture 2.4. *([He02, Conjecture 14.8]) Hertling formulated the following conjecture. If f is an* ihs, *then it's spectral numbers satisfy:*

$$\mathrm{Var}(Sp(f)) := \frac{1}{n} \sum_{j=1}^{n} \left(\alpha_j - \frac{m-1}{2} \right)^2 \leq \frac{\alpha_n - \alpha_1}{12}. \tag{2.3}$$

In view of (2.2), the left-hand side is the *variance* of the spectral numbers.

Theorem 2.5. *The conjecture was proved for all quasihomogeneous* ihs *([He02, Theorem 14.9], and a different proof by Dimca) and for all curve singularities (first by M. Saito for all irreducible curve singularities, later by Brélivet for all curve singularities, see the references in [BH04]).*

2.1.2 *ihs* lattices

This subsection recalls basics of root lattices and describes the gestalt of the Milnor lattices for the relevant *ihs*. For basic facts on root systems, see [Bo68].

Definition 2.6. *(a) A free $\mathbb{Z}$-module L of rank $n \in \mathbb{Z}_{>0}$ is called a* ***lattice***. *Then $L_\mathbb{Q} := L \otimes_\mathbb{Z} \mathbb{Q}$, $L_\mathbb{R} := L \otimes_\mathbb{Z} \mathbb{R}$ and $L_\mathbb{C} := L \otimes_\mathbb{Z} \mathbb{C}$. Let L be a lattice and $(.,.)$ a scalar product on $L_\mathbb{R}$. For $\alpha \in L - \{0\}$ and $b \in L$ define*

$$\langle \beta, \alpha \rangle := \frac{2(\beta, \alpha)}{(\alpha, \alpha)}.$$

Then $s_\alpha : L_\mathbb{R} \to L_\mathbb{R}$, $s_\alpha(x) = x - \langle x, \alpha \rangle \cdot \alpha$ is a reflection.

(b) A ***root lattice*** *is a triple $(L, (.,.), \Phi)$ where L is a lattice, $(.,.) : L_\mathbb{R} \times L_\mathbb{R} \to \mathbb{R}$ is a scalar product, and $\Phi \subset L - \{0\}$ is a finite set such that the following properties hold.*

Φ is a generating set of L as a $\mathbb{Z}$-module.

For any $\alpha \in \Phi : s_\alpha(\Phi) = \Phi$.

$\langle\beta,\alpha\rangle \in \mathbb{Z}$ *for any* $\alpha,\beta\in\Phi$.

For all $\alpha\in\Phi$ *we have* $\Phi\cap\mathbb{R}a=\{\pm\alpha\}$.

The elements of Φ *are the* ***roots****, and* Φ *is a* ***root system****. The finite group*

$$W := \langle s_\alpha \mid \alpha\in\Phi\rangle \subset O(L,(.,.))$$

is the ***Weyl group****.*

(c) Any root lattice is either irreducible (not isomorphic to a sum of several lattices) or isomorphic to an orthogonal sum of several irreducible root lattices. The irreducible ones are classified by:

$$A_n(n\geq 1), B_n(n\geq 2), C_n(n\geq 3),$$

$$D_n(n\geq 4), E_6, E_7, E_8, F_4, G_2.$$

(d) For any element $w\in W$*, and* $(\alpha_1,...,\alpha_k)\in\Phi^k$

$$w = s_{\alpha_1}\circ ...\circ s_{\alpha_k}$$

is a ***presentation of*** w***.*** *Its length is* k*. The* ***length*** $l(w)$ *of* w *is the minimum of all presentations. A presentation with* $k=l(w)$ *is called reduced. An element* w *is called of maximal length if* $l(w)=n:=$ *the rank of the root lattice.*

(e) Define for $w\in W,\lambda\in S^1$*:*

$V_\lambda(w)=\ker(w-\lambda\cdot id)\subset L_{\mathbb{C}}$.

$V_{\neq 1}(w)=\bigoplus_{\lambda\neq 1}V_\lambda(w)\supset V_{\neq 1.\mathbb{R}}(w):=L_{\mathbb{R}}\cap V_{\neq 1}(w)$.

And define $V_{\neq 1,\mathbb{Q}}(w)$ *in the same way. Then the subroot lattice* $L_1:=\sum_{i=1}^k\mathbb{Z}\cdot\alpha_i$ *for a reduced presentation* $(\alpha_1,...,a_k)$ *satisfies*

$$\bigoplus_{i=1}^{l(w)}\mathbb{Q}\cdot\alpha_i = L_{1,\mathbb{Q}} = V_{\neq 1,\mathbb{Q}}(w)$$

and especially $l(w)=dimV_{\neq 1,\mathbb{Q}}(w)$.

(f) We call $w\in W$ *a* ***Coxeter element*** *(in an irreducible root lattice) if it has a presentation* $(\alpha_1,...,\alpha_n)$ *such that* $\alpha_1,...,\alpha_n$ *form a root basis. We call* $w\in W$ *a* ***quasi Coxeter element*** *(in a root lattice) if a reduced presentation of* w *exists whose subroot lattice* L_1 *is the full lattice* L*. Of course then it's of maximal length* $l(w)=n$*.*

Theorem 2.7. *([AGV85, AGV88]) (a) Consider a simple singularity A_n, D_n, E_6, E_7 or E_8 with $m \equiv 0 \mod 4$. Then $(H_{\mathbb{Z},r}, I)$ is an irreducible homogeneous root lattice of the same type as the singularity. The monodromy M is a Coxeter element. The vanishing cycles are the roots,*

$$\Lambda_{van} = \{\delta \in H_{\mathbb{Z},r} \,|\, I(\delta,\delta) = 2\}. \tag{2.4}$$

(b) Consider a simple elliptic singularity $\widetilde{E}_6$, $\widetilde{E}_7$ or $\widetilde{E}_8$ with $m \equiv 0 \mod 4$ Then $(H_{\mathbb{Z},r}, I)$ has a radical $\mathrm{Rad}(I) \subset H_{\mathbb{Z},r}$ of rank 2. The quotient lattice $(H_{\mathbb{Z},r}/\mathrm{Rad}(I), \overline{I})$ with the induced pairing $\overline{I}$ is an irreducible homogeneous root lattice of the type E_6, E_7 respectively E_8.

The radical $\mathrm{Rad}(I)$ is also the eigenspace with eigenvalue 1 of the monodromy M.

The monodromy M induces an automorphism $\overline{M}$ of the quotient lattice $(H_{\mathbb{Z},r}/\mathrm{Rad}(I), \overline{I})$, which does not have the eigenvalue 1. Then $\overline{M}$ has a reduced presentation $\overline{M} = s_{\delta_1} \circ ... \circ s_{\delta_n}$ of maximal length $n = 6, 7$ respectively 8, such that the sublattice $L(\{\delta_1, ..., \delta_n\}) \subset H_{\mathbb{Z},r}/\mathrm{Rad}(I)$ is a sublattice of the following type.

singularity type	$\widetilde{E}_6$	$\widetilde{E}_7$	$\widetilde{E}_8$
type of the lattice $H_{\mathbb{Z},r}/\mathrm{Rad}(I)$	E_6	E_7	E_8
type of the sublattice $L(\{\delta_1, ..., \delta_n\})$	$3A_2$	$A_1 \perp 2A_3$	$A_1 \perp A_2 \perp A_5$

2.2 Stokes matrices and their spectra

The goal of this thesis is to characterize the Stokes matrices of *ihs*, within all upper triangular matrices with ones one the diagonal, in terms of the variance of their spectral numbers. It's inspired by a conjecture made by the physicists Cecotti & Vafa. We define the object of study, the space of Stokes matrices $T(n, \mathbb{R})$ below, introduce the conjecture made by Cecotti and Vafa and comment on it.

In the second subsection, we discuss the geometry of upper triangular matrices in a general context, mainly based on [Eb01]. This is preparation for chapters 3-4.

In the third subsection, we discuss a more refined subspace of Stokes matrices with much nicer properties. Those matrices give rise to Steenbrink polarized mixed Hodge structures, to be discussed in chapters 5-6.

In the fourth and last subsection, we concretize the phrase "spectral recipe" as outlined by Cecotti and Vafa by introducing *Seifert form pairs* and *Seifert form strata*. Building on that we make precise conjectures which we prove (in chapter 6) in the cases $n = 2, 3$ for spectra of Stokes matrices.

2.2.1 $T(n,\mathbb{R})$ and the Cecotti Vafa idea

We define the space

$$T(n,\mathbb{R}) \;:=\; \{S = (s_{ij}) \in M(n\times n,\mathbb{R}) \,|\, s_{ij} = 0 \text{ for } i > j, \tag{2.5}$$
$$s_{ii} = 1, S^{-1}S^t \text{ has eigenvalues in } S^1\} \tag{2.6}$$

Definition 2.8. *A matrix $S \in T(n,\mathbb{R})$ is called a **Stokes matrix**.*

Cecotti and Vafa proposed in [CV93] a beautiful idea how to associate to upper triangular matrices in $T(n,\mathbb{R})$ n *spectral numbers* $\alpha_1,...,\alpha_n \in \mathbb{R}$ such that $e^{-2\pi i\alpha_1},...,e^{-2\pi i\alpha_n}$ are the eigenvalues of $S^{-1}S^t$. Furthermore, they claim to have an almost rigorous proof that the recipe works and that in the case of Landau-Ginzburg models the spectral numbers of its Stokes matrices coincide with the true spectral numbers. We consider the recipe as incomplete and see serious gaps in it and in the arguments that in the case of Landau-Ginzburg models the spectral numbers coincide. We discuss this below. Still, we find the idea fascinating.

This chapter is the result of our efforts to make the recipe work. We succeeded only partially. We have certain subspaces of $T(n,\mathbb{R})$ where the recipe works and which are hopefully big enough to be useful for an extension of the recipe to all of $T(n,\mathbb{R})$. Below we formulate precise conjectures and results. The recipe is as follows.

Recipe 2.9. Start with some matrix $S_1 \in T(n,\mathbb{R})$; we can embed $T(n,\mathbb{R})$ in $\mathbb{R}^{\frac{n(n-1)}{2}}$. Choose a path from the unit matrix E_n to S_1 within $T(n,\mathbb{R})$, i.e. a continuous map $S : [0,1] \to T(n,\mathbb{R})$ with $S(0) = E_n$ and $S(1) = S_1$. Now choose *in a natural way* n continuous functions $\alpha_j : [0,1] \to \mathbb{R}$, $j \in \{1,...,n\}$, such that $\alpha_j(0) = 0$ and $e^{-2\pi i\alpha_1(r)},...,e^{-2\pi i\alpha_n(r)}$ are the eigenvalues of $S(r)^{-1}S(r)^t$. Then $\alpha_1(1),...,\alpha_n(1)$ are defined to be the spectral numbers of S_1.

Conjecture 2.10. *(Proposed by Cecotti-Vafa in [CV93]) For any matrix $S \in T(n,\mathbb{R})$, there is a natural procedure (in the sense of recipe (2.9)) which leads to a well-defined spectrum $Sp(S) = (\alpha_1,...,\alpha_n) \in \mathbb{R}^n$ with $\alpha_1 \le ... \le \alpha_n$. In the case of the Stokes matrix of a distinguished basis of an isolated hypersurface singularity $f(x_0,...,x_m)$ it coincides with the shift $Sp(f) - \frac{m-1}{2}$ of Steenbrink's spectrum $Sp(f)$.*

Remarks 2.11. (i) The recipe assumes that $T(n,\mathbb{R})$ is connected. Cecotti and Vafa conjecture this [CV93, first half of page 590], but have no proof for it. Our conjecture 2.21 (a) below will imply this, but we also have no proof for it. But even if $T(n,\mathbb{R})$ is connected, the spectral numbers might depend on the chosen path.

(ii) Even if a path is given, it might happen that for some $r \in (0,1)$ several eigenvalues of $S(r)^{-1}S(r)^t$ coincide. Then at this parameter r, one can exchange the continuations at r of the functions α_j for these eigenvalues. Then, in general, it is unclear whether and how to make a most natural choice and how to make the phrase *in a natural way* in the recipe 2.9 precise. This holds especially if $\alpha_i(r) - \alpha_j(r) \in 2\mathbb{Z} - \{0\}$.

(iii) Cecotti and Vafa proposed in [CV93, footnote 6 on page 583] to choose the path such that for $r \in (0,1)$ all eigenvalues of $S(r)^{-1}S(r)^t$ are different. This is within $T(n,\mathbb{R})$ for most matrices not possible because the eigenvalue -1 has for all matrices in $T(n,\mathbb{R})$ even multiplicity because $\det(S(r)^{-1}S(r)^t) = 1$.

(iv) Only on the pages 589+590 in [CV93], it is demanded that the path is within $T(n,\mathbb{R})$, not yet on page 583. But if one chooses a path which leaves $T(n,\mathbb{R})$ there are two problems. The resulting spectral numbers might depend on the path. And the arguments with tt^*-geometry for the coincidence of the Stokes matrix spectral numbers with the true spectral numbers of a Landau-Ginzburg model will not work [CV93, first half of page 590]. Because of both problems we restrict to the recipe with paths within $T(n,\mathbb{R})$.

2.2.2 The classification conjecture

Conjecture 2.12. *A matrix $S \in T(n,\mathbb{R}) \cap M(n\times n,\mathbb{Z})$ is the Stokes matrix of a distinguished basis of an isolated hypersurface singularity if and only if it satisfies the following three conditions.*

(A) The CDD of S is connected.

(B) Conjecture 2.10 is true, and the variance of $Sp(S)$ satisfies the inequality (2.7),

$$\mathrm{Var}(Sp(S)) := \frac{1}{n}\sum_{j=1}^{n} \alpha_j^2 \leq \frac{\alpha_n - \alpha_1}{12}. \tag{2.7}$$

(C) The trace satisfies $tr(S^{-1}S^t) = 1$.

The positive (semi-) definite case. If $S+S^t$ is positive semidefinite, there is a straight forward way to define the spectrum (see definition/lemma 4.1 in chapter 4). With that in hand, we will prove the following conjecture in those cases.

Theorem 2.13. *(a) If $S + S^t$ is positive definite, then conjecture 2.12 is true and the associated* ihs *is a simple singularity.*

(b) If $S + S^t$ is positive semidefinite (degenerate) then conjecture 2.12 is true and the associated ihs *is a simple-elliptic singularity.*

The proofs in the subsections 4.4 and 4.5 make use of the well-known identification of the Milnor lattices of the simple singularities with the irreducible homogeneous root lattices, the ADE root lattices, and of the known relation to them in the case of the simple elliptic singularities. The proofs use Carters' classification of the Weyl group conjugacy classes and Deligne's and Kluitmanns' characterization of the set $\mathcal{B}$ of the distinguished bases. For the semidefinite case we need the results in chapter 3 which study nonreduced presentations of Weyl group elements $w = s_{\delta_1} \circ ... s_{\delta_n+2k}$ (with $k > 0$) of the rank n types D_n, E_7 and E_8 such that $\sum_{j=1}^{n+2k} \mathbb{Z} \cdot \delta_j$ is the full root lattice of rank n.

We use the fact that the number

$$k_5(L, \bar{M}) := min\{k \mid \text{a presentation } (\alpha_1, ..., \alpha_{l(\bar{M})+2k}) \text{ of } \bar{M} \text{ with subroot lattice} \\ \text{the full lattice exists}\}$$

defined in definition 3.28, controls the existence of quasi Coxeter elements.

The indefinite case. To follow after the next section. The proof is fundamentally different, and the spectral recipe more involved.

2.2.3 Geometry of Stokes matrices

Stokes matrices induce a rich geometry. On the one hand, they generalize the classical objects of root systems and lattices. On the other hand, they turn up as Stokes matrices in the theory of Stokes structures of irregular poles at 0 of meromorphic differential equations on a disc in $\mathbb{C}$.

The context of *ihs* relates those two concepts via Fourier-Laplace transform. One *ihs* leads to a (homotopy class of a) basic lattice bundle with pairing, to a set $\mathcal{B}$ of *distinguished bases*, and to a set of upper triangular matrices $S \in M(n \times n, \mathbb{Z})$ with 1's on the diagonal. $\mathcal{B}$ and the set of matrices S are orbits under $\mathrm{Br}_n \ltimes \{\pm 1\}^n$. The matrices S can also be encoded in Coxeter-Dynkin diagrams (CDD).

The distinguished bases have been studied a lot by A'Campo, Ebeling, Gabrielov, Gusein-Zade and some (other besides Ebeling) doctoral students of Brieskorn, especially Kluitmann and Voigt. It is an own art to calculate CDD's for special classes of singularities. For the singularities in Arnold's lists and quite many other singularities, especially curve singularities, one has nice CDD's from which one can derive precise information on the Milnor lattice, its intersection form, its monodromy and its monodromy group. But a general idea which upper triangular matrices S turn up in the case of singularities, is missing.

We first define the *distinguished paths* and then build *basic (lattice) bundles* upon them. Basic bundles are geometric objects build on abstract Stokes matrices.

Definition 2.14. *(a) The following data will be fixed: an $n \in \mathbb{Z}_{>0}$, an $r \in \mathbb{R}_{>0}$, a small (compared to r) $\varepsilon \in \mathbb{R}_{>0}$, the disk $\Delta := \{z \in \mathbb{C} \,|\, |z| < r + \varepsilon\}$, and n disjoint points $u_1, ..., u_n \in \Delta$ such that the closures $\overline{\Delta_i}$ of the disks $\Delta_i := \{z \in \mathbb{C} \,|\, |z - u_i| < \varepsilon\}$ are in Δ and do not intersect. We denote $U := \{u_1, ..., u_n\}$ and $\Delta_U := \bigcup_{i=1}^n \Delta_i$.*

(b) For any path (= a continuous map) $\gamma : [0,1] \to \Delta \setminus U$ with $\gamma(0) \in \partial\overline{\Delta_i}$ for some i and with $\gamma(1) = r$, a loop (= a closed path) $\omega : [0,1] : \Delta - U$ is defined and associated to γ as follows. Let $\tau_i : [0,1] \to \Delta \setminus U$ be a path with image in $\partial\overline{\Delta_i}$ and with $\tau_i(0) = \tau_i(1) = \gamma(0)$, which turns around u_i once mathematically positive. Then ω is (up to some reparametrization) the product $\gamma^{-1}\tau\gamma$ (first γ^{-1}, then τ, then γ).

*(c) A **distinguished system of paths** is a family $(\gamma_1, ..., \gamma_n)$ of n paths $\gamma_i : [0,1] \to \Delta \setminus \Delta_U$ with the following properties.*

(i) $\gamma_i(1) = r$. There is a $\sigma \in S_n$ such that $\gamma_i(0) \in \overline{\Delta_{\sigma(i)}}$.

(ii) The paths do not intersect themselves. And they intersect one another only at the point r.

(iii) The paths are numbered such that they arrive in clockwise order at r.

*(d) In the basic situation, a **weakly distinguished system of paths** $(\gamma_1, ..., \gamma_n)$ is a system of paths $\gamma_i : [0,1] \to \Delta \setminus U$ with (i), (ii) in (c) and that additionally generate $\pi(\Delta - U, r)$.*

With this situation in mind, basic (lattice) bundles are the following objects:

Definition 2.15. *(a) A **basic bundle** is a rank n flat complex vector bundle $H \to \Delta \setminus U$ with one element $\delta_i^0 \in H_{u_i+\varepsilon} - \{0\}$ for each $i \in \{1, ..., n\}$ such that the following two properties hold:*

(i) The monodromy $h_i : H_{u_i+\varepsilon} \to H_{u_i+\varepsilon}$ by parallel transport mathematically positive once around u_i on $\partial\overline{\Delta_i}$ satisfies $\mathrm{Im}(h_i - \mathrm{id}) \subset \mathbb{C} \cdot \delta_i^0$

(ii) There exists a weakly distinguished system of paths $(\gamma_1, ..., \gamma_n)$ with $\gamma_i(0) = u_i + \varepsilon$ such that the n elements in H_r obtained for $i = 1, ..., n$ by parallel transport of δ_i^0 along γ_i from $H_{u_i+\varepsilon}$ to H_r form a vector space basis of H_r.

(b) Let $\gamma : [0,1] \to \Delta \setminus U$ be any path. For $\delta \in H_{\gamma(0)}$ denote by $h_\gamma(\delta) \in H_{\gamma(1)}$ the element obtained from δ by parallel shift along γ.

(c) The monodromy group $\Gamma \subset \mathrm{Aut}(H_r)$ of the basic bundle is the group

$$\Gamma := \langle h_{\omega_1}, ..., h_{\omega_n} \rangle$$

where $(\omega_1, ..., \omega_n)$ *is the system of loops associated to a weakly distinguished system of paths* $(\gamma_1, ..., \gamma_n)$.

(d) ([Eb01, 5.4, Satz 5.6]) For any loop ω *in* $\Delta \setminus U$ *with base point* r *the monodromy is* $h_\omega : H_r \to H_r$. *It depends only on the homotopy class in* $\pi_1(\Delta \setminus U, r)$ *of the loop* ω. *One obtains an (anti)homomorphism* $h : \pi_1(\Delta\setminus, r) \to \mathrm{Aut}(H_r)$. *Then for any weakly distinguished system of paths* $(\gamma_1, ..., \gamma_n)$ *with* $\gamma_i(0) = u_i+\varepsilon$, *the* n *elements* $(h_{\gamma_1}(\delta_1^0), ..., h_n(\delta_n^0))$ *in* H_r *form a basis of* H_r. *This basis and any basis* $(\psi_1 h_{\gamma_1}(\delta_1^0), ..., \psi_n h_{\gamma_n}(\delta_n^0))$ *with* $\psi_1, ..., \psi_n \in \{\pm 1\}$ *is called a* ***weakly distinguished basis****. In the case of a distinguished system of paths, these bases are called* ***distinguished bases****.*

(e) A ***basic lattice bundle*** *is a local system* $H_\mathbb{Z} \to \Delta \setminus U$ *of free rank* n *lattices* $H_{\mathbb{Z},t}$, $t \in \Delta \setminus U$, *such that the complexification* $H = H_\mathbb{Z} \otimes \mathbb{C}$ *is a basic bundle and such that for some weakly distinguished system of paths* $(\gamma_1, ..., \gamma_n)$ *with* $\gamma_i(0) = u_i+\varepsilon$ *the basis* $\underline{\delta}$ *with* $\delta_i = h_{\gamma_i}(\delta_i^0)$ *of* H_r *is a* $\mathbb{Z}$*-basis of* $H_{\mathbb{Z},r}$.

In the situation of (a), a ***vanishing cycle*** *is any element* $\delta \in H_{\mathbb{Z},r}$ *such that there exist an* i, *a sign* $\psi \in \{\pm 1\}$ *and a path* $\gamma : [0, 1] \to \Delta \setminus U$ *with* $\gamma(0) = u_i + \varepsilon_i, \gamma(1) = r$ *and* $\delta = \psi \cdot h_\gamma(\delta_i^0)$. *Let* Λ_{van} *be the set of all vanishing cycles.*

Theorem 2.16. *Consider the situation in definition 2.14 (a). Fix a distinguished system of paths* $\underline{\gamma} = (\gamma_1, ..., \gamma_n)$ *with* $\gamma_i(0) = u_{\sigma(i)} + \varepsilon$ *for some* $\sigma \in S_n$. *Fix a number* $m \in \mathbb{Z}$ *(here only* $m \mod 2$ *will be relevant). There is a natural 1:1-correspondence between the following two pieces of data.*

(i) An upper triangular matrix $S = (s_{ij}) \in M(n \times n, \mathbb{C})$ *with 1's on the diagonal, i.e.* $s_{ii} = 1$ *and* $s_{ij} = 0$ *for* $i > j$.

(ii) A basic bundle with pairing $(H \to \Delta \setminus U, \underline{\delta}^0, m, I)$.

One passes from (ii) to (i) as follows. Let $\underline{\delta}$ *be the distinguished basis of* H_r *defined by* $\delta_i := h_{\gamma_i}(\delta^0_{\sigma(i)})$. *Then setting*

$$s_{ij} := I(\delta_i, \delta_j) \quad \text{for} \quad i < j \tag{2.8}$$

defines a unique matrix S *as in (i).*

Below we restrict to matrices S with entries in $\mathbb{Z}$. In that case, the correspondence restricts in (ii) to a *basic lattice bundle*.

Definition 2.17. *(a) For any upper triangular matrix* $S = (s_{ij}) \in M(n \times n, \mathbb{R})$, *with 1's on the diagonal (so* $s_{ii} = 1$, $s_{ij} = 0$ *for* $i > j$*), a* ***Coxeter Dynkin diagram (CDD)*** *is* *defined as follows. It has n vertices which are numbered by $1, ..., n$. The vertices i and j for $i < j$ are connected by an edge if $s_{ij} \neq 0$, and then the edge is equipped with the weight s_{ij}. In the case $s_{ij} = 0$, no edge is drawn.*

(b) If $S \in M(n \times n, \mathbb{Z})$ then an edge with a weight $s_{ij} > 0$ is replaced by s_{ij} dotted edges, and an edge with a weight $s_{ij} < 0$ is replaced by $|s_{ij}|$ edges.

2.2.4 $T_{\mathbf{HOR1}}(n,\mathbb{R})$, $T_{\mathbf{HOR2}}(n,\mathbb{R})$ and a refined spectrum

We have two subfamilies $T_{\mathrm{HOR1}}(n,\mathbb{R})$ and $T_{\mathrm{HOR2}}(n,\mathbb{R}) \subset T(n,\mathbb{R})$ for which the recipe 2.9 works. The families will be presented in section 6.3, but here we give their crucial properties and explain how and why the recipe works for them.

Theorem 2.18. *(a) The subspaces $T_{\mathrm{HOR1}}(n,\mathbb{R})$ and $T_{\mathrm{HOR2}}(n,\mathbb{R}) \subset T(n,\mathbb{R})$ which are defined in definition 6.19 (a) satisfy the following properties.*

(α) $T_{\mathrm{HOR}k}(n,\mathbb{R})$ (for $k \in \{1,2\}$) can be represented by a closed simplex (the convex hull of $\dim T_{\mathrm{HOR}k}(n,\mathbb{R}) + 1$ many points) in $\mathbb{R}^{\dim T_{\mathrm{HOR}k}(n,\mathbb{R})}$. And

$$\begin{array}{lll} & \dim T_{\mathrm{HOR1}}(n,\mathbb{R}) & \dim T_{\mathrm{HOR2}}(n,\mathbb{R}) \\ n \text{ odd} & \frac{n-1}{2} & \frac{n-1}{2} \\ n \text{ even} & \frac{n}{2} & \frac{n-2}{2} \end{array} \tag{2.9}$$

(β) For each $S \in T_{\mathrm{HOR}k}(n,\mathbb{R})$, there is a regular matrix $R^{mat}_{(k)}(S) \in GL(n,\mathbb{R})$ with eigenvalues in S^1 and with

$$(-1)^k \cdot S^{-1}S^t \quad = \quad R^{mat}_{(k)}(S)^n. \tag{2.10}$$

Regular means that $R^{mat}_{(k)}(S)$ has for each eigenvalue only one Jordan block. The map $R^{mat}_{(k)} : T_{\mathrm{HOR}k}(n,\mathbb{R}) \to GL(n,\mathbb{R})$ is as a map to $M(n \times n,\mathbb{R})$ affine linear.

(γ) $R^{mat}_{(k)}(S)$ is semisimple (and thus has pairwise different eigenvalues) if and only if $S \in \mathrm{int}(T_{\mathrm{HOR}k}(n,\mathbb{R}))$.

(δ) $E_n \in \mathrm{int}(T_{\mathrm{HOR}k}(n,\mathbb{R}))$ and $R^{mat}_{(k)}(E_n)$ has the eigenvalues $e^{-2\pi i(j-\frac{k}{2})/n}$, $j \in \{1, ..., n\}$. Furthermore, $\bigcap_{k=1,2} T_{\mathrm{HOR}k}(n,\mathbb{R}) = \{E_n\}$.

(b) The recipe 2.9 works well within $T_{\mathrm{HOR}k}(n,\mathbb{R})$. For $S_1 \in T_{\mathrm{HOR}k}(n,\mathbb{R})$ choose any continuous path $S : [0,1] \to T_{\mathrm{HOR}k}(n,\mathbb{R})$ with $S(0) = E_n, S(1) = S_1$ and $S([0,1)) \subset \mathrm{int}(T_{\mathrm{HOR}k}(n,\mathbb{R}))$. Then for $r \in [0,1)$ the eigenvalues of $R^{mat}_{(k)}(S(r))$ are pairwise different and the paths $\alpha_1, ..., \alpha_n : [0,1] \to \mathbb{R}$ can be chosen uniquely such that $\alpha_j(0) = 0$ and $e^{-2\pi i(\alpha_j(r)+j-\frac{k}{2})/n}$ for $j \in \{1, ..., n\}$ are the eigenvalues of $R^{mat}_{(k)}(S(r))$. The values

$\alpha_1(1), ..., \alpha_n(1)$ *are independent of the chosen path* S *and give the spectrum* $\mathrm{Sp}(S) = \sum_{j=1}^n (\alpha_j(1)) \in \mathbb{Z}_{\geq 0}(\mathbb{R})$.

Proof. (a) This part will be proved in section 6.3.

(b) follows immediately from part (a). In fact, part (a) implies existence and uniqueness of continuous functions $\alpha_j^{(k)} : T_{\mathrm{HOR}k}(n,\mathbb{R}) \to \mathbb{R}$ such that $\alpha_j^{(k)}(E_n) = 0$ and $e^{-2\pi i(\alpha_j^{(k)}(S)+j-\frac{k}{2})/n}$ for $j \in \{1, ..., n\}$ are the eigenvalues of $R_{(k)}^{mat}(S)$ for any $S \in T_{\mathrm{HOR}k}(n,\mathbb{R})$. For any $S \in T_{\mathrm{HOR}k}(n,\mathbb{R})$ the values $\alpha_j^{(k)}(S)$ at S are the spectral numbers of S. The only matrix in $\bigcap_{k=1,2} T_{\mathrm{HOR}k}(n,\mathbb{R})$ is E_n. Both cases $k = 1$ and $k = 2$ associate to E_n the spectrum $\mathrm{Sp}(E_n) = \sum_{j=1}^n (0)$. $\square$

Remarks 2.19. (i) The crucial points are, that the matrices $R_{(k)}^{mat}(S)$ for $S \in \mathrm{int}(T_{\mathrm{HOR}k}(n,\mathbb{R}))$ have pairwise different eigenvalues and the $\alpha_j^{(k)}(S)$ are determined by these eigenvalues and that the values $e^{-2\pi i\alpha_j^{(k)}(S)}$ are the eigenvalues of $S^{-1}S^t$ because of (2.10).

2.2.5 Seifert forms and spectral recipes

That the recipe 2.9 works for the matrices in $\bigcup_{k=1,2} T_{\mathrm{HOR}k}(n,\mathbb{R})$ is good news. It led us to a number of conjectures and results. We hope that they will be useful for a complete positive solution of recipe 2.9.

The rest of this section has two purposes. It fixes notions and proposes the conjectures 2.21, 2.22 and 2.23 which guide us through all of the chapter.

We now recall *Seifert form pairs* to formulate the conjectures 2.21 and 2.22.

Definition 2.20. *(a) A Seifert form pair* $(H_\mathbb{R}, L)$ *consists of a finite dimensional real vector space* $H_\mathbb{R}$ *and a nondegenerate bilinear form* $L : H_\mathbb{R} \times H_\mathbb{R} \to \mathbb{R}$ *(which is in general neither symmetric or antisymmetric). Its monodromy is the (unique) automorphism* $M : H_\mathbb{R} \to H_\mathbb{R}$ *with* $L(Ma, b) = L(b, a)$ *for* $a, b \in H_\mathbb{R}$.

(b) Hermitian Seifert form pairs are classified in [Ne95]. The classification of real Seifert form pairs is done in chapter 5.

(c) Trivial lemma: Any matrix $S \in GL(n,\mathbb{R})$ *gives rise to the Seifert form pair* $\mathrm{Seif}(S) := (M(n \times 1, \mathbb{R}), L)$ *with* $L(a, b) := a^t \cdot S^t \cdot b$. *Its monodromy* M *is given by* $M(a) = S^{-1}S^t \cdot a$.

(d) We define the sets $\mathrm{Seif}(n), \mathrm{Eig}(n)$, *the projection* pr_{SE}, *and the maps* Ψ_{Seif} *and* Ψ_{Eig}

as follows.

$$
\begin{aligned}
\mathrm{Seif}(n) &:= \{\text{isomorphism classes of Seifert form pairs } (H_{\mathbb{R}}, L) \\
&\qquad \text{with } \dim H_{\mathbb{R}} = n \text{ and with eigenvalues of the} \\
&\qquad \text{monodromy } M \text{ in } S^1\}, && (2.11) \\
\mathrm{Eig}(n) &:= \{\text{unordered tuples of numbers } \lambda_1, ..., \lambda_n \in S^1\} \\
&:= (S^1)^n / S_n, && (2.12)
\end{aligned}
$$

$$
\begin{aligned}
pr_{SE} : \mathrm{Seif}(n) \to \mathrm{Eig}(n), &\qquad [(H_{\mathbb{R}}, L)] \mapsto (\text{eigenvalues of } M), && (2.13) \\
\Psi_{Seif} : T(n, \mathbb{R}) \to \mathrm{Seif}(n), &\qquad S \mapsto [\mathrm{Seif}(S)], && (2.14) \\
\Psi_{Eig} := pr_{SE} \circ \Psi_{Seif} \quad : &\quad T(n, \mathbb{R}) \to \mathrm{Eig}(n). && (2.15)
\end{aligned}
$$

(e) The group $G_{sign,n} := \{\pm 1\}^n$ acts on $T(n, \mathbb{R})$ by conjugation,

$$
(\varepsilon_1, ..., \varepsilon_n) : S \mapsto \mathrm{diag}(\varepsilon_1, ..., \varepsilon_n) \cdot S \cdot \mathrm{diag}(\varepsilon_1, ..., \varepsilon_n) \tag{2.16}
$$

for $(\varepsilon_1, ..., \varepsilon_n) \in G_{sign,n}$. The group $G_{sign,n}$ is called sign group. Of course, the maps Ψ_{Seif} and Ψ_{Eig} are $G_{sign,n}$-invariant.

(f) A ***Seifert form stratum*** *in $T(n, \mathbb{R})$ is a union of components of one fiber of Ψ_{Seif} which are permuted transitively by $G_{sign,n}$. An* ***eigenvalue stratum*** *in $T(n, \mathbb{R})$ is a union of components of one fiber of Ψ_{Eig} which are permuted transitively by $G_{sign,n}$.*

(g) The group Br_n is the braid group with n strings (for details see [Eb01, p. 201-204]). The set of distinguished bases of an ihs *forms one orbit of the group $\mathrm{Br}_n \ltimes G_{sign,n}$. Trivial lemma: Let α_j denote the braid operation exchanging the jth and $j+1$th string, $j \in \{1, ..., n-1\}$. Then the operation on the matrix $S = (a_{ij})_{i,j=1,...,n} \in T(n, \mathbb{R})$ may be written as*

$$
\alpha_j : S \mapsto \begin{pmatrix} \ddots & & & \\ & -a_{j,j+1} & 1 & \\ & 1 & 0 & \\ & & & \ddots \end{pmatrix} \cdot S \cdot \begin{pmatrix} \ddots & & & \\ & -a_{j,j+1} & 1 & \\ & 1 & 0 & \\ & & & \ddots \end{pmatrix}.
$$

Even more information than in the spectrum is carried in the spectral pairs, which we will later denote by Spp. The next conjectures include both. The conjectures are a very specific extension of the conjecture 2.9 made by Cecotti & Vafa.

Conjecture 2.21. *(a)* $T_{\mathrm{HOR}1}(n,\mathbb{R})$ *intersects each eigenvalue stratum in* $T(n,\mathbb{R})$.

(b) If $S_1,S_2\in\bigcup_{k=1,2}T_{\mathrm{HOR}k}(n,\mathbb{R})$ *are in the same eigenvalue stratum of* $T(n,\mathbb{R})$ *then* $\mathrm{Sp}(S_1)=\mathrm{Sp}(S_2)$.

(c) If $S_1,S_2\in\bigcup_{k=1,2}T_{\mathrm{HOR}k}(n,\mathbb{R})$ *are in the same Seifert form stratum of* $T(n,\mathbb{R})$ *then* $\mathrm{Spp}(S_1)=\mathrm{Spp}(S_2)$.

If it is true, conjecture 2.21 (a) implies that $T(n,\mathbb{R})$ is connected, conjecture 2.21 (a)+(b) gives spectral numbers $\mathrm{Sp}(S)$ for any matrix $S\in T(n,\mathbb{R})$, and conjecture 2.21 (a)+(c) gives spectral pairs for any matrix S in a Seifert form stratum which is met by $\bigcup_{k=1,2}T_{\mathrm{HOR}k}(n,\mathbb{R})$. But these are not all Seifert form strata, as remark 6.8 (vii) and remark 6.26 (ii) will show. Unfortunately, for the other Seifert form strata, we have no precise idea how to lift $\mathrm{Sp}(S)$ to $\mathrm{Spp}(S)$.

Conjecture 2.22. *Also for the matrices* S *in the Seifert form strata which are not met by* $\bigcup_{k=1,2}T_{\mathrm{HOR}k}(n,\mathbb{R})$, $\mathrm{Sp}(S)$ *lifts in a natural way to* $\mathrm{Spp}(S)$.

Building on the conjectures 2.21 and 2.22, we have a conjecture which embraces the claim of Cecotti and Vafa for Landau-Ginzburg models. We recall details on M-tame functions when needed, in chapter 6.

Conjecture 2.23. *Suppose that the conjectures 2.21 and 2.22 are true. Let* f *be a holomorphic map germ* $f:(\mathbb{C}^{m+1},0)\to(\mathbb{C},0)$ *with an isolated singularity at 0 or an* M*-tame function* $f:X\to\mathbb{C}$ *with* $\dim X=m+1$. *Then any Stokes matrix* S *of* f *satisfies*

$$\mathrm{Spp}(S)=\mathrm{Spp}(f)-(\frac{m-1}{2},m). \tag{2.17}$$

3 Nonreduced presentations of Weyl group elements

This chapter prepares the proof of conjecture 2.12 in the semidefinite case, which is carried out in chapter 4. This preparation consists of a thorough study of what we call *possibly nonreduced root lattices*, extending the definition of a traditional root lattice (standard reference here is [Bo68, ch. VI]), based on the previous works of

Dynkin [Dy57] on subroot lattices of root lattices. In particular, he describes the BDdS algorithm, also appearing in works by Borel and de Siebenthal [BS49].

Carter [Ca72] on presentations of Weyl group elements as products of arbitrary reflections, arbitrary meaning reflections at all possible roots, not just on roots of a fixed root basis, as for instance considered in [Hu90].

The structure of the chapter is as follows. Section 3.1 recalls the classification of irreducible *p.n. root lattices* up to isomorphism, standard types, and Dynkin diagrams. The one new series of types for p.n. root lattices are the types BC_n.

Section 3.2 recalls the BDdS algorithm, and defines the three numbers (k_1, k_2, k_3) relating (root lattice, subroot lattice) pairs in different ways. The main theorem is 3.11, which equates these three numbers.

Section 3.3 gives a proof of the following basic fact which seems to have been unnoticed up to now and which may be of some independent interest: "Any generating set of roots of a root lattice as a $\mathbb{Z}$-module contains a $\mathbb{Z}$-basis of it."

Section 3.4 refines the classification of conjugacy classes by Carter of irreducible root lattices. It provides all reduced presentations as products of reflections. Crucial are the notions of quasi and strict quasi Coxeter elements, the second of which are new in the inhomogeneous case. The number k_4 is defined, which relates the full lattice and the subroot lattice, created by a reduced representation of a Weyl group element.

Section 3.5 defines the last number k_5, which is needed in the application in singularity theory. The main result is theorem 3.29, that $k_5 = k_4$ holds.

Section 3.6 contains the essential application of this chapter. It applies to *extended affine root lattices* (defined by K. Saito [Sa85, (1.2) and (1.3)], see also [AABGP97, Az02]), which arise in singularity theory. One simply replaces in the definition of a p.n. root lattice the scalar product by a positive semidefinite bilinear form $(.,.) : L_{\mathbb{Q}} \times L_{\mathbb{Q}} \to \mathbb{Q}$. Then the quotient $L/\operatorname{Rad}(L)$ becomes in a natural way a *p.n. root lattice*. Any element $w \in W(L)$ induces an element $\bar{w} \in W(L/\mathrm{Rad}(L))$. The simple lemma 3.34 gives for a quasi Coxeter element $w \in W(L)$ the inequalities

$$\begin{aligned} l(\overline{w}) &\leq \operatorname{rank} L - \operatorname{rank} \operatorname{Rad}(L), \\ l(\overline{w}) + 2k_5(L/\operatorname{Rad}(L), \overline{w}) &\leq \operatorname{rank} L. \end{aligned}$$

It gives a constraint on the elements $\overline{w}$ which are induced by quasi Coxeter elements. Theorem 3.29 says $k_5(L/\operatorname{Rad}(L), \overline{w}) = k_4(L/\operatorname{Rad}(L), \overline{w})$, and theorem 3.25 allows to calculate this number. So in essence, k_5 controls the existence of quasi Coxeter elements in those quotient lattices will turn up in the next chapter in the proof of conjecture 2.12.

3.1 Basic facts on (p.n.) root lattices

This section recalls basic facts on root systems and lattices. A standard reference is [Bo68, ch. VI] (there *reduced root systems* denotes what we denote by *root systems*). We call *p.n. root systems* (*p.n.* for *possibly nonreduced*) what is called there *root systems*. We include the *p.n. root lattices* because the condition (3.5) below, which distinguishes root systems, is not necessarily preserved if one goes from an *extended affine root lattice* (see section 3.6) to a quotient lattice. The classification includes only one series of types of p.n. root lattices which are not original root lattices, the types BC_n.

(i) A free $\mathbb{Z}$-module L of rank $n \in \mathbb{Z}_{>0}$ is called a **lattice**. Then $L_{\mathbb{Q}} := L \otimes_{\mathbb{Z}} \mathbb{Q}$, $L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R}$ and $L_{\mathbb{C}} := L \otimes_{\mathbb{Z}} \mathbb{C}$.

(ii) Let L be a lattice and $(.,.)$ be a scalar product on $L_{\mathbb{R}}$. For $\alpha \in L - \{0\}$ and $\beta \in L$ define

$$\langle \beta, \alpha \rangle := \frac{2(\beta, \alpha)}{(\alpha, \alpha)}. \tag{3.1}$$

Then

$$s_\alpha : L_\mathbb{R} \to L_\mathbb{R}, \quad s_\alpha(x) := x - \langle x, \alpha\rangle \cdot \alpha \tag{3.2}$$

is a reflection. Two reflections s_α and s_β satisfy

$$s_\alpha s_\beta = s_\beta s_{s_\beta(\alpha)} = s_{s_\alpha(\beta)} s_\alpha. \tag{3.3}$$

Definition 3.1. *(a) A **p.n. root lattice** is a triple $(L, (.,.), \Phi)$ where L is a lattice, $(.,.) : L_\mathbb{R} \times L_\mathbb{R} \to \mathbb{R}$ is a scalar product, and $\Phi \subset L - \{0\}$ is a finite set such that the following properties hold.*

(1) Φ is a generating set of L as a $\mathbb{Z}$-module.

(2) For any $\alpha \in \Phi$ $s_\alpha(\Phi) = \Phi$.

(3) $\langle \beta, \alpha\rangle \in \mathbb{Z}$ for any $\alpha, \beta \in \Phi$.

*The elements of Φ are the **roots**, and Φ is a **p.n. root system**. The finite group*

$$W := \langle s_\alpha \,|\, \alpha \in \Phi\rangle \subset O(L, (.,.)) \tag{3.4}$$

*is the **Weyl group**.*

*(b) A **root lattice** is a p.n. root lattice $(L, (.,.), \Phi)$ which satisfies additionally the condition:*

$$\text{If } \alpha \in \Phi, \text{ then } \Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}. \tag{3.5}$$

*Then Φ is a **root system**.*

c) (Lemma) The orthogonal sum of several (p.n.) root lattices is (in a most natural way) a (p.n.) root lattice.

*(d) A (p.n.) root lattice is **irreducible** if it is not isomorphic to the orthogonal sum of several (p.n.) root lattices.*

The classification of p.n. root lattices and of root lattices is as follows (a standard reference is [Bo68, ch. VI]).

Theorem 3.2. *(a) Any (p.n.) root lattice is either irreducible or isomorphic to an orthogonal sum of several irreducible (p.n.) root lattices.*

(b) If $(L,(.,.),\Phi)$ is an irreducible (p.n.) root lattice then also $(L,c\cdot(.,.),\Phi)$ for any $c\in\mathbb{R}_{>0}$ is an irreducible (p.n.) root lattice. Two irreducible (p.n.) root lattices are of the same type *if they differ up to isomorphism only by such a scalar c.*

(c) The types of irreducible p.n. root lattices are given by 5 series and 5 exceptional ones with the following names,

$$A_n\ (n\geq 1),\ B_n\ (n\geq 2),\ C_n\ (n\geq 3),\ BC_n\ (n\geq 1),\\ D_n\ (n\geq 4), E_6,\ E_7,\ E_8,\ F_4,\ G_2.$$

All types, except BC_n are ordinary root lattices. *BC_n is the only irreducible type of p.n. root lattice, that is not a root lattice.*

(d) The following list presents one irreducible p.n. root lattice of each type. Always $L_\mathbb{R}\subset\mathbb{R}^m$ for some $m\in\{n,n+1,n+2\}$. Here $(.,.)$ is the restriction to $L_\mathbb{R}$ of the standard scalar product on $\mathbb{R}^m$, and $e_1,...,e_m$ is the standard ON-basis of $\mathbb{R}^m$.

$$\mathbf{A_n}:\quad m=n+1,\quad \Phi=\{\pm(e_i-e_j)\,|\,1\leq i<j\leq n+1\}, \tag{3.6}$$

$$\mathbf{B_n}:\quad m=n,\quad \Phi=\{\pm e_i\,|\,1\leq i\leq n\}\\ \cup\,\{\pm e_i\pm e_j\,|\,1\leq i<j\leq n\}, \tag{3.7}$$

$$\mathbf{C_n}:\quad m=n,\quad \Phi=\{\pm e_i\pm e_j\,|\,1\leq i<j\leq n\}\\ \cup\,\{\pm 2e_i\,|\,1\leq i\leq n\}. \tag{3.8}$$

$$\mathbf{BC_n}:\quad m=n,\quad \Phi=\{\pm e_i\,|\,1\leq i\leq n\}\\ \cup\,\{\pm e_i\pm e_j\,|\,1\leq i<j\leq n\}\cup\{\pm 2e_i\,|\,1\leq i\leq n\}. \tag{3.9}$$

$$\mathbf{D_n}:\quad m=n,\quad \Phi=\{\pm e_i\pm e_j\,|\,1\leq i<j\leq n\}. \tag{3.10}$$

$$\mathbf{E_6}: \quad m=8, \quad \Phi=\{\pm e_i \pm e_j \,|\, 3 \le i < j \le 7\} \cup\{\frac{1}{2}\sum_{i=1}^{8}\varepsilon_i e_i \,|\, \varepsilon_i=\pm 1, \varepsilon_1=\varepsilon_2=\varepsilon_8, \prod_{i=1}^{8}\varepsilon_i=1\} \tag{3.11}$$

$$\mathbf{E_7}: \quad m=8, \quad \Phi=\{\pm e_i \pm e_j \,|\, 2 \le i < j \le 7\} \cup \{\pm(e_1+e_8)\} \cup\{\frac{1}{2}\sum_{i=1}^{8}\varepsilon_i e_i \,|\, \varepsilon_i=\pm 1, \varepsilon_1=\varepsilon_8, \prod_{i=1}^{8}\varepsilon_i=1\} \tag{3.12}$$

$$\mathbf{E_8}: \quad m=8, \quad \Phi=\{\pm e_i \pm e_j \,|\, 1 \le i < j \le 8\} \cup\{\frac{1}{2}\sum_{i=1}^{8}\varepsilon_i e_i \,|\, \varepsilon_i=\pm 1, \prod_{i=1}^{8}\varepsilon_i=1\} \tag{3.13}$$

$$\mathbf{F_4}: \quad m=4, \quad \Phi=\{\pm e_i \,|\, 1 \le i \le 4\} \cup\{\pm e_i \pm e_j \,|\, 1 \le i < j \le 4\} \cup \{\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\}. \tag{3.14}$$

$$\mathbf{G_2}: \quad m=3, \quad \Phi=\{\pm(e_i - e_j) \,|\, 1 \le i < j \le 3\} \cup\{\pm(2e_{\pi(1)}-e_{\pi(2)}-e_{\pi(3)} \,|\, \pi \in S_3\}. \tag{3.15}$$

Remark 3.3. (i) The *p.n. root lattices* above have roots of the following lengths,

A_n	D_n	E_n	B_n	F_4	C_n	G_2	BC_n
$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$1, \sqrt{2}$	$1, \sqrt{2}$	$\sqrt{2}, 2$	$\sqrt{2}, \sqrt{6}$	$1, \sqrt{2}, 2$

The root lattices of types A_n, D_n, E_n have only roots of one length and are therefore called *homogeneous*. The root lattices of types B_n, C_n, F_4 and G_2 have roots of two lengths, *short* and *long* roots. The p.n. root lattices BC_n have roots of three lengths, short, long and *extra long* roots.

(ii) In the tables 3.1 - 3.4 the symbols $A_n, ..., G_2$ will denote root lattices with roots of lengths as above. There we will also consider a few root systems with other lengths, and a few other names for some of the root lattices above:

$A_0 = B_0 = BC_0 = \{0\}$ denotes the rank 0 lattice.
$D_2 := 2A_1 := A_1 \perp A_1, \quad D_3 := A_3.$
$\widetilde{A}_1 = B_1$ denotes a root lattice of type A_1 with roots of length 1.
C_1 denote a root lattice of type A_1 with roots of length 2.
C_2 denotes a root lattice of type B_2 with roots of lengths $\sqrt{2}$ and 2.

In the table 3.5 the roots in the root systems of type C_3 have lengths 1 and $\sqrt{2}$. In the table 3.6, roots in A_2 and A_1 have length $\sqrt{6}$, roots in $\widetilde{A}_1$ have length $\sqrt{2}$.

(iii) The Weyl group $W(A_n)$ of the root lattice above of type A_n acts on the basis $e_1, ..., e_{n+1}$ of $\mathbb{R}^{n+1} \supset L_{\mathbb{R}}$ by permutations, $W(A_n) \cong S_{n+1}$, and $\sigma \in S_{n+1}$ maps e_i to $e_{\sigma(i)}$.

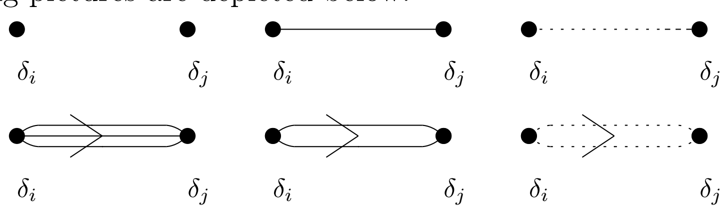
The Weyl groups of the *p.n. root lattices* above of the types B_n, C_n and BC_n coincide and act on the basis $e_1, ..., e_n$ of $\mathbb{R}^n = L_{\mathbb{R}}$ by *signed permutations*, $W(B_n) = W(C_n) = W(BC_n) \cong \{\pm 1\}^n \rtimes S_n$, and $(\varepsilon_1, ..., \varepsilon_n, \sigma) \in \{\pm 1\}^n \rtimes S_n$ maps e_i to $\varepsilon_i e_{\sigma(i)}$.

The Weyl group of the root lattice above of type D_n is the subgroup of index 2 given by the condition $\prod_{i=1}^n \varepsilon_i = 1$.

(iv) Let $(L, (.,.), \Phi)$ be an irreducible root lattice. To any subset $A = \{\delta_1, ..., \delta_l\} \subset \Phi$ with $A \cap (-A) = \emptyset$ we associate a *generalized Dynkin diagram* as follows. It is a graph with l vertices, labelled $\delta_1, ..., \delta_l$. Between vertices δ_i and δ_j with $i \neq j$ there is no edge or an edge with additional information as follows.

$$\begin{aligned}
\text{no edge,} \quad & \text{if } (\delta_i, \delta_j) = 0 \\
\text{a normal edge} \quad & \text{if } \|\delta_i\| = \|\delta_j\| \text{ and } \langle \delta_i, \delta_j \rangle = -1, \\
\text{a dotted edge} \quad & \text{if } \|\delta_i\| = \|\delta_j\| \text{ and } \langle \delta_i, \delta_j \rangle = 1, \\
\text{a double arrow from } \delta_i \text{ to } \delta_j \quad & \text{if } \|\delta_i\| = \sqrt{2}\|\delta_j\| \text{ and } \langle \delta_i, \delta_j \rangle = -2, \\
\text{a double dotted arrow from } \delta_i \text{ to } \delta_j \quad & \text{if } \|\delta_i\| = \sqrt{2}\|\delta_j\| \text{ and } \langle \delta_i, \delta_j \rangle = 2, \\
\text{a triple arrow from } \delta_i \text{ to } \delta_j \quad & \text{if } \|\delta_i\| = \sqrt{3}\|\delta_j\| \text{ and } \langle \delta_i, \delta_j \rangle = -3.
\end{aligned}$$

The corresponding pictures are depicted below.



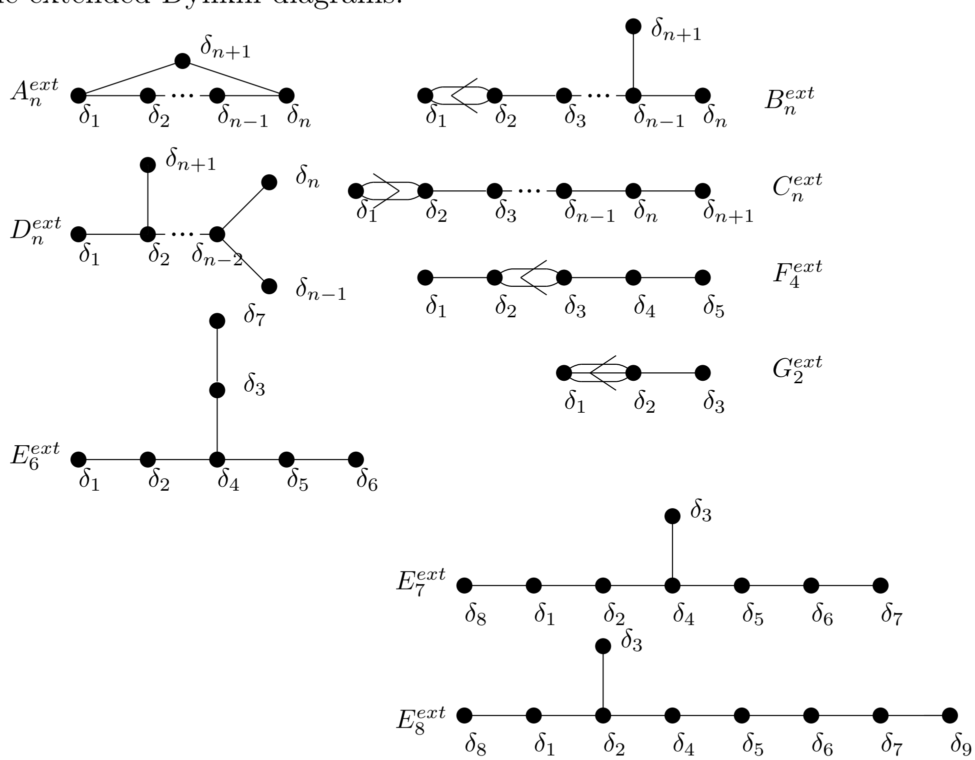
Other cases will not be considered. If A is a $\mathbb{Z}$-basis of L and the diagram is connected, then the diagram encodes up to a common scalar the intersection numbers (δ_i, δ_j), and thus it determines the irreducible root system.

(v) The following list gives for each of the root lattices in theorem 3.2 (d) a *root basis* $\delta_1, ..., \delta_n$ (a $\mathbb{Z}$-basis of L with additional properties [Bo68]) and an additional root δ_{n+1} (which is minus the maximal root with respect to the root basis). The diagram for the root basis is called *Dynkin diagram*, the diagram for $\delta_1, ..., \delta_{n+1}$ is called *extended Dynkin diagram*. The roots $\delta_1, ..., \delta_{n+1}$ satisfy a linear relation. For the cases E_6, E_7, E_8, F_4, G_2, it is given in lemma

3.10 (c). In the case of E_6,$\delta_7 = \frac{1}{2}(-\sum_{i=1,2,3,8} e_i + \sum_{i=4,5,6,7} e_i)$.

type	$\delta_1, ..., \delta_n$	δ_{n+1}
A_n	$e_i - e_{i+1}$ $(i = 1, ..., n)$	$-e_1 + e_{n+1}$
B_n	$-e_1,\ e_i - e_{i+1}$ $(i = 1, .., n-1)$	$e_{n-1} + e_n$
C_n	$-2e_1,\ e_i - e_{i+1}$ $(i = 1, .., n-1)$	$2e_n$
D_n	$e_i - e_{i+1}$ $(i = 1, ..., n-1),\ e_{n-1} + e_n$	$-e_1 - e_2$
E_6	$\frac{1}{2}\sum_{i=1}^{8} e_i,\ -e_3 - e_4,\ e_i - e_{i+1}$ $(i = 3, ..., 6)$	δ_7
E_7	$\frac{1}{2}\sum_{i=1}^{8} e_i,\ -e_2 - e_3,\ e_i - e_{i+1}$ $(i = 2, ..., 6)$	$-e_1 - e_8$
E_8	$\frac{1}{2}\sum_{i=1}^{8} e_i,\ -e_1 - e_2,\ e_i - e_{i+1}$ $(i = 1, ..., 6)$	$e_7 - e_8$
F_4	$\frac{1}{2}\sum_{i=1}^{4} e_i,\ -e_1,\ e_1 - e_2,\ e_2 - e_3$	$e_3 - e_4$
G_2	$e_1 - e_2,\ -e_1 + 2e_2 - e_3$	$-e_1 - e_2 + 2e_3$

The following are the extended Dynkin diagrams.



3.2 Subroot lattices and quotients

We are interested in (root lattice, subroot lattice) pairs denoted by (L, L_1), and minimal ways of reducing L to L_1 and extending L_1 to L. To study these ways, we introduce three numbers. The minimal number of generators of L/L_1, called k_1. The minimal number of roots needed to extend L_1 to L, called k_2. And the minimal number of steps in the BDdS algorithm to extend the Dynkin diagram of L_1 to L, called k_3. This is nontrivial information. We extract it using the BDdS algorithm and marked graphs.

First, we define (p.n.) subroot lattices. The subroot lattices of an irreducible root lattice can be determined up to isomorphism by a recipe due to Dynkin [Dy57] and Borel and de Siebenthal [BS49]. We will extend the list from [Dy57] in two ways. First, we consider also the *p.n. subroot lattices* of the p.n. root lattices of type BC_n. Second, we will calculate for any (isomorphism class of a) pair (L, L_1) where L_1 is a (p.n.) subroot lattice of an irreducible (p.n.) root lattice L the quotient group L/L_1, and the data k_1, k_2, k_3. This will be helpful in section 3.3 and crucial in the sections 3.5 and 3.6.

Definition 3.4. *Let $(L, (.,.), \Phi)$ be a (p.n.) root lattice.*

*(a) A (p.n.) root lattice $(L_1, (.,.)_1, \Phi_1)$ is a **(p.n.) subroot lattice** of $(L, (.,.), \Phi)$ if $L_1 \subset L$ and $(.,.)_1$ is the restriction of $(.,.)$ to L_1 and $\Phi_1 = L_1 \cap \Phi$.*

A notation: Because $(.,.)_1$ and Φ_1 are determined by L_1, we will talk of the subroot lattice L_1.

*(b) A (p.n.) root lattice $(L_1, (.,.)_1, \Phi_1)$ is the (p.n.) root lattice of a **(p.n.) subroot system** if $L_1 \subset L$ and $(.,.)_1$ is the restriction of $(.,.)$ to L_1 and $\Phi_1 \subset L_1 \cap \Phi$.*

*(c) The **index** of a subroot lattice L_1 is $[L \cap L_{1,\mathbb{Q}} : L_1] \in \mathbb{Z}_{\geq 1}$.*

Remark 3.5. Let $(L, (.,.), \Phi)$ be a (p.n.) root lattice.

(i) Let $L_1 \subset L$ be a $\mathbb{Z}$-sublattice. Define $(.,.)_1$ as the restriction of $(.,.)$ to L_1. Define $\Phi_1 := L_1 \cap \Phi$. Then $(L_1, (.,.)_1, \Phi_1)$ is a *(p.n.) subroot lattice* if and only if it is a (p.n.) root lattice, and this holds if and only if L_1 is generated by Φ_1 as a $\mathbb{Z}$-module: (5.5) holds for Φ_1, and $s_\alpha(\Phi_1) \subset L_1$ holds for $\alpha \in \Phi_1$ because of (3.2) and (3.1). This gives (5.4).

(ii) If $A = \{\alpha_1, ..., \alpha_l\} \subset \Phi$ is any nonempty subset, then the data

$$L_1 := \sum_{i=1}^{l} \mathbb{Z} \cdot \alpha_i, \quad (.,.)_1 := (.,.)_{|L_1}, \quad \Phi_1 := L_1 \cap \Phi \tag{3.16}$$

satisfy the conditions in (i) and are a (p.n.) subroot lattice.

(iii) If $(L, (.,.), \Phi)$ is the orthogonal sum of (p.n.) subroot lattices then any other (p.n.) subroot lattice inherits this orthogonal decomposition, though its part in one of the two subroot lattices may be trivial.

(iv) Any (p.n.) subroot lattice is the root lattice of a subroot system. If $(L, (.,.), \Phi)$ is a homogeneous root lattice also the inverse holds. But if $(L, (.,.), \Phi)$ contains orthogonal summands which are of types B_k, C_k, BC_k, F_4 or G_2, then there are subroot lattices $(L_1, (.,.), \Phi_1)$

such that the subsets $\Phi_2 \subsetneqq \Phi_1$ of short roots give rise to root lattices $(L_1,(.,.),\Phi_2)$ of subroot systems Φ_2, such that these root lattices are not subroot lattices. We will not work much with them, but in [Ca72] they are used.

(v) If one erases from any of the extended Dynkin diagrams one vertex, one obtains a disjoint union of Dynkin diagrams. This leads to the following recipe with two kinds of steps with which one obtains easily subroot lattices of a root lattice. It is due to [Dy57] and [BS49], therefore we call the steps (BDdS1) and (BDdS2). Start with a root lattice $(L,(.,.),\Phi)$. Choose a root basis $A \subset \Phi$, that is a $\mathbb{Z}$-basis of L consisting of roots such that its generalized Dynkin diagram (defined in remark 3.3 (iv)) is a disjoint union of Dynkin diagrams. L decomposes uniquely into an orthogonal sum of irreducible subroot lattices, which are called the *summands* of L.

Step (BDdS1): Choose one summand L_1 of L, add to A the unique root $\widetilde{\delta}$ in Φ_1 which gives together with the roots in $A\cap\Phi_1$ an extended Dynkin diagram (it is a linear combination of the roots in $A\cap\Phi_1$) and delete from $A\cup\{\widetilde{\delta}\}$ an arbitrary root in $A\cap\Phi_1$. The new set $\widetilde{A}\subset\Phi$ defines a subroot lattice $\widetilde{L}$ of L of the same rank as L.

Step (BDdS2): Choose one summand L_1 of L and delete from A an arbitrary root in $A\cap\Phi_1$. The new set $\widetilde{A}\subset\Phi$ defines a subroot lattice$\widetilde{L}$ of L with $\operatorname{rank} L_1 = \operatorname{rank} L - 1$.

In both cases $\widetilde{A}$ is a root basis of $\widetilde{L}$. Therefore one can repeat the steps. The change in the Dynkin diagrams is easy to see. In the step (BDdS1) one extends one component to its extended version and then erases one vertex. In the step (BDdS2) one simply erases one vertex.

The following theorem is mainly due to Dynkin [Dy57], the recipe in part (a) is also in [BS49]. The new part is BC_n. That will follow from lemma 3.9 below.

Theorem 3.6. *(a) Let $(L,(.,.),\Phi)$ be a root lattice. Any subroot lattice is obtained by the choice of a suitable root basis of L and by a suitable sequence of the steps (BDdS1) and (BDdS2).*

(b) The first columns of the tables 3.1 - 3.6 list all isomorphism classes of pairs $((L,(.,.),\Phi),L_1)$ where $(L,(.,.),\Phi)$ is an irreducible (p.n.) root lattice with the lengths of the roots as in theorem 3.2 (d) and where L_1 is a subroot lattice.

The tables give the name for the type of L_1, where additionally the lengths of the roots of the summands of L_1 are taken into account. The symbols $A_0,B_0,BC_0,D_2,D_3,\widetilde{A}_1,B_1,C_1,C_2$ from remark 3.3 (ii) are used. The new notations $[...]'$ and $[...]''$ are explained in (d) below.

(c) With one class of exceptions, the following holds. If $((L,(.,.),\Phi),L_1)$ and $((L,(.,.),\Phi),L_2)$ are isomorphic pairs as in (b), then a Weyl group element $w\in W$ with

$w(L_1)=w(L_2)$ exists. The class of exceptions are the sublattices of D_n of types $A_{k_1}+...+A_{k_r}$ with all $k_1,...,k_r$ odd. For each of those types there are two conjugacy classes with respect to W.

(d) The tables 3.3 and 3.4 contain pairs $[H]'$ and $[H]''$ with $H\in\{A_5+A_1,A_5,A_3+2A_1,A_3+A_1,4A_1,3A_1\}$ for E_7 and with $H\in\{A_7,A_5+A_1,2A_3,A_3+2A_1,4A_1\}$ for E_8. Here $[H]'$ and $[H]''$ denote (classes in the sense of (b) of) subroot lattices which are isomorphic if one forgets the embedding into L. But for a subroot lattice $L_1\subset L$ of type $[H]'$ and a subroot lattice $L_2\subset L$ of type $[H]''$, the pairs (L,L_1) and (L,L_2) are not isomorphic. This is an implication of the following properties: A subroot lattice $L_3\subset L$ of type A_7 for E_7 and of type A_8 for E_8 with $L_1\subset L_3\subset L$ exists, but no subroot lattice $L_4\subset L$ of type A_7 for E_7 and of type A_8 for E_8 with $L_2\subset L_4\subset L$ exists.

The information in the following tables 3.1 - 3.6 are treated in theorem 3.6, lemma 3.10 and theorem 3.11. Always L is one of the p.n. subroot lattices in theorem 3.2 (d), and L_1 is a p.n. subroot lattice of the type indicated. In the tables 3.2-3.6 it is $L_1:=\sum_{i\in\{1,...,n\}\cup I-J}\mathbb{Z}\cdot\delta_i$. Here the roots δ_k for $k\geq n+2$ (in the cases E_7,E_8,F_4) are defined in Lemma 3.10. The quotient L/L_1 is given up to isomorphism. Here $\mathbb{Z}_m:=\mathbb{Z}/m\mathbb{Z}$ for $m\in\mathbb{Z}_{>0}$. For $k_1=k_1(L,L_1)$ see theorem 3.11. For the symbols $A_0,B_0,BC_0,D_2,D_3,\widetilde{A}_1,B_1,C_1,C_2$ see remark 3.3 (ii).

Table 3.1 for A_n,B_n,C_n,BC_n,D_n: Here $r\geq 0$, $s\geq 0$, $k_i\geq 0$, $m\geq 0$, in the cases C_{l_j} $l_j\geq 1$, in the cases D_{l_j} $l_j\geq 2$,

$$\sum_{i=1}^{r}(k_i+1)+\sum_{j=1}^{s}l_j+m=n \tag{3.17}$$

(with $m=0$ in the cases A_n,C_n,D_n).

L	L_1	L/L_1	$k_1(L,L_1)$	
A_n	$\sum_{i=1}^r A_{k_i}$	$\mathbb{Z}^{r-1}$	$r-1$	
B_n	$\sum_{i=1}^r A_{k_i}+\sum_{j=1}^s D_{l_j}+B_m$	$\mathbb{Z}^r\times\mathbb{Z}_2^s$	$r+s$	
C_n	$\sum_{i=1}^r A_{k_i}+\sum_{j=1}^s C_{l_j}$	$\mathbb{Z}^r\times\mathbb{Z}_2^{s-1}$	$r+s-1$	if $s\geq 1$
		$\mathbb{Z}^r$	r	if $s=0$
BC_n	$\sum_{i=1}^r A_{k_i}+\sum_{j=1}^s C_{l_j}+BC_m$	$\mathbb{Z}^r\times\mathbb{Z}_2^s$	$r+s$	
D_n	$\sum_{i=1}^r A_{k_i}+\sum_{j=1}^s D_{l_j}$	$\mathbb{Z}^r\times\mathbb{Z}_2^{s-1}$	$r+s-1$	if $s\geq 1$
		$\mathbb{Z}^r$	r	if $s=0$

Table 3.2 for E_6:

L_1	I	J	L/L_1	k_1
E_6	–	–	$\{0\}$	1
A_5+A_1	7	2	$\mathbb{Z}_2$	1
$3A_2$	7	4	$\mathbb{Z}_3$	1
A_5	–	3	$\mathbb{Z}$	1
$2A_2+A_1$	–	4	$\mathbb{Z}$	1
A_4+A_1	–	2	$\mathbb{Z}$	1
D_5	–	1	$\mathbb{Z}$	1
A_3+2A_1	7	$2,3$	$\mathbb{Z}\times\mathbb{Z}_2$	2
A_4	–	$1,2$	$\mathbb{Z}^2$	2
A_3+A_1	–	$2,3$	$\mathbb{Z}^2$	2

L_1	I	J	L/L_1	k_1
$2A_2$	–	–	$\mathbb{Z}^2$	2
A_2+2A_1	–	$4,5$	$\mathbb{Z}^2$	2
$4A_1$	7	$2,3,5$	$\mathbb{Z}^2\times\mathbb{Z}_2$	3
D_4	–	$1,6$	$\mathbb{Z}^2$	2
A_3	–	$1,2,3$	$\mathbb{Z}^3$	3
A_2+A_1	–	$2,3,6$	$\mathbb{Z}^3$	3
$3A_1$	–	$2,3,5$	$\mathbb{Z}^3$	3
A_2	–	$1,2,3,4$	$\mathbb{Z}^4$	4
$2A_1$	–	$1,2,3,5$	$\mathbb{Z}^4$	4
A_1	–	$1,...,5$	$\mathbb{Z}^5$	5

Table 3.3 for E_7:

L_1	I	J	L/L_1	k_1
E_7	–	–	$\{0\}$	1
D_6+A_1	8	5	$\mathbb{Z}_2$	1
A_5+A_2	8	5	$\mathbb{Z}_3$	1
$2A_3+A_1$	8	4	$\mathbb{Z}_4$	1
A_7	8	3	$\mathbb{Z}_2$	1
D_4+3A_1	$8,9$	$1,6$	$\mathbb{Z}_2^2$	2
$7A_1$	$8,9,10$	$1,6,4$	$\mathbb{Z}_2^3$	3
E_6	–	7	$\mathbb{Z}$	1
D_5+A_1	–	6	$\mathbb{Z}$	1
A_4+A_2	–	5	$\mathbb{Z}$	1

L_1	I	J	L/L_1	k_1
$A_3+A_2+A_1$	–	4	$\mathbb{Z}$	1
$[A_5+A_1]'$	8	$1,3$	$\mathbb{Z}\times\mathbb{Z}_2$	2
$[A_5+A_1]''$	–	2	$\mathbb{Z}$	1
D_6	–	1	$\mathbb{Z}$	1
D_4+2A_1	8	$1,6$	$\mathbb{Z}\times\mathbb{Z}_2$	2
A_3+3A_1	8	$1,4$	$\mathbb{Z}\times\mathbb{Z}_2$	2
$3A_2$	8	$2,5$	$\mathbb{Z}\times\mathbb{Z}_3$	2
$2A_3$	8	8	$\mathbb{Z}\times\mathbb{Z}_2$	2
A_6	–	3	$\mathbb{Z}$	1
$6A_1$	–	$1,4,6$	$\mathbb{Z}\times\mathbb{Z}_2^2$	3

D_5	–	$1,7$	$\mathbb{Z}^2$	2
A_4+A_1	–	$3,6$	$\mathbb{Z}^2$	2
$2A_2+A_1$	–	$2,5$	$\mathbb{Z}^2$	2
$[A_5]'$	–	$1,3$	$\mathbb{Z}^2$	2
$[A_5]''$	–	$1,2$	$\mathbb{Z}^2$	2
D_4+A_1	–	$1,6$	$\mathbb{Z}^2$	2
A_3+A_2	–	$3,5$	$\mathbb{Z}^2$	2
$5A_1$	8	$1,4,6$	$\mathbb{Z}^2\times\mathbb{Z}_2$	3
A_2+3A_1	–	$4,6$	$\mathbb{Z}^2$	2
$[A_3+2A_1]'$	8	$1,3,4$	$\mathbb{Z}^2\times\mathbb{Z}_2$	3
$[A_3+2A_1]''$	–	$1,4$	$\mathbb{Z}^2$	2
D_4	–	$1,6,7$	$\mathbb{Z}^3$	3
A_4	–	$1,2,3$	$\mathbb{Z}^3$	3

$[A_3+A_1]'$	–	$1,3,4$	$\mathbb{Z}^3$	3
$[A_3+A_1]''$	–	$1,2,4$	$\mathbb{Z}^3$	3
$2A_2$	–	$1,2,5$	$\mathbb{Z}^3$	3
A_2+2A_1	–	$1,3,5$	$\mathbb{Z}^3$	3
$[4A_1]'$	8	$1,3,4,6$	$\mathbb{Z}^3\times\mathbb{Z}_2$	4
$[4A_1]''$	–	$2,4,6$	$\mathbb{Z}^3$	3
A_3	–	$1,2,3,4$	$\mathbb{Z}^4$	4
A_2+A_1	–	$1,2,3,5$	$\mathbb{Z}^4$	4
$[3A_1]'$	–	$1,3,4,6$	$\mathbb{Z}^4$	4
$[3A_1]''$	–	$1,2,4,6$	$\mathbb{Z}^4$	4
A_2	–	$1,...,5$	$\mathbb{Z}^5$	5
$2A_1$	–	$1,...,4,6$	$\mathbb{Z}^5$	5
A_1	–	$1,...,6$	$\mathbb{Z}^6$	6

Table 3.4 for E_8:

L_1	I	J	L/L_1	k_1
E_8	–	–	$\{0\}$	0
A_8	9	3	$\mathbb{Z}_3$	1
D_8	9	1	$\mathbb{Z}_2$	1
A_7+A_1	9	2	$\mathbb{Z}_4$	1
$A_5+A_2+A_1$	9	4	$\mathbb{Z}_6$	1
$2A_4$	9	5	$\mathbb{Z}_5$	1
$4A_2$	$9,13$	$7,4$	$\mathbb{Z}_3^2$	2

L_1	I	J	L/L_1	k_1
E_6+A_2	9	7	$\mathbb{Z}_3$	1
E_7+A_1	9	8	$\mathbb{Z}_2$	1
D_6+2A_1	$9,10$	$1,8$	$\mathbb{Z}_2^2$	2
D_5+A_3	9	6	$\mathbb{Z}_4$	1
$2D_4$	$9,10$	$1,4$	$\mathbb{Z}_2^2$	2
D_4+4A_1	$9,10,12$	$1,6,4$	$\mathbb{Z}_2^3$	3
$2A_3+2A_1$	$9,10$	$8,4$	$\mathbb{Z}_2\times\mathbb{Z}_4$	2
$8A_1$	$9,10,11,12$	$1,6,8,4$	$\mathbb{Z}_2^4$	4

A_6+A_1	–	2	$\mathbb{Z}$	1
$A_4+A_2+A_1$	–	4	$\mathbb{Z}$	1
A_5+A_2	9	$3,4$	$\mathbb{Z}\times\mathbb{Z}_3$	2
$3A_2+A_1$	9	$4,7$	$\mathbb{Z}\times\mathbb{Z}_3$	2
E_6+A_1	–	7	$\mathbb{Z}$	1
E_7	–	8	$\mathbb{Z}$	1
D_7	–	1	$\mathbb{Z}$	1
D_5+2A_1	9	$6,8$	$\mathbb{Z}\times\mathbb{Z}_2$	2
D_4+3A_1	$9,10$	$1,4,6$	$\mathbb{Z}\times\mathbb{Z}_2^2$	3
$2A_3+A_1$	9	$2,6$	$\mathbb{Z}\times\mathbb{Z}_4$	2

$7A_1$	$9,10,11$	$1,6,8,4$	$\mathbb{Z}\times\mathbb{Z}_2^3$	4
D_6+A_1	9	$1,8$	$\mathbb{Z}\times\mathbb{Z}_2$	2
D_5+A_2	–	6	$\mathbb{Z}$	1
$A_3+A_2+2A_1$	9	$4,6$	$\mathbb{Z}\times\mathbb{Z}_2$	2
D_4+A_3	9	$1,6$	$\mathbb{Z}\times\mathbb{Z}_2$	2
A_3+4A_1	$9,10$	$1,4,8$	$\mathbb{Z}\times\mathbb{Z}_2^2$	3
A_4+A_3	–	5	$\mathbb{Z}$	1
A_5+2A_1	9	$1,4$	$\mathbb{Z}\times\mathbb{Z}_2$	2
$[A_7]'$	–	3	$\mathbb{Z}$	1
$[A_7]''$	9	$1,2$	$\mathbb{Z}\times\mathbb{Z}_2$	2

$3A_2$	9	$3,4,7$	$\mathbb{Z}^2\times\mathbb{Z}_3$	3
E_6	–	$4,8$	$\mathbb{Z}^2$	2
D_6	–	$1,8$	$\mathbb{Z}^2$	2
D_4+2A_1	9	$1,6,8$	$\mathbb{Z}^2\times\mathbb{Z}_2$	3
$[2A_3]'$	–	$3,5$	$\mathbb{Z}^2$	2
$[2A_3]''$	9	$1,2,6$	$\mathbb{Z}^2\times\mathbb{Z}_2$	3
D_5+A_1	–	$1,7$	$\mathbb{Z}^2$	2
A_3+3A_1	9	$1,4,6$	$\mathbb{Z}^2\times\mathbb{Z}_2$	3
D_4+A_2	–	$1,6$	$\mathbb{Z}$	2

$6A_1$	$9,10$	$1,4,6,8$	$\mathbb{Z}^2\times\mathbb{Z}_2^2$	4
A_2+4A_1	9	$4,6,8$	$\mathbb{Z}^2\times\mathbb{Z}_2$	3
A_4+2A_1	–	$1,4$	$\mathbb{Z}^2$	2
A_6	–	$1,3$	$\mathbb{Z}^2$	2
$A_3+A_2+A_1$	–	$4,8$	$\mathbb{Z}^2$	2
$[A_5+A_1]'$	–	$2,3$	$\mathbb{Z}^2$	2
$[A_5+A_1]''$	9	$1,2,4$	$\mathbb{Z}^2\times\mathbb{Z}_2$	3
A_4+A_2	–	$3,4$	$\mathbb{Z}^2$	2
$2A_2+2A_1$	–	$4,6$	$\mathbb{Z}^2$	2

D_5	–	$1,7,8$	$\mathbb{Z}^3$	3
$[A_3+2A_1]'$	–	$2,3,5$	$\mathbb{Z}^3$	3
$[A_3+2A_1]''$	9	$1,2,4,6$	$\mathbb{Z}^3\times\mathbb{Z}_2$	4
A_3+A_2	–	$3,5,8$	$\mathbb{Z}^3$	3
A_5	–	$1,2,8$	$\mathbb{Z}^3$	3
$5A_1$	9	$1,4,6,8$	$\mathbb{Z}^3\times\mathbb{Z}_2$	4
A_4+A_1	–	$1,3,4$	$\mathbb{Z}^3$	3
D_4+A_1	–	$1,6,7$	$\mathbb{Z}^3$	3
A_2+3A_1	–	$1,4,6$	$\mathbb{Z}^3$	3
$2A_2+A_1$	–	$2,3,6$	$\mathbb{Z}^3$	3

D_4	–	$1,6,7,8$	$\mathbb{Z}^4$	4
$[4A_1]'$	–	$2,3,5,7$	$\mathbb{Z}^4$	4
$[4A_1]''$	9	$1,2,4,6,8$	$\mathbb{Z}^4\times\mathbb{Z}_2$	5
A_2+2A_1	–	$1,2,4,6$	$\mathbb{Z}^4$	4
$2A_2$	–	$1,2,3,6$	$\mathbb{Z}^4$	4
A_3+A_1	–	$1,2,3,5$	$\mathbb{Z}^4$	4
A_4	–	$1,2,3,4$	$\mathbb{Z}^4$	4
A_3	–	$1,2,3,4,5$	$\mathbb{Z}^5$	5
A_2+A_1	–	$1,2,3,4,6$	$\mathbb{Z}^5$	5
$3A_1$	–	$1,2,3,5,7$	$\mathbb{Z}^5$	5

A_2	–	$1,...,6$	$\mathbb{Z}^6$	6
$2A_1$	–	$1,...,5,7$	$\mathbb{Z}^6$	6

A_1	–	$1,...,7$	$\mathbb{Z}^7$	7

Table 3.5 for F_4:

L_1	I	J	L/L_1	k_1
F_4	–	–	$\{0\}$	0
B_4	5	1	$\mathbb{Z}_2$	1
$A_3+\widetilde{A}_1$	5	2	$\mathbb{Z}_4$	1
$A_2+\widetilde{A}_2$	5	3	$\mathbb{Z}_3$	1
C_3+A_1	5	4	$\mathbb{Z}_2$	1
D_4	5, 6	1, 2	$\mathbb{Z}_2^2$	2
B_2+2A_1	5, 6	1, 4	$\mathbb{Z}_2^2$	2
$4A_1$	5, 6, 7	1, 2, 4	$\mathbb{Z}_2^3$	3
B_3	–	1	$\mathbb{Z}$	1
B_2+A_1	5	1, 4	$\mathbb{Z}\times\mathbb{Z}_2$	2
$A_2+\widetilde{A}_1$	2	–	$\mathbb{Z}$	1
A_3	5	1, 2	$\mathbb{Z}\times\mathbb{Z}_2$	2
$2A_1+\widetilde{A}_1$	5	2, 4	$\mathbb{Z}\times\mathbb{Z}_2$	2
$A_1+\widetilde{A}_2$	–	3	$\mathbb{Z}$	1
C_3	–	4	$\mathbb{Z}$	1
$3A_1$	5, 6	1, 2, 4	$\mathbb{Z}\times\mathbb{Z}_2^2$	3
A_2	–	1, 2	$\mathbb{Z}^2$	2
B_2	–	1, 4	$\mathbb{Z}^2$	2
$A_1+\widetilde{A}_1$	–	2, 3	$\mathbb{Z}^2$	2
$2A_1$	5	1, 2, 4	$\mathbb{Z}^2\times\mathbb{Z}_2$	3
$\widetilde{A}_2$	–	3, 4	$\mathbb{Z}^2$	2
$\widetilde{A}_1$	–	2, 3, 4	$\mathbb{Z}^3$	3
A_1	–	1, 2, 3	$\mathbb{Z}^3$	3

Table 3.6 for G_2:

L_1	I	J	L/L_1	k_1
G_2	–	–	$\{0\}$	0
A_2	3	1	$\mathbb{Z}_3$	1
$A_1+\widetilde{A}_1$	3	2	$\mathbb{Z}_2$	1
$\widetilde{A}_1$	–	1	$\mathbb{Z}$	1
A_1	–	2	$\mathbb{Z}$	1

Remark 3.7. (i) Let $(L,(.,.),\Phi)$ be an irreducible root lattice. Theorem 3.6 (a)+(b)+(d) tells the following. There is an almost 1:1 correspondence between the set of isomorphism classes of pairs $((L,(.,.),\Phi),L_1)$ with L_1 a subroot lattice and the set of unions of Dynkin diagrams which are obtained by iterations of the graphical versions of the steps (BDdS1) and (BDdS2) in remark 3.5 (iv), namely

(BDdS1): Go from one Dynkin diagram to the extended Dynkin diagram and erase an arbitrary vertex.

(BDdS2): Erase an arbitrary vertex.

The only exceptions are the pairs $[H]'$ and $[H]''$ discussed in theorem 3.6 (d). They have the same Dynkin diagrams.

(ii) In table 11 in [Dy57, ch. II] there are two misprints. A_6+A_2 has to be replaced by E_6+A_2. And one of the two A_7+A_1 has to be replaced by E_7+A_1.

In the cases of the series A_n, B_n, C_n and D_n, one can see the subroot lattices also in a different way, by associating a graph to a generating set $A\subset\Phi_1$ of a subroot lattice L_1. This works also in the case of the series BC_n and will give the proof of the statements in theorem 3.6 for BC_n. The graphs are defined as follows.

Definition 3.8. *Let $(L,(.,.),\Phi)$ be a* p.n. root lattice *in theorem 3.2 (d) of one of the types A_n, B_n, C_n, BC_n, D_n. Let $A = \{\alpha_1, ..., \alpha_l\} \subset \Phi$ be a nonempty subset. It defines a* p.n. subroot lattice $L_1 = \sum_{i=1}^{l} \mathbb{Z}\cdot\alpha_i$. *A graph $\mathcal{G}(A)$ with or without markings of the vertices and with one or two types of edges is defined as follows.*

(a) L of type A_n: The graph $\mathcal{G}(A)$ has $n+1$ vertices which are labelled $1, ..., n+1$. It has l edges. A root $\alpha \in A$ with $\alpha = \pm(e_i - e_j)$ gives an edge between the vertices i and j. So, if $e_i - e_j$ and $e_j - e_i$ are in A, there are two edges between the vertices i and j. The same applies in the cases (b) and (c).

*(b) L of type BC_n: The graph $\mathcal{G}(A)$ has n vertices which are labelled $1, ..., n$. Any root $\pm e_i$ in A leads to a marking of the vertex i which is called a **short marking** (and which may be represented by a circle around the vertex). Any root $\pm 2e_i$ in A leads to a marking of the vertex i which is called a **long marking** (and which may be represented by a square around the vertex). So, depending on how many of the roots $\pm e_i$ and $\pm 2e_i$ are in A, the vertex i has between 0 and 4 markings. Any root $\pm(e_i - e_j)$ gives a normal edge between the vertices i and j. Any root $\pm(e_i + e_j)$ gives a dotted edge between the vertices i and j. So, between the vertices i and j there are between 0 and 4 edges.*

(c) L of type B_n or C_n or D_n: The graph is defined as in the case of type BC_n. (In the case of B_n there are no long markings, in the case of C_n there are no short markings, in the case of D_n there are no markings at all).

The following lemma is obvious.

Lemma 3.9. *Consider the same data as in definition 3.8. The orthogonal irreducible summands of the subroot lattice L_1 can be read off from the graph $\mathcal{G}(A)$ as follows. Each of the following subgraphs yields a summand, which is generated by the roots which contribute via markings or edges to this subgraph.*

A_k : *A component of $\mathcal{G}(A)$ which has no markings and in which any cycle has an even number of dotted edges yields a summand of type A_k. Here $k+1$ is the number of vertices of the component. (An isolated vertex with no markings yields thus the summand $A_0 = \{0\}$).*

B_k ***or*** BC_k : *The union of all components of $\mathcal{G}(A)$ which contain a vertex with a short marking yields a summand of type B_k if L is of type B_n and a summand of type BC_k if L is of type BC_n. Here k is the number of vertices of the union of these* *components. If this union is empty, we write $B_0(=\{0\})$ if L is of type B_n and $BC_0(=\{0\})$ if L is of type BC_n.*

C_k : *A component of $\mathcal{G}(A)$ which does not contain a vertex with a short marking, but which contains a cycle with an odd number of dotted edges or which contains a vertex with a long marking yields a summand of type C_k if L is of type C_n or BC_n. Here k is the number of vertices of this component.*

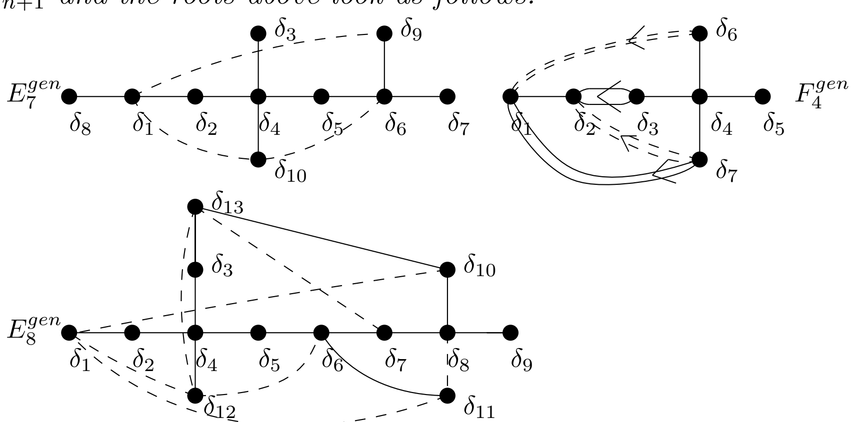
D_k : *A component of $\mathcal{G}(A)$ which does not contain a vertex with a marking, but which contains a cycle with an odd number of dotted edges yields a summand of type D_k if L is of type B_n or D_n. Here k is the number of vertices of this component.*

The statements in theorem 3.6 for the cases A_n, B_n, C_n, BC_n and D_n follow easily from this lemma and from the structure of the Weyl group, which was described in remark 3.3 (iii). For the calculation of the quotients L/L_1 in theorem 3.11, we need in the cases E_6, E_7, E_8, F_4, G_2 a concrete subroot lattice L_1 for each isomorphism class of pairs (L, L_1). This is found in lemma 3.10 by carrying out the recipe with the steps (BDdS1) and (BDdS2).

Lemma 3.10. *Let L be an irreducible root lattice in theorem 3.2 (d) of one of the types E_6, E_7, E_8, F_4, G_2. Additionally to the roots $\delta_1, ..., \delta_{n+1}$ which are defined in remark 3.3 (v), the following roots are considered.*

$$\begin{array}{r|l} E_7 & \delta_9 = e_6 + e_7,\ \delta_{10} = e_4 + e_5. \\ E_8 & \delta_{10} = e_7 + e_8,\ \delta_{11} = e_5 + e_6,\ \delta_{12} = e_3 + e_4, \\ & \delta_{13} = \frac{1}{2}(-e_1 + \sum_{i=2}^{5} e_i - \sum_{i=6}^{8} e_i). \\ F_4 & \delta_6 = e_3 + e_4,\ \delta_7 = -e_1 - e_2. \end{array} \tag{3.18}$$

(a) In the cases E_7, E_8 and F_4, the generalized Dynkin diagrams which take into account the roots $\delta_1, ..., \delta_{n+1}$ and the roots above look as follows.



(The edges which are not horizontal or vertical will be irrelevant except for the dotted edge between δ_1 and δ_{10} in the Dynkin diagram of E_8. It will be used once, in the construction of a subroot lattice of type $2A_3+2A_1$.)

(b) The second and third column in the tables 3.2 - 3.6 encode a realization of the recipe in remark 3.5 (iv), in the following way.

Suppose in the tables 3.2 - 3.6 in the line for one subroot lattice, I is given as the sequence $i_1,...,i_r$ of numbers and J is given as the sequence $j_1,...,j_s$ of numbers, with $0\leq r\leq s$. One carries out r steps (BDdS1): In the k-th step one adds the root δ_{i_k} and then erases the root δ_{j_k}. Afterwards one carries out $s-r$ steps (BDdS2): One erases the roots $\delta_{j_{s-r+1}},...,\delta_{j_s}$. This leads to a subroot lattice of the type indicated in the first column.

(c) The roots δ_k for $k\geq n+1$ are linear combinations of the roots $\delta_1,...,\delta_n$. The linear relations are as follows.

$$
\begin{array}{ll}
\mathbf{E_6}: & 0=\delta_1+2\delta_2+2\delta_3+3\delta_4+2\delta_5+\delta_6+\delta_7.\\
\mathbf{E_7}: & 0=2\delta_1+3\delta_2+2\delta_3+4\delta_4+3\delta_5+2\delta_6+\delta_7+\delta_8,\\
& 0=\delta_2+\delta_3+2\delta_4+2\delta_5+2\delta_6+\delta_7+\delta_9,\\
& 0=\delta_2+\delta_3+2\delta_4+\delta_5+\delta_{10}.\\
\mathbf{E_8}: & 0=2\delta_1+4\delta_2+3\delta_3+6\delta_4+5\delta_5+4\delta_6+3\delta_7+2\delta_8+\delta_9,\\
& 0=2\delta_1+3\delta_2+2\delta_3+4\delta_4+3\delta_5+2\delta_6+\delta_7-\delta_{10},\\
& 0=\delta_2+\delta_3+2\delta_4+2\delta_5+2\delta_6+\delta_7+\delta_{11},\\
& 0=\delta_2+\delta_3+2\delta_4+\delta_5+\delta_{12},\\
& 0=\delta_1+2\delta_2+2\delta_3+3\delta_4+2\delta_5+\delta_6+\delta_{13}.\\
\mathbf{F_4}: & 0=2\delta_1+4\delta_2+3\delta_3+2\delta_4+\delta_5,\\
& 0=2\delta_1+2\delta_2+\delta_3-\delta_6,\\
& 0=2\delta_2+\delta_3-\delta_7.\\
\mathbf{G_2}: & 0=3\delta_1+2\delta_2+\delta_3.
\end{array}
$$

Proof. (a), (c) The proof of the lemma is tedious as there are many cases. The parts (a) and (c) are completely elementary, as is most of part (b). What's left we deal with now.

(b) Here one has to check not only that the result has the correct Dynkin diagram, but also that the steps (BDdS1) work, i.e. that one has after adding a root an extended Dynkin diagram and that then a root of this extended Dynkin diagram is erased. The details are left to the reader.

The only nontrivial part concerns the subroot lattices of types $[H]'$ and $[H]''$ in the cases

E_7 and E_8. One sees that in the cases of E_k, $k = 7, 8$, the constructed subroot lattices of types $[H]'$ are contained in the subroot lattice $\bigoplus_{i\in\{1,2,4,..,k+1\}} \mathbb{Z}\delta_i$ of type A_k. The constructed subroot lattices of type $[H]''$ contain in the case E_7 the roots $\delta_3, \delta_5, \delta_7$ and in the case E_8 the roots $\delta_3, \delta_5, \delta_7, \delta_9$. The following claim shows that the constructed subroot lattices $[H]''$ are not contained in subroot lattices of type A_7 respectively A_8.

Claim: (i) Let L be the lattice of type E_7 in theorem 3.2 (d). There is no subroot lattice L_1 of type A_7 with $\delta_3, \delta_5, \delta_7 \in L_1$.

(ii) Let L be the lattice of type E_8 in theorem 3.2 (d). There is no subroot lattice L_1 of type A_8 with $\delta_3, \delta_5, \delta_7, \delta_9 \in L_1$.

Proof of the claim. **(i)** Suppose that L_1 is a subroot lattice of type A_7 with $\delta_3, \delta_5, \delta_7 \in L_1$. These three roots generate a subroot lattice $L_2 \subset L_1$ of type $3A_1$. By theorem 3.6, up to isomorphism there is only one pair of type $(A_7, 3A_1)$. Therefore a root $\alpha \in L_1$ with $(\alpha, \delta_3) = -1, (\alpha, \delta_5) = 0, (\alpha, \delta_7) = 0$ exists. But now observe $\delta_3 = e_2 - e_3, \delta_5 = e_4 - e_5, \delta_7 = e_6 - e_7$. Neither the roots in $\Phi(L)$ of type $\pm e_i \pm e_j$ nor the roots in $\Phi(L)$ of type $\frac{1}{2}\sum_{i=1}^{8} \varepsilon_i e_i$ can serve as a root α. Contradiction.

Part **(ii)** is analogous to (i). □

This finishes the proof of lemma 3.10. □

Theorem 3.11. *(a) Let $(L, (.,.), \Phi)$ be an irreducible p.n. root lattice, and let L_1 be a p.n. subroot lattice. The third column in table 3.1 and the fourth column in the tables 3.2 - 3.6 gives the isomorphism class of the quotient group L/L_1.*

(b) Let $(L, (.,.), \Phi)$ be a p.n. root lattice, and let L_1 be a p.n. subroot lattice. Define the numbers

$$k_1(L, L_1) := \min\{k \,|\, \text{the group } L/L_1 \text{ has } k \text{ generators}\}, \tag{3.19}$$

$$k_2(L, L_1) := \min\{k \,|\, \exists\ \alpha_1, ..., \alpha_k \in \Phi \text{ s.t. } L = L_1 + \sum_{i=1}^{k} \mathbb{Z}\cdot\alpha_j\}, \tag{3.20}$$

$$k_3(L, L_1) := \min\{k \,|\, L_1 \text{ can be constructed with } k \text{ of the steps (BDdS1) and (BDdS2)}\}. \tag{3.21}$$

Then

$$k_1(L, L_1) = k_2(L, L_1) = k_3(L, L_1). \tag{3.22}$$

The numbers are additive, i.e. if $L = L_2 + L_3$ *and* $L_1 = L_4 + L_5$ *and* $L_2 \supset L_4, L_3 \supset L_5$ *then*

$$k_1(L, L_1) = k_1(L_2, L_4) + k_1(L_3, L_5). \tag{3.23}$$

The last column of the tables 3.1 - 3.6 gives the numbers $k_1(L, L_1)$ *for the pairs with* L *irreducible. Minimal sequences of the steps (BDdS1) and (BDdS2) for the cases with* L *of type* E_6, E_7, E_8, F_4, G_2 *are given in the second and third column of the tables 3.2 - 3.6 (see lemma 3.10).*

Proof. **(a)** First we treat the cases A_n, B_n, C_n, BC_n and D_n. Let A and $\mathcal{G}(A)$ be as in definition 3.8 and lemma 3.9, and let $L_1 \subset L$ be the corresponding subroot lattice. Let $\mathcal{G}(A) = \bigcup_{k\in K} \mathcal{G}_k$ (with $1 \notin K$) be the decomposition into subgraphs $\mathcal{G}_k$ with the properties in lemma 3.9, let L_k be the subroot lattice which corresponds to the subgraph $\mathcal{G}_k$, and let

$$V_k := \bigoplus_{i \text{ is a vertex in } \mathcal{G}_k} \mathbb{Z} \cdot e_i.$$

Then

$$\bigoplus_{k\in K} V_k = \bigoplus_{i=1}^{m} \mathbb{Z} \cdot e_i \supset L \supset L_1 = \bigoplus_{k\in K} L_k$$

(with $m = n + 1$ for A_n and $m = n$ else) and

$$L/L_1 \subset \Big(\bigoplus_{i=1}^{m} \mathbb{Z} \cdot e_i\Big)/L_1 \;\cong\; \bigoplus_{k\in K} V_k/L_k.$$

The following table lists in the second and fourth line the isomorphism classes of the quotients in the first column.

L	A_n	B_n	C_n	BC_n	D_n
$\bigoplus_{i=1}^m \mathbb{Z}e_i/L$	$\mathbb{Z}$	$\{0\}$	$\mathbb{Z}_2$	$\{0\}$	$\mathbb{Z}_2$
L_k	A_l	B_l	C_l	BC_l	D_l
V_k/L_k	$\mathbb{Z}$	$\{0\}$	$\mathbb{Z}_2$	$\{0\}$	$\mathbb{Z}_2$

Finally, in the cases C_n and D_n, for any $k \in K$

$$L \not\subset L_k \oplus \bigoplus_{j\in K-\{k\}} V_k.$$

Therefore L/L_1 has the isomorphism type claimed in the table 3.1.

Now we treat the cases E_6, E_7, E_8, F_4, G_2. Let $L_1 \subset L$ be one of the subroot lattices constructed in lemma 3.10 using the data in the tables 3.2 - 3.6. Let $\delta_{i_1}, ..., \delta_{i_r}$ respectively $\delta_{j_1}, ..., \delta_{j_s}$ with $0 \leq r \leq s$ be the roots in one line in the second respectively third column of these tables. Then

$$L/L_1 = \frac{(\bigoplus_{k=1}^s \mathbb{Z} \cdot \delta_{j_k}) + L_1}{L_1} \cong \frac{\bigoplus_{k=1}^s \mathbb{Z} \cdot \delta_{j_k}}{(\bigoplus_{k=1}^s \mathbb{Z} \cdot \delta_{j_k}) \cap L_1}.$$

The denominator of the right-hand side is a $\mathbb{Z}$-lattice of rank r (because $\operatorname{rank} L_1 = n - s + r$) and is generated by parts of those relations in lemma 3.10 (c) which express the roots $\delta_{i_1}, ..., \delta_{i_r}$ as linear combinations of the roots $\delta_1, ..., \delta_n$.

We give one example: (L, L_1) of type $(E_8, D_4 + 3A_1)$, then $(\delta_{i_1}, ..., \delta_{i_r}) = (\delta_9, \delta_{10})$ and $(\delta_{j_1}, ..., \delta_{j_s}) = (\delta_1, \delta_4, \delta_6)$. The relation for δ_9 gives the element $2\delta_1 + 6\delta_4 + 4\delta_6$ of $(\bigoplus_{k=1}^s \mathbb{Z} \cdot \delta_{j_k}) \cap L_1$, the relation for δ_{10} gives the element $2\delta_1 + 4\delta_4 + 2\delta_6$. Therefore here

$$\frac{L}{L_1} \cong \frac{\mathbb{Z} \cdot \delta_1 \oplus \mathbb{Z} \cdot \delta_4 \oplus \mathbb{Z} \cdot \delta_6}{\mathbb{Z} \cdot (2\delta_1 + 6\delta_4 + 4\delta_6) \oplus \mathbb{Z} \cdot (2\delta_1 + 4\delta_4 + 2\delta_6)} \cong \mathbb{Z} \times \mathbb{Z}_2^2.$$

The calculations for all other cases (L, L_1) are analogous. They are tedious as there are many cases, but elementary.

(b) The additivity of the numbers $k_1(L, L_1), k_2(L, L_1), k_3(L, L_1)$ is obvious. Therefore it is sufficient to prove (3.22) for irreducible L. The last column of the tables 3.1 - 3.6 can be read off from the second to last column immediately. The first of the inequalities

$$k_1(L, L_1) \leq k_2(L, L_1) \leq k_3(L, L_1) \tag{3.24}$$

is obvious. The second inequality follows simply from the fact that in each step (BDdS1) or (BDdS2), one root is erased. In the cases A_n, B_n, C_n, BC_n and D_n, one easily constructs the p.n. subroot lattices in $k_1(L, L_1)$ steps. Therefore then $k_3(L, L_1) \leq k_1(L, L_1)$, and equalities hold in (3.24).

In the cases E_6, E_7, E_8, F_4, G_2 one observes $|J| = k_1(L, L_1)$ in the tables 3.2 - 3.6. In lemma 3.10 $|J|$ steps of type (BDdS1) and (BDdS2) are used. Therefore $k_3(L, L_1) \leq k_1(L, L_1)$, and equalities hold in (3.24). □

3.3 Any generating set of roots contains a $\mathbb{Z}$-basis

The purpose of this section is to prove the next theorem, theorem 3.12. It is crucial in the proof of theorem 3.29. But we extend the proof to an independent result to cases we do not need in this thesis. The proof in the cases A_n, B_n, C_n, BC_n, D_n is an almost trivial application of the graphs in definition 3.8 and lemma 3.9. The proof in the cases E_6, E_7, E_8, F_4, G_2 is more involved.

Theorem 3.12. *Let $(L, (.,.), \Phi)$ be a p.n. root lattice. Let $A \subset \Phi$ be any set of roots which generates the lattice L as a $\mathbb{Z}$-module. Then A contains a $\mathbb{Z}$-basis of L.*

In the case of vector spaces instead of $\mathbb{Z}$-modules, the analogous statement is trivial. For $\mathbb{Z}$-modules it is not true in general, that any generating set contains a basis. For example the lattice $\mathbb{Z}^2$ with standard basis $(1,0),(0,1)$ has the set $\{(1,0),(1,2),(0,3)\}$ as generating set, but any two of these elements generate a proper sublattice.

The rest of this section is devoted to the proof of theorem 3.12. It is obviously sufficient to prove it in the cases where L is an irreducible p.n. root lattice. In the cases A_n, B_n, C_n, BC_n, D_n, theorem 3.12 is an immediate consequence of the following lemma. The lemma follows directly from lemma 3.9.

Lemma 3.13. *Let $(L, (.,.), \Phi)$ be a p.n. root lattice in theorem 3.2 (d) of one of the types A_n, B_n, C_n, BC_n, D_n. Let $L = \{\alpha_1, ..., \alpha_l\} \subset \Phi$ be a nonempty subset. The properties whether A is a generating set of L or a $\mathbb{Z}$-basis of L, will be characterized by properties of the graph $\mathcal{G}(A)$ from definition 3.8.*

(a) L of type A_n:

(i) A generates L as a $\mathbb{Z}$-module $\iff$ $\mathcal{G}(A)$ is connected.

(ii) A is a $\mathbb{Z}$-basis of L $\iff$ $\mathcal{G}(A)$ is a tree.

(b) L of type B_n:

(i) A generates L as a $\mathbb{Z}$-module $\iff$ each component of $\mathcal{G}(A)$ contains at least one vertex with an (automatically short) marking.

(ii) A is a $\mathbb{Z}$-basis of L $\iff$ each component of $\mathcal{G}(A)$ is a tree and contains exactly one marked vertex, and the vertex has only one (automatically short) marking.

(c) L of type C_n:

(i) A generates L as a $\mathbb{Z}$-module $\iff$ $\mathcal{G}(A)$ is connected, and it contains a marking (automatically long) or a cycle with an odd number of dotted edges.

(ii) A is a $\mathbb{Z}$-basis of L $\iff$ either $\mathcal{G}(A)$ is a tree and contains exactly one marked vertex and the marking is simple, or $\mathcal{G}(A)$ is connected and contains no marking, but it contains exactly one cycle and the cycle has an odd number of dotted lines.

(d) L of type BC_n:

(i) A generates L as a $\mathbb{Z}$-module $\iff$ each component of $\mathcal{G}(A)$ contains at least one vertex with a short marking.

(ii) A is a $\mathbb{Z}$-basis of L $\iff$ each component of $\mathcal{G}(A)$ is a tree and contains exactly one marked vertex, and there is only one marking, and the marking is short.

(e) L of type D_n:

(i) A generates L as a $\mathbb{Z}$-module $\iff$ $\mathcal{G}(A)$ is connected, and it contains a cycle with an odd number of dotted edges.

(ii) A is a $\mathbb{Z}$-basis of L $\iff$ $\mathcal{G}(A)$ is connected and contains exactly one cycle, and the cycle has an odd number of dotted lines.

It rests to prove theorem 3.12 in the cases G_2, F_4, E_6, E_7, E_8. Let L be the root lattice of one of these types in theorem 3.2 (d). In each of these cases, it is sufficient to consider a generating set $A = \{\alpha_1, ..., \alpha_{n+1}\} \subset \Phi$ with $n+1$ elements (where n is the rank of the root lattice). The cases of bigger generating sets can be reduced to the case of such a set by an easy inductive argument.

There is an up to the sign unique linear combination

$$0 = \sum_{i=1}^{n+1} \lambda_i \alpha_i \quad \text{with } \lambda_i \in \mathbb{Z},\ \gcd(\lambda_1, ..., \lambda_{n+1}) = 1,\ \text{not all } \lambda_i = 0. \tag{3.25}$$

It has to be shown that an index j with $\lambda_j = \pm 1$ exists.

Denote by $L_i \subset L$, $i \in \{1, ..., n+1\}$, the subroot lattice generated by $A - \{\alpha_i\}$. Then

$$\operatorname{rank} L_i < n \iff \lambda_i = 0. \tag{3.26}$$

If this holds for some i then by induction on the rank of the lattice one can conclude that $A - \{\alpha_i\}$ contains a $\mathbb{Z}$-basis of this subroot lattice. Then this $\mathbb{Z}$-basis together with α_i forms a $\mathbb{Z}$-basis of L.

Thus suppose that all $\lambda_i \notin \{0, \pm 1\}$. Then

$$[L : L_i] = |\lambda_i|, \quad L_i + \mathbb{Z}\alpha_i = L, \quad k_2(L, L_i) = 1.$$

A priori, there are 20 possible cases in the tables 3.2 - 3.6,

L	L_i	$\lvert\lambda_i\rvert$	L	L_i	$\lvert\lambda_i\rvert$
F_4	B_4	2	G_2	$A_1+\widetilde{A}_1$	2
	C_3+A_1	2		A_2	3
	$A_2+\widetilde{A}_2$	3	E_8	D_8	2
	$A_3+\widetilde{A}_1$	4		E_7+A_1	2
E_6	A_5+A_1	2		E_6+A_2	3
	$3A_2$	3		A_8	3
E_7	D_6+A_1	2		D_5+A_3	4
	A_7	2		A_7+A_1	4
	A_5+A_2	3		$2A_4$	5
	$2A_3+A_1$	4		$A_5+A_2+A_1$	6

In the cases G_2, F_4, E_6, E_7, the only possible values for $|\lambda_i|$ are in $\{2,3,4\}$. The condition $\gcd(\lambda_1,...,\lambda_{n+1})=1$ tells that at least one $j\in\{1,...,n+1\}$ with $|\lambda_j|=3$ exists.

The following more complicated argument gives the same conclusion in the case E_8. Assume in the case E_8 that all $|\lambda_i|$ are in $\{2,4,5,6\}$. Define the decomposition

$$I_1 := \{i\,|\,2|\lambda_i\},\quad I_2 := \{i\,|\,\lambda_i=\pm 5\}$$

of $\{1,...,n+1\}$ into two disjoint subset. Because of $\gcd(\lambda_1,...,\lambda_{n+1})=1$, both are nonempty. Define the subroot lattices

$$\widetilde{L}_1 := \sum_{i\in I_1}\mathbb{Z}\alpha_i,\qquad \widetilde{L}_2 := \sum_{i\in I_2}\mathbb{Z}\alpha_i.$$

Then $c:=\gcd(\lambda_i\,|\,i\in I_1)\in\{2,4,6\}$, and $c^{-1}\sum_{i\in I_1}\lambda_i\alpha_i$ is a primitive vector in $\widetilde{L}_1$, and $5^{-1}\sum_{i\in I_2}\lambda_i\alpha_i$ is a primitive vector in $\widetilde{L}_2$. But

$$\sum_{i\in I_1}\lambda_i\alpha_i = -\sum_{i\in I_2}\lambda_i\alpha_i.$$

Therefore the order of the torsion part of $L/\widetilde{L}_1$ is divisible by 5. But table 3.4 contains only one type of subroot lattices with this property, the type $2A_4$. Therefore $|I_1|=8, |I_2|=1, I_2=\{j_0\}$ for some index j_0, and $\frac{5}{2}\alpha_{j_0}\in\widetilde{L}_1\subset L$, which is impossible. Therefore the assumption above that all $|\lambda_i|$ are in $\{2,4,5,6\}$ was wrong.

In the cases G_2, F_4, E_6, E_7 the type of the subroot lattices L_j with $[L:L_j]=3$ is unique, in the case E_8 there are two possibilities,

L	G_2	F_4	E_6	E_7	E_8
L_i	A_2	$A_2+\widetilde{A}_2$	$3A_2$	A_5+A_2	$E_6+A_2,\ A_8$

By renumbering the roots, we can assume $[L:L_{n+1}]=3$.

The case $\mathbf{G_2}$: The roots α_1 and α_2 generate an A_2 lattice and thus are long. Therefore α_3 is short. At least one of α_1 and α_2 is not orthogonal to α_3. That root and α_3 form a $\mathbb{Z}$-basis of L.

The cases $\mathbf{F_4}, \mathbf{E_6}, \mathbf{E_7}$ and the case $(\mathbf{E_8}, \mathbf{E_6}+\mathbf{A_2})$: The sublattice $L_{n+1}\subset L$ (with $\mathbb{Z}$-basis $\alpha_1,...,\alpha_n$) contains one orthogonal summand $\widetilde{L}_1$ of type A_2. Suppose that α_1 and α_2 form a $\mathbb{Z}$-basis of this lattice $\widetilde{L}_1$. Then $\widetilde{L}_2:=\sum_{i\in\{1,2,n+1\}}\mathbb{Z}\alpha_i$ is a subroot lattice with $\widetilde{L}_1\subset\widetilde{L}_2\subset L$ and $\widetilde{L}_2\not\subset L_{n+1}$ and $L_{n+1}\not\subset\widetilde{L}_2$. Because of $0=\sum_{i=1}^{n+1}\lambda_i\alpha_i$ and all $\lambda_i\neq 0$ and $n+1>3$, the root α_{n+1} is not in $\widetilde{L}_{1,\mathbb{Q}}$, so the lattice $\widetilde{L}_2$ has rank 3. The sum

$$\sum_{i\in\{1,2,n+1\}}\lambda_i\alpha_i=-\sum_{i=3}^n\lambda_i\alpha_i$$

is in the sum $\widetilde{L}_{1,\mathbb{R}}^\perp\cap L_{n+1}$ of the other orthogonal summands of L_{n+1} and in the rank one $\mathbb{Z}$-lattice $\widetilde{L}_{1,\mathbb{R}}^\perp\cap\widetilde{L}_2$. In fact, it is a generator of this rank one lattice: This is equivalent to $c_1=1$ where

$$c_1:=\gcd(\lambda_1,\lambda_2,\lambda_{n+1}).$$

If $c_1>1$, then $c_1^{-1}\sum_{i\in\{1,2,n+1\}}\lambda_i\alpha_i$ were in the root lattice $\widetilde{L}_2$. But then also $c_1^{-1}\sum_{i=3}^n\lambda_i\alpha_i$ were in the root lattice L_{n+1}, thus c would divide $\lambda_3,...,\lambda_n$. Then $\gcd(\lambda_1,...,\lambda_{n+1})>1$ which is not true.

The root lattice $\widetilde{L}_1$ is either of type A_3 or of type B_3. In both cases, the following lemma gives the claim.

Lemma 3.14. *In both cases, at least one of λ_1 and λ_2 is equal to ± 1.*

Proof. **(a)** The case $\widetilde{L}_2$ of type A_3: Embed $\widetilde{L}_{2,\mathbb{R}}$ as usual into a Euclidean space $\mathbb{R}_4$ with ON-basis $\widetilde{e}_1,\widetilde{e}_2,\widetilde{e}_3,\widetilde{e}_4$ such that $\Phi(\widetilde{L}_2)=\{\pm(\widetilde{e}_i-\widetilde{e}_j)\,|\,1\leq i<j\leq 4\}$ and $\Phi(\widetilde{L}_1)=\{\pm(\widetilde{e}_i-\widetilde{e}_j),|\,1\leq i<j\leq 3\}$. A generator of the rank one $\mathbb{Z}$-lattice $\widetilde{L}_{1,\mathbb{R}}^\perp\cap\widetilde{L}_2$ is obviously $\widetilde{e}_1+\widetilde{e}_2+\widetilde{e}_3-3\widetilde{e}_4$. Thus

$$\widetilde{e}_1+\widetilde{e}_2+\widetilde{e}_3-3\widetilde{e}_4=\pm\sum_{i\in\{1,2,n+1\}}\lambda_i\alpha_i.$$

Lemma 3.13 (a) applies to the graph $\mathcal{G}(\{\alpha_1,\alpha_2,\alpha_{n+1}\})$. The graph is a tree, as $\alpha_1,\alpha_2,\alpha_{n+1}$ is a $\mathbb{Z}$-basis of $\widetilde{L}_2$. At least two of the four vertices $\widetilde{e}_1,\widetilde{e}_2,\widetilde{e}_3,\widetilde{e}_4$ are leaves, so at least one of the three vertices $\widetilde{e}_1,\widetilde{e}_2,\widetilde{e}_3$ is a leaf. Let α_j with $j\in\{1,2\}$ give the edge which contains this vertex. The coefficient 1 of this vertex must be equal to $\pm\lambda_j$ because the other two terms in the sum $\pm\sum_{i\in\{1,2,n+1\}}\lambda_i\alpha_i$ have no contribution to the coefficient of this vertex.

(b) The case $\widetilde{L}_2$ **of type** B_3: Embed $\widetilde{L}_{2,\mathbb{R}}$ as usual into a Euclidean space $\mathbb{R}_3$ with ON-basis $\widetilde{e}_1,\widetilde{e}_2,\widetilde{e}_3$ such that $\Phi(\widetilde{L}_2)=\{\pm\widetilde{e}_i\pm\widetilde{e}_j\}\cup\{\pm e_i\}$ and $\Phi(\widetilde{L}_1)=\{\pm(\widetilde{e}_i-\widetilde{e}_j),\,|\,1\leq i<j\leq 3\}$. A generator of the rank one $\mathbb{Z}$-lattice $\widetilde{L}_{1,\mathbb{R}}^{\perp}\cap\widetilde{L}_2$ is obviously $\widetilde{e}_1+\widetilde{e}_2+\widetilde{e}_3$. Thus

$$\widetilde{e}_1+\widetilde{e}_2+\widetilde{e}_3=\pm\sum_{i\in\{1,2,n+1\}}\lambda_i\alpha_i. \tag{3.27}$$

Lemma 3.13 applies to the graph $\mathcal{G}(\{\alpha_1,\alpha_2,\alpha_{n+1}\})$. As α_1 and α_2 are long roots, the graph is a tree with one marked vertex. Then at least one vertex $\widetilde{e}_j$ of the 3 vertices $\widetilde{e}_1,\widetilde{e}_2,\widetilde{e}_3$ has no marking and is a leaf. Then $\lambda_j=\pm1$ for the same reason as in (a). □

The case $(\mathbf{E_8},\mathbf{A_8})$: We can choose a root basis $\delta_1,...,\delta_8$ of L and an additional root δ_9 such that they give rise to the extended Dynkin diagram in remark 3.3 (v) and such that the subroot lattice L_{n+1} is generated by $\delta_1,\delta_2,\delta_4,...,\delta_9$. The roots $\delta_1,...,\delta_9$ satisfy the first of the five relations in lemma 3.10 (c) for E_8. Further, we can embed $L_{\mathbb{R}}=L_{n+1,\mathbb{R}}$ into a Euclidean space $\mathbb{R}^9$ with ON-basis $\widetilde{e}_1,...,\widetilde{e}_9$ such that

$$(\delta_1,\delta_2,\delta_4,...,\delta_9)=(\widetilde{e}_1-\widetilde{e}_2,\widetilde{e}_2-\widetilde{e}_3,\widetilde{e}_3-\widetilde{e}_4,...,\widetilde{e}_8-\widetilde{e}_9). \tag{3.28}$$

Part (a) of the following lemma tells how $\Phi(L)$ can be expressed using the $\widetilde{e}_1,...,\widetilde{e}_9$. Part (b) solves the case (E_8,A_8) and finishes the proof of theorem 3.12.

Lemma 3.15. *(a)*

$$\begin{aligned}
\delta_3 &= \frac{1}{3}\left(-2\sum_{i=1}^{3}\widetilde{e}_i+\sum_{i=4}^{9}\widetilde{e}_i\right), && (3.29)\\
\Phi(A_8) &= \{\pm(\widetilde{e}_i-\widetilde{e}_j)\,|\,1\leq i<j\leq 8\},\\
\Phi(E_8) &= \Phi(A_8)\cup\{\pm\frac{1}{3}\left(-2\sum_{i\in I_1}\widetilde{e}_i+\sum_{i\in I_2}\widetilde{e}_i\right)\,|\,I_1\cup I_2=\{1,...,9\},\\
&\qquad |I_1|=3,|I_2|=6\}. && (3.30)
\end{aligned}$$

(b) Above, at least one of the λ_j with $1\leq j\leq 8$ is equal to ±1.

Proof. (a) (3.29) follows from (3.28) and the relation

$$0 = 2\delta_1 + 4\delta_2 + 3\delta_3 + 6\delta_4 + 5\delta_5 + 4\delta_6 + 3\delta_7 + 2\delta_8 + \delta_9,$$

see lemma 3.10 (c). As $\Phi(E_8)$ contains δ_3, it contains the combination of $\widetilde{e}_i$ on the right-hand side of (3.29). As the Weyl group $W(A_8) \subset W(E_8)$ consists of all permutations of $\widetilde{e}_1, ..., \widetilde{e}_9$, the root system $\Phi(E_8)$ contains the right-hand side of (3.30). Counting the size of the right-hand side, one finds

$$2\binom{9}{2} + 2\binom{9}{3} = 72 + 168 = 240 = |\Phi(E_8)|,$$

thus equality holds in (3.30).

(b) The roots $\alpha_1, ..., \alpha_8$ form a $\mathbb{Z}$-basis of L_{n+1}. The root α_9 must be a root in $\Phi(L) - \Phi(L_{n+1})$, so it must be

$$\alpha_9 = \pm\frac{1}{3}\left(-2\sum_{i=1}^{3}\widetilde{e}_{\pi(i)} + \sum_{i=4}^{9}\widetilde{e}_{\pi(i)}\right) \quad \text{for some } \pi \in S_9.$$

The graph $\mathcal{G}(\{\alpha_1, ..., \alpha_8\})$ is a tree by lemma 3.13 (a).

1st case, at least one of the roots $\widetilde{e}_{\pi(4)}, ..., \widetilde{e}_{\pi(9)}$ is a leaf in this graph: Let α_j be the only edge which contains this leaf. Then $\lambda_j\alpha_j$ contains the only contribution to the leaf, in the right-hand side of the following formula,

$$-2\sum_{i=1}^{3}\widetilde{e}_{\pi(i)} + \sum_{i=4}^{9}\widetilde{e}_{\pi(i)} = \pm 3\alpha_9 = \pm\sum_{j=1}^{8}\lambda_j\alpha_j.$$

Thus then $\lambda_j = \pm 1$.

2nd case, none of the roots $\widetilde{e}_{\pi(4)}, ..., \widetilde{e}_{\pi(9)}$ is a leaf in the graph: Then there are two or three leaves, and they form a subset of the set $\{\widetilde{e}_{\pi(1)}, \widetilde{e}_{\pi(2)}, \widetilde{e}_{\pi(3)}\}$. For one of these leaves, the number of vertices on the path from this leaf to the branching vertex (in the case of three leaves) or to the unique inner vertex which is in $\{\widetilde{e}_{\pi(1)}, \widetilde{e}_{\pi(2)}, \widetilde{e}_{\pi(3)}\}$, is maximal. Then the first two edges within this path, which starts at the leaf, have the coefficients λ_j with values $\pm 2, \pm 1$. So ± 1 arises. □

This finishes the proof of theorem 3.12 □

3.4 Reduced presentations of Weyl group elements

Carter studied and classified the conjugacy classes of the elements of the Weyl groups of the irreducible root lattices. Here we will review a part of his results and extend them. Crucial are the in the inhomogeneous cases new, notions of quasi Coxeter elements and strict quasi Coxeter elements. The control of these elements reduces the classification of conjugacy classes of Weyl group elements to the control of subroot lattices in section (3.2). Essential will be the number k_4, defined as the minimal way of extending the "best" subroot lattice, generated by a presentation of a Weyl group element, to the full lattice.

But first, some definitions will be given.

Definition 3.16. *Let $(L,(.,.),\Phi)$ be a p.n. root lattice with Weyl group W.*

(a) For any element $w\in W$ any tuple $(\alpha_1,...,\alpha_k)\in\Phi^k$ with $k\in\mathbb{Z}_{\geq 0}$ and

$$w = s_{\alpha_1}\circ ...\circ s_{\alpha_k} \tag{3.31}$$

*is a **presentation of** w. Its **length** is $k\in\mathbb{Z}_{\geq 0}$. The length $l(w)\in\mathbb{Z}_{\geq 0}$ of w is the minimum of the lengths of all presentations. A presentation with $k=l(w)$ is called **reduced**. The subroot lattice of a presentation $(\alpha_1,...,\alpha_k)$ is $L_1:=\sum_{i=1}^k\mathbb{Z}\cdot\alpha$. The **index** of the presentation is the index $[L\cap L_{1,\mathbb{Q}}:L_1]\in\mathbb{Z}_{\geq 1}$ of the subroot lattice L_1.*

*(b) An element w is of **maximal length** if $l(w)=n:=$ the rank of the root lattice.*

(c) For any element $w\in W$ and any $\lambda\in S^1$ define

$$\begin{aligned} V_\lambda(w) &:= \ker(w-\lambda\cdot\mathrm{id})\subset L_{\mathbb{C}}, & (3.32)\\ V_{\neq 1}(w) &:= \bigoplus_{\lambda\neq 1} V_\lambda(w)\supset V_{\neq 1,\mathbb{R}}:=L_{\mathbb{R}}\cap V_{\neq 1}(w), & (3.33)\end{aligned}$$

and analogously $V_{\neq 1,\mathbb{Q}}(w), V_{\neq 1,\mathbb{Z}}(w)$ $V_{1,\mathbb{R}}(w), V_{1,\mathbb{Q}}(w), V_{1,\mathbb{Z}}(w)$. Of course $V_{\neq 1,\mathbb{R}}=V_{1,\mathbb{R}}^\perp$.

Lemma 3.17. *Let $(L,(.,.),\Phi)$ be a p.n. root lattice with Weyl group W.*

(a) ([Ca72, Lemmata 2 and 3]) A presentation $(\alpha_1,...,\alpha_k)$ of an element $w\in W$ is reduced if and only if $\alpha_1,...,\alpha_k$ are linearly independent (in $L_{\mathbb{Q}}$). The subroot lattice $L_1\subset L$ of a reduced presentation satisfies

$$\begin{aligned} \bigoplus_{i=1}^{l(w)}\mathbb{Q}\cdot\alpha_i = L_{1,\mathbb{Q}} &= V_{\neq 1,\mathbb{Q}}(w) & (3.34)\\ \text{and especially}\qquad l(w) &= \dim V_{\neq 1,\mathbb{Q}}(w). & (3.35)\end{aligned}$$

So, the subroot lattices of all reduced presentations of w generate the same subspace of $L_{\mathbb{C}}$, and it is $V_{\neq 1}(w)$.

(b) ([Kl83, Satz 3.2], [Vo85, Satz 3.2.3]) If $(L,(.,.),\Phi)$ is a homogeneous root lattice, then all reduced presentations of one element $w\in W$ have the same index.

The following definition of a quasi Coxeter element is in the homogeneous cases due to Voigt [Vo85, Def. 3.2.1] and in the inhomogeneous cases new.

Definition 3.18. *Let $(L,(.,.),\Phi)$ be a p.n. root lattice of rank $n\in\mathbb{Z}_{>0}$ with Weyl group W.*

(a) An element $w\in W$ is a ***quasi Coxeter element*** *if a reduced presentation of w exists whose subroot lattice is the full root lattice L. Of course then it is of* ***maximal length*** *$l(w)=n$.*

(b) An element $w\in W$ is a ***strict quasi Coxeter element*** *if the subroot lattice of any reduced presentation is the full root lattice L. Of course then it is a quasi Coxeter element.*

Remark 3.19. (i) An element w in the Weyl group of a p.n. root lattice has many presentations. Often there are several presentations such that the isomorphisms classes of their subroot lattices are different. In the homogeneous cases at least their indices are equal. But in the inhomogeneous cases, even their indices can differ.

(ii) In a homogeneous root lattice, lemma 3.17 (b) implies that there the notions quasi Coxeter element and strict quasi Coxeter element coincide. But in any irreducible inhomogeneous root lattice, there are quasi Coxeter elements which are not strict quasi Coxeter elements. See theorem 3.21.

(iii) Of course, if $(\alpha_1,...,\alpha_k)$ is a reduced presentation of a Weyl group element w, then w is a quasi Coxeter element in the subroot lattice L_1 of this presentation. And of course, any Weyl group element has a reduced presentation such that it is a strict quasi Coxeter element in the subroot lattice L_1 of this presentation.

(iv) Let $L=\bigoplus_{k\in K}L_k$ be the decomposition of a p.n. root lattice into an orthogonal sum of irreducible p.n. root lattices, and let $w\in W$ be a (strict) quasi Coxeter element. Then it decomposes into a product $\prod_{k\in K}w_k$ of commuting elements $w\in W(L_k)$, and w_k is a (strict) quasi Coxeter element in L_k.

(v) Recall that a Coxeter element in an irreducible root lattice is an element $w\in W$ which has a presentation $(\alpha_1,...,\alpha_n)$ such that $\alpha_1,...,\alpha_n$ form a root basis. Because their Dynkin diagram is a tree, lemma 1 in [Bo68, ch. V] implies that the products of $s_{\alpha_1},...,s_{\alpha_n}$ in any order are conjugate. As all root bases are conjugate, all Coxeter elements are conjugate.

Obviously, the Coxeter elements are quasi Coxeter elements. It turns out that they are even strict quasi Coxeter elements, see theorem 3.21.

(vi) Carter's work [Ca72] on the classification of Weyl group elements gives in a direct way the classification of the quasi Coxeter elements in the irreducible homogeneous root lattices and in a less direct way the classification of the strict quasi Coxeter elements in the irreducible inhomogeneous root lattices. In theorem 3.21 these classifications will be given, and also the classification of the quasi Coxeter elements in the irreducible inhomogeneous root lattices.

(vii) Recall the description of the Weyl group W in remark 3.3 (iii) for the root lattices of the types A_n, B_n, C_n, D_n in theorem 3.2 (d): $W(A_n)\cong S_{n+1}$, $W(B_n)=W(C_n)\cong\{\pm 1\}^n\rtimes S_n$. A signed permutation in $\{\pm 1\}^n\rtimes S_n$ will be called *positive* if the number of sign changes in it is even, it will be called *negative* if the number of sign changes in it is odd. The subgroup $W(D_n)\subset W(B_n)=W(C_n)$ consists of the positive signed permutations.

A signed cycle will be written as $(\varepsilon_1 a_1\,\varepsilon_2 a_2 ... \varepsilon_k a_k)$ with $k\geq 1$ and $\varepsilon_1,...,\varepsilon_k\in\{\pm 1\}$, $a_1,...,a_k\in\{1,...,n\}$ with $a_i\neq a_j$ for $i\neq j$. It maps $\pm a_i$ to $\pm\varepsilon_{i+1}a_{i+1}$ for $1\leq i\leq k-1$ and $\pm a_k$ to $\pm\varepsilon_1 a_1$. It is positive if $\prod_j\varepsilon_j=1$ and negative if $\prod_j\varepsilon_j=-1$. Its *support* is defined to be $\{a_1,...,a_k\}$.

Any signed permutation is up to the order a unique product of signed cycles (=cyclic permutations) such that their supports are disjoint and the union of the supports is $\{1,...,n\}$. They are called the signed cycles of the permutation. Here cycles of length one are used. For example $\mathrm{id}=(1)(2)...(n)$ and $-\mathrm{id}=(-1)(-2)...(-n)$.

Remark 3.20. (i) Carter classified in [Ca72] the conjugacy classes of Weyl group elements for all irreducible root lattices. A crucial point was the proof that any element w can be written as a product $w=w_1w_2$ where w_1 and w_2 are involutions with $V_{-1}(w_1)\cap V_{-1}(w_2)=\{0\}$ (proposition 38 and corollary (ii) in [Ca72]).

By [Ca72, lemma 5], any involution has a reduced presentation $(\alpha_1,...,\alpha_k)$ which consists of pairwise orthogonal roots. The composition of two such reduced presentations of two involutions w_1 and w_2 with $V_{-1}(w_1)\cap V_{-1}(w_2)=\{0\}$ is a reduced presentation of $w=w_1w_2$. Its generalized Dynkin diagram is a graph whose cycles (if any exist) have all even length. In [Ca72, theorem A] all graphs are classified which have the following properties: The graph contains cycles, all cycles have even length, the graph is a generalized Dynkin diagram of a presentation of an element $w=w_1w_2$ with w_1 and w_2 as above, the subroot lattice of the presentation is the full lattice, and w is not contained in the Weyl group of a subroot system. The graphs are labeled $D_n(a_k), E_6(a_k), E_7(a_k), E_8(a_k), F_4(a_1)$ and also $D_n(b_{n/2-1})$ if n is even and $E_7(b_2), E_8(b_3), E_8(b_5)$.

In fact, the graphs in [Ca72] are simplified by not distinguishing normal and dotted edges. The generalized Dynkin diagrams are obtained from the graphs in [Ca72] by replacing some edges by dotted edges such that any cycle obtains an odd number of dotted edges. This is possible.

It turns out that the graphs $D_n(a_k), E_6(a_k), E_7(a_k), E_8(a_k), F_4(a_1)$ correspond to conjugacy classes of Weyl group elements, and that these include the elements with presentations giving rise to the graphs $D_n(b_{n/2-1})$ (n even) and $E_7(b_2), E_8(b_3), E_8(b_5)$. These Weyl group elements are strict quasi Coxeter elements, because they are not contained in the Weyl group of a subroot lattice. They are not Coxeter elements [Ca72]. They and the Coxeter elements are the only strict quasi Coxeter elements (theorem 3.21 below).

(ii) Recall that the Coxeter elements in $W(A_n)$ are the cycles of length $n+1$ in S_{n+1}, the Coxeter elements in $W(B_n) = W(C_n) = W(BC_n)$ are the negative cycles of length n in $\{\pm 1\} \rtimes S_n$, and the Coxeter elements in $W(D_n)$ are the products of two negative cycles of lengths 1 and $n-1$. The products of two negative cycles of lengths k and $n-k$ for $2 \leq k \leq [n/2]$ form the conjugacy class $D_n(a_{k-1})$ in $W(D_n)$.

In the second column in the tables 5.1 and 5.2, $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4$ and G_2 denote the conjugacy classes of the Coxeter elements. The root lattice of type F_4 contains subroot lattices of types B_4 and D_4. In $W(F_4)$ the symbols $B_4, C_3 + A_1, D_4(a_1)$ denote the conjugacy classes in $W(F_4)$ of the Coxeter elements in $W(B_4)$ and $W(C_3 + A_1)$ and of the quasi Coxeter elements of type $D_4(a_1)$ in $W(D_4)$. The Coxeter elements of the subroot lattice of type A_2 in G_2 give rise to a conjugacy class in $W(G_2)$ denoted by A_2.

Theorem 3.21 gives the classification of the quasi Coxeter elements and the strict quasi Coxeter elements for the irreducible p.n. root lattices. A good part of it is due to [Ca72].

Theorem 3.21. *Let $(L, (.,.), \Phi)$ be one of the irreducible p.n. root lattices in theorem 3.2 (d). The tables 5.1 and 5.2 list the conjugacy classes of the strict quasi Coxeter elements and in the inhomogeneous cases the conjugacy classes of the quasi Coxeter elements. See the remarks 3.20 for the notations.*

Table 4.1:

	strict quasi Coxeter el. = quasi Coxeter el.
A_n	A_n
D_n	$D_n, D_n(a_1), ..., D_n(a_{[n/2-1]})$
E_6	$E_6, E_6(a_1), E_6(a_2)$
E_7	$E_7, E_7(a_1), E_7(a_2), E_7(a_3), E_7(a_4)$
E_8	$E_8, E_8(a_1), ..., E_8(a_8)$

Table 4.2:

	strict quasi Coxeter el.	quasi Coxeter el.
B_n	B_n	products of negative cycles
BC_n	$-$	products of negative cycles
C_n	C_n	$C_n, D_n, D_n(a_1), ..., D_n(a_{[n/2-1]})$
F_4	$F_4, F_4(a_1)$	$F_4, F_4(a_1), B_4, C_3 + A_1, D_4(a_1)$
G_2	G_2	G_2, A_2

Thus the quasi Coxeter elements of C_n consist of the products of one or two negative cycles.

Proof. It is well known that the Coxeter elements are not elements of some proper Weyl subgroup. Therefore they are strict quasi Coxeter elements. The other elements listed in the second columns are strict quasi Coxeter elements because of the results of Carter [Ca72] discussed in the remarks 3.20 (i).

By the same results, any other element $w \in W$ is in some proper Weyl subgroup. In the homogeneous cases, a proper Weyl subgroup is the Weyl group of a proper subroot lattice. Therefore then w is not a strict quasi Coxeter element. This completes the proof of table 4.1.

In the inhomogeneous cases, the fact that the second column of table 4.2 lists all strict quasi Coxeter elements is a consequence of the third column of table 4.2, in the following way. In the cases of the root lattices of types F_4 and G_2 it is obvious that the quasi Coxeter elements of types $B_4, C_3 + A_1, D_4(a_1)$ and A_2 are not strict quasi Coxeter elements. In the cases of the *p.n. root lattices* B_n, C_n and BC_n, observe

$$\begin{aligned} s_{e_i}s_{e_j} &= s_{e_i+e_j}s_{e_i-e_j} = s_{2e_i}s_{2e_j} \sim (-i)(-j) \qquad \text{and} \\ \mathbb{Z}e_i + \mathbb{Z}e_j &\supsetneqq \mathbb{Z}(e_i + e_j) + \mathbb{Z}(e_i - e_j) \supsetneqq \mathbb{Z}2e_i + \mathbb{Z}2e_j \qquad \text{for } i \neq j. \end{aligned} \tag{3.36}$$

This shows that all permutations whose signed cycles contain at least two negative cycles are not strict quasi Coxeter elements. BC_n has no strict quasi Coxeter elements because of $s_{e_i} = s_{2e_i}$. Therefore the only elements in the third column of table 4.2 which are strict quasi Coxeter elements are those in the second column.

It rests to prove the third column of table 4.2.

The root lattice of type $\mathbf{C_n}$: Because of $L(C_n) = L(D_n)$ the quasi Coxeter elements of D_n are also quasi Coxeter elements of C_n. Let $w \in W(C_n)$ be a quasi Coxeter element of C_n which is not a quasi Coxeter element of D_n. Then it has a presentation $(\alpha_1, ..., \alpha_n)$ such that its subroot lattice is the full root lattice $L(C_n)$ and $A := \{\alpha_1, ..., \alpha_n\} \not\subset \Phi(D_n)$. By lemma

3.13 (c)(ii) and (e)(ii) then the graph $\mathcal{G}(A)$ is a tree and contains exactly one marked vertex. Then w is a negative cycle, so a Coxeter element of C_n. This proves the line for C_n in table 4.2.

The p.n. root lattices of types $\mathbf{B_n}$ and $\mathbf{BC_n}$: Let $w \in W(B_n) = W(BC_n)$ be a quasi Coxeter element of B_n or BC_n, and let $(\alpha_1, ..., \alpha_n)$ be a presentation whose subroot lattice is the full root lattice $L(B_n) = L(BC_n)$. By lemma 3.13 (b)(ii) and (d)(ii) then the graph $\mathcal{G}(A)$ for $A = \{\alpha_1, ..., \alpha_n)$ is a union of trees which have each exactly one marking and which is short. Thus w is a product of negative cycles. This proves the lines for B_n and BC_n in table 4.2. Vice versa, any product$w$ of negative cycles has a presentation $(\alpha_1, ..., \alpha_n)$ such that the graph $\mathcal{G}(\{\alpha_1, ..., \alpha_n\})$ is a union of trees which have each exactly one marking and which is short. Thus w is a quasi Coxeter element.

The root lattice of type $\mathbf{G_2}$: Obviously, its quasi Coxeter elements are the products $s_\alpha s_\beta$ with α short and β long and $\alpha \not\perp \beta$ and the products $s_{\alpha_1} s_{\alpha_2}$ with α_1 and α_2 short and $\alpha_2 \neq \pm\alpha_1$. The elements of the first type are the Coxeter elements of G_2, the elements of the second type can also be written as products $s_{\beta_1} s_{\beta_2}$ with β_1 and β_2 long roots and $\beta_2 \neq \pm\beta_1$. They are the Coxeter elements of the subroot system of long roots, which is of type A_2.

The root lattice of type $\mathbf{F_4}$: See lemma 3.22 (b). The restriction there that in the presentation $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ first the short roots come and then the long roots, is not serious. One can obtain a presentation with this property from an arbitrary presentation using (3.3). $\square$

Lemma 3.22. *Let $(L, (.,.), \Phi)$ be the root lattice of type F_4 in theorem 3.2 (d). Obviously the short roots form a root system of type D_4, which is called $\widetilde{D}_4$, and the long roots form a root system of type D_4, which is called D_4.*

(a) Let $A = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \subset \Phi(F_4)$ be a $\mathbb{Z}$-basis of $L(F_4)$ such that first the short roots come and then the long roots. Then one of the following cases holds.

(i) All four roots are short.

(ii) α_1, α_2 and α_3 are short and α_4 is long. Then an element $w \in W(F_4)$ exist such that $w(\alpha_1), w(\alpha_2), w(\alpha_3)$ generate the subroot system of type $\widetilde{A}_3$ which is also generated by $e_1, \frac{1}{2}\sum_{i=1}^4 e_i, e_2$, and then $w(\alpha_4) = \pm e_i \pm e_j$ with $i \in \{1, 2\}$ and $j \in \{3, 4\}$.

(iii) α_1 and α_2 are short and α_3 and α_4 are long. Then $\langle\alpha_1, \alpha_2\rangle = \pm 1$ and $\langle\alpha_3, \alpha_4\rangle = \pm 1$ and $\mathbb{R}\alpha_1 + \mathbb{R}\alpha_2 \not\perp \mathbb{R}\alpha_3 + \mathbb{R}\alpha_4$.

(b) Let $w \in W(F_4)$ be a quasi Coxeter element, and let $(\alpha_1, ..., \alpha_4)$ be a presentation of w whose subroot lattice is the full lattice $L(F_4)$ and such that first, the short roots come and then the long roots. Then the cases in (a) hold, and w is in each case as follows.

(i) w is in $W(C_3+A_1)$ and is a Coxeter element there, or it is in $W(D_4)$ and is a quasi Coxeter element of type $D_4(a_1)$ there.

(ii) w is in $W(L_1)$ for some subroot lattice L_1 of type B_4 and is a Coxeter element there.

(iii) w is a Coxeter element of type F_4 or a strict quasi Coxeter element of type $F_4(a_1)$.

Proof. (a) The following obvious statements will be used:

(A) The root lattice $L(F_4)$ contains the subroot lattice L_2 of type B_4 with root system

$$\Phi(L_2)=\{\pm e_i\,|\,i\in\{1,2,3,4\}\}\cup\{\pm e_i\pm e_j\,|\,1\leq i<j\leq 4\}.$$

It has the same long roots as $\Phi(F_4)$, but less short roots.

(B) For any short root β_1 a Weyl group element w exists such that $w(\beta_1)=e_1$. If β_2 is a short root with $\beta_1\perp\beta_2$ then $w(\beta_2)\in\{\pm e_2,\pm e_3,\pm e_4\}$, and w can be chosen such that $w(\beta_2)=e_2$.

The case that all four roots $\alpha_1,...,\alpha_4$ are long is impossible because they would only generate the subroot lattice $L(D_4)$. The case that the three roots $\alpha_1,\alpha_2,\alpha_3$ are long and α_4 is short, is also impossible, because by (B) a Weyl group element exists such that $w(\alpha_4)=e_1$, and then all four images $w(\alpha_i)$ are in L_2. Thus either two roots are short and two roots are long, or three roots are short and one root is long, or all four roots are short.

Consider the case that $\alpha_1,\alpha_2,\alpha_3$ are short and α_4 is long. Then α_1,α_2 and α_3 generate a subroot system of rank 3 of $\Phi(\widetilde{D}_4)$. Only the two types $\widetilde{A}_3$ and $3\widetilde{A}_1$ are possible a priori. Here the type $3\widetilde{A}_1$ is not possible, because then by (B) an element $w\in W(F_4)$ exists such that $w(\alpha_i)=e_i$ for $i\in\{1,2,3\}$, and these roots and any long root $w(\alpha_4)$ are in L_2. By theorem 3.6 (b), any two subroot systems of type $\widetilde{A}_3$ of the subroot system $\widetilde{D}_4$ are conjugate by an element of $W(\widetilde{D}_4)$. This shows the first half of part (ii). Obviously $w(\alpha_4)=\pm e_i\pm e_j$ with $i\in\{1,2\}$ and $j\in\{3,4\}$. This gives part (ii).

Consider the case that α_1 and α_2 are short and α_3 and α_4 are long. If $\alpha_1\perp\alpha_2$ then by (B) $\alpha_1,...,\alpha_4$ are mapped by a suitable element $w\in W(F_4)$ into the subroot lattice L_2. Therefore $\langle\alpha_1,\alpha_2\rangle=\pm 1$. Furthermore

$$\mathbb{R}\alpha_1+\mathbb{R}\alpha_2\not\perp\mathbb{R}\alpha_3+\mathbb{R}\alpha_4$$

because else the four roots would generate a reducible subroot lattice. An element $w\in W(F_4)$ exists such that

$$w(\alpha_1)=\pm e_1, w(\alpha_2)=\frac{1}{2}\sum_{i=1}^4 e_i.$$

If $\alpha_3 \perp \alpha_4$ then either

$$w(\alpha_3) = \varepsilon_1(e_i + \varepsilon_2 e_j) \text{ and } w(\alpha_4) = \varepsilon_3(e_i - \varepsilon_2 e_j)$$

for some $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{\pm 1\}$ and some i, j with $1 \leq i < j \leq 4$\$, or

$$w(\alpha_3) = \varepsilon_1(e_i + \varepsilon_2 e_j) \text{ and } w(\alpha_4) = \varepsilon_3(e_k - \varepsilon_4 e_l)$$

for some $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \{\pm 1\}$ and some i, j, k, l with $\{i, j, k, l\} = \{1, 2, 3, 4\}$. One sees easily with some case discussion that in both cases $w(\alpha_1), ..., w(\alpha_4)$ do not generate $L(F_4)$.

(b) Of course, the cases in (a) hold.

The case **(i)**. w is in $W(\widetilde{D}_4)$ and is either a Coxeter element there or a quasi Coxeter element of type $\widetilde{D}_4(a_1)$. In the first case, w is conjugate to

$$s_{\frac{1}{2}\sum_i e_i} s_{e_1} s_{e_2} s_{e_3} = s_{\frac{1}{2}\sum_i e_i} s_{e_1} s_{e_2+e_3} s_{e_2-e_3},$$

which is in $W(C_3 + A_1)$ and which is a Coxeter element there. In the second case, w is conjugate to

$$s_{\frac{1}{2}\sum_i e_i} s_{\frac{1}{2}(e_1+e_2-e_3-e_4)} s_{e_1} s_{e_3} = s_{e_1+e_2} s_{e_3+e_4} s_{e_1+e_3} s_{e_1-e_3},$$

which is in $W(D_4)$ and which is a quasi Coxeter element of type $D_4(a_1)$ there.

The case **(ii)**. The element w is conjugate to

$$s_{\frac{1}{2}\sum_a e_a} s_{e_1} s_{e_2} s_\beta = s_{\frac{1}{2}\sum_a e_a} s_{e_1-e_2} s_{e_1+e_2} s_\beta$$

for some $\beta = e_i + \varepsilon e_k$ with $i \in \{1, 2\}$ and $k \in \{3, 4\}$ and $\varepsilon \in \{\pm 1\}$. This is in $W(L_3)$ for a subroot lattice L_3 of type B_4. In the case $\varepsilon = -1$ the generalized Dynkin diagram of the four roots on the right-hand side is (up to the distinction between dotted and normal edges) the B_4 Dynkin diagram, so then the element is a Coxeter element in $W(L_3)$. In the case $\varepsilon = 1$, the right-hand side is equal to

$$s_{\frac{1}{2}\sum_a e_a} s_{e_1-e_2} s_{-e_j+e_k} s_{e_1+e_2},$$

where j is determined by $\{i, j\} = \{1, 2\}$. This is again a Coxeter element in $W(L_3)$.

The case **(iii)**. Using (3.3) for α_1 and α_2, one can suppose $\mathbb{R}\alpha_2 \not\perp \mathbb{R}\alpha_3 + \mathbb{R}\alpha_4$. After conjugation, one can suppose

$$\alpha_1 = \frac{1}{2}\sum_{i=1}^{4} \pm e_i,\ \alpha_2 = \pm e_1,\ \{\alpha_3, \alpha_4\} \subset \{\pm(e_i - e_j)\,|\,1 \leq i < j \leq 3\}.$$

Using (3.3) for α_3, α_4 and changing possibly some signs and conjugating possibly again, one can suppose

$$\alpha_1 = \frac{1}{2}(e_1 + \varepsilon_2 e_2 + \varepsilon_3 e_3 + e_4),\ \alpha_2 = e_1,\ \alpha_3 = e_2 - e_3,\ \alpha_4 = e_1 - e_2$$

for some $\varepsilon_2, \varepsilon_3 \in \{\pm 1\}$.

In the case $(\varepsilon_2, \varepsilon_3) = (1, 1)$, the generalized Dynkin diagram of the roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ is (up to the distinction between dotted and normal edges) a Dynkin diagram of type F_4. Thus w is a Coxeter element in $W(F_4)$.

In the case $(\varepsilon_2, \varepsilon_3) = (1, -1)$, the element w is conjugate to the product of the two involutions $s_{\alpha_4}s_{\alpha_1}$ and $s_{\alpha_2}s_{\alpha_3}$ with admissible diagram of type $F_4(a_1)$. Thus it is a quasi Coxeter element in $W(F_4)$ of type $F_4(a_1)$.

In the case $(\varepsilon_2, \varepsilon_3) = (-1, -1)$, the element w is

$$w = s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4} = s_{\alpha_2}s_{s_{\alpha_2}(\alpha_1)}s_{s_{\alpha_3}(\alpha_4)}s_{\alpha_3} = s_{e_1}s_{\frac{1}{2}(-e_1-e_2-e_3+e_4)}s_{e_1-e_3}s_{e_2-e_3}.$$

The generalized Dynkin diagram of the roots on the right-hand side is (up to the distinction between dotted and normal edges) a Dynkin diagram of type F_4. Thus w is a Coxeter element in $W(F_4)$.

In the case $(\varepsilon_2, \varepsilon_3) = (-1, 1)$, the element w is

$$w = s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4} = s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}s_{\alpha_4} = s_{\alpha_1}s_{\alpha_3}s_{\alpha_4}s_{s_{\alpha_4}(\alpha_2)} = s_{\alpha_1}s_{\alpha_3}s_{\alpha_4}s_{e_2}.$$

This is conjugate to the element

$$s_{e_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_4} = s_{s_{e_2}(\alpha_1)}s_{e_2}s_{s_{\alpha_3}(\alpha_4)}s_{\alpha_3} = s_{\frac{1}{2}\sum_i e_i}s_{e_2}s_{e_1-e_3}s_{e_2-e_3}.$$

The generalized Dynkin diagram of the roots on the right-hand side is (up to the distinction between dotted and normal edges) a Dynkin diagram of type F_4. Thus w is a Coxeter element in $W(F_4)$. □

Remark 3.23. (i) In the tables 7-11 in [Ca72] all conjugacy classes of elements of the Weyl groups of the root lattices of types G_2, F_4, E_6, E_7 and E_8 are listed in the following form. For any conjugacy class one element and one presentation of it as a strict quasi Coxeter element is chosen. The tables show the isomorphism class of the pair of full lattice and subroot lattice and the type of the strict quasi Coxeter element.

(ii) Theorem 3.25 below gives more information for the root lattices of types G_2, F_4, E_6, E_7 and E_8. For F_4 it lists for any conjugacy class all isomorphism classes of pairs of full lattice and subroot lattice, which turns up as a subroot lattice of a presentation as a quasi Coxeter element, and the type of the quasi Coxeter element. This gives all types of reduced presentations for any element. For G_2, E_6, E_7 and E_8 one can extract the same information from theorem 3.25 and the tables 3.6, 3.2, 3.3 and 3.4.

(iii) For the tables in theorem 3.25, the notations in table 4.2 have to be refined: There are three conjugacy classes of quasi Coxeter elements in F_4 which have also presentations as Coxeter elements in subroot lattices of type B_4 and $C_3 + A_1$ respectively as a quasi Coxeter element of type $D_4(a_1)$ in the subroot lattice of type D_4. The presentations of these elements as quasi Coxeter elements in F_4 are now called $F_4(a_2), F_4(a_3)$ and $F_4(a_4)$.

Analogously, the presentations as quasi Coxeter elements in $W(G_2)$ of those elements which have also presentations as Coxeter elements in A_2 (the subroot lattice of long roots) are denoted by $\widetilde{A}_2$.

For $2 \leq k \leq 4$, the presentations as quasi Coxeter elements in $W(B_k)$ of those elements in $W(B_k)$ which are products of negative cycles of lengths $l_1, .., l_r$ with $l_1 + ... + l_r = k$ are denoted by $B_k(l_1, ..., l_r)$. The case $B_k(k)$ is also denoted by B_k. This will be used in the table 4.4 for F_4. Similarly, $C_3(2, 1)$ is used there.

(iv) The complete control in theorem 3.25 on the reduced presentations of the Weyl group elements of the irreducible root lattices allows determining the number $k_4(L, w)$ in definition 3.24.

Definition 3.24. *Let $(L, (.,.), \Phi)$ be a p.n. root lattice, and let w be a Weyl group element. Recall $k_2(L, L_1)$ from theorem 3.11. Define the number*

$$k_4(L, w) := \min\{k_2(L, L_1) \quad | \quad \text{a reduced presentation of } w \text{ with subroot lattice } L_1 \text{ exists.}\} \tag{3.37}$$

This number will be important in section (3.5). Because of $k_1(L, L_1) = k_2(L, L_1)$ (theorem

3.11 (b)),

$$k_4(L,w) \geq \dim L_{\mathbb{Q}}/V_{\neq 1,\mathbb{Q}}(w) = n - l(w). \tag{3.38}$$

Equality holds if and only if a reduced presentation with subroot lattice $L_1 = V_{\neq 1,\mathbb{Q}} \cap L$ exists. This is the unique primitive subroot lattice L_1 with $L_{1,\mathbb{Q}} = V_{\neq 1,\mathbb{Q}}$. Often equality holds, often not.

Theorem 3.25. *Let $(L,(.,.),\Phi)$ be one of the irreducible p.n. root lattices in theorem 3.2 (d).*

(a) In the cases A_n, B_n and BC_n,

$$k_4(L,w) = n - l(w). \tag{3.39}$$

(b) Consider in the cases C_n and D_n a Weyl group element w which is a product of r positive cycles and s negative cycles with disjoint supports whose union is $\{1,...,n\}$ (remark 3.19 (vii)) (in the case of D_n, s is even.) Then

$$r = n - l(w),$$

and then any reduced presentation with subroot lattice L_1 with minimal $k_2(L,L_1)$ satisfies

$$\begin{aligned} L/L_1 &\cong \mathbb{Z}^r \times \mathbb{Z}_2^{[(s+1)/2]}, && (3.40)\\ k_4(L,w) &= k_2(L,L_1) = n - l(w) + \left[\frac{s+1}{2}\right]. && (3.41)\end{aligned}$$

(c) In the cases G_2, E_6, E_7 and E_8, for the big majority of the Weyl group elements there is only one type of reduced presentations. That means, the pairs (L,L_1) are isomorphic where L_1 runs through the subroot lattices of all reduced presentations.

Table 4.3 lists for the (conjugacy classes of the) exceptions the different ways to write them as quasi Coxeter elements of subroot lattices L_1, and it lists the numbers $k_4(L,w)$.

For the other elements, $k_4(L,w) = k_2(L,L_1)$ for the unique isomorphism class (L,L_1). All these other elements can be found by replacing in the tables 3.6, 3.2, 3.3 and 3.4 L_1 by the possible quasi Coxeter elements with subroot lattice of type L_1. See table 4.1 for the possibilities. (E.g. D_5+A_3 has to be replaced by the two possibilities D_5+A_3 and $D_5(a_1)+A_3$.)

Table 4.3:

L	presentation of w as quasi Coxeter element in $W(L_1)$ for some subroot lattice L_1	$k_4(L,w)$
G_2	$\widetilde{A}_2 \sim A_2$	0
E_7	$D_4(a_1)+2A_1 \sim 2A_3$	2
E_7	$D_4(a_1)+3A_1 \sim 2A_3+A_1$	1
E_7	$D_6(a_1)+A_1 \sim A_7$	1
E_8	$D_4(a_1)+2A_1 \sim [2A_3]''$	3
E_8	$D_4(a_1)+3A_1 \sim 2A_3+A_1$	2
E_8	$D_5(a_1)+2A_1 \sim D_4+A_3$	2
E_8	$D_6(a_1)+A_1 \sim [A_7]''$	2
E_8	$D_4(a_1)+4A_1 \sim 2A_3+2A_1$	2
E_8	$D_4+D_4(a_1) \sim D_5(a_1)+A_3$	1
E_8	$D_5+A_3 \sim A_7+A_1 \sim D_6(a_1)+2A_1$	1
E_8	$D_6(a_2)+2A_1 \sim 2D_4$	2
E_8	$E_6(a_1)+A_2 \sim A_8$	1
E_8	$E_7(a_1)+A_1 \sim D_8$	1
E_8	$E_7(a_3)+A_1 \sim D_8(a_2)$	1

(d) In the case of F_4, the following table 4.4 lists for (the conjugacy classes of) all Weyl group elements all ways to write them as quasi Coxeter elements of subroot lattices. See remark 3.23 (iii) for the notations. It also lists the numbers $k_4(L,w)$.

Table 4.4:

Presentation of w as quasi Coxeter element in $W(L_1)$ for some subroot lattice L_1	$k_4(L,w)$
F_4	0
$F_4(a_1)$	0
$F_4(a_2) \sim B_4$	0
$F_4(a_3) \sim C_3+A_1$	0
$F_4(a_4) \sim D_4(a_1) \sim B_4(2,2)$	0
$B_4(3,1) \sim D_4$	1
$B_4(2,1,1) \sim A_3+\widetilde{A}_1 \sim C_3(2,1)+A_1 \sim B_2+2A_1$	1
$B_4(1,1,1,1) \sim B_2(1,1)+2A_1 \sim 4A_1$	1
$A_2+\widetilde{A}_2$	1

w	$k_4(L,w)$
B_3	1
$B_3(2,1)\sim A_3$	1
$B_3(1,1,1)\sim 2A_1+\widetilde{A}_1$	1
$B_2+A_1\sim C_3(2,1)$	1
$B_2(1,1)+A_1\sim 3A_1$	2
$A_2+\widetilde{A}_1$	1
$A_1+\widetilde{A}_2$	1
C_3	1

w	$k_4(L,w)$
A_2	2
B_2	2
$B_2(1,1)\sim 2A_1$	2
$A_1+\widetilde{A}_1$	2
$\widetilde{A}_2$	2
$\widetilde{A}_1$	3
A_1	3
$\emptyset$	4

Proof. **(a)** In the cases A_n, B_n, BC_n, C_n and D_n, any positive cycle in the Weyl group can be written as a product $s_{\alpha_1}\circ ...\circ s_{\alpha_r}$ where $\alpha_1,...,\alpha_r$ are roots of the type $\pm e_i\pm e_j$ whose graph $\mathcal{G}(\{\alpha_1,...,\alpha_r\})$ is a tree. The subroot lattice $L_1=\sum_{i=1}^r\mathbb{Z}\alpha_i$ is a primitive sublattice.

Any element of $W(A_n)$ is a product of positive cycles with disjoint supports whose union is $\{1,...,n\}$. Because the supports are disjoint, the sum of the subroot lattices of the presentations above of the positive cycles is also a primitive sublattice. Therefore there (3.39) holds.

In the cases B_n and BC_n, any negative cycle can be written as a product $s_{\alpha_1}\circ...\circ s_{\alpha_r}\circ s_{\alpha_{r+1}}$ such that $s_{\alpha_1}\circ ...\circ s_{\alpha_r}$ is a positive cycle with graph a tree, and such that α_{r+1} is a short root which gives a marking of one vertex of the tree. The subroot lattice $L_1=\sum_{i=1}^{r+1}\mathbb{Z}\alpha_i$ is the primitive sublattice, which is generated by all the short roots which correspond to the vertices of the tree.

Any element of $W(B_n)=W(BC_n)$ is a product of positive cycles and/or negative cycles with disjoint supports whose union is $\{1,...,n\}$. Because the supports are disjoint, the sum of the subroot lattices of the presentations above of the positive and/or negative cycles is also a primitive sublattice. Therefore there (3.39) holds.

(b) In the cases C_n and D_n, any pair of negative cycles can be written as a product $s_{\alpha_1}\circ...\circ s_{\alpha_a}\circ s_{\beta_1}\circ...\circ s_{\beta_b}\circ s_{\alpha_{a+1}}\circ s_{\beta_{b+1}}$ such that $s_{\alpha_1}\circ...\circ s_{\alpha_a}$ and $s_{\beta_1}\circ...\circ s_{\beta_b}$ are positive cycles whose graphs are disjoint trees and such that $\alpha_{a+1}=e_i-e_j$ and $\beta_{b+1}=e_i+e_j$ with i a vertex of one tree and j a vertex of the other tree. The subroot lattice $L_1=\sum_{i=1}^{a+1}\mathbb{Z}\alpha_i+\sum_{j=1}^{b+1}\mathbb{Z}\beta_j$ is of type C_{a+b+2} respectively D_{a+b+2}.

In the case C_n, any single negative cycle can be written as a product $s_{\alpha_1}\circ ...\circ s_{\alpha_a}\circ s_{\alpha_{a+1}}$ such that $s_{\alpha_1}\circ ...\circ s_{\alpha_a}$ is a positive cycle with graph a tree, and such that α_{a+1} is a long root of the type $2e_i$ which gives a marking of one vertex of the tree. The subroot lattice $L_1=\sum_{i=1}^{a+1}\mathbb{Z}\alpha_i$ is of type C_{a+1}.

Let w be a Weyl group element which is a product of r positive cycles and s negative cycles with disjoint supports whose union is $\{1, ..., n\}$. One presents the positive cycles as above (in the proof of (a)) and as many pairs of negative cycles as above. At most one (none in the case D_n) single negative cycle is left and is also presented as above. Let L_1 be the subroot lattice of the presentation. Table 3.1 shows (3.40). One sees easily that no reduced presentation with smaller $k_1(L, L_1)$ exists. (3.41) holds.

(c) The tables 7, 9, 10 and 11 in [Ca72] list all conjugacy classes of elements of the Weyl groups of root lattices of the types G_2, E_6, E_7 and E_8. They give in each case one type of presentation as a strict quasi Coxeter element. It is easy to find all presentations as quasi Coxeter elements which are not on the list. One has to find out which elements in the list are given also by these presentations. In most cases, it is sufficient to compare the characteristic polynomials. A table of characteristic polynomials is table 3 in [Ca72].

The only cases where this is not sufficient arise for the E_8 root lattice and there for the presentations as quasi Coxeter elements of types $D_4(a_1) + 2A_1$ and $D_6(a_1) + A_1$. In the first case the presentations as strict quasi Coxeter elements of types $[2A_3]'$ and $[2A_3]''$ have the same characteristic polynomial, in the second case the presentations of types $[A_7]'$ and $[A_7]''$. Because of

$$\begin{aligned} \text{index}(D_4 + 2A_1) = 2 = \text{index}([2A_3]'') \neq \text{index}([2A_3]') = 1, \\ \text{index}(D_6 + A_1) = 2 = \text{index}([A_7]'') \neq \text{index}([A_7]') = 1, \end{aligned}$$

lemma 3.17 (b) tells that $D_4(a_1) + 2A_1$ gives the same conjugacy class as $[2A_3]''$ and that $D_6(a_1) + A_1$ gives the same conjugacy class as $[A_7]''$.

(d) Table 8 in [Ca72] lists 9, 8, 5, 2 and 1 conjugacy classes of elements of the Weyl group of type F_4 of lengths 4, 3, 2, 1 respectively 0. On the other hand there are 19, 12, 6, 2 and 1 types of presentations of elements as quasi Coxeter elements of lengths 4, 3, 2, 1 respectively 0:

length	type of presentation as a quasi Coxeter element
4	$F_4, F_4(a_1), F_4(a_2), F_4(a_3), F_4(a_4), B_4, B_4(3,1), B_4(2,2),$ $B_4(2,1,1), B_4(1,1,1,1), A_3+\widetilde{A}_1, A_2+\widetilde{A}_2, C_3+A_1,$ $C_3(2,1)+A_1, D_4, D_4(a_1), B_2+2A_1, B_2(1,1)+2A_1, 4A_1$
3	$B_3, B_3(2,1), B_3(1,1,1), B_2+A_1, B_2(1,1)+A_1, A_2+\widetilde{A}_1,$ $A_3, 2A_1+\widetilde{A}_1, A_1+\widetilde{A}_2, C_3, C_3(2,1), 3A_1$
2	$B_2, B_2(1,1), \widetilde{A}_2, A_1+\widetilde{A}_1, A_2, 2A_1$
1	$\widetilde{A}_1, A_1$
0	$\emptyset$

For those types of presentations as quasi Coxeter elements in the table above which are not in the table 8 in [Ca72], one has to find out which conjugacy classes they give. In many cases this is determined by the characteristic polynomials. The cases where the characteristic polynomials is not sufficient, can be drawn from lemma 26 in [Ca72]. It lists the presentations as strict quasi Coxeter elements which give different conjugacy classes, but with the same characteristic polynomials. Of the 8 pairs in lemma 26 in [Ca72], only those 4 are relevant here, for which presentations as quasi Coxeter elements exist which are not in table 8 in [Ca72] and which have the same characteristic polynomials. These 4 pairs and their characteristic polynomials are as follows:

D_4 C_3+A_1	A_3 B_2+A_1	$3A_1$ $2A_1+\widetilde{A}_1$	$2A_1$ $A_1+\widetilde{A}_1$
$(t^3+1)(t+1)$	t^3+t^2+t+1	$(t+1)^3(t-1)$	$(t+1)^2(t-1)^2$

The equality

$$s_{e_3}s_{e_4} = s_{e_3+e_4}s_{e_3-e_4}$$

tells that in table 4.4

$$B_2(1,1) \sim 2A_1,\ \ B_2(1,1)+A_1 \sim 3A_1,\ \ B_3(1,1,1) \sim 2A_1+\widetilde{A}_1.$$

The equality

$$s_{e_1}s_{\frac{1}{2}(e_1+e_2-e_3-e_4)}s_{\frac{1}{2}(e_1+e_2+e_3+e_4)} = s_{e_1}s_{e_1+e_2}s_{e_3+e_4}$$

tells that in table 4.4

$$C_3(2,1) \sim B_2 + A_1.$$

The equalities

$$\begin{aligned} s_{e_2-e_3}s_{e_3}s_{e_4} &= s_{e_2-e_3}s_{e_3+e_4}s_{e_3-e_4} \qquad \text{and} \\ s_{e_1-e_2}s_{e_2-e_3}s_{e_3}s_{e_4} &= s_{e_1-e_2}s_{e_2-e_3}s_{e_3+e_4}s_{e_3-e_4} \end{aligned}$$

tell that in table 4.4

$$B_3(2,1) \sim A_3 \text{ and } B_4(3,1) \sim D_4.$$

The equivalence $F_4(a_3) \sim C_3 + A_1$ in table 4.4 holds by definition of $F_4(a_3)$. All other equivalences in table 4.4 follow from the comparison of characteristic polynomials. □

Remark 3.26. (i) From the theorems 3.25, 3.21 and 3.6 (respectively the first columns of the tables 3.1-3.6), one can recover the classification of conjugacy classes of the Weyl group elements of the root lattices of types G_2, F_4, E_6, E_7 and E_8 which is given in the tables 7-11 in [Ca72].

(ii) The proof above of theorem 3.25 had used these tables, but not in a very crucial way. Those few cases where different conjugacy classes have the same characteristic polynomials, can be dealt with by hand. In fact, information on them is given in the lemmas 26 and 27 in [Ca72]. But theorem 3.21 on the (strict) quasi Coxeter elements depends in a crucial way on the results in [Ca72].

(iii) The characteristic polynomials of the strict quasi Coxeter elements in all irreducible root lattices are given in table 3 in [Ca72].

(iv) In [Vo85, (2.3.4)] a table similar to table 4.3 for E_7 and E_8 is given. But one of the cases for E_7 and four of the cases for E_8 are missing there. The case for E_7 which is missing in [Vo85, (2.3.4)], is also missing in [Vo85, (3.2.9)].

Remark 3.27. There is a strange correspondence. Define for any irreducible root lattice $(L, (.,.), \Phi)$ the two numbers

$$\begin{aligned} k_6(L) &:= |\{\text{conjugacy classes of quasi Coxeter elements}\}| - 1, \\ k_7(L) &:= |\{\text{isomorphism classes of pairs } (L, L_1) \text{ with } L_1 \\ &\qquad \text{a subroot lattice of full rank with } k_1(L, L_1) = 1\}|. \end{aligned}$$

Then

$$k_6(L) = k_7(L) \qquad \text{for } A_n, C_n, D_n, F_4, E_6, E_7, E_8 \text{ and } B_2,$$
$$\text{but not for } B_n \ (n \geq 3) \text{ and } G_2,$$

as the following table shows.

	A_n	B_n	C_n	D_n	G_2	F_4	E_6	E_7	E_8
$k_6(L)$	0	$p(n)-1$	$\left[\frac{n}{2}\right]$	$\left[\frac{n}{2}\right]-1$	1	4	2	4	8
$k_7(L)$	0	$n-1$	$\left[\frac{n}{2}\right]$	$\left[\frac{n}{2}\right]-1$	2	4	2	4	8

Here $p(n)$ is the number of partitions of n.

3.5 Nonreduced presentations of Weyl group elements

The last section calculated the number k_4 for Weyl group elements $w \in W$, that was, the minimal way to fill up the "best" subroot lattice, generated by a reduced representation of w, to the full lattice. In this section, we turn to the number which is needed in the singularity theory application. There we want to change a reduced representation to a new (nonreduced) representation such that the new set of roots generates the whole lattice. The key result is, that k_4 is already the answer to that task.

Definition 3.28. *Let $(L, (.,.), \Phi)$ be a p.n. root lattice, and let w be a Weyl group element. Define the number*

$$k_5(L, w) := \min\{k \ \mid \ \text{a presentation } (\alpha_1, ..., \alpha_{l(w)+2k}) \text{ with subroot lattice the full lattice exists}\}. \tag{3.1}$$

Recall the definition (3.37) of the number $k_4(L, w)$ in the same situation. Let $(\alpha_1, ..., \alpha_{l(w)})$ be a reduced presentation of an element w with subroot lattice L_1 such that $k_2(L, L_1)$ is minimal, i.e. $k_2(L, L_1) = k_4(L, w) =: k$. Let $\beta_1, ..., \beta_k$ be roots such that $L_1 + \sum_{j=1}^{k} \mathbb{Z}\beta_j = L$. Then obviously $(\alpha_1, ..., \alpha_{l(w)}, \beta_1, \beta_1, \beta_2, \beta_2, ..., \beta_k, \beta_k)$ is a presentation with root lattice the full root lattice L. Therefore

$$k_5(L, w) \leq k_4(L, w). \tag{3.2}$$

Theorem 3.29. *Let $(L, (.,.), \Phi)$ be a p.n. root lattice, and let w be a Weyl group element. Then*

$$k_5(L, w) = k_4(L, w). \tag{3.3}$$

The proof consists in a reduction to the special case in the following lemma and in the proof of the following lemma. The proof of the lemma is given first.

Lemma 3.30. *Let $(L, (.,.), \Phi)$ be a p.n. root lattice of some rank n, and let w be a Weyl group element of length $n-1$. Then*

$$k_5(L, w) = 1 \iff k_4(L, w) = 1. \tag{3.4}$$

Proof of lemma 3.30. If $(L, (.,.), \Phi)$ is reducible with orthogonal summands $\bigoplus_{k \in K} L_k$, then w decomposes accordingly into a product of commuting elements $w_k \in W(L_k)$, and the numbers $k_4(L, w)$ and $k_5(L, w)$ are additive,

$$k_4(L, w) = \sum_{k \in K} k_4(L_k, w_k), \qquad k_5(L, w) = \sum_{k \in K} k_5(L_k, w_k).$$

Therefore it is sufficient to prove the lemma and also theorem 3.29 for the irreducible p.n. root lattices.

Let $(L, (.,.), \Phi)$ be an irreducible p.n. root lattice of rank n, and let w be a Weyl group element with $l(w) = n-1$. Then $k_5(L, w) \geq 1$. If $k_4(L, w) = 1$ then by (3.2) also $k_5(L, w) = 1$. Thus it is sufficient to prove $k_5(L, w) = 1 \Rightarrow k_4(L, w) = 1$.

The cases $\mathbf{A_n}, \mathbf{D_n}, \mathbf{E_6}, \mathbf{E_7}, \mathbf{E_8}$: Suppose $k_5(L, w) = 1$, and let $(\alpha_1, ..., \alpha_{n+1})$ be a presentation of w whose subroot lattice is the full lattice. By theorem 3.12, the set $\{\alpha_1, ..., \alpha_{n+1}\}$ contains a $\mathbb{Z}$-basis of the full lattice L. Using (3.3), we can suppose that $\alpha_1, ..., \alpha_n$ is a $\mathbb{Z}$-basis of L. Let $(\beta_1, ..., \beta_{n-1})$ be an arbitrary reduced presentation of w. Then

$$s_{\alpha_1} \circ ... \circ s_{\alpha_n} = s_{\beta_1} \circ ... \circ s_{\beta_{n-1}} \circ s_{\alpha_{n+1}}.$$

The subroot lattice of the presentation on the left-hand side is the full lattice, so it has index one. By lemma 3.17 (b), the index of the subroot lattice of the presentation on the right-hand side is the same, so it is also one. Thus

$$\sum_{j=1}^{n-1} \mathbb{Z}\beta_j + \mathbb{Z}\alpha_{n+1} = L.$$

This shows here $k_4(L,w) = 1$.

The cases $\mathbf{B_n}$ and $\mathbf{BC_n}$: Because of theorem 3.25 (a), $k_4(L,w) = n - l(w) = 1$ holds anyway.

The cases $\mathbf{C_n}$: $k_4(L,w) = 1$ holds if and only if a reduced presentation with subroot lattice of type A_{n-1} or of type $A_{k-1} + C_{n-k}$ for some $k \in \{1, 2, ..., n-1\}$ exists. This follows from table 3.1. In the case A_{n-1}, w is a positive cycle of length n. In the case $A_{k-1} + C_{n-k}$, w is a product of a positive cycle of length k and of one or two negative cycles such that the sum of their lengths is $n - k$.

It rests to show that w is such an element if $k_5(L,w) = 1$. Thus suppose $k_5(L,w) = 1$. Let $(\alpha_1, ..., \alpha_{n+1})$ be a presentation of w whose subroot lattice is the full lattice L. By theorem 3.12, the set $\{\alpha_1, ..., \alpha_{n+1}\}$ contains a $\mathbb{Z}$-basis of the full lattice L. Using (3.3), we can suppose that $\alpha_2, ..., \alpha_{n+1}$ is a $\mathbb{Z}$-basis of L. Thus $s_{\alpha_2} \circ ... \circ s_{\alpha_{n+1}} =: v$ is a quasi Coxeter element, so either one negative cycle or the product of two negative cycles.

If α_1 is a long root, multiplying v from the left with s_{α_1} will turn one of the (one or two) negative cycles into a positive cycle.

If α_1 is a short root, so $\alpha_1 = \pm e_i \pm e_j$, then the type of $s_{\alpha_1} \circ v$ depends on the position of the vertices i and j in the supports of the (one or two) negative cycles. If i and j are in the support of the same negative cycle, then it splits into two cycles, one positive and one negative. If i and j are in the supports of different negative cycles, then $s_{\alpha_1} \circ v$ is a positive cycle of length n.

In any case, w is of one of the types which satisfy $k_4(L,w) = 1$.

The case $\mathbf{G_2}$: By table 3.6, all subroot lattices of rank 1 are primitive sublattices. Therefore $k_4(L,w) = n - 1 = 1$ holds anyway.

The case $\mathbf{F_4}$: By table 4.4, the only elements w with $l(w) = 3$ and $k_4(L,w) \geq 2$ are those of type $B_2(1,1) + A_1 \sim 3A_1$, and the elements of this type satisfy $k_4(L,w) = 2$. It rests to show for them $k_5(L,w) \geq 2$.

Suppose that such an element w satisfies $k_5(L,w) = 1$, and let $(\alpha_1, ..., \alpha_5)$ be a presentation of w whose subroot lattice is the full lattice. By theorem 3.12, the set $\{\alpha_1, ..., \alpha_5\}$ contains a $\mathbb{Z}$-basis of the full lattice L. Using (3.3), we can suppose that $\alpha_1, ..., \alpha_4$ is a $\mathbb{Z}$-basis of L. We may suppose $w = s_{e_1-e_2}s_{e_1+e_2}s_{e_3-e_4}$. Then

$$s_{e_1-e_2}s_{e_1+e_2}s_{e_3-e_4}s_{\alpha_5} = s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}.$$

Because of the right-hand side, this is a quasi Coxeter element in $W(F_4)$.

First case, α_5 is a long root $\alpha_5 = \pm e_i \pm e_j$: Then $\{i,j\} = \{1,2\}$ is impossible because else the four roots on the left-hand side were linearly dependent. $|\{i,j\} \cap \{1,2\}| = 1$ is impossible because else the element on the left-hand side were a Coxeter element in $W(D_4)$, and this is not a quasi Coxeter element in $W(F_4)$. Also $\{i,j\} = \{3,4\}$ is impossible because else the left-hand side were an element of type $4A_1$, and this is not a quasi Coxeter element in $W(F_4)$, or the four roots on the left-hand side were linearly dependent. The first case is impossible.

Second case, α_5 is a short root: By conjugation and renumbering of the e_j we can suppose $\alpha_5 = \pm e_i$ for some i. Then $i \in \{1,2\}$ is impossible because else the four roots on the left-hand side were linearly dependent. $i \in \{3,4\}$ is impossible because else the left-hand side were an element of type $B_2 + 2A_1$, and this is not a quasi Coxeter element in $W(F_4)$. The second case is impossible.

Thus $k_5(L,w) \neq 1$, so $k_5(L,w) \geq 2$. This finishes the proof of the case F_4 and the whole proof of lemma 3.30. □

Proof of theorem 3.29. Let $(L, (.,.), \Phi)$ be an irreducible p.n. root lattice of some rank n. At the beginning of the proof of lemma 3.30 it was shown that it is sufficient to prove theorem 3.29 in this case.

Let w be a Weyl group element, and let $(\alpha_1, ..., \alpha_{l(w)+2k})$ be a presentation with subroot lattice the full lattice L and with $k = k_5(L,w)$ minimal with this property.

By theorem 3.12, the set $\{\alpha_1, ..., \alpha_{l(w)+2k})$ contains a $\mathbb{Z}$-basis of the full lattice L. Using (3.3), we can suppose that $\alpha_1, ..., \alpha_n$ is a $\mathbb{Z}$-basis of L. The element $s_{\alpha_1} \circ ... \circ s_{\alpha_n}$ has length n. Thus the element

$$v := s_{\alpha_1} \circ ... \circ s_{\alpha_{n+1}}$$

has length $l(v) = n-1$. And it satisfies $k_5(L,v) = 1$. Lemma 3.30 applies. Therefore a reduced presentation $(\gamma_1, ..., \gamma_{n-1})$ of v and a root γ_0 exist such that $\sum_{i=0}^{n-1} \mathbb{Z}\gamma_i = L$. Let L_1 be the subroot lattice of the presentation$(\gamma_1, ..., \gamma_{n-1}, \alpha_{n+2}, ..., \alpha_{l(w)+2k})$ of w. As k is minimal, $L_1 \subsetneqq L$. Because of $L_1 + \mathbb{Z}\gamma_0 = L$, $k_1(L, L_1) = 1$. The presentation of w in (5.5) shows $k_5(L_1, w) \leq k-1$. If $k_5(L_1,w) < k-1$ then by adding two times γ_0 to a shortest presentation of w with subroot lattice L_1, one obtains also $k_5(L,w) < k$, which contradicts the minimality of k. Thus $k_5(L_1,w)=k-1$. Induction on k gives $k_4(L_1,w)=k-1$. Now

$$k_4(L,w)\leq k_2(L,L_1)+k_4(L_1,w)=1+(k-1)=k=k_5(L,w). \tag{3.5}$$

Together with (3.2), this gives (3.3). □

3.6 Application to extended affine root lattices

The number $k_5(L,w)$ in definition 3.28 and theorem 3.29 controls existence of quasi Coxeter elements in *extended affine root systems*. These had been defined by K. Saito in [Sa85, (1.2) and (1.3)]. In [Az02] the equivalence with an alternative definition in [AABGP97] was shown.

The inequalities in lemma 3.34 below give constraints on a quasi Coxeter element w in an extended affine root system in terms of conditions for a nonreduced presentation of the induced element $\overline{w}$ in the Weyl group of the associated p.n. root lattice $L/\operatorname{Rad}(L)$.

Definition 3.31. *An **extended affine root lattice** is a triple $(L,(.,.),\Phi)$ where L is a lattice, $(.,.):L_{\mathbb{Q}}\times L_{\mathbb{Q}}\to\mathbb{Q}$ is a symmetric positive semidefinite bilinear form, and $\Phi\subset L-\{\alpha\in L\,|\,(\alpha,\alpha)=0\}$ is a subset such that the following properties hold. Here $\langle\beta,\alpha\rangle$ and s_α are defined as in (3.1) and (3.2).*

$$\Phi \text{ is a generating set of } L \text{ as a } \mathbb{Z}\text{-module.} \tag{3.6}$$

$$\text{For any } \alpha\in\Phi\ s_\alpha(\Phi)=\Phi. \tag{3.7}$$

$$\langle\beta,\alpha\rangle\in\mathbb{Z} \text{ for any } \alpha,\beta\in\mathbb{Z}. \tag{3.8}$$

*The elements of Φ are the **roots**, and Φ is an **extended affine root system**.*

$$W:=\langle s_\alpha\,|\,\alpha\in\Phi\rangle\subset O(L,(.,.)) \tag{3.9}$$

*is the **Weyl group of the extended affine root lattice**.*

Remarks 3.32. (i) In [Sa85] the definition of an extended affine root system contains additionally the following irreducibility property: $\Phi=\Phi_1\cup\Phi_2$ with $\Phi_1\perp\Phi_2\Rightarrow\Phi_1=\emptyset$ or $\Phi_2=\emptyset$.

(ii) Let $(L,(.,.),\Phi)$ be an extended affine root lattice. Because $(.,.)$ is positive semidefinite, the radical of $(L_{\mathbb{R}}, (.,.))$ is

$$\begin{aligned} \operatorname{Rad}(L_{\mathbb{R}}) &:= \{\alpha \in L_{\mathbb{R}} \,|\, (\alpha,\beta) = 0 \text{ for all } \beta \in L_{\mathbb{R}}\} \\ &= \{\alpha \in L_{\mathbb{R}} \,|\, (\alpha,\alpha) = 0\}. \end{aligned}$$

Define the radicals $\operatorname{Rad}(L) := \operatorname{Rad}(L_{\mathbb{R}}) \cap L$ and $\operatorname{Rad}(L_{\mathbb{Q}}) := \operatorname{Rad}(L_{\mathbb{R}}) \cap L_{\mathbb{Q}}$. The quotient $L/\operatorname{Rad}(L)$ with the induced bilinear form $(.,.)_{quot}$ and the induced set of roots

$$\Phi_{quot} := (\Phi + \operatorname{Rad}(L))/\operatorname{Rad}(L)$$

is obviously a p.n. root lattice. It is called the *quotient p.n. root lattice*. Any element $w \in O(L,(.,.))$ induces an element $\overline{w} \in O(L/\operatorname{Rad}(L),(.,.)_{quot})$. If $w \in W(L)$, then $\overline{w} \in W(L/\operatorname{Rad}(L))$. If $\alpha \in \Phi$ induces $\overline{\alpha} := [\alpha] \in L/\operatorname{Rad}(L)$, then $\overline{s_\alpha} = s_{\overline{\alpha}} \in W(L/\operatorname{Rad}(L))$.

(iii) The reducedness property (3.5) is not required here. Even if $(L,(.,.),\Phi)$ satisfies it, it does not necessarily hold for the *quotient p.n. root lattice*. That is the reason why in this chapter and the following ***p.n.*** *root lattices* and not only root lattices are considered.

Definition 3.33. *Let $(L,(.,.),\Phi)$ be an extended affine root lattice of rank n.*

*(a) For any element w of its Weyl group, a **presentation** $(\alpha_1,...,\alpha_k)$, the **length of a presentation**, the **subroot lattice of a presentation**, and the length $l(w)$ of the element are defined as in definition 3.16 (a).*

*(b) An element $w \in W$ is a **quasi Coxeter element** if a presentation of length n exists whose subroot lattice is the full lattice (this generalizes definition 3.18 (a)).*

The following simple lemma connects the existence of quasi Coxeter elements with the numbers $k_5(L/\operatorname{Rad}(L),\overline{w})$ from section (3.5). Theorem 3.29 says $k_5(L/\operatorname{Rad}(L),\overline{w})= k_4(L/\operatorname{Rad}(L),\overline{w})$, and theorem 3.25 allows to calculate this number.

Lemma 3.34. *Let $(L,(.,.),\Phi)$ be an extended affine root lattice of rank n with a radical $\operatorname{Rad}(L)$ of rank$r \geq 1$. Let $w \in W$ be a quasi Coxeter element, and let $\overline{w}$ be the induced element in the Weyl group $W(L/\operatorname{Rad}(L))$ of the quotient p.n. root lattice. Then*

$$l(\overline{w}) \leq n - r \quad \text{and} \tag{3.10}$$

$$l(\overline{w}) + 2k_5(L/\operatorname{Rad}(L),\overline{w}) \leq n. \tag{3.11}$$

Proof. (3.10) is a trivial consequence of (3.35), i.e. $l(\overline{w}) = \dim V_{\neq 1}(\overline{w})$. A presentation of length n of w whose subroot lattice is the full lattice L induces a presentation of length n of $\overline{w}$ whose subroot lattice is the full lattice $L/\operatorname{Rad}(L)$. This shows (3.11). □

(i) The classification of the extended affine root lattices whose quotient root lattices are inhomogeneous p.n. root lattices is nontrivial, see [Az02] and references therein.

But if $L/\operatorname{Rad}(L)$ is a homogeneous root lattice, then there is a sublattice $L_1 \subset L$ such that $(L_1, (.,.)|_{L_1}, \Phi \cap L_1)$ is isomorphic to the quotient root lattice and $(L, (.,.), \Phi)$ is equal to $(L_1 \oplus \operatorname{Rad}(L), (.,.), \Phi \cap L_1 + \operatorname{Rad}(L))$. Thus up to isomorphism, $(L, (.,.), \Phi)$ is determined by the isomorphism class of the (homogeneous) quotient root lattice and by the rank r of the radical.

(ii) Let $(L, (.,.), \Phi)$ be an extended affine root lattice of rank n with radical $\operatorname{Rad}(L)$ of rank r. Let $w \in W$ be a quasi Coxeter element such that $\overline{w}$ has maximal length $l(\overline{w}) = n - r$. Then lemma 3.34 and theorem 3.29 give

$$r \geq 2k_5(L/\operatorname{Rad}(L), \overline{w}) = 2k_4(L/\operatorname{Rad}(L), \overline{w}). \tag{3.12}$$

4 Classification in the positive semidefinite case

This chapter proves theorem 2.13, the classification conjecture (conjecture 2.12) in the case that $S + S^t$ is a positive semidefinite matrix. The proof consists of the following steps.

First, in section 4.1, we give the ad-hoc definition of the spectrum in this case.

Second, in section 4.2, we define the variance of the spectrum. We define a spectrum of a complex polynomial via the arguments of its roots, as well as its variance and non-normalized variance ($n \cdot \mathrm{Var}$). If $S + S^t$ is positive (semi-) definite, then the spectrum of S and the characteristic polynomial of $S^{-1}S^t$ determine one another. We deduce rules for calculating with those objects.

Third, in section 4.3, we recall transitivity results on braid group orbits for *ihs*, for the simple and simple elliptic types.

Fourth, in section 4.4, we carry out the proof in the positive definite case. We take any positive definite $S \in T(n, \mathbb{Z})$ and construct the basic lattice bundle from it. Then the monodromy M is a quasi Coxeter element, using calculations we can exclude all but the Coxeter elements. Then the transitivity result from section 4.3 concludes the proof.

Fifth, in section 4.5, we carry out the proof in the semidefinite case. We take any semidefinite $S \in T(n, \mathbb{Z})$, then take the radical and carry out a quotient construction, again constructing a basic lattice bundle. Using the results on the extended affine root lattices from the previous chapter, section 3.6, and a transitivity result from section 4.3, we can exclude all but simple elliptic singularities, which concludes the proof.

4.1 Ad-hoc spectrum for Stokes matrices

We define the spectrum for matrices with an ad-hoc definition for $S \in T(n, \mathbb{R})$ whenever $S + S^t$ is positive (semi-) definite.

Lemma 4.1. *(a) (Lemma) For $S \in T(n, \mathbb{R})$, the multiplicity of -1 as an eigenvalue of $S^{-1}S^t$ is even. If $S + S^t$ is positive definite, -1 is not an eigenvalue of $S^{-1}S^t$.*

(b) (Definition) Let $S \in T(n, \mathbb{R})$ be such that $S + S^t$ is positive definite or positive semidefinite. Then the spectrum is the unique tuple

$$Sp(S) = (\alpha_1, ..., \alpha_n) \in [-\frac{1}{2}, \frac{1}{2}]^n$$

with $\alpha_1 \leq ... \leq \alpha_n$ and $e^{-2\pi i \alpha_1}$,...,$e^{-2\pi i \alpha_n}$ eigenvalues of $S^{-1}S^t$ and such that $-\frac{1}{2}$ and $\frac{1}{2}$ turn up with the same multiplicity (which is half the multiplicity of -1 as an eigenvalue of $S^{-1}S^t$). If $S + S^t$ is positive definite, then by (a) $Sp(f) \in (-\frac{1}{2}, \frac{1}{2})^n$.

Proof of part (a): Denote by H_λ the generalized eigenspace with eigenvalue $\lambda \in S^1 \subset \mathbb{C}^*$ of $S^{-1}S^t$. The property $\mathrm{Rad}(S + S^t) = \ker(S^{-1}S^t + \mathrm{id}) \subset H_{-1}$ (see 5.3) shows that -1 is not an eigenvalue of $S^{-1}S^t$ if $S + S^t$ is positive definite.

The pairing on H_λ defined by $S + S^t$ is $S^{-1}S^t$-invariant. Therefore and because of lemma 5.3 in chapter 5, for $\lambda \neq -1$, the generalized eigenspaces H_λ and $H_{\lambda^{-1}}$ are dual with respect to the pairing. Thus λ and λ^{-1} have the same multiplicity as eigenvalues of $S^{-1}S^t$. As $1 = \det(S^{-1}S^t)$ is the product of all eigenvalues with their multiplicities, the multiplicity of -1 as an eigenvalue is even. □

This ad-hoc definition makes sense because the subspace of $T(n, \mathbb{R})$ where $S + S^t$ is positive (semi-) definite is contractible and thus the definition is unique up to homotopy. This, of course, is a part of conjecture 2.10, but the fact that the space where $S + S^t$ is positive (semi-) definite is bounded (embedded in $\mathbb{R}^{\frac{n \cdot (n-1)}{2}}$), makes it a much different problem compared to the indefinite case (in which case the embedded space is unbound), compare image 6.4.2 in chapter 6.

4.2 Variance of the spectrum

We define the variance of the spectrum of a matrix S. We also define the spectrum of a complex polynomial via the arguments of its roots, the spectrum of such a polynomial, the variance of it and the non-normalized variance of it. If $S + S^t$ is positive (semi-) definite, the spectrum of the characteristic polynomial of $S^{-1}S^t$ and the spectrum of S coincide. We deduce rules for calculating with these objects.

Definition 4.2. *The* ***variance of the spectrum*** *of $S \in T(n, \mathbb{R})$ is defined as*

$$Var(S) = \frac{1}{n} \sum_{j=1}^{n} \alpha_j^2, \text{ for } Sp(S) = (\alpha_1, ..., \alpha_n).$$

Lemma 4.3. *(a) (Definition) For any polynomial* $g \in \mathbb{C}[t]$ *of degree* n *with roots in* S^1 *such that the multiplicity of the zero 1 is even, its spectrum* $Sp(g)$ *is the unique tuple* $(\alpha_1, ..., \alpha_n) \in [-\frac{1}{2}, \frac{1}{2}]^n$ *such that* $e^{-2\pi i(\alpha_1+1/2)}, ..., e^{-2\pi i(\alpha_n+1/2)}$ *are the roots of* g*, and* $\alpha_1 \leq ... \leq \alpha_n$*, and* $-\frac{1}{2}$ *and* $\frac{1}{2}$ *turn up with the same multiplicity (which is half of the multiplicity of 1 as a zero of* g*).*

The ***variance*** $\mathrm{Var}(g)$ *and the* ***non-normalized*** $\mathrm{nnVar}(g)$ *are defined as*

$$\mathrm{Var}(g) := \frac{1}{n}\sum_{j=1}^{n} \alpha_j^2, \qquad \mathrm{nnVar}(g) := n \cdot \mathrm{Var}(g) = \sum_{j=1}^{n} \alpha_j^2. \tag{4.1}$$

(b) (Lemma) If $S \in T(n, \mathbb{R})$*, and* $S + S^t$ *is positive (semi-) definite and* g *is the characteristic polynomial of* $(-1)S^{-1}S^t$*, then* $Sp(S) = Sp(g)$ *and* $\mathrm{Var}(Sp(S)) = \mathrm{Var}(g)$*.*

(c) (Lemma) nnVar *is additive,*

$$\begin{aligned} \mathrm{nnVar}(g_1 \cdot g_1) &= \mathrm{nnVar}(g_1) + \mathrm{nnVar}(g_2), \\ \mathrm{nnVar}(g_1/g_2) &= \mathrm{nnVar}(g_1) - \mathrm{nnVar}(g_2) \quad (\textit{in the case } g_2|g_1). \end{aligned}$$

(d)

$$12 \cdot \mathrm{nnVar}(t^n + 1) = \frac{(n+1)(n-1)}{n}, \tag{4.2}$$

$$12 \cdot \mathrm{nnVar}(\frac{t^n - 1}{t - 1}) = \frac{(n-1)(n-2)}{n}. \tag{4.3}$$

Proof. **(b)** Clear. **(c)** Clear.

(d)

$$\begin{aligned} Sp(t^n + 1) + \frac{1}{2} &= (\frac{2j-1}{2n} \,|\, j = 1, ..., n), \\ Sp(\frac{t^n - 1}{t - 1}) + \frac{1}{2} &= (\frac{j}{n} \,|\, j = 1, ..., n-1). \end{aligned}$$

With the well-known formulas

$$\sum_{j=1}^{n} (2j-1)^2 = \frac{(2n-1)2n(2n+1)}{6} \quad \text{and} \quad \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

one easily calculates

$$
\begin{aligned}
12\cdot \mathrm{nnVar}(t^n+1) &= 12\cdot\sum_{j=1}^{n}(\frac{2j-1}{2n}-\frac{1}{2})^2 = ... = \frac{(n+1)(n-1)}{n},\\
12\cdot \mathrm{nnVar}(\frac{t^n-1}{t-1}) &= 12\cdot\sum_{j=1}^{n-1}(\frac{j}{n}-\frac{1}{2})^2 = ... = \frac{(n-1)(n-2)}{n}.
\end{aligned}
$$

□

4.3 Milnor lattices and Br_n orbits for *ihs*

Deligne studied in a letter to Looijenga [De74] the distinguished bases for the simple singularities in the case $m\equiv 0 \mod 4$. His results were extended by Voigt [Vo85, Vo85b] to quasi Coxeter elements. Kluitmann [Kl83, Kl87] studied the distinguished bases for the simple elliptic singularities in the case $m\equiv 0 \mod 4$. The following theorem puts together their results, see chapter 2 for some of the notation.

Theorem 4.4. *(a) [De74] Consider a simple singularity with $m\equiv 0 \mod 4$. Then the set $\mathcal{B}$ of distinguished bases is*

$$\mathcal{B}=\{(\delta_1,...,\delta_n)\,|\,\delta_j\in\Lambda_{van},\ s_{\delta_1}\circ...\circ s_{\delta_n}=M\}, \tag{4.4}$$

and it forms one orbit under the action of the braid group Br_n (equivalently, any sign change can be realized by a braid).

(b) [Vo85, Vo85b] Consider a homogeneous root lattice (not necessarily irreducible) without an orthogonal summand of type A_1. Consider any quasi Coxeter element $C\in W$. Fix a reduced presentation of it,

$$C=s_{\delta_1^0}\circ...\circ s_{\delta_n^0}.$$

Then $\delta_1^0,...,\delta_n^0$ defines a basic lattice bundle with pairing and is itself a distinguished basis of it. The set $\mathcal{B}$ of all distinguished bases is

$$\mathcal{B}=\{(\delta_1,...,\delta_n)\,|\,\delta_j\in\Phi,\ s_{\delta_1}\circ...\circ s_{\delta_n}=C\}, \tag{4.5}$$

and it forms a single orbit under the braid group action Br_n (equivalently, any sign change can be realized by a braid).

(c) [Kl83, Kl87] Consider a simple elliptic singularity with $m\equiv 0 \mod 4$. The set $\mathcal{B}$ of

distinguished bases is

$$\mathcal{B} = \{(\delta_1, ..., \delta_n) \,|\, \delta_j \in \Lambda_{van},\ s_{\delta_1} \circ ... \circ s_{\delta_n} = M\}, \tag{4.6}$$

and it forms one orbit under the action of the braid group Br_n *(equivalently, any sign change can be realized by a braid).*

Remark 4.5. (i) Voigt's result generalizes Deligne's result, because Coxeter elements are quasi Coxeter elements. The proofs in [Vo85] are motivated by the proof in [De74]. We will not use Voigt's result, but Deligne's result and Kluitmann's result.

4.4 Proof of 2.13 for positive definite $S + S^t$

Let $S \in T(n, \mathbb{R}) \cap M(n \times n, \mathbb{Z})$ be such that the following three properties hold: $S + S^t$ is positive definite, the CDD of S is connected, the variance $\mathrm{Var}(Sp(S))$ of the spectrum $Sp(S)$ (definition 4.1 (b)) satisfies the inequality (2.7) (for this case condition (C): $tr(S^{-1}S^t) = 1$ is not needed). We show that S is the Stokes matrix of a distinguished basis of a simple singularity.

By remark 2.16, S and $m = 0$ induce a basic lattice bundle with pairing $(H_\mathbb{Z} \to \Delta \backslash U, \underline{\delta}^0, m = 0, I)$, see definition 2.15. Here a distinguished system of paths $(\gamma_1, ..., \gamma_n)$ with $\gamma_i(0) = u_{\sigma(i)}$ for some $\sigma \in S_n$ must have been chosen. Then $\underline{\delta}$ with $\delta_i := h_{\gamma_i}(\delta_{\sigma(i)})$ satisfies

$$\begin{aligned} I(\underline{\delta}^t, \underline{\delta}) &= S + S^t \\ M &= s_{\delta_1} \circ ... \circ s_{\delta_n} \\ M(\underline{\delta}) &= \underline{\delta} \cdot (-1) S^{-1} S^t. \end{aligned}$$

$(H_{\mathbb{Z},r}, I)$ is a homogeneous root lattice, because $I : H_{\mathbb{Z},r} \times H_{\mathbb{Z},r} \to \mathbb{Z}$ is positive definite on $H_{\mathbb{R},r}$, the elements $\delta_1, ..., \delta_n$ generate $H_{\mathbb{Z},r}$, and they satisfy $I(\delta_i, \delta_i) = 2$. It is an irreducible root lattice because the CDD is connected. The monodromy M has maximal length and is a quasi Coxeter element because $\delta_1, ..., \delta_n$ generate $H_{\mathbb{Z},r}$.

With lemma 4.3 and the characteristic polynomials of the quasi Coxeter elements, it is easy to calculate the spectrum and the non-normalized variance of the characteristic polynomial of any quasi Coxeter element (given in theorem 3.21). The following first table lists the characteristic polynomials of quasi Coxeter elements (given in Table 3 of [Ca72]). The second lists the relevant results. The second column gives the symbol for a quasi Coxeter element, the sixth column gives the relation $(>, =, <)$ between $12 \cdot \mathrm{nnVar}$ and $n(\alpha_n - \alpha_1)$. Observe $\alpha_n + \alpha_1 = 0$ and $\alpha_n - \alpha_1 = 1 - 2(\alpha_1 + \frac{1}{2})$.

		characteristic polynomial
A_n	A_n	$(t^{n+1}-1)\cdot(t-1)^{-1}$
D_n	D_n	$(t^{n-1}+1)(t+1)$
	$D_n(a_1)$	
	$\vdots$	
	$D_n(a_j)$	$(t^{n-1-j}+1)(t^{j+1}+1)$
	for $1 \leq j < [\frac{n}{2}]$	
E_6	E_6	$(t^6+1)(t^3-1)(t^2+1)^{-1}(t-1)^{-1}$
	$E_6(a_1)$	$(t^9-1)(t^3-1)^{-1}$
	$E_6(a_2)$	$(t^6-1)(t^3+1)(t^2-1)^{-1}(t+1)^{-1}$
E_7	E_7	$(t^9+1)(t+1)(t^3+1)^{-1}$
	$E_7(a_1)$	t^7+1
	$E_7(a_2)$	$(t^6+1)(t^3+1)(t^2+1)^{-1}$
	$E_7(a_3)$	$(t^5+1)(t^3+1)(t+1)^{-1}$
	$E_7(a_4)$	$(t^3+1)^3(t+1)^{-2}$
E_8	E_8	$(t^{15}+1)(t+1)(t^5+1)^{-1}(t^3+1)^{-1}$
	$E_8(a_1)$	$(t^{12}+1)(t^4+1)^{-1}$
	$E_8(a_2)$	$(t^{10}+1)(t^2+1)^{-1}$
	$E_8(a_3)$	$(t^6+1)^2(t^2+1)^{-2}$
	$E_8(a_4)$	$(t^9+1)(t+1)^{-1}$
	$E_8(a_5)$	$(t^{15}-1)(t-1)(t^5-1)^{-1}(t^3-1)^{-1}$
	$E_8(a_6)$	$(t^5+1)^2(t+1)^{-2}$
	$E_8(a_7)$	$(t^6+1)(t^3+1)^2(t^2+1)^{-1}(t+1)^{-2}$
	$E_8(a_8)$	$(t^3+1)^4(t+1)^{-4}$

		$12 \cdot \mathrm{nnVar}$	$\alpha_1 + \frac{1}{2}$	$n \cdot (\alpha_n - \alpha_1)$	
A_n	A_n	$\frac{n(n-1)}{n+1}$	$\frac{1}{n+1}$	$\frac{n(n-1)}{n+1}$	$=$
D_n	D_n	$\frac{n(n-2)}{n-1}$	$\frac{1}{2(n-1)}$	$\frac{n(n-2)}{n-1}$	$=$
	$D_n(a_1)$	$\frac{(n-1)(n-3)}{n-2} + \frac{3\cdot 1}{2}$	$\frac{1}{2(n-2)}$	$\frac{n(n-3)}{n-2}$	$>$
	$\vdots$	$\vdots$	$\vdots$		
	$D_n(a_j)$ for $1 \leq j < [\frac{n}{2}]$	$\frac{(n-j)(n-2-j)}{n-1-j} + \frac{(j+2)j}{j+1}$	$\frac{1}{2(n-1-j)}$	$\frac{n(n-2-j)}{n-1-j}$	$>$
E_6	E_6	5	1/12	5	$=$
	$E_6(a_1)$	50/9	1/9	14/3	$>$
	$E_6(a_2)$	6	1/6	4	$>$
E_7	E_7	56/9	1/18	56/9	$=$
	$E_7(a_1)$	48/7	1/14	6	$>$
	$E_7(a_2)$	7	1/12	35/6	$>$
	$E_7(a_3)$	112/15	1/10	28/5	$>$
	$E_7(a_4)$	8	1/6	14/3	$>$
E_8	E_8	112/15	1/30	112/15	$=$
	$E_8(a_1)$	49/6	1/24	22/3	$>$
	$E_8(a_2)$	42/5	1/20	36/5	$>$
	$E_8(a_3)$	26/3	1/12	20/3	$>$
	$E_8(a_4)$	80/9	1/18	64/9	$>$
	$E_8(a_5)$	136/15	1/15	104/15	$>$
	$E_8(a_6)$	48/5	1/10	32/5	$>$
	$E_8(a_7)$	29/3	1/12	20/3	$>$
	$E_8(a_8)$	8	1/6	16/3	$>$

The variance $\mathrm{Var}(Sp(S))$ satisfies (2.7) by hypothesis at the beginning of this section. And (2.7) is equivalent to $12 \cdot \mathrm{nnVar} \leq n(\alpha_n - \alpha_1)$. But for all quasi Coxeter elements except the Coxeter elements $>$ holds. Thus the monodromy M must be a Coxeter element.

Thus by theorem 2.7 (a), the tuple $(V_{\mathbb{Z}}, I, M)$ is isomorphic to the tuple (Milnor lattice, intersection form, monodromy) of a simple singularity with $m \equiv 0 \mod 4$. $M = s_{\delta_1} \circ \ldots \circ s_{\delta_n}$ and Deligne's result theorem 4.4 (a) together show that the basis $\underline{\delta}$ is a distinguished basis of the basic lattice bundle with pairing of the simple singularity. Therefore this basic lattice bundle with pairing coincides with the basic lattice bundle with pairing associated to S (by theorem 2.16). And the matrix S is a Stokes matrix of the simple singularity. This completes the proof of theorem 2.3 in the positive definite cases.

Remark 4.6. The table above appeared first implicitly in the diploma thesis [Zi09] which was written under the guidance of Prof. Claus Hertling, who had calculated the inequality $12\cdot\text{nnVar} > n(\alpha_n - \alpha_1)$ for some cases and conjectured it for all quasi Coxeter elements which are not Coxeter elements. The diploma thesis [Mo] which reviewed some of the material was also written under the guidance Prof. Claus Hertling.

4.5 Proof of 2.13 for positive semidefinite $S + S^t$

Let $S \in T(\widetilde{n}, \mathbb{R}) \cap M(\widetilde{n} \times \widetilde{n}, \mathbb{Z})$ be such that the following four properties hold: $S + S^t$ is positive semidefinite with a radical $\text{Rad}(I) \supsetneqq \{0\}$, the CDD of S is connected, the variance $\text{Var}(Sp(S))$ of the spectrum $Sp(S)$ (definition 4.1 (b)) satisfies the inequality (2.7), and $tr(S^{-1}S^t) = 1$. We show that S is the Stokes matrix of a distinguished basis of a simple elliptic singularity.

As in (4.4), S and $m = 0$ induce a basic lattice bundle with pairing $(H_\mathbb{Z} \to \Delta \backslash U, \underline{\delta}^0, m = 0, I)$. And a distinguished system of paths with the related properties must have been chosen. $\text{Rad}(I)$ is also the eigenspace with eigenvalue 1 of M (see 5.3). By lemma 4.1 (a) its dimension is even, $\dim \text{Rad}(I) = 2k$ for some $k \in \mathbb{N}$. Write $\widetilde{n} = n + 2k$.

The quotient lattice $(H_{\mathbb{Z},r}/\,\text{Rad}(I), \overline{I})$ with the induced pairing $\overline{I}$ is a homogeneous root lattice of rank n, because $\overline{I} : H_{\mathbb{Z},r}/\,\text{Rad}(I) \times H_{\mathbb{Z},r}/\,\text{Rad}(I) \to \mathbb{Z}$ is positive definite on $H_{\mathbb{R},r}/\,\text{Rad}(I)_\mathbb{R}$, the elements $\delta_1, ..., \delta_{\widetilde{n}}$ generate $H_{\mathbb{Z},r}$, and they satisfy $I(\delta_i, \delta_i) = 2$. It is an irreducible root lattice (thus it is of one of the types A_n, D_n, E_6, E_7, E_8) because the CDD is connected.

The monodromy M induces an automorphism $\overline{M}$ in the Weyl group W which has maximal length because it has no eigenvalue 1. Let $\overline{\delta}_1, ..., \overline{\delta}_{\widetilde{n}}$ denote the classes of $\delta_1, ..., \delta_{\widetilde{n}}$ in $H_{\mathbb{Z},r}/\,\text{Rad}(I)$. Then

$$\overline{M} = s_{\overline{\delta}_1} \circ ... \circ s_{\overline{\delta}_{n+2k}} \tag{4.7}$$

is a non-reduced presentation of $\overline{M}$. If

$$\overline{M} = s_{\alpha_1} \circ ... \circ s_{\alpha_n}$$

is a reduced presentation, the lattice $L(\{\alpha_1, ..., \alpha_n\}) \subset H_{\mathbb{Z},r}/\,\text{Rad}(I)$ is a sub-root lattice of finite index. The a priori possible sublattices (and the isomorphism classes of pairs $(V_\mathbb{Z}, \text{sublattice})$) can be read off from (theorem 3.21, Table 4.1). The a priori possible conjugacy classes of $\overline{M}$ can be read off from (theorem 3.25).

The inequality (2.7) will largely reduce the possibilities. Let $P_M(t)$ and $P_{\overline{M}}(t)$ be the characteristic polynomials of M and $\overline{M}$. Then

$$\begin{aligned}
P_M(t) &= P_{\overline{M}}(t)\cdot(t-1)^{2k},\\
12\cdot \mathrm{Var}(Sp(S)) &= 12\cdot \mathrm{nnVar}(P_M(t))\\
&= 12\cdot \mathrm{nnVar}(P_{\overline{M}}(t))+6k,\\
\alpha_1+\frac{1}{2} &= 0,\quad \alpha_{n+2k}+\frac{1}{2}=1,\\
(n+2k)(\alpha_{n+2k}-\alpha_1) &= n+2k.
\end{aligned}$$

Therefore the inequality (2.7) is here equivalent to

$$4k \le n - 12\cdot \mathrm{nnVar}(P_{\overline{M}}(t)). \tag{4.8}$$

Lemma 4.7. *The inequality (4.8) is for $\overline{M}$ only in the following cases satisfied.*

D_n	2^{l-1}	$(b_1,...,b_{2l})$	with $4k \le \sum_{j=1}^{2l}\frac{1}{b_j}$
E_6	3	$3A_2$	for $k=1$
E_7	4	$A_1 \perp 2A_3 \sim 3A_1 \perp D_4(a_1)$	for $k=1$
	4	$3A_1 \perp D_4$	for $k=1$
	8	$7A_1$	for $k=1$
E_8	6	$A_1 \perp A_2 \perp A_5$	for $k=1$
	8	$4A_1 \perp D_4$	for $k=1$
	8	$4A_1 \perp D_4(a_1) \sim 2A_1 \perp 2A_3$	for $k=1$
	9	$4A_2$	for $k=1$
	16	$8A_1$	for $k=1$ or 2

(4.9)

Proof. One has to calculate nnVar for the characteristic polynomials for all (conjugacy classes of) elements of maximal length in (theorem 3.25). This can be done easily using either lemma 4.3 or the table of nnVar for the quasi Coxeter elements in the proof of lemma 4.3. Then one has to check whether the inequality (4.8) holds. We carry out the details for the cases for D_n.

A conjugacy class of elements of maximal length in the Weyl group of D_n has a characteristic polynomial $\prod_{j=1}^{2l}(t^{b_j}+1)$ with $b_1,...,b_{2l}\in\mathbb{N}$ and $b_1\ge ...\ge b_{2l}$. Then by (4.2)

$$12\cdot \mathrm{nnVar} \;=\; \sum_{j=1}^{2l}\frac{(b_j+1)(b_j-1)}{b_j} = \sum_{j=1}^{2l}(b_j-\frac{1}{b_j}) = n-\sum_{j=1}^{2l}\frac{1}{b_j}.$$

Thus (4.8) is here equivalent to

$$4k \leq \sum_{j=1}^{2l} \frac{1}{b_j}.$$

□

The next theorem finishes the proof by using the number $k_5(L, \bar{M})$, defined in (definition 3.28), to exclude all but the lattices of the simple elliptic singularities.

Theorem 4.8. *Let $(L, (.,.), \Phi)$ be a root latticeof type D_n, E_7 or E_8. Let w be a Weyl group element of maximal length $l(w) = n$. There is no nonreduced presentation*

$$w = s_{\delta_1} \circ ... \circ s_{\delta_{n+2k}}$$

for some $k \in \mathbb{N}$ ($k \geq 1$) such that $L(\{\delta_1, ..., \delta_{n+2k}\}) = V_{\mathbb{Z}}$ is the full lattice of the types below (first column); $L(\{\delta_1, ..., \delta_n\})$ is of the types in the third column and k satisfies the given properties.

D_n	2^{l-1}	$(b_1, ..., b_{2l})$	with $4k \leq \sum_{j=1}^{2l} \frac{1}{b_j}$
E_7	4	$3A_1 \perp D_4$	for $k = 1$
	8	$7A_1$	for $k = 1$
E_8	8	$4A_1 \perp D_4$	for $k = 1$
	8	$4A_1 \perp D_4(a_1) \sim 2A_1 \perp 2A_3$	for $k = 1$
	9	$4A_2$	for $k = 1$
	16	$8A_1$	for $k = 1$ or 2

Except in the case of D_n with $(b_1, ..., b_{2l}) = (1, 1, 1, 1)$ and $k = 1$.

Proof. For the *p. n. root lattice* $(L, (.,.), \Phi)$ and $w \in W$ one can define the numbers $k_4(L, w)$ and $k_5(L, w)$ (definitions 3.24, 3.28). The latter is

$$\begin{aligned} k_5(L, w) :=& \min\{k \mid \text{a presentation } (\alpha_1, ..., \alpha_{l(w)+2k}) \\ & \qquad \text{with subroot lattice the full lattice exists}\} \end{aligned}$$

and as thus controls the existence of quasi Coxeter elements. By lemma 3.34 or the example (ii) after 3.34 we have

$$r \geq 2k_5 = 2k_4.$$

To prove the statements it suffices to calculate the numbers k_4 using theorem 3.25.

(a) The lemma implies for w of type D_n

$$\begin{aligned} k_4(L,w) &:= \min\{k_2(L,L_1) \mid \text{ a reduced presentation of } \bar{M} \text{ with subroot lattice } L_1 \text{ exists.}\} \\ &= l-1 \end{aligned}$$

So

$$\begin{aligned} 4\cdot k &\geq 4\cdot(l-1) \\ &\geq 2\cdot(2l-2) \\ &\geq 2\cdot(\sum_{j=1}^{2l} 1-2) \\ &\geq 2\cdot(\sum_{j=1}^{2l} \frac{1}{b_j}-2). \end{aligned}$$

And thus $4\cdot k > \sum_{j=1}^{2l}\frac{1}{b_j}$ if $(b_1,...,b_{2l})\neq(1,1,1,1)$.

(b)/(c) To calculate the numbers k_4 for (b) and (c) it suffices to calculate $k_2(L,L_1)$ (theorem3.25 (c)) for all subroot lattices $L_1=\{\delta_1,...,\delta_n\}$. Since $k_1(L,L_1)=k_2(L,L_1)=k_3(L,L_1)$ (theorem 3.11) those numbers can simply be read of from the tables 3.3-3.4 in chapter 3 for k_1. With those numbers and $r\geq 2k_4$ the statements (b) and (c) follow. □

Theorem 4.8 allows to exclude all cases except the four cases

$$\begin{aligned} 2D_2\cong 4A_1\subset D_4 &\text{ and } k=1, \\ 3A_2\subset E_6 &\text{ and } k=1, \\ A_1\perp 2A_3\sim 3A_1\perp D_4(a_1)\subset E_7 &\text{ and } k=1, \\ A_1\perp A_2\perp A_5\subset E_8 &\text{ and } k=1. \end{aligned}$$

Remark 4.9. (i) We call the set of all 6×6 matrices S which give rise to a sublattice in the isomorphism class $2D_2\subset D_4$ and to $k=1$ the *class* $\widetilde{D_4}$. Then the matrices $S^{-1}S^t$ have the eigenvalues $-1,-1,1,1,1,1$. The variance inequality is here satisfied as an equality. But the trace of the matrix $S^{-1}S^t$ is $+2$, not $+1$.

(ii) We did not check whether the matrices in the class $\widetilde{D_4}$ satisfy an analog of Kluitmann's theorem 4.4 (c), but we expect it.

(iii) Some matrices in the class $\widetilde{D_4}$ turn up in the construction of Lefschetz thimbles for the pair of functions $(f_1, f_2) : \mathbb{C}^3 \to \mathbb{C}^2$, where both f_1 and f_2 are quadratic (and sufficiently different), in [Eb87, p. 46, the case $n = 1$].

In the three cases $3A_2, A_1 \perp 2A_3 \sim 3A_1 \perp D_4(a_1)$ and $A_1 \perp A_2 \perp A_5$, by theorem 2.7 (b) the tuple $(V_{\mathbb{Z}}, I, M)$ is isomorphic to the tuple (Milnor lattice, intersection form, monodromy) of a simple elliptic singularity with $m \equiv 0 \mod 4$. $M = s_{\delta_1} \circ ... \circ s_{\delta_{\widetilde{n}}}$ and Kluitmann's result theorem 4.4 (c) together show that the basis $\underline{\delta}$ is a distinguished basis of the basic lattice bundle with pairing of the simple elliptic singularity. Therefore this basic lattice bundle with pairing coincides with the basic lattice bundle with pairing associated to S (by theorem 2.16). And the matrix S is a Stokes matrix of the simple elliptic singularity. This completes the proof of theorem 2.3 in the positive semidefinite cases. □

The trace condition (C) might not seem to be essential here, as it is only used to exclude one single case, which belongs to an ICIS. But in fact, the trace condition, later on, will be essential, see for instance the final chapter 7.

5 Real Seifert forms and Steenbrink PMHS

The previous two chapters contained the proof of theorem 2.13 and thus conjecture 2.12 in the case that $S \in T(n, \mathbb{R})$ is positive semidefinite. The other case, that for an $S \in T(n, \mathbb{R})$ the matrix $S + S^t$ is indefinite motivates this and the next chapter. In that case, the ad-hoc spectrum definition is not possible.

Instead, we will use the subspaces T_{HOR1} and T_{HOR2} of $T(n, \mathbb{R})$ in the next chapter to assign spectral numbers, and conjecture an assignment to all of $T(n, \mathbb{R})$. The matrices in those spaces give rise to abstract forms of PMHS, a special form we call Steenbrink PMHS. To establish those results, we first need to study Seifert forms in this chapter, before we go to the explicit case in the next section.

We study the following setting. A real *Seifert form*, that is a nondegenerate bilinear form on a finite-dimensional $\mathbb{R}$ vector space $L : H_{\mathbb{R}} \times H_{\mathbb{R}} \to \mathbb{R}$, which is in general neither symmetric nor antisymmetric. It induces

(1) the *monodromy*: an automorphism $M : H_{\mathbb{R}} \to H_{\mathbb{R}}$, uniquely defined by $L(Ma, b) = L(b, a)$ for all $a, b \in H_{\mathbb{R}}$,

(2) a *symmetric bilinear form* $I_s = L + L^t$,

(3) and an *antisymmetric bilinear form* $I_a = L - L^t$.

course, L, I_s and I_a then are M-invariant.

The structure of the chapter is as follows. We start with the linear algebra study of L, and end with our motivation, the application to the case of *ihs*. The steps are:

Step 1 Section 5.1: we show how the pair $(H_{\mathbb{R}}, L)$ splits uniquely (up to iso.) into an orthogonal sum and discuss the classification of these pairs. The classification is done via so-called *isometric triples.*

Step 2 Section 5.2, 5.3: we connect Seifert forms and isometric triples to (then defined) *Steenbrink PMHS*.

Step 3 Section 5.4: we connect holomorphic vector bundles on $\mathbb{C}^*$ with flat (holomorphic) connection with flat real subbundles with certain conditions using data from Step 2.

Step 4 Section 5.5: we use the previous steps to provide a *Thom-Sebastiani formula* for the *TEZP structure* of an *ihs*.

The steps 1-3 are in a complete abstract setting and as such make it possible to apply the results especially to Landau-Ginzburg models and derived algebraic geometry. In more detail, we classify in section 5.1 the irreducible Seifert form pairs (theorem 5.9) and give the signatures of the form I_s, this is done using the classification of isometric triples by Milnor.

Section 5.2 and 5.3 connects the Seifert form pairs and the isometric triples with Steenbrink polarized mixed Hodge structures (Steenbrink PMHS), an enhancement of mixed Hodge structures which we define in definition 5.15. Section 5.2 reviews mixed Hodge structures, several enhancements by automorphisms and/or polarizing forms, and Steenbrink's notions of spectral pairs and spectral numbers (definition 5.18) of Steenbrink mixed Hodge structures. Theorem 5.20 gives the irreducible isometric triples in a Steenbrink PMHS.

Section 5.3 connects this with Seifert forms. It defines a Seifert form L^{nor} for a Steenbrink PMHS (definition 5.22 (c)) and gives the irreducible Seifert form pairs in a Steenbrink PMHS (theorem 5.24). This theorem recovers also a result of Némethi [Ne95], namely that the spectral pairs modulo $2\mathbb{Z} \times \{0\}$ are equivalent to the Seifert form of a Steenbrink PMHS. The sections 5.1 to 5.3 have some overlap with the paper [Ne95]. A new ingredient which is neither in [Ne95] nor in any other papers except [He03], is an automorphism G.

Section 5.4 works with a holomorphic vector bundle on $\mathbb{C}^*$ with a flat holomorphic connection. It recalls elementary sections, the spaces C^α which they form, the Malgrange-Kashiwara V-filtration, and Brieskorn lattices. It states the correspondence lemma 5.27 between three data: sums of two isometric triples, Seifert form pairs, and holomorphic bundles on $\mathbb{C}^*$ with a flat holomorphic connection and a flat real subbundle and a certain flat pairing P between the fibers at $z \in \mathbb{C}^*$ and $-z$. Theorem 5.28 enhances this correspondence with formulas which express a Fourier-Laplace transformation between elementary sections using G and which connect the pairings P and L^{nor}. Theorem 5.28 and theorem 5.23 give a relation between P and S, which was stated without proof in [He03, Proposition 7.7].

Section 5.5 is the application to the case of *ihs*. The main result is theorem 5.32 which gives the Thom-Sebastiani formula $TEZP(f) \otimes TEZP(g) \cong TEZP(f+g)$. An application is a correction of a Thom-Sebastiani formula in [SS85] for the Hodge filtration $F^\bullet_{St}$ of Steenbrink's MHS. One has to replace in that formula $F^\bullet_{St}$ by $G(F^\bullet_{St})$. So, here the automorphism G is important.

5.1 Isometric structures and real Seifert forms

This section classifies the structure in the middle homology of *ihs*, the *Seifert form pairs*, in general. The main result is 5.9. The classification rests on the classification of *isometric triples* by Milnor [Mi69], but Némethi [Ne95] undertook the classification of hermitian Seifert form pairs. We first define Seifert form pairs and isometric triples. Explain their relation, which is not 1-1, and recall their resp. direct sum decomposition. Then we carry out the classification of Seifert form pairs, based on [Mi69]. One can derive the classification of real Seifert form pairs from [Ne95] as well. But we found it easier to use [Mi69, S 3] directly.

Notations 5.1. In this section, H_K is a finite-dimensional vector space over a field K. If $H_\mathbb{R}$ is given, then $H_\mathbb{C} = H_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} = H_\mathbb{R} \oplus iH_\mathbb{R}$ is the complexification of $H_\mathbb{R}$.

If $L : H_K \times H_K \to K$ is a bilinear form then two subspaces $V_1, V_2 \subset H_K$ are *L-orthogonal* if $L(V_1, V_2) = L(V_2, V_1) = 0$.

If $M : H_K \to H_K$ is an automorphism, then $M_s, M_u, N : H_K \to H_K$ denote its semisimple, its unipotent and its nilpotent part with $M = M_s M_u = M_u M_s$ and $N = \log M_u, e^N = M_u$. If $K = \mathbb{C}$, denote $H_\lambda := \ker(M_s - \lambda \cdot \mathrm{id}) : H_\mathbb{C} \to H_\mathbb{C}$, $H_{\neq 1} := \bigoplus_{\lambda \neq 1} H_\lambda$, $H_{\neq -1} := \bigoplus_{\lambda \neq -1} H_\lambda$.

Definition 5.2. *(a) A Seifert form pair is a pair $(H_\mathbb{R}, L)$ where $L : H_\mathbb{R} \times H_\mathbb{R} \to \mathbb{R}$ is a nondegenerate bilinear form. It is called irreducible if $H_\mathbb{R}$ does not split into two nontrivial (i.e. both $\neq \{0\}$) L-orthogonal subspaces. Its* monodromy *is the unique automorphism below in lemma 5.3, it's eigenvalues are the eigenvalues of its monodromy. A S^1*-Seifert form pair *is a Seifert form pair with eigenvalues in S^1.*

(b) An isometric triple is a triple $(H_\mathbb{R}, M, S)$ where $M : H_\mathbb{R} \to H_\mathbb{R}$ is an automorphism called monodromy, $S : H_\mathbb{R} \times H_\mathbb{R} \to \mathbb{R}$ is a nondegenerate and (for some $m \in \{0, 1\}$) $(-1)^m$-symmetric bilinear form and M is an isometry of S. The triple is called irreducible if $H_\mathbb{R}$ does not split into two nontrivial S-orthogonal and M-invariant subspaces.

The following two lemmata show that one can go from Seifert form pairs to isometric triples and vice versa, though the relation is not 1-1. Starting with $(H_\mathbb{R}, L)$, one has a fixed monodromy M on $H_\mathbb{R}$, but there are several possible choices of a suitable subspace $H'_\mathbb{R}$ and a bilinear form S such that $(H'_\mathbb{R}, M, S)$ is an isometric triple. Below I_s and I_a are most prominent, but $I_s^{(2)}, I_a^{(2)}, I_s^{(3)}$ and $I_a^{(3)}$ play a role in the PMHS's of *ihs*.

Lemma 5.3. *A Seifert form pair $(H_\mathbb{R}, L)$ comes equipped with the following data.*

(a) Its monodromy $M : H_\mathbb{R} \to H_\mathbb{R}$ *is the unique automorphism with*

$$L(Ma, b) = L(b, a) \qquad \text{for all } a, b \in H_\mathbb{R}. \tag{5.1}$$

(b) Define bilinear forms I_s *and* I_a *on* $H_\mathbb{R}$, $I_s^{(2)}$ *on* $H_\mathbb{R} \cap H_{\neq -1}$, $I_a^{(2)}$ *on* $H_\mathbb{R} \cap H_{\neq 1}$, $I_s^{(3)}$ *on* $H_\mathbb{R} \cap H_1$ *and* $I_a^{(3)}$ *on* $H_\mathbb{R} \cap H_{-1}$ *by*

$$\begin{aligned}
I_s(a,b) &:= L(b,a) + L(a,b) = L((M+\mathrm{id})a, b), \qquad (5.2)\\
I_a(a,b) &:= L(b,a) - L(a,b) = L((M-\mathrm{id})a, b),\\
I_s^{(2)}(a,b) &:= L(a, \frac{1}{M+\mathrm{id}} b) = I_s(\frac{1}{M+\mathrm{id}} a, \frac{1}{M+\mathrm{id}} b),\\
I_a^{(2)}(a,b) &:= L(a, \frac{1}{M-\mathrm{id}} b) = I_a(\frac{1}{M-\mathrm{id}} a, \frac{1}{M-\mathrm{id}} b),\\
I_s^{(3)}(a,b) &:= L(a, \frac{N}{M-\mathrm{id}} b),\\
I_a^{(3)}(a,b) &:= L(a, \frac{N}{M+\mathrm{id}} b),
\end{aligned}$$

where $\frac{N}{M-\varepsilon\,\mathrm{id}}$ *on* $H_\mathbb{R} \cap H_\varepsilon$ *for* $\varepsilon \in \{\pm 1\}$ *is the inverse of the automorphism*

$$\frac{M - \varepsilon\,\mathrm{id}}{N} \quad := \quad \frac{\varepsilon e^N - \varepsilon\,\mathrm{id}}{N} := \varepsilon \cdot \sum_{k=1}^{\dim H_\mathbb{R}} \frac{1}{k!} \cdot N^{k-1}. \tag{5.3}$$

(Remark that for example in the case $N = 0$ *we have* $\frac{M-\varepsilon\,\mathrm{id}}{N} = \varepsilon\,\mathrm{id}$ *.)*

The bilinear forms $I_s, I_s^{(2)}$ *and* $I_s^{(3)}$ *are symmetric, the bilinear forms* $I_a, I_a^{(2)}$ *and* $I_a^{(3)}$ *are antisymmetric.* $I_s^{(2)}, I_a^{(2)}, I_s^{(3)}$ *and* $I_a^{(3)}$ *are nondegenerate (on their respective definition domains). The radical of* I_s *is* $\ker(M+\mathrm{id}) \subset H_{-1}$*, so* I_s *is nondegenerate on* $H_{\neq -1}$*. The radical of* I_a *is* $\ker(M-\mathrm{id}) \subset H_1$*, so* I_a *is nondegenerate on* $H_{\neq 1}$*.*

The automorphisms M, M_s *and* M_u *are isometries of* $L, I_s, I_a, I_s^{(2)}, I_a^{(2)}, I_s^{(3)}$*,* $I_a^{(3)}$ *and* N *is an infinitesimal isometry of them.*

Proof. (a) M is well defined and unique because L is nondegenerate.

(b) M is an isometry of L because applying two times (5.1) gives

$$L(Ma, Mb) = L(Mb, a) = L(a, b).$$

$I_s^{(3)}$ is symmetric and $I_a^{(3)}$ is antisymmetric because for $\varepsilon \in \{\pm 1\}$ and $a, b \in H_\varepsilon$

$$\begin{aligned} L(a, \frac{N}{M - \varepsilon\,\mathrm{id}} b) &= L(M \frac{N}{M - \varepsilon\,\mathrm{id}} b, a) = \varepsilon L(\frac{-N}{M^{-1} - \varepsilon\,\mathrm{id}} b, a) \\ &= \varepsilon L(b, \frac{N}{M - \varepsilon\,\mathrm{id}} a). \end{aligned}$$

The rest is elementary linear algebra. □

Lemma 5.4. *From an isometric triple one can obtain in different ways a Seifert form pair. Let $\delta \in \{\pm 1\}$. Let $(H_\mathbb{R}, M, S)$ be an isometric triple with S δ-symmetric and $H_{-\delta} = \{0\}$, so $H = H_{\neq -\delta}$ and $M + \delta\,\mathrm{id}$ is invertible. Define the Seifert forms $L^{(1)}$ and $L^{(2)}$ by*

$$\begin{aligned} L^{(1)}(a, b) &:= S(\frac{1}{M + \delta\,\mathrm{id}} a, b), \\ L^{(2)}(a, b) &:= S(a, (M + \delta\,\mathrm{id}) b). \end{aligned} \tag{5.4}$$

If $H = H_\delta$, define the Seifert form $L^{(3)}$ by

$$L^{(3)}(a, b) := S(a, \frac{M - \delta\,\mathrm{id}}{N} b).$$

For any of these Seifert forms, the monodromy M in lemma 5.3 (a) is the monodromy M here. The following table says which bilinear form in lemma 5.3 (b) is the S here.

	$L^{(1)}$	$L^{(2)}$	$L^{(3)}$	
$\delta = 1$	I_s	$I_s^{(2)}$	$I_s^{(3)}$	$= S$
$\delta = -1$	I_a	$I_a^{(2)}$	$I_a^{(3)}$	$= S$

(5.5)

Proof. M here and M in lemma 5.3 (a) coincide because the M here satisfies

$$\begin{aligned} L^{(1)}(Ma, b) &= S(\frac{M}{M + \delta\,\mathrm{id}} a, b) = \delta \cdot S(b, \frac{M}{M + \delta\,\mathrm{id}} a) \\ &= \delta \cdot S(\frac{M^{-1}}{M^{-1} + \delta\,\mathrm{id}} b, a) = S(\frac{\mathrm{id}}{\delta\,\mathrm{id} + M} b, a) = L^{(1)}(b, a), \end{aligned}$$

and similarly $L^{(2)}(Ma, b) = L^{(2)}(b, a)$, $L^{(3)}(Ma, b) = L^{(3)}(b, a)$. The table follows from comparison of the formulas in lemma 5.3 (b) and in lemma 5.4. □

Because in a Seifert form pair $(H_\mathbb{R}, L)$ and in an isometric triple $(H_\mathbb{R}, M, S)$, the monodromy M is an isometry, the subspace H_λ is L-dual respectively S-dual to $H_{\lambda^{-1}}$ and L-orthogonal respectively S-orthogonal to all subspaces H_κ with $\kappa \neq \lambda^{-1}$. Therefore $H_\mathbb{R}$ splits canonically into the M-invariant and L-orthogonal respectively S-orthogonal summands

$$H_{\mathbb{R}} \cap H_1, \quad H_{\mathbb{R}} \cap H_{-1}, \tag{5.6}$$

$$H_{\mathbb{R}} \cap (H_\lambda \oplus H_{\overline{\lambda}}) \qquad \text{for } \lambda \in \{\zeta \in S^1 \mid \operatorname{Im}\zeta > 0\}, \tag{5.7}$$

$$H_{\mathbb{R}} \cap (H_\lambda \oplus H_{\lambda^{-1}}) \qquad \text{for } \lambda \in \mathbb{R}_{>1} \cup \mathbb{R}_{<-1}, \tag{5.8}$$

$$H_{\mathbb{R}} \cap (H_\lambda \oplus H_{\lambda^{-1}} \oplus H_{\overline{\lambda}} \oplus H_{\overline{\lambda}^{-1}}) \tag{5.9}$$
$$\text{for } \lambda \in \{\zeta \in \mathbb{C} \mid |\zeta| > 1, \operatorname{Im}\zeta > 0\}.$$

In the case of a Seifert form pair, one can choose on each of these summands a bilinear form S in lemma 5.3 (b) such that (the summand, M, S) becomes an isometric triple. Then a splitting of this summand into (irreducible) Seifert form pairs is a splitting into (irreducible) isometric triples and vice versa.

Milnor classified isometric triples in [Mi69, S 3] and proved part (a) of the following theorem. Part (b) is a consequence of part (a) and the lemmata 5.3 and 5.4.

Theorem 5.5. *(a) Any isometric triple splits into a direct sum (the summands are S-orthogonal and M-invariant) of irreducible isometric triples. The splitting is unique up to isomorphism.*

(b) Any Seifert form pair splits into a direct sum (the summands are L-orthogonal) of irreducible Seifert form pairs. The splitting is unique up to isomorphism.

It rests to classify the irreducible isometric triples and via this the irreducible Seifert form pairs. The irreducible isometric triples had been classified in an implicit way in [Mi69, S 3]. Némethi [Ne95] classified the *hermitian Seifert form pairs*, building on [Mi69, S 3], and one can derive from [Ne95] also the irreducible real Seifert form pairs. But we will use [Mi69, S 3] directly. We start with examples which in fact will contain all irreducible isometric triples.

Examples 5.6. (i) For $n \in \mathbb{Z}_{\geq 1}$, the following $n \times n$-matrices will be useful.

$$E_n = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}, \quad J_n = (\delta_{j,k+1})_{j,k=1,\dots,n} = \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix},$$

$$E_n^{per} = ((-1)^{j-1}\delta_{j,n+1-k})_{j,k=1,\dots,n} = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \ddots & & \\ (-1)^{n-1} & & & \end{pmatrix}.$$

(ii) Choose $n\in\mathbb{Z}_{\geq 1}$, $\lambda\in\{\pm 1\}$ and $\varepsilon\in\{\pm 1\}$. Let $\dim H_\mathbb{R}=n$, and let $\underline{a}=(a_1,...,a_n)$ be a basis of $H_\mathbb{R}$. Then the monodromy M and the $(-1)^{n-1}$-symmetric pairing S with

$$M_s=\lambda\cdot\mathrm{id},\quad N\underline{a}=\underline{a}\cdot J_n,\quad S(\underline{a}^t,\underline{a})=\varepsilon\cdot E_n^{per} \tag{5.10}$$

give an isometric triple $(H_\mathbb{R},M,S)$, which is called $\mathrm{Tr}(\lambda,1,n,\varepsilon)$. It is irreducible because the monodromy has only one Jordan block.

(iii) Choose $n\in\mathbb{Z}_{\geq 1}$, $\lambda\in S^1$, $\varepsilon\in\{\pm 1\}$ and $m\in\{0,1\}$. Let $\dim H_\mathbb{R}=2n$. Choose a complex subspace $H^{(1)}\subset H_\mathbb{C}$ such that $H_\mathbb{C}=H^{(1)}\oplus\overline{H^{(1)}}$. Let $\underline{a}=(a_1,...,a_n)$ be a basis of $H^{(1)}$. Then $\underline{\overline{a}}=(\overline{a}_1,...,\overline{a}_n)$ is a basis of $\overline{H^{(1)}}$. Then the monodromy M and the $(-1)^m$-symmetric pairing S with

$$\begin{aligned} M_s=\lambda\cdot\mathrm{id}|_{H^{(1)}}\oplus\overline{\lambda}\cdot\mathrm{id}|_{\overline{H^{(1)}}},&\quad N\underline{a}=\underline{a}\cdot J_n,\quad N\underline{\overline{a}}=\underline{\overline{a}}\cdot J_n,\\ S(\begin{pmatrix}\underline{a}^t\\ \underline{\overline{a}}^t\end{pmatrix},(\underline{a},\underline{\overline{a}}))=i^{n+m+1}&\cdot\varepsilon\cdot\begin{pmatrix}0 & E_n^{per}\\ (-1)^{n+m+1}E_n^{per} & 0\end{pmatrix}\end{aligned} \tag{5.11}$$

give an isometric triple $(H_\mathbb{R},M,S)$, which is called $\mathrm{Tr}(\lambda,2,n,m,\varepsilon)$. Using the basis $(\underline{\overline{a}},\underline{a})$ instead of the basis $(\underline{a},\underline{\overline{a}})$, one finds

$$\mathrm{Tr}(\lambda,2,n,m,\varepsilon)\cong\mathrm{Tr}(\overline{\lambda},2,n,m,(-1)^{n+m+1}\varepsilon). \tag{5.12}$$

If $\lambda\neq\pm 1$ it is irreducible because the two generalized eigenspaces $H^{(1)}$ and $\overline{H^{(1)}}$ are S-dual (that they are complex conjugate, serves equally well) and the monodromy has on each of them only one Jordan block. For $\lambda=\pm 1$ see lemma 5.7.

(iv) Choose $n\in\mathbb{Z}_{\geq 1}$, $\lambda\in\mathbb{R}_{>1}\cup\mathbb{R}_{<-1}$ and $m\in\{0,1\}$. Let $\dim H_\mathbb{R}=2n$. Choose a splitting $H_\mathbb{R}=H^{(1)}\oplus H^{(2)}$ into two n-dimensional subspaces. Let $\underline{a}=(a_1,...,a_n)$ be a basis of $H^{(1)}$, and let $\underline{b}=(b_1,...,b_n)$ be a basis of $H^{(2)}$. Then the monodromy M and the $(-1)^m$-symmetric pairing S with

$$\begin{aligned} M_s=\lambda\cdot\mathrm{id}|_{H^{(1)}}\oplus\lambda^{-1}\cdot\mathrm{id}|_{H^{(2)}},&\quad N\underline{a}=\underline{a}\cdot J_n,\quad N\underline{b}=\underline{b}\cdot J_n,\\ S(\begin{pmatrix}\underline{a}^t\\ \underline{b}^t\end{pmatrix},(\underline{a},\underline{b}))&=\begin{pmatrix}0 & E_n^{per}\\ (-1)^{n+m+1}E_n^{per} & 0\end{pmatrix}\end{aligned} \tag{5.13}$$

give an isometric triple $(H_\mathbb{R},M,S)$, which is called $\mathrm{Tr}(\lambda,2,n,m)$. It is irreducible because the two generalized eigenspaces $H^{(1)}$ and $H^{(2)}$ are S-dual and the monodromy has on each of them only one Jordan block.

(v) Choose $n\in\mathbb{Z}_{\geq 1}$, $\lambda\in\{\zeta\in\mathbb{C}\,|\,|\zeta|>1,\mathrm{Im}\,\zeta>0\}$, $\varepsilon\in\{\pm 1\}$ and $m\in\{0,1\}$. Let

$\dim H_{\mathbb{R}} = 4n$. Choose two n-dimensional complex subspaces $H^{(1)}, H^{(2)} \subset H_{\mathbb{C}}$ such that $H_{\mathbb{C}} = H^{(1)} \oplus H^{(2)} \oplus \overline{H^{(1)}} \oplus \overline{H^{(2)}}$. Let $\underline{a} = (a_1, ..., a_n)$ be a basis of $H^{(1)}$, and let $\underline{b} = (b_1, ..., b_n)$ be a basis of $H^{(2)}$. Then the monodromy M and the $(-1)^m$-symmetric pairing S with

$$\begin{aligned} M_s &= \lambda \cdot \mathrm{id}\,|_{H^{(1)}} \oplus \lambda^{-1} \cdot \mathrm{id}\,|_{H^{(2)}} \oplus \overline{\lambda} \cdot \mathrm{id}\,|_{\overline{H^{(1)}}}, \overline{\lambda}^{-1} \cdot \mathrm{id}\,|_{\overline{H^{(2)}}}, \\ N\underline{a} &= \underline{a} \cdot J_n, \quad N\underline{b} = \underline{b} \cdot J_n, \quad N\overline{\underline{a}} = \overline{\underline{a}} \cdot J_n, \quad N\overline{\underline{b}} = \overline{\underline{b}} \cdot J_n, \end{aligned} \tag{5.14}$$

$$S(\begin{pmatrix} \underline{a}^t \\ \underline{b}^t \\ \overline{\underline{a}}^t \\ \overline{\underline{b}}^t \end{pmatrix}, (\underline{a}, \underline{b}, \overline{\underline{a}}, \overline{\underline{b}})) = \begin{pmatrix} 0 & E_n^{per} & 0 & 0 \\ (-1)^{n+m+1} E_n^{per} & 0 & 0 & 0 \\ 0 & 0 & 0 & E_n^{per} \\ 0 & 0 & (-1)^{n+m+1} E_n^{per} & 0 \end{pmatrix}$$

give an isometric triple $(H_{\mathbb{R}}, M, S)$, which is called $\mathrm{Tr}(\lambda, 4, n, m)$. It is irreducible because the monodromy has on each of the four generalized eigenspaces only one Jordan block, $H^{(2)}$ is S-dual to $H^{(1)}$, $\overline{H^{(1)}}$ is the complex conjugate of $H^{(1)}$ and $\overline{H^{(2)}}$ is S-dual to $\overline{H^{(1)}}$ (and the complex conjugate of $H^{(2)}$).

Lemma 5.7. *Consider $\lambda \in \{\pm 1\}$. The types $\mathrm{Tr}(\lambda, 1, n, \varepsilon)$ in the examples 5.6 (ii) are irreducible and pairwise non-isomorphic. If $n + m + 1 \equiv 1(2)$ then by (5.12)*

$$\mathrm{Tr}(\lambda, 2, n, m, 1) \;\; \cong \;\; \mathrm{Tr}(\lambda, 2, n, m, -1). \tag{5.15}$$

This type is irreducible. If $n + m + 1 \equiv 0(2)$ then $\mathrm{Tr}(\lambda, 2, n, m, 1)$ and $\mathrm{Tr}(\lambda, 2, n, m, -1)$ are not isomorphic and are reducible,

$$Tr(\lambda, 2, n, m, \varepsilon) \;\; \cong \;\; 2 \cdot \mathrm{Tr}(\lambda, 1, n, (-1)^{\frac{n+m+1}{2}} \varepsilon). \tag{5.16}$$

Proof. The ε in $\mathrm{Tr}(\lambda, 1, n, \varepsilon)$ is an invariant of the isomorphism class because $S(b, N^{n-1}b) \in \varepsilon \cdot \mathbb{R}_{>0}$ for any $b \in H_{\mathbb{R}} - \mathrm{Im}\, N$. Therefore the $\mathrm{Tr}(\lambda, 1, n, \varepsilon)$ are pairwise non-isomorphic.

Now we turn to the examples (5.6) (iii). For the proof of (5.16), work with the real basis $(\underline{a} + \overline{\underline{a}}, i(\underline{a} - \overline{\underline{a}}))$. One has to calculate the matrix of S for the new basis. Details are left to the reader.

Irreducibility of $\mathrm{Tr}(\lambda, 2, n, m, 1)$ in the case $n + m + 1 \equiv 1(2)$: Indirect proof. Suppose $H_{\mathbb{R}} = V_1 \oplus V_2$ is an S-orthogonal and M-invariant splitting. Then each of V_1 and V_2 consists of one Jordan block of M. Choose a basis $\underline{c} = (c_1, ..., c_n)$ of V_1 with $N\underline{c} = \underline{c} \cdot J_n$. Use that S

is here $(-1)^n$-symmetric and that N is an infinitesimal isometry. It gives

$$\begin{aligned} S(c_j, c_{n+1-j}) &\overset{N}{=} (-1)^{(n+1-j)-j} \cdot S(c_{n+1-j}, c_j) \overset{S}{=} -S(c_j, c_{n+1-j}), \text{ so } =0, \\ S(c_j, c_{n+1-k}) &\overset{N}{=} 0 \quad \text{for } k<j \text{ anyway.} \end{aligned}$$

Then S is degenerate on V_1, a contradiction. □

Theorem 5.8. *[Mi69, S 3] The irreducible isometric triples are given by the following types, which are all non-isomorphic.*

$$\mathrm{Tr}(\lambda, 1, n, \varepsilon) \quad \text{with} \quad \lambda \in \{\pm 1\}, \tag{5.17}$$

$$\mathrm{Tr}(\lambda, 2, n, m, 1) \quad \text{with} \quad \lambda \in \{\pm 1\} \ \&\ m \equiv n(2), \tag{5.18}$$

$$\mathrm{Tr}(\lambda, 2, n, m, \varepsilon) \quad \text{with} \quad \lambda \in \{\zeta \in S^1 \mid \mathrm{Im}\,\zeta > 0\}, \tag{5.19}$$

$$\mathrm{Tr}(\lambda, 2, n, m) \quad \text{with} \quad \lambda \in \mathbb{R}_{>1} \cup \mathbb{R}_{<-1}, \tag{5.20}$$

$$\mathrm{Tr}(\lambda, 4, n, m) \quad \text{with} \quad \lambda \in \{\zeta \in \mathbb{C} \mid |\zeta| > 1, \mathrm{Im}\,\zeta > 0\}. \tag{5.21}$$

Here $n \in \mathbb{Z}_{\geq 1}, \varepsilon \in \{\pm 1\}, m \in \{0, 1\}$.

Proof. As this is only implicit in [Mi69, S 3], we provide additional arguments.

The cases $\lambda \in \mathbb{R}_{>1} \cup \mathbb{R}_{<-1}$ and $\lambda \in \{\zeta \in \mathbb{C} \mid |\zeta| > 1, \mathrm{Im}\,\zeta > 0\}$ are subsumed in [Mi69, S 3] as "case 3" and are the easiest cases. Consider $\lambda \in \{\zeta \in \mathbb{C} \mid |\zeta| > 1, \mathrm{Im}\,\zeta > 0\}$, and consider an isometric triple $(H_{\mathbb{R}}, M, S)$ with $H_{\mathbb{C}} = H_\lambda \oplus H_{\lambda^{-1}} \oplus H_{\overline{\lambda}} \oplus H_{\overline{\lambda}^{-1}}$ and S $(-1)^m$-symmetric. Choose a basis $\underline{a} = (a_1, ..., a_n)$ of H_λ which is adapted to the Jordan block structure of N on $H^{(1)}$, so

$$\underline{a} = (\underline{a}^{(1)}, ..., \underline{a}^{(r)}) \text{ with } N\underline{a}^{(j)} = \underline{a}^{(j)} \cdot J_{n_j} \quad \text{for some } r, n_1, ..., n_r \in \mathbb{Z}_{\geq 1}$$

(so $n_1 + ... + n_r = n$). Let $\underline{c} = (\underline{c}^{(1)}, ..., \underline{c}^{(r)})$ be the S-dual basis of $H_{\lambda^{-1}}$. Define

$$\begin{aligned} \underline{b}^{(j)} &:= ((-1)^{n_j-1} b^{(j)}_{n_j}, (-1)^{n_j-2} b^{(j)}_{n_j-1}, ..., -b^{(j)}_2, b^{(j)}_1) \quad \text{and} \\ \underline{b} &:= (\underline{b}^{(1)}, ..., \underline{b}^{(r)}). \end{aligned}$$

Then $H_{\mathbb{C}}$ splits into the S-orthogonal and M-invariant subspaces $\langle \underline{a}^{(j)}, \underline{b}^{(j)}, \overline{\underline{a}}^{(j)}, \overline{\underline{b}}^{(j)} \rangle$ for $j = 1, ..., r$, and the j-th space is with this basis of the type $\mathrm{Tr}(\lambda, 4, n_j, m)$.

The case $\lambda \in \mathbb{R}_{>1} \cup \mathbb{R}_{<-1}$ is similar.

The cases $\lambda \in \{\zeta \in S^1 \mid \mathrm{Im}\,\zeta > 0\}$ and $\lambda = \pm 1$ are called "case 1" respectively "case 2" in [Mi69, S 3]. For such a value λ let $(H_{\mathbb{R}}, M, S)$ be an isometric triple with S $(-1)^m$-symmetric for some $m \in \{0, 1\}$ and with $H_{\mathbb{C}} = H_\lambda \oplus H_{\overline{\lambda}}$ in case 1 and $H_{\mathbb{C}} = H_\lambda$ in case 2.

Theorem 3.2 in [Mi69] says that the isometric triple splits into isometric triples such that on each summand all Jordan blocks have the same length and that the summands are unique up to isomorphism. Therefore suppose that on $H_\mathbb{C}$ all Jordan blocks have the same length n.

Now consider first case 1, so $\lambda \in \{\zeta \in S^1 \,|\, \operatorname{Im}\zeta > 0\}$. The sesquilinear (=linear×semilinear) form $S_{res,1}$ on $H_\lambda/(H_\lambda \cap \operatorname{Im} N)$ with

$$S_{res,1}([a],[b]) \;:=\; (-i)^{n+m+1}\cdot S(a, N^{n-1}\overline{b}) \quad \text{for } a,b\in H_\lambda \tag{5.22}$$

is well defined and nondegenerate and hermitian: It is well defined and nondegenerate because N is an infinitesimal isometry and all Jordan blocks have the same length n, so that especially $\ker N = \operatorname{Im} N^{n-1}$ and $S(\operatorname{Im} N, \ker N) = 0$. The following calculation shows that it is hermitian,

$$\begin{aligned} S_{res,1}([b],[a]) &= (-i)^{n+m+1}\cdot S(b, N^{n-1}\overline{a}) \\ &= (-1)^{n+m+1}(-1)^m \cdot S(N^{n-1}\overline{a}, b) \\ &= (-i)^{n+m+1}(-1)^{n+m+1}\overline{S(a, N^{n-1}\overline{b})} \\ &= \overline{S_{res,1}([a],[b])}. \end{aligned}$$

Theorem 3.3 in [Mi69] implies that the isomorphism class of the isometric triple $(H_\mathbb{R}, M, S)$ is determined by the signature of $S_{res,1}$.

In the case $\operatorname{Tr}(\lambda, 2, n, m, \varepsilon)$ we have $H_\lambda/(H_\lambda\cap \operatorname{Im} N) = \mathbb{C}\cdot[a_1]$ and

$$S_{res,1}([a_1],[a_1]) = (-i)^{n+m+1}\cdot S(a_1, N^{n-1}\overline{a_1}) = \varepsilon.$$

Therefore in the general case above, the isometric triple $(H_\mathbb{R}, M, S)$ is isomorphic to a sum of triples $\operatorname{Tr}(\lambda, 2, n, m, \varepsilon_j)$ for $j = 1, 2, ..., \frac{1}{2n}\dim H_\mathbb{R}$ where the $\varepsilon_j \in \{\pm 1\}$ are determined by the signature of $S_{res,1}$.

Finally consider case 2, so $\lambda = \pm 1$. The bilinear form $S_{res,2}$ on $H_\mathbb{R}/\operatorname{Im} N$ with

$$S_{res,2}([a],[b]) \;:=\; S(a, N^{n-1}b) \quad \text{for } a,b\in H_\mathbb{R} \tag{5.23}$$

is well defined and nondegenerate and $(-1)^{n+m+1}$-symmetric: It is well defined and nondegenerate for the same reasons as $S_{res,1}$. The following calculation shows that it is $(-1)^{n+m+1}$-

symmetric,

$$\begin{aligned} S_{res,2}([b],[a]) &= S(b, N^{n-1}a) = (-1)^m \cdot S(N^{n-1}a, b) \\ &= (-1)^{n+m+1} S(a, N^{n-1}b) = (-1)^{n+m+1} S_{res,2}([a],[b]). \end{aligned}$$

Theorem 3.4 in [Mi69] implies that the isomorphism class of the isometric triple $(H_\mathbb{R}, M, S)$ is determined by the signature of $S_{res,2}$ if $n+m+1 \equiv 0(2)$ and that it is independent of any additional data if $n+m+1 \equiv 1(2)$.

In the cases $\mathrm{Tr}(\lambda, 1, n, \varepsilon)$ with $\lambda = \pm 1$ and $n+m+1 \equiv 0(2)$ we have $H_\mathbb{R}/\operatorname{Im} N = \mathbb{R} \cdot [a_1]$ and

$$S_{res,2}([a_1],[a_1]) = S(a_1, N^{n-1}a_1) = \varepsilon.$$

Therefore in the general case above, the isometric triple $(H_\mathbb{R}, M, S)$ is in the case $n+m+1 \equiv 0(2)$ isomorphic to a sum of triples $\mathrm{Tr}(\lambda, 1, n, \varepsilon_j)$ for $j = 1, 2, ..., \frac{1}{n}\dim H_\mathbb{R}$ where the $\varepsilon_j \in \{\pm 1\}$ are determined by the signature of $S_{res,2}$. In the case $n+m+1 \equiv 1(2)$, the isometric triple $(H_\mathbb{R}, M, S)$ is isomorphic to a sum of triples $\mathrm{Tr}(\lambda, 2, n, m, 1)$. □

Theorem 5.8 together with the lemmata 5.3 and 5.4 gives also the classification of the irreducible Seifert form pairs in theorem 5.9. The proof of theorem 5.9 states which isometric triples give rise to which Seifert form pairs.

Theorem 5.9. *The irreducible Seifert form pairs are given by the types with the following names.*

$$\begin{aligned} \mathrm{Seif}(\lambda, 1, n, \varepsilon) \quad &\text{with} \quad (\lambda = 1 \ \& \ n \equiv 1(2)) && (5.24)\\ &\text{or} \quad (\lambda = -1 \ \& \ n \equiv 0(2)), \\ \mathrm{Seif}(\lambda, 2, n) \quad &\text{with} \quad (\lambda = 1 \ \& \ n \equiv 0(2)) && (5.25)\\ &\text{or} \quad (\lambda = -1 \ \& \ n \equiv 1(2)), \\ \mathrm{Seif}(\lambda, 2, n, \zeta) \quad &\cong \quad \mathrm{Seif}(\overline{\lambda}, 2, n, \overline{\zeta}) && (5.26)\\ &\text{with} \quad \lambda, \zeta \in S^1 - \{\pm 1\}, \zeta^2 = \overline{\lambda} \cdot (-1)^{n+1}, \\ \mathrm{Seif}(\lambda, 2, n) \quad &\text{with} \quad \lambda \in \mathbb{R}_{>1} \cup \mathbb{R}_{<-1}, && (5.27)\\ \mathrm{Seif}(\lambda, 4, n) \quad &\text{with} \quad \lambda \in \{\zeta \in \mathbb{C} \mid |\zeta| > 1, \operatorname{Im}\zeta > 0\}. && (5.28) \end{aligned}$$

Here $n \in \mathbb{Z}_{\geq 1}, \varepsilon \in \{\pm 1\}$. The types are uniquely determined by the properties above of λ and n and the following properties.

(5.24) $\mathrm{Seif}(\lambda,1,n,\varepsilon)$: $\dim H_{\mathbb{R}}=n$, $H_{\mathbb{C}}=H_\lambda$, *one Jordan block, for each* $a\in H_{\mathbb{R}}-\mathrm{Im}\,N$

$$L(a,N^{n-1}a)\in\varepsilon\cdot\mathbb{R}_{>0}.$$

(5.25) $\mathrm{Seif}(\lambda,2,n)$: $\dim H_{\mathbb{R}}=2n$, $H_{\mathbb{C}}=H_\lambda$, *two Jordan blocks of size* n.

(5.26) $\mathrm{Seif}(\lambda,2,n,\zeta)$: $\dim H_{\mathbb{R}}=2n$, $H_{\mathbb{C}}=H_\lambda\oplus H_{\overline{\lambda}}$, *two Jordan blocks, for each* $a\in H_\lambda-\mathrm{Im}\,N$

$$L(a,N^{n-1}\overline{a})\in\zeta\cdot\mathbb{R}_{>0}.$$

(5.27) $\mathrm{Seif}(\lambda,2,n)$: $\dim H_{\mathbb{R}}=2n$, $H_{\mathbb{C}}=H_\lambda\oplus H_{\lambda^{-1}}$, *two Jordan blocks of size* n.

(5.28) $\mathrm{Seif}(\lambda,4,n)$: $\dim H_{\mathbb{R}}=4n$, $H_{\mathbb{C}}=H_\lambda\oplus H_{\lambda^{-1}}\oplus H_{\overline{\lambda}}\oplus H_{\overline{\lambda}^{-1}}$, *four Jordan blocks of size* n.

Proof. The following table lists irreducible isometric triples and chosen Seifert forms from lemma 5.4 which give rise to irreducible Seifert form pairs. In the cases (5.24) and (5.26), calculations after the table show that the Seifert form pairs have the stated properties. In all cases (5.24)–(5.28), one sees that the stated properties characterize the Seifert form pairs uniquely by going back via lemma 5.3 to isometric triples and comparing their classification in theorem 5.8.

Lemma 5.4 will be applied now. The δ in lemma 5.4 is here in the table in the case (5.24) $\delta=\lambda=(-1)^{n-1}$, in the other cases $\delta=(-1)^m$.

	L from lemma 5.4	
$\mathrm{Tr}(\lambda,1,n,\varepsilon)$	$L^{(1)}$ or $L^{(2)}$ or $L^{(3)}$	$\mathrm{Seif}(\lambda,1,n,\lambda\cdot\varepsilon)$
$\mathrm{Tr}(\lambda,2,n,m,1)$	$L^{(1)}$ or $L^{(2)}$ or $L^{(3)}$	$\mathrm{Seif}(\lambda,2,n)$
with $m\equiv n(2)$		
$\mathrm{Tr}(\lambda,2,n,m,\varepsilon)$	$L^{(1)}$ or $L^{(2)}$	$\mathrm{Seif}(\lambda,2,n,\frac{\overline{\lambda}+1}{\lvert\lambda+1\rvert}i^{n+1}\varepsilon)$
$\mathrm{Tr}(\lambda,2,n,m)$	$L^{(1)}$ or $L^{(2)}$	$\mathrm{Seif}(\lambda,2,n)$
$\mathrm{Tr}(\lambda,4,n,m)$	$L^{(1)}$ or $L^{(2)}$	$\mathrm{Seif}(\lambda,4,n)$

The calculation for the case (5.24) with $L^{(1)}$ ($L^{(2)}$ and $L^{(3)}$ are analogous):

$$\begin{aligned}L^{(1)}(a,N^{n-1}a) &= S(\frac{1}{M+\lambda\,\mathrm{id}}a,N^{n-1}a)\\ &= S(\frac{1}{2\lambda}a(+\text{something in }\mathrm{Im}\,N),N^{n-1}a)\\ &= \frac{1}{2}\lambda\cdot S(a,N^{n-1}a)=\frac{1}{2}\cdot\lambda\cdot\varepsilon.\end{aligned}$$

The calculation for the case (5.26) with $L^{(2)}$ ($L^{(1)}$ is analogous):

$$\begin{aligned} L^{(2)}(a, N^{n-1}a) &= S(a, (M + (-1)^m \operatorname{id})N^{n-1}\overline{a}) \\ &= S(a, (\overline{\lambda} + (-1)^m)N^{n-1}\overline{a}) \\ &= (\overline{\lambda} + (-1)^m) \cdot i^{n+m+1} \cdot \varepsilon \\ &\in (\overline{\lambda} + 1) \cdot i^{n+1} \cdot \varepsilon \cdot \mathbb{R}_{>0}. \end{aligned}$$

In the last line $\operatorname{Im}(\lambda) > 0$ (in (5.19) for $\operatorname{Tr}(\lambda, 2, n, m, \varepsilon)$) is used. □

The next lemma gives for each irreducible Seifert form pair the signature of I_s. This is useful if one wants to determine the irreducible pieces of a given Seifert form pair. Here the signature (p, q, r) means $p := \max(\dim U \,|\, U$ pos. def. subspace of $H_\mathbb{R})$, $q := \dim \operatorname{Rad} I_s$, $r = n - p - q = \max(\dim U \,|\, U$ neg. def. subspace of $H_\mathbb{R})$.

Lemma 5.10. *The following table lists for the irreducible Seifert form pairs in theorem 5.9 the signature of I_s and for all cases with $\operatorname{Rad} I_s = \{0\}$ the type of the irreducible isometric triple.*

type of a Seifert	form pair	signature of I_s	isometric str.
$\operatorname{Seif}(1, 1, n, \varepsilon)$	with $n \equiv \varepsilon(4)$	$(\frac{n+1}{2}, 0, \frac{n-1}{2})$	$\operatorname{Tr}(1, 1, n, \varepsilon)$
$\operatorname{Seif}(1, 1, n, \varepsilon)$	with $n \equiv -\varepsilon(4)$	$(\frac{n-1}{2}, 0, \frac{n+1}{2})$	$\operatorname{Tr}(1, 1, n, \varepsilon)$
$\operatorname{Seif}(-1, 1, n, \varepsilon)$	with $n - 1 \equiv \varepsilon(4)$	$(\frac{n}{2}, 1, \frac{n-2}{2})$	
$\operatorname{Seif}(-1, 1, n, \varepsilon)$	with $n - 1 \equiv -\varepsilon(4)$	$(\frac{n-2}{2}, 1, \frac{n}{2})$	
$\operatorname{Seif}(1, 2, n)$	(with $n \equiv 0(2)$)	$(n, 0, n)$	$\operatorname{Tr}(1, 2, n, 0, 1)$
$\operatorname{Seif}(-1, 2, n)$	(with $n \equiv 1(2)$)	$(n - 1, 2, n - 1)$	
$\operatorname{Seif}(\lambda, 2, n, \zeta\varepsilon)$	with $n \equiv 0(2)$	$(n, 0, n)$	$\operatorname{Tr}(\lambda, 2, n, 0, \varepsilon)$
	(and $\lambda \in S^1 - \{\pm 1\}$)		
$\operatorname{Seif}(\lambda, 2, n, \zeta)$	with $n \equiv 1(2)$	$(n - 1, 0, n + 1)$	$\operatorname{Tr}(\lambda, 2, n, 0, 1)$
	(and $\lambda \in S^1 - \{\pm 1\}$)		
$\operatorname{Seif}(\lambda, 2, n, -\zeta)$	with $n \equiv 1(2)$	$(n + 1, 0, n - 1)$	$\operatorname{Tr}(\lambda, 2, n, 0, -1)$
	(and $\lambda \in S^1 - \{\pm 1\}$)		
$\operatorname{Seif}(\lambda, 2, n)$	with $\lambda \in \mathbb{R}_{>1} \cup \mathbb{R}_{<-1}$	$(n, 0, n)$	$\operatorname{Tr}(\lambda, 2, n, 0)$
$\operatorname{Seif}(\lambda, 4, n)$	with $\lambda \in \{\zeta \in \mathbb{C} \,|$	$(2n, 0, 2n)$	$\operatorname{Tr}(\lambda, 4, n, 0)$
	$\|\zeta\| > 1, \operatorname{Im} \zeta > 0\}$		

Here $n \in \mathbb{Z}_{\geq 1}, \varepsilon \in \{\pm 1\}$, and in the lines 7–9 $\zeta := \frac{\overline{\lambda}+1}{|\lambda+1|} \cdot i^{n+1}$.

Proof. For all cases except those in the lines 3, 4 and 6, $(H_\mathbb{R}, M, I_s)$ is an irreducible isometric triple, and the proof of theorem 5.9 tells which it is. Then one can read off the signature of I_s from the examples 5.6.

The least easy cases are in the lines 8 and 9. We treat the case in line 9 and leave the other cases to the reader. The case in line 9 is a special case of example 5.6 (iii). Here I_s has the same signature as the hermitian matrix

$$S(\begin{pmatrix}\underline{a}^t\\ \overline{\underline{a}}^t\end{pmatrix}, (\overline{\underline{a}}, \underline{a})) = i^{n-1}\cdot\begin{pmatrix}E_n^{per} & 0\\ 0 & E_n^{per}\end{pmatrix}.$$

The signature is $(n+1, 0, n-1)$.

In the cases in the lines 3, 4 and 6, lemma 5.3 says $\mathrm{Rad}\, I_s = \ker(M+\mathrm{id}) = \ker N$. The induced isometric triple $(H_{\mathbb{R}}/\mathrm{Rad}\, I_s, M, I_s)$ has eigenvalue -1 and in the cases in the lines 3 and 4 only one Jordan block of size $n-1$ and in the cases in the line 6 two Jordan blocks of sizes $n-1$. Theorem 5.5 and 5.8 tell us: The isometric triple $(H_{\mathbb{R}}/\mathrm{Rad}\, I_s, M, I_s)$ is in all cases irreducible. It is of the type $\mathrm{Tr}(-1, 1, n-1, \widetilde{\varepsilon})$ with a suitable $\widetilde{\varepsilon}$ in the lines 3 and 4 and of the type $\mathrm{Tr}(-1, 2, n-1, 0, 1) \cong \mathrm{Tr}(-1, 2, n-1, 0, -1)$ (with $n-1 \equiv 0(2)$) in line 6. The type $\mathrm{Tr}(-1, 2, n-1, 0, \pm 1)$ has signature $(n-1, 0, n-1)$. This gives $(n-1, 2, n-1)$ in line 6.

The cases in the lines 3 and 4: $\widetilde{\varepsilon}$ has to be determined. For each $a \in H_{\mathbb{R}} - \mathrm{Im}\, N$ we have $L(a, N^{n-1}a) \in \varepsilon \cdot \mathbb{R}_{>0}$.

$$\begin{aligned}
& I_s(a, N^{n-2}a) = L(a, N^{n-2}a) + L(N^{n-2}a, a)\\
= \;& 2L(a, N^{n-2}a) = 2L(N^{n-2}a, a) = 2L(Ma, N^{n-2}a)\\
= \;& 2L(-e^N a, N^{n-2}a) = -2L(a + Na, N^{n-2}a)\\
= \;& -2L(a, N^{n-2}a) + 2L(a, N^{n-1}a), \quad \text{thus it is}\\
= \;& L(a, N^{n-1}a) \in \varepsilon \cdot \mathbb{R}_{>0},
\end{aligned}$$

so $\widetilde{\varepsilon} = \varepsilon$. The signature of I_s on $H_{\mathbb{R}}/\mathrm{Rad}(I_s)$ is the signature of $\varepsilon \cdot E_{n-1}^{per}$. □

We finish this section with some elementary statements on induced structures on the dual space.

Notations 5.11. Let H_K be a finite-dimensional K-vector space. $H_K^\vee := \mathrm{Hom}(H_K, K)$ is the dual space, and $\langle , \rangle : H_K^\vee \times H_K \to K$ denotes the natural pairing. If $M : H_K \to H_K$ is an automorphism, then $M^\vee : H_K^\vee \to H_K^\vee$ is defined by $\langle M^\vee a, Mb\rangle = \langle a, b\rangle$. If $L : H_K \times H_K \to K$ is a nondegenerate pairing, let $L^{lin} : H_K^\vee \to H_K$ be the induced isomorphism with $L(a, b) = \langle (L^{lin})^{-1}(a), b\rangle$, and define $L^\vee : H_K^\vee \times H_K^\vee \to K$ by $L^\vee(a, b) = \langle a, L^{lin}b\rangle = L(L^{lin}a, L^{lin}b)$.

Lemma 5.12. *(a) If $(H_\mathbb{R}, L)$ is a Seifert form pair with monodromy m, then $(L^{lin})^{-1} : (H_\mathbb{R}, L, M) \to (H_\mathbb{R}^\vee, L^\vee, M^\vee)$ is an isomorphism of Seifert form pairs with monodromies.*

(b) If $(H_\mathbb{R}, M, S)$ is an isometric triple, then $(S^{lin})^{-1} : (H_\mathbb{R}, M, S) \to (H_\mathbb{R}^\vee, M^\vee, S^\vee)$ is an isomorphism of isometric triples.

(c) Let $(H_\mathbb{R}, L)$ be a Seifert form pair with $H = H_{\neq -\delta}$ for some $\delta \in \{\pm 1\}$. Denote $S := I_s$ if $\delta = 1$ and $S := I_a$ if $\delta = -1$. Then

$$L^{lin} \circ M^\vee = M \circ L^{lin}, \tag{5.29}$$

$$S^{lin} = L^{lin} \circ \frac{1}{M^\vee + \delta\,\mathrm{id}} = \frac{1}{M + \delta\,\mathrm{id}} \circ L^{lin}, \tag{5.30}$$

$$S^\vee = S^{L^\vee,(2)} \quad \text{with } S^{L^\vee,(2)}(a,b) := L^\vee(a, \frac{1}{M^\vee + \delta\,\mathrm{id}}), \tag{5.31}$$

so $S^{L^\vee,(2)}$ is the pairing $I_s^{(2)}$ respectively $I_a^{(2)}$ in lemma 5.3 (b), but for $L^\vee$ instead of L.

Proof. Elementary. □

5.2 Polarized mixed Hodge structures

Steenbrink defined mixed Hodge structures for *ihs* and their spectral pairs. These mixed Hodge structures are special in several aspects. They come equipped with an automorphism of the vector space which induces the weight filtration and which is essential for the spectral pairs. And they come equipped with a natural polarization. Though the spectral pairs are defined without using the polarization.

Usually, a $\mathbb{Z}$-lattice or a $\mathbb{Q}$-vector space underly a mixed Hodge structure. They give a rigidity and richness which are usually precious. But we do not want this rigidity here, so we will not consider a $\mathbb{Z}$-lattice or a $\mathbb{Q}$-vector space here.

Notations 5.13. The notations 5.1 will be used again. All filtrations in this paper are finite and exhaustive. An upper index means a decreasing filtration, a lower index means an increasing filtration. The Gauss bracket is denoted $\lfloor . \rfloor : \mathbb{R} \to \mathbb{Z}$. The upper Gauss bracket is denoted $\lceil . \rceil : \mathbb{R} \to \mathbb{Z}$. The following two functions will allow to treat several cases simultaneously:

$$\begin{aligned} [.]_2 : \mathbb{Z} \to \{0,1\} &\qquad \text{with} \quad n \equiv [n]_2 \bmod 2, \\ \theta : S^1 \to \{0,1\} &\qquad \text{with } \theta(1) := 1 \text{ and } \theta(\lambda) := 0 \text{ for } \lambda \neq 1. \end{aligned}$$

The following lemma from [Sch73, Lemma 6.4] (see also e.g. [He99, Lemma 2.1]) prepares definition 5.15. It is stated in [Sch73] with $H_\mathbb{Q}$ instead of $H_\mathbb{R}$.

Lemma 5.14. *Let $m \in \mathbb{Z}$, $H_\mathbb{R}$ a finite-dimensional $\mathbb{R}$-vector space, $S : H_\mathbb{R} \times H_\mathbb{R} \to \mathbb{R}$ a nondegenerate $(-1)^m$-symmetric bilinear form, and $N : H_\mathbb{R} \to H_\mathbb{R}$ a nilpotent endomorphism which is an infinitesimal isometry of S.*

(a) There exists a unique increasing filtration $W_\bullet \subset H_\mathbb{R}$ such that $N(W_l) \subset W_{l-2}$ and such that $N^l : \mathrm{Gr}^W_{m+l} \to \mathrm{Gr}^W_{m-l}$ is an isomorphism. Sometimes it will be called $W_\bullet^{(N,m)}$.

(b) $S(W_k, W_l) = 0$ if $k + l < 2m$.

(c) A nondegenerate $(-1)^{m+l}$-symmetric bilinear form S_l is well defined on Gr^W_{m+l} for $l \geq 0$ by the requirement: $S_l(a, b) = S(\tilde{a}, N^l\tilde{b})$ if $\tilde{a}, \tilde{b} \in W_{m+l}$ represent $a, b \in \mathrm{Gr}^W_{m+l}$.

(d) The primitive subspace P_{m+l} of Gr^W_{m+l} is defined by

$$P_{m+l} = \ker(N^{l+1} : \mathrm{Gr}^W_{m+l} \to \mathrm{Gr}^W_{m-l-2})$$

if $l \geq 0$ and $P_{m+l} = 0$ if $l < 0$. Then

$$\mathrm{Gr}^W_{m+l} = \bigoplus_{i\geq 0} N^i P_{m+l+2i},$$

and this decomposition is orthogonal with respect to S_l if $l \geq 0$.

Definition 5.15. *(a) A mixed Hodge structure (short: MHS) is a tuple $(H_\mathbb{R}, H_\mathbb{C}, F^\bullet, W_\bullet)$ with $F^\bullet \subset H_\mathbb{C}$ a decreasing Hodge filtration and $W_\bullet \subset H_\mathbb{R}$ an increasing weight filtration such that $F^\bullet \mathrm{Gr}^W_k$ gives a pure Hodge structure of weight k on Gr^W_k, i.e.*

$$\mathrm{Gr}^W_k = F^p \,\mathrm{Gr}^W_k \oplus \overline{F^{k+1-p}\,\mathrm{Gr}^W_k}. \tag{5.1}$$

(b) A Steenbrink MHS of weight $m \in \mathbb{Z}$ is an MHS $(H_\mathbb{R}, H_\mathbb{C}, F^\bullet, W_\bullet)$ together with an automorphism M (called monodromy) of $(H_\mathbb{R}, H_\mathbb{C}, W_\bullet)$ with the following properties: Its semisimple part maps F^p to F^p, its nilpotent part N maps F^p to F^{p-1}, and N determines $W_\bullet$ as follows.

$$W_\bullet|_{H_{\neq 1}} = W_\bullet^{(N,m)} \text{ on } H_{\neq 1}, \quad \text{and } W_\bullet|_{H_1} = W_\bullet^{(N,m+1)} \text{ on } H_1. \tag{5.2}$$

(c) [CK82][He99] A polarized mixed Hodge structure (short: PMHS) of weight $m \in \mathbb{Z}$ is a tuple $(H_\mathbb{R}, H_\mathbb{C}, F^\bullet, W_\bullet, N, S)$ with $(m, H_\mathbb{R}, H_\mathbb{C}, S, N, W_\bullet)$ as in lemma 5.14 and

(i) $(H_\mathbb{R}, H_\mathbb{C}, F^\bullet, W_\bullet)$ is an MHS.

(ii) $N(F^p) \subset F^{p-1}$.

(iii) $S(F^p, F^{m+1-p}) = 0$.

(iv) The pure Hodge structure $F^\bullet P_{m+l}$ of weight $m + l$ on P_{m+l} is polarized by S_l, i.e.

(α) $S_l(F^p P_{m+l}, F^{m+l+1-p} P_{m+l}) = 0$.

(β) $i^{2p-m-l} \cdot S_l(a, \overline{a}) > 0$ for $a \in F^p P_{m+l} \cap \overline{F^{m+l-p} P_{m+l}} - \{0\}$.

(d) A Steenbrink PMHS of weight $m \in \mathbb{Z}$ is a Steenbrink MHS together with a nondegenerate pairing S such that the restriction to $H_{\neq 1}$ is a PMHS of weight m and the restriction to H_1 is a PMHS of weight $m+1$ (especially, S is $(-1)^m$-symmetric on $H_{\neq 1}$ and $(-1)^{m+1}$-symmetric on H_1).

Remark 5.16. In [CK82] condition (c) (iii) is omitted. Condition (c) (iii) implies condition (iv) (α) (therefore we could have omitted condition (iv) (α)). In the case of an *ihs*, the polarization on H_1 was not considered by Steenbrink, only later in [He99].

Deligne defined subspaces $I^{p,q}$ of an MHS which split the Hodge filtration and the weight filtration in a natural way [De71]. They also behave well with respect to morphisms and a polarizing form [CK82][He99].

Lemma 5.17. *For an MHS define*

$$I^{p,q} \quad := \quad (F^p \cap W_{p+q}) \cap \left(\overline{F}^q \cap W_{p+q} + \sum_{j>0} \overline{F}^{q-j} \cap W_{p+q-j-1} \right).$$

Then

$$F^p \quad = \quad \bigoplus_{i,q:\, i \geq p} I^{i,q}, \tag{5.3}$$
$$W_l \quad = \quad \bigoplus_{p+q \leq l} I^{p,q}, \tag{5.4}$$
$$I^{q,p} \quad \cong \quad \overline{I^{p,q}} \bmod W_{p+q-2}. \tag{5.5}$$

If $W = W^{(N,m)}$ for a nilpotent endomorphism $N : H_{\mathbb{R}} \to H_{\mathbb{R}}$ and a weight $m \in \mathbb{Z}$, then define additionally for $p + q \geq m$

$$I_0^{p,q} \quad := \quad \ker(N^{p+q-m+1} : I^{p,q} \to I^{m-q-1,m-p-1}).$$

Then

$$N(I^{p,q}) \quad \subset \quad I^{p-1,q-1}, \tag{5.6}$$
$$I^{p,q} \quad = \quad \bigoplus_{j \geq 0} N^j I_0^{p+j,q+j}, \tag{5.7}$$
$$I_0^{q,p} \quad \cong \quad \overline{I_0^{p,q}} \bmod W_{p+q-2}. \tag{5.8}$$

In the case of a PMHS of weight m with polarizing form S

$$S(I^{p,q},I^{r,s}) = 0 \quad \text{for } (r,s)\neq(m-p,m-q), \tag{5.9}$$
$$S(N^iI_0^{p,q},N^jI_0^{r,s}) = 0 \quad \text{for } (r,s,i+j)\neq(q,p,p+q-m). \tag{5.10}$$

Steenbrink's spectral pairs provide a very intuitive picture which allows to see and understand the discrete data in a Steenbrink MHS well.

Definition 5.18. *[St77] Let $(H_{\mathbb{R}},H_{\mathbb{C}},F^\bullet,W_\bullet,M)$ be a Steenbrink MHS of weight m with $n:=\dim H_{\mathbb{R}}\geq 1$. The spectral pairs are n pairs $(\alpha,k)\in\mathbb{R}\times\mathbb{Z}$ with multiplicities $d(\alpha,k)\in\mathbb{Z}_{\geq 0}$,*

$$\begin{aligned} \mathrm{Spp} &= \sum_{(\alpha,k)} d(\alpha,k)\cdot(\alpha,k)\in\mathbb{Z}_{\geq 0}[\mathbb{R}\times\mathbb{Z}], \\ d(\alpha,k) &:= \dim \mathrm{Gr}_F^{\lfloor m-\alpha\rfloor}\mathrm{Gr}^W_{k+\theta(\lambda)}H_\lambda \quad \text{for } e^{-2\pi i\alpha}=\lambda \end{aligned} \tag{5.11}$$

($\theta(\lambda)$ was defined in the notations 5.13). The spectral numbers are the first entries in the spectral pairs,

$$\begin{aligned} \mathrm{Sp} &= \sum_{\alpha} d(\alpha)\cdot(\alpha)\in\mathbb{Z}_{\geq 0}[\mathbb{R}], \\ d(\alpha) &:= \sum_k d(\alpha,k)=\dim \mathrm{Gr}_F^{\lfloor m-\alpha\rfloor}H_\lambda \quad \text{for } e^{-2\pi i\alpha}=\lambda. \end{aligned} \tag{5.12}$$

Now we will discuss the geometry in the spectral pairs. Lemma 3.20 will be crucial. Consider some $p,q\in\mathbb{Z}$ and $\lambda\in S^1$ such that the space $(I_0^{p,q})_\lambda:=I_0^{p,q}\cap H_\lambda$ is not $\{0\}$. Then $p+q=m+\theta(\lambda)+l$ for some $l\in\mathbb{Z}_{\geq 0}$, and the spaces in the two sequences

$$(I_0^{p,q})_\lambda, N(I_0^{p,q})_\lambda, ..., N^l(I_0^{p,q})_\lambda, \tag{5.13}$$
$$(I_0^{q,p})_{\overline{\lambda}}, N(I_0^{q,p})_{\overline{\lambda}}, ..., N^l(I_0^{q,p})_{\overline{\lambda}}, \tag{5.14}$$

have all the same dimension. They give rise to the following *ordered pair of spectral pair ladders*, where each spectral pair has the same multiplicity $\dim(I_0^{p,q})_\lambda$:

$$(\alpha,m+l),(\alpha+1,m+l-2),...,(\alpha+l,m-l), \tag{5.15}$$
$$\begin{aligned} (m-l-1-\alpha,m+l),(m-l-\alpha+1,m+l-2),...,& \\ (m-1-\alpha,m-l).& \end{aligned} \tag{5.16}$$

In one row the first entry is increasing by 1, the second entry is decreasing by 2. Here

$\alpha \in \mathbb{R}$ is determined by $e^{-2\pi i\alpha} = \lambda$ and $p = \lfloor m - \alpha \rfloor = m - \lceil \alpha \rceil$. The first spectral pair $(\alpha, m + l)$ in the first spectral pair ladder (5.15) comes from $(I_0^{p,q})_\lambda$. The first spectral pair $(m - l - 1 - \alpha, m + l)$ in the second spectral pair ladder (5.16) comes from $(I_0^{q,p})_{\overline{\lambda}}$, because $q + p = m + \theta(\lambda) + l$, $e^{-2\pi i(m-l-1-\alpha)} = \overline{\lambda}$, and

$$\begin{aligned} \lfloor m - (m - l - 1 - \alpha) \rfloor &= l + 1 + \lfloor \alpha \rfloor = l + \theta(\lambda) + \lceil \alpha \rceil \\ &= l + \theta(\lambda) + m - p = q. \end{aligned}$$

The other spectral pairs follow from the first ones by applying (5.6) repeatedly.

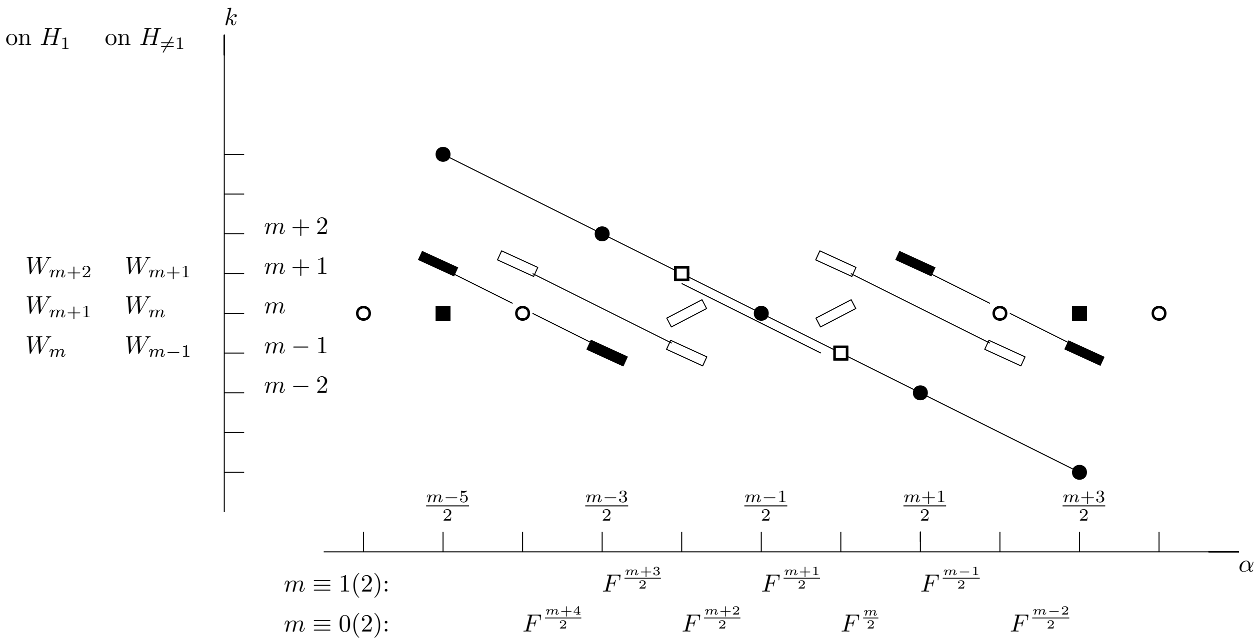
If $(p, \lambda) = (q, \overline{\lambda})$ (so $\lambda \in \{\pm 1\}$) then $(I_0^{p,q})_\lambda = (I_0^{q,p})_{\overline{\lambda}}$ and then there is only one spectral pair ladder, i.e. (5.15) and (5.16) agree and their multiplicity is $\dim(I_0^{p,p})_\lambda$. Then the spectral pair ladder is its own partner. By (5.7) Spp consists completely of spectral pair ladders, namely pairs of them and (for $(p, \lambda) = (q, \overline{\lambda})$) single ones. Each pair of spectral pair ladders and also the single ones are invariant under the Kleinian group $\mathrm{id}, \pi_1, \pi_2, \pi_3 : \mathbb{R} \times \mathbb{Z} \to \mathbb{R} \times \mathbb{Z}$ with

$$\begin{aligned} \pi_1 : (\frac{m-1}{2} + \alpha, m + l) &\mapsto (\frac{m-1}{2} - \alpha, m - l), \\ \pi_2 : (\frac{m-1-l}{2} + \alpha, m + l) &\mapsto (\frac{m-1-l}{2} - \alpha, m + l), \\ \pi_3 = \pi_1 \circ \pi_2 = \pi_2 \circ \pi_1 : (\alpha, m + l) &\mapsto (\alpha + l, m - l). \end{aligned} \tag{5.17}$$

Obviously, the decomposition of Spp into ordered pairs of spectral pair ladders and single ones with these symmetries is unique up to changing the order of the ordered pairs. If a spectral pair ladder starts at $(\alpha, m + l)$, its *length* is $l + 1$ and the *distance* to its partner is $2\alpha + l + 1 - m$. The single ones have distance 0. Thus for the single ones

$$\begin{gathered} \alpha = \frac{m-l-1}{2} \in \frac{1}{2}\mathbb{Z} \quad \text{and} \\ l \equiv 0(2) \text{ if } \alpha \in \frac{m-1}{2} + \mathbb{Z}, \quad l \equiv 1(2) \text{ if } \alpha \in \frac{m}{2} + \mathbb{Z}. \end{gathered} \tag{5.18}$$

The following picture gives an example. Each dot stands for a spectral pair or the corresponding space $N^j(I_0^{p,q})_\lambda$. Dots of the same shape indicate spectral pairs in one pair of spectral pair ladders. Only in order not to overload the picture, we restrict in the picture to $\alpha \in \frac{1}{2}\mathbb{Z}$. Also, $F^\bullet$ and $W_\bullet$ can be read off.



In the definition 5.18 of Spp, only a Steenbrink MHS is needed, not a Steenbrink PMHS. But if we have a Steenbrink PMHS, then it makes sense to study the underlying isometric structures on $H_{\neq 1}$ and on H_1. Theorem 5.20 studies the isometric triples $(H_\mathbb{R} \cap H_{\neq 1}, M, S)$ and $(H_\mathbb{R} \cap H_1, M, S)$ The following observation from [He99] simplifies this study.

Remark 5.19. Starting with a reference PMHS $(H_\mathbb{R}, H_\mathbb{C}, F_0^\bullet, W_\bullet, N, S)$ of some weight m, a classifying space D_{PMHS} for all Hodge filtrations $F^\bullet$ on $H_\mathbb{C}$ such that $(H_\mathbb{R}, H_\mathbb{C}, F^\bullet, W_\bullet, N, S)$ is a PMHS with the same spectral pairs as the reference PMHS was constructed in [He99]. It contains a filtration $F_1^\bullet$ such that the $I^{pq}(F_1^\bullet)$ satisfy $I^{q,p}(F_1^\bullet) = \overline{I^{p,q}(F_1^\bullet)}$. Such a PMHS is called *split*. All this holds also for Steenbrink PMHS.

Theorem 5.20. *Let $(H_\mathbb{R}, H_\mathbb{C}, F^\bullet, W_\bullet, M, S)$ be a Steenbrink PMHS of weight m. Because of remark 5.19 we can suppose that it is split, i.e. $I^{q,p} = \overline{I^{p,q}}$.*

(a) The sum $(H_\mathbb{R} \cap H_{\neq 1}, M, S) \oplus (H_\mathbb{R} \cap H_1, M, S)$ of isometric triples decomposes into the isometric triples

$$\sum_{j=0}^{l} N^j (I_0^{p,q})_\lambda + \sum_{j=0}^{l} N^j (I_0^{q,p})_{\overline{\lambda}} \tag{5.19}$$
$$\text{for } p, q, \lambda \text{ with } (I_0^{p,q})_\lambda \neq \{0\},\ (p,\lambda) \neq (q, \overline{\lambda}),\ \mathrm{Im}((-1)^{m+1}\lambda) \geq 0$$

and the isometric triples

$$\sum_{j=0}^{l} N^j (I_0^{p,p})_\lambda \quad \text{for } p, \lambda \text{ with } \lambda = \pm 1,\ (I_0^{p,p})_\lambda \neq \{0\}. \tag{5.20}$$

(b) Each of the isometric triples in (5.19) decomposes into $\dim(I_0^{p,q})_\lambda$ *many copies of the isometric triple*

$$\mathrm{Tr}(\lambda, 2, l+1, [m+\theta(\lambda)]_2, (-1)^{\lceil\alpha\rceil-1-\frac{1}{2}(m-\theta(\lambda)+[m+\theta(\lambda)]_2)}). \tag{5.21}$$

Each of the isometric triples in (5.20) decomposes into $\dim(I_0^{p,p})_\lambda$ *many copies of the isometric triple*

$$\mathrm{Tr}(\lambda, 1, l+1, (-1)^{\lceil\alpha\rceil-\frac{1}{2}(m-\theta(\lambda)-l)}), \tag{5.22}$$
$$\text{and then } (-1)^{m+1}\lambda = (-1)^l.$$

Proof. **(a)** $I^{q,p} = \overline{I^{p,q}}$ implies $I_0^{q,p} = \overline{I_0^{p,q}}$ and $(I_0^{q,p})_{\overline{\lambda}} = \overline{(I_0^{p,q})_\lambda}$. Therefore the spaces in (5.19) and (5.20) are complexifications of real subspaces.

The polarizing form S is M-invariant. The decomposition is S-orthogonal by (5.10) and the M-invariance of S. It is obviously M-invariant.

(b) Formula (5.10) and the M-invariance of S show that the isometric triples in (5.19) and (5.20) are sums of isometric triples of the types $\mathrm{Tr}(\lambda, 2, l{+}1, [m{+}\theta(\lambda)]_2, \varepsilon)$ and $\mathrm{Tr}(\lambda, 1, l{+}1, \varepsilon)$ for suitable ε. Here S is $(-1)^{[m+\theta(\lambda)]_2}$-symmetric. Therefore in the case $\mathrm{Tr}(\lambda, 1, l+1, \varepsilon)$ $l \equiv [m+\theta(\lambda)]_2 \bmod 2$, i.e. $(-1)^{m+1}\lambda = (-1)^l$.

It rests to determine ε. Choose $a \in (I_0^{p,q})_\lambda - \{0\}$. The polarizing condition (c)(iv)(β) in definition 5.15 says

$$0 < i^{p-q} \cdot S_l(a, \overline{a}) = i^{2p-m-\theta(\lambda)-l} \cdot S(a, N^l\overline{a}).$$

The following calculations use also $p = m - \lceil\alpha\rceil$.

Consider first the case (5.19). The definition in example 5.6 (iii) says

$$S(a, N^l\overline{a}) \in i^{(l+1)+[m+\theta(\lambda)]_2+1} \cdot \varepsilon \cdot \mathbb{R}_{>0}.$$

Then

$$\begin{aligned} \varepsilon &= i^{-(2p-m-\theta(\lambda)-l)-(l+1+[m+\theta(\lambda)]_2+1)} \\ &= (-1)^{-p-1+\frac{1}{2}(m+\theta(\lambda)-[m+\theta(\lambda)]_2)} \\ &= (-1)^{\lceil\alpha\rceil-1-\frac{1}{2}(m-\theta(\lambda)+[m+\theta(\lambda)]_2)}. \end{aligned}$$

Consider now the case (5.20). The isometric triple must be one in example 5.6 (ii), so then

$m+\theta(\lambda)\equiv l(2)$ and $S(a,N^l a)\in\varepsilon\cdot\mathbb{R}_{>0}$ and

$$\begin{aligned}\varepsilon &= i^{-(2p-m-\theta(\lambda)-l)} \\ &= (-1)^{-p+\frac{1}{2}(m+\theta(\lambda)+l)} = (-1)^{\lceil\alpha\rceil-\frac{1}{2}(m-\theta(\lambda)-l)}.\end{aligned}$$

□

5.3 Seifert forms and Steenbrink PMHS

The purpose of this and the next section is to compare and relate several bilinear forms: the polarizing form of a Steenbrink PMHS, a Seifert form and, in section 5.4, a pairing on a flat bundle on $\mathbb{C}^*$. They all arise in the case of an *ihs*. But here we consider them abstractly.

Lemma 5.21 starts with a Seifert form and gives a family of together symmetric forms and a hermitian form.

Definition 5.22 and the theorems 5.23 and 5.24 start from a Steenbrink PMHS. A (normalized) Seifert form is defined, and also an automorphism G, which seems to have received less attention than it deserves. Its significance will become fully transparent only in section 5.4 when a Fourier-Laplace transformation is considered.

Theorem 5.23 fixes the relations between the polarizing form, the Seifert form and, this automorphism. Theorem 5.24 classifies the irreducible Seifert form pairs in a Steenbrink PMHS. It recovers the result of Némethi [Ne95] that the spectral pairs $\mathrm{Spp} \bmod 2\mathbb{Z}\times\{0\}$ are equivalent to the Seifert form (and the weight m, which we need as our Seifert form is normalized, but Spp is not). Finally, theorem 5.26 gives for a Steenbrink PMHS a square root of a Tate twist. This uses the automorphism G. It is modeled after the suspension of an *ihs*.

Lemma 5.21. *Let $(H_\mathbb{R},L)$ be a Seifert form pair.*

(a) For λ with $H_\lambda\neq\{0\}$ and κ with $\kappa^2=\lambda$ define a pairing

$$L_\kappa^{sym}: H_\lambda\times H_{1/\lambda}\to\mathbb{C},\quad L_\kappa^{sym}(a,b):=\kappa\cdot L(a,e^{-N/2}b). \tag{5.1}$$

Then L_κ^{sym} and $L_{1/\kappa}^{sym}$ satisfy together the symmetry condition

$$L_{1/\kappa}^{sym}(b,a)=L_\kappa^{sym}(a,b). \tag{5.2}$$

(b) For $\lambda\in S^1$ with $H_\lambda\neq\{0\}$ and for κ with $\kappa^2=\lambda$ define a sesquilinear pairing

$$L_\kappa^{herm}: H_\lambda\times H_\lambda\to\mathbb{C},\quad L_\kappa^{herm}(a,b):=L_\kappa^{sym}(a,\overline{b}). \tag{5.3}$$

It is hermitian.

Proof. (a)

$$\begin{aligned} L^{sym}_{1/\kappa}(b,a) &= \kappa^{-1}L(b,e^{-N/2}a) = \kappa^{-1}L(Me^{-N/2}a,b) \\ &= \kappa^{-1}L(\lambda e^{N/2}a,b) = \kappa^{-1}\lambda L(a,e^{-N/2}b) = L^{sym}_\kappa(a,b). \end{aligned}$$

(b) Here $\kappa\in S^1$, so $\kappa^{-1}=\overline{\kappa}$.

$$\begin{aligned} L^{herm}_\kappa(b,a) &= L^{sym}_\kappa(b,\overline{a}) = L^{sym}_{1/\kappa}(\overline{a},b) = \overline{\kappa}L(\overline{a},e^{-N/2}b) \\ &= \overline{\kappa L(a,e^{-N/2}\overline{b})} = \overline{L^{sym}_\kappa(a,\overline{b})} = \overline{L^{herm}_\kappa(a,b)}. \qquad \square \end{aligned}$$

Definition 5.22. *Let $(H_\mathbb{R},H_\mathbb{C},F^\bullet,W_\bullet,M,S)$ be a Steenbrink PMHS of weight m.*

(a) Then each eigenvalue of M is in S^1. Define an automorphism $G:H_\mathbb{C}\to H_\mathbb{C}$ as follows.

$$\begin{aligned} G &:= \bigoplus_{\alpha\in(0,1]} G^{(\alpha)} \text{ with } G^{(\alpha)}:H_{e^{-2\pi i\alpha}}\to H_{e^{-2\pi i\alpha}}, \\ G^{(\alpha)} &:= \sum_{k\geq 0}\frac{1}{k!}\Gamma^{(k)}(\alpha)\left(\frac{-N}{2\pi i}\right)^k = \Gamma(\alpha\cdot\mathrm{id}-\frac{N}{2\pi i}). \end{aligned} \tag{5.4}$$

G does not respect $H_\mathbb{R}$ if $N\neq 0$. But it commutes with M and M_s and N and it respects $W_\bullet$.

(b) The normalized monodromy is $M^{nor}:=(-1)^{m+1}M$, so $M_s^{nor}=(-1)^{m+1}M_s$, $N^{nor}=N$. But in (c) and in theorem 5.23, H_λ and $H_{\neq 1}$ still refer to M, not to M^{nor}.

(c) The normalized Seifert form $L^{nor}:H_\mathbb{R}\times H_\mathbb{R}\to\mathbb{R}$ is defined as follows. First define an automorphism $\nu:H_\mathbb{R}\to H_\mathbb{R}$ by

$$\nu := \begin{cases} \frac{1}{M-\mathrm{id}} & \text{on } H_{\neq 1}, \\ \frac{-N}{M-\mathrm{id}} & \text{on } H_1. \end{cases} \tag{5.5}$$

Now define L^{nor} by

$$L^{nor}(a,b) := -S(a,\nu^{-1}b) = \begin{cases} (-1)^m\cdot L^{(2)}(a,b) & \text{on } H_{\neq 1}, \\ (-1)^{m+1}\cdot L^{(3)}(a,b) & \text{on } H_1. \end{cases} \tag{5.6}$$

Here $L^{(2)}$ and $L^{(3)}$ come from lemma 5.4 and the isometric triples $(H_\mathbb{R}\cap H_{\neq 1},M^{nor},S)$ (with $\delta=(-1)^m$) and $(H_\mathbb{R}\cap H_1,M^{nor},S)$ (with $\delta=(-1)^{m+1}$). Thus M^{nor} is the monodromy of L^{nor}.

Theorem 5.23. *Let $(H_{\mathbb{R}}, H_{\mathbb{C}}, F^\bullet, W_\bullet, M, S)$ be a Steenbrink PMHS of weight m.*

(a) For $a \in H_\lambda, b \in H_{\overline{\lambda}}$ with $\lambda = e^{-2\pi i\alpha}$ and $0 < \alpha < 1$

$$S(a,b) = \frac{-1}{2\pi i} \cdot e^{-\pi i\alpha} \cdot L^{nor}(Ga, e^{-N/2}Gb). \tag{5.7}$$

For $a, b \in H_1$

$$S(a,b) = L^{nor}(Ga, e^{-N/2}Gb). \tag{5.8}$$

(b) $(H_{\mathbb{R}}, H_{\mathbb{C}}, G(F^\bullet), W_\bullet, M)$ is a Steenbrink MHS of weight m with

$$L^{sym}_{\kappa\cdot i^{m-1}}(G(F^p)H_\lambda, G(F^{m+1-p})H_{\overline{\lambda}}) = 0 \qquad \text{for } \lambda \neq 1 \text{ and } \kappa \text{ with } \kappa^2 = \lambda, \tag{5.9}$$

$$L^{sym}_{i^{m-1}}(G(F^p)H_1, G(F^{m+2-p})H_1) = 0. \tag{5.10}$$

Remark that $(\kappa \cdot i^{m-1})^2 = \lambda(-1)^{m-1}$ is the eigenvalue of M^{nor} on H_λ.

(c) Recall the relation between $(I_0^{p,q})_\lambda$ and the first spectral pair $(\alpha, m+l)$ in the spectral pair ladder in (5.15): $p+q = m+\theta(\lambda)+l$, $p = \lfloor m-\alpha \rfloor$, $\lambda = e^{-2\pi i\alpha}$. Recall also that $m-l-1-\alpha$ is the first spectral number of the partner spectral pair ladder and that $2\alpha+l+1-m$ is the distance from the spectral pair ladder to its partner.

For $a \in (I_0^{p,q})_\lambda - \{0\}$ as well as for $a \in G((I_0^{p,q})_\lambda) - \{0\}$

$$L^{nor}(a, N^l\overline{a}) \in e^{\frac{1}{2}\pi i(2\alpha+l+1-m)} \cdot \mathbb{R}_{>0}. \tag{5.11}$$

Proof. **(a)** Recall the following identities of the Gamma function (they are equivalent if one uses $\Gamma(x+1) = x\Gamma(x)$).

$$\begin{aligned}\Gamma(x)\Gamma(1-x) &= \frac{\pi}{\sin\pi x} = e^{\pi i x}\frac{2\pi i}{e^{2\pi i x}-1},\\ \Gamma(1+x)\Gamma(1-x) &= \frac{\pi x}{\sin\pi x} = e^{\pi i x}\frac{2\pi i x}{e^{2\pi i x}-1}.\end{aligned}$$

They imply for $0 < \alpha < 1$

$$\begin{aligned}\Gamma(\alpha\,\mathrm{id} + \frac{N}{2\pi i})\Gamma(\mathrm{id} - (\alpha\,\mathrm{id} + \frac{N}{2\pi i})) &= e^{\pi i\alpha}e^{N/2}\frac{2\pi i}{e^{2\pi i\alpha}e^N - \mathrm{id}},\\ \Gamma(\mathrm{id} + \frac{N}{2\pi i})\Gamma(\mathrm{id} - \frac{N}{2\pi i}) &= e^{N/2}\frac{N}{e^N - \mathrm{id}}.\end{aligned}$$

Now calculate for $a \in H_\lambda, b \in H_{\overline{\lambda}}$ with $\lambda = e^{-2\pi i\alpha}$ and $0 < \alpha < 1$

$$\begin{aligned} L^{nor}(Ga, e^{-N/2}Gb) &= L^{nor}(\Gamma(\alpha\,\mathrm{id} - \frac{N}{2\pi i})a, e^{-N/2}\Gamma((1-\alpha)\,\mathrm{id} - \frac{N}{2\pi i})b) \\ &= L^{nor}(a, e^{-N/2}\Gamma(\alpha\,\mathrm{id} + \frac{N}{2\pi i})\Gamma(\mathrm{id} - (\alpha\,\mathrm{id} + \frac{N}{2\pi i}))b) \\ &= L^{nor}(a, e^{\pi i\alpha}\frac{2\pi i}{M - \mathrm{id}}b) \\ &= e^{\pi i\alpha} \cdot 2\pi i \cdot (-1)S(a,b). \end{aligned}$$

And calculate for $a, b \in H_1$

$$\begin{aligned} L^{nor}(Ga, e^{-N/2}Gb) &= L^{nor}(\Gamma(\mathrm{id} - \frac{N}{2\pi i})a, e^{-N/2}\Gamma(\mathrm{id} - \frac{N}{2\pi i})b) \\ &= L^{nor}(a, e^{-N/2}\Gamma(\mathrm{id} + \frac{N}{2\pi i})\Gamma(\mathrm{id} - \frac{N}{2\pi i})b) \\ &= L^{nor}(a, \frac{N}{M - \mathrm{id}}b) = S(a,b). \end{aligned}$$

(b) Definition 5.15 (c)(iii)+(d) says here

$$S(F^pH_\lambda, F^{m+\theta(\lambda)+1-p}H_{\overline{\lambda}}) = 0$$

(recall $\theta(\lambda)$ in the notation 5.13). Part (b) follows from this, from part (a) and from lemma 5.21 (a).

(c) The spectral number α and the number $\beta \in (0,1]$ with $e^{-2\pi i\beta} = \lambda$ satisfy $\alpha = m - p - 1 + \beta$. The positivity condition in definition 5.15 (c)(iv) (β) says for $a \in (I_0^{p,q})_\lambda - \{0\}$ as well as for $a \in G((I_0^{pq})_\lambda) - \{0\}$

$$0 \;<\; i^{2p-m-\theta(\lambda)-l} \cdot S(a, N^l\overline{a}).$$

In the case $\lambda \neq 1$ this is because of (5.7)

$$\begin{aligned} & i^{2p-m-l} \cdot \frac{-1}{2\pi i} \cdot e^{-\pi i\beta} \cdot L^{nor}(Ga, e^{-N/2}GN^l\overline{a}) \\ = \; & \frac{\Gamma(\beta)\Gamma(1-\beta)}{2\pi} e^{\pi i(p - \frac{1}{2}(m+l-1))} e^{-\pi i(\alpha - m + p + 1)} L^{nor}(a, N^l\overline{a}) \\ = \; & \frac{\Gamma(\beta)\Gamma(1-\beta)}{2\pi} e^{\frac{1}{2}\pi i(m-l-1-2\alpha)} L^{nor}(a, N^l\overline{a}). \end{aligned}$$

In the case $\lambda = 1$ it is because of (5.8)

$$\begin{aligned} & i^{2p-m-1-l} \cdot L^{nor}(Ga, e^{-N/2}GN^l\overline{a}) \\ = \;& e^{\pi i(p-\frac{1}{2}(m+1+l))} L^{nor}(a, N^l\overline{a}) \\ = \;& e^{\frac{1}{2}\pi i(m-l-1-2\alpha)} L^{nor}(a, N^l\overline{a}). \qquad \square \end{aligned}$$

The Hodge filtration $F^\bullet$ is self-isotropic with respect to S by definition 5.15 (c)(iii)&(d). The Hodge filtration $G(F^\bullet)$ is self-isotropic with respect to the pairings L^{sym}_κ in definition 5.21 (a) by theorem 5.23 (b).

In section 5.5 we will see that $G(F^\bullet)$ behaves well with respect to a Thom-Sebastiani formula in the *ihs* case. Below in theorem 5.26 a special case is formalized and gives a certain square root of a Tate twist for Steenbrink PMHS.

If $N \neq 0$ then the automorphism G in definition 5.22 (a) does not respect $H_\mathbb{R}$. Then Deligne's $I^{p,q}(G(F^\bullet))$ for $G(F^\bullet)$ are not equal to the images $G(I^{p,q}(F^\bullet))$ under G of Deligne's $I^{p,q}(F^\bullet)$ for $F^\bullet$. In view of (5.9), the images $G(I^{p,q}(F^\bullet))$ satisfy isotropy conditions for the pairings L^{sym}_κ, probably contrary to the $I^{p,q}(G(F^\bullet))$. Therefore we worked in theorem 5.23 (c) with the $I^{p,q}(F^\bullet)$ and the images $G(I^{p,q}(F^\bullet))$.

The next theorem 5.24 gives the classification of the irreducible pieces in the Seifert form pair $(H_\mathbb{R}, L^{nor})$ of a Steenbrink PMHS. The proof uses theorem 5.20 and theorem 5.23. Part (b) recovers Némethi's result [Ne95] that the isomorphism class of the Seifert form pair $(H_\mathbb{R}, L^{nor})$ together with the number m on one side and the spectral pairs modulo $2\mathbb{Z}$ in the first entry, i.e. the data $\mathrm{Spp} \bmod 2\mathbb{Z} \times \{0\}$, on the other side, determine each other.

Theorem 5.24. *Let $(H_\mathbb{R}, H_\mathbb{C}, F^\bullet, W_\bullet, M, S)$ be a Steenbrink PMHS of weight m. Recall that $2\alpha + l + 1 - m$ is the distance from a spectral pair ladder to its partner. It is an integer if and only if $\lambda \in \{\pm 1\}$.*

(a) For each ordered pair of spectral pair ladders or a single spectral pair ladder the first spectral pair is called $(\alpha, m + l)$, and then $\lambda := e^{-2\pi i\alpha}$. The Seifert form pair $(H_\mathbb{R}, L^{nor})$ (from definition 5.22 (c)) decomposes as follows.

It contains for each ordered pair of spectral pair ladders with $\lambda \in S^1 - \{0\}$ a Seifert form pair of the type

$$\mathrm{Seif}((-1)^{m+1}\lambda, 2, l + 1, e^{\frac{1}{2}\pi i(2\alpha+l+1-m)}). \tag{5.12}$$

It contains for each pair of spectral pair ladders with odd distance an irreducible Seifert form *pair of the type*

$$\begin{gathered}\mathrm{Seif}((-1)^{m+1}\lambda, 2, l+1), \\ \text{and then } (-1)^{m+1}\lambda = (-1)^{l+1}.\end{gathered} \tag{5.13}$$

It contains for each pair of spectral pair ladders with even distance and each single spectral pair ladder (then the distance $2\alpha+l+1-m$ is 0) two respectively one Seifert form pair(s) of the type

$$\begin{gathered}\mathrm{Seif}((-1)^{m+1}\lambda, 1, l+1, (-1)^{\frac{1}{2}(2\alpha+l+1-m)}), \\ \text{and then } (-1)^{m+1}\lambda = (-1)^{l}.\end{gathered} \tag{5.14}$$

(b) $\mathrm{Spp} \bmod 2\mathbb{Z}\times\{0\}$ *and the isomorphism class of* $(H_\mathbb{R}, L^{nor})$ *together with* m *determine one another.*

Proof. (a) By remark 5.19 we can suppose as in theorem 5.20 that the Steenbrink PMHS is split. Then we can consider the isometric triples in theorem 5.20 and the corresponding Seifert form pairs with L^{nor}.

Remark that the monodromy of L^{nor} is $M^{nor} = (-1)^{m+1}M$, so the eigenvalues change from λ to $(-1)^{m+1}\lambda$.

If $\lambda \neq \pm 1$, the isometric triple, and the corresponding Seifert form pair are both irreducible. Then (5.11) and theorem 5.9 give (5.12).

In the other cases $\lambda \in \{\pm 1\}$. Then by lemma 5.7, the isometric triple in (5.21) is irreducible if and only if

$$(l+1) + [m+\theta(\lambda)]_2 + 1 \equiv 1(2), \quad \text{i.e. } l+m+\theta(\lambda) \equiv 1(2).$$

But

$$\begin{aligned} 2\alpha &\equiv \theta(\lambda)+1 \bmod 2 \quad \text{and then} \\ 2\alpha+l+1-m &\equiv l+m+\theta(\lambda) \bmod 2. \end{aligned}$$

So, in the case of an odd distance $2\alpha+l+1-m$, the isometric triple in (5.21) is irreducible. By the proof of theorem 5.9 then also the corresponding Seifert form pair is irreducible. This gives (5.13).

In the case of an even distance $2\alpha+l+1-m$, the isometric triple and the Seifert form pair are both reducible. Each pair of spectral pair ladders and each single spectral pair ladder give two respectively one Seifert form pair $\mathrm{Seif}((-1)^{m+1}\lambda, 1, l+1, \varepsilon)$. Here $\varepsilon = (-1)^{\frac{1}{2}(2\alpha+l+1-m)}$

because of (5.11) and theorem 5.9. This shows (5.14).

(b) It is rather easy to see that $\mathrm{Spp} \bmod 2\mathbb{Z}\times\{0\}$ is equivalent to the spectral pair ladders modulo $2\mathbb{Z}\times\{0\}$. Part (a) shows that these are equivalent to the union of the corresponding Seifert form pairs in $(H_\mathbb{R}, L)$ together with the number m. The number m is used to fix the symmetry point $(\frac{m-1}{2}, m)$ of Spp. $\square$

Remark 5.25. In the case $N = 0$ a Steenbrink PMHS can also be called a (pure) Steenbrink PHS. Then $G(F^\bullet) = F^\bullet$ and $I_0^{p,q} = I^{p,q} = H^{p,q}$ with $q = m + \theta(\lambda) - p$.

Then the ν in (5.5) is $-\,\mathrm{id}$ on H_1. Define L^{nor} and M^{nor} as in definition 5.22. M^{nor} has on H_λ the eigenvalue $\lambda\cdot(-1)^{m+1}$.

Then on H_λ the hermitian form L^{herm}_κ from lemma 5.21 (b) for any (of the two) κ with $\kappa^2 = \lambda\cdot(-1)^{m+1}$ is up to a constant equal to the hermitian form $i^{-m-\theta(\lambda)}S(.,\bar{.})$.

The Hodge decomposition $\bigoplus_p (H^{p,q})_\lambda$ is then orthogonal with respect to L^{herm}_κ. The positivity condition in (5.11) can then be written as

$$L^{herm}_{\exp(-\pi i(\alpha-\frac{m-1}{2}))}(a,a) > 0 \tag{5.15}$$

for $a \in H_\lambda^{p,m+\theta(\lambda)-p} - \{0\}$ and the spectral number α with $e^{-2\pi i\alpha} = \lambda$ and $\lfloor m-\alpha\rfloor = p$.

The following theorem 5.26 constructs from a Steenbrink PMHS of weight m a Steenbrink PMHS of weight $m+1$, with the same underlying normalized Seifert form pair $(H_\mathbb{R}, L^{nor})$. In the *ihs* case it corresponds to a suspension: one goes from an *ihs* $f(x_0,...,x_m)$ to an *ihs* $f(x_0,...,x_m)+x_{m+1}^2$, see remark 5.34 (iii). It can be seen as a square root of a Tate twist.

Theorem 5.26. *Let $(H_\mathbb{R}, H_\mathbb{C}, F^\bullet, W_\bullet, M, S)$ be a Steenbrink PMHS of weight m. We construct a Steenbrink PMHS $(\widetilde{H}_\mathbb{R}, \widetilde{H}_\mathbb{C}, \widetilde{F}^\bullet, \widetilde{W}_\bullet, \widetilde{M}, \widetilde{S})$ of weight $\widetilde{m} = m+1$ as follows:*

$$\begin{gathered}
\widetilde{H}_\mathbb{R} = H_\mathbb{R}, \quad \widetilde{H}_\mathbb{C} = H_\mathbb{C}, \quad \widetilde{M} = -M, \quad \widetilde{M}_s = -M_s, \quad \widetilde{N} = N, \\
\widetilde{H}_\lambda = H_{-\lambda}, \quad \widetilde{W}_\bullet \widetilde{H}_{\neq 1} = W_\bullet^{(N,m+1)}\widetilde{H}_{\neq 1}, \quad \widetilde{W}_\bullet \widetilde{H}_1 = W_\bullet^{(N,m+2)}\widetilde{H}_1,
\end{gathered}$$

$$\begin{aligned}
\widetilde{\nu} \;&:=\; \begin{cases} \frac{1}{\widetilde{M}-\mathrm{id}} & \text{on } \widetilde{H}_{\neq 1} \quad (\text{eigenvalues } \neq 1 \text{ w.r.t. } \widetilde{M}) \\ \frac{-N}{\widetilde{M}-\mathrm{id}} & \text{on } \widetilde{H}_1 \quad (\text{eigenvalue } 1 \text{ w.r.t. } \widetilde{M}), \end{cases} \\
\widetilde{S}(a,b) \;&:=\; -L^{nor}(a, \widetilde{\nu}b),
\end{aligned}$$

$$\begin{aligned}
F^p\widetilde{H}_\lambda \;&:=\; (G^{(\alpha+\frac{1}{2})})^{-1}G^{(\alpha)}F^{p+1}\widetilde{H}_\lambda \qquad \text{if } -\lambda = e^{-2\pi i\alpha}, 0 < \alpha \leq \frac{1}{2}, \\
F^p\widetilde{H}_\lambda \;&:=\; (G^{(\alpha-\frac{1}{2})})^{-1}G^{(\alpha)}F^{p}\widetilde{H}_\lambda \qquad \text{if } -\lambda = e^{-2\pi i\alpha}, \frac{1}{2} < \alpha \leq 1.
\end{aligned}$$

Then $\widetilde{\text{Spp}} = \text{Spp} + (\frac{1}{2}, 1)$. *The two Steenbrink PMHS have the same underlying Seifert form pair* $(H_\mathbb{R}, L^{nor})$. *Carrying out the construction twice, leaves* $(H_\mathbb{R}, H_\mathbb{C}, M, S)$ *invariant and gives* $\widetilde{\widetilde{m}} = m+2$, $\widetilde{\widetilde{W}}_\bullet = W_{\bullet+2}$, $\widetilde{\widetilde{F}}^\bullet = F^{\bullet+1}$, *so it is a Tate twist.*

Proof. The proof uses theorem 5.23. We leave the details to the reader. Compare also corollary 5.33 and remark 5.34 (iii). □

5.4 Fourier-Laplace transformation and pairings

We will present an equivalence between three types of pairings and additional data, a polarizing form plus a monodromy, a Seifert form, and a pairing on a flat bundle on $\mathbb{C}^*$. Then we will consider holomorphic sections with moderate growth in the bundle and study a Fourier-Laplace transformation on them.

This will make the meaning of the automorphism G in definition 5.22 transparent. Theorem 5.28 will also fill the equivalence with life, by nice formulas which connect the pairings. Theorem 5.28 was stated in [He03] as proposition 7.7, but the proof was essentially omitted.

Lemma 5.27. *The following three data are equivalent.*

(α) $(H_\mathbb{R}, M, S, m)$. *Here* $H_\mathbb{R}$ *is a finite-dimensional* $\mathbb{R}$*-vector space,* M *is an automorphism on it with eigenvalues in* S^1*, called monodromy.* $m \in \mathbb{Z}$. *And* S *is an* M*-invariant nondegenerate bilinear form. On* $H_{\neq 1}$ *it is* $(-1)^m$*-symmetric. On* H_1 *it is* $(-1)^{m+1}$*-symmetric.*

(β). $(H_\mathbb{R}, L, m)$. *Here* $(H_\mathbb{R}, L)$ *is a Seifert form pair such that the eigenvalues of* L *are in* S^1*, and* $m \in \mathbb{Z}$.

(γ) $(H_\mathbb{R}^{bun} \to \mathbb{C}^*, \nabla, P, m)$. *Here* $H_\mathbb{R}^{bun} \to \mathbb{C}^*$ *is a bundle of* $\mathbb{R}$*-vector spaces on* $\mathbb{C}^*$ *with flat connection* ∇*, whose monodromy has eigenvalues in* S^1*. Its complexification is a holomorphic flat bundle and is denoted* $H_\mathbb{C}^{bun} \to \mathbb{C}^*$*. Again* $m \in \mathbb{Z}$*. And* P *is a flat and nondegenerate and* $(-1)^{m+1}$*-symmetric pairing*

$$P : H_{\mathbb{R},z}^{bun} \times H_{\mathbb{R},-z}^{bun} \to i^{m+1} \cdot \mathbb{R} \quad \text{for } z \in \mathbb{C}^*. \tag{5.1}$$

From (α) *to* (β)*:* $L := L^{nor}$ *in (5.6), using (5.5).*

From (β) *to* (γ)*: Define a flat bundle* $H_\mathbb{R}^{bun} \to \mathbb{C}^*$ *with monodromy* $(-1)^{m+1}M$*. Then* $L : H_{\mathbb{R},z}^{bun} \times H_{\mathbb{R},z}^{bun} \to \mathbb{R}$ *is defined on each fiber and is flat. Define* P *by*

$$P(a,b) := \frac{1}{(2\pi i)^{m+1}} \cdot L(a, \gamma_{-\pi} b). \tag{5.2}$$

Here $\gamma_{-\pi} : H_{\mathbb{R},z}^{bun} \to H_{\mathbb{R},-z}^{bun}$ *is the isomorphism by flat shift in mathematically negative direction.*

One goes from (γ) *to* (β) *and from* (β) *to* (α) *by inverting these constructions.*

Proof. Lemma 5.3 and lemma 5.4 show the equivalence of (α) and (β).

From (β) **to** (γ)**:** P is well defined and nondegenerate and flat because L has these properties. It is $(-1)^{m+1}$-symmetric because of $L(b,a) = L(Ma,b)$: For $a \in H^{bun}_{\mathbb{R},z}$, $b \in H^{bun}_{\mathbb{R},-z}$

$$\begin{aligned} & (2\pi i)^{m+1} \cdot P(b,a) = L(b,\gamma_{-\pi}a) = L(M\gamma_{-\pi}a,b) \\ = \; & (-1)^{m+1} L((-1)^{m+1} M\gamma_{-\pi}a,b) = (-1)^{m+1} L(\gamma_\pi a,b) \\ = \; & (-1)^{m+1} L(a,\gamma_{-\pi}b) = (2\pi i)^{m+1} \cdot (-1)^{m+1} \cdot P(a,b). \end{aligned} \tag{5.3}$$

Here $\gamma_\pi : H^{bun}_{\mathbb{R},z} \to H^{bun}_{\mathbb{R},-z}$ is the isomorphism by flat shift in mathematically positive direction.

From (γ) **to** (β)**:** Define L on any fiber of H^{bun} by $L(a,b) := (2\pi i)^{m+1} \cdot P(a,\gamma_\pi b)$. The $(-1)^{m+1}$-symmetry of P gives $L(Ma,b) = L(b,a)$, by inverting the calculation (5.3). Take $H_\mathbb{R} := H^{bun}_{\mathbb{R},z}$ for an arbitrary $z \in \mathbb{C}^*$. □

Now we consider all the data in lemma 5.27. Before we come to theorem 5.28, we have to describe the *elementary sections* and the *Kashiwara-Malgrange* V*-filtration*. Of course, this is standard and can be found in many places, e.g. [SS85], [AGV88] or [He02].

The space of flat multivalued global sections in $H^{bun}_\mathbb{C} \to \mathbb{C}^*$ is denoted $H^\infty_\mathbb{C}$. It can be identified in a non-unique way with $H_\mathbb{C}$. It comes with a monodromy which is then identified with $(-1)^{m+1}M$. Now H^∞_λ means the generalized eigenspace with respect to this monodromy. H^∞ also comes with a real subspace $H^\infty_\mathbb{R}$.

Any global flat multivalued section $A \in H^\infty_\lambda$ and any choice of $\alpha \in \mathbb{R}$ with $e^{-2\pi i\alpha} = \lambda$ leads to a holomorphic univalued section with specific growth condition at $0 \in \Delta$, the *elementary section* $es(A,\alpha)$ with

$$es(A,\alpha)(\tau) := e^{\log\tau(\alpha - \frac{N}{2\pi i})} \cdot A(\log\tau).$$

Denote by C^α the $\mathbb{C}$-vector space of all elementary sections with fixed α and λ. The map $es(.,\alpha) : H^\infty_\lambda \to C^\alpha$ is an isomorphism. The space $V^{mod} := \bigoplus_{\alpha\in(-1,0]} \mathbb{C}\{\tau\}[\tau^{-1}] \cdot C^\alpha$ is the space of all germs at 0 of the sheaf of holomorphic sections on the flat cohomology bundle with moderate growth at 0. The Kashiwara-Malgrange V-filtration is given by the subspaces

$$V^\alpha := \bigoplus_{\beta\in[\alpha,\alpha+1)} \mathbb{C}\{\tau\} \cdot C^\beta, \quad V^{>\alpha} := \bigoplus_{\beta\in(\alpha,\alpha+1]} \mathbb{C}\{\tau\} \cdot C^\beta.$$

It is a decreasing filtration by free $\mathbb{C}\{\tau\}$-modules of rank μ with $\mathrm{Gr}_V^\alpha = V^\alpha / V^{>\alpha} \cong C^\alpha$. And

$$\begin{aligned} \tau : C^\alpha \to C^{\alpha+1} \text{ bijective}, &\qquad \tau \cdot es(A,\alpha) = es(A,\alpha+1), \\ \partial_\tau : C^\alpha \to C^{\alpha-1} \text{ bijective}, &\qquad \text{if } \alpha \neq 0, \\ \tau\partial_\tau - \alpha : C^\alpha \to C^\alpha \text{ nilpotent}, &\qquad (\tau\partial_\tau - \alpha) es(A,\alpha) = es(\frac{-N}{2\pi i}A,\alpha). \end{aligned}$$

Theorem 5.28. *[He03, Proposition 7.7]*

(a) Let τ and z both be coordinates on $\mathbb{C}$. For $\alpha > 0$ and $A \in H^\infty_{e^{-2\pi i\alpha}}$, the Fourier-Laplace transformation FL with

$$FL(es(A,\alpha-1)(\tau))(z) := \int_0^{\infty\cdot z} e^{-\tau/z} \cdot es(A,\alpha-1)(\tau) d\tau \tag{5.4}$$

is well defined and maps the elementary section $es(A,\alpha-1)(\tau)$ in τ to the elementary section

$$FL(es(A,\alpha-1)(\tau))(z) = es(G^{(\alpha)}A,\alpha)(z) \tag{5.5}$$

in z.

(b) For $0 < \alpha < 1$ and $A \in H^\infty_{e^{-2\pi i\alpha}}$, $B \in H^\infty_{e^{2\pi i\alpha}}$,

$$\begin{aligned} & P(es(G^{(\alpha)}A,\alpha)(z), es(G^{(1-\alpha)}B,1-\alpha)(-z)) \\ = & \frac{z}{(2\pi i)^{m+1}} \cdot e^{\pi i(1-\alpha)} \cdot L^{nor}(G^{(\alpha)}A(\log z), e^{-N/2}G^{(1-\alpha)}B(\log z)) \\ = & \frac{z}{(2\pi i)^m} \cdot S(A,b). \end{aligned} \tag{5.6}$$

For $A, B \in H_1^\infty$,

$$\begin{aligned} & P(es(G^{(1)}A,1)(z), es(G^{(1)}B,1)(-z) \\ = & \frac{-z^2}{(2\pi i)^{m+1}} \cdot L^{nor}(G^{(1)}A(\log z), e^{-N/2}G^{(1)}B(\log z)) \\ = & \frac{-z^2}{(2\pi i)^{m+1}} \cdot S(A,b). \end{aligned} \tag{5.7}$$

Proof. As the proof was not carried out in [He03], we give it here.

(a) The Gamma function satisfies for $\alpha > 0$ the identity

$$\left(\frac{d}{d\alpha}\right)^k (\Gamma(\alpha)z^\alpha) = \int_0^{\infty\cdot z} e^{-\tau/z} \cdot \tau^{\alpha-1}(\log\tau)^k d\tau.$$

Then

$$\begin{aligned}
& FL(es(A,\alpha-1)(\tau))(z) \\
= & \int_0^{\infty\cdot z} e^{-\tau/z}\cdot\tau^{\alpha-1}\cdot\sum_{k\geq 0}\frac{1}{k!}(\log\tau)^k\left(\frac{-N}{2\pi i}\right)^k A(\log\tau)d\tau \\
= & \sum_{k\geq 0}\frac{1}{k!}\left(\frac{-N}{2\pi i}\right)^k A(\log z)\cdot\left(\frac{d}{d\alpha}\right)^k(\Gamma(\alpha)z^\alpha) \\
= & \sum_{k\geq 0}\frac{1}{k!}\left(\frac{-N}{2\pi i}\right)^k A(\log z)\cdot\sum_{l=0}^{k}\binom{k}{l}\Gamma^{(l)}(\alpha)\cdot(\log z)^{k-l}\cdot z^\alpha \\
= & \sum_{j+l=k\, j,l\geq 0} z^\alpha\frac{1}{j!}\left(\log z\cdot\frac{-N}{2\pi i}\right)^j\cdot\frac{1}{l!}\Gamma^{(l)}(\alpha)\left(\frac{-N}{2\pi i}\right)^l A(\log z) \\
= & e^{\log z(\alpha-\frac{N}{2\pi i})}\Gamma(\alpha-\frac{N}{2\pi i})A(\log z) = es(G^{(\alpha)}A,\alpha)(z).
\end{aligned}$$

(b) For $0<\alpha<1$ and $A\in H^\infty_{e^{-2\pi i\alpha}}$, $B\in H^\infty_{e^{2\pi i\alpha}}$

$$\begin{aligned}
& (2\pi i)^{m+1}\cdot P(es(A,\alpha)(z),es(B,1-\alpha)(-z)) \\
= & L^{nor}(es(A,\alpha)(z),\gamma_{-\pi}es(B,1-\alpha)(e^{\pi i}z)) \\
= & L^{nor}(e^{\log z(\alpha-\frac{N}{2\pi i})}A(\log z),\gamma_{-\pi}e^{(\pi i+\log z)(1-\alpha-\frac{N}{2\pi i})}B(\pi i+\log z)) \\
= & L^{nor}(A(\log z),e^{\log z(\alpha+\frac{N}{2\pi i})}e^{(\pi i+\log z)(1-\alpha-\frac{N}{2\pi i})}B(\log z)) \\
= & z\cdot e^{\pi i(1-\alpha)}\cdot L^{nor}(A(\log z),e^{-N/2}B(\log z)).
\end{aligned}$$

For $A,B\in H_1^\infty$,

$$\begin{aligned}
& (2\pi i)^{m+1}\cdot P(es(A,1)(z),es(B,1)(-z)) \\
= & L^{nor}(es(A,1)(z),\gamma_{-\pi}es(B,1)(e^{\pi i}z)) \\
= & L^{nor}(e^{\log z(1-\frac{N}{2\pi i})}A(\log z),\gamma_{-\pi}e^{(\pi i+\log z)(1-\frac{N}{2\pi i})}B(\pi i+\log z)) \\
= & L^{nor}(A(\log z),e^{\log z(1+\frac{N}{2\pi i})}e^{(\pi i+\log z)(1-\frac{N}{2\pi i})}B(\log z)) \\
= & (-z^2)\cdot L^{nor}(A(\log z),e^{-N/2}B(\log z)).
\end{aligned}$$

The equalities involving S follow now with theorem 5.23 (a). □

5.5 Application to *ihs*

Our main motivation for this chapter is the study of *ihs*, which we will continue in the next chapter 6. Each comes with its Milnor lattice, a $\mathbb{Z}$-lattice with an integer-valued Seifert form.

We recall the relevant data and define normalizations to get rid of some sign problems. We use two examples, one of M-tame functions and one of T_{pqr} singularities to illustrate the structures studied so far in this chapter. We then go on to recall the idea of the Brieskorn lattice of a *ihs*. And finally, discuss $TEZP$ structures.

We alter the notation from chapter 2 slightly. In particular, let us denote the Milnor number by μ, and the (for $m = 0$ reduced) middle homology groups by $H_m^{(red)}(f^{-1}(\tau), \mathbb{Z}) \cong \mathbb{Z}^\mu$ for $\tau \in \Delta'$. Each comes equipped with an intersection form I, which is a datum of one fiber, a monodromy M and a Seifert form L, which come from the Milnor fibration, see [AGV88, I.2.3] for their definitions. M is a quasiunipotent automorphism, I and L are bilinear forms with values in $\mathbb{Z}$, I is $(-1)^m$-symmetric, and L is unimodular. L determines M and I because of the formulas [AGV88, I.2.3]

$$\begin{aligned} L(Ma, b) &= (-1)^{m+1} L(b, a), & (5.1) \\ I(a, b) &= -L(a, b) + (-1)^{m+1} L(b, a) = L((M - \mathrm{id})a, b). & (5.2) \end{aligned}$$

The Milnor lattices $H_m(f^{-1}(\tau), \mathbb{Z})$ for all Milnor fibrations $f : X' \to \Delta'$ and then all $\tau \in \mathbb{R}_{>0} \cap T'$ are canonically isomorphic, and the isomorphisms respect M, I and L. This follows from Lemma 2.2 in [LR73]. These lattices are identified and called *Milnor lattice* $Ml(f)$.

A result of Thom and Sebastiani compares the Milnor lattices and monodromies of the *ihs* $f = f(x_0, ..., x_m), g = g(y_0, ..., y_n)$ and $f + g = f(x_0, ..., x_m) + g(x_{m+1}, ..., x_{m+n+1})$. There is an extension by Deligne for the Seifert form [AGV88, I.2.7]. It is restated here. There is a canonical isomorphism

$$\begin{aligned} \Phi : Ml(f + g) &\xrightarrow{\cong} Ml(f) \otimes Ml(g), & (5.3) \\ \text{with } M(f + g) &\cong M(f) \otimes M_h(g) & (5.4) \\ \text{and } L(f + g) &\cong (-1)^{(m+1)(n+1)} \cdot L(f) \otimes L(g). & (5.5) \end{aligned}$$

This motivates the definition of the *normalized Seifert form* and the *normalized monodromy* on the Milnor lattice $Ml(f)$

$$\begin{aligned} L^{hnor}(f) &:= (-1)^{(m+1)(m+2)/2} \cdot L(f), & (5.6) \\ M^{hnor}(f) &:= (-1)^{m+1} \cdot M(f) & (5.7) \end{aligned}$$

because then

$$L^{hnor}(f+g) \cong L^{hnor}(f)\otimes L^{hnor}(g), \tag{5.8}$$
$$M^{hnor}(f+g) \cong M^{hnor}(f)\otimes M^{hnor}(g) \tag{5.9}$$

and M^{hnor} is the monodromy of L^{hnor} in the sense of lemma 5.3 (a).

In the special case $g = x_{m+1}^2$, the function germ $f+g = f(x_0,...,x_m)+x_{m+1}^2 \in \mathcal{O}_{\mathbb{C}^{m+2},0}$ is called *stabilization* or *suspension* of f. As there are only two isomorphisms $Ml(x_{m+1}^2)\to\mathbb{Z}$, and they differ by a sign, there are two equally canonical isomorphisms $Ml(f)\to Ml(f+x_{m+1}^2)$, and they differ just by a sign. Therefore automorphisms and bilinear forms on $Ml(f)$ can be identified with automorphisms and bilinear forms on $Ml(f+x_{m+1}^2)$. In this sense [AGV88, I.2.7]

$$L(f+x_{m+1}^2) = (-1)^m\cdot L(f), \tag{5.10}$$
$$M(f+x_{m+1}^2) = -M(f), \tag{5.11}$$
$$L^{hnor}(f+x_{m+1}^2) = L^{hnor}(f), \tag{5.12}$$
$$M^{hnor}(f+x_{m+1}^2) = M^{hnor}(f). \tag{5.13}$$

Denote by $H^\infty_\mathbb{C}$ the μ-dimensional vector space of global flat multivalued sections in the flat cohomology bundle $\bigcup_{\tau\in\Delta'} H^m(f^{-1}(\tau),\mathbb{C})$ (reduced cohomology for $m=0$). It comes equipped with a $\mathbb{Z}$-lattice $H^\infty_\mathbb{Z}$ and a real subspace $H^\infty_\mathbb{R}$ and a monodromy which is also denoted by M.

There is a natural *signed* Steenbrink PMHS (*signed*: definition 5.29 below) $(H^\infty_\mathbb{C}, H^\infty_\mathbb{R}, F^\bullet_{St}, W_\bullet, M, S)$ on $H^\infty_\mathbb{C}$. The weight filtration is $W^{(N,m)}_\bullet H^\infty_{\neq 1}$ on $H^\infty_{\neq 1}$ and $W^{(N,m+1)}_\bullet H^\infty_1$ on H^∞_1 (see lemma 5.14 (a) for $W^{(N,m)}_\bullet$). The Hodge filtration was defined first by Steenbrink using resolution of singularities [St77]. Then Varchenko [Va80] constructed a closely related Hodge filtration $F^\bullet_{Va}$ from the Brieskorn lattice $H_0''(f)$ (definition below). Scherk and Steenbrink [SS85] and M. Saito [SaM82] modified this construction to recover $F^\bullet_{St}$. Below we explain the Brieskorn lattice and this modified construction.

But first we give the polarizing form S. The lattice $H^\infty_\mathbb{Z}$ can be identified with the dual $Ml(f)^\vee = \mathrm{Hom}(Ml(f),\mathbb{Z})$ of the Milnor lattice $Ml(f)$, and thus it comes equipped with the dual Seifert form $L^\vee$ (using the notations 5.11) of the Seifert form L on $Ml(f)$. Define the normalized Seifert form L^{nor} on $H^\infty_\mathbb{Z}$ by

$$L^{nor} := (-1)^{(m+1)(m+2)/2} L^\vee = (L^{hnor})^\vee, \tag{5.14}$$

an M-invariant automorphism $\nu : H^\infty_\mathbb{Q} \to H^\infty_\mathbb{Q}$

$$\nu := \begin{cases} \frac{1}{M-\mathrm{id}} & \text{on } H^\infty_{\neq 1}, \\ \frac{-N}{M-\mathrm{id}} & \text{on } H^\infty_1, \end{cases} \tag{5.15}$$

and the M-invariant polarizing form $S : H^\infty_\mathbb{Q} \times H^\infty_\mathbb{Q} \to \mathbb{Q}$ by

$$S(a,b) := -L^{nor}(a, \nu b). \tag{5.16}$$

L^{nor} and S are related by the equivalence in lemma 5.27. Therefore S is $(-1)^m$-symmetric on $H^\infty_{\neq 1}$ and $(-1)^{m+1}$-symmetric on H^∞_1. The restriction to $H^\infty_{\neq 1}$ is $(-1)^{m(m+1)/2} \cdot I^\vee$, where $I^\vee$ on $H^\infty_{\neq 1}$ is dual to I (which is non-degenerate on $Ml(f)_{\neq 1}$). This follows from (5.31) in corollary 5.12.

Steenbrink had this restriction to $H^\infty_{\neq 1}$ of S, but not the part on H^∞_1. That part was defined with a sign mistake in [He99] and correctly in [He02]. The same sign mistake led to the claim in [He99] that $(H^\infty_\mathbb{R}, H^\infty_\mathbb{C}, F^\bullet_{St}, W_\bullet, M, S)$ is a Steenbrink PMHS of weight m. But it is (as stated correctly in [He02]) a *signed* Steenbrink PMHS of weight m.

Definition 5.29. *A tuple $(H_\mathbb{R}, H_\mathbb{C}, F^\bullet, W_\bullet, M, S)$ is a signed Steenbrink PMHS of weight m if $(H_\mathbb{R}, H_\mathbb{C}, F^\bullet, W_\bullet, M_s \cdot e^{-N}, S)$ is a Steenbrink PMHS of weight m.*

Remarks 5.30. (i) The only difference between a Steenbrink PMHS and a signed Steenbrink PMHS is that the positivity condition in definition 5.15 (c)(iv)(β) (see the notations 5.13 for $\theta(\lambda)$)

$$i^{2p-m-\theta(\lambda)-l} \cdot S(a, N^l \overline{a}) > 0$$

for $a \in (F^p P_{m+l} \cap \overline{F^{m+\theta(\lambda)+l-p} P_{m+l}})_\lambda - \{0\}$ has to be replaced by the positivity condition

$$i^{2p-m-\theta(\lambda)-l} \cdot S(a, (-N)^l \overline{a}) > 0.$$

This changes the sign in the case of a Jordan block of even size, i.e. in the case of a pair of spectral pair ladders (or a single one) of even length $l+1$.

(ii) This leads to obvious variants of the theorems 5.20, 5.23 and 5.24 for signed Steenbrink PMHS: In (5.21) and (5.22) the last entry $\varepsilon \in \{\pm 1\}$ in the isometric triples has to be replaced by $-\varepsilon$ if $l+1$ is even. The factor in (5.11) and the last entry in the Seifert form pairs in (5.12) and (5.14) have to be multiplied by -1 if $l+1$ is even.

(iii) We did not work from the beginning only with signed Steenbrink PMHS because also Steenbrink PMHS naturally appear. An M-tame function on an affine manifold of dimension

$m+1$ leads by work of Sabbah [Sa98][NS99] to a Steenbrink MHS $(H_{\mathbb{R}}, H_{\mathbb{C}}, F^\bullet_{Sa}, W_\bullet, M)$ of weight m. Then for S defined as above, the tuple $(H_{\mathbb{R}}, H_{\mathbb{C}}, G^{-1}F^\bullet_{Sa}, W_\bullet, M, S)$ is a Steenbrink PMHS of weight m [HS07, theorem 7.3].

(iv) A Steenbrink MHS of weight m with polarizing form S is a Steenbrink PMHS respectively a signed Steenbrink PMHS if and only if $e^{zN}F^\bullet$ is for $\operatorname{Im} z \gg 0$ respectively for $\operatorname{Im} z \ll 0$ the Hodge filtration of a sum of pure polarized Hodge structures (of weight m on $H_{\neq 1}$ and of weight $m+1$ on H_1) [CKS86] (see also [He03][HS07]). This was lifted in [HS07] to Sabbah orbits respectively nilpotent orbits of TERP-structures.

Examples 5.31. (i) The Laurent polynomial $x_0 + \frac{1}{x_0}$ (so $m=0$) is an M-tame function on $\mathbb{C}^*$. It is the mirror partner of the quantum cohomology of $\mathbb{P}^1$. It has two A_1 *ihs*, so the global Milnor number is $\mu = 2$. Here the Milnor lattice has to be replaced by a $\mathbb{Z}$-lattice of rank 2 of Lefschetz thimbles. This comes equipped with a Seifert form. For a suitable basis the matrix of the Seifert form is $-\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} =: -S^t$. Here the normalized Seifert form is $L^{hnor} = -L$, and the normalized monodromy is $M^{hnor} = -M$. It has the matrix

$$S^{-1}S^t = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix} = -\exp \begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix}$$

with one 2×2 Jordan block and eigenvalue -1 and matrix $\begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix}$ of its nilpotent part N. For the vector a represented by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ one finds

$$L^{hnor}(a, Na) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2 > 0. \tag{5.17}$$

Thus by theorem 5.9 the Seifert form pair $(H_{\mathbb{R}}, L^{hnor})$ is of type $\mathrm{Seif}(-1,1,2,1)$. This is in accordance with the fact that here we have a Steenbrink PMHS of weight one and with (5.14), which predicts this type $\mathrm{Seif}(-1,1,2,1)$.

(ii) Each hyperbolic surface (so $m=2$) singularity T_{pqr} (with $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$) has a rank 2 $\mathbb{Z}$-sublattice $Ml(f)_1 \cap Ml(f)$ of the Milnor lattice $Ml(f)$. For a suitable basis $\underline{a} = (a_1, a_2)$

of this sublattice, the matrix of the Seifert form L is by [GH17, (29)]

$$\begin{aligned} L(\underline{a}^t,\underline{a}) &= \begin{pmatrix} 0 & -\chi \\ \chi & \frac{\chi^2}{2}(\kappa-1) \end{pmatrix} =: S^t \\ \text{where } \kappa &:= \frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1, \quad \chi := \mathrm{lcm}(p,q,r). \end{aligned}$$

Here the normalized Seifert form is $L^{hnor}=L$ and is given by the matrix S^t. Its monodromy is $M^{hnor}=-M$ and has on $Ml(f)_1$ the matrix

$$\begin{aligned} S^{-1}S^t &= \frac{1}{\chi^2}\begin{pmatrix} \frac{\chi^2}{2}(\kappa-1) & -\chi \\ \chi & 0 \end{pmatrix}\begin{pmatrix} 0 & -\chi \\ \chi & \frac{\chi^2}{2}(\kappa-1) \end{pmatrix} \\ &= -\exp\begin{pmatrix} 0 & \chi(\kappa-1) \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Its nilpotent part N on $Ml(f)_1$ has the matrix $\begin{pmatrix} 0 & \chi(\kappa-1) \\ 0 & 0 \end{pmatrix}$. For the vector a represented by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ one finds

$$\begin{aligned} L^{hnor}(a,Na) &= \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -\chi \\ \chi & \frac{\chi^2}{2}(\kappa-1) \end{pmatrix}\begin{pmatrix} 0 & \chi(\kappa-1) \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \chi^2(\kappa-1)<0. \end{aligned} \tag{5.18}$$

Thus by theorem 5.9 the Seifert form pair $(Ml(f)_1\cap Ml(f)_{\mathbb{R}},L^{hnor})$ is of type $\mathrm{Seif}(-1,1,2,-1)$. This is in accordance with the fact that here we have a signed Steenbrink PMHS of weight one and with the variant of (5.14), which predict this type $\mathrm{Seif}(-1,1,2,-1)$.

Now we apply the notations from section 5.4 to the cohomology bundle $\bigcup_{\tau\in\Delta'}H^m(f^{-1}(\tau),\mathbb{C})$, i.e. the V-filtration and the spaces C^α and the isomorphisms $es(.,\alpha):H^\infty_{e^{-2\pi i\alpha}}\to C^\alpha$.

The *Brieskorn lattice* is a free $\mathbb{C}\{\tau\}$-module $H_0''(f)\subset V^{>-1}$ which had first been defined by Brieskorn [Br70]. The name *Brieskorn lattice* is due to [SaM89], the notation H_0'' is from [Br70]. The Brieskorn lattice is generated by germs of sections $s[\omega]$ from holomorphic $(m+1)$-forms $\omega\in\Omega_X^{m+1}$: Integrating the Gelfand-Leray form $\frac{\omega}{df}|_{f^{-1}(\tau)}$ over cycles in $H_m(f^{-1}(\tau),\mathbb{C})$ gives a holomorphic section $s[\omega]$ in the cohomology bundle, whose germ $s[\omega]_0$ at 0 is in fact in $V^{>-1}$ (this was proved first by Malgrange).

Steenbrink's Hodge filtration $F^\bullet_{St}H^\infty$ can be recovered as follows [SS85][SaM82]. Consider for $\lambda \in S^1$ the unique $\alpha \in (0,1]$ with $\lambda = e^{-2\pi i\alpha}$. Then

$$F^p_{St}H^\infty_\lambda = es(.,\alpha-1)(\tau)^{-1}\Big(\partial_\tau^{m-p}\,\mathrm{Gr}_V^{m-p+\alpha-1}\,H_0''(f)\Big). \tag{5.19}$$

The Brieskorn lattice is invariant under ∂_τ^{-1}, which is well defined as an isomorphism $\partial_\tau^{-1} : V^{>-1} \to V^{>0}$. Thus $H_0''(f)$ is a free $\mathbb{C}\{\{\partial_\tau^{-1}\}\}$-module of rank μ. The Fourier-Laplace transform in theorem 5.2 (a) can be described algebraically by

$$\partial_\tau^{-1} \mapsto z, \quad \partial_\tau \mapsto z^{-1}, \quad \tau \mapsto -\partial_{z^{-1}} = z^2\partial_z. \tag{5.20}$$

The Fourier-Laplace transform $FL(H_0''(f))$ is a free $\mathbb{C}\{z\}$-module in $V^{>0}_{(z)}$ (the index (z) indicates that here we use the coordinate z) and is invariant under $z^2\partial_z$. Theorem 5.28 (a) and (5.20) give

$$G^{(\alpha)}F^p_{St}H^\infty_\lambda = es(.,\alpha)(z)^{-1}\left(z^{-(m-p)}\,\mathrm{Gr}_{V_{(z)}}^{m-p+\alpha}\,FL(H_0''(f))\right). \tag{5.21}$$

Theorem 5.28 (b) and theorem 5.23 (a) say roughly

$$\begin{aligned} & F^\bullet_{St} \text{ has good isotropy properties w.r.t. } S \\ \iff\quad & GF^\bullet_{St} \text{ has good isotropy properties w.r.t. } L^{sym}_\kappa \\ \iff\quad & \mathrm{Gr}^\bullet_V\, FL(H_0''(f)) \text{ has good isotropy properties w.r.t. } P. \end{aligned} \tag{5.22}$$

In fact, a stronger compatibility of $FL(H_0''(f))$ and P holds [He02, Theorem 10.28], [He03, 8.1]:

$$P : FL(H_0''(f)) \times FL(H_0''(f)) \to z^{m+1}\mathbb{C}\{z\}, \tag{5.23}$$

and the induced symmetric pairing, which is the z^{m+1}-coefficient,

$$FL(H_0''(f)/z\cdot FL(H_0'') \times FL(H_0''(f))/z\cdot FL(H_0''(f)) \to \mathbb{C} \tag{5.24}$$

is nondegenerate. Here the μ-dimensional space $FL(H_0''(f)/z\cdot FL(H_0'')$ is the 0-fiber of a canonical extension to 0 of a bundle on $\mathbb{C}^*$ dual to a bundle of Lefschetz thimbles.

The tuple

$$TEZP(f) := (H^\infty_{\mathbb{Z}}, L^{nor}, V^{mod}_{(z)}, P, FL(H_0''(f))) \tag{5.25}$$

(the index (z) in $V^{mod}_{(z)}$ indicates that the coordinate on $\mathbb{C}$ is here z, not τ) is a TERP-structure of weight $m+1$ in the sense of [He03, definition 2.12]. Because of the lattice structure $H^\infty_{\mathbb{Z}}$ (instead of just the real structure $H^\infty_{\mathbb{R}}$) we even call it a $TEZP$ structure.

Finally, consider two *ihs* $f(x_0, ..., x_m)$ and $g(x_{m+1}, ..., x_{m+n+1})$ and their Thom-Sebastiani sum $f+g$. The canonical isomorphism

$$(Ml(f), L^{hnor}) \otimes (Ml(g), L^{hnor}) \cong (Ml(f+g), L^{hnor}) \tag{5.26}$$

induces a canonical isomorphism

$$\begin{aligned} & \bigcup_{z\in\Delta'} H^m(f^{-1}(z), \mathbb{Z}) \otimes H^n(g^{-1}(z), \mathbb{Z}) \\ \to & \bigcup_{z\in\Delta'} H^{m+n+1}((f+g)^{-1}(z), \mathbb{Z}) \end{aligned} \tag{5.27}$$

which respects the pairings P (the L in (5.2) is here L^{nor}).

The following theorem was essentially shown in [SS85, Lemma 8.7]. But see the remarks after it for some critic.

Theorem 5.32. *The $TEZP$ structures satisfy for f and g as above the following Thom-Sebastiani formula:*

$$TEZP(f+g) \cong TEZP(f) \otimes TEZP(g). \tag{5.28}$$

Proof. The isomorphism for the data $(H^\infty_{\mathbb{Z}}, L^{nor})$ is the classical Thom-Sebastiani result in (5.3) and (5.8). The isomorphism for P follows from its definition with L^{nor} in (5.2). The isomorphism for $V^{mod}_{(z)}$ is trivial.

The isomorphism

$$FL(H''_0(f+g)) \cong FL(H''_0(f)) \otimes FL(H''_0(g)) \tag{5.29}$$

is not so difficult to see, when one looks at the (imitations of) oscillating integrals behind the sections in $FL(H''_0(f))$. If $\sigma_1 \in H''_0(f) \cap \bigoplus_{-1<\alpha<N_1} C^\alpha_\tau$ for some arbitrarily large N_1 and if $\delta_1(z) \in H_n(f^{-1}(z), \mathbb{C})$, then

$$\begin{aligned} FL(\sigma_1)(\delta(z)) &= \int_0^{z\cdot(+\infty)} e^{-\tau_1/z} \cdot \sigma_1(\delta_1(\tau_1)) d\tau_1, \\ \text{where } \sigma_1(\delta_1(\tau_1)) &= \sum_{-1<\alpha<N_1} \sum_{k=0}^{n} a(\sigma_1, \alpha, k) \cdot \tau_1^\alpha \cdot (\log \tau_1)^k \\ \text{for some} \quad & a(\sigma_1, \alpha, k) \in \mathbb{C}, \end{aligned} \tag{5.30}$$

and analogously for $FL(\sigma_2)(\delta_2(z))$ if $\sigma_2 \in H_0''(g)\cap\bigoplus_{-1<\alpha<N_2} C_\tau^\alpha$ and $\delta_2(z) \in H_m(g^{-1}(z),\mathbb{C})$. The construction of the topological isomorphism (5.8) of Milnor lattices in [AGV88, I.2.7] gives for $\tau \in]0, z\cdot(+\infty)[$

$$(\sigma_1\otimes\sigma_2)((\delta_1\otimes\delta_2)(\tau)) \quad = \quad \int_0^\tau \sigma_1(\delta_1(\tau_1))\cdot\sigma_2(\delta_2(\tau-\tau_1))d\tau_1. \tag{5.31}$$

We obtain (with $\tau = \tau_1+\tau_2$ in the second equality)

$$\begin{aligned} & FL(\sigma_1\otimes\sigma_2)((\delta_1\otimes\delta_2)(z)) \\ = & \int_0^{z\cdot(+\infty)} e^{-\tau/z}\cdot(\sigma_1\otimes\sigma_2)((\delta_1\otimes\delta_2)(\tau))d\tau \\ = & FL(\sigma_1)(\delta_1(z))\cdot FL(\sigma_2)(\delta_2(z)). \end{aligned} \tag{5.32}$$

This proves the isomorphism (5.29). □

Corollary 5.33. *Steenbrink's Hodge filtration satisfies for f and g as above the following Thom-Sebastiani formula:*

$$\begin{aligned} & G(F^p_{St})H^\infty_{e^{-2\pi i\alpha}}(f+g) \\ = & \sum_{\beta,\gamma,q,r:(*)} G(F^q_{St})H^\infty_{e^{-2\pi i\beta}}(f)\otimes G(F^r_{St})H^\infty_{e^{-2\pi i\gamma}}(g) \end{aligned} \tag{5.33}$$

where $0<\alpha\leq 1$ and

$$\begin{aligned} (*) \quad : \quad & 0<\beta,\gamma\leq 1,\ \beta+\gamma = (\alpha \text{ or } \alpha+1), \\ & (m-q+\beta)+(n-r+\gamma) = m+n+1-p+\alpha. \end{aligned}$$

Proof. Apply theorem 5.32 and (5.21). □

Remarks 5.34. (i) The isomorphism (5.29) for H_0'' was essentially proved in [SS85, Lemma (8.7)]. Though Scherk and Steenbrink did not make the compatibility with the topological Thom-Sebastiani isomorphism between the cohomology bundles precise, and they avoided the use of the Fourier-Laplace transformation. They obtained a ∂_τ^{-1}-linear isomorphism $H_0''(f+g)\cong H_0''(f)\otimes H_0''(g)$.

(ii) They applied this isomorphism together with (5.19) in order to obtain a Thom-Sebastiani formula for Steenbrink's Hodge filtration $F^\bullet_{St}$ [SS85, Theorems (8.2) and (8.11)]: It is (5.33) without the twists by G. But in the cases with $N\neq 0$, this twist is necessary, in these cases their formula is not correct.

In their proof, they mixed ∂_τ^{-1}-linearity and τ-linearity. They extracted from the isomorphism $H_0''(f+g) \cong H_0''(f) \otimes H_0''(g)$ maps $C^\beta(f) \otimes C^\gamma(g) \to C^{\beta+\gamma}(f+g)$ in the variable τ [SS85, Lemma (8.8)] and went with them into the defining formula (5.19) of $F^\bullet_{St}$.

Of course, in the case $N = 0$, the isomorphism G in definition 5.22 is just a rescaling, and then $G(F^\bullet_{St}) = F^\bullet_{St}$, so then their Thom-Sebastiani formula is correct.

(iii) In the case $g = x^2_{m+1}$, the sum $f + g = f + x^2_{m+1}$ is a suspension of f. Theorem 5.32 gives in that case an isomorphism

$$TEZP(f + x^2_{n+1}) \cong TEZP(f) \otimes TEZP(x^2_{n+1}), \tag{5.34}$$

and formula (5.33) boils down to theorem 5.26.

(iv) We expect that the following generalization of the theorem 5.26 holds: *For any two (signed or not) Steenbrink PMHS, the formula (5.32) gives a (signed or not) Steenbrink PMHS.*

6 Spectra for the spaces $T_{\mathrm{HOR1}}, T_{\mathrm{HOR2}}$ and *ihs*

This chapter introduces the subspaces $T_{\mathrm{HOR1}}(n,\mathbb{R})$ and $T_{\mathrm{HOR2}}(n,\mathbb{R})$ of $T(n,\mathbb{R})$. Building on chapter 5, we endow HOR matrices with a Steenbrink polarized mixed Hodge structure and a spectrum. Recall that we have already stated the important properties of these spaces in theorem 2.18 and defined a part of the objects here, in definition 2.20. The main result of this chapter is:

Theorem 6.1. *(a) (Section 6.4) In the cases $n = 2$ and $n = 3$, the conjectures 2.21 and 2.22 and the conjecture 2.23 for function germs are true.*

(b) (Section 6.6) In the case of any chain type singularity $f(x_0, ..., x_m)$, the matrix $S \in T_{\mathrm{HOR}k}(\mu,\mathbb{Z})$ with $k \equiv m(2)$ which is considered in [OR77, (4.1) Conjecture], satisfies $\mathrm{Sp}(S) = \mathrm{Sp}(f) - \frac{m-1}{2}$.

Section 6.1 discusses spectral pairs from an abstract point of view. This is elementary, but must be provided. Essential is the notion of an enhancement of a real Seifert form pair. This is a decomposition of the pair, together with spectral pairs for each composing piece, such that the first spectral numbers control the pieces' Seifert form pair type.

Section 6.2 prepares the introduction of the HOR spaces. It introduces isomorphic subspaces $T^{scal}_{\mathrm{HOR}k}(n,\mathbb{R})$ and it formalizes and studies the recipe

$$(\text{eigenvalues of } R^{mat}_{(k)}(S)) \mapsto (\text{spectral numbers } \alpha_1, ..., \alpha_n \text{ of } S), \tag{6.1}$$

which is implicit in the proof of theorem 2.18 (b). This is elementary, but worth to be studied for itself. Properties of these spectral numbers give, combined with conjecture 2.23 on the spectral numbers of holomorphic functions, new features of these spectral numbers.

The recipe (6.1) will also be extended to a recipe for spectral pairs $\mathrm{Spp}(S) = \sum_{j=1}^{n}(\alpha_j, k_j) \in \mathbb{Z}_{\geq 0}(\mathbb{R} \times \mathbb{Z})$.

Section 6.3 introduces the subfamilies $T_{\mathrm{HOR}k}(n, \mathbb{R}) \subset T(n, \mathbb{R})$ for $k \in \{1, 2\}$ of HOR matrices and proves theorem 2.18 (a). And it adds more precise information, especially, that the spectral pairs and the eigenspace decompositions of such a matrix give rise to a natural split polarized mixed Hodge structure.

Section 6.4 contains the study of $n = 2$ and $n = 3$, result mentioned above.

In section 6.5 we will review some facts on holomorphic map germs $f : (\mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at 0 and on M-tame functions $f : X \to \mathbb{C}$ with $\dim X = m + 1$. Especially, we will discuss the following. In both cases, there is a Milnor number $\mu = \mu(f) \in \mathbb{Z}_{\geq 1}$. In both cases, there is a $\mathrm{Br}_\mu \ltimes G_{sign,\mu}$ orbit of matrices $S \in T(\mu, \mathbb{Z}) := T(\mu, \mathbb{R}) \cap GL(\mu, \mathbb{Z})$. Here Br_μ is the braid group with μ strings. We call these matrices Stokes matrices. Then $(-1)^{m+1} S^{-1} S^t$ is a matrix of the (classical global) monodromy. In both cases, there are μ spectral pairs $\mathrm{Spp}(f) = \sum_{j=1}^{\mu}(\alpha_j(f), l(f)) \in \mathbb{Z}_{\geq 0}(\mathbb{Q} \times \mathbb{Z})$ which come from natural mixed Hodge structures. The first entries are the spectral numbers $\mathrm{Sp}(f) = \sum_{j=1}^{\mu}(\alpha_j(f)) \in \mathbb{Z}_{\geq 0}(\mathbb{Q})$. In a suitable numbering, the spectral numbers satisfy the symmetry $\alpha_j(f) + \alpha_{\mu+1-j}(f) = \frac{m-1}{2}$.

Section 6.6 contains the study of chain type singularities.

6.1 Enhanced real Seifert form pairs and spectral pairs

In this section we make precise what it means to associate to a matrix $S \in T(n, \mathbb{R})$ a spectrum. This is done in lemma 6.7 and the remarks afterwards where those matrices are related to Seifert form pairs. Seifert form pairs in turn with a certain enhancement are equivalent to split Steenbrink PMHS. We first introduce those concepts, then recover the result of Némethi [Ne95] for *ihs* that the real Seifert form isomorphism class, together with an enhancement is equivalent to $\mathrm{Spp}(f) \mod 2\mathbb{Z} \times \{0\}$. Finally we turn to lemma 6.7, in which we show that we have three equivalent kinds of describing the data here, as flags, decompositions, or as matrices $S \in T(n, \mathbb{R})$.

Enhancements of Seifert form pairs are decompositions of the pair, together with spectral pairs for each composing piece. The spectral pairs must be such, that the first spectral numbers control the pieces' Seifert form pair type.

We start with spectral pairs, first in an elementary abstract setting. Compare to definition 5.18, in which we already discussed the geometry in the Steenbrink spectrum.

Definition 6.2. *(a) A* **spectral pair** *is a pair* $(\alpha, k) \in \mathbb{R} \times \mathbb{Z}$. *An unordered tuple of* n *spectral pairs is denoted by*

$$\mathrm{Spp} = \sum_{(\alpha,k)\in\mathbb{R}\times\mathbb{Z}} d(\alpha,k)(\alpha,k) \in \mathbb{Z}_{\geq 0}(\mathbb{R}\times\mathbb{Z}) \subset \mathbb{Z}(\mathbb{R}\times\mathbb{Z})$$

with $|\,\mathrm{Spp}\,| := \sum_{(\alpha,k)} d(\alpha,k) = n$. *Here* $\mathbb{Z}(\mathbb{R}\times\mathbb{Z})$ *is the group ring over* $\mathbb{R}\times\mathbb{Z}$. *The number* $d(\alpha,k)$ *is the multiplicity of* (α,k) *as a spectral pair. Any numbering of the* n *spectral pairs gives* $\mathrm{Spp} = \sum_{j=1}^n (\alpha_j, k_j)$.

(b) (i) A **spectral pair ladder** *(short: spp-ladder) consists of* $l+1$ *spectral pairs*

$$(\alpha + k, m + l - 2k) \quad \text{with } k \in \{0, 1, ..., l\}. \tag{6.1}$$

Here $m \in \mathbb{Z}$ *and* $l \in \mathbb{Z}_{\geq 0}$. *The numbers* m *and* l *are uniquely determined by the spectral pair ladder.* $l+1$ *is its length, and* m *is its* **center**. *Its* **first spectral pair** *is the pair* $(\alpha, m+l)$. *Its* **first spectral number** *is* α. *The spp-ladder is determined by* m *and* l *and its first spectral number.*

(ii) The **partner spp-ladder** *is the spp-ladder*

$$(m - l - 1 - \alpha + k, m + l - 2k) \quad \text{with } k \in \{0, 1, ..., l\}. \tag{6.2}$$

It has the same length and center. The **distance** *of an spp-ladder to its partner is* $2\alpha + l + 1 - m$.

(iii) A spp-ladder is **single** *if it is its own partner, i.e. if the distance to its partner is 0, i.e. if* $\alpha = \frac{m-l-1}{2}$.

(c) An unordered pair of spp-ladders (short: sppl-pair) consists of two spp-ladders which are partners of one another and which have distance $\neq 0$.

Lemma 6.3. *(a) Each sppl-pair and each single spp-ladder are invariant under the Kleinian group* $\mathrm{id}, \pi_1, \pi_2, \pi_3 : \mathbb{R}\times\mathbb{Z} \to \mathbb{R}\times\mathbb{Z}$ *with*

$$\begin{aligned} \pi_1 : (\frac{m-1}{2} + \alpha, m+k) &\mapsto (\frac{m-1}{2} - \alpha, m-k), \\ \pi_2 : (\frac{m-1-k}{2} + \alpha, m+k) &\mapsto (\frac{m-1-k}{2} - \alpha, m+k), \\ \pi_3 = \pi_1 \circ \pi_2 = \pi_2 \circ \pi_1 : (\alpha, m+k) &\mapsto (\alpha + k, m-k). \end{aligned} \tag{6.3}$$

In the case of a sppl-pair, π_3 *maps each spp-ladder to itself,* π_1 *and* π_2 *map the two spp-ladders to one another.*

(b) Suppose that a tuple $\mathrm{Spp} \in \mathbb{Z}_{\geq 0}(\mathbb{R}\times\mathbb{Z})$ *of* n *spectral pairs is built from sppl-pairs and single spp-ladders. Then the sppl-pairs and the single spp-ladders are uniquely determined by* Spp.

(c) Suppose that a tuple $\mathrm{Spp} \in \mathbb{Z}_{\geq 0}(\mathbb{R}\times\mathbb{Z})$ *of* n *spectral pairs is built from sppl-pairs and single spp-ladders. Then* $\mathrm{Spp} \mod 2\mathbb{Z}\times\{0\}$ *determines each spp-ladder uniquely up to a simultaneous shift of its members by elements of* $2\mathbb{Z}\times\{0\}$*, so it determines the lengths, the centers and the first spectral numbers modulo* $2\mathbb{Z}$ *of all spp-ladders.*

Proof. Trivial. □

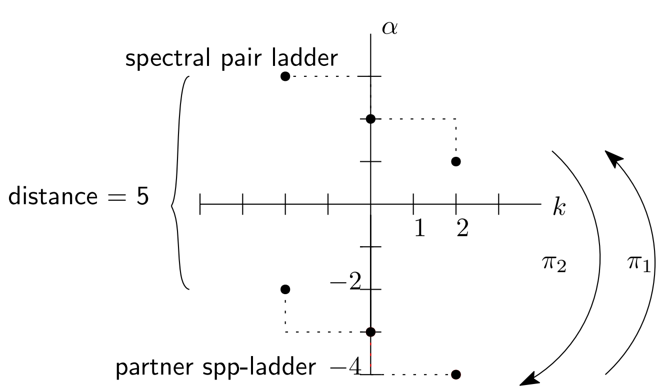
Example. (i) The spectral pairs $(3.5, 1)$, $(7, 1)$, $(7, 1) \in \mathbb{R}\times\mathbb{Z}$ as an unordered 3-tuple are written as

$$1 \cdot (3.5, 1) + 2 \cdot (7, 1) \in \mathbb{Z}_{\geq 0}(\mathbb{R}\times\mathbb{Z}).$$

(ii) For first spectral number $\alpha = 1$, center $m = 0$, and length $l = 2$, the spectral pair ladder with then 3 pairs is

$$(1, 2), (2, 0), (3, -2)$$

and the *partner spp-ladder* is: $$(-4, 2), (-3, 0), (-2, -2).$$



For the definition of an enhancement we need the irreducible types of Seifert pair from the last chapter, for ease of reading, we recall them here. The irreducible Seifert form pairs (cf. theorem 5.9) are given by the types with the following names.

$$
\begin{aligned}
\mathrm{Seif}(\lambda,1,n,\varepsilon) \quad &\text{with} \quad (\lambda=1\ \&\ n\equiv 1(2)) && (6.4)\\
&\text{or} \quad (\lambda=-1\ \&\ n\equiv 0(2)),\\
\mathrm{Seif}(\lambda,2,n) \quad &\text{with} \quad (\lambda=1\ \&\ n\equiv 0(2)) && (6.5)\\
&\text{or} \quad (\lambda=-1\ \&\ n\equiv 1(2)),\\
\mathrm{Seif}(\lambda,2,n,\zeta) \quad &\cong \quad \mathrm{Seif}(\overline{\lambda},2,n,\overline{\zeta}) && (6.6)\\
&\text{with} \quad \lambda,\zeta\in S^1-\{\pm 1\},\zeta^2=\overline{\lambda}\cdot(-1)^{n+1},\\
\mathrm{Seif}(\lambda,2,n) \quad &\text{with} \quad \lambda\in\mathbb{R}_{>1}\cup\mathbb{R}_{<-1}, && (6.7)\\
\mathrm{Seif}(\lambda,4,n) \quad &\text{with} \quad \lambda\in\{\zeta\in\mathbb{C}\,|\,|\zeta|>1,\mathrm{Im}\,\zeta>0\}. && (6.8)
\end{aligned}
$$

Definition 6.4. *Let $(H_\mathbb{R},L)$ be an S^1-Seifert form pair, i.e. a Seifert form pair with eigenvalues of its' monodromy in S^1, check definition 5.2.*

(a) An ***enhancement*** *of it is a decomposition of $(H_\mathbb{R},L)$ into a direct and L-orthogonal sum of Seifert form pairs $(H_\mathbb{R}^{(j)},L^{(j)})$ with $j\in\{1,...,r\}$ for some $r\in\mathbb{Z}_{\geq 1}$ together with spectral pairs $\mathrm{Spp}^{(j)}\in\mathbb{Z}_{\geq 0}(\mathbb{R}\times\mathbb{Z})$ with the following properties.*

(i) $\mathrm{Spp}^{(j)}$ consists of finitely many copies of the same sppl-pair or the same single spp-ladder. Its length is called l_j. All sppl-pairs and spp-ladders in $\mathrm{Spp}:=\sum_{j=1}^r\mathrm{Spp}^{(j)}$ have the same center $m\in\mathbb{Z}$. This is also called the ***center of the enhancement****. The first spectral number of the/one of the two spp-ladders is called α_j (if there are two, it does not matter which one). $|\,\mathrm{Spp}^{(j)}\,|=\dim H_\mathbb{R}^{(j)}$.*

(ii) $(H_\mathbb{R}^{(j)},L^{(j)})$ decomposes into copies of one irreducible Seifert form pair

$$
\begin{aligned}
\mathrm{Seif}((-1)^{m+1}e^{-2\pi i\alpha_j},2,l_j+1,\zeta_j) \quad &\text{if} \quad 2\alpha_j+l_j+1-m\in\mathbb{R}-\mathbb{Z},\\
\mathrm{Seif}((-1)^{m+1}e^{-2\pi i\alpha_j},2,l_j+1) \quad &\text{if} \quad 2\alpha_j+l_j+1-m\in\mathbb{Z}-2\mathbb{Z},\\
\mathrm{Seif}((-1)^{m+1}e^{-2\pi i\alpha_j},1,l_j+1,\varepsilon_j) \quad &\text{if} \quad 2\alpha_j+l_j+1-m\in 2\mathbb{Z},
\end{aligned}
$$

in equations (6.4), (6.5) and (6.6).

(b) An enhancement is ***polarized*** *if in (a)(ii)*

$$(\varepsilon_j \text{ resp. } \zeta_j)=e^{\frac{1}{2}\pi i(2\alpha_j+l_j+1-m)}. \tag{6.9}$$

An enhancement is ***signed polarized*** *if in (a)(ii)*

$$(\varepsilon_j \text{ resp. } \zeta_j)=(-1)^{l_j}e^{\frac{1}{2}\pi i(2\alpha_j+l_j+1-m)}. \tag{6.10}$$

Remarks 6.5. (i) Claim: An S^1-Seifert form pair $(H_\mathbb{R}, L)$ with (signed) polarized enhancement gives rise to and is equivalent to a *split (signed) Steenbrink polarized mixed Hodge structure* on $H_\mathbb{C}$.

The notions mixed Hodge structure and split mixed Hodge structure where recalled in definition 5.15 (a) and remark 5.19. The notion *Steenbrink polarized mixed Hodge structure* is defined in definition 5.15 (d). The signed version is defined in definition 5.29. The signed version turns up in the case of *ihs*. The unsigned version turns up in M-tame functions. For both cases see section 6.5.

The claim follows easily from the results in chapter 5, especially theorem 5.24. It builds on Deligne's $I^{p,q}$ of a mixed Hodge structure, on the polarizing form of a polarized mixed Hodge structure, and on the relation between Seifert form pairs and isometric triples, which is developed in the last chapter 5. Theorem 5.23 shows that a Steenbrink PMHS gives rise to the enhancement described above.

(ii) Némethi [Ne95] considered the case of an *ihs* f and studied there the relationship between the spectral pairs $\mathrm{Spp}(f)$ of Steenbrink's mixed Hodge structure and the real Seifert form. He found that $\mathrm{Spp}(f) \mod 2\mathbb{Z}\times\{0\}$ is equivalent to the isomorphism class of the real Seifert form. The following lemma recovers this result modulo the claim above in (i).

(iii) But for this result, one has to know a priori that $\mathrm{Spp}(f)$ comes from a signed Steenbrink polarized mixed Hodge structure, or that $\mathrm{Spp}(f)$ is part of a *signed polarized* enhancement of the real Seifert form.

Lemma 6.6. *Two S^1-Seifert form pairs $(H^i_\mathbb{R}, L^i)$ for $i \in \{1,2\}$ with polarized enhancements (or with signed polarized enhancements) with centers m and spectral pairs* Spp^i *satisfy*

$$(H^1_\mathbb{R}, L^1) \cong (H^2_\mathbb{R}, L^2) \iff \mathrm{Spp}^1 \equiv \mathrm{Spp}^2 \mod 2\mathbb{Z}\times\{0\}. \tag{6.11}$$

Proof. One can refine the decompositions of $(H^1_\mathbb{R}, L^1)$ and $(H^2_\mathbb{R}, L^2)$ in their enhancements to decompositions into sums of irreducible Seifert form pairs such that each comes equipped with a single spp-ladder or a sppl-pair. Then the irreducible Seifert form pair determines the length l of the single spp-ladder or of each spp-ladder in the sppl-pair.

The first spectral number α of the single spp-ladder or the first spectral numbers α and $\widetilde{\alpha}$ of the two spp-ladders in the sppl-pair are determined modulo $\mathbb{Z}$ by $e^{-2\pi i\alpha} = (-1)^{m+1}\lambda$ and $e^{-2\pi i\widetilde{\alpha}} = (-1)^{m+1}\overline{\lambda}$ where λ and $\overline{\lambda}$ are the eigenvalue(s) of the irreducible Seifert form pair.

α and $\widetilde{\alpha}$ are determined modulo $2\mathbb{Z}$ by the condition (6.9) respectively (6.10) in the cases (6.4) and (6.6). In the case (6.5), they satisfy $\alpha \in \frac{1}{2}\mathbb{Z}$ and $\widetilde{\alpha} \equiv \alpha + 1(2)$.

Therefore the isomorphism class of $(H^i_{\mathbb{R}}, L^i)$ determines the union Spp^i of all spp-ladders in the enhancement modulo $2\mathbb{Z} \times \{0\}$. This proves $\Rightarrow$.

$\Leftarrow$: Let (α_j, m_j, l_j) for $j \in \{1, ..., \rho^1\}$ be the first spectral numbers, the centers and the lengths minus one of the spectral pair ladders in Spp^1. By lemma 6.3 (c), the triples (α_j mod $2\mathbb{Z}, m_j, l_j$) are determined by $\mathrm{Spp}^1 \mod 2\mathbb{Z} \times \{0\}$. Definition 6.4 and (6.4) and (6.5) show that each such triple determines a unique irreducible Seifert form pair in $(H^1_{\mathbb{R}}, L^1)$. In the case of a sppl-pair, the triples of the two spp-ladders determine the same irreducible Seifert form pair. This shows $\Leftarrow$. $\square$

Finally, we put the matrices in $T(n, \mathbb{R})$ into the frame of Seifert form pairs.

Lemma 6.7. *Let $(H_{\mathbb{R}}, L)$ be an S^1-Seifert form pair with $\dim H_{\mathbb{R}} = n \in \mathbb{Z}_{\geq 1}$. The following data are equivalent.*

(A) A basis $\underline{v} = (v_1, ..., v_n)$ with $L(\underline{v}^t, \underline{v}) \in T(n, \mathbb{R})$ up to the signs of the basis vectors v_j.

(B) A splitting $H_{\mathbb{R}} = \bigoplus_{j=1}^n H^{(j)}_{\mathbb{R}}$ with $\dim H^{(j)}_{\mathbb{R}} = 1$, $L(H^{(i)}_{\mathbb{R}}, H^{(j)}_{\mathbb{R}}) = 0$ for $i < j$ and $L(H^{(j)}_{\mathbb{R}}, H^{(j)}_{\mathbb{R}}) = \mathbb{R}_{\geq 0}$.

(C) A complete flag $\{0\} \subset U_0 \subset U_1 \subset U_2 \subset ... \subset U_n = H_{\mathbb{R}}$ (complete flag means $\dim U_j = j$) with

$$H_{\mathbb{R}} = \bigoplus_{j=1}^n H^{(j)}_{\mathbb{R}} \quad \text{where } H^{(j)}_{\mathbb{R}} := U_j \cap U_{j-1}^{\perp R}, \tag{6.12}$$

$$L(H^{(j)}_{\mathbb{R}}, H^{(j)}_{\mathbb{R}}) = \mathbb{R}_{\geq 0}. \tag{6.13}$$

Proof. (A)$\Rightarrow$(B): Put $H^{(j)}_{\mathbb{R}} := \mathbb{R} \cdot v_j$.

(B)$\Rightarrow$(A): For each j choose a basis vector v_j of $H^{(j)}_{\mathbb{R}}$ with $L(v_j, v_j) = 1$. It exists and is unique up to the sign.

(B)$\Rightarrow$(C): Put $U_j := \bigoplus_{i \leq j} H^{(i)}_{\mathbb{R}}$. Then $U_{j-1}^{\perp R} = \bigoplus_{i \geq j} H^{(i)}_{\mathbb{R}}$.

(C)$\Rightarrow$(B): $H^{(j)}_{\mathbb{R}}$ has because of $\dim U_j + \dim U_{j-1}^{\perp R} = n + 1$ at least dimension 1. By (6.12) it has dimension 1. $\square$

Remarks 6.8. (i) A splitting as in (B) can be called a *semiorthogonal decomposition*. Such splittings are considered in a much richer context in derived algebraic geometry.

(ii) The complete flag in (C) and the positivity condition (6.13) might remind one of Hodge structures. But there is no close relationship.

(iii) In the case of isolated hypersurfaces, the data in lemma 6.7 come from a distinguished basis, a refinement of the $\mathbb{Z}$-lattice structure. Steenbrink's mixed Hodge structure is of a transcendent origin and has a clear relationship with the real structure, but no known relationship with distinguished bases.

(iv) Nevertheless, the wish to associate to matrices $S\in T(n,\mathbb{R})$ spectral pairs, can now be interpreted as the wish to see in the data in lemma 6.7 a shadow of mixed Hodge structures.

(v) Let $(H_\mathbb{R},L)$ be a real Seifert form pair. The set of all complete flags in $H_\mathbb{R}$ is a real projective algebraic manifold M^{flags}. For any complete flag $U_\bullet$, the condition (6.12) is equivalent to the condition

$$U_j\oplus U_j^{\perp R}=H_\mathbb{R}\quad\text{for any } j\in\{1,...,n\}. \tag{6.14}$$

Let us call complete flags which do not satisfy (6.12) *degenerate*. They form a Zariski closed subvariety M^{degen} in M^{flags}, which separates the complement into components. For each component a tuple $(\varepsilon_1,...,\varepsilon_n)\in\{\pm 1\}^n$ with

$$L(H_\mathbb{R}^{(j)},H_\mathbb{R}^{(j)})=\varepsilon_j\cdot\mathbb{R}_{\geq 0} \tag{6.15}$$

exists, where $U_\bullet$ is in the component and $H_\mathbb{R}^{(j)}$ is defined as in (6.12). This follows from the nondegeneracy of L. The components with $(\varepsilon_1,...,\varepsilon_n)=(1,...,1)$ give by (B)⇒(A) sets of matrices $L(\underline{v}^t,\underline{v})$ in $T(n,\mathbb{R})$. The wish to associate to matrices $S\in T(n,\mathbb{R})$ spectral pairs, is the wish to associate to each such component spectral pairs.

(vi) A refinement of it is the wish to associate to each complete flag in $M^{flags}-M^{degen}$ in a component with $(\varepsilon_1,...,\varepsilon_n)=(1,...,1)$ an enhancement of $(H_\mathbb{R},L)$. In the case of $S\in\bigcup_{k=1,2}T_{\mathrm{HOR}k}(n,\mathbb{R})$, we will obtain such an enhancement.

(vii) There are Seifert form pairs $(H_\mathbb{R},L)$ for which $M^{flags}-M^{degen}$ has no components with $(\varepsilon_1,...,\varepsilon_n)=(1,...,1)$, i.e. which are not isomorphic to $(M(n\times 1,\mathbb{R}),\widetilde{L})$ with $\widetilde{L}(a,b)=a^t\cdot S^t\cdot b$ for any $S\in T(n,\mathbb{R})$. Any sum of irreducible Seifert form pairs

$$\mathrm{Seif}(1,1,1,-1),\ \mathrm{Seif}(-1,1,2,-1),\ \mathrm{Seif}(-1,2,1),\ \mathrm{Seif}(\lambda,2,1,\zeta)$$

(with $\lambda\in S^1-\{\pm 1\}$ and $\zeta=\frac{\overline{\lambda}+1}{|\lambda+1|}\cdot i^{n+1}$) has this property because then I_s is negative (semi-) definite by lemma 5.9. In the cases $n\in\{2,3\}$ the only other Seifert form pairs with this property are those which contain $\mathrm{Seif}(1,1,1,-1)$ or $\mathrm{Seif}(1,1,3,-1)$, see remark 6.26 (ii).

6.2 A recipe for spectral pairs

Section 6.3 will present the subspaces $T_{\mathrm{HOR}k}(n,\mathbb{R})$ of $T(n,\mathbb{R})$ for $k\in\{1,2\}$ and study the properties of the matrices in these subspaces. Here we prepare this and define the spectral recipe, the assignment of spectral pairs and a spectrum. For that, we introduce the *spaces of root arguments*, T^{scal}_{HOR1}, T^{scal}_{HOR2}. Both are subsets of the n-dimensional unit cube $[0,1]^n$, both are isomorphic to a simplex. As the name suggests they serve as building blocks, as arguments for the roots of polynomials in the spaces T^{pol}_{HOR1}, T^{pol}_{HOR2}. Those spaces will in the next section 6.3 give rise to HOR matrices inside $T(n,\mathbb{R})$. In the *spectral recipe 6.11* we assign spectral pairs Spp and thus a spectrum Sp first to a tuple in $T^{scal}_{\mathrm{HOR}k}$, via that to a polynomial in $T^{pol}_{\mathrm{HOR}k}$ and thus in the next section to an upper triangular matrix. Recall the notation in 2.18, it will be used below, especially $R^{mat}_{(k)}(S)^n$. We start with the definition of the *spaces of root arguments*.

Definition 6.9. *For $n\in\mathbb{Z}_{\geq 1}$ define the spaces*

$$\begin{aligned} T^{scal}_{\mathrm{HOR1}}(n,\mathbb{R}) &:= \{(\beta_1,...,\beta_n)\in[0,1]^n\,|\,\beta_1\leq ...\leq\beta_n, \\ &\qquad \beta_j+\beta_{n+1-j}=1\}, && (6.1)\\ T^{scal}_{\mathrm{HOR2}}(n,\mathbb{R}) &:= \{(\beta_1,...,\beta_n)\in[0,1]^n\,|\,0=\beta_1\leq ...\leq\beta_n, \\ &\qquad \beta_j+\beta_{n+2-j}=1 \text{ for } j\geq 2\}, && (6.2)\\ T^{simp}(n) &:= \{(\beta_1,...,\beta_n)\in[0,\tfrac{1}{2}]^n\,|\,\beta_1\leq ...\leq\beta_n\}. && (6.3)\end{aligned}$$

Define the map

$$\begin{aligned}\Pi:\bigcup_{k=1,2}T^{scal}_{\mathrm{HOR}k}(n,\mathbb{R}) &\to \mathbb{R}[x]_{\deg=n} \qquad (6.4)\\ \underline{\beta}=(\beta_1,...,\beta_n) &\mapsto \prod_{j=1}^{n}(x-e^{-2\pi i\beta_j})\end{aligned}$$

and the spaces

$$T^{pol}_{\mathrm{HOR}k}(n,\mathbb{R}):=\Pi(T^{scal}_{\mathrm{HOR}k}(n,\mathbb{R}))\subset\mathbb{R}[x]_{\deg=n}\quad\text{for } k\in\{1,2\}. \qquad (6.5)$$

Lemma 6.10. *(a) $T^{simp}(n)$ is the n-simplex in $\mathbb{R}^n$ with the $n+1$ corners $(x_{1j},...,x_{nj})$ for $j\in\{0,1,...,n\}$ with $x_{ij}=0$ for $i\leq j$ and $x_{ij}=\frac{1}{2}$ for $i>j$.*

(b) The following maps are affine linear isomorphisms. For odd n

$$\begin{aligned} T^{scal}_{\mathrm{HOR1}}(n,\mathbb{R})\to T^{simp}(\frac{n-1}{2}), &\qquad (\beta_1,...,\beta_n)\mapsto(\beta_1,...,\beta_{\frac{n-1}{2}}),\\ T^{scal}_{\mathrm{HOR2}}(n,\mathbb{R})\to T^{simp}(\frac{n-1}{2}), &\qquad (\beta_1,...,\beta_n)\mapsto(\beta_2,...,\beta_{\frac{n+1}{2}}).\end{aligned}$$

For even n

$$T^{scal}_{\mathrm{HOR1}}(n,\mathbb{R}) \to T^{simp}(\frac{n}{2}), \qquad (\beta_1,...,\beta_n)\mapsto(\beta_1,...,\beta_{\frac{n}{2}}),$$
$$T^{scal}_{\mathrm{HOR2}}(n,\mathbb{R}) \to T^{simp}(\frac{n-2}{2}), \qquad (\beta_1,...,\beta_n)\mapsto(\beta_2,...,\beta_{\frac{n}{2}}).$$

(c) The map Π in (6.4) is injective, and

$$\begin{aligned} T^{pol}_{\mathrm{HOR1}}(n,\mathbb{R}) &= \{p\in\mathbb{R}[x]\,|\,\deg p = n, p_n=1, p_j=p_{n-j}, \\ &\qquad \text{all zeros of } p \text{ are in } S^1\}, \end{aligned} \tag{6.6}$$
$$\begin{aligned} T^{pol}_{\mathrm{HOR2}}(n,\mathbb{R}) &= \{p\in\mathbb{R}[x]\,|\,\deg p = n, p_n=1, p_j=-p_{n-j}, \\ &\qquad \text{all zeros of } p \text{ are in } S^1\}. \end{aligned} \tag{6.7}$$

If $p\in T^{pol}_{\mathrm{HOR}k}(n,\mathbb{R})$ then $p_0=(-1)^{k-1}$, $p_j=p_0p_{n-j}$, $x^np(x^{-1})=p_0\cdot p(x)$, $\lambda\in S^1$ and $\overline{\lambda}$ have the same multiplicity as zeros of p, and the multiplicity of 1 as a zero of p is even for $k=1$ and odd for $k=2$.

Proof. **(a)** Trivial.

(b) For $\underline{\beta}\in T^{scal}_{\mathrm{HOR1}}(n,\mathbb{R})$ the symmetry $\beta_j+\beta_{n+1-j}$ is used. For odd n it implies $\beta_{\frac{n+1}{2}}=\frac{1}{2}$. For $\underline{\beta}\in T^{scal}_{\mathrm{HOR2}}(n,\mathbb{R})$ $\beta_1=0$ and the symmetry $\beta_j+\beta_{n+2-j}=1$ for $j\geq 2$ are used. For even n the symmetry implies $\beta_{\frac{n+2}{2}}=\frac{1}{2}$.

(c) Trivial. □

The following is the formalization of the recipe

$$\Big(\text{eigenvalues of } R^{mat}_{(k)}(S)\Big)\mapsto\Big(\text{spectral numbers } \operatorname{Sp}(S)\Big)$$

which is implicit in the proof of theorem 2.18 (b). In definition 6.19 (c) and theorem 6.20 (d) it is connected with theorem 2.18 (b). It is completely elementary, but interesting in its own right.

Recipe 6.11. (a) The following recipe associates to any tuple $\underline{\beta}\in T^{scal}_{\mathrm{HOR}k}(n,\mathbb{R})$ for $k\in\{1,2\}$ a spectrum $\operatorname{Sp}(\underline{\beta})=\sum_{j=1}^n(\alpha_j)\in\mathbb{Z}_{\geq 0}(\mathbb{R})$. Define for $j\in\{1,...,n\}$

$$\gamma_j := \frac{1}{n}(j-\frac{k}{2}) = \begin{cases} \frac{1}{n}(j-\frac{1}{2}) & \text{if } k=1, \\ \frac{1}{n}(j-1) & \text{if } k=2, \end{cases} \tag{6.8}$$
$$\alpha_j := n(\beta_j-\gamma_j) = \begin{cases} n\beta_j-j+\frac{1}{2} & \text{if } k=1, \\ n\beta_j-j+1 & \text{if } k=2. \end{cases} \tag{6.9}$$

(b) The following extends the recipe in (a) to a recipe for spectral pairs $\mathrm{Spp}(\underline{\beta}) := \sum_{j=1}^n (\alpha_j, k_j) \in \mathbb{Z}_{\geq 0}(\mathbb{R}\times\mathbb{Z})$. See lemma 6.12 for the properties of $\mathrm{Spp}(\underline{\beta})$. Consider $\kappa \in S^1$ with $\{\beta_j \,|\, e^{-2\pi i\beta_j} = \kappa\} \neq \emptyset$. Then the recipe gives in fact

$$\sum_{j:\, \exp(-2\pi i\beta_j)=\kappa} (\alpha_j) = \sum_{j=0}^{l} (\alpha + j) \text{ for some } \alpha \in \mathbb{R}, l \in \mathbb{Z}_{\geq 0} \tag{6.10}$$

(if $k = 1$ and $\beta_1 = 0$ then $(\alpha_1, \alpha_n) = (\frac{-1}{2}, \frac{1}{2})$, and if $k = 2$ and $\beta_2 = 0$ then $(\alpha_1, \alpha_2, \alpha_n) = (0, -1, 1)$). Extend this to the spp-ladder $\sum_{j=0}^l (\alpha + j, 1 + l - 2j)$ of length $l + 1$ and center $m = 1$ as in definition 6.2 (b), and define $\mathrm{Spp}(\underline{\beta})$ as the sum of these spp-ladders.

(c) For a polynomial $p \in T^{pol}_{\mathrm{HOR}k}(n, \mathbb{R})$ define the spectrum and the spectral pairs as follows,

$$\mathrm{Sp}(p) := \mathrm{Sp}(\Pi^{-1}(p)), \quad \mathrm{Spp}(p) := \mathrm{Spp}(\Pi^{-1}(p)). \tag{6.11}$$

The spectral numbers $\alpha_1, ..., \alpha_n$ in this recipe are usually not ordered by size. But they satisfy the symmetry in part (b) of the following lemma. The lemma states also properties of the spectral pairs.

Lemma 6.12. *(a) Denote $\underline{\gamma} := (\gamma_1, ..., \gamma_n)$ in both cases $k = 1$ and $k = 2$. Then*

$$\underline{\gamma} \in T^{scal}_{\mathrm{HOR}k}(n, \mathbb{R}), \quad \Pi(\underline{\gamma}) = x^n - (-1)^k, \tag{6.12}$$

$$\mathrm{Spp}(\underline{\gamma}) = n \cdot (0, 1), \quad \mathrm{Sp}(\underline{\gamma}) = n \cdot (0). \tag{6.13}$$

(b) The spectral numbers $\alpha_1, ..., \alpha_n$ in the recipe satisfy the symmetry

$$\alpha_j + \alpha_{n+1-j} = 0 \quad \text{for } k = 1, \tag{6.14}$$

$$\alpha_1 = 0, \alpha_j + \alpha_{n+2-j} = 0 \quad \text{for } k = 2 \text{ and } j \geq 2. \tag{6.15}$$

$\mathrm{Spp}(\underline{\beta})$ consists of sppl-pairs and single spp-ladders with center $m = 1$, for each value $\kappa \in S^1$ with $\{\beta_j \,|\, e^{-2\pi i\beta_j} = \kappa\} \neq \emptyset$ one spp-ladder. The partner of the spp-ladder from κ is the one from $\overline{\kappa}$. The single spp-ladders are those which come from $\kappa \in \{\pm 1\}$, so there are at most two of them.

(c) If $p \in T^{pol}_{\mathrm{HOR}k}(n, \mathbb{R})$ then $(-1)^n p(-x) \in T^{pol}_{\mathrm{HOR}\widetilde{k}}(n, \mathbb{R})$ with $\widetilde{k} \equiv k + n(2)$, and then

$$\mathrm{Spp}(p) = \mathrm{Spp}((-1)^n p(-x)). \tag{6.16}$$

Proof. (a) Trivial.

(b) $\underline{\beta}$ and $\underline{\gamma}$ are both in $T^{scal}_{\mathrm{HOR}k}(n,\mathbb{R})$ and satisfy the same symmetry in (6.1) or (6.2). Thus the tuple $\frac{1}{n}\underline{\alpha}=\underline{\beta}-\underline{\gamma}$ and the tuple $\underline{\alpha}$ satisfy the symmetry in (6.14) or (6.15). Consider as in part (b) of the recipe 6.11 $\kappa\in S^1$ with $\{\beta_j\,|\,e^{-2\pi i\beta_j}=\kappa\}\neq\emptyset$ and its spp-ladder. One sees easily with the symmetries (6.14) and (6.15) that the spp-ladders for κ and $\overline{\kappa}$ are partners. Especially, those for $\kappa\in\{\pm 1\}$ are single spp-ladders.

(c) Write

$$\widetilde{p}(x):=(-1)^n p(-x),\ \underline{\beta}:=\Pi^{-1}(p),\ \underline{\widetilde{\beta}}=\Pi^{-1}(\widetilde{p}),\ \widetilde{p}\in T^{pol}_{\mathrm{HOR}\widetilde{k}}(n,\mathbb{R}).$$

Then $p_0=(-1)^{k-1}$ and $\widetilde{p}_0=(-1)^{n+k-1}$ show the first line of (c).

For even n

$$\widetilde{\beta}_{\frac{n}{2}+j}=\frac{1}{2}+\beta_j\quad\text{ and }\quad\widetilde{\beta}_j=-\frac{1}{2}+\beta_{\frac{n}{2}+j}\text{ for }j=1,...,\frac{n}{2}.$$

For odd n and $k=1$

$$\begin{aligned}\widetilde{\beta}_{\frac{n+1}{2}+j}&=\frac{1}{2}+\beta_j\ \text{ for }\ j=1,...,\frac{n-1}{2},\\ \widetilde{\beta}_j&=-\frac{1}{2}+\beta_{\frac{n-1}{2}+j}\ \text{ for }\ j=1,...,\frac{n+1}{2}.\end{aligned}$$

For odd n and $k=2$

$$\begin{aligned}\widetilde{\beta}_{\frac{n-1}{2}+j}&=\frac{1}{2}+\beta_j\ \text{ for }\ j=1,...,\frac{n+1}{2},\\ \widetilde{\beta}_j&=-\frac{1}{2}+\beta_{\frac{n+1}{2}+j}\ \text{ for }\ j=1,...,\frac{n-1}{2}.\end{aligned}$$

Observe that

$$\underline{\widetilde{\gamma}}=_{def}\Pi^{-1}(\widetilde{x^n-(-1)^k})\stackrel{!}{=}\Pi^{-1}(x^n-(-1)^{\widetilde{k}})$$

is the $\underline{\gamma}$-vector for $\widetilde{k}$. As $\underline{\widetilde{\gamma}}$ is obtained from $\underline{\gamma}$ as any $\underline{\widetilde{\beta}}$ from $\underline{\beta}$, the tuples of differences $\underline{\widetilde{\beta}}-\underline{\widetilde{\gamma}}$ and $\underline{\beta}-\underline{\gamma}$ coincide up to reordering. Therefore $\mathrm{Sp}(\widetilde{p})=\mathrm{Sp}(p)$. Its extension to $\mathrm{Spp}(\widetilde{p})=\mathrm{Spp}(p)$ is rather obvious. □

It is interesting to ask about the images in $\mathbb{Z}_{\geq 0}(\mathbb{R}\times\mathbb{Z})$ and in $\mathbb{Z}_{\geq 0}(\mathbb{R})$ of the maps Spp and Sp from $T^{pol}_{\mathrm{HOR}k}(n,\mathbb{R})$. The answer is not difficult, it is given in the following corollary. We omit the rather trivial proof.

Corollary 6.13. *An unordered tuple $\sum_{\alpha\in\mathbb{R}}d(\alpha)(\alpha)\in\mathbb{Z}_{\geq 0}(\mathbb{R})$ of n numbers (so $\sum_{\alpha\in\mathbb{R}}d(\alpha)=n$) is in $\mathrm{Sp}(T^{pol}_{\mathrm{HOR}k}(n,\mathbb{R}))$ if and only if the numbers can be ordered as $\alpha_1,...,\alpha_n$ such that the* *symmetry in (6.14) respectively (6.15) holds and* $\alpha_{j+1} \geq \alpha_j - 1$*, and in the case* $k = 1$ *also* $\alpha_1 \geq -\frac{1}{2}$.

An unordered tuple $\sum_{(\alpha,k)\in\mathbb{R}\times\mathbb{Z}} d(\alpha,k)(\alpha,k) \in \mathbb{Z}_{\geq 0}(\mathbb{R}\times\mathbb{Z})$ *of* n *pairs is in* $\mathrm{Spp}(T^{pol}_{\mathrm{HOR}k}(n,\mathbb{R}))$ *if and only if the pairs can be ordered as* $(\alpha_1,k_1),...,(\alpha_n,k_n)$ *such that the conditions above hold and the tuple* $\sum_{j=1}^{n}(\alpha_j,k_j)$ *is obtained from the tuple* $\sum_{j=1}^{n}(\alpha_j)$ *by part (b) of recipe 6.11.*

Remark 6.14. The corollary 6.13 is relevant, because conjecture 2.23 implies that $\mathrm{Spp}(f)$ and $\mathrm{Sp}(f)$ satisfy the properties in this corollary. This implies especially that there is no gap of size > 1 if the spectral numbers are ordered by size. This is not a very strong claim in the case of *ihs* (there usually the gaps between spectral numbers are much smaller), but it is new in any case.

Examples 6.15. (i) It is also interesting to ask about the preimages in $T^{pol}_{\mathrm{HOR}k}(n,\mathbb{R})$ of spectral numbers or spectral pairs, especially for $\mathrm{Sp}(f)$ with f an *ihs* or an M-tame function. In these cases $\mathrm{Spp}(f) \in \mathbb{Z}_{\geq 0}(\mathbb{Q}\times\mathbb{Z})$, and, even stronger, the characteristic polynomial $p_{ch,M}(x) := \prod_{j=1}^{\mu}(x - e^{-2\pi i\alpha_j})$ is in $\mathbb{Z}[x]$, i.e. it is a product of cyclotomic polynomials.

(ii) In most cases, the preimages, the polynomials $p \in T^{pol}_{\mathrm{HOR}k}(\mu,\mathbb{R})$ with the correct spectrum $Sp(p) = \mathrm{Sp}(f) - \frac{m-1}{2}$, are not in $\mathbb{Z}[x]$. If one looks only at the correct eigenvalues, and not at the correct spectral numbers, one obtains the possibly bigger set

$$\begin{aligned}\{p \in T^{pol}_{\mathrm{HOR}k}(\mu,\mathbb{R}) \quad | \quad & p(x) = \prod_{j=1}^{\mu}(x-\kappa_j), \\ & p_{ch,M}(x) = \prod_{j=1}^{\mu}(x - (-1)^{k+m-1}\kappa_j^{\mu})\}.\end{aligned} \tag{6.17}$$

Even this set does often not contain polynomials in $\mathbb{Z}[x]$, for example for the *ihs* of type E_6, see below (v).

(iii) Remarkable exceptions are the chain type singularities, which are treated in section 6.6. For them distinguished polynomials $p \in \mathbb{Z}[x]$ with the correct spectrum $\mathrm{Sp}(p) = Sp(f) - \frac{m-1}{2}$ exist. This will be proved in theorem 6.33. The polynomials p are given in (6.6). In fact, in the moment, the chain type singularities are the only candidates within *ihs* for which we know polynomials p in $\mathbb{Z}[x] \cap T^{pol}_{\mathrm{HOR}k}(\mu,\mathbb{R})$ with the correct spectrum.

(iv) If $f(x_0,x_1)$ (so $m = 1$) is one of the ADE-type *ihs*, then the spectral numbers satisfy $\frac{-1}{2} < \alpha_1 \leq ... \leq \alpha_\mu < \frac{1}{2}$. Then the number of $\underline{\beta} \in T^{scal}_{\mathrm{HOR}k}(\mu,\mathbb{R})$ with $\mathrm{Sp}(\underline{\beta}) = \mathrm{Sp}(f)$ is (here

$(2N)!! := 2^N N!)$

$$\begin{array}{rll} \mu!! & \text{if} & \mu \text{ is even and the singularity is not } D_\mu, \\ (\mu-1)!! & \text{if} & \mu \text{ is odd}, \\ \mu!! \cdot \dfrac{1}{2} & \text{if} & \text{the singularity is } D_\mu \text{ and } \mu \text{ is even}. \end{array}$$

The numbers β_j must satisfy the symmetry in (6.1) or (3.17) including $\beta_1 = 0$ in (6.1), as well as

$$\begin{aligned} &\beta_j = \gamma_j + \frac{1}{\mu}\alpha_{\sigma(j)}, \\ &0 \leq \beta_1 \leq ... \leq \beta_\mu \leq 1, \end{aligned} \tag{6.18}$$

here $\sigma \in S_\mu$ is a permutation. Because of

$$\max_j |\alpha_j| < \frac{1}{2} = 1 - \mu\gamma_\mu = \frac{\mu}{2}(\gamma_j - \gamma_{j-1}) = \mu\gamma_1 - 0,$$

one can choose $\sigma \in S_\mu$ almost arbitrarily. Only the symmetry in (6.1) or (6.2) has to be observed. For all ADE-type *ihs* except D_μ with μ even, the spectral numbers are pairwise different. For D_μ with μ even, $\alpha_{\frac{\mu}{2}} = \alpha_{\frac{\mu+2}{2}} = 0$.

Though most of the polynomials $p = \Pi(\underline{\beta})$ are not in $\mathbb{Z}[x]$. The types A_μ, D_μ and E_7 can be written as chain type singularities. Therefore by theorem 6.33 at least the polynomial in (6.6) is in $\mathbb{Z}[x]$.

(v) But the *ihs* of type E_6 and many other *ihs* have characteristic polynomials $p_{ch,M}$ such that not even the set in (6.17) contains any polynomial in $\mathbb{Z}[x]$. For E_6 as a curve singularity $p_{ch,M} = \Phi_{12}\Phi_6$. For E_8 as a curve singularity $p_{ch,M} = \Phi_{15}$, and the set in (6.17) with $k = 1$ contains the polynomial $p = \Phi_{15} \in \mathbb{Z}[x]$. But $\mathrm{Sp}(p) \neq \mathrm{Sp}(f)$.

6.3 HOR matrices

HOR matrices, named after the authors Horocholyn, Orlik and Randel (see subsection 6.7.1), are the matrices in the spaces $T_{\mathrm{HOR1}}(n, \mathbb{R})$ and $T_{\mathrm{HOR2}}(n, \mathbb{R})$, the essential object in this whole chapter. Any real invertible matrix S, not necessarily upper triangular, defines a generic, a symmetric and an antisymmetric bilinear form, as well as a "monodromy" automorphism. We can define, based on this data, reflections and pseudo-reflections and deduce, for the monodromy matrices $S^{-1}S^t$ and $-S^{-1}S^t$, a decomposition into reflections resp. pseudo-reflections. This property is crucial for the HOR spaces. We define the HOR spaces via the previous spaces $T^{pol}_{\mathrm{HOR}k}$. Take a polynomial p and it's coefficients, but not the one of degree zero, p_0. Then the coefficients are used to generate an upper triangular matrix $\tilde{S}$ that is also a Toeplitz matrix. If $p_0 = +1$, that is if $p \in T_{\mathrm{HOR1}}$ we show that the characteristic polynomial $\tilde{S}^{-1}\tilde{S}^t$ is p, whereas for $p_0 = -1$, $p \in T_{\mathrm{HOR2}}$ we show p is the characteristic polynomial of $-\tilde{S}^{-1}\tilde{S}^t$. This, in turn, enables us to extend the spectrum from the previous section to the HOR matrices, and finally study the signature of the HOR bilinear forms. We start with the generic construction and decomposition.

Theorem 6.16. *Let $n \in \mathbb{Z}_{\geq 1}$, let $H_\mathbb{R}$ be an $\mathbb{R}$-vector space with a basis $\underline{e} = (e_1, ..., e_n)$, and let $S \in GL(n, \mathbb{R})$.*

(a) The matrix S defines on $H_\mathbb{R}$ a bilinear form L, which is called Seifert form, a symmetric bilinear form I_s, an antisymmetric bilinear form I_a and an automorphism M, which is called monodromy, by the formulas

$$\begin{aligned} L(\underline{e}^t, \underline{e}) &= S^t, && (6.1)\\ I_s(\underline{e}^t, \underline{e}) &= S + S^t, \quad \text{so } I_s(a,b) = L(a,b) + L(b,a), && (6.2)\\ I_a(\underline{e}^t, \underline{e}) &= S - S^t, \quad \text{so } I_a(a,b) = L(b,a) - L(a,b), && (6.3)\\ M\,\underline{e} &= \underline{e} \cdot S^{-1}S^t, \quad \text{so } L(Ma, b) = L(b,a). && (6.4) \end{aligned}$$

L determines I_s, I_a and M. The monodromy M respects all three bilinear forms L, I_s and I_a.

(b) Define endomorphisms $s^{(1)}_a$ and $s^{(2)}_b$ on $H_\mathbb{R}$ for $a \in H_\mathbb{R}$ with $I_s(a,a) = 2$ and for arbitrary $b \in H_\mathbb{R}$ by

$$s^{(1)}_a(c) := c - I_s(a,c) \cdot a, \quad s^{(2)}_b(c) := c - I_a(b,c) \cdot b. \tag{6.5}$$

Then $s^{(1)}_a$ respects I_s and is a reflection (semisimple, eigenvalues $1, ..., 1, -1$). And $s^{(2)}_b$ respects I_a and is a pseudo-reflection ($s^{(2)}_b = \mathrm{id}$ or $s^{(2)}_b - \mathrm{id}$ nilpotent with one single 2×2 Jordan block).

(c) Now let $S = (s_{ij}) \in T(n, \mathbb{R})$ (so $s_{ij} = 0$ for $i > j$, $s_{jj} = 1$, and the eigenvalues of $S^{-1}S^t$ are in S^1). Then

$$(-1)^k \cdot M = s^{(k)}_{e_1} \circ ... \circ s^{(k)}_{e_n} \qquad \text{for } k \in \{1, 2\}. \tag{6.6}$$

Proof. **(a)** $L(b,a)=L(Ma,b)$ is equivalent to $L(M\underline{e}^t,\underline{e})=L(\underline{e}^t,\underline{e})^t$ which holds:

$$L(M\underline{e}^t,\underline{e})=L((\underline{e}\cdot S^{-1}S^t)^t,\underline{e})=SS^{-t}\cdot S^t=S=L(\underline{e}^t,\underline{e})^t.$$

M respects L because of

$$L(Ma,Mb)=L(Mb,a)=L(a,b).$$

M respects I_s and I_a because of their relation to L in (6.2) and (6.3).

(b) $s_a^{(1)}$ respects I_s because of

$$\begin{aligned} I_s(s_a^{(1)}(b),s_a^{(1)}(c)) &= I_s(b-I_s(a,b)a,c-I_s(a,c)a)\\ &= I_s(b,c)-I_s(a,b)I_s(a,c)-I_s(a,c)I_s(b,a)\\ &\quad +I_s(a,b)I_s(a,c)I_s(a,a)\\ &= I_s(b,c). \end{aligned}$$

$s_a^{(1)}$ is a reflection because its restriction to $\{c\in H_{\mathbb{R}}\,|\,I_s(a,c)=0\}$ is id and because of $s_a^{(1)}(a)=-a$.

$s_b^{(2)}$ respects I_a because of

$$\begin{aligned} & I_a(s_b^{(2)}(c),s_b^{(2)}(d))=I_a(c-I_a(b,c)b,d-I_a(b,d)b)\\ =\ & I_a(c,d)-I_a(b,c)I_a(b,d)-I_a(b,d)I_a(c,b)+I_a(b,c)I_a(b,d)I_a(b,b)\\ =\ & I_a(c,d). \end{aligned}$$

$s_b^{(2)}$ is a pseudo-reflection because its restriction to $\{c\in H_{\mathbb{R}}\,|\,I_a(b,c)=0\}$ is id and this space has dimension $n-1$ or n and contains b.

(c) Denote $D_{kl}:=(\delta_{ik}\cdot\delta_{jl})_{i,j=1,\dots,n}\in M(n\times n,\mathbb{Z})$. Denote by $E_n:=(\delta_{ij})=\sum_{j=1}^n D_{jj}$ the $n\times n$ unit matrix. Observe

$$D_{ij}D_{kl}=0\quad\text{if } j\neq k,$$

which implies

$$\begin{aligned} (E_n+D_{ij})(E_n+D_{kl}) &= E_n+D_{ij}+D_{kl}\quad\text{if } j\neq k,\\ (E_n+D_{ij})^{-1} &= E_n-D_{ij}\quad\text{if } i\neq j. \end{aligned}$$

These identities are applied often in the following calculations. Empty places mean zeros.

$$
\begin{aligned}
S &= \begin{pmatrix} 1 &&& \\ & \ddots && \\ && 1 & s_{n-1,n} \\ &&& 1 \end{pmatrix}
\begin{pmatrix} 1 &&&& \\ & \ddots &&& \\ && 1 & s_{n-2,n-1} & s_{n-2,n} \\ &&& 1 & 0 \\ &&&& 1 \end{pmatrix} \dots
\begin{pmatrix} 1 & s_{12} & \dots & s_{1n} \\ & 1 && \\ && \ddots & \\ &&& 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 &&& s_{1n} \\ & \ddots && \vdots \\ && 1 & s_{n-1,n} \\ &&& 1 \end{pmatrix}
\begin{pmatrix} 1 &&& s_{1,n-1} & \\ & \ddots && \vdots & \\ && 1 & s_{n-2,n-1} & \\ &&& 1 & \\ &&&& 1 \end{pmatrix} \dots
\begin{pmatrix} 1 & s_{12} && \\ & 1 && \\ && \ddots & \\ &&& 1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
& S^{-1} S^t \\
&= \begin{pmatrix} 1 & -s_{12} & \dots & -s_{1n} \\ & 1 && \\ && \ddots & \\ &&& 1 \end{pmatrix} \dots
\begin{pmatrix} 1 &&&& \\ & \ddots &&& \\ && 1 & -s_{n-2,n-1} & -s_{n-2,n} \\ &&& 1 & 0 \\ &&&& 1 \end{pmatrix}
\begin{pmatrix} 1 &&& \\ & \ddots && \\ && 1 & -s_{n-1,n} \\ &&& 1 \end{pmatrix} \\
& \quad \begin{pmatrix} 1 &&& \\ s_{12} & 1 && \\ && \ddots & \\ &&& 1 \end{pmatrix} \dots
\begin{pmatrix} 1 &&&& \\ & \ddots &&& \\ &&& 1 & \\ s_{1,n-1} & \dots & s_{n-2,n-1} & 1 & \\ &&&& 1 \end{pmatrix}
\begin{pmatrix} 1 &&& \\ & \ddots && \\ &&& 1 & \\ s_{1n} & \dots & s_{n-1,n} & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & -s_{12} & \dots & -s_{1n} \\ & 1 && \\ && \ddots & \\ &&& 1 \end{pmatrix}
\begin{pmatrix} 1 &&&& \\ s_{12} & 1 & -s_{23} & \dots & -s_{2n} \\ && 1 && \\ &&& \ddots & \\ &&&& 1 \end{pmatrix} \dots \\
& \quad \begin{pmatrix} 1 &&&& \\ & \ddots &&& \\ && 1 && \\ s_{1,n-1} & \dots & s_{n-2,n-1} & 1 & -s_{n-1,n} \\ &&&& 1 \end{pmatrix}
\begin{pmatrix} 1 &&& \\ & \ddots && \\ && 1 & \\ s_{1n} & \dots & s_{n-1,n} & 1 \end{pmatrix}
\end{aligned}
$$

$$= (s^{(2)}_{e_1})^{mat} \cdot (s^{(2)}_{e_2})^{mat} \cdot \dots \cdot (s^{(2)}_{e_{n-1}})^{mat} \cdot (s^{(2)}_{e_n})^{mat}$$

where the $n \times n$-matrix

$$
(s^{(2)}_{e_j})^{mat} := \begin{pmatrix}
1 &&&&&& \\
& \ddots &&&&& \\
&& 1 &&&& \\
s_{1j} & \dots & s_{j-1,j} & 1 & -s_{j,j+1} & \dots & -s_{jn} \\
&&&& 1 && \\
&&&&& \ddots & \\
&&&&&& 1
\end{pmatrix}
$$

satisfies

$$s_{e_j}^{(2)}\,\underline{e} = \underline{e}\cdot (s_{e_j}^{(2)})^{mat}.$$

This shows $M = s_{e_1}^{(2)}\circ ...\circ s_{e_n}^{(2)}$. Define the matrix $(s_{e_j}^{(1)})^{mat}$ by

$$s_{e_j}^{(1)}\,\underline{e} = \underline{e}\cdot (s_{e_j}^{(1)})^{mat}.$$

Observe

$$\begin{aligned}
(s_{e_j}^{(1)})^{mat} &= \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ -s_{1j} & ... & -s_{j-1,j} & -1 & -s_{jj+1} & ... & -s_{jn} \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \\
&= \Big(-\sum_{i=1}^{j-1} D_{ii} + \sum_{i=j}^{n} D_{ii}\Big)\cdot (s_{e_j}^{(2)})^{mat}\cdot \Big(-\sum_{i=1}^{j} D_{ii} + \sum_{i=j+1}^{n} D_{ii}\Big)
\end{aligned}$$

This shows

$$-S^{-1}S^t = (s_{e_1}^{(1)})^{mat}\cdot ...\cdot (s_{e_n}^{(1)})^{mat} \quad \text{and} \quad -M = s_{e_1}^{(1)}\circ ...\circ s_{e_n}^{(1)}.$$

□

Corollary 6.17. *Consider the same situation as in theorem 6.16. Define the cyclic automorphism C by*

$$C\,\underline{e} = \underline{e}\cdot \left(\begin{array}{c|c} & 1 \\ \hline E_{n-1} & \end{array}\right) = \underline{e}\cdot C^{mat}, \tag{6.7}$$

$$\text{so } C\,e_j = e_{j+1} \text{ for } 1\leq j\leq n-1,\ C\,e_n = e_1,\ \text{and } C^n = \mathrm{id}\,. \tag{6.8}$$

Define the automorphisms $R_{(kj)}$ for $k\in\{1,2\}, j\in\{1,...,n\}$ of $H_{\mathbb{R}}$ by

$$R_{(kj)} := C^{-(j-1)}\circ s_{e_j}^{(k)}\circ C^j. \tag{6.9}$$

Then

$$R_{(kj)}\,\underline{e} = \underline{e}\cdot R^{mat}_{(kj)}$$

with

$$R^{mat}_{(1j)} = \left(\begin{array}{ccccccc|c} -s_{j,j+1} & \dots & -s_{jn} & -s_{1j} & \dots & -s_{j-1,j} & & -1 \\ \hline & & & & & & & \\ & & & E_{n-1} & & & & \end{array}\right), \tag{6.10}$$

$$R^{mat}_{(2j)} = \left(\begin{array}{cccccc|c} -s_{j,j+1} & \dots & -s_{jn} & s_{1j} & \dots & s_{j-1,j} & 1 \\ \hline & & & & & & \\ & & & E_{n-1} & & & \end{array}\right), \tag{6.11}$$

and

$$\begin{aligned} (-1)^k\cdot S^{-1}S^t &= R^{mat}_{(k1)}\circ\dots\circ R^{mat}_{(kn)} \\ \text{and}\quad (-1)^k\cdot M &= R_{(k1)}\circ\dots\circ R_{(kn)}. \end{aligned} \tag{6.12}$$

Proof. $(-1)^kM = R_{(k1)}\circ\dots\circ R_{(kn)}$ is an immediate consequence of $(-1)^kM = s^{(k)}_{e_1}\circ\dots\circ s^{(k)}_{e_n}$ and the definition of $R_{(kj)}$ and $C^n = \mathrm{id}$. The formulas for $R^{mat}_{(kj)}$ follow from the formulas for $(s^{(k)}_{e_j})^{mat}$. $\square$

Remarks 6.18. The matrices $R^{mat}_{(kj)}$ are *companion matrices*. A companion matrix is here a matrix (empty places mean zeros)

$$R^{mat} = \left(\begin{array}{cccc|c} -p_{n-1} & -p_{n-2} & \dots & -p_1 & -p_0 \\ \hline & & & & \\ & & E_{n-1} & & \end{array}\right) \tag{6.13}$$

with $p_{n-1},\dots,p_0\in\mathbb{C}$. Its characteristic polynomial is $p(x) = x^n + p_{n-1}x^{n-1} + \dots + p_1x + p_0$. For each eigenvalue $\kappa\in\mathbb{C}$, it has only one Jordan block. A basis of a Jordan block of size

$l+1$ with eigenvalue κ is

$$v_j = \begin{pmatrix} (n-1)_j \cdot \kappa^{n-1} \\ (n-2)_j \cdot \kappa^{n-2} \\ \vdots \\ (j)_j \cdot \kappa^j \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{for } j = 0, 1, ..., l, \tag{6.14}$$

with

$$(a)_b := a(a-1) \cdot ... \cdot (a-b+1) \quad \text{for } a \in \mathbb{C}, b \in \mathbb{Z}_{\geq 0}, \tag{6.15}$$

(and $(a)_0 = 1$) and

$$\Big(\kappa^{-1} R^{mat} - E_n\Big) v_j = j \cdot v_{j-1} \qquad (\text{with } v_{-1} = 0). \tag{6.16}$$

Here we used that κ is a zero of $p^{(j)}(x) = (n)_j x^{n-j} + p_{n-1}(n-1)_j x^{n-1-j} + ... + p_j (j)_j x^0$ for $0 \leq j \leq l$, and we used

$$(a)_b - (a-1)_b = b \cdot (a-1)_{b-1} \qquad \text{for } b \in \mathbb{Z}_{\geq 1}. \tag{6.17}$$

Definition 6.19. *Fix $n \in \mathbb{Z}_{\geq 1}$ and $k \in \{1, 2\}$.*

(a) The space of polynomials $T^{pol}_{\mathrm{HOR}k}(n, \mathbb{R}) \subset \mathbb{R}[x]_{\deg = n}$ was defined in definition 6.9. Define the map

$$\begin{aligned} S^{(k)} : T^{pol}_{\mathrm{HOR}k}(n, \mathbb{R}) &\to GL(n, \mathbb{R}) \cap T(n, \mathbb{R}) \\ p(x) = x^n + p_{n-1}x^{n-1} + ... + p_0 &\mapsto \begin{pmatrix} 1 & p_{n-1} & \dots & p_2 & p_1 \\ & \ddots & \ddots & & p_2 \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & p_{n-1} \\ & & & & 1 \end{pmatrix}. \end{aligned} \tag{6.18}$$

Define its image as $T_{\mathrm{HOR}k}(n, \mathbb{R}) := S^{(k)}(T^{pol}_{\mathrm{HOR}k}(n, \mathbb{R}))$ (theorem 6.20 (a) will show that it is *a subspace of* $T(n,\mathbb{R})$*). Define the map*

$$\begin{array}{rcl} R^{mat}_{(k)}:T_{\mathrm{HOR}k}(n,\mathbb{R}) & \to & GL(n,\mathbb{R}) \\ S=S^{(k)}(p) & \mapsto & \left(\begin{array}{cccc|c} -p_{n-1} & -p_{n-2} & \dots & -p_1 & -p_0 \\ \hline & & & & \\ & & E_{n-1} & & \end{array}\right) \end{array} \tag{6.19}$$

(recall $p_0=(-1)^{k-1}$*).* $R^{mat}_{(k)}(S)$ *is a companion matrix, and its characteristic polynomial is* $p(x)$ *by remark 6.18.*

(b) For $S\in T_{\mathrm{HOR}k}(n,\mathbb{R})$ *take up the data in theorem 6.16. Define an automorphism* $R_{(k)}(S):H_\mathbb{R}\to H_\mathbb{R}$ *by* $R_{(k)}(S)\,\underline{e}:=\underline{e}\cdot R^{mat}_{(k)}(S)$*.*

(c) For $S\in T_{\mathrm{HOR}k}(n,\mathbb{R})$ *define* $\mathrm{Spp}(S):=\mathrm{Spp}(p)$ *and* $\mathrm{Sp}(S):=\mathrm{Sp}(p)$ *where* $p\in T^{pol}_{\mathrm{HOR}k}(n,\mathbb{R})$ *is the characteristic polynomial of* $R^{mat}_{(k)}(S)$ *(or, equivalently, of* $R_{(k)}(S)$*), and where* $\mathrm{Spp}(p)$ *and* $\mathrm{Sp}(p)$ *are defined in recipe 6.11 (c).*

Definition 6.19 (a) and the next formula (6.20) are essentially due to Horocholyn [Ho17, ch. 2] (he considered half of the cases). He also studied the signature of $S+S^t$. Theorem 6.20 and corollary 6.21 encompass his results. In cases relevant for chain type singularities (see section 6.6), the matrices S and $R^{mat}_{(k)}(S)$ are also given in [OR77]. But there (6.20) is not even mentioned, although the authors are certainly aware of it.

Theorem 6.20. *Choose* $S\in T_{\mathrm{HOR}k}(n,\mathbb{R})$ *and take up the data in theorem 6.16.*

(a)

$$(-1)^k\cdot S^{-1}S^t=R^{mat}_{(k)}(S)^n \quad\text{and}\quad (-1)^k\cdot M=R_{(k)}(S)^n. \tag{6.20}$$

The generalized eigenspaces of $R_{(k)}(S)$ *are the spaces* $H^{(R)}_\kappa:=\ker((R_{(k)}(S)-\kappa\cdot\mathrm{id})^n)\subset H_\mathbb{C}$ *with* $p(\kappa)=0$*. The generalized eigenspaces of* M *are the spaces* $H_\lambda=\bigoplus_{\kappa:(-1)^k\kappa^n=\lambda}H^{(R)}_\kappa$*. Especially,* $T_{\mathrm{HOR}k}(n,\mathbb{R})\subset T(n,\mathbb{R})$*. The monodromy* M *and the automorphism* $R_{(k)}(S)$ *have a single Jordan block on* $H^{(R)}_\kappa$ *(because of remark 6.18).*

(b) $R_{(k)}(S)$ *respects* L*. Therefore* $H_\mathbb{R}$ *decomposes* L*-orthogonally into the Seifert form pairs* $(H^{(R)}_1\cap H_\mathbb{R},L)$*,* $(H^{(R)}_{-1}\cap H_\mathbb{R},L)$*, and* $((H^{(R)}_\kappa\oplus H^{(R)}_{\overline{\kappa}})\cap H_\mathbb{R})$ *for each* $\kappa\in S^1$ *with* $\mathrm{Im}\,\kappa>0$ *and* $H^{(R)}_\kappa\neq\{0\}$*.*

(c) $\mathrm{Spp}(S)$ *and the decomposition of* $(H_\mathbb{R},L)$ *in (b) give a polarized enhancement of* $(H_\mathbb{R},L)$ *(definition 6.4):* $\mathrm{Spp}(S)$ *consists of spp-ladders, one for each eigenvalue* κ *of* $R_{(k)}(S)$*. The spp-ladder for* κ *has length* $l+1=\dim H^{(R)}_\kappa$*, center* $m=1$*, and first spectral number* α

with $e^{-2\pi i\alpha} = \kappa$ *Furthermore*

$$L(a, N^l\overline{a}) \in e^{\frac{1}{2}\pi i(2\alpha+l)} \cdot \mathbb{R}_{>0} \quad \text{for } a \in H_\kappa^{(R)} - N(H_\kappa^{(R)}). \tag{6.21}$$

If $\kappa = \pm 1$*, it is a single spp-ladder. If* $\kappa \neq \pm 1$*, the partner spp-ladder is the one for* $\overline{\kappa}$*.*

(d) The underlying spectrum $\mathrm{Sp}(S)$ *is the one which recipe 2.9 gives for* S *if it is applied to* $T_{\mathrm{HOR}k}(n, \mathbb{R})$ *(see part (c) of theorem 2.18).*

Proof. (a) The coefficients $p_{n-1}, ..., p_1$ in the matrix

$$S = \begin{pmatrix} 1 & p_{n-1} & \dots & p_1 \\ & \ddots & \ddots & \vdots \\ & & \ddots & p_{n-1} \\ & & & 1 \end{pmatrix} \in T_{\mathrm{HOR}k}(n, \mathbb{R})$$

satisfy $p_{n-j} = (-1)^{k-1}p_j$. Therefore the matrices $R^{mat}_{(kj)}$ for $j \in \{1, ..., n\}$ in corollary 6.17 are all equal to one another and to $R^{mat}_{(k)}(S)$. Thus $(-1)^k \cdot M = R_{(k)}(S)^n$ and (6.20). The other statements are immediate consequences of (6.20).

(b) We have to prove $R^{mat}_{(k)}(S)^t \cdot S^t \cdot R^{mat}_{(k)}(S) = S^t$. Equivalent is $S \cdot R^{mat}_{(k)}(S) = R^{mat}_{(k)}(S)^{-t} \cdot S$. Recall $p_{n-j} = p_0 \cdot p_j$ and observe

$$\begin{aligned} R^{mat}_{(k)}(S)^{-1} &= \left(\begin{array}{c|ccc} & & E_{n-1} & \\ \hline -p_0 & -p_1 & \dots & -p_{n-1} \end{array}\right), \\ R^{mat}_{(k)}(S)^{-t} &= \left(\begin{array}{c|c} & -p_0 \\ \hline & -p_1 \\ E_{n-1} & \vdots \\ & -p_{n-1} \end{array}\right). \end{aligned}$$

One calculates $S \cdot R^{mat}_{(k)}(S)$ and $R^{mat}_{(k)}(S)^{-t} \cdot S$ and finds in both cases

$$\left(\begin{array}{cccc|c} 0 & \dots & \dots & 0 & -p_0 \\ \hline 1 & p_{n-1} & \dots & p_2 & 0 \\ & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & p_{n-1} & \vdots \\ & & & 1 & 0 \end{array}\right).$$

(c) All statements in part (c) except that the enhancement is polarized follow immediately from part (b) and from lemma 6.12 (b).

It rests to show that the enhancement is polarized, i.e. (6.21).

$(-1)^k \cdot M = R_{(k)}(S)^n$ gives $N = n \cdot (\text{nilpotent part of } R_{(k)}(S))$. On $H_\kappa^{(R)}$

$$N^l = n^l \cdot (\text{nilpotent part of } R_{(k)}(S))^l = n^l \cdot (\kappa^{-1} R_{(k)}(S) - \text{id})^l.$$

The vector v_l in remark 6.18 corresponds to an element $a \in H_\kappa^{(R)} - N(H_\kappa^{(R)})$. We have to calculate the phase of

$$L(a, N^l \overline{a}) = v_l^t \cdot S^t \cdot (\overline{\kappa}^{-1} R_{(k)}^{mat}(S) - E_n)^l \cdot \overline{v_l} = v_l^t \cdot S^t \cdot l! \cdot \overline{v_0}$$

and want to find $e^{\frac{1}{2}\pi i(2\alpha + l)}$. We denote $p_n := 1$.

$$v_l^t \cdot S^t \cdot \overline{v_0} = \begin{pmatrix} (n-1)_l \kappa^{n-1} \\ (n-2)_l \kappa^{n-2} \\ \vdots \\ (l)_l \kappa^l \\ 0 \\ \vdots \\ 0 \end{pmatrix}^t \begin{pmatrix} 1 & & & \\ p_{n-1} & \ddots & & \\ \vdots & \ddots & \ddots & \\ p_1 & \dots & p_{n-1} & 1 \end{pmatrix} \begin{pmatrix} \overline{\kappa}^{n-1} \\ \overline{\kappa}^{n-2} \\ \vdots \\ \overline{\kappa}^0 \end{pmatrix}$$

$$\begin{aligned}
&= (n-1)_l \cdot \kappa^{n-1} \cdot \overline{\kappa}^{n-1} \\
&+ (n-2)_l \cdot \kappa^{n-2} \cdot (p_{n-1} \cdot \overline{\kappa}^{n-1} + p_n \cdot \overline{\kappa}^{n-2}) \\
&+ \dots \\
&+ (l)_l \cdot \kappa^l \cdot (p_{l+1} \cdot \overline{\kappa}^{n-1} + p_{l+2} \cdot \overline{\kappa}^{n-2} + \dots + p_n \cdot \overline{\kappa}^l) \\
&= ((n-1)_l + (n-2)_l + \dots + (l)_l) \cdot p_n \cdot \overline{\kappa}^0 \\
&+ ((n-2)_l + \dots + (l)_l) \cdot p_{n-1} \cdot \overline{\kappa}^1 + \dots + (l)_l \cdot p_{l+1} \cdot \overline{\kappa}^{n-l-1} \\
&= \frac{1}{l+1} \Big[(n)_{l+1} \cdot p_n \cdot \overline{\kappa}^0 + (n-1)_{l+1} \cdot p_{n-1} \cdot \overline{\kappa}^1 \\
&+ \dots + (l+1)_{l+1} \cdot p_{l+1} \cdot \overline{\kappa}^{n-l-1} \Big] \\
&= \frac{1}{l+1} \cdot \overline{\kappa}^{n-l-1} \cdot p^{(l+1)}(\kappa).
\end{aligned}$$

Now write $\underline{\beta} = (\beta_1, \dots, \beta_n) := \Pi^{-1}(p(x))$ and $\kappa_j := e^{-2\pi i \beta_j}$. Then $p(x) = \prod_{j=1}^n (x - \kappa_j)$

and κ is a zero of it of order $l+1$. Thus

$$p^{(l+1)}(\kappa) = (l+1)! \cdot \prod_{j:\,\kappa_j\neq\kappa} (\kappa-\kappa_j).$$

If $\kappa=\pm 1$ then a single spp-ladder is associated to $H^{(R)}_\kappa$. It satisfies $2\alpha+l=0$, so then (6.21) predicts $L(a, N^l\overline{a})>0$, so $v_l^t\cdot S^t\cdot\overline{v_0}>0$. Indeed, if $\kappa=1$ then the $\kappa_j\neq\kappa$ come in complex conjugate pairs or are equal to -1, so $p^{(l+1)}(1)>0$ and $v_l^t\cdot S^t\cdot\overline{v_0}>0$. If $\kappa=-1$ then the $\kappa_j\neq\kappa$ come in complex conjugate pairs or are equal to 1. Thus the multiplicity of 1 is congruent to $n-l-1 \mod 2$. Therefore $p^{(l+1)}(-1)\in(-1)^{n-l-1}\cdot\mathbb{R}_{>0}$ and $v_l^t\cdot S^t\cdot\overline{v_0}>0$.

It rests to consider the case $\kappa\neq\pm 1$. We can suppose $\operatorname{Im}\kappa<0$. Then an index a exists with $\beta_{a-1}<\beta_a=...=\beta_{a+l}<\beta_{a+l+1}$ and $a+l\leq\frac{n}{2}$ and $\kappa=\kappa_a=...=\kappa_{a+l}$.

We have the four cases $(k=1\,\&\,n\equiv 0(2))$, $(k=1\,\&\,n\equiv 1(2))$, $(k=2\,\&\,n\equiv 0(2))$ and $(k=2\,\&\,n\equiv 1(2))$. We treat only the case $(k=2\,\&\,n\equiv 0(2))$. The other cases are analogous. Then $\kappa_1=1$, $\kappa_{\frac{n+2}{2}}=-1$ and

$$\begin{aligned}
&\overline{\kappa}^{n-l-1}\cdot\prod_{j:\,\kappa_j\neq\kappa}(\kappa-\kappa_j)\\
=\ &\overline{\kappa}^{n-l-1}(\kappa-\kappa_1)(\kappa-\kappa_{\frac{n+2}{2}})\cdot\prod_{2\leq j\leq\frac{n}{2},\,\kappa_j\neq\kappa}(\kappa^2-\kappa(\kappa_j+\overline{\kappa_j})+1)\\
=\ &\overline{\kappa}^{n-l-1-(\frac{n-2}{2}-l-1)-1}\cdot(\kappa-\overline{\kappa})^{l+2}\cdot\prod_{2\leq j\leq\frac{n}{2},\,\kappa_j\neq\kappa}(\kappa+\overline{\kappa}-(\kappa_j+\overline{\kappa_j}))\\
\in\ &\overline{\kappa}^{n/2}\cdot(-i)^{l+2}\cdot(-1)^{a-2}\cdot\mathbb{R}_{>0}.
\end{aligned}$$

Here $\alpha=\alpha_{a+l}$ by the recipe 6.11, and

$$\begin{aligned}
\overline{\kappa}^{n/2} = (e^{2\pi i\beta_{a+l}})^{n/2} = e^{\pi i n\cdot\beta_{a+l}} + e^{\pi i(\alpha_{a+l}+n\cdot\gamma_{a+l})} + e^{\pi i(\alpha+a+l-1)},\\
\overline{\kappa}^{n/2}\cdot(-i)^{l+2}\cdot(-1)^{a-2} = e^{\pi i\frac{1}{2}(2\alpha+l)}\cdot\mathbb{R}_{>0}.
\end{aligned}$$

(d) This was essentially proved in the proof of theorem 2.18 (b). Define

$$\underline{\beta}^{(k)}=(\beta_1^{(k)},...,\beta_n^{(k)}):=(S^{(k)}\circ\Pi)^{-1}:T_{\mathrm{HOR}k}(n,\mathbb{R})\to T^{scal}_{\mathrm{HOR}k}(n,\mathbb{R}).$$

Then the functions $\beta_j^{(k)}:T_{\mathrm{HOR}k}(n,\mathbb{R})\to[0,1]$ and the function $\alpha_k^{(k)}$ in the proof of theorem 2.18 (b) are related by the recipe 6.11 (a), i.e. by $\alpha_j^{(k)}=\frac{1}{n}\beta_j^{(k)}(S)-j+\frac{k}{2}$. □

The following corollary of theorem 6.20 gives an example, what is in the polarized enhancement in theorem 6.20 (c). It was proved in a more elementary way in [Ho17] (for the cases considered there).

Corollary 6.21. *Choose a matrix $S \in T_{\mathrm{HOR}k}(n,\mathbb{R})$ and take up the data in theorem 6.16. The symmetric form I_s is nondegenerate on $H_{\neq -1}$. Its signature on $H_\mathbb{R} \cap H_{\neq -1}$ is (s_+, s_0, s_-) with*

$$\begin{aligned} s_+ &= |\{\alpha_j \,|\, \alpha_j \in (\frac{-1}{2}, \frac{1}{2}) \mod 2\mathbb{Z}\}|, \\ s_- &= \dim H_{\neq -1} - s_+, \quad s_0 = 0. \end{aligned} \tag{6.22}$$

Proof. The polarized enhancement of $(H_\mathbb{R}, L)$ in theorem 6.20 (c) is (by remark 6.5) a split Steenbrink polarized mixed Hodge structure on $H_\mathbb{R} \cong M(n\times 1, \mathbb{R})$ of weight $m = 1$. Such structures are studied in chapter 5. Theorem 5.26 gives a square root of a Tate twist, which allows going from weight $m = 1$ to an arbitrary weight $\widetilde{m} \in \mathbb{Z}$. In [CKS86, Corollary 3.13] (see also [He03, Theorem 7.5]) an equivalence between a polarized mixed Hodge structure and a nilpotent orbit of polarized pure Hodge structures is given. Especially, they have the same spectral numbers and the same polarizing form. Therefore we can work with a polarized pure Hodge structure of even weight $\widetilde{m}$. In that case, the polarizing form on $H_{\neq -1}$ is I_s, and (6.22) is an immediate consequence of the polarization. □

Remark 6.22. In corollary 6.21, when is I_s positive definite on $H_\mathbb{R}$? Only if all spectral numbers are in $(\frac{-1}{2}, \frac{1}{2}) \mod 2\mathbb{Z}$. But by corollary 6.13, the gaps between subsequent spectral numbers (if they are ordered by size) are ≤ 1. This enforces that all spectral numbers are in $(\frac{-1}{2}, \frac{1}{2})$. And this implies that the numbers β_j in $\underline{\beta} = (\Pi \circ S^{(k)})^{-1}(S) \in T^{scal}_{\mathrm{HOR}k}(n,\mathbb{R})$ are *interlacing* with the numbers $\gamma_1, ..., \gamma_n$: Their pairwise distances are $|\beta_j - \gamma_j| < \frac{1}{2n}$. Such an interlacing is also discussed in [Ho17].

Remark 6.23. For $S \in T_{\mathrm{HOR}k}(n,\mathbb{R})$ take up the data in theorem 6.16 and define $H_\mathbb{Z} := M(n \times 1, \mathbb{Z})$. Then $L : H_\mathbb{Z} \times H_\mathbb{Z} \to \mathbb{Z}$ is unimodular, and $R_{(k)}(S)$ and $M = (-1)^k R_{(k)}(S)^n$ are L-orthogonal automorphisms of $H_\mathbb{Z}$. For $R_{(k)}(S)$ this follows from theorem 6.20 (b).

Furthermore, let $\underline{e}^*$ be the $\mathbb{Z}$-basis of $H_\mathbb{Z}$ which is left L-dual to the standard basis $\underline{e}$, i.e. with $L((\underline{e}^*)^t, \underline{e}) = E_n$. Then the matrix $R^{mat*}_{(k)}(S)$ of $R_{(k)}(S)$ with respect to $\underline{e}^*$, so with $R_{(k)}(S)(\underline{e}^*) = \underline{e}^* \cdot R^{mat*}_{(k)}(S)$, is

$$R^{mat*}_{(k)}(S) = R^{mat}_{(k)}(S)^{-t} = \left(\begin{array}{c|c} & -p_0 \\ \hline & -p_1 \\ E_{n-1} & \vdots \\ & -p_{n-1} \end{array}\right) \tag{6.23}$$

by the proof of theorem 6.20 (b).

This implies $R_{(k)}(S)(e_j^*) = e_{j+1}^*$ for $j \in \{1, ..., n-1\}$. So $R_{(k)}(S)$ is a cyclic automorphism of $H_{\mathbb{Z}}$. This applies to the chain type singularities and is a remarkable fact there (remark 6.31 (iv)).

6.4 Eigenvalue and Seifert form strata for the spaces $T(2,\mathbb{R})$ and $T(3,\mathbb{R})$

After introducing HOR matrices, we make things for $n = 2, 3$ very explicit. We prove conjectures 2.21 and 2.23 for $T_{\mathrm{HOR1}}(2,\mathbb{R})$ (conjecture 2.22 is empty). And conjectures 2.21 , 2.22 and 2.23 for $T_{\mathrm{HOR1}}(3,\mathbb{R})$.

We discuss all eigenvalue and Seifert form strata for the complete spaces $T(n,\mathbb{R})$ for $n = 2, 3$.

6.4.1 The case $n = 2$

Consider an upper triangular matrix $S = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ with $a \in \mathbb{R}$ and consider the matrix $R^{mat}_{(1)}(S) = \begin{pmatrix} -a & -1 \\ 1 & 0 \end{pmatrix}$. By the proof of theorem 6.20 (a) (or a direct calculation)

$$-S^{-1}S^t = R^{mat}_{(1)}(S)^2. \tag{6.1}$$

The characteristic polynomial of $R^{mat}_{(1)}(S)$ is $p(x) = x^2 + ax + 1$. Thus $R^{mat}_{(1)}$ and $S^{-1}S^t$ have eigenvalues in S^1 if and only if $|a| \le 2$. Therefore

$$\begin{aligned} T(2,\mathbb{R}) &= T_{\mathrm{HOR1}}(2,\mathbb{R}) = \{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \,|\, a \in [-2, 2]\} \cong [-2, 2] \\ &\cong T^{scal}_{\mathrm{HOR1}}(2,\mathbb{R}) = \{(\beta_1, \beta_2) \,|\, \beta_1 \in [0, \tfrac{1}{2}], \beta_2 = 1 - \beta_1\} \cong [0, \tfrac{1}{2}]. \end{aligned} \tag{6.2}$$

The recipe 6.11 gives for $p(x) = x^2 + ax + 1$ with $|a| \le 2$

$$\beta_1 \in [0, \frac{1}{2}], \quad \beta_2 = 1 - \beta_1 \in [\frac{1}{2}, 1] \quad \text{with } 2\cos(2\pi\beta_1) = -a, \tag{6.3}$$

$$\gamma_1 = \frac{1}{4}, \quad \gamma_2 = \frac{3}{4}, \tag{6.4}$$

$$\alpha_1 = 2\beta_1 - \frac{1}{2} \in [-\frac{1}{2}, \frac{1}{2}], \quad \alpha_2 = 2\beta_2 - \frac{3}{2} = -\alpha_1. \tag{6.5}$$

α_1 is determined by $2\sin(\pi\alpha_1) = a$. $R^{mat}_{(1)}(S)$ and $S^{-1}S^t$ are not semisimple precisely at the boundary of $T_{\rm HOR1}(2,\mathbb{R})$. There they have a 2×2 Jordan block and the following eigenvalues, and the spectral pairs are:

	$a=-2$	$a=2$
β_1	0	$\frac{1}{2}$
eigenvalue of $R^{mat}_{(1)}(S)$	$e^{-2\pi i\beta_1}=1$	$e^{-2\pi i\beta_1}=-1$
α_1	$-\frac{1}{2}$	$\frac{1}{2}$
eigenvalue of $S^{-1}S^t$	$e^{-2\pi i\alpha_1}=-1$	$e^{-2\pi i\alpha_1}=-1$
$\mathrm{Spp}(S)$	$(-\frac{1}{2},2),(\frac{1}{2},0)$	$(-\frac{1}{2},2),(\frac{1}{2},0)$

(6.6)

The following table lists the types of the Seifert form pairs which one obtains by theorem 6.20 (c) for each $a\in[-2,2]$.

$a=0$	$2\cdot\mathrm{Seif}(1,1,1,1)$
$a\in]-2,2[-\{0\}$	$\mathrm{Seif}(e^{-2\pi i\alpha_1},2,1,e^{\pi i\alpha_1})$ $\cong \mathrm{Seif}(e^{2\pi i\alpha_1},2,1,e^{-\pi i\alpha_1})$
$a=\pm 2$	$\mathrm{Seif}(-1,1,2,1)$

(6.7)

The eigenvalue strata and the Seifert form strata (definition 2.20 (f)) in $T(2,\mathbb{R})$ coincide. One is $\{E_2\}$, the others are $\{\begin{pmatrix}1&a\\0&1\end{pmatrix},\begin{pmatrix}1&-a\\0&1\end{pmatrix}\}$ for $a\in[-2,2]-\{0\}$.

The set $T_{\rm HOR2}(2,\mathbb{R})$ has dimension 0 by (2.9). It is $T_{\rm HOR2}(2,\mathbb{R})=\{E_2\}$, and

$$R^{mat}_{(2)}(E_2)=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad R^{mat}_{(2)}(E_2)^2=E_2. \tag{6.8}$$

Recipe 6.11 gives in the case $k=2$ for $S=E_2$

$$\beta_1=0,\ \beta_2=\frac{1}{2},\ \gamma_1=0,\ \gamma_2=\frac{1}{2},\ \alpha_1=0,\ \alpha_2=0. \tag{6.9}$$

In the case $n=2$ conjecture 2.21 is satisfied (and conjecture 2.22 is empty). The only singularity up to suspension with $\mu=2$ is A_2. It is a chain type singularity. Theorem 6.33 implies for $n=2$ conjecture 2.23 for function germs.

6.4.2 The case $n = 3$

The following theorem 6.24 describes the set $T(3,\mathbb{R})$, its Seifert form strata and its eigenvalue strata (definition 2.20 (f)). Define

$$\begin{aligned} &f^{\mathbb{C}} : \mathbb{C}^3 \to \mathbb{C}, \quad f(a_1,a_2,a_3) := 4 + a_1a_2a_3 - (a_1^2 + a_2^2 + a_3^2), \\ &f := f^{\mathbb{C}}|_{\mathbb{R}} : \mathbb{R}^3 \to \mathbb{R}, \end{aligned} \tag{6.10}$$

$$\begin{aligned} S^{[3]} : \mathbb{R}^3 &\to M(3\times 3,\mathbb{R}), \\ a = (a_1,a_2,a_3) &\mapsto S^{[3]}(a) = \begin{pmatrix} 1 & a_1 & a_3 \\ & 1 & a_2 \\ & & 1 \end{pmatrix}, \\ M(3,\mathbb{R})_{tri} &:= S^{[3]}(\mathbb{R}^3) \subset M(3\times 3,\mathbb{R}), \end{aligned} \tag{6.11}$$

$$\mathrm{ray}(S) := S^{[3]}(\mathbb{R}\cdot a) \text{ for } S = S^{[3]}(a) \neq E_3 \text{ (i.e. for } a \neq 0). \tag{6.12}$$

Theorem 6.24. *$T(3,\mathbb{R})$ is the closed semialgebraic subset of $M(3,\mathbb{R})_{tri}$*

$$T(3,\mathbb{R}) = \{S^{[3]}(a) \in M(3,\mathbb{R})_{tri} \,|\, 0 \leq f(a_1,a_2,a_3) \leq 4\}. \tag{6.13}$$

Consider the subsets

$$\begin{aligned} T(3,\mathbb{R})_{pos} &:= \{S \in T(3,\mathbb{R}) \,|\, S + S^t \text{ pos. def. or pos. semidefinite}\}, \\ T(3,\mathbb{R})_{exc} &:= \{S^{[3]}(2,2,2), S^{[3]}(-2,-2,2), \\ &\qquad S^{[3]}(-2,2,-2), S^{[3]}(2,-2,-2)\}, \\ T(3,\mathbb{R})_{ind} &:= T(3,\mathbb{R}) - T(3,\mathbb{R})_{pos}. \end{aligned} \tag{6.14}$$

$T(3,\mathbb{R})_{pos}$ is homeomorphic to a 3-ball and $G_{\mathrm{sign},3}$-invariant ($G_{sign,n}$: definition 2.20 (e)).

$$\begin{aligned} \overline{T(3,\mathbb{R})_{ind}} &= T(3,\mathbb{R})_{ind} \cup T(3,\mathbb{R})_{exc}, \\ \overline{T(3,\mathbb{R})_{ind}} \cap T(3,\mathbb{R})_{pos} &= T(3,\mathbb{R})_{exc}. \end{aligned} \tag{6.15}$$

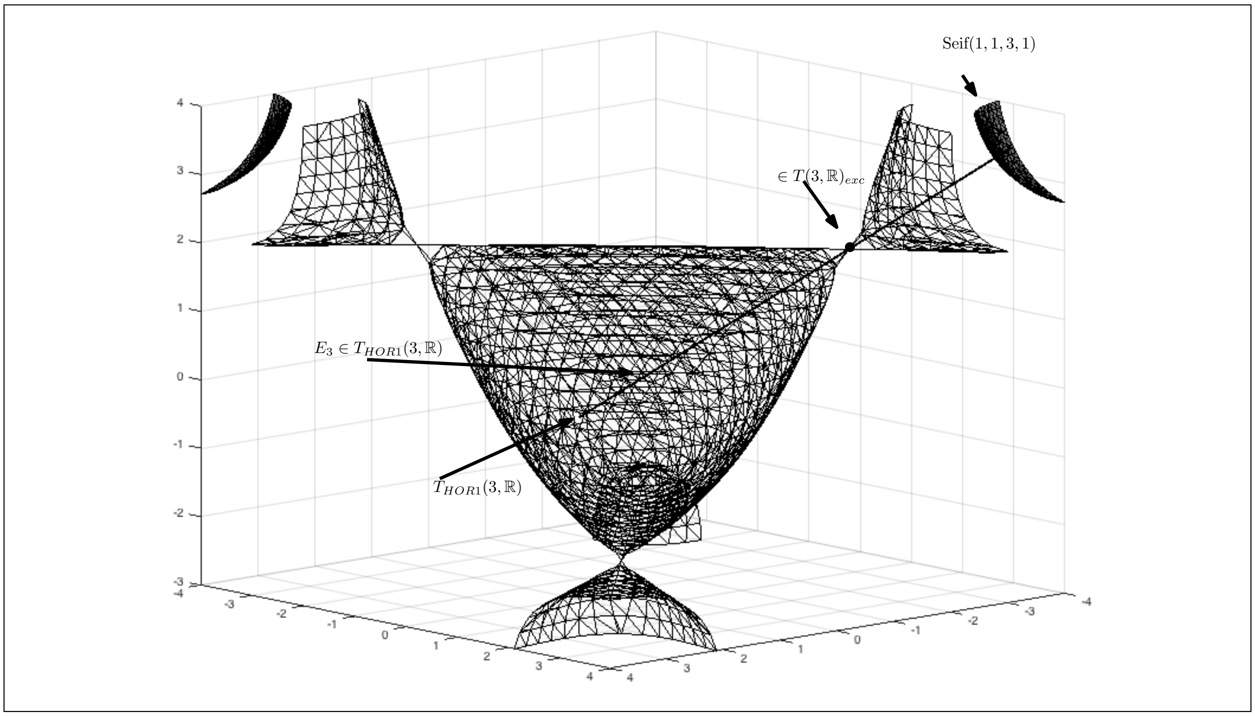
$\overline{T(3,\mathbb{R})_{ind}}$ is homeomorphic to four copies of $[0,1]\times\mathbb{R}^2$. These components are permuted by the group $G_{\mathrm{sign},3}$. Each component is in the open quadrant in $M(3,\mathbb{R})_{tri} \cong \mathbb{R}^3$ which contains one of the points in $T(3,\mathbb{R})_{exc}$. The boundary $\partial T(3,\mathbb{R})$ is smooth and transversal to the rays $\mathrm{ray}(S)$ for $S \in M(3,\mathbb{R})_{tri} - \{E_3\}$ except at the 4 points in $T(3,\mathbb{R})_{exc}$. At each of the 4 points

in $T(3,\mathbb{R})_{exc}$ *it is isomorphic to a cone.*

For each type of a Seifert form pair of rank 3, at most one Seifert form stratum exists. The following table lists those which exist.

Type of a Seifert form pair	description of Seifert form stratum
$3\cdot\mathrm{Seif}(1,1,1,1)$	$\{E_3\}$
$\mathrm{Seif}(1,1,1,1)$ $+\,\mathrm{Seif}(e^{-2\pi i\alpha_1},2,1,e^{\pi i\alpha_1})$	diffeomorphic to a 2-sphere in $\mathrm{int}(T(3,\mathbb{R})_{pos})$
$\mathrm{Seif}(1,1,1,1)+$ $+\,\mathrm{Seif}(-1,1,2,1)$	$\partial T(3,\mathbb{R})_{pos}-T(3,\mathbb{R})_{exp}$ $\approx$ 2-sphere $-$ 4 points
$\mathrm{Seif}(1,1,1,1)$ $+\,\mathrm{Seif}(-1,2,1)$	$T(3,\mathbb{R})_{exc}$
$\mathrm{Seif}(1,1,1,1)$ $+\,\mathrm{Seif}(-1,1,2,-1)$	the 4 components of $\partial T(3,\mathbb{R})_{ind}$ whose closures contain points of $T(3,\mathbb{R})_{exp}$ $\approx$ 4 copies of $\mathbb{R}^2-\{0\}$
$\mathrm{Seif}(1,1,1,1)$ $+\,\mathrm{Seif}(e^{-2\pi i\alpha_1},2,1,-e^{\pi i\alpha_1})$	diffeomorphic to 4 copies of $\mathbb{R}^2$, one in each component of $\mathrm{int}(T(3,\mathbb{R})_{ind})$
$\mathrm{Seif}(1,1,3,1)$	the 4 components of $\partial\overline{T(3,\mathbb{R})_{ind}}$ which do not intersect $T(3,\mathbb{R})_{exc}$ $\approx$ 4 copies of $\mathbb{R}^2$

The three Seifert form strata with eigenvalues $(1,-1,-1)$ *form one eigenvalue stratum. It is one component of* $\partial T(3,\mathbb{R})$*. The other Seifert form strata are eigenvalue strata. The following is a rough picture of a part of* $T(3,\mathbb{R})$*. The thick line indicates* $T_{\mathrm{HOR1}}(3,\mathbb{R})$*, which will be discussed below.*



Proof. The characteristic polynomial of $S^{-1}S^t$ is

$$\begin{aligned} p_{ch,S}(x) &= \det(xE_3 - S^{-1}S^t) = \det(xS - S^t) \\ &= x^3 - (f(a)-1)x^2 + (f(a)-1)x - 1 \\ &= (x-1)(x^2 - (f(a)-2)x + 1). \end{aligned} \tag{6.16}$$

This shows (6.13). The boundary $\partial T(3,\mathbb{R})$ of $T(3,\mathbb{R})$ is $\{S \in M(3,\mathbb{R})_{tri} \,|\, f(a) = 0 \text{ or } f(a) = 4\} - \{E_3\}$. For any $S = S^{[3]}(a) \in M(3,\mathbb{R})_{tri} - \{E_3\}$, consider the function

$$\begin{aligned} g^{\mathrm{ray},S} &: \mathbb{R}_{\geq 0} \to \mathbb{R}, \\ g^{\mathrm{ray},S}(r) &:= f(r\cdot a) = 4 + r^3 \cdot a_1a_2a_3 - r^2(a_1^2 + a_2^2 + a_3^2). \end{aligned}$$

Claim 1.

(i) If $a_1a_2a_3 \leq 0$, then $g^{\mathrm{ray},S}$ is strictly decreasing with the limit $-\infty$, so then $\mathrm{ray}(S)$ intersects $\partial T(3,\mathbb{R})$ only in one point.

(ii) If $a_1a_2a_3 > 0$ and $S \notin \mathbb{R}_{>0}\cdot T(3,\mathbb{R})_{exc}$, then $g^{\mathrm{ray},S}$ is first strictly decreasing to a minimum < 0 and then strictly increasing with limit $+\infty$. Then $\mathrm{ray}(S)$ intersects $\partial T(3,\mathbb{R})$ at three points.

(iii) If $S \in T(3,\mathbb{R})_{exc}$, then $g^{\mathrm{ray},S}$ is first strictly decreasing with minimum $= 0$ at S and then strictly increasing with limit $+\infty$. Then $\mathrm{ray}(S)$ intersects $\partial T(3,\mathbb{R})$ at S and at one other point.

Proof of claim 1. (i) is clear.

(ii), (iii)

$$\begin{aligned}
(g^{\mathrm{ray},S})'(r) &= r\cdot(3r\cdot a_1a_2a_3 - 2(a_1^2+a_2^2+a_3^2)),\\
r_0 &:= 2(a_1^2+a_2^2+a_3^2)/(3a_1a_2a_3), \quad \text{so that } (g^{\mathrm{ray},S})'(r_0)=0,\\
g^{\mathrm{ray},S}(r_0) &= \frac{4}{27(a_1a_2a_3)^2}(27a_1^2a_2^2a_3^2 - (a_1^2+a_2^2+a_3^2)^3)\\
&\begin{cases} =0 & \text{for } S\in T(3,\mathbb{R})_{exc},\\ \overset{(*)}{<} 0 & \text{for } a_1a_2a_3>0, S\notin \mathbb{R}_{>0}\cdot T(3,\mathbb{R})_{exc}.\end{cases}
\end{aligned}$$

$\overset{(*)}{<}$ is an easy exercise. This finishes the proof of claim 1. $(\Box)$

The eigenvalue map $\Psi_{Eig} : T(3,\mathbb{R}) \to \mathrm{Eig}(3)$ has the same fibers as the map $T(3,\mathbb{R}) \to \mathbb{R}$, $S(a) \to f(a)$. Claim 1 shows that the fibers are smooth and transversal to the rays $\mathrm{ray}(S)$, except at the point E_3 and the four points in $T(3,\mathbb{R})_{exc}$. At each of these four points the fiber is locally diffeomorphic to a cone, because f has at $(a_1,a_2,a_3)\in\{(2,2,2),(-2,-2,2),(-2,2,-2),(2,-2,-2)\}$ an A_1-singularity, and the signature of the Hessian

$$\mathrm{Hess}(f)(a) = \left(\frac{\partial^2 f}{\partial a_i\partial a_j}\right)(a) = \begin{pmatrix} -2 & a_3 & a_2\\ a_3 & -2 & a_1\\ a_2 & a_1 & -2\end{pmatrix}$$

is $(1,0,2)$, because $\det\mathrm{Hess}(f)(a)=32>0$ and $-2<0$.

Claim 1 shows that $M(3,\mathbb{R})_{tri} - \{S(a)\,|\,f(a)=0\}$ has six components: the component C_1 which contains E_3, the component C_2 which contains all of the four quadrants with $a_1a_2a_3<0$ except their intersection with $\overline{C_1}$, and the four components C_3, C_4, C_5, C_6 which contain each one of the partial rays in $\mathbb{R}_{>1}\cdot T(3,\mathbb{R})_{exc}$.

(6.16) implies

$$\det(S+S^t) = 2\cdot f(a)\begin{cases} >0 & \text{on } C_1, C_3, C_4, C_5 \text{ and } C_6,\\ <0 & \text{on } C_2.\end{cases}$$

E_3 has signature $(3,0,0)$, any matrix $S+S^t$ with S in $\mathbb{R}_{>1}\cdot T(3,\mathbb{R})_{exc}$ has signature $(1,0,2)$

because $\det \begin{pmatrix} 2 & a_1 \\ a_1 & 2 \end{pmatrix} < 0$ for such matrices. Therefore

$$\text{signature}(S+S^t) = \begin{cases} (3,0,0) & \text{on } C_1, \\ (2,0,1) & \text{on } C_2, \\ (1,0,2) & \text{on } C_3, C_4, C_5 \text{ and } C_6, \end{cases}$$

$$\text{signature}(S+S^t) = \begin{cases} (2,1,0) & \text{on the part of } \partial T(3,\mathbb{R}) \\ & \text{between } C_1 \text{ and } C_2, \\ (1,2,0) & \text{on } T(3,\mathbb{R})_{exc}, \\ (1,1,1) & \text{on the part of } \partial T(3,\mathbb{R}) \\ & \text{between } C_2 \text{ and } C_3, C_4, C_5, C_6. \end{cases}$$

Thus $\dim \mathrm{Rad}(S+S^t) = 1$ for S in $\{S(a)\,|\,f(a)=0\} - T(3,\mathbb{R})_{exc}$. Therefore $S^{-1}S^t$ has for such an S a 2×2 Jordan block with eigenvalues -1. For $S \in T(3,\mathbb{R})_{exc}$ it is semisimple with eigenvalues $1, -1, -1$.

Finally, consider the set $\{S(a)\,|\,f(a)=4\} - \{E_3\}$. It is the union of the four boundary components of $T(3,\mathbb{R})$ which do not contain $T(3,\mathbb{R})_{exc}$. For $S \in \{S(a)\,|\,f(a)=4\} - \{E_3\}$, claim 1 gives $a_1a_2a_3 > 0$. This implies $\mathrm{rk}(S^t - S) = \mathrm{rk} \begin{pmatrix} 0 & -a_3 \\ a_3 & 0 \end{pmatrix} = 2$ and $\dim \mathrm{Rad}(S^t - S) = 1$ and that $S^{-1}S^t$ has a single 3×3 Jordan block with eigenvalue 1.

The proof up to now gives all statements in theorem 6.24 except the table with Seifert form pairs and Seifert form strata. The proof gives also the eigenvalue strata and the signature of I_s at each point of $M(3,\mathbb{R})_{tri}$. The table with Seifert form pairs and Seifert form strata can now be deduced from the eigenvalues of $S^{-1}S^t$, its Jordan block structure, claim 1, the signature of $S+S^t$, and from chapter 5 lemma 5.9. $\square$

Now we study the subvarieties $T_{\mathrm{HOR}k}(3,\mathbb{R}) \subset T(3,\mathbb{R})$ for $k \in \{1,2\}$.

$$\begin{aligned} p(x) &= x^3 + p_2x^2 + p_1x + p_0 \\ &= x^3 + (-1)^{k-1}p_1x^2 + p_1x + (-1)^{k-1} \in T^{pol}_{\mathrm{HOR}k}(3,\mathbb{R}) \\ &= \begin{cases} x^3 + p_1x^2 + p_1x + 1 \\ \qquad = (x+1)(x^2 + (p_1-1)x + 1) \text{ for } k=1, \\ x^3 - p_1x^2 + p_1x - 1 \\ \qquad = (x-1)(x^2 - (p_1-1)x + 1) \text{ for } k=2. \end{cases} \end{aligned}$$

$$
\begin{aligned}
T_{\mathrm{HOR1}}(3,\mathbb{R}) &= \begin{pmatrix} 1 & p_1 & p_1 \\ & 1 & p_1 \\ & & 1 \end{pmatrix} \,|\, p_1 \in [-1,3]\}, \\
T_{\mathrm{HOR2}}(3,\mathbb{R}) &= \begin{pmatrix} 1 & -p_1 & p_1 \\ & 1 & -p_1 \\ & & 1 \end{pmatrix} \,|\, p_1 \in [-1,3]\}.
\end{aligned}
$$

The element $g = (1,-1,1) \in G_{sign,3}$ exchanges $T_{\mathrm{HOR1}}(3,\mathbb{R})$ and $T_{\mathrm{HOR2}}(3,\mathbb{R})$. In the picture after theorem 6.24, the thick line indicates $T_{\mathrm{HOR1}}(3,\mathbb{R})$.

By lemma 6.12 (c) $\mathrm{Spp}(g(S)) = \mathrm{Spp}(S)$ for $S \in \bigcup_{k=1,2} T_{\mathrm{HOR}k}(3,\mathbb{R})$. Therefore we restrict in the following to $T_{\mathrm{HOR1}}(3,\mathbb{R})$.

Theorem 6.25. *(a) $T_{\mathrm{HOR1}}(3,\mathbb{R})$ intersects the Seifert form stratum of type $\mathrm{Seif}(1,1,1,1) + \mathrm{Seif}(e^{-2\pi i\alpha_1}, 2, 1, e^{\pi i\alpha_1})$ twice, the Seifert form stratum of type $\mathrm{Seif}(1,1,1,1) + \mathrm{Seif}(-1,1,2,-1)$ not at all and each other Seifert form stratum once.*

(b) Recipe 6.11 gives for $T_{\mathrm{HOR1}}(3,\mathbb{R})$ numbers $\beta_j, \gamma_j, \alpha_j$ for $j = 1,2,3$ with

$$
\begin{aligned}
\beta_1 \in [0,\frac{1}{2}],\ \beta_2 = \frac{1}{2},\ \beta_3 = 1-\beta_1 \in [\frac{1}{2},1], \\
\gamma_1 = \frac{1}{6},\ \gamma_2 = \frac{1}{2},\ \gamma_3 = \frac{5}{6}, \\
\alpha_1 = 3\beta_1 - \frac{1}{2} \in [-\frac{1}{2},1],\ \alpha_2 = 3\beta_2 - \frac{3}{2} = 0, \\
\alpha_3 = 3\beta_3 - \frac{5}{2} = -\alpha_1 \in [-1,\frac{1}{2}].
\end{aligned}
\tag{6.17}
$$

β_1 is determined by $\beta_1 \in [0,\frac{1}{2}]$ and $\cos(2\pi\beta_1) = \frac{1-p_1}{2}$. Thus β_1 and α_1 are monotonically increasing with $p_1 \in [-1,3]$.

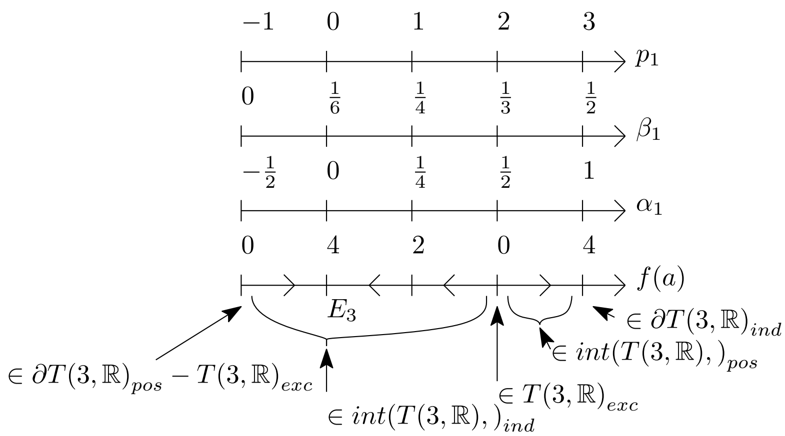


Diagram of α_1, β_1, $f(a)$ and p_1

(c) The conjectures [2.21] and [2.22] hold. The Seifert form strata in $T(3,\mathbb{R})_{pos}$ have spectral numbers in $[-\frac{1}{2},\frac{1}{2}]$, the Seifert form strata in $\overline{T(3,\mathbb{R})_{ind}}$ have spectral numbers in $[-1,-\frac{1}{2}]\cup\{0\}\cup[\frac{1}{2},1]$. The two Seifert form strata with eigenvalues $1,-1,-1$ and a 2×2 Jordan block for the eigenvalue -1 have the same spectral pairs $(0,1),(-\frac{1}{2},2),(\frac{1}{2},0)$. The Seifert form stratum $\{S^{[3]}(a)\,|\,f(a)=4\}-\{E_3\}$ with a 3×3 Jordan block has the spectral pairs $(-1,3),(0,1),(1,-1)$.

(d) Conjecture 2.23 for function germs holds in the case $n=3$.

Proof: (a) $T_{\mathrm{HOR1}}(3,\mathbb{R})$ is the intersection of $T(3,\mathbb{R})$ with the line through $E_3 = S^{[3]}(0,0,0)$ and $S^{[3]}(2,2,2)\in T(3,\mathbb{R})_{exc}$. This and theorem 6.24 show part (a).

(b) β_1 in recipe 6.11 for $S\in T_{\mathrm{HOR1}}(3,\mathbb{R})$ is determined by $\beta_1\in[0,\frac{1}{2}]$ and $(x-e^{2\pi i\beta_1})(x-e^{-2\pi i\beta_1}) = x^2+(p_1-1)x+1$, which is $\cos(2\pi\beta_1)=\frac{1-p_1}{2}$. This shows all of (b).

(c) This follows from (a) and (b) and the following observation. At the boundary points of $T_{\mathrm{HOR1}}(3,\mathbb{R})$, the monodromy $S^{-1}S^t$ and $R^{mat}_{(1)}(S)$ have for each eigenvalue of $R^{mat}_{(1)}(S)$ one Jordan block.

(d) The only singularity up to suspension with $\mu=3$ is A_3. It is a chain type singularity. Theorem 6.33 implies for $n=3$ conjecture 2.23 for function germs. □

Remarks 6.26. (i) By theorem 6.25 (a), $T_{\mathrm{HOR}k}(3,\mathbb{R})$ for $k\in\{1,2\}$ does not intersect the Seifert form stratum of type $\mathrm{Seif}(1,1,1,1)+\mathrm{Seif}(-1,1,2,-1)$. This is consistent with theorem 6.20 (c): On this Seifert form stratum, the single spp-ladder $(\frac{-1}{2},2),(\frac{1}{2},0)$ has first spectral number $\alpha=\frac{-1}{2}$ and $l=1$, and

$$L(a,N^la)\in(-1)\cdot\mathbb{R}_{>0}=(-1)\cdot e^{\frac{1}{2}\pi i(2\alpha+l)}\cdot\mathbb{R}_{>0}.$$

Theorem 6.20 (c) forbids the existence of a matrix in $T_{\mathrm{HOR}k}(n,\mathbb{R})$ and in this Seifert form stratum.

(ii) The table (6.5) for $n=2$ and the table in theorem 6.25 for $n=3$ show that precisely the following S^1-Seifert form pairs have no realization as $(M(n\times 1,\mathbb{R}),L)$ with $L(a,b)=a^tS^tb$ with $S\in T(n,\mathbb{R})$): All those for which I_s is negative semidefinite (cf. chapter 5 lemma 5.9 and remark 6.8 (vii)), and all those which contain $\mathrm{Seif}(1,1,1,-1)$ or $\mathrm{Seif}(1,1,3,-1)$. It is an interesting question what holds for $n\geq 4$.

6.5 M-tame functions

The purpose of this section is to give references and facts on M-tame functions, introduce their spectrum and mention the relation to the *ihs* spectrum. In particular, that an M-tame $f:X\to\mathbb{C}$ come equipped with $\mathrm{Br}_\mu\ltimes G_{sign,\mu}$-orbits of *Stokes matrices* in $T(\mu,\mathbb{Z})$ and with spectral pairs $\mathrm{Spp}(f)$. The very similar material for *ihs* is contained in the last chapter in section 5.5.

First, let us recall the definition of M-tameness.

Definition 6.27. *[NZ90][NS99] A function $f:X\to\mathbb{C}$ is M-tame if X is an affine manifold (of some dimension $m+1$) and if for some closed embedding $X\hookrightarrow\mathbb{C}^N$ the following holds. For any $\eta>0$ an $R(\eta)>0$ exists such that the fibers $f^{-1}(\tau)$ with $|\tau|<\eta$ are transversal to all spheres $S_R^{2N+1}=\{z\in\mathbb{C}^N\,|\,|z|=R\}$ with $R\geq R(\eta)$.*

M-tameness is analogous to the existence of a similar fibration as was introduced in the case of *ihs* (See [NS99] for a discussion on M-tameness). We may also note, that Thom-Sebastiani formulas hold for two M-tame functions.

Set $\Delta_\eta:=\{\tau\in\mathbb{C}\,|\,|\tau|<\eta\}$ and construct a good representative $f:Y\to\Delta_\eta$ for $\eta>0$ sufficiently large. The Milnor number μ is the sum of the Milnor numbers of all singularities of $f:Y\to\Delta_\eta$, which are all singularities of $f:X\to\mathbb{C}$. The relative homology groups (reduced if $m=0$) $Ml(f,\zeta):=H_{m+1}(Y,f^{-1}(\zeta\eta),\mathbb{Z})$ with $\zeta\in S^1$ are isomorphic to $\mathbb{Z}^\mu$ [Lo84, (5.11)] [AGV88, ch. 2], and some generators of them can be called (classes of) *Lefschetz thimbles*. They form a flat $\mathbb{Z}$-lattice bundle on S^1. An intersection form for Lefschetz thimbles is well defined on relative homology groups with different boundary parts. It is for any $\zeta\in S^1$ a $(-1)^{m+1}$ symmetric unimodular bilinear form

$$I_{Lef}:Ml(f,\zeta)\times Ml(f,-\zeta)\to\mathbb{Z} \tag{6.18}$$

Let γ_π (respectively $\gamma_{-\pi}$) be the isomorphism $Ml(f,-\zeta)\to Ml(f,\zeta)$ by flat shift in mathematically positive (respectively negative) direction. Then the classical Seifert form is given by

$$L : Ml(f,\zeta) \times Ml(f,\zeta) \to \mathbb{Z}, \quad L(a,b) := (-1)^{m+1} I_{Lef}(a, \gamma_{-\pi} b). \tag{6.19}$$

The classical monodromy M and the intersection form I on $Ml(f,\zeta)$ are given by

$$L(Ma,b) = (-1)^{m+1} L(b,a), \tag{6.20}$$
$$I(a,b) = -L(a,b) + (-1)^{m+1} L(b,a) = L((M - \mathrm{id})a, b). \tag{6.21}$$

We define a *normalized* Seifert form L^{hnor} and a *normalized* monodromy M^{hnor} by

$$L^{hnor} := (-1)^{(m+1)(m+2)/2} \cdot L, \tag{6.22}$$
$$M^{hnor} := (-1)^{m+1} M. \tag{6.23}$$

Thus M^{hnor} is the monodromy of L and of L^{hnor} in the sense of definition 5.2 (b). Distinguished bases of $Ml(f,\zeta)$ are defined as usual, see chapter 2. Recall, the set of distinguished bases forms one orbit of the group $\mathrm{Br}_\mu \ltimes G_{sign,\mu}$. Here Br_μ is the braid group with μ strings, both are defined in definition 5 (e). The group $G_{sign,\mu}$ acts componentwise by sign changes. Each distinguished basis $\underline{\delta}$ gives rise to one matrix

$$S := L^{hnor}(\underline{\delta}^t, \underline{\delta})^t \in T(\mu, \mathbb{Z}). \tag{6.24}$$

We call these matrices *Stokes matrices* because some of them encode certain Stokes structures (which will not be discussed here). These matrices form also one $\mathrm{Br}_\mu \ltimes G_{sign,\mu}$-orbit. In the case of the *ihs*, this orbit is finite only for the simple and the simple elliptic singularities, and the orbit of distinguished bases is finite only for the simple singularities [Eb16].

Now we come to the spectral pairs. In the case of an ihs f, spectral pairs $\mathrm{Spp}(f)$ were first defined by Steenbrink [St77] as invariants of his natural mixed Hodge structure on the space dual to $Ml(f,1)$ (see also [AGV88]). It is in the notation of the last chapter a signed Steenbrink polarized mixed Hodge structure of weight m. In the case of an M-tame function f, the spectral pairs are defined in the same way as invariants of Sabbah's natural mixed Hodge structure [Sa98] on the space dual to $Ml(f,\zeta)$. A certain twist of Sabbah's Hodge filtration is a part of a Steenbrink polarized mixed Hodge structure of weight m [HS07, Corollary 11.4] (in the notation of [**?**]).

In both cases, f ihs or f M-tame function, $\mathrm{Spp}(f)$ is a union of single spp-ladders and sppl-pairs with center m, as the spectral pairs of any Steenbrink mixed Hodge structure in the sense of chapter 5.

6.6 Chain type singularities and their spectra

We introduced HOR matrices, visualized their space for $n = 2, 3$ and provided them with a spectrum in the previous three sections. In this section, we prove theorem 6.20 that this spectrum $\mathrm{Sp}(S)$ coincides with the spectrum $\mathrm{Sp}(f)$ of chain type singularities, up to a shift.

This is positive evidence for conjecture 2.23. What we do not show is that the HOR matrix actually belongs to a distinguished basis for f, which was conjectured by conjecture 6.30.

We first recall *chain type singularities*, then make a reduction on the first exponent a_0 of it, and finally prove the main theorem 6.20. For that, we construct a canonical Jacobi algebra basis for chain type singularities and then are able to compare the spectra.

Definition 6.28. *(a) A* **chain type singularity** *is a function germ on* $(\mathbb{C}^{m+1}, 0)$ *which is defined by a polynomial*

$$f(x_0, ..., x_m) = x_0^{a_0} + x_0 x_1^{a_1} + ... + x_{m-1} x_m^{a_m} = x_0^{a_0} + \sum_{j=1}^{m} x_{j-1} x_j^{a_j}$$

with $a_0 \in \mathbb{Z}_{\geq 2}, a_1, ..., a_m \in \mathbb{Z}_{\geq 1}$.

(b) Define the function

$$\begin{aligned} \rho : \bigcup_{k=0}^{\infty} \mathbb{Z}^k &\to \mathbb{Z}, \qquad (6.1) \\ \rho(a_0, a_1, ..., a_{k-1}) &:= a_0 ... a_{k-1} - a_1 ... a_{k-1} + ... + (-1)^{k-1} a_{k-1} + (-1)^k \end{aligned}$$

(the case $k = 0$ *is* $\rho(\emptyset) = 1$*).*

Lemma 6.29. *Consider f in definition 6.28 (a). It has indeed an isolated singularity at 0. It is a quasihomogeneous polynomial of weighted degree 1 with respect to weights* $(w_0, ..., w_m)$ *which are determined as follows. Define*

$$r_{-1} := 1,\ r_k := a_0 ... a_k = r_{k-1} a_k \quad \text{for } 0 \leq k \leq m, \qquad (6.2)$$

$$\mu_{-1} := 1,\ \mu_k := \rho(a_0, ..., a_k) = r_k - \mu_{k-1} \quad \text{for } 0 \leq k \leq m, \qquad (6.3)$$

$$w_{-1} := 0,\ w_k := \frac{\mu_{k-1}}{r_k} = \frac{1 - w_{k-1}}{a_k} \quad \text{for } 0 \leq k \leq m. \qquad (6.4)$$

Its Milnor number is $\mu = \mu_m$.

Proof. The partial derivatives of f are

$$\begin{aligned}
\frac{\partial f}{\partial x_0} &= a_0 x_0^{a_0-1} + x_1^{a_1}, \\
\frac{\partial f}{\partial x_1} &= a_1 x_0 x_1^{a_1-1} + x_2^{a_2}, ..., \\
\frac{\partial f}{\partial x_{m-1}} &= a_{m-1} x_{m-2} x_{m-1}^{a_{m-1}-1} + x_m^{a_m}, \\
\frac{\partial f}{\partial x_m} &= a_m x_{m-1} x_m^{a_m-1}.
\end{aligned} \tag{6.5}$$

Suppose that $x \in \mathbb{C}^{m+1}$ is a zero of all partial derivatives. Then

$$\begin{aligned}
x_0 \neq 0 \Rightarrow x_1 \neq 0 \Rightarrow x_2 \neq 0 \Rightarrow ... \Rightarrow x_m \neq 0 \\
\Rightarrow \frac{\partial f}{\partial x_m}(x) \neq 0, \text{ a contradiction.} \\
x_0 = 0 \Rightarrow x_1 = 0 \Rightarrow x_2 = 0 \Rightarrow ... \Rightarrow x_m = 0.
\end{aligned}$$

Therefore the singularity $x = 0$ of f is the only singularity in $\mathbb{C}^{m+1}$. The weights $(w_0, ..., w_m)$ are uniquely determined by

$$\begin{aligned}
w_0 &= \frac{1}{a_0} = \frac{\mu_{-1}}{r_0}, \\
w_k &= \frac{1 - w_{k-1}}{a_k} = \frac{1 - \frac{\mu_{k-2}}{r_{k-1}}}{a_k} = \frac{r_{k-1} + \mu_{k-2}}{r_{k-1} a_k} = \frac{\mu_{k-1}}{r_k}.
\end{aligned}$$

In the following calculation of the Milnor number, $\overset{(*)}{=}$ is a well-known formula for all quasihomogeneous singularities.

$$\mu \overset{(*)}{=} \prod_{k=0}^{m} \left(\frac{1}{w_k} - 1 \right) = \prod_{k=0}^{m} \frac{r_k - \mu_{k-1}}{\mu_{k-1}} = \prod_{k=0}^{m} \frac{\mu_k}{\mu_{k-1}} = \mu_m. \qquad \square$$

Conjecture 6.30. *[OR77, Conjecture (4.1)] The chain type singularity $f = x_0^{a_0} + x_0 x_1^{a_1} + ... + x_{m-1} x_m^{a_m}$ has a distinguished basis whose Stokes matrix S is the HOR matrix S (definition 6.19 (a)) with polynomial*

$$p(x) = x^{\mu} + p_{\mu-1} x^{\mu-1} + ... + p_0 = \prod_{k=-1}^{m} (x^{r_k} - 1)^{(-1)^{m-k}}. \tag{6.6}$$

Remarks 6.31. (i) In conjecture 6.30 $p(x)$ has only simple eigenvalues, namely all zeros of $x^{r_m} - 1$ minus certain gaps, which are most zeros of $x^{r_{m-1}} - 1$.

(ii) In conjecture 6.30 $p_0=(-1)^{m+1}$ and $S\in T_{\mathrm{HOR},k}(\mu,\mathbb{R})\cap T(\mu,\mathbb{Z})$ with $k\equiv m(2)$.

(iii) Theorem (2.11) in [OR77] says that for a suitable basis of $Ml(f)$, the monodromy matrix is $R^{mat}_{(k)}(S)^\mu$ with $k\equiv m(2)$. This is compatible with conjecture 6.30 and theorem 6.20 (a), which give this for a distinguished basis with Stokes matrix S. Here recall that in the *ihs* case the monodromy in theorem 6.16 is the normalized monodromy M^{nor} in (6.23) and that the true monodromy is $(-1)^{m+1}M^{nor}$.

(iv) Conjecture 6.30 and definition 6.19 (b) give the automorphism $R_{(k)}(S):Ml(f)\to Ml(f)$ (with $k\equiv m(2)$) with characteristic polynomial $p(x)$. It respects L by theorem 6.20 (b), it satisfies $R_{(k)}(S)^\mu=M$ by theorem 6.20 (a), and it is cyclic by remark 6.23.

Remarks 6.32. Here we will argue that it is almost always (and especially in the proof of theorem 6.6) sufficient to consider chain type singularities $f=x_0^{a_0}+x_0x_1^{a_1}+...+x_{m-1}x_m^{a_m}$ with $a_0\in\mathbb{Z}_{\geq 3},a_1,...,a_m\in\mathbb{Z}_{\geq 2}$, and the A_1-type *ihs* x_0^2.

(i) $f(x)$ is right equivalent to $c_0\cdot x_0^{a_0}+c_1\cdot x_0x_1^{a_1}+...+c_m\cdot x_{m-1}x_m^{a_m}$ for arbitrary $c_0,...,c_m\in\mathbb{C}^*$.

(ii) Let $f(x_0,...,x_m)$ be a chain type singularity with $a_0=2$. Consider the new coordinates $\widetilde{x}_0=x_0+\frac{1}{2}x_1^{a_1},\widetilde{x}_k=x_k$ for $1\leq k\leq m$. Then

$$\begin{aligned} f(x_0,...,x_m) &= (x_0+\frac{1}{2}x_1^{a_1})^2-\frac{1}{4}x_1^{2a_1}+x_1x_2^{a_2}+...+x_{m-1}x_m^{a_m} \\ &= \widetilde{x}_0^2-\frac{1}{4}\widetilde{x}_1^{2a_1}+\widetilde{x}_1\widetilde{x}_2^{a_2}+...+\widetilde{x}_{m-1}\widetilde{x}_m^{a_m}. \end{aligned} \tag{6.7}$$

This is (up to a rescaling in $\widetilde{x}_1$) a 1-fold suspension of the chain type singularity $\widetilde{f}(y_0,...,y_{m-1})=y_0^{2a_1}+y_0y_1^{a_2}+...+y_{m-2}y_{m-1}^{a_m}$ with

$$\begin{aligned} \widetilde{a}_0 &= 2a_1,\quad \widetilde{a}_k=a_{k+1}\quad\text{for } 1\leq k\leq m-1, \\ \widetilde{r}_k &= r_{k+1}\quad\text{for } 0\leq k\leq m-1,\quad\text{all } \widetilde{r}_k\equiv 0(2), \\ p(x) &= (x+1)^{(-1)^{m-1}}\cdot\prod_{k=1}^m(x^{r_k}-1)^{(-1)^{m-k}}, \\ \widetilde{p}(x) &= (x-1)^{(-1)^{m-1}}\cdot\prod_{k=1}^m(x^{r_k}-1)^{(-1)^{m-k}}=(-1)^\mu\cdot p(-x), \\ \mathrm{Sp}(\widetilde{f}) &= \mathrm{Sp}(f)-\frac{1}{2}. \end{aligned}$$

Lemma 6.12 (c) implies $\mathrm{Sp}(\widetilde{p}(x))=\mathrm{Sp}(p(x))$.

(iii) Let $f(x_0,...,x_m)$ be a chain type singularity with $a_0=3$. Suppose that it has an exponent $a_j=1$ and that $a_1,...,a_{j-1}\geq 2$. Consider the new coordinates $\widetilde{x}_{j-1}=x_{j-1}+x_{j+1}^{a_{j+1}}$

and $\widetilde{x}_k = x_k$ for $k \neq j-1$. Then

$$\begin{aligned}
& f(x_0, ..., x_m) \\
= \ & x_0^{a_0} + x_0 x_1^{a_1} + ... + x_{j-2} x_{j-1}^{a_{j-1}} + (x_{j-1} + x_{j+1}^{a_{j+1}}) x_j \\
+ \ & x_{j+1} x_{j+2}^{a_{j+2}} + ... + x_{m-1} x_m^{a_m} \\
= \ & \widetilde{x}_0^{a_0} + \widetilde{x}_0 \widetilde{x}_1^{a_1} + ... + (-1)^{a_{j-1}} \widetilde{x}_{j-2} \widetilde{x}_{j+1}^{a_{j-1} a_{j+1}} + \widetilde{x}_{j+1} \widetilde{x}_{j+2}^{a_{j+2}} + ... + \widetilde{x}_{m-1} \widetilde{x}_m^{a_m} \\
+ \ & \widetilde{x}_{j-1} \widetilde{x}_j + \widetilde{x}_{j-2} \cdot \Big(\sum_{k=0}^{a_{j-1}-1} (-1)^k \binom{a_{j-1}}{k} \widetilde{x}_{j-1}^{a_{j-1}-k} (\widetilde{x}_{j+1}^{a_{j+1}})^k \Big).
\end{aligned} \tag{6.8}$$

The first line of (6.8) is a chain type singularity $\widetilde{f}(y_0, ..., y_{m-2})$ with

$$\begin{aligned}
\widetilde{a}_k &= a_k \text{ for } 0 \leq k \leq j-2, \widetilde{a}_{j-1} = a_{j-1} a_{j+1}, \\
& \quad \widetilde{a}_k = a_{k+2} \text{ for } j \leq k \leq m-2, \\
\widetilde{r}_k &= r_k \text{ for } 0 \leq k \leq j-2, \ \widetilde{r}_k = r_{k+2} \text{ for } j-1 \leq k \leq m-2, \\
\widetilde{p}(x) &= p(x), \\
\mathrm{Sp}(\widetilde{f}) &= \mathrm{Sp}(f) - 1.
\end{aligned}$$

The first monomial $\widetilde{x}_{j-1} \widetilde{x}_j$ in the second line of (6.8) gives a 2-fold suspension of $\widetilde{f}$. The second part $\widetilde{x}_{j-2} \cdot (...)$ consists of monomials of weighted degree > 1 if one associates to $\widetilde{x}_{j-1}$ and to $\widetilde{x}_j$ the degree $\frac{1}{2}$, because for $k = \alpha_j - 1$

$$\begin{aligned}
& a_{j+1} a_{j-1} \widetilde{w}_{j+1} = 1 - \widetilde{w}_{j-2} \text{ and} \\
& \widetilde{w}_{j-2} + \widetilde{w}_{j-1} + a_{j+1}(a_{j-1} - 1) \widetilde{w}_{j+1} \\
= \ & \frac{1}{2} + 1 - a_{j+1} \widetilde{w}_{j+1} = \frac{1}{2} + 1 - \frac{1 - \widetilde{w}_{j-2}}{a_{j-1}} > 1 \quad \text{because } a_{j-1} \geq 2.
\end{aligned}$$

(iv) One transforms a chain type singularity with $a_0 = 2$ with (ii) to a 1-fold suspension of a chain type singularity with one variable less. One repeats (ii) until one arrives either at the A_1-type *ihs* x_0^2 or at a chain type singularity with $a_0 \geq 3$. Then one repeats (iii) until one arrives at a chain type singularity with $a_0 \geq 3$, $a_1, ..., a_{\widetilde{m}} \geq 2$. Then $\mathrm{Sp}(\widetilde{p}(x)) = \mathrm{Sp}(p(x))$.

Theorem 6.33. *Consider a chain type singularity $f(x) = x_0^{a_0} + x_0 x_1^{a_1} + ... + x_{m-1} x_m^{a_m}$. The spectrum of the HOR matrix S in conjecture 6.30 (see definition 6.19 (c) for $\mathrm{Sp}(S)$) satisfies*

$$\mathrm{Sp}(S) = \mathrm{Sp}(f) - \frac{m-1}{2}. \tag{6.9}$$

Proof. For the A_1-type *ihs* x_0^2 $S = (1)$ and $\mathrm{Sp}(S) = (0)$ and $\mathrm{Sp}(f) = (-\frac{1}{2})$ and $m = 0$, so (6.9) holds. Because of this and the remarks 6.32 (ii)–(iv), it is sufficient to prove theorem 6.33 for the cases $a_0 \in \mathbb{Z}_{\geq 3}, a_1, ..., a_m \in \mathbb{Z}_{\geq 2}$. The spectrum $\mathrm{Sp}(f) = (\alpha_1(f), ..., \alpha_\mu(f))$ (with an arbitrary numbering) of a quasihomogeneous singularity with weights $w_0, ..., w_m \in \mathbb{Q} \cap (0,1)$ such that $\deg_w f = 1$ can be given in several ways:

(A) By the generating function

$$\sum_{j=1}^{\mu} t^{\alpha_j(f)+1} = \prod_{k=0}^{m} \frac{t - t^{w_k}}{t^{w_k} - 1}. \tag{6.10}$$

(B) If $m_1, ..., m_\mu \in \mathbb{C}[x]$ are weighted homogeneous polynomials which represent a basis of the Jacobi algebra then

$$\alpha_j(f) = -1 + \sum_{k=0}^{m} w_k + \deg_w m_j \quad \text{for } j = 1, ..., \mu. \tag{6.11}$$

Here (B) is more convenient than (A). Claim 1 is the first of four steps of the main part of the proof.

Step 1 = Claim 1. The following monomials represent a basis of the Jacobi algebra:

$$\begin{aligned}
&x_0^{b_0} x_1^{b_1} \cdot ... \cdot x_m^{b_m} \quad \text{with} \quad \begin{cases} 0 \leq b_j \leq a_j - 1 \text{ for} \\ \qquad j \in \{0, 1, ..., m-1\} \\ \text{and } 0 \leq b_m \leq a_m - 2, \end{cases} \\
&x_0^{b_0} x_1^{b_1} \cdot ... \cdot x_{m-2}^{b_{m-2}} x_m^{a_m - 1} \quad \text{with} \quad \begin{cases} 0 \leq b_j \leq a_j - 1 \text{ for} \\ \qquad j \in \{0, 1, ..., m-3\} \\ \text{and } 0 \leq b_{m-2} \leq a_{m-2} - 2, \end{cases} \\
&\qquad \vdots \\
&\text{for } m \equiv 0(2): \\
&x_0^{b_0} x_2^{a_2 - 1} x_4^{a_4 - 1} \cdot ... \cdot x_m^{a_m - 1} \quad \text{with} \quad 0 \leq b_0 \leq a_0 - 2, \\
&\text{for } m \equiv 1(2): \\
&x_0^{b_0} x_1^{b_1} x_3^{a_3 - 1} \cdot ... \cdot x_m^{a_m - 1} \quad \text{with} \quad \begin{cases} 0 \leq b_0 \leq a_0 - 1 \\ \text{and } 0 \leq b_1 \leq a_1 - 2 \end{cases} \\
&x_1^{a_1 - 1} x_3^{a_3 - 1} \cdot ... \cdot x_m^{a_m - 1}.
\end{aligned} \tag{6.12}$$

Their number is

$$\begin{aligned}\mu &= a_0 ... a_{m-1}(a_m - 1) + a_0 ... a_{m-3}(a_{m-2} - 1) \\ &\quad + ... + \begin{cases} a_0 - 1 & \text{for } m \equiv 0(2) \\ a_0(a_1 - 1) + 1 & \text{for } m \equiv 1(2) \end{cases}\end{aligned}$$

Therefore for claim 1 it is sufficient to prove that any monomial in $\mathbb{C}\{x\}$ is a linear combination of the monomials above and of an element of the Jacobi ideal $J_f = \left(\frac{\partial f}{\partial x_0}, ..., \frac{\partial f}{\partial x_m}\right)$. The generators $\frac{\partial f}{\partial x_j}$ of J_f are given in (6.5). Obviously also

$$x_{m-1}x_m^{a_m},\ x_{m-2}x_{m-1}^{a_{m-1}}, ..., x_1 x_2^{a_2},\ x_0 x_1^{a_1},\ x_0^{a_0}$$

are in J_f. Start with any monomial in $\mathbb{C}\{x\}$. Using $\frac{\partial f}{\partial x_{m-1}}, \frac{\partial f}{\partial x_{m-2}}, ..., \frac{\partial f}{\partial x_0}$ and $x_0^{a_0}$ (in this order), one can reduce it modulo J_f to 0 or to a monomial $x_0^{b_0} \cdot ... \cdot x_m^{b_m}$ with $0 \leq b_j \leq a_j - 1$ for all j.

If $b_m \leq a_m - 2$ stop here. Suppose $b_m = a_m - 1$. If $b_{m-1} \geq 1$ the monomial is in J_f. Suppose $b_{m-1} = 0$. If $b_{m-2} \leq a_{m-2} - 2$ stop here. Suppose $b_{m-2} = a_{m-2} - 1$. If $b_{m-3} \geq 1$, the monomial is modulo $\mathbb{C} \cdot \frac{f}{\partial x_{m-2}}$ congruent to a monomial $x_0^{\widetilde{b}_0} \cdot ... \cdot x_m^{\widetilde{b}_m}$ with $b_{m-1} \geq a_{m-1}$, $b_m = a_m - 1$, so it is in J_f. Suppose $b_{m-3} = 0$. The claim is proved by repeating these arguments. $(\square)$

Step 2. The second step is the definition of a directed graph G whose vertices are labelled by the monomials in (6.12). Before defining the directed edges, consider the following $m+1$ Laurent monomials in $\mathbb{C}[x_0^{\pm 1}, ..., x_m^{\pm 1}]$:

$$\begin{aligned}
\underline{x}^{\underline{g}(m)} &:= x_m^{-1}, \\
\underline{x}^{\underline{g}(m-1)} &:= x_{m-1}x_m^{a_m-2}, \\
\underline{x}^{\underline{g}(m-2)} &:= x_{m-2}^{-1}x_{m-1}^{-(a_{m-1}-1)}x_m^{a_m-1}, \\
\underline{x}^{\underline{g}(m-3)} &:= x_{m-3}^{1}x_{m-2}^{a_{m-2}-1}x_{m-1}^{-(a_{m-1}-1)}x_m^{a_m-2}, \\
\underline{x}^{\underline{g}(m-4)} &:= x_{m-4}^{-1}x_{m-3}^{-(a_{m-3}-1)}x_{m-2}^{a_{m-2}-1}x_{m-1}^{-(a_{m-1}-1)}x_m^{a_m-1}, \\
\underline{x}^{\underline{g}(m-5)} &:= x_{m-5}^{1}x_{m-4}^{a_{m-4}-1}x_{m-3}^{-(a_{m-3}-1)}x_{m-2}^{a_{m-2}-1}x_{m-1}^{-(a_{m-1}-1)}x_m^{a_m-2}, \\
&\ \vdots \\
&\text{for } m \equiv 0(2): \\
\underline{x}^{\underline{g}(0)} &:= x_0^{-1}x_1^{-(a_1-1)}...x_{m-4}^{a_{m-4}-1}x_{m-3}^{-(a_{m-3}-1)}x_{m-2}^{a_{m-2}-1}x_{m-1}^{-(a_{m-1}-1)}x_m^{a_m-1}, \\
&\text{for } m \equiv 1(2): \\
\underline{x}^{\underline{g}(0)} &:= x_0^{1}x_1^{a_1-1}...x_{m-4}^{a_{m-4}-1}x_{m-3}^{-(a_{m-3}-1)}x_{m-2}^{a_{m-2}-1}x_{m-1}^{-(a_{m-1}-1)}x_m^{a_m-2}.
\end{aligned}$$

Now an edge labelled by $\underline{g}(j)$ goes from $\underline{x}^{\underline{b}} = x_0^{b_0} \cdot ... \cdot x_m^{b_m}$ to $\underline{x}^{\underline{c}} = x_0^{c_0} \cdot ... \cdot x_m^{c_m}$ if $\underline{x}^{\underline{b}} \cdot \underline{x}^{\underline{g}(j)} = \underline{x}^{\underline{c}}$.

This defines a directed graph G with vertices labelled by the monomials in (6.12) and edges labelled by $\underline{g}(0),...,\underline{g}(m)$.

Claim 2. (a) The graph is a chain. If $m \equiv 0(2)$ it starts at $x_0^{a_0-1}x_2^{a_2-1}\cdot ... \cdot x_m^{a_m-2}$ and ends at $x_1^{a_1-1}x_3^{a_3-1}\cdot ... \cdot x_{m-1}^{a_{m-1}-1}$. If $m \equiv 1(2)$ it starts at $x_1^{a_1-1}x_3^{a_3-1}\cdot ... \cdot x_m^{a_m-1}$ and ends at $x_0^{a_0-1}x_2^{a_2-1}\cdot ... \cdot x_{m-1}^{a_{m-1}-1}$.

(b) The weight of the Laurent monomial $\underline{x}^{\underline{g}(j)}$ is

$$\deg_w \underline{x}^{\underline{g}(j)} = \begin{cases} -w_m & \text{if } j \equiv m(2), \\ 1-2w_m & \text{if } j \equiv m+1(2). \end{cases} \tag{6.13}$$

Proof of claim 2. (a) Careful inspection of the set of monomials in (6.12).

(b) In both cases use $w_{k-1} + a_k w_k = 1$. □

Step 3. The third step consists in making precise the recipe 6.11 in the case of the HOR matrix S respectively its polyonomial $p(x)$ in conjecture 6.30. Because $p_0 = (-1)^{m+1}$, the case $m \equiv 0(2)$ is the case $k = 1$ in recipe 6.11, and the case $m \equiv 1(2)$ is the case $k = 2$ in recipe 6.11. Then

$$\begin{aligned} \alpha_j &= \mu(\beta_j - \gamma_j) \quad \text{for } j = 1,...,\mu \\ \mu \cdot \gamma_j &= \begin{cases} j - \frac{1}{2} & \text{in the case } m \equiv 0(2) \\ j-1 & \text{in the case } m \equiv 1(2) \end{cases} \\ \mu \cdot \underline{\beta} &= \frac{\mu}{r_m}(\delta_1,...,\delta_\mu) \quad \text{with } \delta_{j+1} = \delta_j + 1 \text{ or } \delta_{j+1} = \delta_j + 2, \\ & \quad \{\delta_1,...,\delta_\mu\} \subset \{0,1,2,...,r_m - 1\}, \text{ namely} \end{aligned}$$

$$\prod_{j=1}^{\mu}(x - e^{-2\pi i \delta_j / r_m}) = p(x) = \prod_{l=0}^{m}(x^{r_l} - 1)^{(-1)^{m-l}}.$$

$\alpha_1,...,\alpha_\mu$ denote now the spectral numbers in $\mathrm{Sp}(S)$ with the order from recipe 6.11. We find:

$$\begin{array}{llll} \text{If } \delta_{j+1} = \delta_j + 1 & \text{then} & \alpha_{j+1} - \alpha_j = \frac{\mu}{r_m} - 1 = \frac{\mu - r_m}{r_m} = \frac{-\mu_{m-1}}{r_m} = -w_m \\ \text{If } \delta_{j+1} = \delta_j + 2 & \text{then} & \alpha_{j+1} - \alpha_j = 2\frac{\mu}{r_m} - 1 = 1 - 2w_m. \end{array} \tag{6.14}$$

We have to show that $\alpha_1,...,\alpha_\mu$ coincide up to the shift by $\frac{m-1}{2}$ with the spectral numbers of f which are given by (6.11) and (6.12).

Step 4 = Claim 3. Denote the monomials in (6.12) by $m_1,...,m_\mu$ with the numbering as the chain G prescribes it. Denote $\alpha_1(f),...,\alpha_\mu(f)$ according to (6.11). Then

$$\alpha_j = \alpha_j(f) - \frac{m-1}{2}, \quad \text{so } \operatorname{Sp}(S) = \operatorname{Sp}(f) - \frac{m-1}{2}.$$

Proof of claim 3. If the vertices m_j and m_{j+1} in the chain G are connected by an edge of type $\underline{g}(l)$ then

$$\begin{aligned} \alpha_{j+1}(f) - \alpha_j(f) &= \deg_w m_{j+1} - \deg_w m_j = \deg_w \underline{x}^{\underline{g}(l)} \\ &= \begin{cases} -w_m & \text{if } l \equiv m(2) \\ 1 - 2w_m & \text{if } l \equiv m+1(2) \end{cases} \end{aligned}$$

Therefore it rests to see two points:

$$\begin{aligned} \alpha_1(f) &= \frac{m-1}{2} + \alpha_1, \\ \delta_{j+1} = \delta_j + 2 &\iff \text{the edge from } m_j \text{ to } m_{j+1} \text{ is of type} \\ &\qquad \gamma_l \text{ with } l \equiv m+1(2). \end{aligned}$$

We carry out the first point in both cases $m \equiv 0(2)$ and $m \equiv 1(2)$ and leave the second point to the reader.

The case $m \equiv 0(2)$: Then

$$\begin{aligned} \alpha_1 &= \mu(\beta_1 - \gamma_1) = \frac{\mu}{r_m} - \frac{1}{2} = \frac{1}{2} - w_m, \\ \alpha_1(f) &= -1 + \sum_{k=0}^{m} w_k + \deg_w x_0^{a_0-1} x_2^{a_2-1} ... x_m^{a_m-2} \\ &= -1 + \deg_w x_0^{a_0} x_1 x_2^{a_2} x_3 ... x_{m-1} x_m^{a_m-1} \\ &= \frac{m}{2} - w_m = \frac{m-1}{2} + \alpha_1. \end{aligned}$$

The case $m \equiv 1(2)$: Then

$$\begin{aligned} \alpha_1 &= \mu(\beta_1 - \gamma_1) = \frac{0}{r_m} - 0 = 0, \\ \alpha_1(f) &= -1 + \sum_{k=0}^{m} w_k + \deg_w x_1^{a_1-1} x_3 a_3 - 1 ... x_m^{a_m-1} \\ &= -1 + \deg_w x_0 x_1^{a_1} x_2 x_3^{a_3} ... x_{m-2}^{a_{m-2}} x_{m-1} x_m^{a_m} \\ &= \frac{m-1}{2} = \frac{m-1}{2} + \alpha_1. \qquad (\Box) \end{aligned}$$

This finishes the proof of theorem 6.33 □

6.7 Remarks and speculations

In the following three subsections, we offer a critical discussion of some arguments in [CV93] with a counterexample, we comment on flat vector bundles and Thom-Sebastiani formulas.

6.7.1 Historical remarks

HOR are the initials of the authors Horocholyn, Orlik and Randell of [Ho17] and [OR77]. In [Ho17, ch. 2] half of the matrices in $\bigcup_{k=1,2} T_{\mathrm{HOR}k}(n,\mathbb{R})$ were studied and the crucial equation (2.10) was proved for them. In [OR77, (4.1) Conjecture] it was conjectured that special matrices S in $\bigcup_{k=1,2} T_{\mathrm{HOR}k}(n,\mathbb{Z})$ turn up as Stokes matrices of the chain type singularities (sections 6.5 and 6.6). The main result Theorem (2.11) in [OR77] is that $(-1)^k S^{-1}S^t$ is a monodromy matrix for such an *ihs*. See the beginning of section 6.3 for [Ho17].

The conjecture of Orlik & Randell and theorem 6.20 (a) would imply that the matrix of the monodromy for this distinguished basis is $(R^{mat}_{(k)})^\mu$ with $k \equiv m(2)$ (remark 6.31 (iii)). The main result theorem (2.11) in [OR77] says that the matrix of the monodromy for some basis of the Milnor lattice is this matrix. The evidence we presented with theorem 6.20, that up to a shift of $\frac{m-1}{2}$ we have the correct spectrum of an *ihs*, $\mathrm{Sp}(S) = \mathrm{Sp}(f) - \frac{m-1}{2}$ is positive for conjecture 2.23 Of course, the evidence would be stronger if somebody would prove conjecture (4.1) in [OR77].

6.7.2 Cecotti & Vafa arguments for conjecture 2.23

The arguments in [CV93] concern the case of M-tame functions respectively Landau-Ginzburg models. They are given precisely in [CV93, pages 589 and 590]. They use tt^*-geometry.

Indeed, any matrix $S \in T(n,\mathbb{R})$ gives together with arbitrary values $(u_1,...,u_n)$ with $u_i \neq u_j$ for $i \neq j$ and a sufficiently generic value $\xi \in S^1$ rise to a $TERP$ structure in the sense of [He03], more precisely, it gives a semisimple mixed $TERP$ structure of weight 1 [HS07, Lemma 10.1], which we call now $TERP(S,(u_1,...,u_n),\xi)$.

But for conjecture 2.23, Cecotti and Vafa want to consider a limit $TERP$ structure for $(u_1,...,u_n) \to (0,...,0)$. This should be the UV limit (ultraviolet limit). They *assume* that it exists and that it has good properties, especially it should be pure and polarized and have the correct spectrum. In [CV93, ch. 5, page 601], they *conclude* that the UV limit is well defined and nondegenerate (in a certain sense), if $S^{-1}S^t$ is semisimple.

We agree neither with the *assumption* nor with the *conclusion*. The following example serves for both as a counterexample.

Therefore we do not consider conjecture 2.23 (for the M-tame case respectively the Landau-Ginzburg models) as proved in [CV93]. Though we do believe that tt^*-geometry is a promising road. But a much more precise analysis of the limit behavior seems to be needed.

Consider a family of exceptional unimodal singularities, e.g. the family E_{12}:

$$f_{t_\mu}(x,y) = x^3 + y^7 + t_\mu \cdot xy^5 \quad \text{with } \mu = 12. \tag{6.1}$$

f_0 is quasihomogeneous of weighted degree 1 with respect to the weights $(w_x, w_y) = (\frac{1}{3}, \frac{1}{7})$, and f_{t_μ} for $t_\mu \neq 0$ is semiquasihomogeneous.

The $TERP$ structures $TERP(f_{t_\mu})$ were studied in [He03, 8.3 (C)]: There is a bound $r_2 \in \mathbb{R}_{>0}$ such that $TERP(f_{t_\mu})$ is not pure for $|t_\mu| = r_2$, it is pure and polarized for $|t_\mu| < r_2$, and it is pure, but not polarized for $|t_\mu| > r_2$. The spectral numbers (from Steenbrink's MHS) are called $\alpha_1, ..., \alpha_\mu$ and satisfy here

$$\begin{gathered}\alpha_j + \alpha_{\mu+1-j} = 0,\\ \alpha_1 = \frac{-11}{21} < \frac{-1}{2} < \alpha_2 = \frac{-8}{21} < ... < \alpha_{\mu-1} = \frac{8}{21} < \frac{1}{2} < \frac{11}{21} = \alpha_\mu.\end{gathered} \tag{6.2}$$

The eigenvalues of the supersymmetric index $\mathcal{Q}$ are for $|t_\mu| \neq r_2$

$$\alpha_2, ..., \alpha_{\mu-1} \text{ and } \pm\Big(1 - \frac{|t_\mu|^2}{r_2^2}\Big)^{-1}\Big(\alpha_1 - \frac{|t_\mu|^2}{r_2^2}(-1-\alpha_1)\Big). \tag{6.3}$$

The last two eigenvalues of $\mathcal{Q}$ tend for $|t_\mu| \to 0$ to $\pm\alpha_1 = \mp\frac{11}{21}$ and for $|t_\mu| \to \infty$ to $\pm(-1-\alpha_1) = \mp\frac{10}{21}$.

Now consider a universal unfolding

$$F_t(x,y) = f_{t_\mu}(x,y) + \sum_{j=1}^{\mu-1} t_j m_j, \quad t \in M \subset \mathbb{C}^\mu, \tag{6.4}$$

for suitable monomials m_j with weighted degree $\deg_w(m_j) < 1$. Here $M \subset \mathbb{C}^\mu$ is an open set which contains $\mathbb{C}^{\mu-1} \times \{0\} \cup \{(0,...,0)\} \times \mathbb{C}$ and which is invariant under the flow of the Euler field $E = \sum_{j=1}^{\mu} \deg_w(t_j) \cdot t_j \frac{\partial}{\partial t_j}$.

Choose $\xi \in S^1$ and choose for any $(u_1, ..., u_\mu) \in \mathbb{C}^\mu$ with $\mathrm{Re}\Big(\frac{u_i - u_j}{\xi}\Big) \neq 0$ for $i \neq j$ a special distinguished system of paths: They shall go straight in the direction ξ to $\partial\Delta_\eta$ and then turn on $\partial\Delta_\eta$ to $\xi\cdot\eta$. The set

$$\begin{aligned}\{t\in M \quad | \quad & \text{the critical values } u_1,...,u_\mu \text{ of } F_t \text{ satisfy} \\ & \mathrm{Re}\Big(\frac{u_i-u_j}{\xi}\Big)\neq 0 \text{ for } i\neq j\}\end{aligned} \tag{6.5}$$

consists of finitely many regions, the Stokes regions. Each Stokes region gives one $G_{sign,\mu}$-orbit of Stokes matrices S. For t in one region

$$TERP(F_t)=TERP(S,(u_1,...,u_\mu),\xi), \tag{6.6}$$

and rescaling $(u_1,...,u_\mu)$ to $(r\cdot u_1,...,r\cdot u_\mu)$ with $r>0, r\to 0$, corresponds to moving t along $-\mathrm{Re}\,E$. There are now two severe problems.

(I) For $t\in M$ in one region $TERP(F_t)$ tends to $TERP(f_0)$ only if $t_\mu=0$. If $t_\mu\neq 0$ then for $r\to 0$ $TERP(F_t)$ approximates $TERP(f_{t_\mu})$ for larger and larger t_μ, so it will become pure, but not polarized, and the eigenvalues of its supersymmetric index $\mathcal{Q}$ will tend to $\alpha_2,...,\alpha_{\mu-1},\pm(-1-\alpha_1)$.

(II) The $\mathrm{Br}_\mu\ltimes G_{sign,\mu}$-orbit of all Stokes matrices is infinite [Eb16]. The $G_{sign,\mu}$-orbits of the Stokes matrices from the finitely many Stokes regions in M form only a finite subset. For S not in this subset it is not at all clear how $TERP(S,(u_1,...,u_\mu),\xi)$ will behave for $r\to 0$.

Both problems show that the assumption and the conclusion about existence and good properties of the UV limit are not justified in the generality in which they are claimed in [CV93].

6.7.3 Harmonic vector bundles

We hope that the conjectures 2.21, 2.22 and 2.23 are true and will be proved in the future. The special cases of the HOR matrices made crucial use of the formulas (6.20) $(-1)^k\cdot S^{-1}S^t=R^{mat}_{(k)}(S)^n$ for $k\in\{1,2\}$. They are special cases of the formulas (6.12) $(-1)^k\cdot S^{-1}S^t=R^{mat}_{(k1)}\circ...\circ R^{mat}_{(kn)}$. Here the matrices $R^{mat}_{(kj)}$ are obtained by a certain twist from matrices for Picard-Lefschetz transformations, and they are companion matrices (remark 6.18). We hope that the formulas (6.12) will be useful for an approach to the conjectures 2.21, 2.22 and 2.23.

Certainly, it will also be useful to consider the flat vector bundle on $\mathbb{C}-\{u_1,...,u_n\}$ of rank n whose monodromy is given by these matrices $R^{mat}_{(kj)}$ at u_j (for $j\in\{1,...,n\}$) and by $(-1)^kS^{-t}S$ at ∞. The vector bundle whose monodromy is given by the Picard-Lefschetz transformations and $(-1)^kS^{-t}S$ is very familiar, it arises as homology bundle of a suitable function with A_1-type *ihs* only. We hope that the local monodromies given by the companion matrices R^{mat}_{kj} will become useful beyond the special case of HOR matrices.

In the special case of HOR matrices, the flat bundle decomposes because of (6.20) into flat subbundles, for each eigenvalue κ of $R^{mat}_{(k)}(S)$ one. In the semisimple case, these are flat line bundles. Then one can understand the $\underline{\beta}$ and $Sp(S)$ in terms of natural holomorphic extensions of these line bundles on $\mathbb{C}-\{u_1,...,u_n\}$ to $\mathbb{P}^1\mathbb{C}$.

But how this observation might extend to the general case of arbitrary matrices $S\in T(n,\mathbb{R})$ is not clear to us. Possibly work on harmonic bundles, tame or wild at $\{u_1,...,u_n,\infty\}$, by Biquard, Boalch, Mochizuki and Sabbah might be useful. And this might have connections to the $TERP$ structures.

6.7.4 Thom-Sebastiani formulas

In the case of *ihs*, an important technique for obtaining new ihs is, to consider the sum $f(x_0,...,x_m)+g(x_{m+1},...,x_{m+n+1})$ of two *ihs* f and g in different variables. This is discussed in [AGV88, I.2.7] and reviewed (in notations closer to this thesis) in [GH17]. There is a canonical isomorphism

$$\begin{aligned} \Phi: Ml(f+g,1) &\xrightarrow{\cong} Ml(f,1)\otimes Ml(g,1), & (6.7)\\ \text{with } M(f+g) &\cong M(f)\otimes M(g) & (6.8)\\ \text{and } L^{hnor}(f+g) &\cong L^{hnor}(f)\otimes L^{hnor}(g). & (6.9)\end{aligned}$$

If $\underline{\delta}=(\delta_1,...,\delta_{\mu(f)})$ and $\underline{\gamma}=(\gamma_1,...,\gamma_{\mu(g)})$ are distinguished bases of f and g with Stokes matrices $S(f)$ and $S(g)$, then

$$\Phi^{-1}(\delta_1\otimes\gamma_1,...,\delta_1\otimes\gamma_{\mu(g)},\delta_2\otimes\gamma_1,...,\delta_2\otimes\gamma_{\mu(g)},...,\delta_{\mu(f)}\otimes\gamma_1,...,\delta_{\mu(f)}\otimes\gamma_{\mu(g)})$$

is a distinguished basis of $Ml(f+g,1)$, that means, one takes the vanishing cycles $\Phi^{-1}(\delta_i\otimes\gamma_j)$ in the lexicographic order. Then by (6.24) and (6.9), the matrix

$$S(f+g)=S(f)\otimes S(g) \tag{6.10}$$

(where the tensor product is defined so that it fits to the lexicographic order) is the Stokes matrix of this distinguished basis.

In [SS85, ch. 8] a Thom-Sebastiani for Steenbrink's mixed Hodge structure is stated. It is fine if the monodromy is semisimple, but it needs a correction in the general case. That correction is an interesting and nontrivial twist (corollary 5.33), which comes from a Fourier-

Laplace transformation. Anyway, the resulting Thom-Sebastiani formula in [SS85] for the spectral pairs of f, g and $f+g$ is correct.

The set of HOR matrices is not invariant under the tensor product of matrices. It might be a good idea to check whether there are natural modifications for the recipe how the HOR matrices give rise to spectral numbers, which are compatible with the Thom-Sebastiani formulas.

7 Classification in the general context

This chapter serves two different purposes. As thus, we have structured the sections accordingly.

Section 7.1 is based on conjecture 6.30. If this conjecture were right (see section 7.2 for some comments on it) then we would have a particular class of CDDs for chain type singularities. Using this we conjecturally calculate their spectrum, based on the recipe in chapter 6. Now, turns out, that not all of the (conjectured) CDDs that fulfill the variance inequality come from a chain type singularity. But a closer look at the higher Bernoulli moments makes clear that the "spread" of all Bernoulli moments in those cases is the hint to separating the "wrong" cases. We provide bounds that work up to $\mu = 30$ but do not generalize properly, though we conjecture that a proper generalization exists.

Section 7.2 contains a list of the unproven conjectures with comments, merely as a collection, which, if proven, would complete the classification based on Stokes matrices and Bernoulli moments.

7.1 Classification with bounds on higher moments

The main result of this section is, under the assumption that conjecture 6.30 is correct, that an integral HOR matrix, with μ up to 30, is the matrix of an *ihs*, if the Hertling variance inequality is satisfied and certain bounds on the higher moments hold. The result is verified via (computer-based) calculations. We first recall higher Bernoulli moments, explain the terminology of *cyclo products* and then formulate the main result, theorem 7.2. The proof involves computer calculations checking long lists of cyclo products. Of course, we do not provide the complete list but rather data that illustrates the point made here: higher Bernoulli moments matter.

Bernoulli moments for the spectrum of an *ihs*, were introduced and investigated in [BH04]. Let $\alpha_1 \leq ... \leq \alpha_\mu$ be the ordered spectral numbers of an *ihs* f with $\alpha_i + \alpha_{\mu+1-i} = n-1$. To study their distribution, we can look at the numbers

$$V_{2k}(f) := \sum_{i=1}^{\mu}(\alpha_i - \frac{n-1}{2})^{2k}, k \geq 0.$$

In those terms, the Hertling inequality 2.4 becomes

$$V_2(f) \leq V_0(f) \cdot \frac{w(f)}{12}.$$

The numbers $V_{2k}(f)$ or simply V_{2k}, are called higher moments. The *Bernoulli moments* are certain linear combinations of these higher moments. We list some higher Bernoulli moments, given an *ihs* f, its higher moments $V_0, V_2, ...$ and its spectral width $w(f) = \alpha_\mu - \alpha_1$. The general definition of Bernoulli moments is contained e.g. in [BH04, p.2-3].

$$\begin{aligned}
\Gamma_0 &:= V_0 \\
\Gamma_2 &:= V_2 - V_0 \cdot \frac{1}{12} \cdot w(f) \\
\Gamma_4 &:= V_4 - V_2 \cdot \frac{1}{2} \cdot w(f) + V_0(\frac{1}{120} w(f) + \frac{1}{48}(w(f))^2) \\
\Gamma_6 &:= V_6 - V_4 \cdot \frac{5}{4} \cdot w(f) + V_2 \cdot (\frac{1}{8} w(f) + \frac{5}{16} w(f)) \\
&\quad - V_0 \cdot (\frac{1}{252} w(f) + \frac{1}{96}(w(f))^2 + \frac{5}{576}(w(f))^3)
\end{aligned}$$

The zeroth Bernoulli moment is always equal to μ, and the Hertling inequality then simply becomes $\Gamma_2 \leq 0$. The aforementioned infinite series of inequalities is

Conjecture 7.1. *([BH04, Conjecture 1.2.] strong form) Let f be an* ihs*. Then for all $k \in \mathbb{Z}_{\geq 0}$ we have*

$$(-1)^k \Gamma_{2k} \geq 0.$$

Equally, we can define these data if instead of the spectrum of an *ihs* we have the spectrum of a matrix $S \in T(n, \mathbb{R})$. We simply denote those by $\Gamma_{2k}(S)$, $k \geq 0$.

begindefinition

Let $p(x) \in \mathbb{R}[x]$ be the product of cyclotomic polynomials. We call those polynomials **cyclo products**. The maps $S^{(1)}, S^{(2)}$ (6.19) take a cyclo product to a matrix in $T(\mu, \mathbb{R})$. We usually will denote the degree of a cyclo product by μ. And we write $p(x) = \Pi_{j=1}^{\mu}(x-\lambda_j) = \sum_{j=0}^{\mu} p_j x^j$.

Cyclo products appear as characteristic polynomials of the monodromy of the Seifert form pairs belonging to *ihs*. They have a couple of important properties.

Remarks 7.1. (i) We have $p(x) \in T^{pol}_{\mathrm{HOR1}} \cup T^{pol}_{\mathrm{HOR2}}$ and $(-1)^n p(-x) \in T^{pol}_{\mathrm{HOR1}} \cup T^{pol}_{\mathrm{HOR2}}$.

(ii) The matrices $S^{(k)}(p)$ belong to $T(\mu, \mathbb{Z})$. As such, it makes sense to say: the associated CDD (definition 2.17) of $S^{(k)}(p)$ is connected.

Theorem 7.2. *Assume that conjecture 6.30 is right. Take a cyclo product p with $\mu \leq 30$. Then the matrix $S := S^{(k)}(\tilde{p}) \in T_{HOR1}(\mu, \mathbb{Z}) \cup T_{HOR2}(\mu, \mathbb{Z})$ belongs to an* ihs *(specifically, to a* chain type singularity*) if the following conditions hold:*

(1) (trace condition for chain-types) $\sum_{j=1}^{\mu} \lambda^{\mu} = -p_0$.
(2) $\Gamma_4(S) \leq \frac{1}{30} \cdot \mu \cdot w(S) \cdot \frac{1}{8}$.
(3) $-\Gamma_6(S) \leq \frac{1}{42} \cdot \mu \cdot w(S) \cdot \frac{13}{96}$.

If these three conditions are true, then we can construct a normal form for the *ihs*.

Remark 7.3. We do not want to include $\Gamma_2(f) = 0$, although it would be true. We also know that the bounds (2) and (3) are too sharp for singularities which are not quasihomogeneous. But what we want to suggest is, there might be a smart way to generalize these bounds to the complete series of higher Bernoulli moments and include the case of non-quasihomogeneous singularities.

Let us first note that those conditions are true for quasihomogeneous singularities.

Lemma 7.4. *Let $f(x_0, ..., x_n)$ be a quasihomogeneous singularity with weights $w_0, ..., w_n \in (0, \frac{1}{2}]$, then*

$$\Gamma_4(f) \leq \frac{1}{30} \cdot \mu(f) \cdot w(f) \cdot \frac{1}{8}$$
$$-\Gamma_6(f) \leq \frac{1}{42} \cdot \mu(f) \cdot w(f) \cdot \frac{13}{96}.$$

Proof. For the quasihomogeneous singularity case we have formulas for the first Bernoulli moments in terms of their weights, see [BH04, Theorem 5.4]. Notice also that $\sum_{i=0}^{n}(\frac{1}{2} - w_i) = \frac{1}{2}(\alpha_n - \alpha_1) = \frac{1}{2} \cdot w(f)$ holds.

(Γ_4). For the 4th Bernoulli moment, we have

$$\Gamma_4(f) = \frac{1}{30} \cdot \mu(f) \cdot \sum_{i=0}^{n} (\frac{1}{2} - w_i) w_i (1 - w_i).$$

The term $w_i(1-w_i)$ is smaller than $\frac{1}{4}$, for $w_i \neq \frac{1}{2}$. And its maximal value is $\frac{1}{2}$ at $w_i = \frac{1}{2}$. So we have

$$\begin{aligned}
\Gamma_4(f) &= \frac{1}{30} \cdot \mu(f) \cdot \sum_{i=0}^{n} (\frac{1}{2} - w_i) w_i (1 - w_i) \\
&\leq \frac{1}{30} \cdot \mu(f) \cdot \sum_{i=0}^{n} (\frac{1}{2} - w_i) \cdot \frac{1}{4} \\
&= \frac{1}{30} \cdot \mu \cdot w(f) \cdot \frac{1}{8}.
\end{aligned}$$

(Γ_6). For the 6th Bernoulli moment, we have

$$-\Gamma_6(f) = \frac{1}{42} \cdot \mu(f) \cdot \sum_{i=0}^{n} (\frac{1}{2} - w_i) w_i (1 - w_i)(\frac{4}{3} - w_i(1 - w_i)).$$

Set $t = w_i(1-w_i)$. The term $t \cdot (\frac{4}{3} - t)$ has its maximal value of $\frac{13}{48}$ at $t = \frac{1}{4}$ for $t \in (0, \frac{1}{4}]$. Thus

$$\begin{aligned}
-\Gamma_6(f) &= \frac{1}{42} \cdot \mu(f) \cdot \sum_{i=0}^{n} (\frac{1}{2} - w_i) w_i (1 - w_i)(\frac{4}{3} - w_i(1 - w_i)) \\
&\leq \frac{1}{42} \cdot \mu(f) \cdot \sum_{i=0}^{n} (\frac{1}{2} - w_i) \cdot \frac{13}{48} \\
&= \frac{1}{42} \cdot \mu \cdot w(f) \cdot \frac{13}{96}.
\end{aligned}$$

□

Definition 7.5. *A polynomial $p(x) \in \mathbb{C}[x]$ stems from a chain type singularity if there are numbers $m \in \mathbb{Z}_{\geq 0}$,*

$a_0 \in \mathbb{Z}_{\geq 3}, a_1, ..., a_m \in \mathbb{Z}_{\geq 2}$ with

$$\begin{aligned}
r_{-1} &:= 1 & r_0 &:= a_0, r_1 := a_0 \cdot a_1, ..., r_m := a_0 \cdot ... \cdot a_m \\
div(p(x)) &= & &\sum_{j=0}^{m} (-1)^{m-j} \cdot \Lambda_{r_j}
\end{aligned}$$

where $\operatorname{div}(p(x))$ is the divisor of $p(x) \in \mathbb{C}[x]$ and $\Lambda_m := \operatorname{div}(x^m - 1)$.

Remark 7.6. Table 7.1 below contains cyclo products which stem from chain type singularities. In that case, we can calculate the divisor of a cyclo product $p(x)$. We then get the series $r_1, r_0, r_1, \dots$. We then can calculate $a_0, \dots, a_m$ and thus get a normal form $f(x_0, \dots, x_m) = x_0^{a_0} + x_0 x_1^{a_1} + \dots + x_{m-1} x_m^{a_m}$, that is a quasihomogeneous polynomial of chain type which has an isolated singularity at 0. According to [OR77, (2.11), (2.12)] this *ihs* has a monodromy action and $p(x)$ is the characteristic polynomial of $R^{mat}_{(k)}(S)$ at $S = S^{(k)}(p)$.

Computer calculations, without conditions (2) and (3). With those definitions and calculations in hand, we can iterate through all products of cyclotomic polynomials, which we call *cyclo products*, degree by degree. Let us call the degrees up to and including 21 the **lower Milnor numbers**. If μ is a lower Milnor number, then the condition $\Gamma_2 \geq 0$, and the trace condition are enough to identify the cyclo products that belong to a chain-type singularity. And indeed, for lower Milnor numbers, we are able to recover and identify by their normal forms, all singularities known with those Milnor numbers.

However, once we get to higher degrees, namely 22 and larger, we run into examples where we need both additional boundaries. The tables below start at 22 and end at degree 30. Table 7.1 contains polynomials which stem from a chain type singularity, and table 7.2 contains the ones which do not. The spectral numbers are assigned based on the recipe 6.11 (a)-(c).

Table 7.1:

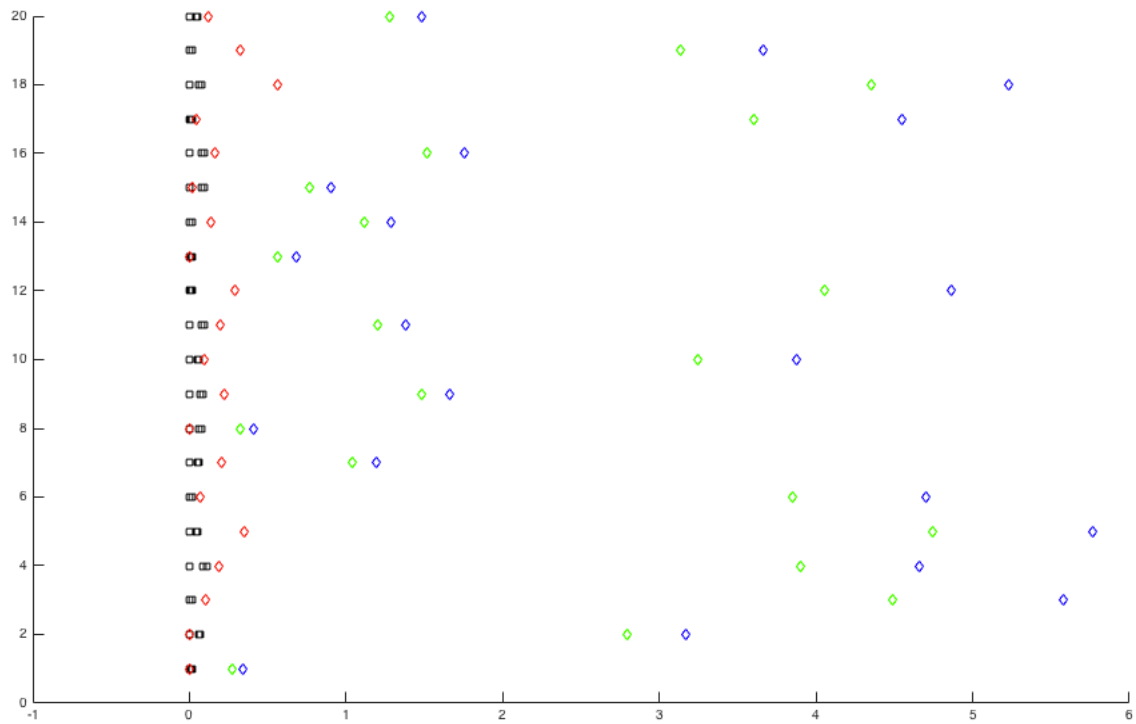
product representation, cyclo products	μ
Φ_{46}	22
$\Phi_1\Phi_2\Phi_3\Phi_7\Phi_{21}$	22
$\Phi_1\Phi_2\Phi_4\Phi_{28}$	22
$\Phi_1\Phi_2\Phi_6\Phi_{14}\Phi_{42}$	22
$\Phi_1\Phi_2\Phi_4\Phi_6\Phi_{12}\Phi_{36}$	22
$\Phi_1\Phi_2\Phi_4\Phi_6\Phi_8\Phi_{12}\Phi_{24}$	22
$\Phi_1\Phi_3\Phi_{33}$	23
$\Phi_1\Phi_4\Phi_{44}$	23
$\Phi_1\Phi_4\Phi_8\Phi_{32}$	23
$\Phi_2\Phi_4\Phi_8\Phi_{40}$	23
$\Phi_1\Phi_3\Phi_6\Phi_7\Phi_{21}$	23
$\Phi_1\Phi_3\Phi_4\Phi_6\Phi_{12}\Phi_{36}$	23
$\Phi_1\Phi_3\Phi_4\Phi_6\Phi_8\Phi_{12}\Phi_{24}$	23
$\Phi_5\Phi_{25}$	24
$\Phi_1\Phi_2\Phi_{23}$	23
$\Phi_5\Phi_8\Phi_{40}$	24
$\Phi_3\Phi_5\Phi_6\Phi_{15}\Phi_{30}$	24
$\Phi_1\Phi_9\Phi_{27}$	25
$\Phi_1\Phi_{13}\Phi_{26}$	25
$\Phi_1\Phi_{16}\Phi_{32}$	25
$\Phi_1\Phi_{16}\Phi_{48}$	25
$\Phi_2\Phi_{13}\Phi_{26}$	25
$\Phi_2\Phi_{16}\Phi_{32}$	25
$\Phi_2\Phi_{16}\Phi_{48}$	25
$\Phi_2\Phi_{18}\Phi_{54}$	25
$\Phi_1\Phi_9\Phi_{18}\Phi_{36}$	25
$\Phi_2\Phi_7\Phi_{14}\Phi_{28}$	25
$\Phi_2\Phi_4\Phi_{12}\Phi_{18}\Phi_{36}$	25
$\Phi_2\Phi_5\Phi_{10}\Phi_{15}\Phi_{30}$	25

product representation, cyclo products	μ
$\Phi_1\Phi_2\Phi_3\Phi_6\Phi_{10}\Phi_{15}\Phi_{30}$	26
$\Phi_1\Phi_2\Phi_5\Phi_8\Phi_{40}$	26
$\Phi_1\Phi_2\Phi_{10}\Phi_{50}$	26
$\Phi_4\Phi_6\Phi_{12}\Phi_{18}\Phi_{36}$	26
$\Phi_5\Phi_6\Phi_{10}\Phi_{15}\Phi_{30}$	26
$\Phi_6\Phi_9\Phi_{18}\Phi_{36}$	26
$\Phi_6\Phi_8\Phi_{54}$	26
$\Phi_6\Phi_{16}\Phi_{48}$	26
$\Phi_2\Phi_4\Phi_7\Phi_{14}\Phi_{28}$	27
$\Phi_2\Phi_4\Phi_9\Phi_{18}\Phi_{36}$	27
$\Phi_2\Phi_6\Phi_7\Phi_{14}\Phi_{42}$	27
$\Phi_2\Phi_4\Phi_{16}\Phi_{32}$	27
$\Phi_2\Phi_4\Phi_{16}\Phi_{48}$	27
$\Phi_2\Phi_4\Phi_{52}$	27
$\Phi_2\Phi_6\Phi_{78}$	27
$\Phi_1\Phi_2\Phi_4\Phi_6\Phi_{12}\Phi_{18}\Phi_{36}$	28
$\Phi_1\Phi_2\Phi_5\Phi_6\Phi_{10}\Phi_{15}\Phi_{30}$	28
$\Phi_1\Phi_2\Phi_6\Phi_9\Phi_{18}\Phi_{36}$	28
$\Phi_1\Phi_2\Phi_6\Phi_{18}\Phi_{54}$	28
$\Phi_1\Phi_2\Phi_6\Phi_{16}\Phi_{48}$	28
$\Phi_2\Phi_3\Phi_4\Phi_6\Phi_{12}\Phi_{18}\Phi_{36}$	29
$\Phi_2\Phi_3\Phi_5\Phi_6\Phi_{10}\Phi_{15}\Phi_{30}$	29
$\Phi_2\Phi_3\Phi_6\Phi_9\Phi_{18}\Phi_{36}$	29
$\Phi_2\Phi_3\Phi_6\Phi_{18}\Phi_{54}$	29
$\Phi_2\Phi_3\Phi_6\Phi_{16}\Phi_{48}$	29
$\Phi_2\Phi_3\Phi_6\Phi_{21}\Phi_{42}$	29
$\Phi_2\Phi_5\Phi_8\Phi_{10}\Phi_{40}$	29
$\Phi_2\Phi_5\Phi_{10}\Phi_{50}$	29
$\Phi_2\Phi_8\Phi_{16}\Phi_{32}$	29
$\Phi_2\Phi_8\Phi_{16}\Phi_{48}$	29
$\Phi_2\Phi_8\Phi_{56}$	29
$\Phi_2\Phi_{10}\Phi_{70}$	29
$\Phi_1\Phi_2\Phi_{70}$	29
$\Phi_1\Phi_2\Phi_{58}$	30
Φ_{62}	30

Table 7.2:

product representation, cyclo products	μ
$\Phi_1\Phi_2\Phi_3\Phi_4\Phi_8\Phi_{12}\Phi_{16}$	22
$\Phi_5\Phi_8\Phi_{15}\Phi_{30}$	24
$\Phi_1\Phi_6\Phi_{11}\Phi_{28}$	25
$\Phi_1\Phi_2\Phi_3\Phi_4\Phi_{10}\Phi_{12}\Phi_{21}$	26
$\Phi_1\Phi_2\Phi_6\Phi_8\Phi_{11}\Phi_{30}$	26
$\Phi_1\Phi_2\Phi_4\Phi_{10}\Phi_{22}\Phi_{30}$	26
$\Phi_1\Phi_2\Phi_{14}\Phi_{27}$	26
$\Phi_6\Phi_{24}\Phi_{48}$	26
$\Phi_{14}\Phi_{25}$	26
$\Phi_1\Phi_2^2\Phi_3\Phi_4\Phi_6\Phi_{10}\Phi_{14}\Phi_{20}$	27
$\Phi_1\Phi_2^2\Phi_4\Phi_{10}\Phi_{14}\Phi_{28}$	27
$\Phi_1\Phi_2^2\Phi_4\Phi_{10}\Phi_{16}\Phi_{22}$	27
$\Phi_2\Phi_3\Phi_6^2\Phi_{66}$	27
$\Phi_9\Phi_{10}\Phi_{27}$	28
$\Phi_{22}\Phi_{27}$	28
$\Phi_1\Phi_2^2\Phi_3\Phi_6^2\Phi_{10}\Phi_{15}\Phi_{30}$	29
$\Phi_1\Phi_2^2\Phi_3\Phi_6\Phi_9\Phi_{14}\Phi_{22}$	29
$\Phi_1\Phi_2^2\Phi_6\Phi_8\Phi_{16}\Phi_{26}$	29
$\Phi_2\Phi_3\Phi_4\Phi_6^2\Phi_{12}\Phi_{24}\Phi_{30}$	29
$\Phi_2\Phi_4\Phi_6\Phi_8\Phi_{12}\Phi_{24}\Phi_{30}$	29
$\Phi_2\Phi_3\Phi_6\Phi_{12}\Phi_{50}$	29
$\Phi_2\Phi_3\Phi_6\Phi_{15}\Phi_{16}\Phi_{30}$	29
$\Phi_2\Phi_3\Phi_6\Phi_{56}$	29
$\Phi_2\Phi_5\Phi_8\Phi_{10}\Phi_{15}\Phi_{30}$	29
$\Phi_1\Phi_2\Phi_3\Phi_6^2\Phi_{12}\Phi_{14}\Phi_{42}$	30

Computer calculations, with conditions (2) and (3). Looking at the distribution of higher Bernoulli moments, we can visualize the problematic higher Bernoulli moments. Here we have illustrated the case by visualizing the first 20 cyclo products from each table. They are simply numbered from $1-20$. The y-axis denotes their index. The black squares indicate $\Gamma_2, \Gamma_4, \Gamma_6$ for table 7.1, the good cases. The red diamonds indicate Γ_2, the blue ones Γ_4, the green ones Γ_6, for table 7.2. Those are the bad cases. Of course for all good cases, $\Gamma_2 = 0$.



This picture led us to conclude, that higher Bernoulli moments, for higher μ are the key to distinguish matrices in $S \in T(n,\mathbb{Z})$ which belong to *ihs* from others. Using conditions (2) and (3), we can exclude all but the polynomials in table 7.1 and thus prove theorem 7.2. $\square$

7.2 List of unproven conjectures

Here we list the conjectures either made in this thesis and related articles, or in previous papers by the authors Cecotti & Vafa, Orlik & Randell or Hertling. We make some comments and notes. This section serves merely as a "collection".

7.2.1 Hertling variance inequality

Conjecture 7.2. *(Conjecture 2.4, [He02, Conjecture 14.8]) Let $f(x_0, ..., x_m)$ be an* ihs *with Milnor number n. Denote the spectral numbers by $\alpha_1 \leq ... \leq \alpha_n$. Then*

$$\mathrm{Var}(Sp(f)) := \frac{1}{n}\sum_{j=1}^{n}\left(\alpha_j - \frac{m-1}{2}\right)^2 \leq \frac{\alpha_n - \alpha_1}{12}. \tag{7.1}$$

7.2.2 OR conjecture on chain type Stokes matrices

The next conjecture is from [OR77, Conjecture (4.1)], more evidence has amounted since then, we list it below.

Conjecture 7.7. *(Conjecture 6.30, [OR77, Conjecture (4.1)]) The chain type singularity $f = x_0^{a_0} + x_0 x_1^{a_1} + ... + x_{m-1} x_m^{a_m}$ has a distinguished basis whose Stokes matrix S is the HOR matrix S (definition 6.19 (a)) with polynomial*

$$p(x) = x^\mu + p_{\mu-1}x^{\mu-1} + ... + p_0 = \prod_{k=-1}^{m} (x^{r_k} - 1)^{(-1)^{m-k}}. \tag{7.2}$$

Chapter 6 provide much more evidence for this conjecture in combination with the works of Horocholyn. We have calculated a few cases in lower Milnor numbers for the series A_n, D_n and found the conjecture to be true for them. We consider that of course weak evidence as these cases are simple, positive definite and thus have finite braid group orbits, as opposed to the generic conjecture.

How could one go about proving this conjecture: We already have a canonical choice of Jacobi algebra basis. What is missing is a canonical CDD (Coxeter Dynkin diagram). With a canonical CDD for a chain type singularity (by our remark 6.32, it would suffice to consider the ones with leading exponent ≥ 3) one could try to use a series of braids to transform it into an upper triangular upper Toeplitz matrix, a HOR matrix. One then could investigate the braid group operation on the HOR matrices, and conclude. One important question in this context would be, can any two matrices in the HOR spaces be braid equivalent?

A second possibility is to prove some kind of transitivity result, but that, judged based on the works of Kluitmann, Voigt, Deligne seems to be harder.

7.2.3 The CV natural spectral recipe

Conjecture 7.8. *(Proposed by Cecotti & Vafa in [CV93], here 2.10)*
For any matrix $S \in T(n, \mathbb{R})$, there is a natural procedure (in the sense of recipe (2.9)) which leads to a well-defined spectrum $Sp(S) = (\alpha_1, ..., \alpha_n) \in \mathbb{R}^n$ with $\alpha_1 \leq ... \leq \alpha_n$ and $\alpha_i + \alpha_{n+1-i} = 0$ and such that $e^{-2\pi i \alpha_1}, ..., e^{-2\pi i \alpha_n}$ are the eigenvalues of the matrix $S^{-1}S^t$. In the case of the Stokes matrix of a distinguished basis of an isolated hypersurface singularity $f(x_0, ..., x_m)$ it coincides with the shift $Sp(f) - \frac{m-1}{2}$ of Steenbrink's spectrum $Sp(f)$.

This conjecture can be proved, by proving the next three, more specific, conjectures.

7.2.4 HOR matrices are enough

The following conjecture has at its basis the idea that the HOR spaces might serve as a skeleton for $T(n,\mathbb{R})$. To assign a spectrum and spectral pairs, take any $S \in T(n,\mathbb{R})$, go to the HOR matrix inside the same eigenvalue spectrum and take its spectral numbers and pairs.

Conjecture 7.9. *(Conjecture 2.21) (a) $T_{\mathrm{HOR1}}(n,\mathbb{R})$ intersects each eigenvalue stratum in $T(n,\mathbb{R})$.*

(b) If $S_1, S_2 \in \bigcup_{k=1,2} T_{\mathrm{HOR}k}(n,\mathbb{R})$ are in the same eigenvalue stratum of $T(n,\mathbb{R})$ then $\mathrm{Sp}(S_1) = \mathrm{Sp}(S_2)$.

(c) If $S_1, S_2 \in \bigcup_{k=1,2} T_{\mathrm{HOR}k}(n,\mathbb{R})$ are in the same Seifert form stratum of $T(n,\mathbb{R})$ then Spp$(S_1) = \mathrm{Spp}(S_2)$.

7.2.5 HOR spectrum lifts

Spectral pairs carry more information than just a spectrum. In the cases $n = 2, 3$ we know we can simply lift everything up to all of $T(n,\mathbb{R})$.

Conjecture 7.10. *(Conjecture 2.22) Also for the matrices S in the Seifert form strata which are not met by $\bigcup_{k=1,2} T_{\mathrm{HOR}k}(n,\mathbb{R})$,* $\mathrm{Sp}(S)$ *lifts in a natural way to* $\mathrm{Spp}(S)$.

7.2.6 HOR spectrum coincides with Steenbrink

Conjecture 7.11. *(Conjecture 2.23) Suppose that the conjectures 2.21 and 2.22 are true. Let f be a holomorphic map germ $f : (\mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at 0 or an M-tame function $f : X \to \mathbb{C}$ with $\dim X = m + 1$. Then any Stokes matrix S of f satisfies*

$$\mathrm{Spp}(S) = \mathrm{Spp}(f) - (\frac{m-1}{2}, m). \tag{7.3}$$

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