Random Fields on General Domains - Analytic Properties, Models, and Simulations

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Abstract

The present thesis addresses two aspects of random fields: sample continuity and the simulation of random fields.

In the first part of the thesis we formulate and prove a local and a global variant of the Kolmogorov-Chentsov theorem in [70] for random fields on metric spaces. From this we obtain a theorem for random fields on Riemannian manifolds that is easy to apply and yields the existence of a modification which is locally uniformly sample continuous or locally Hölder sample continuous.

In the second part we present a model of a random field on a topological space that unifies well-known models such as the Poisson hyperplane tessellation model, the random token model, and the dead leaves model. In addition to generalizing these submodels from $\mathbb{R}^d$ to other spaces such as the d-dimensional unit sphere $\mathbb{S}^d$, our construction also extends the classical models themselves, e.g. by replacing the Poisson distribution by an arbitrary discrete distribution. Moreover, the method of construction directly produces an exact and fast simulation procedure. By investigating the covariance structure of the general model we recover various explicit correlation functions on $\mathbb{R}^d$ and $\mathbb{S}^d$ and obtain several new ones.

This second part also contains a proof of the spectral turning bands method on $\mathbb{S}^d$, which has the same properties as its analogue in $\mathbb{R}^d$.

Zusammenfassung

In der vorliegenden Arbeit werden zwei Aspekte von Zufallsfeldern untersucht: Die Stetigkeit der Realisationen und die Simulation von Zufallsfeldern.

Im ersten Teil der Arbeit formulieren und beweisen wir eine lokale und eine globale Version des Kolmogorov-Chentsov Theorems in [70] für Zufallsfelder auf metrischen Räumen. Dies führt uns zu einem Theorem für Zufallsfelder auf Riemannschen Mannigfaltigkeiten, welches einfach anzuwenden ist und die Existenz einer Modifikation mit lokal gleichmäßig stetigen oder lokal Hölderstetigen Realisationen liefert.

Im zweiten Teil präsentieren wir ein Modell für Zufallsfelder auf topologischen Räumen, das bekannte Modelle wie das Poisson-Hyperplane-Tessellation-Modell, das Random-Token-Modell und das Dead-Leaves-Modell vereint. Unser Modell verallgemeinert diese Modelle von $\mathbb{R}^d$ auf andere Räume wie zum Beispiel die d-dimensionale Einheitssphäre $\mathbb{S}^d$. Außerdem erweitern wir die bekannten Modelle auch in $\mathbb{R}^d$, beispielsweise durch die Möglichkeit, allgemeine diskrete Verteilungen anstelle der Poisson-Verteilung zu betrachten. Aus der Konstruktion unseres Modells ergibt sich direkt eine schnelle und genaue Simulationsmethode. Indem wir die Kovarianzstruktur unseres Modells untersuchen, erhalten wir viele bekannte und einige neue Korrelationsfunktionen auf $\mathbb{R}^d$ und auf $\mathbb{S}^d$.

Schließlich beweisen wir im zweiten Teil der Arbeit noch die Spectral-Turning-Bands-Methode auf $\mathbb{S}^d$, die die selben Eigenschaften wie ihr Analogon in $\mathbb{R}^d$ hat.

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Introduction

For any set M, a real-valued random field Z on M may be defined as a family of real-valued random variables $(Z(x), x \in M)$ on a common probability space $(\Omega, \mathscr{A}, P)$. In the classical theory [2, 53, 20], the set M is usually assumed to be a subset of $\mathbb{R}^d$ and the corresponding random field can be interpreted as a tool to model spatial uncertainty. Many applications for example in geostatistics, cosmology, and material sciences show that the choice $M \subseteq \mathbb{R}^d$ may be inappropriate or even too restrictive in general. This led to an increased interest in the study of random fields on general spaces [3, 5, 39, 57, 59, 69, 70, 71] and it is the aim of the present thesis to contribute to this development.

The assumptions on the space M will vary throughout the thesis depending on the particular subject. However, one example that will accompany us all the way is $M = \mathbb{S}^2$. The reason for this is twofold. First, the sphere may be seen as the simplest non-trivial example of a space with non-Euclidean geometry, which makes random fields on the sphere interesting from a theoretical point of view. Second, random fields on the two-dimensional sphere are of particular interest for applications. In geosciences, spatial data collected by satellites or obtained as output from climate models often cover a large portion of the globe. Examples for such data include sea surface temperature [14], sea level pressure [4], and total column ozone levels [65]. The analysis of such data sets requires random fields and covariance models on $\mathbb{S}^2$, as can be seen in [13, 36, 41, 44, 45, 47]. Furthermore, random fields on the sphere have applications in material sciences [29], serve as radial functions of star-shaped random sets [37], and can be used to model cosmic microwave background radiation [15, 59].

The present thesis addresses two aspects of random fields on general domains: sample continuity and the simulation of random fields.

Part I: Sample Continuity

One of the key theorems in the theory of stochastic processes is the Kolmogorov–Chentsov theorem (the classical references are [82] and [17]), which establishes the existence of a continuous modification of a given stochastic process based on tail or moment estimates of its increments. The core of the method leading to this theorem is the application of the Borel-Cantelli lemma in order to conclude from statistical properties of the random field to continuity properties of its realizations. The method has been generalized by a number of authors to the case of a random field on $\mathbb{R}^d$, see [2, 6, 21, 46] and references therein. It can be seen from the proofs that a crucial ingredient of this method is the existence of a dense subset with certain properties. This dense subset is naturally available in $\mathbb{R}^d$ by the (multivariate) dyadic rationals, but there is no natural analogue in more general spaces.

In the article [70] the author identified and generalized the properties of this dense subset which enable a proof of the method, resulting in a Kolmogorov-Chentsov type theorem for random fields on metric spaces. This paper is the starting point for the work that is presented in part I.

In an attempt to apply the results of [70] to the metric space $(\mathbb{S}^2, d_{\mathbb{S}^2})$, where $d_{\mathbb{S}^2}$ denotes the great circle metric on $\mathbb{S}^2$, we identified two obstacles that prevent a direct application. These obstacles are explained in more detail in Section 1.2. In summary, we found that there are two assumptions in [70] on a metric space that are not compatible with the geometry of $\mathbb{S}^2$.

This problem led us to two different solutions, both of which are presented in Chapter 2. The first one is presented in Section 2.4 and consists of a direct generalization of the method given in [70]. We formulate weakened assumptions on the underlying space and show that the proof given in [70] can be retained in its core and eventuates in a theorem that yields the existence of a uniformly continuous modification under the classical condition on the random field. In Section 3.1, we show that it is possible to apply this generalized theorem to $(\mathbb{S}^2, d_{\mathbb{S}^2})$, resulting in a criterion for the uniform sample continuity of a random field on the sphere.

The second approach, which we present first, is based on joint work with Annika Lang, Jürgen Potthoff, and Martin Schlather [51]. In this approach, we prove in a first step a local variant of the Kolmogorov-Chentsov theorem in [70]. This local formulation of the theorem reduces the validation of the assumptions in any application from the whole space to local domains and is therefore less restrictive. On each of these domains we then obtain a local modification by applying the theorem in [70] and we show that it is possible to glue these modifications together in order to obtain a global modification with the desired properties. We then apply our theorem to the important case of M being a finite-dimensional Riemannian manifold. The key here is that by utilizing the structure of a Riemannian manifold we are able to show that it is possible to construct a local coordinatization of the underlying space which is such that we can apply the local Kolmogorov–Chentsov theorem. In particular, our construction can be made for every finite-dimensional Riemannian manifold in the same manner, such that for the theorem that we obtain this way there are no additional assumptions on the underlying space of the random field. Furthermore, the condition on the random field in the theorem is formulated in terms of the topological metric of the Riemannian manifold and is in particular independent of any choice of coordinatization of the manifold.

We then proceed and discuss the existence of modifications which are locally sample Hölder continuous and provide sufficient conditions on the moments or tails of the increments. Additionally, we apply our findings to the special case of Gaussian random fields.

The application of this second method to our example $M = \mathbb{S}^2$ is trivial as there are no further assumptions on the underlying space other than that M is a finite-dimensional Riemannian manifold. In Section 3.2, we compare both approaches and discuss their results.

Part II: Random Field Models and Simulations

For many of the applications mentioned above it is essential to have a simulation procedure which generates samples of a random field. So far, many simulation methods have been developed in $\mathbb{R}^d$ (see [75] for an overview), while for any other space M, e.g. the sphere, only few methods are available [22, 52, 56]. In the second part of this thesis we define and investigate two random field models that lead to direct simulation procedures.

The first model, mosaic random fields, are piecewise constant random fields that are build upon a random tessellation of the underlying space. These random fields have applications for example in material sciences [29, 42, 84], cosmology [23], and geosciences [18, 19]. In Chapter 4, we present a general class of mosaic random fields that unifies well-known models in $\mathbb{R}^d$ and generalizes them to other spaces. The idea behind the generalization is best explained by considering two classical models in $\mathbb{R}^d$: the mosaic random field that is build from a Poisson hyperplane tessellation [62, 63, 66] and the random token field [81, 20, 53].

Suppose we are given a Poisson point process Π in $\mathbb{S}_+^{d-1} \times \mathbb{R}$ and that for each point (x, r) of a realization of Π a hyperplane with normal vector $\operatorname{sgn}(r)x$ pointing from the origin to the hyperplane and distance $|r|$ from the origin is drawn. The polytopes that are delimited by this network of random hyperplanes form the Poisson hyperplane tessellation and a random field is obtained by the assignment of a different random variable from an independent and identically distributed sequence $(U_i, i \in \mathbb{N})$ to each cell. For the random token field, bounded subsets or tokens are placed at the points of a Poisson point process in $\mathbb{R}^d$ and to each token B_i a random variable is associated from an independent and identically distributed sequence $(U_i, i \in \mathbb{N})$. At each location x the random field is then defined as the sum of all random variables U_i that are associated to tokens containing x.

The first step that led to the construction in Section 4.2 is to distinguish between the underlying mosaic and the procedure which assigns random variables to the cells of the random mosaic. It is natural to think of both, the Poisson hyperplane tessellation and the mosaic which is constructed from random tokens, as particular examples of a mosaic which is build from the intersections of a random number of general random closed sets. In more detail, if $(B_1, \ldots, B_N)$ are independent and identically distributed random closed sets in M and N is an $\mathbb{N}_0$-valued random variable, then the family of cells $(C_I, I \subseteq \{1, \ldots, N\})$ defined by

$$C_I = \Bigl(\bigcap_{i \in I} B_i\Bigr) \cap \Bigl(\bigcap_{j \in \{1,\ldots,N\} \setminus I} B_j^c\Bigr), \quad I \subseteq \{1, \ldots, N\},$$

forms a partition of M. If we choose a Poisson distributed N and random half-spaces for the random closed sets B_i, then we obtain the Poisson hyperplane tessellation, and if we take random tokens as random closed sets we obtain the mosaic of the random token field.

Given such a generalized mosaic, we obtain an associated random field Z by defining $Z(x) = V_I$ for all x that are contained in the cell C_I and V_I is a real-valued random variable that is associated with the cell C_I. As a result, the random field Z is constant on every cell. The random variables $V_I, I \subseteq \{1, \ldots, N\}$, neither have to be independent nor identically distributed. In order to investigate the random field Z, it is nonetheless of advantage to impose some restrictions on the random variables V_I. In the present thesis we assume that the random variables V_I are given as

$$V_I = \sum_{i \in \mathbb{I}_I} U_{g(I),i}, \quad I \subseteq \{1, \ldots, N\},$$

where $(U_{i,j}, i, j \in \mathbb{N})$ is an independent and identically distributed sequence of real-valued random variables, g is a function mapping finite subsets of $\mathbb{N}$ to positive integers, and $(\mathbb{I}_I, I \subset \{1, \ldots, N\})$ is a family of finite subsets of $\mathbb{N}$. This way, our model includes the examples from above, as can be seen by taking $\mathbb{I}_I = \{1\}$ and an injective function g for the Poisson hyperplane tessellation model, and $\mathbb{I}_I = I$ and $g \equiv 1$ for the random token model. There are many more examples of mosaic random fields like the dead leaves random field [61] or mixtures of classical models that are included in this class of mosaic random fields.

The advantages of this construction, apart from its unifying character, are twofold. The most obvious advantage, which was also the motivation for this approach, is that the formulation of the mosaic in terms of intersections of random closed sets allows to formulate the model on topological spaces. In practice, whenever there is a method available to simulate random closed sets in a topological space M, we obtain through our construction a method to simulate a random field on M. To give an example, in Section 4.5 we take random hemispheres and random spherical caps on $M = \mathbb{S}^d$ and obtain multiple explicit random field models on $\mathbb{S}^d$ by the combination of different attributes of the general mosaic random field.

Additionally, our approach generalizes the present models in $\mathbb{R}^d$ themselves. First, our construction allows to combine any choice of random closed sets with different assignment procedures of random variables to the cells, so that the general model includes for example a mosaic random field which combines the assignment procedure of the Poisson hyperplane tessellation model with random closed balls. Second, our approach allows to avoid the assumption of a Poisson distributed number of random closed sets whenever the underlying space $M \subseteq \mathbb{R}^d$ is bounded. This turns out to be especially beneficial and results ultimately in a considerable number of covariance models that are either new or have not been associated with mosaic random fields yet.

Section 4.2 contains the formulation of the general mosaic random field. We show that the formulation there allows obtaining a general formula for the mean and the first mixed moment of the mosaic random field. From this, we obtain formulae for the correlation functions of the most important submodels. In Sections 4.3, 4.5, and 4.7, we give examples of random closed sets in $\mathbb{R}^d$, on $\mathbb{S}^d$, on a cylinder, and on the torus, which lead to multiple explicit correlation functions of the mosaic random field that are presented in Tables 4.1, 4.2, 4.3, and 4.4. The interested reader will find the necessary information for simulating the mosaic random fields corresponding to the correlation functions in these tables in Appendix B.1.

Our step-by-step analysis of the covariance structure can also be used to find mosaic random fields that correspond to given correlation functions. This approach is used for example in Section 4.9, where it is shown that the power correlation function [40] is a correlation function on $\mathbb{S}^d$ for all dimensions $d \in \mathbb{N}$.

The results of Chapter 4 are accompanied by many examples, which illustrate the techniques that produce explicit models and explicit correlation functions. The examples include visualizations in form of simulated realizations of mosaic random fields and approximate Gaussian random fields that were built from mosaic random fields.

The results of Chapter 4 are based on joint work with Martin Schlather and Jürgen Potthoff [78].

The second method, which is presented in Chapter 5, is the spectral turning bands method on $\mathbb{S}^d$. In 1938 and 1942, Schoenberg published two papers [76, 77] in which he showed that every continuous, stationary, and isotropic covariance function C on $\mathbb{R}^d$ or $\mathbb{S}^d$ admits a spectral representation with a non-negative spectral measure. From the representation in $\mathbb{R}^d$, a simulation method has emerged [58] that is known as the spectral turning bands method. For $\mathbb{S}^d$ however, to the best of the author's knowledge, a corresponding simulation method is missing.

In Section 5.1 we show by recalling the known method in $\mathbb{R}^d$, that the key for the formulation of the spectral turning bands method on $\mathbb{S}^d$ is to find a measurable space $(W, \mathscr{W})$ and a function $f : M \to W$ such that Schoenberg's representation becomes $C(x, y) = \langle f(x), f(y)\rangle_{L^2(W)}$ for all $x, y \in \mathbb{S}^d$. This representation readily leads to a corresponding random field by a well-known result which is in the spirit of Karhunen's spectral representation of random fields [48, 10]. In Section 5.2, we prove that Schoenberg's representation on $\mathbb{S}^d$ can indeed be reformulated in the form $C(x, y) = \langle f(x), f(y)\rangle_{L^2(W)}$ by utilizing a particular orthogonality property of Gegenbauer polynomials.

For every continuous and isotropic covariance function on $\mathbb{S}^d$, the spectral turning bands method produces a corresponding random field. When it comes to practice, the method is subject to the same limitations as the method in $\mathbb{R}^d$. In order to simulate the spectral turning bands random field one has to simulate a random variable that is distributed according to the normalized spectral measure of the covariance function, so that this spectral measure must be given in a form that makes it possible to sample from it. In Section 5.3 we give examples on $\mathbb{S}^2$ for which the correspondence between covariance function and spectral measure is explicit and for which the spectral turning bands method therefore is applicable. We illustrate these examples with simulations.

Part I.

Sample Continuity

1. Motivation

1.1. Preliminaries

Throughout this thesis we assume that we are given a set M and a probability space $(\Omega, \mathscr{A}, P)$. In order to avoid trivialities, let us assume that M in non-empty here and henceforth. We call a family of real-valued random variables $(Z(x), x \in M)$ on $(\Omega, \mathscr{A}, P)$ that is indexed by the elements of M a *random field on M*. When there is no danger of confusion we denote the family $(Z(x), x \in M)$ simply by Z. In the course of this thesis we will have different assumptions on the structure that is given on the index set of the random field, depending on our particular purposes. In this part of the thesis we assume M to be either a metric space or a Riemannian manifold and we study the particular case of M being a two-dimensional sphere in more detail.

Let us begin by briefly introducing the necessary terminology of Riemannian geometry and the objects that are associated to Riemannian manifolds and in particular to the sphere, setting up our notation at the same time. For further background the interested reader is referred to the standard literature, e.g. [38, 43, 54, 68].

Assume that $d \in \mathbb{N}$ and that (M, g) is a d–dimensional Riemannian manifold as defined in [38]. That is, M is a connected, d–dimensional C^∞–manifold together with a symmetric, strictly positive definite tensor field g of type $(0, 2)$. For each $x \in M$, the Riemannian metric g determines an inner product $g_x(\cdot, \cdot)$ on the tangent space T_xM at x:

$$g_x : \begin{cases} T_xM \times T_xM & \longrightarrow \quad \mathbb{R}, \\ (X, Y) & \longmapsto \quad g_x(X, Y). \end{cases}$$

The corresponding norm on T_xM is given by

$$\|X\| = g_x(X, X)^{1/2}, \qquad X \in T_xM.$$

Let $\beta : [a, b] \to M$ be a smooth curve in M. Then its derivative $\beta'(t)$ at $t \in (a, b)$ belongs to $T_{\beta(t)}M$, and the length of β is given by

$$L(\beta) = \int_a^b \|\beta'(t)\| \, dt.$$

The Riemannian distance $d_M(x, y)$ of two points x, $y \in M$ is defined as the infimum of the lengths of curve segments joining x and y. Indeed, d_M is a metric on M and it can be shown that under the given assumptions on M the metric space (M, d_M) is separable, locally compact and connected [38, Proposition I.9.6]. Furthermore, the original topology and the topology defined by d_M coincide [38, Corollary I.9.5].

As explained in the introduction we are particularly interested in random fields that are indexed by the elements of $\mathbb{S}^d$, $d \in \mathbb{N}$. Here and henceforth let

$$\mathbb{S}^d = \left\{ z \in \mathbb{R}^{d+1} \,|\, \langle z, z \rangle = 1 \right\}$$

denote the *d-dimensional unit sphere in* $\mathbb{R}^{d+1}$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product of $\mathbb{R}^{d+1}$. Sometimes it is also useful to consider the case $d = 0$, for which the sphere $\mathbb{S}^0$ is just the set $\{-1, 1\}$. If $\bar{g}$ denotes the standard Riemannian metric on $\mathbb{R}^{d+1}$, i.e.

$$\bar{g}_x(X, Y) = \sum_{i=1}^{d+1} X_i Y_i = \langle X, Y \rangle, \quad x \in \mathbb{R}^{d+1}, X, Y \in T_x\mathbb{R}^{d+1},$$

then the inclusion $\iota : \mathbb{S}^d \hookrightarrow \mathbb{R}^{d+1}$ defines a Riemannian metric $\mathring{g}$ on $\mathbb{S}^d$ (for instance [54, Example 13.16]), so that we may think of the sphere as a Riemannian manifold. The Riemannian metric $\mathring{g}$ is called the *round Riemannian metric* or the *standard Riemmanian metric* on $\mathbb{S}^d$ (for instance [54]). Let us denote the induced distance function on $\mathbb{S}^d$ by $d_{\mathbb{S}^d}$. On $(\mathbb{S}^d, \mathring{g})$ a shortest curve segment that joins two points is a segment of a great circle (e.g. [68, Example 30]). Therefore, the distance $d_{\mathbb{S}^d}(x, y)$ between $x, y \in \mathbb{S}^d$ is given by the length of a shortest great circle segment that joins x and y. Note that in case x and y are antipodal points, there are infinitely many shortest great circle segments that join x and y, all of which have the same length. Because we can identify each $x \in \mathbb{S}^d$ with a vector in $\mathbb{R}^{d+1}$ and because the length of a shortest great circle segment that starts in $x \in \mathbb{S}^d$ and ends in $y \in \mathbb{S}^d$ is given by the angle between x and y in radians, it follows that in Euclidean coordinates the metric $d_{\mathbb{S}^d}$ admits the representation

$$d_{\mathbb{S}^d}(x, y) = \arccos\big(\langle x, y \rangle\big), \quad x, y \in \mathbb{S}^d.$$

The metric $d_{\mathbb{S}^d}$ is called the *geodesic* or *great circle metric*.

Often it is convenient to work with spherical coordinates on $\mathbb{S}^d$, which are given by the map $\phi_d : [0, 2\pi) \times [0, \pi]^{d-1} \to \mathbb{S}^d$ recursively defined by

$$\begin{aligned} \phi_1(\varphi) &= \big(\cos(\varphi), \sin(\varphi)\big), \\ \phi_k(\varphi, \theta_1, \dots, \theta_{k-1}) &= \big(\phi_{k-1}(\varphi, \theta_1, \dots, \theta_{k-2}) \sin(\theta_{k-1}), \cos(\theta_{k-1})\big), \quad k \geq 2. \end{aligned} \tag{1.1}$$

In order for ϕ_d to be one-to-one, the domain of ϕ_d has to be restricted but this can be neglected for our purposes.

Let $\mathscr{B}(\mathbb{S}^d)$ be the σ-algebra on $\mathbb{S}^d$ that is generated by the topology of the metric space $(\mathbb{S}^d, d_{\mathbb{S}^d})$. We denote the surface measure of $\mathbb{S}^d$ by σ_d. In spherical coordinates, the surface measure σ_d admits the representation

$$\sigma_d(B) = \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} \mathbb{1}_B\big(\phi_d(\varphi, \theta_1, \dots, \theta_{d-1})\big) \prod_{k=1}^{d-1} \sin^k(\theta_k)\, d\theta_{d-1} \dots d\theta_1 d\varphi \tag{1.2}$$

for every $B \in \mathscr{B}(\mathbb{S}^d)$. For further reference we note that the total mass of σ_d is given by

$$\sigma_d(\mathbb{S}^d) = \frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)}, \quad d \in \mathbb{N}, \tag{1.3}$$

and we let $\bar{\sigma}_d = 2^{-1}\pi^{-(d+1)/2}\Gamma((d+1)/2)\sigma_d$ denote the uniform probability measure on $\mathbb{S}^d$.

1.2. Motivation

Suppose we are given a random field $(Z(x), x \in M)$ on some space M. If M is a subset of $\mathbb{R}^d$, the question of the regularity of the samples of Z is a well-studied subject. However, there are only few results that address this question in more general settings [5, 39, 69, 70, 71]. In [70], a generalized Kolmogorov–Chentsov theorem for random fields on metric spaces is proved, which supplies the existence of a (Hölder) sample continuous modification of Z based on tail or moment estimates of its increments. Because our results originated from the work in [70] and in order to explain the necessity for further work on this topic, let us begin by stating the main result of [70].

Let (M, d_M) be a metric space and suppose that $(D_n, n \in \mathbb{N})$ is an increasing sequence of finite subsets of M. We may think of each set D_n as a grid in the space M, where each grid D_n is a refinement of the previous grid D_{n-1}. For each $n \in \mathbb{N}$, the set D_n will be called *n-grid* henceforth. Given a sequence of grids, the sequence $(\delta_n^0, n \in \mathbb{N})$ is defined by

$$\delta_n^0 = \begin{cases} \min\{d_M(x,y) | x, y \in D_n, x \neq y\}, & |D_n| \geq 2, \\ +\infty, & \text{otherwise}, \end{cases}$$

and we assume that we are given an associated sequence $(\delta_n, n \in \mathbb{N})$ of real numbers such that $\delta_n^0 \leq \delta_n$ for all $n \in \mathbb{N}$. The object $\mathcal{D} = ((D_n, \delta_n), n \in \mathbb{N})$ is called a *scale of* (M, d_M) in [70] and we adopt this terminology. Given such a scale, we call $x, y \in D_n$ *neighbors in* D_n if $d_M(x, y) \leq \delta_n$. For $n \in \mathbb{N}$ and $x \in D_n$ let

$$C_n(x) = \{y \in D_n | d_M(x,y) \leq \delta_n\}.$$

The set $C_n(x)$ is called a *clique of x in D_n*. Furthermore, let π_n denote the set consisting of all unordered pairs of neighbors in D_n. Using the notation $\langle x, y\rangle_{up}$ for an unordered pair of elements $x, y \in M$,

$$\pi_n = \big\{\langle x, y\rangle_{up} \,|\, x, y \in D_n, d_M(x,y) \leq \delta_n\big\}, \quad n \in \mathbb{N}. \tag{1.4}$$

Suppose c is a positive constant and let us assume that we are given two monotonically increasing functions $r : [0, c) \to \mathbb{R}_+$ and $q : [0, c) \to \mathbb{R}_+$, such that $r(0) = q(0) = 0$. The second part of Theorem 2.8 in [70] may now be stated as follows.

Theorem 1.2.1 (Potthoff, 2009). *Suppose that $(M, d_M, ((D_n, \delta_n), n \in \mathbb{N}))$, r, q, and the random field Z satisfy the following conditions:*

(a) $D = \bigcup_{n \in \mathbb{N}} D_n$ *is dense in* (M, d_M)*;*

(b) $\limsup_n \delta_n / \delta_n^0 < +\infty$*;*

(c) For almost all $n \in \mathbb{N}$ and all $x, y \in D_{n+1}$ there exist $x', y' \in D_n$ such that $\langle x, x'\rangle_{up}, \langle y, y'\rangle_{up} \in \pi_{n+1}$, and $d_M(x', y') \leq d_M(x, y)$;

(d) $\sum_{n \in \mathbb{N}: \delta_n < c} |\pi_n| q(\delta_n) < +\infty$*;*

(e) $\sum_{n \in \mathbb{N}: \delta_n < c} r(\delta_n) < +\infty$*;*

(f) *For all $x, y \in M$ such that $d_M(x,y) < c$ the inequality*

$$P\Big(|Z(x) - Z(y)| > r\big(d_M(x,y)\big)\Big) \leq q\big(d_M(x,y)\big) \tag{1.5}$$

holds true.

Then Z has a modification which is uniformly sample continuous.

Let us now turn to our recurrent example $(M, d_M) = (\mathbb{S}^2, d_{\mathbb{S}^2})$. A natural choice for the grids on $\mathbb{S}^2$ is

$$D_n = \Big\{\phi_2\Big(k\frac{\pi}{2^{n-1}}, l\frac{\pi}{2^{n-1}}\Big)\Big| k = 0, 1, \ldots, 2^{n-1}, l = 0, 1, \ldots, 2^n - 1\Big\}, \quad n \in \mathbb{N}. \tag{1.6}$$

Then clearly $(D_n, n \in \mathbb{N})$ is increasing and since ϕ_2 is continuous, $D = \bigcup_{n\in\mathbb{N}} D_n$ is dense in $\mathbb{S}^2$. The next step is to define the sequence $(\delta_n, n \in \mathbb{N})$. By definition (1.4), the number of elements in the sets $(\pi_n, n \in \mathbb{N})$ is determined by the choice of the sequence $(\delta_n, n \in \mathbb{N})$. In view of Condition (d), it is desirable that $(|\pi_n|, n \in \mathbb{N})$ does not grow too fast to ∞. This is because a rapid growth of $(|\pi_n|, n \in \mathbb{N})$ must be absorbed with a function q that decreases fast to 0 as its argument decreases to 0 and such a choice of q implies a restrictive bound on the right hand side in (1.5). In consequence, we want δ_n to be as small as possible. On the other hand, each δ_n must be in particular large enough such that Condition (c) holds true. From the application of the Theorem 1.2.1 in [70] to $M = \mathbb{R}^d$, it can be seen that δ_n must be at least as large as the distance of one grid point $x \in D_n$ to any of its adjacent grid points in D_n in order for Condition (c) to be satisfied. It is easy to find examples of grid points which show, that this is also true on $\mathbb{S}^2$, and from Lemma A.1.2 in Appendix A we have therefore the restriction

$$\delta_n \geq \arccos\Big(\cos^2\Big(\frac{\pi}{2^{n-1}}\Big)\Big), \quad n \in \mathbb{N}. \tag{1.7}$$

Furthermore, it follows from Lemma A.1.1 that

$$\delta_n^0 = \arccos\Big(\sin^2\Big(\frac{\pi}{2^{n-1}}\Big)\cos\Big(\frac{\pi}{2^{n-1}}\Big) + \cos^2\Big(\frac{\pi}{2^{n-1}}\Big)\Big), \quad n \in \mathbb{N}. \tag{1.8}$$

In consequence, the combination of (1.7), (1.8), and the statement of Corollary A.1.4 show that Condition (b) is violated, so that we can not apply Theorem 1.2.1 to the sphere $\mathbb{S}^2$ directly in case the grids are (1.6).

At this point we may either try to find a suitable grid or adjust the assumptions of Theorem 1.2.1 in such a way that the theorem is also applicable to $(M, d_M) = (\mathbb{S}^2, d_{\mathbb{S}^2})$. Because we think that the application of Theorem 1.2.1 should not amount to a creative construction of suitable grids, we seek to adjust the assumptions of Theorem 1.2.1 such that it is applicable also to $\mathbb{S}^2$.

In what follows, we present two different approaches. The first one is based on the idea, that we may exploit the manifold structure of $\mathbb{S}^2$ in order to apply Theorem 1.2.1 locally. We prove a local variant of Theorem 1.2.1 and show, that this local variant can be applied not only to $\mathbb{S}^2$ but also to arbitrary Riemannian manifolds. Despite the local approach, we show that Condition (f) can be retained, so that in particular the condition on the random field is independent of any choice of coordinatization. Our second approach in Section 2.4 consists of a direct generalization of Theorem 1.2.1 which is motivated by the proof given in [70]. While retaining the assertion of Theorem 1.2.1, we show that it is possible to weaken the assumptions (b) and (c) in such a way that the new theorem is also applicable to $\mathbb{S}^2$. Section 3.2 consists of a discussion of both approaches.

Remark 1.2.2. Actually, the failure of Condition (b) does not constitute a serious obstacle for the application of Theorem 1.2.1 to $\mathbb{S}^2$. It can be seen from the proof in [70], that Condition (b) can be replaced with the weaker assumption of $(\delta_n, n \in \mathbb{N})$ being a null sequence. However, in this case $(|\pi_n|, n \in \mathbb{N})$ may grow very fast. On top of that, there are more delicate issues concerning the validity of Condition (c) on $\mathbb{S}^2$. We choose to not go into detail at this point to not distract the reader from the main results.

2. A Kolmogorov-Chentsov Theorem for Random Fields on Metric Spaces and Riemannian Manifolds

2.1. A Local Kolmogorov–Chentsov Theorem for Metric Spaces

In this section we give a variant of the Kolmogorov–Chentsov type theorem in [70], which follows rather directly from it, and in some sense sharpens that result.

Suppose that (M, d_M) is a separable metric space, that $(\Omega, \mathscr{A}, P)$ is a probability space, and that $Z = \big(Z(x),\, x \in M\big)$ is a real-valued random field on this probability space indexed by M.

Assume furthermore that r and q are two strictly increasing functions on an interval $[0, c)$, $c > 0$, such that $r(0) = q(0) = 0$. Throughout this section we suppose that for all x, $y \in M$ with $d_M(x, y) < c$, we have the bound

$$P\Big(\big|Z(x) - Z(y)\big| > r\big(d_M(x, y)\big)\Big) \leq q\big(d_M(x, y)\big). \tag{2.1}$$

We make the following assumptions on the metric space (M, d_M):

Assumptions 2.1.1.

(a) There exists an at most countable open cover $(U_n,\, n \in \mathbb{N})$ of M, and for every $n \in \mathbb{N}$, there exists a metric d_n on U_n so that $\alpha_n\, d_n(x, y) \leq d_M(x, y) \leq d_n(x, y)$ for all x, $y \in U_n$ and some $\alpha_n \in (0, 1]$;

(b) for every $n \in \mathbb{N}$, (U_n, d_n) is well-separable in the sense of [70], i.e.:

  (i) there exists an increasing sequence $(D_{n,k},\, k \in \mathbb{N})$ of finite subsets of U_n such that $D_n = \bigcup_k D_{n,k}$ is dense in (U_n, d_n), and for $x \in D_{n,k}$, let $C_{n,k}(x) = \{y \in D_{n,k} \,|\, d_n(x, y) \leq \delta_{n,k}\}$, where $\delta_{n,k}$ denotes the minimal distance of distinct points in $D_{n,k}$ with respect to d_n;

  (ii) every $z \in U_n$ has a neighborhood $V \subset U_n$ so that for almost all $k \in \mathbb{N}$ and all x, $y \in D_{n,k+1} \cap V$, there exist x', $y' \in D_{n,k} \cap V$ with $x' \in C_{n,k+1}(x)$, $y' \in C_{n,k+1}(y)$, and $d_n(x', y') \leq d_n(x, y)$;

(c) for n, $k \in \mathbb{N}$, let $\pi_{n,k}$ be the set of all unordered pairs $\langle x, y\rangle_{up}$, x, $y \in D_{n,k}$ with $d_n(x, y) \leq \delta_{n,k}$, and let $|\pi_{n,k}|$ denote the number of elements in this set, then

$$\sum_{k \in \mathbb{N}: \delta_{n,k} < c} |\pi_{n,k}|\, q(\delta_{n,k}) < +\infty, \tag{2.2}$$

$$\sum_{k \in \mathbb{N}: \delta_{n,k} < c} r(\delta_{n,k}) < +\infty \tag{2.3}$$

hold true.

We remark that due to the assumption on the metrics d_M and d_n, $n \in \mathbb{N}$, in (a) above, the relative topology on U_n generated by d_M coincides with the topology generated by d_n.

For x, $y \in U_n$ with $d_M(x,y) < \alpha_n c$, we can estimate as follows

$$\begin{aligned} P\Big(\big|Z(x) - Z\big| > r\big(d_n(x,y)\big)\Big) \\ &\leq P\Big(\big|Z(x) - Z(y)\big| > r\big(d_M(x,y)\big)\Big) \\ &\leq q\big(d_M(x,y)\big) \\ &\leq q\big(d_n(x,y)\big), \end{aligned}$$

because r and q are both increasing. Theorem 2.8 in [70] shows that from this estimate, together with the assumptions (a), (b), and (c) above, it follows that for every $n \in \mathbb{N}$, the restriction Z_n of Z to U_n has a locally uniformly continuous modification $\tilde{Z}_n$ which is such that Z_n, $\tilde{Z}_n$, and Z coincide on D_n. In more detail we have that for every $n \in \mathbb{N}$, there exists a random field $\tilde{Z}_n$ indexed by U_n such that

(i) for every $\omega \in \Omega$, the mapping $\tilde{Z}_n(\cdot,\omega) : U_n \to \mathbb{R}$ is locally uniformly continuous;

(ii) for every $x \in U_n$, there exists a P–null set $N_{x,n}$ so that $\tilde{Z}_n(x,\omega) = Z(x,\omega)$ for all ω in the complement of $N_{x,n}$, and if $x \in D_n$, $N_{x,n}$ can be chosen as the empty set.

In order to get for $x \in M$ a universal P-null set N_x, we set $N_x = \bigcup_{n'} N_{x,n'}$, where the union is over all $n' \in \mathbb{N}$ such that $x \in U_{n'}$. Since this is a countable union, N_x is indeed a P–null set.

From the modifications $\tilde{Z}_n$ of Z_n, $n \in \mathbb{N}$, we construct a locally uniformly continuous modification $\tilde{Z}$ of Z. We show

Lemma 2.1.2.

$$P\big(\tilde{Z}_n(x) = \tilde{Z}_{n'}(x),\, x \in U_n \cap U_{n'},\, n, n' \in \mathbb{N}\big) = 1.$$

Proof. Assume that $x \in U_n \cap U_{n'}$. Since $\tilde{Z}_n$ and $\tilde{Z}_{n'}$ are modifications of Z when all these random fields are restricted to $U_n \cap U_{n'}$, we get $\tilde{Z}_n(x) = \tilde{Z}_{n'}(x)$ on the complement of the P–null set N_x. Since (M, d_M) is separable so is $(U_n \cap U_{n'}, d_M)$, and letting x range over a countable dense subset $E_{n,n'}$ and taking the union of all associated P–null sets, we get the existence of a P–null set $N_{n,n'}$ such that for all $x \in E_{n,n'}$, we have $\tilde{Z}_n(x) = \tilde{Z}_{n'}(x)$ on the complement of $N_{n,n'}$. $\tilde{Z}_n$ and $\tilde{Z}_{n'}$ are continuous on $U_n \cap U_{n'}$, and hence we obtain for all $x \in U_n \cap U_{n'}$ the equality $\tilde{Z}_n(x) = \tilde{Z}_{n'}(x)$ on the complement of $N_{n,n'}$. Finally, we set $N = \bigcup_{n,n'} N_{n,n'}$ so that we find for all n, n', and all $x \in U_n \cap U_{n'}$ the equality $\tilde{Z}_n(x) = \tilde{Z}_{n'}(x)$ on the complement of the P–null set N. □

On the exceptional set N of the last lemma we define $\tilde{Z}(x) = 0$ for all $x \in M$. On its complement we set $\tilde{Z}(x) = \tilde{Z}_n(x)$ whenever $x \in U_n$, and the last lemma shows that this makes $\tilde{Z}$ well-defined. For $x \in M$, define the P–null set $N'_x = N_x \cup N$ where N_x and N are the P–null sets defined above. We have $x \in U_n$ for some $n \in \mathbb{N}$, and for all ω in the complement of N'_x, we find that $\tilde{Z}(x,\omega) = \tilde{Z}_n(x,\omega) = Z(x,\omega)$. Thus $\tilde{Z}$ is a modification of Z. We have proved the following

Theorem 2.1.3. *Under Condition* (2.1) *on the random field Z and under the above assumptions 2.1.1 on (M, d_M), r and q, Z has a locally uniformly continuous modification.*

2.2. A Kolmogorov–Chentsov Theorem for Riemannian Manifolds

Assume that $d \in \mathbb{N}$ and that (M, g) is an d–dimensional Riemannian manifold as defined in Section 1.1. We denote the open ball of radius $R > 0$ centered at $x \in M$ relative to the metric d_M by $B_R^M(x)$, while the ball of radius R in T_xM with center at $X \in T_xM$ with respect to the norm $\|\cdot\|$ is denoted by $B_R(X)$.

With the Riemannian metric g there is canonically associated – via the notions of parallel transport and geodesics – the exponential map (Exp_x, $x \in M$), which for each $x \in M$ is a mapping from T_xM into a neighborhood of x in M. It can be shown that for each $x \in M$, there exists a radius $R(x) > 0$ such that Exp_x maps $B_{R(x)}(0)$ diffeomorphically onto $B_{R(x)}^M(x)$ [38, Theorem I.9.9, Proposition I.9.4]. Moreover, for all Y, $Z \in B_{R(x)}(0)$ such that $\mathrm{Exp}_x(Y) = y$, $\mathrm{Exp}_x(Z) = z$, the quotient $\|Y - Z\|/d_M(y, z)$ converges to 1 as $(y, z) \to (x, x)$ [38, Proposition I.9.10].

In view of Theorem 2.1.3 in Section 2.1, we construct a countable cover (U_n, $n \in \mathbb{N}$) of M as follows. The separability of M (see above) allows us to fix a countable dense subset $\{x_n \,|\, n \in \mathbb{N}\}$ of M. For every $n \in \mathbb{N}$, choose $R_n \in \big(0, 1/(2\sqrt{d})\big]$ in such a way that:

1. the exponential map Exp_{x_n} is a diffeomorphism from $B_{R_n}(0) \subseteq T_{x_n}M$ onto $B_{R_n}^M(x_n) \subseteq M$,
2. for all $X, Y \in B_{R_n}(0)$ such that $\mathrm{Exp}_{x_n}(X) = x$, $\mathrm{Exp}_{x_n}(Y) = y$, $x, y \in B_{R_n}^M(x_n)$,

$$2^{-1}\|X - Y\| \le d_M(x, y) \le 2\|X - Y\|. \tag{2.4}$$

The existence of a strictly positive R_n for each $n \in \mathbb{N}$ with these properties follows from the facts mentioned before.

The idea is now to use the exponential map in order to define a convenient coordinatization of U_n and to use inequality (2.4) for the definition of a suitable metric d_n on U_n. To this end, we fix an orthonormal basis $(X_{n,1}, \dots, X_{n,d})$ of $(T_{x_n}M, g_{x_n})$ so that every $X \in T_{x_n}M$ can be written in a unique way as

$$X = \sum_{i=1}^{d} a_i X_{n,i}$$

with $a = (a_1, \dots, a_d) \in \mathbb{R}^d$. Let us denote the so defined linear mapping from $\mathbb{R}^d$ onto $T_{x_n}M$ by L_{x_n}. The orthonormality of $(X_{n,1}, \dots, X_{n,d})$ entails that L_{x_n} is an isometric isomorphism if $\mathbb{R}^d$ is equipped with the standard Euclidean metric. In particular, the ball $B_{R_n}(0)$ is under L_{x_n} in one-to-one correspondence with the Euclidean ball $B_{R_n}^d(0)$ in $\mathbb{R}^d$. Define

$$\varphi_n(x) = L_{x_n}^{-1} \circ \mathrm{Exp}_{x_n}^{-1}(x), \qquad x \in B_{R_n}^M(x_n),$$

then φ_n is a C^∞–coordinatization of $B_{R_n}^M(x_n)$ which maps this ball onto $B_{R_n}^d(0) \subseteq \mathbb{R}^d$.

For $x,\, y \in B^M_{R_n}(x_n)$, define

$$d_n(x,y) = 2\sqrt{d}\, \max_{i=1,\dots,d} \big|\varphi^i_n(x) - \varphi^i_n(y)\big|, \tag{2.5}$$

where $\varphi^i_n(x)$ denotes the i–th Cartesian coordinate of $\varphi_n(x)$. Set $\alpha_n = 1/(4\sqrt{d})$. If $\|\cdot\|_2$ denotes the usual Euclidean norm on $\mathbb{R}^d$, we obtain from (2.4)

$$\begin{aligned}
\alpha_n\, d_n(x,y) &= 2^{-1} \max_i \big|\varphi^i_n(x) - \varphi^i_n(y)\big| \\
&\le 2^{-1} \|\varphi_n(x) - \varphi_n(y)\|_2 \\
&= 2^{-1} \|\mathrm{Exp}^{-1}_{x_n}(x) - \mathrm{Exp}^{-1}_{x_n}(y)\| \\
&\le d_M(x,y) \\
&\le 2\|\mathrm{Exp}^{-1}_{x_n}(x) - \mathrm{Exp}^{-1}_{x_n}(y)\| \\
&= 2\|\varphi_n(x) - \varphi_n(y)\|_2 \\
&\le d_n(x,y).
\end{aligned}$$

Consider the open hypercube $H^d_{R_n}(0)$

$$H^d_{R_n}(0) = \big\{x \in \mathbb{R}^d \,|\, \max_{i=1,\dots,d} |\, x_i| < d^{-1/2} R_n\big\}$$

in $\mathbb{R}^d$ of side length $2d^{-1/2}R_n$ centered at the origin. Clearly we have $H^d_{R_n}(0) \subseteq B^d_{R_n}(0)$. Set

$$U_n = \varphi_n^{-1}\big(H^d_{R_n}(0)\big)$$

so that $(U_n,\, n \in \mathbb{N})$ is an open cover of M.

For each $k \in \mathbb{N}$, define the following subset $G_{n,k}$ of the hypercube $H^d_{R_n}(0)$

$$G_{n,k} = \Big\{a \in \mathbb{R}^d \,\Big|\, a = -\frac{R_n}{\sqrt{d}} + \frac{lR_n}{2^k\sqrt{d}},\; l \in \big\{1,\dots,2^{k+1}-1\big\}^d\Big\}.$$

By construction, for each $n \in \mathbb{N}$, $(G_{n,k},\, k \in \mathbb{N})$ is an increasing sequence of finite subsets of $H^d_{R_n}(0)$, and the union of these sets is dense in $H^d_{R_n}(0)$. Next set $D_{n,k} = \varphi_n^{-1}(G_{n,k})$. Then for each $n \in \mathbb{N}$, $(D_{n,k},\, k \in \mathbb{N})$ is an increasing sequence of subsets of U_n, its limit being dense in U_n. Moreover, it is easy to see that Condition (b.ii) of Assumptions 2.1.1 holds true for the sequence $(D_{n,k},\, k \in \mathbb{N})$, where for every $z \in U_n$, we may choose the neighborhood V in this condition as U_n itself. (For an explicit argument, see also [70].)

By construction we have (in terms of the notation of Section 2.1)

$$\delta_{n,k} = \min\big\{d_n(x,y) \,|\, x, y \in D_{n,k},\, x \neq y\big\} = 2^{-k+1} R_n.$$

Recall that for every $n, k \in \mathbb{N}$

$$\begin{aligned}
\pi_{n,k} &= \{\langle x,y\rangle_{up} \,|\, x,y \in D_{n,k}, d_n(x,y) \le \delta_{n,k}\} \\
\text{and}\quad C_{n,k}(x) &= \{y \in D_{n,k} \,|\, d_n(x,y) \le \delta_{n,k}\}, \quad x \in D_{n,k}.
\end{aligned}$$

By definition of $D_{n,k}$ and the metric d_n,

$$|C_{n,k}(x)| = \bigg|\Big\{\varphi_n(y) \in G_{n,k} \,\Big|\, \max_{i=1,\dots,d} |\varphi^i_n(y) - \varphi^i_n(x)| \le \frac{R_n}{2^k\sqrt{d}}\Big\}\bigg|, \quad x \in D_{n,k},$$

so that $|C_{n,k}|$ is bounded by 3^d. Concerning the number $|\pi_{n,k}|$ of unordered pairs in $\pi_{n,k}$ (cf. Assumption 2.1.1.(c)), we can estimate as follows:

$$|\pi_{n,k}| \le \sum_{x\in D_{n,k}} \sum_{y\in D_{n,k}} \mathbb{1}_{C_{n,k}(x)}(y) \le |G_{n,k}|\,3^d = (2^{k+1}-1)^d\,3^d \le K_d\,2^{dk} \tag{2.6}$$

for some constant $K_d > 0$.

Now let $c \in (0,1]$, and make the usual choices of the functions r, and q:

$$r(h) = \log_2(h^{-1})^{-\alpha}, \tag{2.7}$$
$$q(h) = K\,\log_2(h^{-1})^{-\tilde{\alpha}}\,h^d, \tag{2.8}$$

for $h \in (0,c)$, and $r(0) = q(0) = 0$. Here $K > 0$ is an arbitrary constant, and α, $\tilde{\alpha} > 1$. Let us define $K_{n,c} = \big(\log(R_n) - \log(c) + \log(2)\big)/\log(2)$, then Condition (2.3) follows from

$$\sum_{k\in\mathbb{N}:\delta_{n,k}<c} r(\delta_{n,k}) = \log(2)^{\alpha} \sum_{k\in\mathbb{N}:k>K_{n,c}} \frac{1}{\big((k-1)\log(2)-\log(R_n)\big)^{\alpha}}, \quad n\in\mathbb{N}.$$

For the Condition (2.2) we may use (2.6) to get

$$\sum_{k\in\mathbb{N}:\delta_{n,k}<c} |\pi_{n,k}|\,q(\delta_{n,k}) \le \log(2)^{\tilde{\alpha}}(2R_n)^d K_d \sum_{k\in\mathbb{N}:k>K_{n,c}} \frac{1}{\big((k-1)\log(2)-\log(R_n)\big)^{\tilde{\alpha}}}$$

for all $n\in\mathbb{N}$. Thus we can apply Theorem 2.1.3 and obtain

Theorem 2.2.1. *Suppose that Z is a random field defined on a d–dimensional Riemannian manifold (M,g) with topological metric d_M such that for all x, $y \in M$ with $d_M(x,y) < c$,*

$$P\Big(\big|Z(x)-Z(y)\big| > r\big(d_M(x,y)\big)\Big) \le q\big(d_M(x,y)\big)$$

holds true, where the functions r, q are defined as in (2.7), (2.8) for some constants $K > 0$, α, $\tilde{\alpha} > 1$ and $c \in (0,1]$. Then Z has a locally uniformly sample continuous modification.

2.3. Hölder Continuity and Moment Conditions

While a locally uniformly sample continuous modification was constructed in the previous section, the goal here is to show higher regularity in terms of orders of Hölder continuity under additional assumptions.

Therefore, let (M,d_M) be a metric space and Z be as in Section 2.1, Assumptions 2.1.1. For the sequences of minimal distances $(\delta_{n,k}, k\in\mathbb{N})$ of distinct points in the grids $(D_{n,k}, k\in\mathbb{N})$ and the function r, we make the stronger assumptions

Assumptions 2.3.1.

(a) For every $n \in \mathbb{N}$, there exist constants $\eta_n \in (0,1)$, $C_n > 0$ such that for almost all $k \in \mathbb{N}$,
$$\frac{1}{C_n}\,\eta_n^k \leq \delta_{n,k} \leq C_n\,\eta_n^k \tag{2.9}$$
holds true;

(b) there exist constants $\tau \in (0,1)$, $K_\tau > 0$ so that for all $h \in [0,c)$, the inequality
$$r(h) \leq K_\tau h^\tau \tag{2.10}$$
is valid.

(Note that Assumption (b) on r above is stronger than the requirement of inequality (2.3).) Then, on every (U_n, d_n), we are in the situation of Theorem 2.9 in [70] and get the existence of a modification $\tilde{Z}_n$ which is locally Hölder continuous of order τ, i.e., for every $\omega \in \Omega$ and every $z \in U_n$, there exists a neighborhood $V(\omega)$ of z in (U_n, d_n) and a constant $\alpha_{\tau,n}$ such that
$$\sup_{x,y\in V(\omega),\,x\neq y} \left|\frac{\tilde{Z}_n(x,\omega) - \tilde{Z}_n(y,\omega)}{d_n(x,y)^\tau}\right| \leq \alpha_{\tau,n}.$$
Actually, the constant $\alpha_{\tau,n}$ was explicitly calculated in [70] and is given by
$$\alpha_{\tau,n} = 2K_\tau\,\frac{C_n^{2\tau}}{\eta_n^\tau(1-\eta_n^\tau)}.$$

Again we can glue these modifications together to get a modification $\tilde{Z}$ of Z on (M, d_M) which is locally Hölder continuous of order τ.

Corollary 2.3.2. *Assume that Condition* (2.1) *on the random field Z holds true. Suppose furthermore that the assumptions 2.1.1 are valid, together with the additional stronger properties given in Assumptions 2.3.1. Then Z has a modification which is locally Hölder continuous of order τ.*

We return to the case where M is a d–dimensional Riemannian manifold with topological metric d_M. Let the open cover $((U_n, d_n), n \in \mathbb{N})$, and the sequences of grids $((D_{n,k}, \delta_{n,k}), k \in \mathbb{N})$, $n \in \mathbb{N}$, be defined as in Section 2.2. Recall that $\delta_{n,k} = 2^{-k+1}R_n$ and $R_n \in \big(0, 1/(2\sqrt{d})\big]$, $n, k \in \mathbb{N}$. Set $\eta_n = 1/2$, and choose $C_n \geq 1/(2R_n)$. Then Condition (2.9) is fulfilled. As before let c be in $(0,1]$. Define q as in (2.8), and
$$r(h) = h^\tau \tag{2.11}$$
for some $\tau \in (0,1)$. Then Condition (2.10) is valid as well, and so we arrive at

Corollary 2.3.3. *Let Z be a random field defined on a d–dimensional Riemannian manifold (M, g), $d \in \mathbb{N}$, with topological metric d_M, such that for all x, $y \in M$ with $d_M(x,y) < c$,*
$$P\Big(\big|Z(x) - Z(y)\big| > r\big(d_M(x,y)\big)\Big) \leq q\big(d_M(x,y)\big)$$
holds true, where the functions r, q are defined as in (2.11), (2.8) *for some constants $K > 0$, $\alpha > 1$, $\tau \in (0,1)$ and $c \in (0,1]$. Then Z has a locally Hölder continuous modification of order τ.*

The standard application of Chebychev's inequality yields sufficient conditions in terms of moments:

Corollary 2.3.4. *Suppose that Z is a random field defined on a d–dimensional Riemannian manifold M, $d \in \mathbb{N}$, with topological metric d_M.*

(a) If there exist $c \in (0,1]$, $l \geq 1$, $\kappa \geq d$, $\nu > l+1$, and $K > 0$ such that

$$\mathbb{E}\big(|Z(x) - Z(y)|^l\big) \leq K \log_2\big(d_M(x,y)^{-1}\big)^{-\nu} d_M(x,y)^\kappa \tag{2.12}$$

for all x, $y \in M$ with $d_M(x,y) < c$, then Z has a modification which is locally uniformly sample continuous.

(b) If there are $c \in (0,1]$, $l \geq 1$, $\tau \in (0,1)$, and $\alpha > 1$ such that

$$\mathbb{E}\big(|Z(x) - Z(y)|^l\big) \leq K \log_2\big(d_M(x,y)^{-1}\big)^{-\alpha} d_M(x,y)^{d+l\tau}$$

for all x, $y \in M$ with $d_M(x,y) < c$, the modification can be chosen to have locally Hölder continuous sample paths of order τ.

Proof. Let r be the function

$$r(0) = 0, \quad r(h) = \log_2(h^{-1})^{-\alpha}, \quad h \in (0,c),$$

defined in (2.7), where we can choose any $\alpha \in (1, (\nu-1)/l)$. If $x, y \in M$, $x \neq y$, are such that $d_M(x,y) < c$, we have from Chebychev's inequality and (2.12)

$$\begin{aligned} P\Big(|Z(x) - Z(y)| > r\big(d_M(x,y)\big)\Big) &\leq \frac{\mathbb{E}\big(|Z(x) - Z(y)|^l\big)}{r\big(d_M(x,y)\big)} \\ &\leq K \log_2\big(d_M(x,y)\big)^{-(\nu-\alpha l)} d_M(x,y)^\kappa \\ &\leq q\big(d_M(x,y)\big) \end{aligned}$$

where q is the function defined in (2.8) with $\tilde{\alpha} = \nu - \alpha l > 1$. The assertion in (a) thus follows from Theorem 2.2.1. For part (b) we use the function r defined in (2.11) and Corollary 2.3.3 and proceed analogously. □

In case of a Gaussian random field, Corollary 2.3.4 leads to a condition which can be formulated in terms of the semivariogram of the random field:

Corollary 2.3.5. *Assume that Z is a Gaussian random field defined on a d–dimensional Riemannian manifold M, $d \in \mathbb{N}$, with topological metric d_M and semivariogram $\gamma(x,y) = \mathbb{E}\big((Z(x) - Z(y))^2\big)$. If there exist $c \in (0,1]$, $\eta \in (0,1)$, and $C > 0$ such that*

$$\gamma(x,y) \leq C\, d_M(x,y)^\eta \tag{2.13}$$

for all x, $y \in M$ with $d_M(x,y) < c$, then Z has a modification which is locally Hölder continuous of order τ for all $\tau < \eta/2$.

Proof. Let $\hat{Z}(x) = Z(x) - \mathbb{E}(Z(x))$, $x \in M$, be the associated centered random field. Because $\mathbb{E}\big((\hat{Z}(x) - \hat{Z}(y))^2\big) \leq \gamma(x,y)$, we have for every $n \in \mathbb{N}$ from (2.13)

$$\mathbb{E}\Big(\big(\hat{Z}(x) - \hat{Z}(y)\big)^{2n}\Big) = (2n-1)!!\Big(\mathbb{E}\Big(\big(\hat{Z}(x) - \hat{Z}(y)\big)^2\Big)\Big)^n \leq K_n d_M(x,y)^{n\eta}$$

with $K_n = (2n-1)!!\, C^n$. Let $\alpha > 1$, $\tau \in (0, \eta/2)$, and let n be larger than $d/(\eta - 2\tau)$. Then there is a constant $C' > 0$ such that for all $x, y \in M$ with $d_M(x,y) < c$ the inequality

$$d_M(x,y)^{n\eta} \leq C' \log_2\bigl(d_M(x,y)^{-1}\bigr)^{-\alpha} d_M(x,y)^{d+2n\tau}$$

holds true. It follows therefore from (b) in Corollary 2.3.4 that $\hat{Z}$ has a modification $\tilde{Z}$ which is locally Hölder sample continuous of order τ for all $\tau \in (0, \eta/2)$. On the other hand, we have from (2.13)

$$|\mathbb{E}(Z(x)) - \mathbb{E}(Z(y))| \leq \Bigl(\mathbb{E}\Bigl(\bigl(Z(x) - Z(y)\bigr)^2\Bigr)\Bigr)^{1/2} \leq \sqrt{C}\, d_M(x,y)^{\eta/2}$$

for all $x, y \in M$ with $d_M(x,y) < c$, so that the function $x \mapsto \mathbb{E}(Z(x))$ is locally Hölder continuous of order $\eta/2$. Therefore the random field $\tilde{Z}(x) + \mathbb{E}(Z(x))$, $x \in M$, is a modification of Z which is locally Hölder continuous of every order $\tau \in (0, \eta/2)$. $\square$

2.4. A Global Kolmogorov-Chentsov Theorem for Metric Spaces

Throughout this chapter we assume that (M, d_M) is a metric space, $(\Omega, \mathscr{A}, P)$ is a probability space, and $Z = (Z(x), x \in M)$ is a real-valued random field on M. Let us assume again that we are given two monotonically increasing functions r and q on $[0, c)$ with $r(0) = q(0) = 0$ and some $c > 0$. In addition, we assume that $(D_n, n \in \mathbb{N})$ is an increasing sequence of finite subsets of M. The interpretation is again that each set D_n, $n \in \mathbb{N}$, is a grid in M and the grids become finer as n increases.

In Section 1.2 we have seen that one problem with the application of Theorem 1.2.1 to $\mathbb{S}^2$ is given by the assumption of a constant grid width δ_n for the grids D_n. On $\mathbb{S}^2$, the distance of neighboring grid points at the poles converges faster to 0 than the distance of neighboring points at the equator. The idea for a generalization is therefore to allow the grid width to vary over M. For the following theorem we go one step further and do not assume at all, that the notion of neighborhood in the grids D_n is determined by some condition involving the distance of grid points. That is, we will think of $x, y \in D_n$ as neighbors if their unordered pair $\langle x, y\rangle_{up}$ belongs to the set π_n, where we do not specify the sets π_n and just assume that $\pi_n \subseteq \{\langle x, y\rangle_{up} \,|\, x, y \in D_n\}$. In any application we can therefore *define* which grid points we want to identify as neighbors, for example by using a more convenient metric than the given one. With this abstract notion of neighborhood, we define Δ_n to be the maximal distance of neighboring points in D_n, i.e.

$$\Delta_n = \max_{\langle x,y\rangle_{up} \in \pi_n} d_M(x,y), \quad n \in \mathbb{N}.$$

As we have mentioned in Remark 1.2.2, Condition (c) in Theorem 1.2.1 causes also problems in the application to $\mathbb{S}^2$. We replace this condition therefore with the following *neighborhood condition*:

(NC) For almost all $n \in \mathbb{N}$ there exists $\tilde{\delta}_n > 0$ such that for all $m \geq n$ and all $x, y \in D_m$ with $d_M(x,y) \leq \tilde{\delta}_n$ there are $x_1, y_1 \in D_{m-1}, \ldots, x_{m-n}, y_{m-n} \in D_n$ such that

$$\langle x, x_1\rangle_{up}, \langle y, y_1\rangle_{up} \in \pi_m, \ldots, \langle x_{m-n-1}, x_{m-n}\rangle_{up}, \langle y_{m-n-1}, y_{m-n}\rangle_{up} \in \pi_{n+1},$$
$$\text{and} \quad \langle x_{m-n}, y_{m-n}\rangle_{up} \in \pi_n.$$

The interpretation of Condition (NC) is that grid points which are sufficiently close can be iteratively connected through the transition to neighbors. In case the sets π_n are defined as in (1.4), Condition (c) of Theorem 1.2.1 entails the validity of Condition (NC). Indeed, let us define $\tilde{\delta}_n = \delta_n$, where δ_n is the grid width which defines π_n in (1.4). Let $n \in \mathbb{N}$ be sufficiently large and pick arbitrary $m \geq n$ and $x, y \in D_m$ with $d_M(x,y) \leq \tilde{\delta}_n$. Then applying (c) of Theorem 1.2.1 iteratively results in grid points $x_1, y_1 \in D_{m-1}, \ldots, x_{m-n}, y_{m-n} \in D_n$ such that $\langle x, x_1\rangle_{up}, \langle y, y_1\rangle_{up} \in \pi_m, \ldots, \langle x_{m-n-1}, x_{m-n}\rangle_{up}, \langle y_{m-n-1}, y_{m-n}\rangle_{up} \in \pi_{n+1}$. Furthermore, we have $d_M(x_{m-n}, y_{m-n}) \leq \cdots \leq d_M(x,y) \leq \tilde{\delta}_n$, which implies $\langle x_{m-n}, y_{m-n}\rangle_{up} \in \pi_n$ by definition of π_n. The important difference between the conditions is that under Condition (c) in Theorem 1.2.1 the distance is monotonically decreasing throughout the iteration, while we do not assume such a monotonicity in Condition (NC).

Theorem 2.4.1. *Suppose that $(M, d_M, ((D_n, \pi_n), n \in \mathbb{N}))$, r, q, and the random field Z satisfy the following:*

(a) $D = \bigcup_{n\in\mathbb{N}} D_n$ is dense in (M, d_M);

(b) $(\Delta_n, n \in \mathbb{N})$ is a null sequence;

(c) Condition (NC) is valid;

(d) $\displaystyle\sum_{n\in\mathbb{N}:\Delta_n<c} |\pi_n|\, q(\Delta_n) < +\infty$*;*

(e) $\displaystyle\sum_{n\in\mathbb{N}:\Delta_n<c} r(\Delta_n) < +\infty$*;*

(f) For all $x, y \in M$ with $d_M(x,y) < c$ it is true that

$$P\Big(|Z(x) - Z(y)| > r\big(d_M(x,y)\big)\Big) \leq q\big(d_M(x,y)\big).$$

Then Z has a modification which is uniformly sample continuous.

The proof of Theorem 2.4.1 follows the lines of the corresponding proof given in [70] with subtle adjustments. The first lemma shows that it is sufficient to show uniform continuity on the dense subset D in case Z is continuous in probability.

Lemma 2.4.2. *Suppose that the random field Z is continuous in probability and that almost surely the restriction of Z to D is uniformly sample continuous, i.e., there exists a P-null set $N \in \mathscr{A}$ such that for every $\omega \in N^c$ the mapping $Z(\cdot,\omega)|_D$ is uniformly continuous. Then Z has a modification $\tilde{Z}$ which is uniformly sample continuous and furthermore $Z(x,\omega) = \tilde{Z}(x,\omega)$ for every $(\omega, x) \in N^c \times D$.*

Proof. Let $\tilde{Z}$ be defined as follows: For every $(\omega, x) \in N \times M$ we define $\tilde{Z}(x,\omega) = 0$, and if $(\omega, x) \in N^c \times D$ we let $\tilde{Z}(x,\omega) = Z(x,\omega)$. If $\omega \in N^c$ and $x \in D^c$, we choose a sequence $(x_n, n \in \mathbb{N})$ in D which converges to x. Since $Z(\cdot,\omega)|_D$ is uniformly continuous for every $\omega \in N^c$ and $(x_n, n \in \mathbb{N})$ is a Cauchy sequence, the sequence $(Z(x,\omega), n \in \mathbb{N})$ is Cauchy, thus convergent. Hence we can define $\tilde{Z}(x,\omega) = \lim_{n\to\infty} Z(x_n,\omega)$ for $\omega \in N^c$ and $x \in D^c$ and have now defined the random field $\tilde{Z}$

entirely. A standard $\varepsilon/3$-argument shows that $\tilde{Z}$ is uniformly sample continuous. In order to show that $\tilde{Z}$ is a modification of Z, let $x \in M$ and let $(x_n, n \in \mathbb{N})$ be a sequence in D converging to x. By construction, the sequence $(Z(x_n), n \in \mathbb{N})$ converges almost surely to $\tilde{Z}(x)$. On the other hand, $(Z(x_n), n \in \mathbb{N})$ converges in probability to $Z(x)$ because Z is continuous in probability by assumption. Hence $P\big(Z(x) = \tilde{Z}(x)\big) = 1$. □

Lemma 2.4.3. *The assumptions of Theorem 2.4.1 entail the continuity in probability of Z.*

Proof. By Assumption (a) D is dense in (M, d_M) and because we have excluded the case $M = \emptyset$, D can not be empty. Since $(D_n, n \in \mathbb{N})$ is an increasing sequence, D_n has at least one element for almost all $n \in \mathbb{N}$. Furthermore, for almost all $n \in \mathbb{N}$, there exists a $\tilde{\delta}_n > 0$ such that Condition (NC) is true. We choose an $n \in \mathbb{N}$ which is large enough such that $\tilde{\delta}_n$ exists and $D_n \neq \emptyset$. Then there is an $x \in D_n$ and it is true that $d_M(x, x) \leq \tilde{\delta}_n$ so that Assumption (NC) implies that $\langle x, x\rangle_{up} \in \pi_n$ for almost all $n \in \mathbb{N}$. In consequence, we have $|\pi_n| \geq 1$ for almost all $n \in \mathbb{N}$ and in view of Assumption (d) the sequence $(q(\Delta_n), n \in \mathbb{N})$ has to be a null sequence. The monotonicity of q and Assumption (b) imply $\lim_{n\to\infty} q(a_n) = 0$ for any null sequence $(a_n, n \in \mathbb{N})$ in $[0, c)$. The same holds true with similar arguments for the function r.

Now let $x \in M$. In case x is isolated, there is nothing to show. Thus let $(x_n, n \in \mathbb{N})$ be a sequence converging to x. Then there exists an $n_0 \in \mathbb{N}$, such that $d_M(x, x_n) < c$ for all $n \geq n_0$. Since $\big(r\big(d_M(x, x_n)\big), n \geq n_0\big)$ is converging to 0, we can find for all $\varepsilon > 0$ an $n_1 \geq n_0$ such that $r\big(d_M(x, x_n)\big) \leq \varepsilon$ for all $n \geq n_1$. With the inequality given in Assumption (f) it follows that

$$P\big(|Z(x) - Z(x_n)| > \varepsilon\big) \leq P\Big(|Z(x) - Z(x_n)| > r\big(d_M(x, x_n)\big)\Big) \leq q\big(d_M(x, x_n)\big)$$

for all $n \geq n_1$. Because $\big(q\big(d_M(x, x_n)\big), n \in \mathbb{N}\big)$ converges to 0, the lemma is proven. □

The next lemma uses the Borel–Cantelli lemma and shows that Assumptions (d) to (f) entail a continuity property of Z which is the first step in showing that the assumptions of Lemma 2.4.2 are satisfied.

Lemma 2.4.4. *Under the assumptions of Theorem 2.4.1 there exists a P-null set $N \in \mathscr{A}$ such that for every $\omega \in N^c$ there exists an $n(\omega) \in \mathbb{N}$ with*

$$\max_{\langle x,y\rangle_{up} \in \pi_n} |Z(x, \omega) - Z(y, \omega)| \leq r(\Delta_n) \tag{2.14}$$

for all $n \geq n(\omega)$.

Proof. The sequence $(\Delta_n, n \in \mathbb{N})$ is a null sequence by Assumption (b) so that there exists an $n_0 \in \mathbb{N}$ with $\Delta_n < c$ for all $n \geq n_0$. By the arguments given in Lemma 2.4.3 there is an $n_1 \in \mathbb{N}$ such that π_n is not empty for all $n \geq n_1$. Let $n \geq \max\{n_0, n_1\}$ and $\langle x, y\rangle_{up} \in \pi_n$, then by Assumption (f) we find

$$\begin{aligned} P\big(|Z(x) - Z(y)| > r(\Delta_n)\big) &\leq P\Big(|Z(x) - Z(y)| > r\big(d_M(x, y)\big)\Big) \\ &\leq q\big(d_M(x, y)\big) \\ &\leq q(\Delta_n), \end{aligned}$$

where we have used the monotonicity of r and q and the fact that $\langle x,y\rangle_{up} \in \pi_n$ implies $d_M(x,y) \le \Delta_n$. Consequently,

$$\begin{aligned} P\Big(\max_{\langle x,y\rangle_{up}\in\pi_n} |Z(x)-Z(y)| > r(\Delta_n)\Big) &= P\Big(\bigcup_{\langle x,y\rangle_{up}\in\pi_n} \big\{|Z(x)-Z(y)| > r(\Delta_n)\big\}\Big) \\ &\le \sum_{\langle x,y\rangle_{up}\in\pi_n} P\big(|Z(x)-Z(y)| > r(\Delta_n)\big) \\ &\le \sum_{\langle x,y\rangle_{up}\in\pi_n} q(\Delta_n) \\ &= |\pi_n|\, q(\Delta_n). \end{aligned}$$

Hence Assumption (d) entails that

$$\sum_{n\ge\max\{n_0,n_1\}} P\Big(\max_{\langle x,y\rangle_{up}\in\pi_n} |Z(x)-Z(y)| > r(\Delta_n)\Big)$$

converges, so that by the Borel–Cantelli lemma

$$P\Big(\limsup_{n\ge\max\{n_0,n_1\}} \Big\{\max_{\langle x,y\rangle_{up}\in\pi_n} |Z(x)-Z(y)| > r(\Delta_n)\Big\}\Big) = 0.$$

If we denote this P-null set by N, then $\omega \in N^c$ implies the existence of $n(\omega) \in \mathbb{N}$ such that for all $n \ge n(\omega)$ inequality (2.14) is true. $\square$

Now let us take Condition (NC) into account.

Lemma 2.4.5. *Under the assumptions of Theorem 2.4.1 there exists a P-null set $N \in \mathscr{A}$ and for every $\omega \in N^c$ there exists $n(\omega) \in \mathbb{N}$ such that the following statement is true for all $n \ge n(\omega)$: There exists a $\tilde\delta_n > 0$ such that for all $m \ge n$ and all $x,y \in D_m$ with $d_M(x,y) \le \tilde\delta_n$ we have*

$$|Z(x,\omega) - Z(y,\omega)| \le 2\sum_{i=n}^{m} r(\Delta_i). \tag{2.15}$$

Proof. Let N, ω, and $n(\omega)$ be as in Lemma 2.4.4. Because the statement of Lemma 2.4.4 is true for all $n \ge n(\omega)$, we may without loss of generality assume that $n(\omega)$ is large enough such that for all $n \ge n(\omega)$ we are in the situation of Assumption (NC). In particular, $n \ge n(\omega)$ implies that $\Delta_n < c$ (see Lemma 2.4.4) so that the bound in (2.15) is well-defined. Let us now pick an arbitrary $n \ge n(\omega)$ and consider the case $m = n$ first. In that case Assumption (NC) implies the existence of a $\tilde\delta_n$ such that for all $x,y \in D_n$ with $d_M(x,y) \le \tilde\delta_n$ we have $\langle x,y\rangle_{up} \in \pi_n$. Thus by Lemma 2.4.4 we have

$$|Z(x,\omega)-Z(y,\omega)| \le \max_{\langle x,y\rangle_{up}\in\pi_n} |Z(x,\omega)-Z(y,\omega)| \le r(\Delta_n) \le 2\sum_{i=n}^{m} r(\Delta_i)$$

and (2.15) is true in case $m = n$. If $m > n$, Assumption (NC) entails the existence of $x_1, y_1 \in D_{m-1}, \dots, x_{m-n}, y_{m-n} \in D_n$ such that

$$\begin{gathered} \langle x,x_1\rangle_{up}, \langle y,y_1\rangle_{up} \in \pi_m, \dots, \langle x_{m-n-1},x_{m-n}\rangle_{up}, \langle y_{m-n-1},y_{m-n}\rangle_{up} \in \pi_{n+1} \\ \text{and } \langle x_{m-n},y_{m-n}\rangle_{up} \in \pi_n. \end{gathered}$$

Let us define $x_0 = x$ and $y_0 = y$, then we can estimate with Lemma 2.4.4 as follows:

$$\begin{aligned}
|Z(x,\omega) - Z(y,\omega)| &\leq \sum_{i=0}^{m-n-1} \Big(|Z(x_i,\omega) - Z(x_{i+1},\omega)| + |Z(y_i,\omega) - Z(y_{i+1},\omega)| \Big) \\
&\qquad + |Z(x_{m-n},\omega) - Z(y_{m-n},\omega)| \\
&\leq r(\Delta_n) + \sum_{i=0}^{m-n-1} 2r(\Delta_{m-i}) \\
&\leq 2 \sum_{i=0}^{m-n} r(\Delta_{m-i}) \\
&= 2 \sum_{i=n}^{m} r(\Delta_i).
\end{aligned}$$

This concludes the proof. □

The statement of Lemma 2.4.5 can now be utilized to show that the assumptions of Lemma 2.4.2 are satisfied.

Corollary 2.4.6. *Under the assumptions of Theorem 2.4.1 there exists a P-null set $N \in \mathscr{A}$ such that for every $\omega \in N^c$ the mapping $Z(\cdot,\omega)|_D$ is uniformly continuous.*

Proof. Let N, $\omega \in N^c$, and $n(\omega)$ be as in Lemma 2.4.5 and fix an arbitrary $\varepsilon > 0$. Because of Assumption (e) there exists an $n_0 \in \mathbb{N}$, such that $\sum_{i=n}^{\infty} r(\Delta_i) \leq \varepsilon/2$ for all $n \geq n_0$. Choose

$$\delta(\omega) = \tilde{\delta}_{\max\{n_0, n(\omega)\}}.$$

If $x, y \in D$, then there is an $m \in \mathbb{N}$ such that $x, y \in D_m$ because $(D_n, n \in \mathbb{N})$ is increasing and furthermore we may assume that $m \geq \max\{n_0, n(\omega)\}$. For any $x, y \in D$ with $d_M(x,y) \leq \delta(\omega)$ we have therefore by Lemma 2.4.5

$$|Z(x,\omega) - Z(y,\omega)| \leq 2 \sum_{i=\max\{n_0,n(\omega)\}}^{m} r(\Delta_i) \leq 2 \sum_{i=n_0}^{\infty} r(\Delta_i) \leq \varepsilon.$$

□

The proof of Theorem 2.4.1 follows now from the preceding statements.

Proof of Theorem 2.4.1. Lemma 2.4.3 shows, that Z is continuous in probability under the given assumptions. Corollary 2.4.6 provides the continuity property that is presumed in Lemma 2.4.2. The assertion then follows by Lemma 2.4.2. □

Remark 2.4.7. It is tempting to follow [70] and formulate a variant of Theorem 2.4.1 which yields higher regularity in terms of orders of Hölder continuity under the more restrictive assumptions

$$\begin{aligned}
&\frac{1}{\alpha}\,\eta^n \leq \Delta_n \leq \alpha\,\eta^n, \quad n \in \mathbb{N}, \alpha > 0, \eta \in (0,1), \\
&r(0) = 0, \quad r(h) \leq h^\tau, \quad h \in (0,c), \tau, K_\tau > 0.
\end{aligned}$$

In order to transfer the proof given in [70] to our scenario we have to assume that the quantities $\tilde{\delta}_n$ of Assumption (NC) and Δ_n are equal ($\tilde{\delta}_n \leq \Delta_n$ is not sufficient). However, the application of Theorem 2.4.1 to $\mathbb{S}^2$ in the coming chapter shows that such an assumption would not be reasonable. As there seems to be no apparent workaround, the author considers the formulation and the proof of a variant of Theorem 2.4.1 which yields a Hölder continuous modification as an open problem.

3. Application and Comparison

3.1. Application to $\mathbb{S}^2$

Let us begin by showing that both Theorem 2.1.3 and Theorem 2.4.1 can be applied to the metric space $(\mathbb{S}^2, d_{\mathbb{S}^2})$. We start with the global version of Section 2.4.

As it was done in Section 1.2, we choose the spherical coordinate grids D_n, $n \in \mathbb{N}$, defined by

$$D_n = \Big\{\phi_2\Big(k\frac{\pi}{2^{n-1}}, l\frac{\pi}{2^{n-1}}\Big)\Big| k = 0, 1, \dots, 2^{n-1}, l = 0, 1, \dots, 2^n - 1\Big\}, \quad n \in \mathbb{N},$$

so that the continuity of ϕ_2 entails Condition (a) of Theorem 2.4.1.

In what follows we will work with polar coordinates θ_x, φ_x associated to $x \in \mathbb{S}^2$. For any $x \in \mathbb{S}^2$ which is neither the North Pole nor the South Pole, there are unique $\theta_x \in (0, \pi), \varphi_x \in [0, 2\pi)$ such that $\phi_2(\theta_x, \varphi_x) = x$. In case $x = (0, 0, 1)$ or $x = (0, 0, -1)$, θ_x is either 0 or π and we can pick an arbitrary $\tilde{\varphi}_x \in [0, 2\pi)$ such that $\phi_2(\theta_x, \tilde{\varphi}_x) = x$ holds true. Every statement that we make here and henceforth is independent of this particular choice of the azimuthal angle $\tilde{\varphi}_x$, and therefore - with a slight abuse of language - we will speak of (θ_x, φ_x) as *the* polar coordinates of $x \in \mathbb{S}^2$.

From the examples in [70] it can be seen that it is convenient to work with the maximum metric in the grids in $\mathbb{R}^d$. Since we have transformed such grids to $\mathbb{S}^2$ with the help of the spherical coordinate map ϕ_2, it is therefore natural to define a similar maximum metric on $\mathbb{S}^2$. The idea in the following definition of a maximum metric d_∞ on $\mathbb{S}^2$ is that whenever one of the points x, y is a pole the distance $d_\infty(x, y)$ is independent of the azimuthal angles φ_x, φ_y. Furthermore, one has to pay attention with the identification of points for which the azimuthal angle is at the boundaries of $[0, 2\pi)$. We define

$$\begin{aligned} d_\infty(x, y) &= \min\{d^1_\infty(x, y), d^2_\infty(x, y)\} \qquad (3.1)\\ \text{with} \quad d^1_\infty(x, y) &= \max\big\{|\theta_x - \theta_y|, \min\{|\varphi_x - \varphi_y|, 2\pi - |\varphi_x - \varphi_y|\}\big\} \\ \text{and} \quad d^2_\infty(x, y) &= \min\{\theta_x + \theta_y, 2\pi - (\theta_x + \theta_y)\}. \end{aligned}$$

Note that if x is either the North Pole or the South Pole, we have $d_\infty(x, y) = d^2_\infty(x, y)$ for all $y \in \mathbb{S}^2$ so that the distance $d_\infty(x, y)$ is independent of the choice of the azimuthal angle $\tilde{\varphi}_x$ in case $\theta_x \in \{0, \pi\}$.

With a little effort it can be shown that (3.1) does indeed define a metric on $\mathbb{S}^2$, but this statement is not necessary for our purposes. Let us now define the sets π_n which define the notion of neighborhood in the grids D_n by

$$\pi_n = \Big\{\langle x, y\rangle_{up} | x, y \in D_n, d_\infty(x, y) \le \frac{\pi}{2^{n-1}}\Big\}, \quad n \in \mathbb{N}. \qquad (3.2)$$

It follows now from Lemma A.1.2 that

$$\Delta_n = \max_{\langle x,y\rangle_{up} \in \pi_n} d_{\mathbb{S}^2}(x, y) = \arccos\Big(\cos^2\Big(\frac{\pi}{2^{n-1}}\Big)\Big), \quad n \ge 2, \qquad (3.3)$$

and we see that $(\Delta_n, n \in \mathbb{N})$ is a null sequence, so that Condition (b) of Theorem 2.4.1 is also satisfied. Let us turn to Condition (NC) of Theorem 2.4.1. For almost all $n \in \mathbb{N}$, we have to find a critical distance $\tilde{\delta}_n > 0$ such that all grid points x, y in finer grids D_m, $m \geq n$, which satisfy $d_{\mathbb{S}^2}(x,y) \leq \tilde{\delta}_n$ can be connected through the transition to neighbors. The following lemma shows, that we can find such a $\tilde{\delta}$ if we would haven taken the metric d_∞ instead of $d_{\mathbb{S}^2}$.

Lemma 3.1.1. *Let $n \geq 3$, $m \geq n$, and $x, y \in D_m$. Then $d_\infty(x,y) \leq \pi/2^{n-1}$ implies the existence of $x_1, y_1 \in D_{m-1}, \dots, x_{m-n}, y_{m-n} \in D_n$ such that*

$$\langle x, x_1\rangle_{up}, \langle y, y_1\rangle_{up} \in \pi_m, \dots, \langle x_{m-n-1}, x_{m-n}\rangle_{up}, \langle y_{m-n-1}, y_{m-n}\rangle_{up} \in \pi_{n+1},$$
$$\text{and} \quad \langle x_{m-n}, y_{m-n}\rangle_{up} \in \pi_n.$$

The proof of this lemma can be found in Appendix A.2. The next lemma shows, that there is a sufficient condition in terms of $d_{\mathbb{S}^2}$ such that $d_\infty(x,y) \leq \pi/2^{n-1}$ holds true. The proof is also deferred to Appendix A.2.

Lemma 3.1.2. *Let $n \geq 2$, $m \geq n$, and $x, y \in D_m$. Then*

$$d_{\mathbb{S}^2}(x,y) \leq \arccos\Big(\sin^2\Big(\frac{\pi}{2^n}\Big)\cos\Big(\frac{\pi}{2^{n-1}}\Big) + \cos^2\Big(\frac{\pi}{2^n}\Big)\Big)$$

implies $d_\infty(x,y) \leq \pi/2^{n-1}$.

Hence if we choose

$$\tilde{\delta}_n = \arccos\Big(\sin^2\Big(\frac{\pi}{2^n}\Big)\cos\Big(\frac{\pi}{2^{n-1}}\Big) + \cos^2\Big(\frac{\pi}{2^n}\Big)\Big), \quad n \in \mathbb{N},$$

it follows from Lemmas 3.1.1 and 3.1.2 that Condition (NC) in Theorem 2.4.1 is satisfied.

Remark 3.1.3. The formulation in Lemma 3.1.2 is tailored to fit our purposes. However, a glance at the proof in Appendix A.2 shows, that although we have used the representation $x = \phi_2(k_x\pi/2^{m-1}, l_x\pi/2^{m-1})$ and $y = \phi_2(k_y\pi/2^{m-1}, l_y\pi/2^{m-1})$ for $x, y \in D_m$, the proof does not require k_x, k_y, l_x, l_y to be integer-valued. From this we conclude that the statement of Lemma 3.1.2 holds true for all $x, y \in M$ rather than just for grid points $x, y \in D_m$. Let us furthermore observe, that the sequences $(\tilde{\delta}_n, n \in \mathbb{N})$ and $(\pi/2^{n-1}, n \in \mathbb{N})$ are null sequences. Then it is easy to see that Lemma 3.1.2 implies the following statement:

> For all $x \in M$ and for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $y \in M$
> $d_{\mathbb{S}^2}(x,y) < \delta$ implies $d_\infty(x,y) < \varepsilon$.

With other words, for each $x \in M$ and every $\varepsilon > 0$ there exists a $\delta > 0$ such that an open $d_{\mathbb{S}^2}$-ball with radius δ and centered at x is contained in an open d_∞-ball with radius ε and centered at x. Therefore, every subset of M that is open in the topology induced by d_∞ is also open in the topology induced by $d_{\mathbb{S}^2}$. With similar arguments as in the proof of Lemma 3.1.2 it can be shown that the reverse statement is also true, and it follows that $d_{\mathbb{S}^2}$ and d_∞ are *topologically equivalent* (for instance [11]). Note that this property is weaker than the conventional equivalence of metrics. Indeed, for $n \in \mathbb{N}$ let $x_n = \phi_2(1/n, 1/n)$ and $y_n = \phi_2(1/n, 0)$, then

$$\begin{aligned}\frac{d_{\mathbb{S}^2}(x_n,y_n)}{d_\infty(x_n,y_n)} &= \frac{\arccos\big(\sin^2(1/n)\cos(1/n) + \cos^2(1/n)\big)}{1/n}\\ &= \frac{\arccos\big(\sin^2(1/n)\cos(1/n) + \cos^2(1/n)\big)}{\arccos(\cos(1/n))},\end{aligned}$$

and an application of inequality (A.11) just as in Corollary A.1.4 shows that the sequence $(d_{\mathbb{S}^2}(x_n, y_n)/d_\infty(x_n, y_n), n \in \mathbb{N})$ converges to 0. Therefore there can not be constants $c, C > 0$ such that

$$c\, d_\infty(x, y) \leq d_{\mathbb{S}^2}(x, y) \leq C\, d_\infty(x, y)$$

for all $x, y \in \mathbb{S}^2$.

At this point one may think of applying Theorem 2.4.1 to the metric space $(\mathbb{S}^2, d_\infty)$ instead of $(\mathbb{S}^2, d_{\mathbb{S}^2})$, since it is easier then to validate Conditions (b) and (NC) of Theorem 2.4.1. We chose not do so because in this case the condition on the random field (f) in Theorem 2.4.1 must be formulated also in terms of d_∞, and we believe that on $\mathbb{S}^2$ it is more natural to have a condition which is stated in terms of the great circle metric $d_{\mathbb{S}^2}$.

In view of Condition (d) of Theorem 2.4.1 we estimate the number of elements in the sets π_n, $n \in \mathbb{N}$. As in Section 1.2, let $C_n(x) = \{y \in D_n \mid \langle x, y\rangle_{up} \in \pi_n\}$ be the clique of a grid point $x \in D_n$, then again

$$|\pi_n| \leq \sum_{x \in D_n} \sum_{y \in D_n} \mathbb{1}_{C_n(x)}(y).$$

If x is either the North Pole or the South Pole, its clique $C_n(x)$ contains the pole itself and any grid point on the nearest circle of latitude. The cliques of grid points on the first and last circle of latitude contain 7 grid points, and any other clique contains 3^2 grid points. Hence it is true that

$$|\pi_n| \leq 2\,(2^n + 1) + 2\,2^n\,7 + (2^{n-1} - 3)\,2^n\,3^2 \leq K_2\,2^{2n}$$

with $K_2 = 3^2$.

Let $c \in (0, \pi/2]$ and define the function s by

$$s(h) = \frac{1}{2\pi} \arccos\big(\sqrt{\cos(h)}\big), \quad h \in [0, c), \tag{3.4}$$

then we get from (3.3)

$$s(\Delta_n) = 2^{-n} \tag{3.5}$$

for all $n \in \mathbb{N}$ such that $\Delta_n < c$. Let us now choose r and q as in Section 2.2, i.e., for $h \in \big(0, \arccos\big(\sqrt{\cos(c)}\big)/(2\pi)\big)$ and some constants $K > 0$, $\alpha, \tilde{\alpha} > 1$ we define

$$r(h) = \log_2(h^{-1})^{-\alpha}, \quad q(h) = K \log_2(h^{-1})^{-\tilde{\alpha}} h^2,$$

and for $h = 0$ we let $r(0) = q(0) = 0$. If we now define

$$\tilde{r} = r \circ s, \quad \tilde{q} = q \circ s, \tag{3.6}$$

then it follows from (3.5) and the arguments given in Section 2.2 that Conditions (d) and (e) of Theorem 2.4.1 are satisfied for $\tilde{r}$ and $\tilde{q}$. We can thus apply Theorem 2.4.1 to $(\mathbb{S}^2, d_{\mathbb{S}^2})$ and obtain the following

Corollary 3.1.4. *Suppose Z is a random field on $\mathbb{S}^2$ such that for all $x, y \in \mathbb{S}^2$ with $d_{\mathbb{S}^2}(x, y) < c$ the inequality*

$$P\Big(|Z(x) - Z(y)| > \tilde{r}\big(d_{\mathbb{S}^2}(x, y)\big)\Big) \leq \tilde{q}\big(d_{\mathbb{S}^2}(x, y)\big)$$

holds true, where the functions $\tilde{r}$ and $\tilde{q}$ are defined as in (3.6) *for some constants $K > 0$, $\alpha, \tilde{\alpha} > 1$ and $c \in (0, \pi/2]$. Then Z has a modification which is uniformly sample continuous.*

For the application of Theorem 2.1.3 to $(\mathbb{S}^2, d_{\mathbb{S}^2})$ we have done all the work already in Section 2.2 in showing that Theorem 2.1.3 can be applied to any Riemannian manifold. Hence the choice $(M, g) = (\mathbb{S}^2, \mathring{g})$ in Theorem 2.2.1 and Corollary 2.3.2 yields immediately the following corollaries:

Corollary 3.1.5. *Suppose Z is a random field on $\mathbb{S}^2$ such that for all $x, y \in \mathbb{S}^2$ with $d_{\mathbb{S}^2}(x, y) < c$ the inequality*

$$P\Big(|Z(x) - Z(y)| > r\big(d_{\mathbb{S}^2}(x, y)\big)\Big) \leq q\big(d_{\mathbb{S}^2}(x, y)\big)$$

holds true, where the functions r and q are defined as in (2.7) *and* (2.8) *for some constants $K > 0$, $\alpha, \tilde{\alpha} > 1$ and $c \in (0, 1]$. Then Z has a modification which is locally uniformly sample continuous.*

Corollary 3.1.6. *Suppose Z is a random field on $\mathbb{S}^2$ such that for all $x, y \in \mathbb{S}^2$ with $d_{\mathbb{S}^2}(x, y) < c$ we have the inequality*

$$P\Big(|Z(x) - Z(y)| > r\big(d_{\mathbb{S}^2}(x, y)\big)\Big) \leq q\big(d_{\mathbb{S}^2}(x, y)\big),$$

where the functions r and q are defined as in (2.10) *and* (2.8) *for some constants $K > 0$, $\alpha > 1$, $\tau \in (0, 1)$ and $c \in (0, 1]$. Then Z has a locally Hölder sample continuous modification of order τ.*

3.2. Comparison

From the application of Theorems 2.1.3 and 2.4.1 it is evident, that the major difference between the approaches of Sections 2.1 and 2.4 lies in their applicability. The local approach in Section 2.1 is tailored to produce an easy to validate criterion for the existence of a continuous modification of random fields on any Riemannian manifold. The application of Theorem 2.4.1 in Section 3.1 required the definition of a suitable grid on the sphere, the identification of neighboring grid points, and in order to validate Conditions (b) and (NC) of Theorem 2.4.1 we had to compare and compute distances in the grids and find the necessary techniques that enable us to do so. An application to a different Riemannian manifold than $\mathbb{S}^2$ is immediate from Theorem 2.2.1, but we can not hope that any of the constructions made in Section 3.1 transfer to different examples and a thorough understanding of the underlying geometry would be necessary in order to apply Theorem 2.4.1. Therefore it is evident that from a practical point of view the local approach of Section 2.1 should be preferred.

Thus let us discuss the results that both approaches provide. In theory, Theorem 2.4.1 may provide the existence of a uniformly sample continuous modification of a random field while Theorem 2.1.3 can provide the existence of a locally uniformly sample continuous modification. However, in our example $M = \mathbb{S}^2$ this disparity vanished because the sphere is compact and any locally uniformly continuous function thereon is also uniformly continuous. Furthermore, as in both approaches the continuity-δ is non-explicit because it is constructed from an index $n(\omega)$ which comes from the Borel–Cantelli lemma, the difference between local uniform continuity and uniform continuity is only of theoretical nature. Nonetheless it would be interesting to know whether there is actually a non-trivial example of a Riemannian manifold for which the application of Theorem 2.4.1 produces a uniformly sample continuous modification while the modification provided in Theorem 2.2.1 is only locally uniformly sample continuous. This may be an area of further research.

A significant drawback of the approach given in Section 2.4 is its non-compatibility with the conventional techniques that produce a criterion for Hölder continuity. This non-compatibility suggests that the generalization of the assumptions in [70] to that in Theorem 2.4.1 may not be the most natural one.

Finally, let us discuss the different conditions on the random field in case $M = \mathbb{S}^2$. The functions r and q in Corollary 3.1.5 and the functions $\tilde{r}$ and $\tilde{q}$ in Corollary 3.1.4 have been chosen such that the series

$$\sum_{k\in\mathbb{N}:\delta_{n,k}<c} r(\delta_{n,k}) \quad \text{and} \quad \sum_{k\in\mathbb{N}:\delta_{n,k}<c} |\pi_{n,k}|\, q(\delta_{n,k})$$

and

$$\sum_{n\in\mathbb{N}:\Delta_n<c} r(\Delta_n) \quad \text{and} \quad \sum_{n\in\mathbb{N}:\Delta_n<c} |\pi_n|\, q(\Delta_n)$$

have the same convergence behaviour. The different conditions on the random field are therefore comparable. It is easy to see that the function s defined in (3.4) is strictly smaller than the identity for all sufficiently small arguments. Therefore we have for all sufficiently small $h > 0$ the inequalities $\tilde{r}(h) < r(h)$ and $\tilde{q}(h) < q(h)$, so that

$$\begin{aligned} P\Big(|Z(x) - Z(y)| > r\big(d_{\mathbb{S}^2}(x,y)\big)\Big) &\le P\Big(|Z(x) - Z(y)| > \tilde{r}\big(d_{\mathbb{S}^2}(x,y)\big)\Big) \\ &\le \tilde{q}\big(d_{\mathbb{S}^2}(x,y)\big) \\ &\le q\big(d_{\mathbb{S}^2}(x,y)\big). \end{aligned}$$

This shows that the condition on the random field which is formulated in Corollary 3.1.4 is more restrictive than the condition in Corollary 3.1.5.

Altogether we conclude, that the local approach in Section 2.1 should be preferred to the approach given in Section 2.4.

Part II.

Random Field Models

4. A General Class of Mosaic Random Fields

4.1. Preliminaries

4.1.1. Covariance Functions

Let M be a non-empty set. It is well-known that a function $C : M \times M \to \mathbb{R}$ is *symmetric* and *positive definite*, i.e.

$$C(x, y) = C(y, x), \quad x, y \in M, \tag{4.1}$$

$$\text{and} \quad \sum_{i,j=1}^{n} \alpha_i \alpha_j C(x_i, x_j) \geq 0, \quad n \in \mathbb{N}, \{x_1, \ldots, x_n\} \subseteq M, \alpha_1, \ldots, \alpha_n \in \mathbb{R}, \tag{4.2}$$

if and only if there is a probability space $(\Omega, \mathscr{A}, P)$ and a real-valued random field $Z = (Z(x), x \in M)$ thereon such that C is the covariance function of Z. If a covariance function C satisfies $C(x, x) = 1$ for all $x \in M$, the covariance function and the correlation function of the corresponding random field coincide. We use this characterization as a definition.

Definition 4.1.1. 1. A function $C : M \times M \to \mathbb{R}$ which is symmetric and positive definite is called a real-valued *covariance function* on M.

2. A real-valued covariance function ρ on M which satisfies $\rho(x, x) = 1$ for all $x \in M$ is called a real-valued *correlation function* on M.

3. A real-valued covariance function (correlation function) on M is called *strictly positive definite*, if the inequality in (4.2) is strict whenever at least one of the $\alpha_1, \ldots, \alpha_n$ does not vanish.

Because only real-valued covariance functions are considered in this thesis, we will omit the prefix *real-valued* henceforth.

From [7, page 69] we have the inequality $|C(x, y)|^2 \leq C(x, x)\, C(y, y)$, $x, y \in M$, which is valid for any covariance function on M. Therefore the range of a correlation function is contained in $[-1, 1]$. The set of all covariance functions (correlation functions) on M has the following closure properties (e.g. [7, Chapter 3, §1]).

Lemma 4.1.2. *1. If C_1 and C_2 are covariance functions on M and $\alpha, \beta \geq 0$, then $\alpha\, C_1 + \beta\, C_2$ is a covariance function on M.*

2. If ρ_1 and ρ_2 are correlation functions on M and $\lambda \in [0, 1]$, then the function $\lambda\, \rho_1 + (1 - \lambda)\, \rho_2$ is a correlation function on M.

3. If C_1 and C_2 are covariance functions (correlation functions) on M, then $C_1 \cdot C_2$ is a covariance function (correlation function) on M.

4. *If $(C_n, n \in \mathbb{N})$ is a sequence of covariance functions (correlation functions) on M which converges pointwise to a function C on $M \times M$, then C is a covariance function (correlation function) on M.*

A function $C : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is called *stationary* if

$$C(x, y) = C(x + h, y + h), \quad x, y, h \in \mathbb{R}^d, \tag{4.3}$$

and it is called *isotropic* if

$$C(x, y) = C(\mathcal{R}x, \mathcal{R}y), \quad x, y \in \mathbb{R}^d, \mathcal{R} \in \mathrm{SO}(d). \tag{4.4}$$

Here $\mathrm{SO}(d)$ denotes the special orthogonal group in dimension d, i.e.

$$\mathrm{SO}(d) = \{\mathcal{R} \in \mathbb{R}^{d\times d} \,|\, \mathcal{R}'\mathcal{R} = I, \det(\mathcal{R}) = 1\},$$

and the group operation is matrix multiplication. Note that $\mathrm{SO}(d)$ acts also on $\mathbb{S}^{d-1}$ and that this action is transitive, i.e., for all $x, y \in \mathbb{S}^{d-1}$ there exists an $\mathcal{R} \in \mathrm{SO}(d)$ such that $\mathcal{R}x = y$.

If $C : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is stationary and isotropic, $x, y \in \mathbb{R}^d$, $x \neq y$, and if we take $h = -y$ in (4.3) and the rotation $\mathcal{R}_{xy}$ which maps $(x - y)/\|x - y\| \in \mathbb{S}^{d-1}$ to the unit vector $e_d = (0, \ldots, 0, 1)$ in (4.4), then

$$C(x, y) = C(x - y, 0) = C\big(\|x - y\|e_d, 0\big).$$

If $x = y$ we have already after the translation $C(x, x) = C(0, 0) = C\big(\|x - x\|e_d, 0\big)$. Hence a stationary and isotropic covariance function is a function of the distance of x and y only. If on the other hand there is a function $\tilde{C} : [0, \infty) \to \mathbb{R}$ such that for $C : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$

$$C(x, y) = \tilde{C}\big(\|x - y\|\big), \quad x, y \in \mathbb{R}^d, \tag{4.5}$$

is true, then C is clearly stationary and isotropic. Therefore in $\mathbb{R}^d$ Condition (4.5) is true if and only if both Conditions (4.3) and (4.4) are satisfied.

On an arbitrary set M stationarity or isotropy may not be well-defined. However, we can use (4.5) as a definition for stationary and isotropic functions on a metric space (M, d_M). Recall that the diameter of a metric space (M, d_M) is defined by

$$\operatorname{diam} M = \sup\{d_M(x, y) \,|\, x, y \in M\}.$$

Definition 4.1.3. Let (M, d_M) be a metric space and define

$$I_M = \begin{cases} [0, \operatorname{diam} M), & \operatorname{diam} M = \infty, \\ [0, \operatorname{diam} M], & \operatorname{diam} M < \infty. \end{cases}$$

1. A function $C : M \times M \to \mathbb{R}$ is called *stationary and isotropic* if there is a function $\tilde{C} : I_M \to \mathbb{R}$ such that

$$C(x, y) = \tilde{C}\big(d_M(x, y)\big), \quad x, y \in M.$$

2. A function $\tilde{C} : I_M \to \mathbb{R}$ is said to be a (*strictly positive definite*) *covariance function* on M, if the stationary and isotropic function $C : M \times M \to \mathbb{R}$ defined by $C(x, y) = \tilde{C}\big(d_M(x, y)\big)$ is a (strictly positive definite) covariance function on M.

The metric space $(\mathbb{S}^d, d_{\mathbb{S}^d})$ is somewhat special. Because $\mathrm{SO}(d+1)$ acts on $\mathbb{S}^d$, we may call a function $C : \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{R}$ in analogy to (4.4) *isotropic* if

$$C(x, y) = C(\mathcal{R}x, \mathcal{R}y), \quad x, y \in \mathbb{S}^d, \mathcal{R} \in \mathrm{SO}(d+1). \tag{4.6}$$

This definition is in line with the definition of weakly isotropic random fields on $\mathbb{S}^d$ in [59, 52] and that of homogeneous random fields on homogeneous spaces in [87]. It follows from the definition of the metric $d_{\mathbb{S}^d}$ that a function $C : \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{R}$ which is stationary and isotropic in the sense of Definition 4.1.3, is also isotropic in the sense of (4.6). The reverse implication is also true but not as obvious. For convenience of the reader a proof is given in Appendix B.2. Because of this equivalence, we call a function $C : \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{R}$ which is stationary and isotropic in the sense of Definition 4.1.3 simply isotropic.

4.1.2. Spectral Representation of Stationary and Isotropic Covariance Functions on $\mathbb{R}^d$ and $\mathbb{S}^d$

Let $M = \mathbb{R}^d$, $d \geq 1$. In this chapter we denote the Euclidean inner product on $\mathbb{R}^d$ by $\langle \cdot, \cdot \rangle$ and the corresponding norm is denoted by $\| \cdot \|$. For any subset A of $\mathbb{R}^d$ we write $\mathscr{B}^d(A)$ for the trace σ-algebra of A in $\mathscr{B}(\mathbb{R}^d)$ where $\mathscr{B}(\mathbb{R}^d)$ denotes the Borel σ-algebra over $\mathbb{R}^d$. The Lebesgue-measure on $\big(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d)\big)$ is denoted by λ^d.

The following characterization of continuous, stationary, and isotropic covariance functions on $\mathbb{R}^d$ due to Schoenberg [76] is based on Bochner's characterization of characteristic functions [12]. The *Bessel function of the first kind* J_ν is defined by (for instance [35, Section 8.4])

$$J_\nu(z) = \frac{z^\nu}{2^\nu} \sum_{n \in \mathbb{N}_0} (-1)^n \frac{z^{2n}}{2^{2n} n! \Gamma(\nu + n + 1)}, \quad \nu \in \mathbb{C}, |\arg z| < \pi. \tag{4.7}$$

Theorem 4.1.4 (Schoenberg, 1938)**.** *Every continuous, stationary, and isotropic covariance function $C : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is of the form*

$$C(x, y) = \int_{[0,\infty)} \Omega_d\big(r\|x - y\|\big)\, d\mu(r), \quad x, y \in \mathbb{R}^d, \tag{4.8}$$

where μ is a uniquely determined finite measure on $\big([0,\infty), \mathscr{B}^1\big([0,\infty)\big)\big)$ and for all $d \in \mathbb{N}$

$$\begin{aligned} \Omega_d(r) &= 1 - \frac{r^2}{2d} + \frac{r^4}{2 \cdot 4 \cdot d(d+2)} - \frac{r^6}{2 \cdot 4 \cdot 6 \cdot d(d+2)(d+4)} + \dots && (r \geq 0) \\ &= \Gamma\Big(\frac{d}{2}\Big)\Big(\frac{2}{r}\Big)^{(d-2)/2} J_{(d-2)/2}(r). && (r > 0) \end{aligned}$$

If ρ is a correlation function we have $\rho(x, x) = 1 = \mu\big([0,\infty)\big)$ from (4.8) for the corresponding measure μ, therefore the measure μ associated with a correlation function is a probability measure. The set consisting of all continuous correlation functions $\tilde{\rho} : [0,\infty) \to [-1, 1]$ on $\mathbb{R}^d$ is denoted by Φ_d. In view of Definition 4.1.3 we may identify Φ_d with the class of all continuous, stationary, and isotropic correlation functions $\rho : \mathbb{R}^d \times \mathbb{R}^d \to [-1, 1]$ on $\mathbb{R}^d$ and Theorem 4.1.4 shows that this set is parametrized by the set of all probability measures on $[0,\infty)$. It can be shown (see [33] and references therein) that

$$\Phi_1 \supset \Phi_2 \supset \cdots \supset \Phi_\infty = \bigcap_{d\in\mathbb{N}} \Phi_d,$$

with the inclusions being strict.

In 1942, Schoenberg published a paper [77] in which an analogous result was given for the sphere $M=\mathbb{S}^d$. The base functions on the sphere are the *Gegenbauer polynomials* C_n^λ, which are defined by the expansion (e.g., [26, Formula 18.12.4])

$$\sum_{n\in\mathbb{N}_0} C_n^\lambda(x) z^n = \frac{1}{\big(1-2xz+z^2\big)^\lambda}, \quad n\in\mathbb{N}_0, \lambda>0, x\in[-1,1], |z|<1. \tag{4.9}$$

For $\lambda=0$ the functions C_n^0 are defined in [77] as

$$C_n^0\big(\cos(\theta)\big) = \cos(n\theta), \quad \theta\in[0,\pi], \tag{4.10}$$

and we adopt this definition. Note that for $\lambda = 1/2$ the generating function in (4.9) is the generating function of the classical Legendre polynomials P_n so that $C_n^{1/2} = P_n$. The representation of continuous and isotropic covariance functions on the sphere is as follows.

Theorem 4.1.5 (Schoenberg, 1942)**.** *Every continuous and isotropic covariance function $C:\mathbb{S}^d\times\mathbb{S}^d\to\mathbb{R}$ is of the form*

$$C(x,y) = \sum_{n\in\mathbb{N}_0} a_{n,d}\, C_n^{(d-1)/2}\Big(\cos\big(d_{\mathbb{S}^d}(x,y)\big)\Big), \quad x,y\in\mathbb{S}^d, \tag{4.11}$$

with non-negative, uniquely determined coefficients $(a_{n,d}, n\in\mathbb{N}_0)$ such that the series $\sum_{n\in\mathbb{N}_0} a_{n,d}\, C_n^{(d-1)/2}(1)$ converges.

From Theorem 4.1.5 it follows that every continuous and isotropic correlation function ρ on the sphere $\mathbb{S}^d$ admits the representation

$$\rho(x,y) = \sum_{n\in\mathbb{N}_0} b_{n,d} \frac{C_n^{(d-1)/2}\Big(\cos\big(d_{\mathbb{S}^d}(x,y)\big)\Big)}{C_n^{(d-1)/2}(1)}, \quad x,y\in\mathbb{S}^d, \tag{4.12}$$

with a uniquely determined sequence $(b_{n,d}, n\in\mathbb{N}_0)$ of probabilities $b_{n,d}\in[0,1]$, $\sum_{n\in\mathbb{N}_0} b_{n,d} = 1$. In analogy to the case $M=\mathbb{R}^d$, the set consisting of all continuous correlation functions $\tilde{\rho}:[0,\pi]\to[-1,1]$ on $\mathbb{S}^d$ is denoted by Ψ_d and the sets Ψ_d are non-increasing in d (see [33])

$$\Psi_1 \supset \Psi_2 \supset \cdots \supset \Psi_\infty = \bigcap_{d\in\mathbb{N}} \Psi_d$$

with the inclusions being strict. Theorem 4.1.5 in the form (4.12) shows that Ψ_d is parametrized by the set of all probability distributions on $\mathbb{N}_0$. The coefficients $b_{n,d}$ are referred to in the literature [33, 88, 24] as *d-Schoenberg coefficients* and we adopt this terminology.

The Gegenbauer polynomials are orthogonal on $[-1,1]$ with respect to the weight function $w(x)=(1-x^2)^{\lambda-1/2}$ (for instance [26, Table 18.3.1]), in detail:

$$\int_{-1}^{1} C_n^\lambda(x) C_m^\lambda(x)(1-x^2)^{\lambda-1/2}dx = \delta_{n,m}\frac{2^{1-2\lambda}\pi\Gamma(n+2\lambda)}{(n+\lambda)\big(\Gamma(\lambda)\big)^2 n!}, \quad n,m\in\mathbb{N}_0, \lambda>0,$$
$$\int_0^\pi C_n^0\big(\cos(\theta)\big)C_m^0\big(\cos(\theta)\big)\,d\theta = \delta_{n,m}\frac{\pi}{\min\{n+1,2\}}, \quad n,m\in\mathbb{N}_0.$$

Because for all $n\in\mathbb{N}_0$ it is true that (e.g. [26, Formula 18.6.1])

$$C_n^\lambda(1) = \frac{\Gamma(n+2\lambda)}{n!\Gamma(2\lambda)}, \quad \lambda>0, \quad C_n^0(1)=1, \tag{4.13}$$

it follows from the orthogonality of the Gegenbauer polynomials that the Schoenberg-coefficients in the expansion (4.12) are given by

$$b_{n,d} = \frac{2n+d-1}{2^{3-d}\pi}\frac{\Gamma\big((d-1)/2\big)^2}{\Gamma(d-1)}\int_0^\pi C_n^{(d-1)/2}\big(\cos(\theta)\big)\sin^{d-1}(\theta)\tilde{\rho}(\theta)\,d\theta \tag{4.14}$$

for $d\geq 2$ and $n\in\mathbb{N}_0$ and in case $d=1$

$$b_{n,1} = \frac{\min\{n+1,2\}}{\pi}\int_0^\pi \cos(n\theta)\tilde{\rho}(\theta)\,d\theta, \quad n\in\mathbb{N}_0, \tag{4.15}$$

where $\tilde{\rho}:[0,\pi]\to[-1,1]$ is the function associated to $\rho:\mathbb{S}^d\times\mathbb{S}^d\to[-1,1]$ via Definition 4.1.3. Theorem 4.1.5 may also be used to check whether a given continuous function $\tilde{\rho}:[0,\pi]\to[-1,1]$ is a correlation function on $\mathbb{S}^d$, since $\tilde{\rho}$ is a correlation function on $\mathbb{S}^d$ if and only if its d-Schoenberg coefficients (4.14) are non-negative and sum up to 1. To give an example, it is well-known that the spherical correlation function

$$\rho(x,y) = \Big(1-\frac{3d_M(x,y)}{2a}+\frac{d_M(x,y)^3}{2a^3}\Big)\mathbb{1}_{d_M(x,y)\leq a}, \quad x,y\in M, \tag{4.16}$$

is a valid correlation function on $(M,d_M)=(\mathbb{R}^d,\|\cdot-\cdot\|)$ for $d\leq 3$ (e.g., [20]). In [40], the authors used sine expansions of Legendre polynomials to obtain the following representation of the 2-Schoenberg coefficients $(b_{n,2}, n\in\mathbb{N}_0)$ of the spherical correlation function (4.16), which we state here for later reference:

$$b_{n,2} = \frac{2n+1}{2}\sum_{k=0}^{\infty} C_{n,k}a_{n,k}, \quad n\in\mathbb{N}_0, \tag{4.17}$$
$$\text{with } C_{n,k} = \frac{4}{\pi}\frac{(2n)!!}{(2n+1)!!}\frac{(2k-1)!!}{k!}\frac{(n+1)\cdots(n+k)}{(2n+3)\cdots(2n+2k+1)}, \quad n,k\in\mathbb{N}_0,$$
$$\text{and } a_{n,k} = \int_0^\pi \tilde{\rho}(\theta)\sin\big((n+2k+1)\theta\big)\sin(\theta)\,d\theta, \quad n,k\in\mathbb{N}_0.$$

Using this representation (4.17), the authors of [40] were able to show that $b_{n,2}\geq 0$ for all $n\in\mathbb{N}_0$ and $\sum_{n\in\mathbb{N}_0} b_{n,2}=1$ hold true, implying the validity of the spherical correlation function (4.16) on $(M,d_M)=(\mathbb{S}^2,d_{\mathbb{S}^2})$. Gneiting [33] used a different method and showed that (4.16) is a correlation function on the sphere $\mathbb{S}^d$ for all $d\leq 3$.

Another interesting result, which connects the Schoenberg-coefficients with strict positive definiteness, is the following one [16]:

Theorem 4.1.6 (Chen, Menegatto, Sun, 2003). *A function $\tilde{\rho} : [0, \pi] \to [-1, 1]$ is a strictly positive definite correlation function on $\mathbb{S}^d$, $d \geq 2$, if and only if $b_{n,d} > 0$ for infinitely many even and infinitely many odd $n \in \mathbb{N}_0$.*

4.1.3. Miscellaneous

Definition 4.1.7. A function $\psi : [-1, 1] \to [-1, 1]$ of the form

$$\psi(t) = \sum_{n \in \mathbb{N}_0} p_n t^n, \quad p_n \in [0, 1], \sum_{n \in \mathbb{N}_0} p_n = 1, \quad t \in [-1, 1], \tag{4.18}$$

is called a *probability generating function*. If N is a $\mathbb{N}_0$-valued random variable on some probability space $(\Omega, \mathscr{A}, P)$, the *probability generating function of N ψ_N* is defined as

$$\psi_N(t) = \mathbb{E}\big(t^N\big), \quad t \in [-1, 1].$$

If ν is a probability measure on $\big(\mathbb{N}_0, \mathcal{P}(\mathbb{N}_0)\big)$, the *probability generating function of ν ψ_ν* is defined as

$$\psi_\nu(t) = \sum_{n \in \mathbb{N}_0} \nu\big(\{n\}\big) t^n, \quad t \in [-1, 1].$$

It is evident from the definitions that the probability generating function of a random variable and its distribution coincide so that there is no reason to distinguish between them. Because the coefficients in the series above are non-negative and because their sum is 1, the series above are well-defined. Probability generating functions possess many useful properties. We state their stability under compositions as a lemma for later reference. The proof of the lemma is evident.

Lemma 4.1.8. *Suppose $L, K_1, K_2, \dots$ are independent and $\mathbb{N}_0$-valued random variables. If the random variables $(K_l, l \in \mathbb{N})$ are identically distributed, then the $\mathbb{N}_0$-valued random variable*

$$N = \sum_{l=1}^{L} K_l$$

has the probability generating function $\psi_L \circ \psi_{K_1}$.

Here and henceforth, we make the convention that an empty sum equals 0 and an empty product equals 1.

Since there are two geometric distributions and because we are using both, let us also mention that the abbreviation $\mathrm{Geo}_{\mathbb{N}}(p)$ stands for the distribution

$$\sum_{n \in \mathbb{N}} p(1-p)^{n-1} \varepsilon_n, \quad p \in (0, 1],$$

counting the number of independent and identically distributed Bernoulli trials needed to get a success and the distribution which counts the number of failures until a success occurs,

$$\sum_{n \in \mathbb{N}_0} p(1-p)^n \varepsilon_n, \quad p \in (0, 1],$$

is denoted by $\mathrm{Geo}_{\mathbb{N}_0}(p)$.

4.2. The Model

We consider a random field $Z = (Z(x), x \in M)$ on a second countable locally compact Hausdorff space M equipped with its Borel σ-algebra. Let N be an $\mathbb{N}_0$-valued random variable, not almost surely equal to zero, and $(U_{i,j}, i, j \in \mathbb{N})$ a doubly indexed i.i.d. sequence of real-valued random variables with finite variances. Let $(B_n, n \in \mathbb{N})$ be an i.i.d. sequence of random closed sets in M [63]. We assume that the family formed by N, $(U_{i,j}, i, j \in \mathbb{N})$, and $(B_n, n \in \mathbb{N})$ is independent. The random variables U and B refer to a generic member of the sequences $(U_{i,j}, i, j \in \mathbb{N})$ and $(B_n, n \in \mathbb{N})$, respectively.

Let $\mathcal{P}_n$ denote the power set of $\{1, \ldots, n\}$. For every $n \in \mathbb{N}$ we define the family $(C_I, I \in \mathcal{P}_n)$ of disjoint random subsets of M by

$$C_I = \Big(\bigcap_{i \in I} B_i\Big) \bigcap \Big(\bigcap_{j \in \{1,\ldots,n\}\setminus I} B_j^c\Big).$$

If $n = 0$ we define $(C_I, I \in \mathcal{P}_0) = (C_\emptyset)$ by $C_\emptyset = M$. We call C_I a *random cell of M*.

Let $\mathcal{P}^*(\mathbb{N})$ denote the set consisting of all finite subsets of $\mathbb{N}$ and let a function $g : \mathcal{P}^*(\mathbb{N}) \to \mathbb{N}$ be given. Suppose $(\mathbb{I}_I, I \in \mathcal{P}_n)$, $n \in \mathbb{N}_0$, are families of elements of $\mathcal{P}^*(\mathbb{N})$. We generalize the Poisson hyperplane tessellation model, the random token model, and the dead leaves model as follows:

$$Z(x) = \sum_{I \in \mathcal{P}_N} \Big(\sum_{j \in \mathbb{I}_I} U_{g(I),j}\Big) \mathbb{1}_{x \in C_I}, \quad x \in M. \tag{4.19}$$

We call the random field Z *simple mosaic random field* and we write Z_M instead of Z, when g is an injection of $\mathcal{P}^*(\mathbb{N})$ and $\mathbb{I}_I = \{1\}$ for all $I \in \mathcal{P}^*(\mathbb{N})$. In this case there exists an i.i.d. sequence $\big(U_I, I \in \mathcal{P}^*(\mathbb{N})\big)$ such that we have

$$Z_M(x) \overset{d}{=} \sum_{I \in \mathcal{P}_N} U_I \, \mathbb{1}_{x \in C_I}, \quad x \in M, \tag{4.20}$$

where the equality is in the sense of distribution. If $M = \mathbb{R}^d$, the sets B_n are half-spaces determined by random hyperplanes in $\mathbb{R}^d$, and N is taken to be Poisson distributed, then Z_M is the Poisson hyperplane tessellation model in $\mathbb{R}^d$.

The choices $\mathbb{I}_I = I$, $I \in \mathcal{P}^*(\mathbb{N})$, and $g \equiv 1$ in (4.19) lead to the field

$$Z_{RT}(x) \overset{d}{=} \sum_{I \in \mathcal{P}_N} \Big(\sum_{i \in I} U_i\Big) \mathbb{1}_{x \in C_I}, \quad x \in M, \tag{4.21}$$

where in this case $(U_i, i \in \mathbb{N})$ is an i.i.d. sequence. Here each random closed set B_n is associated with a random variable and to each point $x \in M$ we assign the sum of all random variables associated to random closed sets containing x. The random field Z_{RT} is called *random token field* if $M = \mathbb{R}^d$ and we keep the name for general M as above.

In a third example, we take g from the simple mosaic random field and $(\mathbb{I}_I, I \in \mathcal{P}_n)$, $n \in \mathbb{N}_0$, from the random token field and get

$$Z_{MRT} = \sum_{I \in \mathcal{P}_N} \Big(\sum_{i \in I} U_{g(I),i}\Big) \mathbb{1}_{x \in C_I}, \quad x \in M. \tag{4.22}$$

Figure 4.1.: From left to right: simple Mosaic with random half-spaces on $[-1,1]^2$, random token with random balls on the sphere, dead leaves with random balls on the torus.

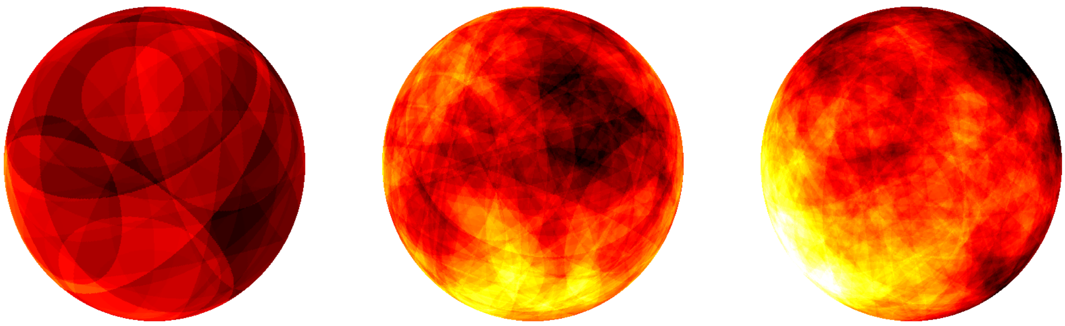


Figure 4.2.: Weighted sum of $n = 10, 100$, and 200 realizations of a random token field on the sphere.

We call Z_{MRT} *mixture random field.*

In the *dead leaves model* [61] the random sets $(B_n, n \in \mathbb{N})$ are placed sequentially in M, partially overlapping previously placed random sets. The corresponding random field Z_{DL} is defined at each $x \in M$ as $Z_{DL}(x) = U$ for the random variable U associated to the latest random set covering x. In our setup, this random field corresponds to the choices $\mathbb{I}_I = \{1\}$ and $g(I) = \mathbb{1}_{I \neq \emptyset} \max I$ for all $I \in \mathcal{P}^*(\mathbb{N})$, such that

$$Z_{DL}(x) \stackrel{d}{=} \sum_{I \in \mathcal{P}_N} U_{g(I)}\, \mathbb{1}_{x \in C_I}, \quad x \in M, \tag{4.23}$$

for an i.i.d. sequence $(U_i, i \in \mathbb{N}_0)$.

Realizations of different mosaic random fields on $[-1,1]^2$, on the sphere, and on the torus are illustrated in Figure 4.1. Figure 4.2 displays the weighted sum of $n = 10, 100$, and 200 realizations of a mosaic random field on the sphere.

In order to get reasonable analytic formulae for the covariance function of Z, we assume that there exist functions $f_n : \mathbb{N}_0^2 \to \mathbb{N}_0$, $n \in \mathbb{N}_0$, such that for all $n \in \mathbb{N}_0$

$$|\mathbb{I}_I \cap \mathbb{I}_J| = f_n\big(|I \cap J|, |I \triangle J|\big) \quad \text{for all } I, J \in \mathcal{P}_n, \tag{4.24}$$

holds, where $\triangle$ denotes the symmetric difference of two sets. In the following we present a class of functions f_n for which we can construct families $(\mathbb{I}_I, I \in \mathcal{P}_n)$, $n \in \mathbb{N}_0$, such that (4.24) holds. The functions corresponding to Z_M, Z_{RT}, Z_{MRT}, and Z_{DL} above are given by $f_n(i,j) = 1$, and $f_n(i,j) = i$, respectively, and they are included in the following class.

Lemma 4.2.1. *Suppose that $(a_n, n \in \mathbb{N}_0)$, $(b_n, n \in \mathbb{N}_0)$, and $(c_n, n \in \mathbb{N}_0)$ are sequences such that for every $n \in \mathbb{N}_0$, $a_n \in \mathbb{Z}$, $b_n, c_n \in \mathbb{N}_0$ holds. Assume furthermore that for all $n \in \mathbb{N}_0$, $a_n \geq -b_n$, $c_n \geq nb_n$ holds true, and set $f_n(i,j) = a_n i - b_n j + c_n$, $i, j \in \mathbb{N}_0$. Then there are families $(\mathbb{I}_I, I \in \mathcal{P}_n)$, $n \in \mathbb{N}_0$, such that (4.24) holds true.*

Proof. Fix $n \in \mathbb{N}_0$. Let A and B_i, C_i, $i = 1, \dots, n$, be disjoint subsets of $\mathbb{N}$ such that $|A| = c_n - nb_n$, and $|B_i| = b_n$, $|C_i| = a_n + b_n$ holds true for $i = 1, \dots, n$. Set

$$\mathbb{I}_I = A \bigcup \Big(\bigcup_{i \in \{1,\dots,n\} \setminus I} B_i \Big) \bigcup \Big(\bigcup_{j \in I} C_j \Big), \quad I \in \mathcal{P}_n,$$

then we get for all $I, J \in \mathcal{P}_n$

$$\begin{aligned} |\mathbb{I}_I \cap \mathbb{I}_J| &= |A| + \Bigg| \bigcup_{i \in \{1,\dots,n\} \setminus I} \ \bigcup_{j \in \{1,\dots,n\} \setminus J} (B_i \cap B_j) \Bigg| + \Bigg| \bigcup_{i \in I} \bigcup_{j \in J} (C_i \cap C_j) \Bigg| \\ &= |A| + \big(n - |I \cup J|\big)|B_1| + |I \cap J||C_1| = f_n\big(|I \cap J|, |I \triangle J|\big), \end{aligned}$$

and the lemma is proved. $\square$

For $x, y \in M$ and $n \in \mathbb{N}_0$, let

$$p_x = P(x \in B) \quad \text{and} \quad p_{xy} = P(x, y \in B),$$

and let

$$V_{xy,n} = (V^1_{xy,n}, V^2_{xy,n}, V^3_{xy,n}, V^4_{xy,n})$$

be a multinomial distributed random vector with parameters n, p_{xy}, $p_x - p_{xy}$, $p_y - p_{xy}$, and $1 - p_x - p_y + p_{xy}$. In the case $n = 0$, the vector $V_{xy,n}$ equals the zero vector almost surely.

Theorem 4.2.2. *Suppose that there are functions $(f_n, n \in \mathbb{N}_0)$ such that (4.24) holds true for the families $(\mathbb{I}_I, I \in \mathcal{P}_n)$, $n \in \mathbb{N}_0$, of the random field $(Z(x), x \in M)$ defined in (4.19). Then for all $x, y \in M$ it is true that*

$$\mathbb{E}\big(Z(x)\big) = \mathbb{E}(U)\,\mathbb{E}\big(f_N(V^1_{xx,N}, 0)\big) \tag{4.25}$$

and

$$\begin{aligned} \mathbb{E}\big(Z(x)Z(y)\big) = {} & \operatorname{Var}(U)\,\mathbb{E}\big(f_N(V^1_{xy,N}, V^2_{xy,N} + V^3_{xy,N})\big) \\ & + \mathbb{E}(U)^2\,\mathbb{E}\big(f_N(V^1_{xy,N} + V^2_{xy,N}, 0) f_N(V^1_{xy,N} + V^3_{xy,N}, 0)\big) - G_{xy} \operatorname{Var}(U) \end{aligned} \tag{4.26}$$

with

$$G_{xy} = \mathbb{E}\Bigg(\sum_{\substack{I, J \in \mathcal{P}_N \\ g(I) \neq g(J)}} P(x \in C_I, y \in C_J) f_N\big(|I \cap J|, |I \triangle J|\big) \Bigg). \tag{4.27}$$

Proof. By definition of the cells, independence and identity of the distributions of the B_i, $i \in \mathbb{N}$, we have for every $n \in \mathbb{N}_0$, $I \in \mathcal{P}_n$, and every $x \in M$

$$P(x \in C_I) = \prod_{i \in I} P(x \in B_i) \prod_{j \in \{1,\dots,n\} \setminus I} P(x \notin B_j) = p_x^{|I|}(1-p_x)^{n-|I|}.$$

Using this we get

$$\begin{aligned}
\mathbb{E}\big(Z(x)\big) &= \sum_{n \in \mathbb{N}_0} P(N=n)\,\mathbb{E}\Bigg(\sum_{I \in \mathcal{P}_n}\Big(\sum_{j \in \mathbb{I}_I} U_{g(I),j}\Big)\mathbb{1}_{x \in C_I}\Bigg) \\
&= \sum_{n \in \mathbb{N}_0} P(N=n) \sum_{I \in \mathcal{P}_n} P(x \in C_I) \sum_{i \in \mathbb{I}_I} \mathbb{E}(U_{g(I),i}) \\
&= \mathbb{E}(U) \sum_{n \in \mathbb{N}_0} P(N=n) \sum_{I \in \mathcal{P}_n} p_x^{|I|}(1-p_x)^{n-|I|}|\mathbb{I}_I|.
\end{aligned}$$

By assumption $|\mathbb{I}_I| = f_n(|I|, 0)$, and as there are $\binom{n}{k}$ subsets of $\{1,\dots,n\}$ with k elements, we find

$$\begin{aligned}
\mathbb{E}\big(Z(x)\big) &= \mathbb{E}(U) \sum_{n \in \mathbb{N}_0} P(N=n) \sum_{k=0}^{n} \binom{n}{k} p_x^k (1-p_x)^{n-k} f_n(k,0) \\
&= \mathbb{E}(U) \sum_{n \in \mathbb{N}_0} P(N=n)\,\mathbb{E}\big(f_n(V^1_{xx,n},0)\big| N=n\big) = \mathbb{E}(U)\,\mathbb{E}\big(f_N(V^1_{xx,N},0)\big),
\end{aligned}$$

proving formula (4.25). Regarding the mixed moment, we have

$$\sum_{i \in \mathbb{I}_I, j \in \mathbb{I}_J} \mathbb{E}\big(U_{g(I),i}U_{g(J),j}\big) = \mathbb{E}(U)^2\big(|\mathbb{I}_I||\mathbb{I}_J| - |\mathbb{I}_I \cap \mathbb{I}_J|\big) + \sum_{i \in \mathbb{I}_I \cap \mathbb{I}_J} \mathbb{E}\big(U_{g(I),i}U_{g(J),i}\big)$$

for all $I, J \in \mathcal{P}^*(\mathbb{N})$. Since the second indices of the random variables in the last sum are equal, the last sum equals $\mathbb{E}(U^2)|\mathbb{I}_I \cap \mathbb{I}_J|$ in case $g(I) = g(J)$ and $\mathbb{E}(U)^2|\mathbb{I}_I \cap \mathbb{I}_J|$ otherwise. Hence

$$\sum_{i \in \mathbb{I}_I} \sum_{j \in \mathbb{I}_J} \mathbb{E}\big(U_{g(I),i}U_{g(J),j}\big) = \mathbb{E}(U)^2\,|\mathbb{I}_I||\mathbb{I}_J| + \mathrm{Var}(U)\,|\mathbb{I}_I \cap \mathbb{I}_J|\,\mathbb{1}_{g(I)=g(J)}.$$

This yields

$$\begin{aligned}
\mathbb{E}\big(Z(x)Z(y)\big) &= \sum_{n \in \mathbb{N}_0} P(N=n) \sum_{I,J \in \mathcal{P}_n} P(x \in C_I, y \in C_J) \sum_{i \in \mathbb{I}_I} \sum_{j \in \mathbb{I}_J} \mathbb{E}\big(U_{g(I),i}U_{g(J),j}\big) \\
&= \mathrm{Var}(U) \sum_{n \in \mathbb{N}_0} P(N=n) \sum_{I,J \in \mathcal{P}_n} P(x \in C_I, y \in C_J)\,|\mathbb{I}_I \cap \mathbb{I}_J| \\
&\qquad + \mathbb{E}(U)^2 \sum_{n \in \mathbb{N}_0} P(N=n) \sum_{I,J \in \mathcal{P}_n} P(x \in C_I, y \in C_J)\,|\mathbb{I}_I||\mathbb{I}_J| \\
&\qquad - \mathrm{Var}(U)\,G_{xy}.
\end{aligned}$$

Furthermore, for $n \in \mathbb{N}_0$,

$$\begin{aligned}
&P(x \in C_I, y \in C_J) \\
&\quad = P(x,y \in B)^{|I \cap J|} P(x \in B, y \notin B)^{|I \setminus J|} P(x \notin B, y \in B)^{|J \setminus I|} P(x,y \notin B)^{n-|I \cup J|} \\
&\quad = p_{xy}^{|I \cap J|}(p_x - p_{xy})^{|I \setminus J|}(p_y - p_{xy})^{|J \setminus I|}(1 - p_x - p_y + p_{xy})^{n-|I \cup J|}.
\end{aligned}$$

With the assumptions on $(\mathbb{I}_I, I \in \mathcal{P}_n)$, $n \in \mathbb{N}_0$, and the multinomial distribution we get

$$\begin{aligned}
\sum_{I,J\in\mathcal{P}_n} & P(x \in C_I, y \in C_J)\, |\mathbb{I}_I \cap \mathbb{I}_J| \\
&= \sum_{\substack{k_1,k_2,k_3,k_4\in\mathbb{N}_0\\ k_1+k_2+k_3+k_4=n}} \binom{n}{k_1,k_2,k_3,k_4} p_{xy}^{k_1}(p_x - p_{xy})^{k_2}(p_y - p_{xy})^{k_3} \\
&\qquad\qquad \times (1 - p_x - p_y + p_{xy})^{k_4} f_n(k_1, k_2 + k_3) \\
&= \mathbb{E}\big(f_n\big(V^1_{xy,n}, V^2_{xy,n} + V^3_{xy,n}\big)\big|N = n\big). \qquad (4.28)
\end{aligned}$$

Similarly, the sum $\sum_{I,J\in\mathcal{P}_n} P(x \in C_I, y \in C_J)\,|\mathbb{I}_I||\mathbb{I}_J|$ reduces to the expression (4.28) where $f_n\big(V^1_{xy,n}, V^2_{xy,n} + V^3_{xy,n}\big)$ is replaced by $f_n\big(V^1_{xy,n} + V^2_{xy,n}, 0\big) f_n\big(V^1_{xy,n} + V^3_{xy,n}, 0\big)$, yielding formula (4.26). □

Remark 4.2.3. As the proof of Theorem 4.2.2 shows, the restriction to mosaic random fields for which the associated sets $(\mathbb{I}_I, I \in \mathcal{P}_n)$, $n \in \mathbb{N}_0$, satisfy Condition (4.24) allows us to utilize combinatorial arguments in order to obtain a rather simple formula for the mean and the first mixed moment of the mosaic random field. Lemma 4.2.1 gives a class of functions $(f_n, n \in \mathbb{N}_0)$ such that there are always associated sets $(\mathbb{I}_I, I \in \mathcal{P}_n)$, $n \in \mathbb{N}_0$. In general, there are functions $(f_n, n \in \mathbb{N}_0)$ such that there can not exist sets $(\mathbb{I}_I, I \in \mathcal{P}_n)$, $n \in \mathbb{N}_0$, for which (4.24) is satisfied. An example is given by $f_n(i, j) = j$ for all $n, i, j \in \mathbb{N}_0$. Indeed, if there would exist associated sets $(\mathbb{I}_I, I \in \mathcal{P}_n)$, $n \in \mathbb{N}_0$, then $n = 2$, $I = \emptyset$ and $J = \{1, 2\}$ results in

$$|\mathbb{I}_I \cap \mathbb{I}_J| = f_2(0, 2) > f_2(0, 0) = |\mathbb{I}_I|,$$

which can not be true. On the other hand, there are sets $(\mathbb{I}_I, I \in \mathcal{P}_n)$, $n \in \mathbb{N}_0$, for which there can not exist function $(f_n, n \in \mathbb{N}_0)$ such that (4.24) holds true. This can be seen from the example $\mathbb{I}_I = I \cup \{1\}$ for all $I \in \mathcal{P}_n$ and $n \in \mathbb{N}_0$, which yields for $n = 5$, $I = \{1, 2, 3\}$, and $J = \{1, 2, 4\}$

$$f_5(2, 2) = |\mathbb{I}_I \cap \mathbb{I}_J| = 2,$$

and for $\tilde{I} = \{2, 3, 4\}$ and $\tilde{J} = \{3, 4, 5\}$ Condition (4.24) implies

$$f_5(2, 2) = |\mathbb{I}_{\tilde{I}} \cap \mathbb{I}_{\tilde{J}}| = 3.$$

For sets $(\mathbb{I}_I, I \in \mathcal{P}_n)$, $n \in \mathbb{N}_0$, that do not satisfy Condition (4.24), the expectation $\mathbb{E}\big(Z(x)\big)$ and the first mixed moment $\mathbb{E}\big(Z(x)Z(y)\big)$ of the corresponding mosaic random field can not be computed as in Theorem 4.2.2, but there can be of course similar arguments that lead to similar formulae. The author considers the formulation of a more general conditions than (4.24) and a proof of Theorem 4.2.2 with such a condition as an area of further research.

We write ρ_i, $i = M, RT, MRT$, and DL, for the correlation function of Z_i. Furthermore, recall that ψ_N denotes the probability generating function of the $\mathbb{N}_0$-valued random variable N.

Corollary 4.2.4. *Let U, N, and B be such that* $\operatorname{Var}(U) > 0$, $\mathbb{E}(N) > 0$, *and* $p_x > 0$ *for all* $x \in M$. *Then for all* $x, y \in M$

$$\rho_M(x,y) = \psi_N(1 + 2p_{xy} - p_x - p_y), \tag{4.29}$$

$$\rho_{RT}(x,y) = \frac{ap_{xy} + bp_xp_y}{\sqrt{(a+bp_x)(a+bp_y)p_xp_y}} \tag{4.30}$$

with $a = \mathbb{E}(U^2)\,\mathbb{E}(N)$ *and* $b = \mathbb{E}(U)^2\big(\operatorname{Var}(N) - \mathbb{E}(N)\big)$,

$$\rho_{MRT}(x,y) = \frac{p_{xy}\big(c\psi_N'(1+2p_{xy}-p_x-p_y) - d\big)}{\sqrt{(a+bp_x)(a+bp_y)p_xp_y}} + \rho_{RT}(x,y) \tag{4.31}$$

with $c = \operatorname{Var}(U)$ *and* $d = \operatorname{Var}(U)\,\mathbb{E}(N)$, *and*

$$\rho_{DL}(x,y) = \frac{p_{xy} + (p_x + p_y - 2p_{xy})\psi_N(1 - p_x - p_y + p_{xy})}{p_x + p_y - p_{xy}} \tag{4.32}$$

hold true.

Proof. For the simple mosaic random field (4.20) we have $\mathbb{I}_I = \{1\}$ for all $I \in \mathcal{P}^*(\mathbb{N})$, hence the functions f_n in (4.24) can by taken to be identically 1. Consequently, we get

$$\mathbb{E}\big(Z_M(x)\big) = \mathbb{E}(U) \tag{4.33}$$

for all $x \in M$ from (4.25). Since g is injective for this field, we have for all $x, y \in M$ for the variable G_{xy} from (4.27) with the same reasoning as in the proof of Theorem 4.2.2

$$\begin{aligned} G_{xy} &= \sum_{n\in\mathbb{N}_0} P(N=n) \sum_{I\neq J} P(x \in C_I, y \in C_J) \\ &= 1 - \sum_{n\in\mathbb{N}_0} P(N=n) \sum_{I\in\mathcal{P}_n} P(x, y \in C_I) \\ &= 1 - \psi_N(1 + 2p_{xy} - p_x - p_y). \end{aligned}$$

Thus formula (4.26) yields

$$\mathbb{E}\big(Z_M(x)Z_M(y)\big) = \operatorname{Var}(U)\,\psi_N(1 + 2p_{xy} - p_x - p_y) + \mathbb{E}(U)^2. \tag{4.34}$$

From this we can compute the variance of $Z_M(x)$, the covariance of $Z_M(x)$ and $Z_M(y)$, and then (4.29) follows. In case of the random token field (4.21), we have $\mathbb{I}_I = I$ for all $I \in \mathcal{P}^*(\mathbb{N})$, and we can choose f_n to be the projection on the first coordinate. Therefore

$$\mathbb{E}\big(Z_{RT}(x)\big) = \mathbb{E}(U)\,\mathbb{E}(V^1_{xx,N}) = \mathbb{E}(U)\,\mathbb{E}(N)p_x \tag{4.35}$$

by Theorem 4.2.2. The function g is identically 1 in case of the random token field, hence $G_{xy} = 0$ and

$$\mathbb{E}\big(Z_{RT}(x)Z_{RT}(y)\big) = \operatorname{Var}(U)\,\mathbb{E}(V^1_{xy,N}) + \mathbb{E}(U)^2\,\mathbb{E}\big((V^1_{xy,N} + V^2_{xy,N})(V^1_{xy,N} + V^3_{xy,N})\big)$$

by (4.26). The covariance of the components of a multinomial distributed random vector is well-known and a straightforward computation yields

$$\begin{aligned}\mathbb{E}\big(Z_{RT}(x)Z_{RT}(y)\big) &= \mathbb{E}(U^2)\mathbb{E}(N)p_{xy} + \mathbb{E}(U)^2\big(\mathbb{E}(N^2)-\mathbb{E}(N)\big)p_xp_y \\ &= ap_{xy} + bp_xp_y + \mathbb{E}\big(Z_{RT}(x)\big)\mathbb{E}\big(Z_{RT}(y)\big),\end{aligned} \tag{4.36}$$

which implies (4.30). Now consider the mixture random field (4.22). Again, we have $\mathbb{I}_I = I$ for all $I \in \mathcal{P}^*(\mathbb{N})$ and we can choose the same f_n as above. Consequently, $\mathbb{E}\big(Z_{MRT}(x)\big) = \mathbb{E}\big(Z_{RT}(x)\big)$. But in contrast to the random token field, g is injective for the mixture random field. Reasoning as in the proof of Theorem 4.2.2 we obtain

$$\begin{aligned}G_{xy} &= \sum_{n\in\mathbb{N}_0} P(N=n) \sum_{I,J\in\mathcal{P}_n} P(x\in C_I, y\in C_J)|I\cap J| \\ &\qquad - \sum_{n\in\mathbb{N}_0} P(N=n) \sum_{I\in\mathcal{P}_n} P(x,y\in C_I)|I| \\ &= \mathbb{E}(N)p_{xy} - \sum_{n\in\mathbb{N}_0} P(N=n)\left(1+2p_{xy}-p_x-p_y\right)^{n-1} np_{xy} \\ &= \mathbb{E}(N)p_{xy} - p_{xy}\psi_N'(1+2p_{xy}-p_x-p_y)\end{aligned}$$

and then with (4.26) and (4.36)

$$\begin{aligned}&\mathbb{E}\big(Z_{MRT}(x)Z_{MRT}(y)\big) \\ &\quad = \mathbb{E}\big(Z_{RT}(x)Z_{RT}(y)\big) - G_{xy}\operatorname{Var}(U) \\ &\quad = cp_{xy}\psi_N'(1+2p_{xy}-p_x-p_y) + (a-d)p_{xy} + bp_xp_y + \mathbb{E}\big(Z_{MRT}(x)\big)\mathbb{E}\big(Z_{MRT}(y)\big).\end{aligned}$$

This shows (4.31). For the dead leaves model (4.23) we have $f_n \equiv 1$ for all $n \in \mathbb{N}_0$ and hence $\mathbb{E}\big(Z_{DL}(x)\big) = \mathbb{E}(U)$. In order to compute G_{xy} we let

$$A_n = \sum_{\substack{I,J\in\mathcal{P}_n \\ g(I)=g(J)}} P(x\in C_I, y\in C_J), \quad n\in\mathbb{N}_0.$$

Writing $\mathcal{P}_{n+1} = \mathcal{P}_n \cup \{I\cup\{n+1\} \mid I\in\mathcal{P}_n\}$ and using $g(I) = \mathbb{1}_{I\neq\emptyset}\max I$ we get the recurrence relation

$$A_{n+1} = P(x,y\notin B)A_n + P(x,y\in B), \quad n\in\mathbb{N}_0,$$

which leads to

$$A_n = \frac{P(x,y\in B)}{1-P(x,y\notin B)} + \frac{P(x\in B, y\notin B)+P(x\notin B, y\in B)}{1-P(x,y\notin B)}P(x,y\notin B)^n$$

for all $n\in\mathbb{N}_0$, and then with (4.27)

$$\begin{aligned}G_{xy} &= 1 - \sum_{n\in\mathbb{N}_0} P(N=n)A_n \\ &= \frac{P(x\in B, y\notin B)+P(x\notin B, y\in B)}{1-P(x,y\notin B)}\big(1-\psi_N\big(P(x,y\notin B)\big)\big).\end{aligned}$$

Collecting terms we get with (4.26)

$$\mathbb{E}\big(Z_{DL}(x)Z_{DL}(y)\big) = \mathbb{E}(U^2) - \operatorname{Var}(U)\big(1-\psi_N(1-p_x-p_y+p_{xy})\big)\frac{p_x+p_y-2p_{xy}}{p_x+p_y-p_{xy}}$$

and then (4.32) follows. $\square$

Remark 4.2.5. Let us consider the special case where N is a Poisson random variable. In this case we write ρ_i^*, $i = M, RT$, and MRT for the correlation function of Z_i. Plugging in the moments and the probability generating function of the Poisson distribution into formulae (4.29), (4.30), and (4.31), yields the relation

$$\rho_{MRT}^* = \lambda\, \rho_{RT}^*\, \rho_M^* + (1-\lambda)\, \rho_{RT}^* \quad \text{with} \quad \lambda = \frac{\mathrm{Var}(U)}{\mathbb{E}(U^2)} \in (0,1].$$

In view of the simulations in Sections 4.4 and 4.6 the absolute third moment $\mathbb{E}\big(|Z(x)|^3\big)$ appearing in the Berry-Esseen Theorem (4.55) is also of interest. The next proposition provides bounds for this absolute third moment for the general mosaic random field and its submodels.

Proposition 4.2.6. *Under the assumptions of Theorem 4.2.2 we have for the mosaic random field Z in* (4.19)

$$\mathbb{E}\big(|Z(x)|^3\big) \le \mathbb{E}\big(|U|^3\big)\mathbb{E}\big(f_N(V_{xx,N}^1, 0)^3\big), \quad x \in M. \tag{4.37}$$

If the mosaic random field is a simple mosaic random field (4.20) *or a dead leaves random field* (4.23) *we have*

$$\mathbb{E}\big(|Z_M(x)|^3\big) = \mathbb{E}\big(|Z_{DL}(x)|^3\big) = \mathbb{E}\big(|U|^3\big), \quad x \in M. \tag{4.38}$$

In case we have a random token field (4.21) *or a mixture random field* (4.22),

$$\begin{aligned} \mathbb{E}\big(|Z_{RT}(x)|^3\big) &= \mathbb{E}\big(|Z_{MRT}(x)|^3\big) \\ &\le p_x \mathbb{E}\big(|U|^3\big)\big(p_x^2\mathbb{E}(N^3) + 3p_x(1-p_x)\mathbb{E}(N^2) + (1-2p_x)(1-p_x)\mathbb{E}(N)\big) \end{aligned} \tag{4.39}$$

holds true for all $x \in M$.

Proof. Let $x \in M$ and let Z be the general mosaic random field (4.19). By disjointedness of the cells $(C_I, I \in \mathcal{P}_n)$ for every $n \in \mathbb{N}_0$ we have

$$\begin{aligned} \mathbb{E}\big(|Z(x)|^3\big) &= \mathbb{E}\Bigg(\sum_{I\in\mathcal{P}_N}\Big|\sum_{i\in\mathbb{I}_I} U_{g(I),i}\Big|^3 \mathbb{1}_{x\in C_I}\Bigg) \\ &= \sum_{n\in\mathbb{N}_0} P(N=n) \sum_{I\in\mathcal{P}_n} P(x\in C_I)\mathbb{E}\Bigg(\Big|\sum_{i\in\mathbb{I}_I} U_{g(I),i}\Big|^3\Bigg). \end{aligned} \tag{4.40}$$

In case Z is a simple mosaic random field or a dead leaves random field we have $\mathbb{I}_I = \{1\}$ for all $I \in \mathcal{P}^*(\mathbb{N})$. Since the random variables $U_{i,j}$, $i, j \in \mathbb{N}$, are identically distributed, the equality (4.38) follows from (4.40).

In the general case, for every choice of the functions g, for each $n \in \mathbb{N}_0$, and all $I \in \mathcal{P}_n$, the sum $\sum_{i\in\mathbb{I}_I} U_{g(I),i}$ is just a sum of $|\mathbb{I}_I| = f_n(|I|, 0)$ independent and identically distributed random variables. Minkowski's inequality yields

$$\mathbb{E}\Bigg(\Big|\sum_{i\in\mathbb{I}_I} U_{g(I),i}\Big|^3\Bigg) \le \Bigg(\sum_{i\in\mathbb{I}_I}\mathbb{E}\big(|U_{g(I),i}|^3\big)^{1/3}\Bigg)^3 = f_n(|I|,0)^3\mathbb{E}\big(|U|^3\big),$$

so that with the random variables $V^1_{xx,n} \sim \text{Bin}(n, p_x)$, $n \in \mathbb{N}_0$, $x \in M$, of Theorem 4.2.2, we get from (4.40)

$$\begin{aligned}
\mathbb{E}\big(|Z(x)|^3\big) &\leq \mathbb{E}\big(|U|^3\big) \sum_{n\in\mathbb{N}_0} P(N=n) \sum_{I\in\mathcal{P}_n} P(x \in C_I) f_n(|I|, 0)^3 \\
&= \mathbb{E}\big(|U|^3\big) \sum_{n\in\mathbb{N}_0} P(N=n) \sum_{k=0}^{n} \binom{n}{k} P(x\in B)^k P(x \notin B)^{n-k} f_n(k,0)^3 \\
&= \mathbb{E}\big(|U|^3\big) \mathbb{E}\big(f_N(V^1_{xx,N}, 0)^3\big),
\end{aligned}$$

i.e. the bound in (4.37) for the general mosaic random field holds true.

The bound in (4.39) follows from (4.40), $f_n(i,j) = i$ for all $i, j \in \mathbb{N}, n \in \mathbb{N}_0$, and

$$\mathbb{E}\big((V^1_{xx,n})^3\big) = p_x\big(p_x^2 n^3 + 3p_x(1-p_x)n^2 + (1-2p_x)(1-p_x)n\big), \quad n \in \mathbb{N}_0, x \in M,$$

since $V^1_{xx,n} \sim \text{Bin}(n, p_x)$. □

4.3. Random Closed Sets in $\mathbb{R}^d$

The formulae in Corollary 4.2.4 depend on the law of the random closed set B through the probabilities $p_x = P(x \in B)$ and $p_{xy} = P(x, y \in B)$. Observe that for every $x \in M$ we have $p_x = p_{xx}$ so that it suffices to compute p_{xy} for all $x, y \in M$. In what follows we give examples for B and compute these probabilities to obtain explicit correlation functions in the next section. In order to get reasonable formulae we require that the random sets are in some sense uniformly placed in $\mathbb{R}^d$. In the pertinent literature this is typically done by placing the random sets at the points of a Poisson point process. The drawback of this method is that the number of random sets N must follow a Poisson distribution. As the formulae in Theorem 4.2.2 and Corollary 4.2.4 indicate, different distributions for N may lead to different types of correlation functions, depending on the concrete choices determining a submodel. In the sequel we let M be a bounded subset of $\mathbb{R}^d$, and it is convenient - and without any serious loss of generality - to assume furthermore that M is closed or open. In this way it is possible to place the random sets uniformly on M and have an arbitrary distribution on $\mathbb{N}_0$ for the number of random sets.

As a first example we take a half-space delimited by random hyperplanes for the random closed set B. A hyperplane $P(x, r)$ in $\mathbb{R}^d$, given in normal form, is the set of all $z \in \mathbb{R}^d$ with $\langle z, x\rangle = r$, where $x \in \mathbb{S}^{d-1}$, $r \in \mathbb{R}$, and rx is the vector from the origin perpendicular to $P(x, r)$. The hyperplane $P(x, r)$ divides $\mathbb{R}^d$ into two half-spaces, consider the half-space that is given by $H(x, r) = \{z \in \mathbb{R}^d \mid \langle z, x\rangle \geq r\}$. Let $(X_n, n \in \mathbb{N})$ be an independent sequence of uniformly distributed random variables on $\mathbb{S}^{d-1}$ (e.g., [67, 60]) and let $(R_n, n \in \mathbb{N})$ be an independent sequence of uniformly distributed random variables on the interval $[-C_M, C_M]$ for a constant $C_M > 0$ large enough such that M is contained in a closed ball with radius C_M centered at the origin. Furthermore, let $(X_n, n \in \mathbb{N})$ and $(R_n, n \in \mathbb{N})$ be independent. Then $(H_n, n \in \mathbb{N})$ defined by

$$H_n = H(X_n, R_n) \cap M = \{z \in M \mid \langle z, X_n\rangle \geq R_n\}, \quad n \in \mathbb{N},$$

is a sequence of random closed sets in M.

For the second example we fix $a > 0$ and let $(Y_n, n \in \mathbb{N})$ be an independent sequence of random variables, uniformly distributed on the ball $B_{C_M+a/2}(0)$ of radius $C_M + a/2$ centered at the origin. Furthermore, let $(D_n, n \in \mathbb{N})$ be an i.i.d. sequence of $[0,a]$-valued random variables, independent of $(Y_n, n \in \mathbb{N})$. Then $(B_n, n \in \mathbb{N})$ defined by

$$B_n = B_{D_n/2}(Y_n) \cap M = \Big\{ z \in M \,\Big|\, \|z - Y_n\| \le \frac{D_n}{2} \Big\}, \quad n \in \mathbb{N},$$

is an i.i.d. sequence of random closed sets in M. Since Y is uniformly distributed and independent of the diameter D, we have

$$P\big(x, y \in B_{D/2}(Y)\big) = P\big(Y \in B_{D/2}(x) \cap B_{D/2}(y)\big) = \frac{\mathbb{E}\Big(\lambda^d\big(B_{D/2}(x) \cap B_{D/2}(y)\big)\Big)}{\lambda^d\big(B_{C_M+a/2}(0)\big)} \tag{4.41}$$

for all $x, y \in M$. If for example D is taken to be deterministic, this reduces to a normalized geometric covariogram of a ball (e.g., [53]).

The intersection of two balls in $\mathbb{R}^d$ can be represented as the union of two equally sized hyperspherical caps. Hence, if D is equal to some $0 < t \le a$ it follows from (4.41) and [55] that

$$P\big(x, y \in B_{t/2}(Y)\big) = \frac{\Gamma(d/2+1)t^d}{\sqrt{\pi}(2C_M + a)^d \Gamma((d+1)/2)} B_{1-d_{xy}^2/t^2}\left(\frac{d+1}{2}, \frac{1}{2}\right) \mathbb{1}_{d_{xy} \le t}, \tag{4.42}$$

where we define $d_{xy} = \|x - y\|$ and

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\, dt, \quad x \in [0,1], a, b > 0,$$

is the incomplete Beta function (for instance [26, Section 8.17]). For example in dimension 2, we can use [26, Formula 8.17.20] and [35, Formulae 8.391 and 9.121.26] to obtain for $d_{xy} \le t$

$$\begin{aligned} B_{1-d_{xy}^2/t^2}\left(\frac{3}{2}, \frac{1}{2}\right) &= \frac{1}{2} B_{1-d_{xy}^2/t^2}\left(\frac{1}{2}, \frac{1}{2}\right) - \frac{d_{xy}}{t}\left(1 - \frac{d_{xy}^2}{t^2}\right)^{1/2} \\ &= \left(1 - \frac{d_{xy}^2}{t^2}\right)^{1/2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; 1 - \frac{d_{xy}^2}{t^2}\right) - \frac{1}{t^2} d_{xy}(t^2 - d_{xy}^2)^{1/2} \\ &= \arcsin\left(\left(1 - \frac{d_{xy}^2}{t^2}\right)^{1/2}\right) - \frac{1}{t^2} d_{xy}(t^2 - d_{xy}^2)^{1/2} \\ &= \arccos\left(\frac{d_{xy}}{t}\right) - \frac{1}{t^2} d_{xy}(t^2 - d_{xy}^2)^{1/2}. \end{aligned} \tag{4.43}$$

Here ${}_pF_q$ is the *hypergeometric function* which is formally defined by

$$\begin{aligned} {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) &= \sum_{k \in \mathbb{N}_0} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k} \frac{z^k}{k!} \\ &\text{with} \quad (a)_0 = 1, \quad (a)_k = a(a+1)\cdots(a+k-1), \quad k \in \mathbb{N}, \end{aligned}$$

see [35, Section 9.14] and [26, Section 16.2] for more information.

From (4.42) and (4.43) it follows for $d = 2$ that

$$P\big(x, y \in B_{t/2}(Y)\big) = \frac{2}{\pi(2C_M + a)^2}\bigg(t^2 \arccos\Big(\frac{d_{xy}}{t}\Big) - d_{xy}\sqrt{t^2 - d_{xy}^2}\bigg)\mathbb{1}_{d_{xy} \le t}.$$

If the diameter D is chosen to be a continuously distributed random variable, equation (4.42) has to be integrated with respect to the distribution of D. Sironvalle showed in [81], that for $d = 2$ the choice

$$F(x) = \frac{1}{a}\big(a - \sqrt{a^2 - x^2}\big)\mathbb{1}_{0 \le x \le a} + \mathbb{1}_{x > a}, \quad x \in \mathbb{R}, \tag{4.44}$$

for the distribution function of the diameter D results in $P(x, y \in B)$ being proportional to the spherical correlation function

$$\rho(x, y) = \Big(1 - \frac{3d_{xy}}{2a} + \frac{d_{xy}^3}{2a^3}\Big)\mathbb{1}_{d_{xy} \le a}. \tag{4.45}$$

In Proposition 4.3.1 below we consider the case of uniformly distributed diameter.

An example for random sets which lead to a stationary but anisotropic correlation function is given by hyperrectangles of the form

$$E_n = E(Z_n) \cap M = \{z \in M \,|\, |z_1 - Z_n^1| \le a_1, \ldots, |z_d - Z_n^d| \le a_d\}, \quad n \in \mathbb{N},$$

for $a_1, \ldots, a_d > 0$ and an i.i.d. sequence $(Z_n, n \in \mathbb{N})$ such that $Z = \big(Z^1, \ldots, Z^d\big)$ is uniformly distributed on $\prod_{k=1}^d [-(R_k + a_k), R_k + a_k]$ where $R = \prod_{k=1}^d [-R_k, R_k]$ is a hyperrectangle large enough such that $M \subseteq R$.

Proposition 4.3.1. *Suppose that $M \subset \mathbb{R}^d$ is as above, fix $x, y \in M$, and let $d_{xy} = \|x - y\|$. Then for $H = H(X, R) \cap M$*

$$P(x, y \in H) = \frac{1}{2}\Big(1 - c_d \frac{d_{xy}}{2C_M}\Big) \tag{4.46}$$

holds with $c_d = \Gamma(d/2)/\big(\sqrt{\pi}\Gamma((d + 1)/2)\big)$. For $B = B_{D/2}(Y) \cap M$ with D being uniformly distributed on $[0, a]$ the following formula holds true

$$P(x, y \in B) = \tilde{c}_d\Big(a^d B_{1 - d_{xy}^2/a^2}\Big(\frac{d + 1}{2}, \frac{1}{2}\Big) - \frac{d_{xy}^{d+1}}{a} B_{1 - d_{xy}^2/a^2}\Big(\frac{d + 1}{2}, -\frac{d}{2}\Big)\Big)\mathbb{1}_{d_{xy} \le a}, \tag{4.47}$$

where $d_{xy}^{d+1} B_{1 - d_{xy}^2/a^2}\big((d+1)/2, -d/2\big)$ is defined as zero for $d_{xy} = 0$, and the constant is $\tilde{c}_d = \Gamma(d/2 + 1)/\big((d + 1)\sqrt{\pi}(2C_M + a)^d\Gamma((d + 1)/2)\big)$. For $E = E(Z) \cap M$

$$P(x, y \in E) = \prod_{k=1}^d \frac{1}{2(R_k + a_k)}\big(2a_k - |x_k - y_k|\big)_+ \tag{4.48}$$

holds true.

Proof. The point $(x,r) \in \mathbb{S}^{d-1} \times \mathbb{R}$ defines the same hyperplane as the point $(-x,-r)$, but due to the opposite direction of the normal vector x we have the relation $H(x,r) \setminus P(x,r) = H(-x,-r)^c$ for the half-spaces. By construction, X and R have the same distribution as $-X$ and $-R$, respectively. Thus

$$\begin{aligned} P\big(x \in H(X,R)\big) &= P\big(x \in H(-X,-R)\big) \\ &= P\big(x \in H(-X,-R) \setminus P(-X,-R)\big) = P\big(x \notin H(X,R)\big), \end{aligned}$$

which implies $P(x \in H) = P(x \notin H) = 1/2$ for all $x \in M$. Now let $x \neq y$ and $d \geq 2$, then

$$\begin{aligned} P(x \in H, y \notin H) &= P\big(\langle X, y\rangle < R \leq \langle X, x\rangle\big) \\ &= \int_{\mathbb{S}^{d-1}} \frac{1}{2C_M}\big(\langle z,x\rangle - \langle z,y\rangle\big) \mathbb{1}_{\langle z,x\rangle > \langle z,y\rangle}\, d\bar{\sigma}_{d-1}(z) \\ &= \frac{d_{xy}}{2C_M} \mathbb{E}\Big(\Big\langle X, \frac{x-y}{d_{xy}}\Big\rangle\Big) \mathbb{1}_{\langle X,(x-y)/d_{xy}\rangle > 0}. \end{aligned}$$

Let $\mathcal{R}$ be a rotation which maps $(x-y)/d_{xy} \in \mathbb{S}^{d-1}$ to the point $\big(0,\dots,0,1\big)$, then $\langle X, (x-y)/d_{xy}\rangle = \big\langle \mathcal{R}X, \big(0,\dots,0,1\big)\big\rangle$. Since $\mathcal{R}X$ and X have the same distribution, we have using (1.1) and (1.2)

$$\mathbb{E}\Big(\Big\langle X, \frac{x-y}{d_{xy}}\Big\rangle\Big) \mathbb{1}_{\langle X,(x-y)/d_{xy}\rangle > 0} = \frac{1}{2\pi}\int_0^{2\pi} \sin(\varphi)\, \mathbb{1}_{\sin(\varphi)>0}\, d\varphi = \frac{1}{\pi}$$

for $d = 2$ and for $d \geq 3$

$$\begin{aligned} &\mathbb{E}\Big(\Big\langle X, \frac{x-y}{d_{xy}}\Big\rangle\Big) \mathbb{1}_{\langle X,(x-y)/d_{xy}\rangle > 0} \\ &\quad= \frac{\Gamma(d/2)}{2\pi^{d/2}} \int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi} \cos(\theta_{d-2}) \prod_{k=1}^{d-2} \sin^k(\theta_k)\, \mathbb{1}_{\cos(\theta_{d-2})>0}\, d\theta_{d-2}\dots d\theta_1 d\varphi \\ &\quad= \frac{\Gamma(d/2)}{2\sqrt{\pi}\Gamma((d+1)/2)}, \end{aligned}$$

where we used [35, Formulae 3.621.1, 3.621.5, 8.384.1, and 8.335.1]. For $d = 1$, one can do the same computation without spherical coordinates since the uniform distribution on $\mathbb{S}^0$ is just the two-point distribution on $\{-1,1\}$ which assigns both values probability 1/2. For $x = y$, the probability $P(x \in H, y \notin H)$ is zero. Hence, we have for all $x, y \in M$ and $d \geq 1$

$$P(x \in H, y \notin H) = c_d \frac{1}{2} \frac{d_{xy}}{2C_M}$$

and formula (4.46) is then obtained from

$$P(x, y \in H) = P(x \in H) - P(x \in H, y \notin H).$$

In the case of the random set $B_{D/2}(Y) \cap M$, it follows from (4.41) and (4.42) that

$$\begin{aligned} P\big(x, y \in B_{D/2}(Y) \cap M\big) &= \frac{\mathbb{1}_{d_{xy} \leq a}}{a\, \tilde{c}_d} \int_{d_{xy}}^{a} (d+1)\, t^d\, B_{1-d_{xy}^2/t^2}\Big(\frac{d+1}{2}, \frac{1}{2}\Big)\, dt \\ &= \frac{\mathbb{1}_{d_{xy} \leq a}}{a\, \tilde{c}_d} \int_{d_{xy}}^{a} \int_0^{1-d_{xy}^2/t^2} (d+1)\, t^d\, s^{(d-1)/2}\, (1-s)^{-1/2}\, ds\, dt. \end{aligned}$$

An application of Fubini's theorem yields for the integral

$$\int_0^{1-d_{xy}^2/a^2} s^{(d-1)/2}(1-s)^{-1/2} \int_{d_{xy}/\sqrt{1-s}}^{a} (d+1)\, t^d \, dt\, ds$$
$$= a^{d+1} B_{1-d_{xy}^2/a^2}\Big(\frac{d+1}{2}, \frac{1}{2}\Big) - \int_0^{1-d_{xy}^2/a^2} d_{xy}^{d+1} s^{(d-1)/2}(1-s)^{-(d+2)/2}\, ds.$$

The last integral is 0 if $d_{xy} = 0$, and we can write it as $d_{xy}^{d+1} B_{1-d_{xy}^2/a^2}\big((d+1)/2, -d/2\big)$ if $0 < d_{xy} \leq a$. Collecting terms we obtain (4.47).

Regarding (4.48), the components of $Z = \big(Z^1, \ldots, Z^d\big)$ are independent and uniformly distributed on $[-(R_k + a_k), R_k + a_k]$ and (4.48) follows from

$$P(x, y \in E) = \prod_{k=1}^{d} P\big(|x_k - Z^k| \leq a_k, |y_k - Z^k| \leq a_k\big)$$

and

$$P\big(|x_k - Z^k| \leq a_k, |y_k - Z^k| \leq a_k\big) = \frac{1}{2(R_k + a_k)}\big(2a_k - |x_k - y_k|\big)_+$$

for $k = 1, \ldots, d$. $\square$

Remark 4.3.2. The correlation functions in Corollary 4.2.4 depend on x and y only through the probabilities $p_{xy} = P(x, y \in B)$. Proposition 4.3.1 and equation (4.42) show, that the functions $(x, y) \mapsto p_{xy}$ are stationary and isotropic on M if the random set B is a random half-space $H(X, R)$ or a random closed ball $B_{D/2}(Y)$. Therefore, the correlation functions that we get from the combination of Corollary 4.2.4, Proposition 4.3.1, and equation (4.42), will be stationary and isotropic. To get less regular correlation functions, more general random closed sets B have to be considered.

A very simple choice for the distribution of the random variable N in (4.29) on page 58 is $N \equiv 1$. This leads to the correlation function

$$\rho(x, y) = P(x, y \in B) + P(x, y \notin B), \quad x, y \in M, \tag{4.49}$$

for the simple mosaic random field which is build from one random closed set. Taking random half-spaces H as random closed sets B, we obtain with (4.46)

$$P(x, y \in H) = P(x \in H) - P(y \notin H) + P(x, y \notin H) = P(x, y \notin H),$$

and therefore (4.49) becomes with (4.46)

$$\rho(x, y) = 1 - c_d \frac{d_{x,y}}{2C_M}, \quad x, y \in M,$$

with the constant $c_d = \Gamma(d/2)/\big(\sqrt{\pi}\Gamma((d+1)/2)\big)$. This *base correlation* will be used hereafter to produce more evolved correlation functions. Before doing so, let us take a closer look at the behaviour of the constant c_d which governs the decay of the base correlation (4.49). For $d = 1, 2$, and 3 the factor c_d equals $1, 2/\pi$, and $1/2$, respectively. Figure 4.3 suggests, that c_d is strictly decreasing as the dimension increases to ∞. The following lemma shows, that this is indeed true and that c_d converges to 0.

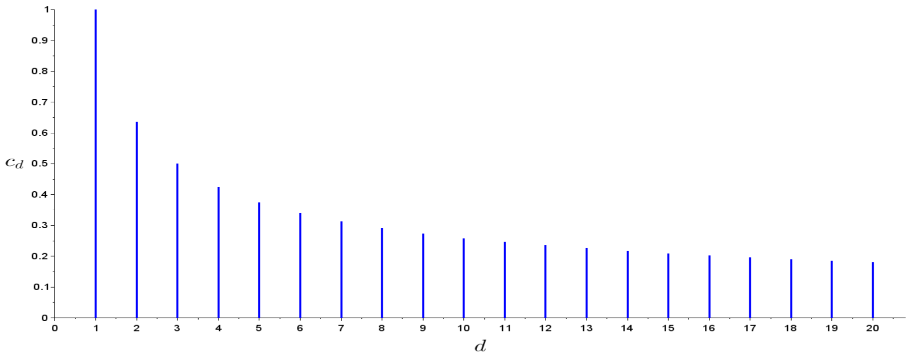


Figure 4.3.: The factor c_d for $d = 1, \ldots, 20$.

Lemma 4.3.3. *The sequence* $(c_d, d \in \mathbb{N})$ *is strictly decreasing to* 0 *as* d *increases to* ∞.

Proof. From [50] we have the following inequality for the quotient of gamma functions:

$$e^{(1-\lambda)\Psi(x+\lambda)} < \frac{\Gamma(x+1)}{\Gamma(x+\lambda)} < e^{(1-\lambda)\Psi(x+1)}, \quad x > 0, \lambda \in (0,1). \tag{4.50}$$

Here $\Psi(x) = \frac{d}{dx}\log(\Gamma(x))$, $x > 0$, denotes the *digamma function* (e.g., [26, Formula 5.5.2]). Using (4.50) with $\lambda = 1/2$, we have for all $x > 1/2$

$$\frac{\Gamma(x)}{\Gamma(x+1/2)}\frac{\Gamma(x+1)}{\Gamma(x+1/2)} > e^{-1/2\Psi(x+1/2)}e^{1/2\Psi(x+1/2)} = 1,$$

hence

$$\frac{\Gamma(x)}{\sqrt{\pi}\Gamma(x+1/2)} > \frac{\Gamma(x+1/2)}{\sqrt{\pi}\Gamma(x+1)}, \quad x > \frac{1}{2},$$

implying $c_d > c_{d+1}$ for all $d > 1$. For $d = 1$ this inequality is also true so that $(c_d, d \in \mathbb{N})$ is strictly decreasing. From [26, Formula 5.15.1] we have

$$\Psi'(x) = \sum_{k=0}^{\infty} \frac{1}{(x+k)^2}, \quad x > 0,$$

so that Ψ is strictly increasing. The mean value theorem entails that for all $x > 0$

$$\log\big(\Gamma(x+1)\big) - \log\big(\Gamma(x)\big) = \Psi(x_0)$$

for some $x_0 \in (x, x+1)$. Since Ψ is strictly increasing we get

$$\log\Big(\frac{\Gamma(x+1)}{\Gamma(x)}\Big) = \log x < \Psi(x+1)$$

so that Ψ increases to ∞ as $x \to \infty$. Another application of (4.50) yields

$$\frac{\Gamma(x)}{\sqrt{\pi}\Gamma(x+1/2)} < \frac{1}{\sqrt{\pi}}e^{-1/2\Psi(x)}, \quad x > \frac{1}{2},$$

which implies that $(c_d, d \in \mathbb{N})$ converges to 0. $\square$

4.4. Explicit Correlation Functions on Bounded Subsets of $\mathbb{R}^d$

In this section we give examples of correlation functions on bounded subsets M of $\mathbb{R}^d$ which can be obtained for the mosaic random field (4.19) by the combination of Proposition 4.3.1 and Corollary 4.2.4.

Before we begin with the examples, let us mention a lesser-known probability distribution and its probability generating function which we will use frequently in the following. In [80] it was shown, that for every $\alpha \in (0,1)$,

$$p_n = \frac{\alpha\,\Gamma(n-\alpha)}{\Gamma(1-\alpha)\,n!}, \quad n \in \mathbb{N}, \tag{4.51}$$

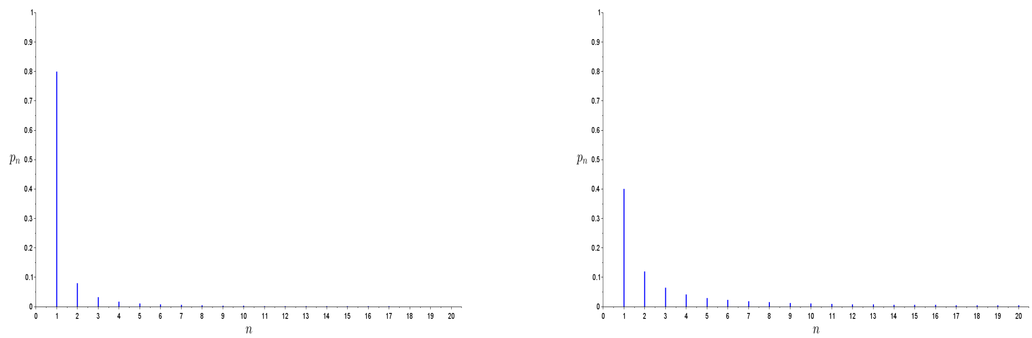
defines a probability distribution $\sum_{n\in\mathbb{N}} p_n \varepsilon_n$ on $\big(\mathbb{N}, \mathcal{P}(\mathbb{N})\big)$. This distribution can be extended to $\alpha \in (0,1]$ by letting $p_1 = 1$ and $p_n = 0$, $n \geq 2$, in case $\alpha = 1$. In the following, this distribution will be called Sibuya(α) distribution after [80]. By the functional equation of the gamma function we have for every $\alpha \in (0,1)$

$$p_n = -\frac{\Gamma(n-\alpha)}{\Gamma(-\alpha)\,n!}, \quad n \in \mathbb{N}.$$

Using this and [73, Formula 7.3.1.27] we have for the probability generating function ψ of the Sibuya(α) distribution

$$\psi(t) = \sum_{n=1}^{\infty} p_n t^n = 1 - \sum_{n=0}^{\infty} \frac{\Gamma(n-\alpha)}{\Gamma(-\alpha)} \frac{t^n}{n!} = 1 - (1-t)^\alpha, \quad t \in [-1,1], \alpha \in (0,1). \tag{4.52}$$

Since $1-(1-t)^\alpha$ is also the probability generating function of the Sibuya(α) distribution in case $\alpha = 1$, the result of (4.52) holds true for all $\alpha \in (0,1]$.



(a) $\alpha = 0.8$. (b) $\alpha = 0.4$.

Figure 4.4.: The probabilities $p_1, \dots, p_{20}$ in (4.51).

Figure 4.4 displays the probabilities $p_1, \dots, p_{20}$ for different α. The Sibuya(α) distribution is heavy-tailed for $\alpha \in (0,1)$. To see this, we write (4.51) as

$$p_n = \frac{\alpha}{\Gamma(1-\alpha)(n-\alpha)} \frac{\Gamma(n+1-\alpha)}{\Gamma(n+1)}, \quad n \in \mathbb{N}.$$

An application of Gautschi's inequality [26, Formula 5.6.4] for the quotient of gamma functions yields the bounds

$$\frac{\alpha}{\Gamma(1-\alpha)(n-\alpha)(n+1)^\alpha} < p_n < \frac{\alpha}{\Gamma(1-\alpha)(n-\alpha)n^\alpha}, \quad n \in \mathbb{N}, \alpha \in (0,1). \quad (4.53)$$

From this it follows that

$$\sum_{n\in\mathbb{N}} p_n n \geq \frac{\alpha}{\Gamma(1-\alpha)} \sum_{n\in\mathbb{N}} \frac{1}{n^\alpha},$$

and we conclude that the Sibuya(α) distribution does not have a finite mean if $\alpha \in (0,1)$.

For the simulation of Sibuya(α) distributed random numbers for $\alpha \in (0,1)$ we use the following result which is taken from [80] (see also [25]): If $X_1 \sim \mathrm{Exp}(1)$, $X_2 \sim \mathrm{Gamma}(1-\alpha, 1)$, and $X_3 \sim \mathrm{Gamma}(\alpha, 1)$ are independent random variables and N is Poisson distributed with random parameter $\lambda = (X_1X_2)/X_3$, then

$$1 + N \sim \mathrm{Sibuya}(\alpha).$$

In case $\alpha = 1$, we have by definition Sibuya(α) $= \varepsilon_1$.

The next lemma is used for our simulations of approximate Gaussian random fields below. It is a direct consequence of the multivariate central limit theorem (e.g., [6, Satz 30.3]).

Lemma 4.4.1. *Let M be a set, let $Z = (Z(x), x \in M)$ be a real-valued and square-integrable random field on M with correlation function ρ, and let $(Z_i, i \in \mathbb{N})$ be a sequence of independent copies of Z. Let the sequence of random fields $(S_n, n \in \mathbb{N})$ on M be defined by*

$$S_n(x) = \frac{1}{\sqrt{n \operatorname{Var}\big(Z(x)\big)}} \Big(\sum_{i=1}^n Z_i(x) - n\mathbb{E}\big(Z(x)\big) \Big), \quad n \in \mathbb{N}, x \in M. \quad (4.54)$$

If $Y = (Y(x), x \in M)$ is a centered Gaussian random field with the covariance function ρ, then for each $m \in \mathbb{N}$ and any $x_1, \dots, x_m \in M$, the random vector $\big(S_n(x_1), \dots, S_n(x_m)\big)$ converges in distribution to $\big(Y(x_1), \dots, Y(x_m)\big)$ as $n \to \infty$.

Henceforth, the random field (4.54) will be called an *approximate Gaussian random field.* If we can simulate the random field Z in Lemma 4.4.1, then we can simulate an approximate Gaussian random field with the correlation function ρ of Z as its covariance function by means of (4.54).

There are many approaches which have been developed in order to decide from which number n of superpositions in (4.54) onwards, (4.54) can be considered as a Gaussian random field for practical purposes (for an overview see [53, Section 15.2.5]). One of these approaches is given by considering the Kolmogorov distance

$$\sup_{y\in\mathbb{R}} |P(S_n(x) \leq y) - \Phi(y)|, \quad x \in M,$$

between the marginal distribution of S_n and the standard normal distribution (here Φ denotes the distribution function of the standard normal distribution). We may then think of $(S_n(x), x \in M)$ as an adequate approximation to a Gaussian random field if this distance is small. The Berry–Esseen theorem [9, 27] provides an upper bound for this distance:

$$\sup_{y\in\mathbb{R}} |P(S_n(x) \le y) - \Phi(y)| \le \frac{C_{BE}\,\mathbb{E}\big(|Z(x)|^3\big)}{\operatorname{Var}\big(Z(x)\big)^{3/2}\sqrt{n}}, \quad x \in M. \tag{4.55}$$

The constant C_{BE} in the bound is valid for any distribution of the random variable $Z(x)$ defining the sum $S_n(x)$. In the original work [27], the value of C_{BE} was given as 7.59. It has been lowered successively over the years up to a value of 0.4784 in [49] more recently. For simplicity we assume $C_{BE} = 1/2$ henceforth.

This criterion is certainly not the best possible, since (4.55) does only involve the marginal distributions and the convergence that we aim for is the convergence in the sense of the finite-dimensional distributions. However, the advantage of this approach is that the information that is necessary in order to apply the criterion is available for the random fields that are considered in this thesis (see Proposition 4.2.6 for the mosaic random field and Proposition 5.2.4 on page 110 for the spectral turning bands random field on $\mathbb{S}^d$ that is presented in Chapter 5). Therefore, we stick to this criterion and keep in mind that the bound on n that is obtained from (4.55) will be too small in general.

All simulations in the present thesis have been performed with the Scilab software package [79, Version 6.0.1].

Example 4.4.2. The generalized Cauchy correlation function [32, 34] in $\mathbb{R}^d$ is of the form

$$\rho(x,y) = \left(1 + \left(\frac{d_{xy}}{c}\right)^{\alpha}\right)^{-\beta/\alpha}, \quad x, y \in M. \tag{4.56}$$

Here $c > 0$ is a scale parameter and $\alpha \in (0,2]$, $\beta > 0$, are shape parameters. Suppose M is a bounded and closed subset of $\mathbb{R}^d$, then a mosaic random field having the correlation function (4.56) can be obtained as follows. The negative binomial distribution $\operatorname{NegBin}(r,p)$ with parameter $r > 0$, $p \in (0,1)$, is defined by the probability mass function

$$p_n = \frac{\Gamma(n+r)}{n!\,\Gamma(r)} p^n (1-p)^r, \quad n \in \mathbb{N}_0.$$

The probability generating function ψ_1 of the negative binomial distribution is of the form

$$\psi_1(t) = \left(\frac{1}{p} - \frac{1-p}{p} t\right)^{-r}, \quad t \in [-1,1].$$

Composing this probability generating function with the probability generating function ψ_2 of a Sibuya(α) distribution (4.52) yields

$$(\psi_1 \circ \psi_2)(t) = \left(1 + \frac{1-p}{p}(1-t)^{\alpha}\right)^{-r}, \quad t \in [-1,1]. \tag{4.57}$$

At this point we have to make the restriction $\alpha \in (0,1]$, otherwise (4.51) would not define a probability distribution (this can be seen by considering the first probability p_1). Concerning this restriction, see also Remark 4.6.2 in Section 4.6. If now L is an $\mathrm{NegBin}(r,p)$ distributed random variable and if $(K_l, l \in \mathbb{N})$ are independent $\mathrm{Sibuya}(\alpha)$ distributed random variables, independent of L, then by Lemma 4.1.8 $N = \sum_{l=1}^{L} K_l$ has the probability generating function (4.57). By (4.29), Corollary 4.2.4, the correlation function of a simple mosaic random field for which the underlying mosaic is build from N random sets has the correlation function

$$\rho(x,y) = \Big(1 + \frac{1-p}{p}(p_x + p_y - 2p_{xy})^{\alpha}\Big)^{-r}, \quad x, y \in M.$$

Let us now choose for this simple mosaic random field random half-spaces $H(X,R)$ as random sets. Here $X \sim \mathcal{U}(\mathbb{S}^{d-1})$, $R \sim \mathcal{U}([-C_M, C_M])$, and $C_M > 0$ is a constant large enough such that $M \subseteq B_{C_M}(0) \subseteq \mathbb{R}^d$. From (4.46) in Proposition 4.3.1 we get

$$\rho(x,y) = \Big(1 + \frac{1-p}{p}\Big(c_d \frac{d_{xy}}{2C_M}\Big)^{\alpha}\Big)^{-r}, \quad x, y \in M.$$

For any $\alpha \in (0,1]$ and all $\beta > 0$ the choice $r = \beta/\alpha$ is a valid choice for the parameter r of the negative binomial distribution. Furthermore, for all $\alpha \in (0,1]$, $c > 0$, any constant $C_M > 0$, and all $d \in \mathbb{N}$,

$$p = \Big(1 + \Big(\frac{2C_M}{c_d c}\Big)^{\alpha}\Big)^{-1}$$

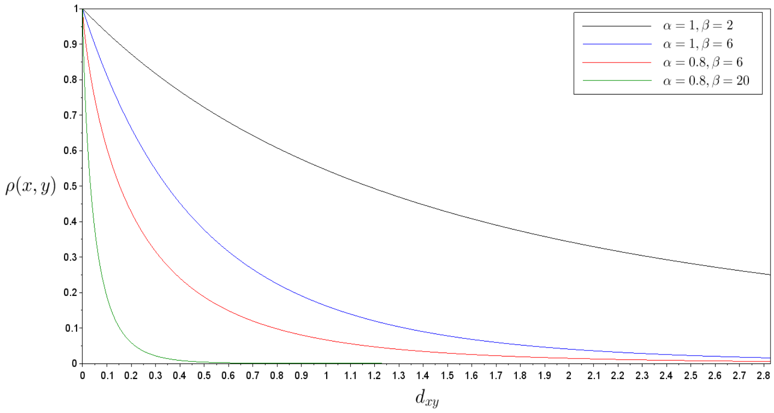
is a valid choice for the parameter p of the negative binomial distribution. With these values of r and p, the correlation function of the simple mosaic random field becomes

$$\rho(x,y) = \Big(1 + \Big(\frac{d_{xy}}{c}\Big)^{\alpha}\Big)^{-\beta/\alpha}, \quad \alpha \in (0,1], \beta, c > 0, x, y \in M. \tag{4.58}$$

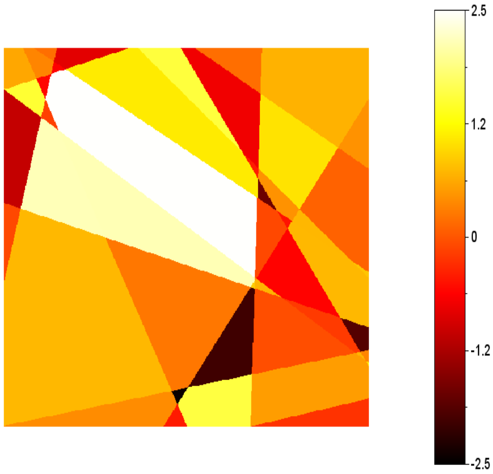
The generalized Cauchy correlation function, realizations of the corresponding simple mosaic random field, and realizations of an approximate Gaussian random field are given in Figure 4.5. For the simulation we have chosen $M = [-1,1]^2$, $C_M = \sqrt{2}$, $c = 2\sqrt{2}$, and standard normally distributed marginals $U_{i,j}, i, j \in \mathbb{N}$. By (4.33) and (4.34) we have $\mathbb{E}\big(Z_M(x)\big) = \mathbb{E}(U) = 0$ and $\mathrm{Var}\big(Z(x)\big) = \mathrm{Var}(U) = 1$. Hence (4.38) in Proposition 4.2.6 yields the bound $C_{BE}\,\mathbb{E}(|U^3|)/\sqrt{n}$ on the Kolmogorov distance between the marginal distribution of the approximate Gaussian random field and a standard normal distribution. The absolute third moment of a standard normal distribution can be found from integration by parts and it is equal to $4/\sqrt{2\pi}$. With $C_{BE} = 1/2$ the bound in (4.55) equals $\sqrt{2/(n\pi)}$ and $n = 300$ superpositions yields a bound lower than a critical value of 0.05. Figure 4.5d (4.5e) presents a realization of an approximate Gaussian random field which is build from from $n = 300$ ($n = 1000$) realizations of a simple mosaic random field with the generalized Cauchy correlation function (4.58).

Example 4.4.3. In this example we consider a random token field on a bounded closed set $M \subseteq \mathbb{R}^2$. We assume that the number of random closed sets N defining the underlying mosaic of the random token field is Poisson distributed, so that $\mathrm{Var}(N) = \mathbb{E}(N)$ and we get from (4.30) in Corollary 4.2.4 the correlation function

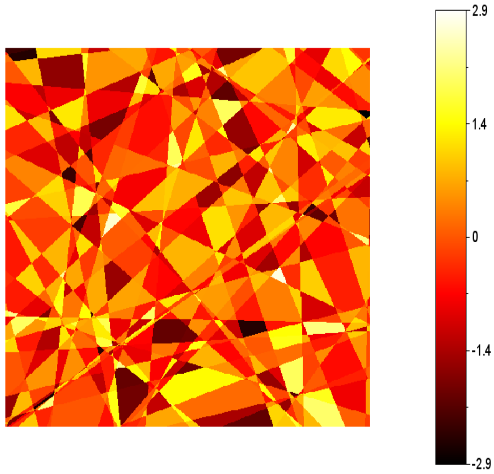
$$\rho(x,y) = \frac{p_{xy}}{\sqrt{p_x p_y}}, \quad x, y \in [-1,1]^2.$$



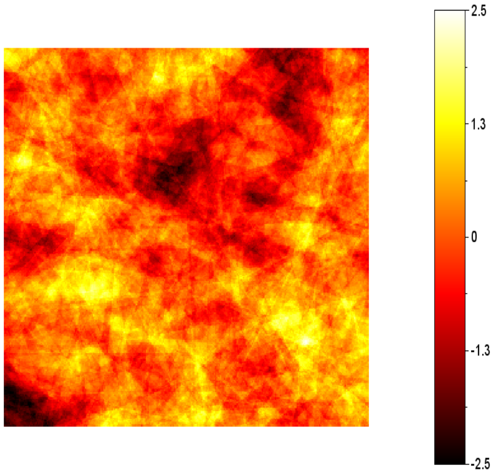
(a) Correlation function (4.58) for different α, β. The scale parameter is $c = 2\sqrt{2}$.



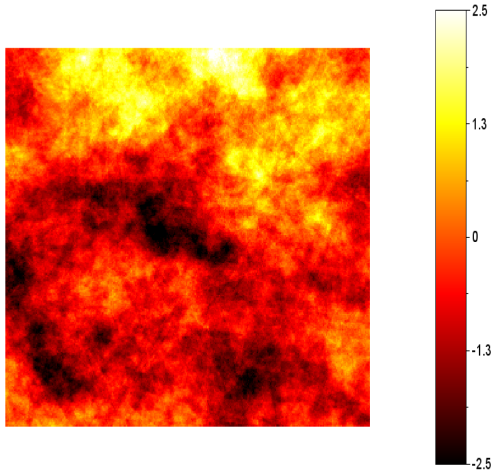
(b) Simple mosaic random field with $\alpha = 0.8,\ \beta = 6,\ c = 2\sqrt{2}$.



(c) Simple mosaic random field with $\alpha = 0.8,\ \beta = 20,\ c = 2\sqrt{2}$.



(d) Approximate Gaussian random field with $\alpha = 0.8,\ \beta = 6,\ c = 2\sqrt{2},\ n = 300$.



(e) Approximate Gaussian random field with $\alpha = 0.8,\ \beta = 6,\ c = 2\sqrt{2},\ n = 1000$.

Figure 4.5.: Simple mosaic random field with the Cauchy correlation function (4.58).

Hence the choice of the distribution of the marginals $(U_{i,j}, i, j \in \mathbb{N})$ and the parameter of the Poisson distribution for N has no impact on the correlation function, as long as $\mathrm{Var}(U) > 0$. As random closed sets we take random balls $(B_n, n \in \mathbb{N})$ defined by $B_n = B_{D_n/2}(Y_n) \cap M$, where the Y_n are uniformly distributed on $B_{C_M+a/2}(0) \subseteq \mathbb{R}^2$ and $D_n \sim \mathcal{U}([0, a])$. Here $a > 0$ is a cutoff-parameter and $C_M > 0$ is a constant depending on the set M. Then we are in the situation of (4.47) in Proposition 4.3.1 with $d = 2$ and get

$$p_{xy} = \frac{2}{3\pi(2C_M + a)^2}\Big(a^2 B_{1-d_{xy}^2/a^2}\Big(\frac{3}{2}, \frac{1}{2}\Big) - \frac{d_{xy}^3}{a} B_{1-d_{xy}^2/a^2}\Big(\frac{3}{2}, -1\Big)\Big)\mathbb{1}_{d_{xy} \leq a}$$

for $x, y \in M$. An explicit form for the first incomplete Beta function has been given in (4.43). For the second incomplete Beta function we can use [35, Formula 8.391] and [73, Formula 7.3.2.210] such that for $d_{xy} \leq a$

$$\begin{aligned} B_{1-d_{xy}^2/a^2}\Big(\frac{3}{2}, -1\Big) &= \Big(1 - \frac{d_{xy}^2}{a^2}\Big)^{3/2} {}_2F_1\Big(\frac{3}{2}, 2; \frac{5}{2}; 1 - \frac{d_{xy}^2}{a^2}\Big) \\ &= \Big(1 - \frac{d_{xy}^2}{a^2}\Big)^{1/2}\Big(\frac{a^2}{d_{xy}^2} - \Big(1 - \frac{d_{xy}^2}{a^2}\Big)^{-1/2} \operatorname{artanh}\Big(\Big(1 - \frac{d_{xy}^2}{a^2}\Big)^{1/2}\Big)\Big). \end{aligned}$$

The probability p_{xy} then becomes

$$\begin{aligned} p_{xy} = \frac{2}{3\pi(2\sqrt{2} + a)^2}\Big(& a^2 \arccos\Big(\frac{d_{xy}}{a}\Big) - 2d_{xy}\sqrt{a^2 - d_{xy}^2} \\ &+ \frac{d_{xy}^3}{a} \operatorname{artanh}\Big(\Big(1 - \frac{d_{xy}^2}{a^2}\Big)^{1/2}\Big)\Big)\mathbb{1}_{d_{xy} \leq a}, \quad x, y \in M. \end{aligned} \tag{4.59}$$

From this we also get $p_x = a^2/\big(3(2C_M + a)^2\big)$, so that the correlation function of the random token field is

$$\begin{aligned} \rho(x, y) = \Big(& \frac{2}{\pi} \arccos\Big(\frac{d_{xy}}{a}\Big) - \frac{4}{\pi a^2} d_{xy}\sqrt{a^2 - d_{xy}^2} \\ &+ \frac{2}{\pi a^3} d_{xy}^3 \operatorname{artanh}\Big(\Big(1 - \frac{d_{xy}^2}{a^2}\Big)^{1/2}\Big)\Big)\mathbb{1}_{d_{xy} \leq a} \end{aligned} \tag{4.60}$$

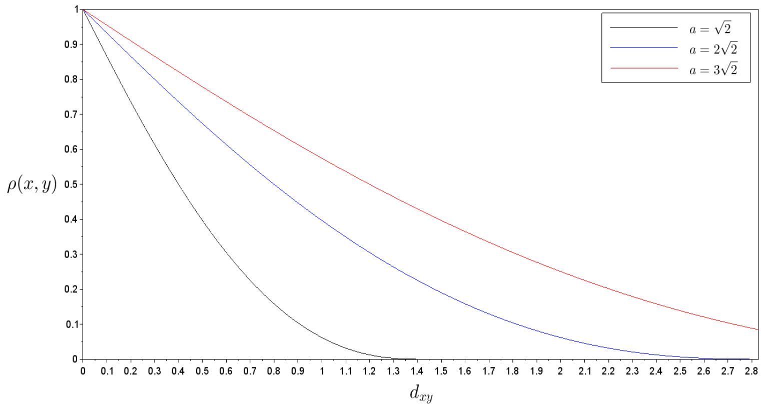
for $x, y \in M$.

Figure 4.6 displays this correlation function, realizations of the random token field, and realizations of the corresponding approximate Gaussian random field. For the simulations we have chosen $M = [-1, 1]^2$ $(C_M = \sqrt{2})$, $\lambda = 50$ for the Poisson distribution, and standard normally distributed marginals $U_{i,j}$, $i, j \in \mathbb{N}$. For the approximate Gaussian random field we took the cutoff parameter $a = \sqrt{2}$. By formulae (4.35) and (4.36) the variance of a random token field is given by

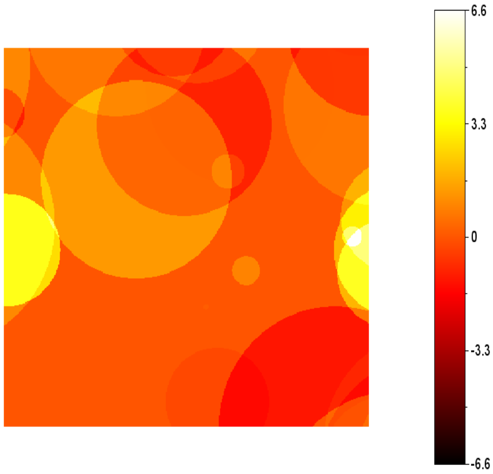
$$\mathrm{Var}(Z_{RT}(x)) = \mathbb{E}(U^2)\mathbb{E}(N)p_x + \mathbb{E}(U)^2\big(\mathrm{Var}(N) - \mathbb{E}(N)\big)p_x^2, \quad x \in M,$$

and for the absolute third moment we have from (4.39) in Proposition 4.2.6 the bound

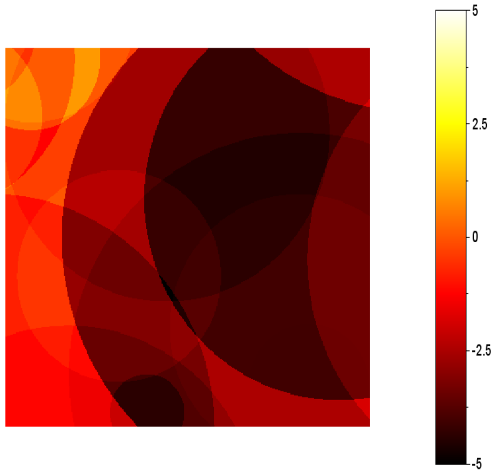
$$\mathbb{E}(|Z_{RT}(x)|^3) \leq p_x\mathbb{E}(|U|^3)\big(p_x^2\mathbb{E}(N^3) + 3p_x(1 - p_x)\mathbb{E}(N^2) + (1 - 2p_x)(1 - p_x)\mathbb{E}(N)\big).$$



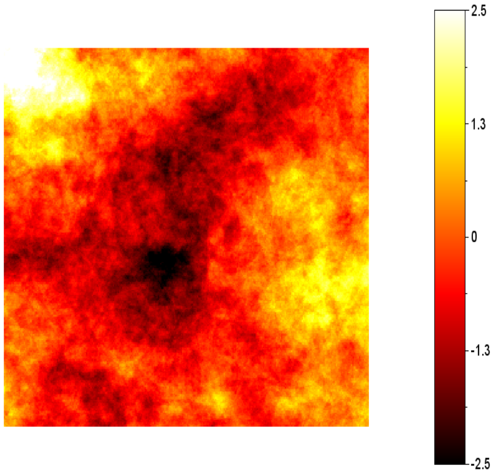
(a) Correlation function (4.60) for different a.



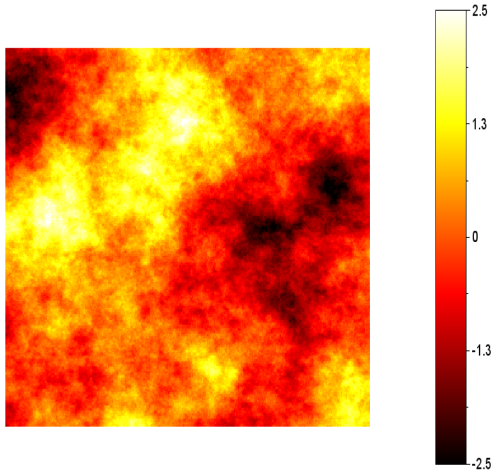
(b) Random token field with $\lambda = 50$, $a = \sqrt{2}$.



(c) Random token field with $\lambda = 50$, $a = 2\sqrt{2}$.



(d) Approximate Gaussian random field with $\lambda = 50$, $a = \sqrt{2}$, $n = 500$.



(e) Approximate Gaussian random field with $\lambda = 50$, $a = \sqrt{2}$, $n = 14000$.

Figure 4.6.: Random token field with correlation function (4.60).

Plugging in the corresponding values in (4.55), we get an upper bound on the Kolmogorov distance which is smaller than $5.86/\sqrt{n}$. We choose $n = 14000$ superpositions in the approximate Gaussian random field such that this bound is smaller than 0.05. Figure 4.6e displays the result of the simulation. Figure 4.6d displays a simulation with $n = 500$ superpositions.

Example 4.4.4. Let Z be a dead leaves random field on a bounded subset M of $\mathbb{R}^d$. We take random half-spaces $H(X, R)$ as random sets, such that formulae (4.46) and (4.32) yield the correlation function

$$\rho(x,y) = \frac{1/2\big(1 - c_d\, d_{xy}/(2C_M)\big) + c_d\, d_{xy}/(2C_M)\psi_N\Big(1/2\big(1 - c_d\, d_{xy}/(2C_M)\big)\Big)}{1/2\big(1 + c_d\, d_{xy}/(2C_M)\big)}$$

for $x, y \in M$. The probability generating function of the $\mathrm{Geo}_{\mathbb{N}}(p)$ distribution is given by

$$\psi(t) = \frac{p\,t}{1 - (1-p)\,t}, \quad p \in (0,1), t \in [-1,1].$$

Taking this distribution for the number N of random half-spaces, we obtain after some manipulations the correlation function

$$\rho(x,y) = \frac{1 - c_d\, d_{xy}/(2C_M)}{1 + (1-p)/(1+p)\, c_d\, d_{xy}/(2C_M)}, \quad x, y \in M.$$

If $c > 2C_M/c_d$, we may choose $p = (c\,c_d - 2C_M)/(c\,c_d + 2C_M)$ as the success probability for the geometric distribution and obtain the correlation function

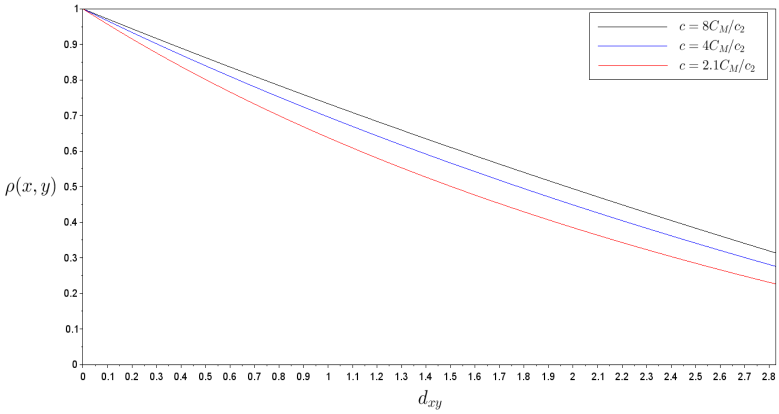
$$\rho(x,y) = \frac{1 - c_d\, d_{xy}/(2C_M)}{1 + d_{xy}/c}, \quad x, y \in M, c > \frac{2C_M}{c_d}. \tag{4.61}$$

Figure 4.7 displays this correlation function on $M = [-1,1]^2$ for different c. Simulations of the dead leaves random field and the corresponding approximate Gaussian random field are depicted in Figure 4.7. For the simulations we took $M = [-1,1]^2$, $C_M = \sqrt{2}$, and standard normally distributed marginals. In Figure 4.7b there are $N = 3$ simulated half-spaces, while in Figure 4.7c there are $N = 13$. However, in both figures only 3 half-spaces are visible because in Figure 4.7c earlier simulated half-spaces are hidden under later simulated ones. This also explains the minor influence of the parameter c on the correlation function (4.61), which can be observed in Figure 4.7a. Concerning the approximation to a Gaussian random field, we have by (4.38) the same bound on the Kolmogorov distance as in Example 4.4.2. Figure 4.7d displays a simulation of the approximate Gaussian random field with $n = 300$ simulated dead leaves random fields and in Figure 4.7e we have chosen $n = 5000$.

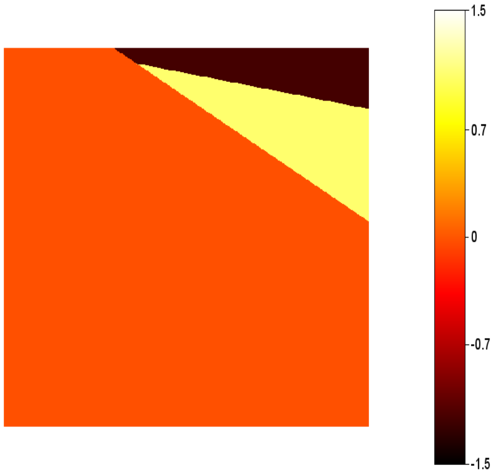
More correlation functions on bounded subsets of $\mathbb{R}^d$ for which there are associated mosaic random fields are given in Tables 4.1 and 4.2. The objects which determine the respective mosaic random fields are given in Appendix B.1 .

4.5. Random Closed Sets on $\mathbb{S}^d$

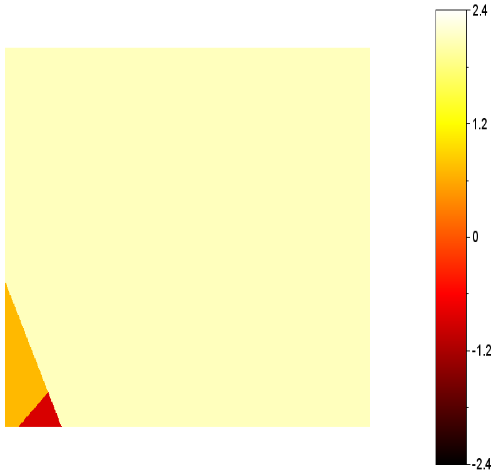
In this section we let $M = \mathbb{S}^d$ be the d-dimensional unit sphere, σ_d the surface measure on $\mathbb{S}^d$ defined in (1.2), and ϕ_d the spherical coordinate map defined in (1.1).



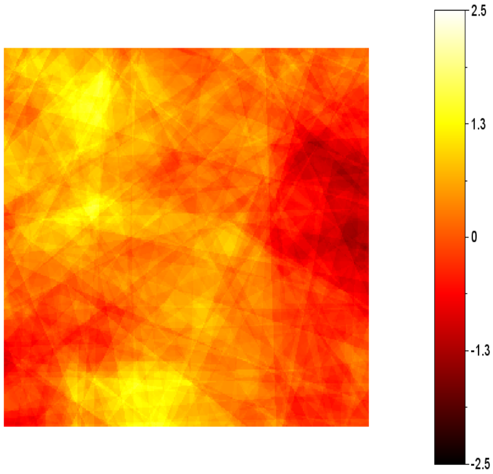
(a) Correlation function (4.61) for different c.



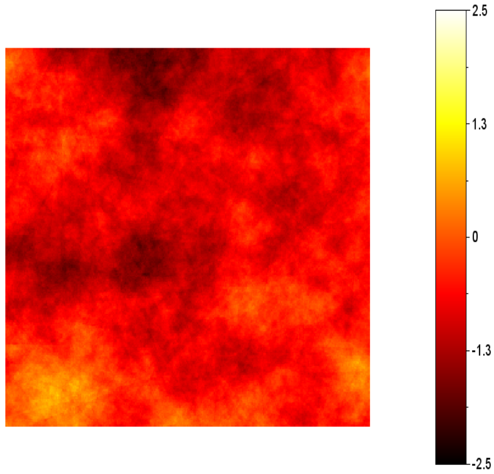
(b) Dead leaves field with $c = 4C_M/c_2$.



(c) Dead leaves field with $c = 2.1C_M/c_2$.



(d) Approximate Gaussian random field with $c = 2.1C_M/c_2$, $n = 300$.



(e) Approximate Gaussian random field with $c = 2.1C_M/c_2$, $n = 5000$.

Figure 4.7.: Dead leaves random field with correlation function (4.61).

	Correlation function	Parameter	Lit.
1.	$\rho(x,y) = \left(1 + \left(\frac{d_{xy}}{c}\right)^{\alpha}\right)^{-\beta/\alpha}$	$\alpha \in (0,1]$, $\beta, c > 0$	[32, 34]
2.	$\rho(x,y) = e^{-(d_{xy}/c)^{\alpha}}$	$\alpha \in (0,1]$, $c > 0$	[20, 53]
3.	$\rho(x,y) = 1 - \left(\frac{(d_{xy}/c)^{\alpha}}{1+(d_{xy}/c)^{\alpha}}\right)^{\beta}$	$\alpha, \beta \in (0,1]$, $c > 0$	[7]
4.	$\rho(x,y) = 1 - \left(\frac{(1+(d_{xy}/c)^{\alpha})^{\gamma}-1}{(1+(d_{xy}/c)^{\alpha})^{\gamma}}\right)^{\beta}$	$\alpha, \beta \in (0,1]$, $\gamma, c > 0$	
5.	$\rho(x,y) = \left(1 - \left(\frac{d_{xy}}{c}\right)^{\alpha}\right)^{n}$	$\alpha \in (0,1]$, $c \geq \frac{2C_M}{c_d}, n \in \mathbb{N}_0$	[75]*
6.	$\rho(x,y) = \left(1 - c_d \frac{d_{xy}}{2C_M}\right)^{n} e^{-(d_{xy}/c)^{\alpha}}$	$\alpha \in (0,1]$, $c > 0$, $n \in \mathbb{N}_0$	[20, 53]*
7.	$\rho(x,y) = \frac{1-(c_d\, d_{xy}/(2C_M))^{\alpha}}{1-c_d\, d_{xy}/(2C_M)}$	$\alpha \in (0,1]$	
8.	$\rho(x,y) = \frac{1-e^{-(d_{xy}/c)^{\alpha}}}{(d_{xy}/c)^{\alpha}}$	$\alpha \in (0,1]$, $c > 0$	
9.	$\rho(x,y) = \frac{\log(1+(d_{xy}/c)^{\alpha})}{(d_{xy}/c)^{\alpha}}$	$\alpha \in (0,1]$, $c > 0$	
10.	$\rho(x,y) = \left(1 + \beta - \beta e^{-(d_{xy}/c)^{\alpha}}\right)^{-\gamma}$	$\alpha \in (0,1]$, $\beta, \gamma, c > 0$	
11.	$\rho(x,y) = e^{-\beta (d_{xy}/c)^{\alpha}/(1+(d_{xy}/c)^{\alpha})}$	$\alpha \in (0,1], \beta, c > 0$	
12.	$\rho(x,y) = \frac{1+(d_{xy}/c)^{\alpha}}{1+\beta(d_{xy}/c)^{\alpha}}$	$\alpha \in (0,1]$, $\beta \geq 1$, $c > 0$	
13.	$\rho(x,y) = e^{-\beta(1-e^{-(d_{xy}/c)^{\alpha}})}$	$\alpha \in (0,1]$, $\beta, c > 0$	
14.	$\rho(x,y) = 1 - \left(\frac{\gamma\,(d_{xy}/c)^{\alpha}}{1+(1+\gamma)(d_{xy}/c)^{\alpha}}\right)^{\beta}$	$\alpha, \beta \in (0,1], \gamma \geq 0, c > 0$	

Table 4.1.: Correlation functions of simple mosaic random fields on bounded subsets of $\mathbb{R}^d$. A '*' at the reference indicates that the given correlation function is new, but can be obtained as convex combinations or products of known correlation functions.

	Correlation function	Parameter	Lit.
1.	$\rho(x,y) = \lambda\big(1 - c_d \frac{d_{xy}}{2C_M}\big) + 1 - \lambda$	$\lambda \in (0,2)$	[20, 53]
2.	$\rho(x,y) = \frac{B_{1-d_{xy}^2/a^2}((d+1)/2,1/2)}{B((d+1)/2,1/2)} \mathbb{1}_{d_{xy}\le a}$	$a > 0$	[20, 53]
3.	$\rho_2(x,y) = \big(\frac{2}{\pi}\arccos\big(\frac{d_{xy}}{a}\big)$ $- \frac{2}{\pi a^2} d_{xy}\sqrt{a^2 - d_{xy}^2}\big)\mathbb{1}_{d_{xy}\le a}$	$a > 0$	[20, 53]
4.	$\rho_{2/3}(x,y) = \big(1 - \frac{3d_{xy}}{2a} + \frac{d_{xy}^3}{2a^3}\big)\mathbb{1}_{d_{xy}\le a}$	$a > 0$	[81]
5.	$\rho(x,y) = \Big(\frac{B_{1-d_{xy}^2/a^2}((d+1)/2,1/2)}{B((d+1)/2,1/2)}$ $- \frac{d_{xy}^{d+1}}{a^{d+1}} \frac{B_{1-d_{xy}^2/a^2}((d+1)/2,-d/2)}{B((d+1)/2,1/2)}\Big)\mathbb{1}_{d_{xy}\le a}$	$a > 0$	
6.	$\rho_2(x,y) = \Big(\frac{2}{\pi}\arccos\big(\frac{d_{xy}}{a}\big) - \frac{4}{\pi a^2} d_{xy}\sqrt{a^2 - d_{xy}^2}$ $+ \frac{2}{\pi a^3} d_{xy}^3 \operatorname{artanh}\Big(\big(1 - \frac{d_{xy}^2}{a^2}\big)^{1/2}\Big)\Big)\mathbb{1}_{d_{xy}\le a}$	$a > 0$	
7.	$\rho(x,y) = \frac{1 - c_d\, d_{xy}/(2C_M)}{1 + d_{xy}/c}$	$c > \frac{2C_M}{c_d}$	
8.	$\rho(x,y) = \lambda\big(1 - c_d \frac{d_{xy}}{2C_M}\big)e^{-d_{xy}/c}$ $+ (1-\lambda)\big(1 - c_d \frac{d_{xy}}{2C_M}\big)$	$c > 0,\ \lambda \in (0,1)$	[20, 53]*
9.	$\rho(x,y) = 1 - 2^{1-\alpha} \frac{c_d\, d_{xy}/(2C_M)}{(1 + c_d\, d_{xy}/(2C_M))^{1-\alpha}}$	$\alpha \in (0,1]$	
10.	$\rho(x,y) = \frac{\prod_{k=1}^{n}(2a_k - \lvert x_k - y_k\rvert)_+}{2^d \prod_{k=1}^{d} a_k}$	$a_1, \ldots, a_d > 0$	[20, 53]*

Table 4.2.: Correlation functions of random token, dead leaves, and mixture random fields on bounded subsets of $\mathbb{R}^d$. Correlation functions 3. and 6. are correlation functions on $M \subseteq \mathbb{R}^2$, 4. is valid on $M \subseteq \mathbb{R}^d$ with $d = 2$ or $d = 3$. A '*' at the reference indicates that the given correlation function is new, but can be obtained as convex combinations or products of known correlation functions.

Furthermore, we denote the great circle metric on $\mathbb{S}^d$ by $d_{\mathbb{S}^d}(x,y) = \arccos\big(\langle x,y\rangle\big)$, $x,y \in \mathbb{S}^d$.

Let $B_r(x) = \{z \in \mathbb{S}^d \,|\, d_{\mathbb{S}^d}(x,z) \leq r\}$ denote a closed ball or spherical cap on $\mathbb{S}^d$, centered at $x \in \mathbb{S}^d$ and with radius $r \in [0,\pi]$. Let $(X_n, n \in \mathbb{N})$ be an independent sequence of random variables uniformly distributed on $\mathbb{S}^d$ (e.g., [67, 60]) and let $(R_n, n \in \mathbb{N})$ be an i.i.d. sequence of random variables with values in $[0,\pi]$, independent of $(X_n, n \in \mathbb{N})$. Then $B_n = B_{R_n}(X_n)$ defines an i.i.d. sequence of random closed sets in $\mathbb{S}^d$. As in the previous section, we have

$$P(x,y \in B) = \frac{\Gamma((d+1)/2)}{2\pi^{(d+1)/2}} \mathbb{E}\Big(\sigma_d\big(B_R(x) \cap B_R(y)\big)\Big),$$

i.e. $P(x,y \in B)$ is proportional to the mean surface volume of the intersection of two spherical caps with random but equal radius.

For a deterministic radius $R = r \in [0,\pi]$ and $d = 1$, an elementary geometric consideration yields

$$P\big(x,y \in B_r(X)\big) = \left(\frac{r}{\pi} - \frac{d_{\mathbb{S}^1}(x,y)}{2\pi}\right)_+, \quad x,y \in \mathbb{S}^1.$$

Tovchigrechko and Vakser [85] used spherical trigonometry to obtain a formula for $\sigma_d\big(B_r(x) \cap B_r(y)\big)$ in case $d = 2$, which results in

$$\begin{aligned} P\big(x,y \in B_r(X)\big) = \Bigg(&\frac{1}{2\pi} \arccos\Big(\frac{\cos^2(r) - \cos(d_{\mathbb{S}^2}(x,y))}{\sin^2(r)}\Big) \\ &- \frac{\cos(r)}{\pi} \arccos\Big(\frac{\cos(r)\big(1 - \cos(d_{\mathbb{S}^2}(x,y))\big)}{\sin(r)\sin(d_{\mathbb{S}^2}(x,y))}\Big)\Bigg) \mathbb{1}_{d_{\mathbb{S}^2}(x,y) \leq 2r} \end{aligned} \tag{4.62}$$

for all $x \neq y \in \mathbb{S}^2$ and $r \in (0,\pi/2]$. For higher dimension, Estrade and Istas [28] provide the recursive formula

$$\sigma_d\big(B_r(x) \cap B_r(y)\big) = \int_{-\sin r}^{\sin r} (1-a^2)^{(d-2)/2} \sigma_{d-1}\big(B_{r(a)}(x') \cap B_{r(a)}(y')\big)\, da \tag{4.63}$$

for all $d \geq 2$, $x,y \in \mathbb{S}^d$, and $r \in [0,\pi/2]$, where $r(a) = \arccos\big(\cos(r)/\sqrt{1-a^2}\big)$, and x', y' are arbitrary points in $\mathbb{S}^{d-1}$ satisfying $d_{\mathbb{S}^d}(x,y) = d_{\mathbb{S}^{d-1}}(x',y')$ (there appears to be a misprint in [28] regarding formula (4.63)). This recursion is particularly useful if the balls are hemispheres, i.e. $r = \pi/2$, yielding for all $d \geq 1$

$$P\big(x,y \in B_{\pi/2}(X)\big) = \frac{1}{2} - \frac{d_{\mathbb{S}^d}(x,y)}{2\pi}, \quad x,y \in \mathbb{S}^d. \tag{4.64}$$

From these formulae it is possible to compute $P(x,y \in B)$ for a discretely distributed radius R, although the formulae become quickly lengthy. In what follows we consider a family of continuous distributions for R which results in rather simple formulae for $P(x,y \in B)$. A hyperplane in $\mathbb{R}^{d+1}$ that intersects $\mathbb{S}^d$ divides $\mathbb{S}^d$ into two spherical caps. If $r \in [0,\pi]$ is the radius of one such spherical cap, the distance of the hyperplane to the origin is given by the absolute value of $\cos(r)$. We assume henceforth, that $\cos(R)$ is continuously distributed with a distribution function of the form

$$F_Q(t) = \left(\frac{1}{2} + \sum_{q=0}^{Q} p_q\, t^{2q+1}\right)\mathbb{1}_{[-1,1]}(t) + \mathbb{1}_{(1,\infty)}(t), \quad t \in \mathbb{R}, \tag{4.65}$$

for $Q \in \mathbb{N}_0$ and $p_0, \ldots, p_Q \in \mathbb{R}_+$ with $\sum_{q=0}^{Q} p_q = 1/2$. If $Q = 0$ and $p_0 = 1/2$, this is the distribution function of the uniform distribution on $[-1, 1]$.

Proposition 4.5.1. *Assume that* $\cos(R)$ *is continuously distributed with the distribution function* F_Q *given in* (4.65) *and set* $d_{xy} = d_{\mathbb{S}^d}(x, y)$. *Then for all* $d \geq 1$ *and all* $x, y \in \mathbb{S}^d$

$$P(x, y \in B) = \frac{1}{2} - \sum_{q=0}^{Q}\sum_{l=1}^{q+1} p_q\, C_{q,l,d} \sin^{2l-1}\Big(\frac{d_{xy}}{2}\Big) \cos^{2(q-l+1)}\Big(\frac{d_{xy}}{2}\Big) \tag{4.66}$$

with

$$C_{q,l,d} = 2^{-(2q+1)} \frac{\Gamma(2q+2)\Gamma((d+1)/2)}{\Gamma((2l+1)/2)\Gamma(q-l+2)\Gamma((2q+d+2)/2)} \tag{4.67}$$

holds true.

Proof. The distribution function F_Q in (4.65) fulfills $F_Q(t) + F_Q(-t) = 1$ for all $t \in \mathbb{R}$, which is equivalent to $\cos(R) \overset{d}{=} -\cos(R)$ or $R \overset{d}{=} \pi - R$. With the symmetry of X and the definition of $d_{\mathbb{S}^d}$ this gives for all $x \in \mathbb{S}^d$

$$\begin{aligned} P\big(d_{\mathbb{S}^d}(x, X) \leq R\big) &= P\big(d_{\mathbb{S}^d}(x, -X) \leq R\big) \\ &= P\big(d_{\mathbb{S}^d}(x, X) \geq \pi - R\big) \\ &= P\big(d_{\mathbb{S}^d}(x, X) > R\big) \end{aligned}$$

and consequently $P(x \in B) = 1/2$. Thus

$$P(x, y \in B) = \frac{1}{2} - P(x \in B, y \notin B) = \frac{1}{2} - P\big(d_{\mathbb{S}^d}(x, X) \leq R < d_{\mathbb{S}^d}(y, X)\big). \tag{4.68}$$

The surface measure (1.2) is rotational invariant and we can therefore replace x and y in (4.68) by any points $x_+, x_- \in \mathbb{S}^d$, which satisfy $d_{\mathbb{S}^d}(x_+, x_-) = d_{xy}$. A convenient choice is

$$x_\pm = \phi_d\Big(\pi \mp \frac{\pi}{2}, \frac{\pi}{2}, \ldots, \frac{\pi}{2}, \frac{d_{xy}}{2}\Big) = \Big(0, \pm\sin\Big(\frac{d_{xy}}{2}\Big), 0, \ldots, 0, \cos\Big(\frac{d_{xy}}{2}\Big)\Big). \tag{4.69}$$

By independence of X and R we have

$$\begin{aligned} &P\big(d_{\mathbb{S}^d}(x_+, X) \leq R < d_{\mathbb{S}^d}(x_-, X)\big) \\ &\quad= \int_{\mathbb{S}^d} P\big(d_{\mathbb{S}^d}(x_+, z) \leq R < d_{\mathbb{S}^d}(x_-, z)\big)\mathbb{1}_{d_{\mathbb{S}^d}(x_+,z)\leq d_{\mathbb{S}^d}(x_-,z)}\, d\bar{\sigma}_d(z) \\ &\quad= \int_{\mathbb{S}^d} P\big(\langle x_-, z\rangle < \cos(R) \leq \langle x_+, z\rangle\big)\mathbb{1}_{\langle x_+ - x_-, z\rangle \geq 0}\, d\bar{\sigma}_d(z) \\ &\quad= \sum_{q=0}^{Q} p_q \int_{\mathbb{S}^d} \big(\langle x_+, z\rangle^{2q+1} - \langle x_-, z\rangle^{2q+1}\big)\mathbb{1}_{\langle x_+ - x_-, z\rangle \geq 0}\, d\bar{\sigma}_d(z). \end{aligned}$$

Passing to spherical coordinates (1.1), the difference $\langle x_+, z\rangle^{2q+1} - \langle x_-, z\rangle^{2q+1}$ becomes

$$\sum_{l=0}^{2q+1} \binom{2q+1}{l} \big(1-(-1)^l\big) \sin^l\Big(\frac{d_{xy}}{2}\Big) \sin^l(\varphi) \prod_{i=1}^{d-1} \sin^l(\theta_i) \cos^{2q+1-l}\Big(\frac{d_{xy}}{2}\Big) \cos^{2q+1-l}(\theta_{d-1})$$
$$= 2\sum_{l=1}^{q+1} \binom{2q+1}{2l-1} \sin^{2l-1}\Big(\frac{d_{xy}}{2}\Big) \sin^{2l-1}(\varphi)$$
$$\times \prod_{i=1}^{d-1} \sin^{2l-1}(\theta_i) \cos^{2(q-l+1)}\Big(\frac{d_{xy}}{2}\Big) \cos^{2(q-l+1)}(\theta_{d-1}).$$

Besides, the condition $\langle x_+ - x_-, z\rangle \geq 0$ becomes in spherical coordinates $\varphi \in [0,\pi]$, and this in fact explains the choice of $x_\pm$ in (4.69). Altogether we obtain

$$P\big(d_{\mathbb{S}^d}(x_+, X) \leq R < d_{\mathbb{S}^d}(x_-, X)\big)$$
$$= \sum_{q=0}^{Q} \sum_{l=1}^{q+1} p_q \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}} \binom{2q+1}{2l-1} \sin^{2l-1}\Big(\frac{d_{xy}}{2}\Big) \cos^{2(q-l+1)}\Big(\frac{d_{xy}}{2}\Big)$$
$$\times \int_0^\pi \sin^{2l-1}(\varphi)\, d\varphi \prod_{i=1}^{d-2} \int_0^\pi \sin^{2l-1+i}(\theta_i)\, d\theta_i$$
$$\times \int_0^\pi \sin^{2l+d-2}(\theta_{d-1}) \cos^{2(q-l+1)}(\theta_{d-1})\, d\theta_{d-1}.$$

For the last integral we can use [35, Formulae 3.621.5 and 8.384.1] because the exponent of the cosine is even. The other integrals can be evaluated with [35, Formulae 3.621.1, 8.384.1, and 8.335.1]. We obtain

$$P\big(d_{\mathbb{S}^d}(x_+, X) \leq R < d_{\mathbb{S}^d}(x_-, X)\big)$$
$$= \sum_{q=0}^{Q} \sum_{l=1}^{q+1} p_q \binom{2q+1}{2l-1} \frac{\Gamma((d+1)/2)\Gamma(l)\Gamma((2q-2l+3)/2)}{\pi\Gamma((2q+d+2)/2)}$$
$$\times \sin^{2l-1}\Big(\frac{d_{xy}}{2}\Big) \cos^{2(q-l+1)}\Big(\frac{d_{xy}}{2}\Big).$$

Writing the binomial coefficient in terms of the gamma function and applying [35, Formula 8.335.1] two times we find formula (4.66). □

Remark 4.5.2. Remark 4.3.2 applies also here. The correlation functions that result from the combination of Corollary 4.2.4, Proposition 4.5.1, and equations (4.62), (4.64), are isotropic functions on $\mathbb{S}^d$.

Remark 4.5.3. Suppose Z is a random token field on $\mathbb{S}^2$ which is build from a Poisson distributed number N of random hemispheres $B_{\pi/2}(X)$. Then it follows from Corollary 4.2.4 and (4.64) that the correlation function of Z is given by

$$\rho(x,y) = 1 - \frac{d_{\mathbb{S}^2}(x,y)}{\pi}, \quad x, y \in \mathbb{S}^2. \tag{4.70}$$

This correlation function is an example of a positive definite function on $\mathbb{S}^2$ that is not strictly positive definite, and there are at least two ways to see this. First, it follows from (4.14) that the even Schoenberg-coefficients $b_{2n,2}$, $n \in \mathbb{N}$, of this correlation function satisfy

$$\begin{aligned}\frac{2}{4n+1} b_{2n,2} &= \int_0^\pi P_{2n}\big(\cos(\theta)\big)\sin(\theta)\Big(1-\frac{\theta}{\pi}\Big)\,d\theta \\ &= \int_{-1}^1 P_{2n}(x)\Big(\frac{1}{2}-\frac{\arccos(x)}{\pi}\Big)\,dx \\ &= \int_{-1}^1 P_{2n}(x)\Big(\frac{1}{2}-\frac{\arccos(x)}{\pi}\Big)\,dx + \frac{1}{2}\int_{-1}^1 P_{2n}(x)\,dx.\end{aligned}$$

The first integral in the last line vanishes because P_{2n} is even and the function $x \mapsto 1/2 - \arccos(x)/\pi$ is odd and the second integral equals zero by the orthogonality of the Legendre polynomials. Therefore it follows from Theorem 4.1.6 on page 52 that the correlation function (4.70) is not strictly positive definite.

Another way to see this is to consider the random token field Z corresponding to the correlation function (4.70). Let x and y be arbitrary points in $\mathbb{S}^2$ and let x_A and y_A denote their antipodal points, respectively. Because the random closed sets B are hemispheres, we have $x \in B$ if and only if $x_A \notin B$ and $y \in B$ if and only if $y_A \notin B$. It follows therefore from the definition of the random token field (4.21) that

$$Z(x) + Z(x_A) = Z(y) + Z(y_A) = \sum_{i=1}^N U_{1,i},$$

which implies

$$Z(x) + Z(x_A) - \big(Z(y) + Z(y_A)\big) = 0.$$

We have thus found a linear combination of random variables in $(Z(x), x \in \mathbb{S}^2)$ which has a variance of 0, implying that (4.70) is not strictly positive definite.

4.6. Explicit Correlation Functions on $\mathbb{S}^d$

Example 4.6.1. The Dagum correlation function [33, 8] on $M = \mathbb{S}^d$ is of the form

$$\rho(x,y) = 1 - \Big(\frac{(d_{xy}/c)^\alpha}{1+(d_{xy}/c)^\alpha}\Big)^\beta, \quad x, y \in \mathbb{S}^d. \tag{4.71}$$

(There appears to be a misprinted bracket in [33] concerning (4.71)). It is shown in [33], that $c > 0$, $\alpha \in (0,1]$, and $\beta \in (0,1)$ are sufficient such that (4.71) is a valid correlation function on $\mathbb{S}^d$ for any $d \in \mathbb{N}$. In fact, the case $\beta = 1$ can be included since the Dagum correlation function with $\beta = 1$ equals the generalized Cauchy correlation function with $\beta = \alpha$. The construction of a mosaic random field Z having (4.71) as its correlation function is similar to the construction made for the Cauchy correlation function in Example 4.4.2 as we will show now.

Let ψ_2 be the probability generating function of the Sibuya(α) distributed random variable in Example 4.4.2 and let ψ_1 be the probability generating function of the NegBin(r, p) distributed random variable in Example 4.4.2. This time, we take $r = 1$, hence a geometric distribution on $\mathbb{N}_0$, so that

$$(\psi_1 \circ \psi_2)(t) = \Big(1 + \frac{1-p}{p}(1-t)^\alpha\Big)^{-1}, \quad t \in [-1,1].$$

Putting additionally a Sibuya(β) probability generating function in front, we obtain

$$\psi_N(t) = 1 - \Big(\frac{(1-t)^\alpha(1-p)/p}{1+(1-t)^\alpha(1-p)/p}\Big)^\beta, \quad t \in [-1,1],$$

as the probability generating function of the compound random variable

$$N = \sum_{m=1}^{M}\sum_{l=1}^{L_m} K_{l,m}$$

for independent random variables $M \sim$ Sibuya(β), $L_m \sim \mathrm{Geo}_{\mathbb{N}_0}(p)$, $m \in \mathbb{N}$, and $K_{l,m} \sim$ Sibuya(α), $l, m \in \mathbb{N}$ (see Lemma 4.1.8). If we take random hemispheres $B_{\pi/2}(X)$, $X \sim \mathcal{U}(\mathbb{S}^d)$, as random sets, we have by (4.64) and (4.29)

$$\rho(x,y) = 1 - \Big(\frac{(d_{xy}/\pi)^\alpha(1-p)/p}{1+(d_{xy}/\pi)^\alpha(1-p)/p}\Big)^\beta, \quad x, y \in M,$$

as the correlation function of the corresponding simple mosaic random field. Choosing $p = \big(1+\big(\pi/c\big)^\alpha\big)^{-1}$, $c > 0, \alpha \in (0,1]$, as the success probability for the geometric distribution, we arrive at (4.71).

For the simulations we choose $d = 2$ and standard normally distributed marginals $(U_{i,j}, i, j \in \mathbb{N})$. Figure 4.8a displays the Dagum correlation function on the sphere for a scale of $c = \pi/2$ and different shape parameter α and β. Simulations of the associated simple mosaic random field are given in Figures 4.8b and 4.8c. The bound on the Kolmogorov distance of the marginal distribution of Z and a standard normal distribution is again equal to $C_{BE}\,\mathbb{E}(|U|^3)/\sqrt{n} = \sqrt{2/(n\pi)}$ and we have chosen $n = 300$ superpositions of simple mosaic random field in Figure 4.8d and $n = 1000$ in Figure 4.8e.

Remark 4.6.2. Example 4.6.1 illustrates, that the correlation function of a mosaic random field is widely independent of the particular choice of the index set M of the mosaic random field. This independence can also be used to explain the restriction to $\alpha \in (0,1]$ in Example 4.4.2 concerning the generalized Cauchy correlation function.

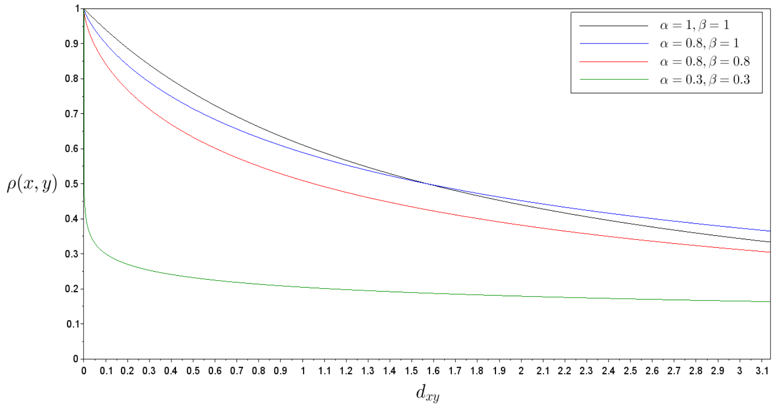
Suppose Z is a simple mosaic random field on $M = [-1,1]^d$ for $d \geq 2$ with random half-spaces as random sets and let us assume for the moment, that we can find a distribution for the number N of random sets such that for the correlation function ρ of Z

$$\rho(x,y) = \Big(1 + \Big(\frac{d_{xy}}{c}\Big)^\alpha\Big)^{-\beta/\alpha}, \quad x, y \in [-1,1]^d, \tag{4.72}$$

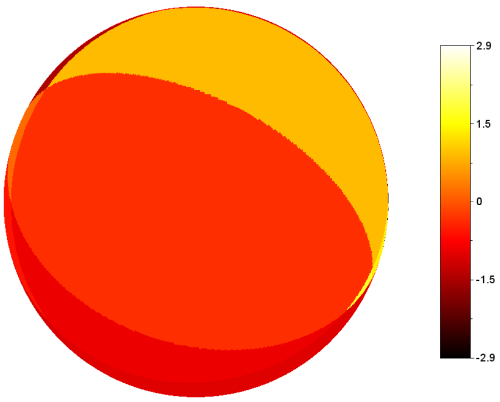
holds true also for $\alpha \in (1,2]$. Let us now choose a simple mosaic random field $\tilde{Z}$ on $\mathbb{S}^d$ with random hemispheres as random sets and a compound number $\tilde{N}$ of random sets of the form

$$\tilde{N} = \sum_{n=1}^{N} A_n,$$

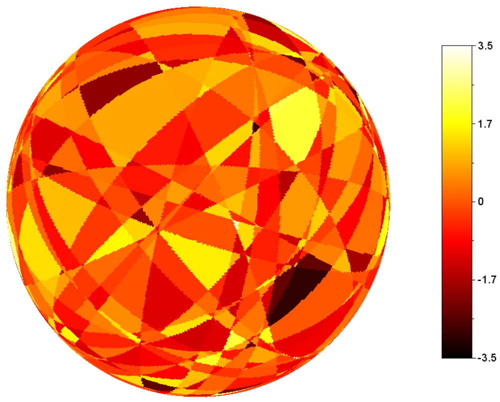
where N is the number of random sets for the simple mosaic random field on $[-1,1]^d$ from above and $(A_n, n \in \mathbb{N})$ is an independent sequence of Bernoulli random variables with success probability $p = (c_d\,\pi)/(2\sqrt{d})$, which we assume to be independent



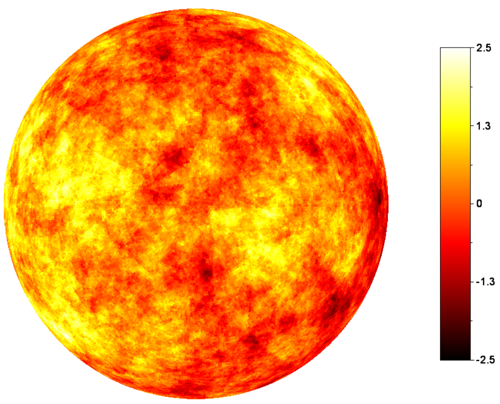
(a) Dagum Correlation function (4.71) for different α, β. The scale parameter is $c = \pi/2$.



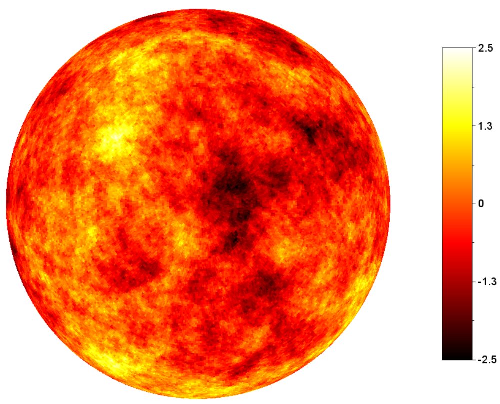
(b) Simple mosaic random field with $\alpha = 0.8$, $\beta = 1$, $c = \pi/2$.



(c) Simple mosaic random field with $\alpha = \beta = 0.8$, $c = \pi/2$.



(d) Approximate Gaussian random field with $\alpha = \beta = 0.8$, $c = \pi/2$, $n = 300$.



(e) Approximate Gaussian random field with $\alpha = \beta = 0.8$, $c = \pi/2$, $n = 1000$.

Figure 4.8.: Simple mosaic random field with the Dagum correlation function (4.71).

of N. For all $d \geq 2$ we have $p \in (0,1)$ since $c_2 = 2/\pi$ and c_d is strictly decreasing by Lemma 4.3.3. The probability generating function ψ_A of a Bernoulli random variable A is

$$\psi_A(t) = 1 - p + p\,t, \quad t \in [-1,1].$$

Using (4.29), Lemma 4.1.8, and (4.64), we see that the correlation function $\tilde{\rho}$ of $\tilde{Z}$ is

$$\tilde{\rho}(x,y) = (\psi_N \circ \psi_A)\Big(1 - \frac{d_{\mathbb{S}^d}(x,y)}{\pi}\Big) = \psi_N\Big(1 - c_d \frac{d_{\mathbb{S}^d}(x,y)}{2\sqrt{d}}\Big), \quad x, y \in \mathbb{S}^d.$$

Our assumption (4.72) implies

$$\psi_N\Big(1 - c_d \frac{\theta}{2\sqrt{d}}\Big) = \Big(1 + \Big(\frac{\theta}{c}\Big)^{\alpha}\Big)^{-\beta/\alpha}$$

for all $\theta \in \big[0, 2\sqrt{d}\big]$. Because $\psi_{\tilde{N}} = \psi_N \circ \psi_A$ is a power series, we have with the identity theorem for power series (e.g. [30, Theorem III.3.2]) that

$$\psi_{\tilde{N}}\Big(1 - \frac{\theta}{\pi}\Big) = \Big(1 + \Big(\frac{\theta}{c}\Big)^{\alpha}\Big)^{-\beta/\alpha}$$

holds true for all $\theta \in [0,\pi]$. In other words, the generalized Cauchy correlation function with $\alpha \in (1,2]$ is the correlation function of the hypothetical random field $\tilde{Z}$ on $\mathbb{S}^d$ that we have constructed. But [33, Example 3] shows that the generalized Cauchy correlation function with $\alpha \in (1,2]$ is not a covariance function on $\mathbb{S}^d$. We conclude that there can not be a simple mosaic random field on $[-1,1]^d$, $d \geq 2$, with half-spaces as random sets, such that the correlation function of this mosaic random field is given by the Cauchy correlation function for $\alpha \in (1,2]$. A similar construction for the powered exponential correlation function (2. in Table 4.1) and [33, Example 1] shows that the same holds true for the powered exponential function and $\alpha \in (1,2]$.

Example 4.6.3. In this example we want to illustrate the application of formula (4.66) in Proposition 4.5.1.

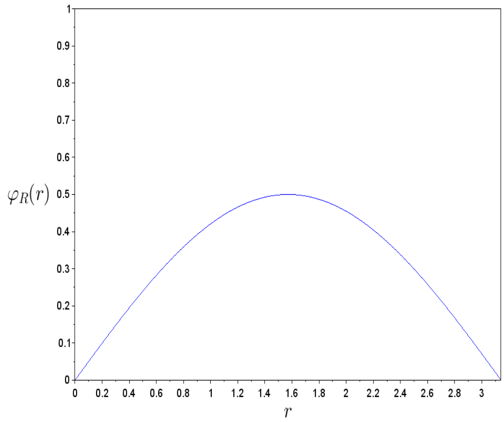
Suppose Z is a random token field on $\mathbb{S}^d$ with random closed balls $B_R(X)$ as random sets and $X \sim \mathcal{U}(\mathbb{S}^d)$. Let $\cos(R)$ be distributed according to the distribution function F_Q in (4.65). In case $Q = 0$ and $p_0 = 1/2$, the function F_Q is the distribution function of a uniform distribution on $[-1,1]$. Accordingly, the density φ_R of R is

$$\varphi_R(r) = \frac{1}{2}\sin(r)\mathbb{1}_{[0,\pi]}(r), \quad r \in \mathbb{R}. \tag{4.73}$$

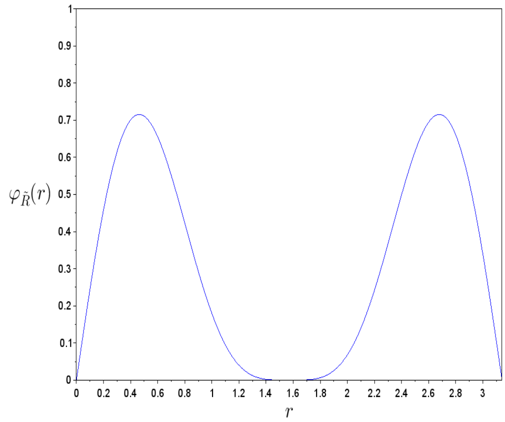
The density $\varphi_{\tilde{R}}$ of a radius $\tilde{R}$ such that $\cos(\tilde{R})$ is distributed according to the distribution function F_Q with $Q = 2$, $p_0 = p_1 = 0$, and $p_2 = \frac{1}{2}$, is

$$\varphi_{\tilde{R}}(r) = \frac{5}{2}\sin(r)\cos^4(r)\mathbb{1}_{[0,\pi]}(r), \quad r \in \mathbb{R}. \tag{4.74}$$

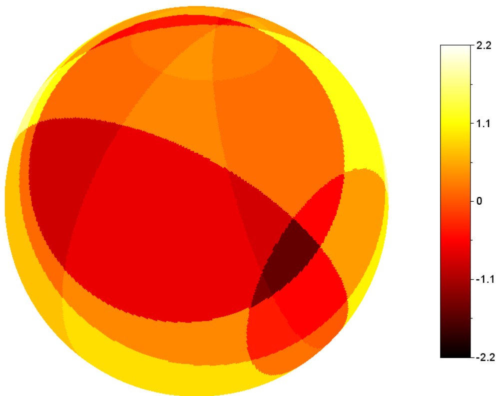
Both densities and simulations of random token fields with random balls $B_R(X)$ and $B_{\tilde{R}}(X)$, respectively, are displayed in Figure 4.9. For the simulations in Figures 4.9c and 4.9d we took the same simulated values for the number of random sets N, the



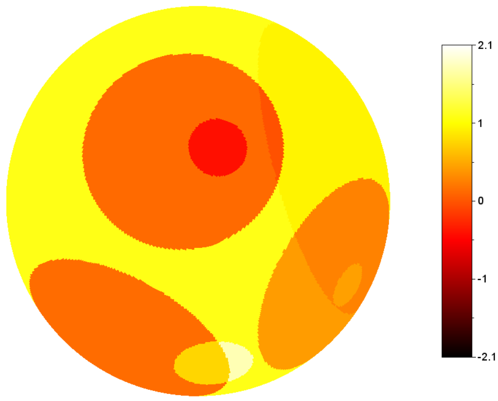
(a) Density φ_R in (4.73).



(b) Density $\varphi_{\tilde{R}}$ in (4.74).



(c) Random token field with random balls $B_R(X)$.



(d) Random token field with random balls $B_{\tilde{R}}(X)$.

Figure 4.9.: Comparison of random token fields with different distributions of the radius R.

centers of the balls X_n, and the marginals $U_{i,j}$. Figure 4.9 illustrates, how the choice of Q and that of the probabilities $p_0, \ldots, p_Q$, determines the size of the random balls which are visible in the simulations. While R is likely to attain values near $\pi/2$, therefore realizing random balls $B_R(X)$ which are nearly as large as hemispheres, the realizations of the random variable $\tilde{R}$ are small or large, resulting in random balls $B_{\tilde{R}}(X)$ which are either small or very large (larger than hemispheres).

By formulae (4.66) and (4.67) in Proposition 4.5.1 we have

$$\begin{aligned} P\big(x, y \in B_R(X)\big) &= \frac{1}{2} - \frac{1}{2} C_{0,1,d} \sin\Big(\frac{d_{xy}}{2}\Big) \\ &= \frac{1}{2}\left(1 - \frac{\Gamma\big((d+1)/2\big)}{\sqrt{\pi}\,\Gamma\big((d+2)/2\big)} \sin\Big(\frac{d_{xy}}{2}\Big)\right) \\ &= \frac{1}{2}\Big(1 - c_{d+1} \sin\Big(\frac{d_{xy}}{2}\Big)\Big), \quad x, y \in \mathbb{S}^d, \end{aligned} \tag{4.75}$$

and

$$\begin{aligned}P\big(x,y\in B_{\tilde R}(X)\big) = \frac{1}{2}\bigg(1 - C_{2,1,d}\sin\Big(\frac{d_{xy}}{2}\Big)\cos^4\Big(\frac{d_{xy}}{2}\Big) - C_{2,2,d}\sin^3\Big(\frac{d_{xy}}{2}\Big)\cos^2\Big(\frac{d_{xy}}{2}\Big)\\ - C_{2,3,d}\sin^5\Big(\frac{d_{xy}}{2}\Big)\bigg), \quad x,y\in\mathbb{S}^d. \end{aligned} \tag{4.76}$$

The appearance of the factor c_d of Lemma 4.3.3 in (4.75) is no coincidence. Because of $d_{\mathbb{S}^d}(x,y) = \arccos\big(\langle x,y\rangle\big)$ we have

$$\begin{aligned}P\big(x,y\in B_R(X)\big) &= P\big(d_{\mathbb{S}^d}(x,X)\le R, d_{\mathbb{S}^d}(y,X)\le R\big)\\ &= P\big(\langle x,X\rangle \ge \cos(R), \langle y,X\rangle\ge\cos(R)\big)\\ &= P\Big(x,y\in H\big(X,\cos(R)\big)\Big)\end{aligned}$$

with random half-spaces $H\big(X,\cos(R)\big)$ in $\mathbb{R}^{d+1}$. Since $\cos(R)\sim\mathcal{U}\big([-1,1]\big)$ we have by (4.46)

$$P\big(x,y\in B_R(X)\big) = \frac{1}{2}\Big(1 - c_{d+1}\frac{\|x-y\|}{2}\Big).$$

Basis trigonometry shows that the *chordal distance* $\|x-y\|$ of two points $x,y\in\mathbb{S}^d$ is related to the great circle distance $d_{\mathbb{S}^d}(x,y)$ by

$$\|x-y\| = 2\sin\Big(\frac{d_{\mathbb{S}^d}(x,y)}{2}\Big), \quad x,y\in\mathbb{S}^d,$$

so that (4.75) also follows from (4.46).

If ρ denotes the correlation function of a random token field with radius R and $\tilde\rho$ the correlation function in case the radius is $\tilde R$, plugging in formulae (4.75) and (4.76) into formula (4.30) for the correlation function of a random token field yields

$$\rho(x,y) = \lambda\Big(1 - c_{d+1}\sin\Big(\frac{d_{xy}}{2}\Big)\Big) + 1 - \lambda, \quad x,y\in\mathbb{S}^d \tag{4.77}$$

and

$$\begin{aligned}\tilde\rho(x,y) = \lambda\bigg(1 - C_{2,1,d}\sin\Big(\frac{d_{xy}}{2}\Big)\cos^4\Big(\frac{d_{xy}}{2}\Big) - C_{2,2,d}\sin^3\Big(\frac{d_{xy}}{2}\Big)\cos^2\Big(\frac{d_{xy}}{2}\Big)\\ - C_{2,3,d}\sin^5\Big(\frac{d_{xy}}{2}\Big)\bigg) + 1 - \lambda, \quad x,y\in\mathbb{S}^d,\end{aligned} \tag{4.78}$$

with

$$\lambda = \frac{a}{a+b/2}, \quad a = \mathbb{E}(U^2)\mathbb{E}(N), \quad b = \mathbb{E}(U)^2\big(\operatorname{Var}(N) - \mathbb{E}(N)\big).$$

We observe that the correlation function of the random token field is a linear combination of the correlation function $\rho_1(x,y) = 2P\big(x,y\in B_R(X)\big)$ ($\tilde{\rho_1}(x,y) = 2P\big(x,y\in B_{\tilde R}(X)\big)$, respectively) and the degenerate correlation function $\rho_2\equiv 1$. To see that ρ_1 and $\tilde{\rho_1}$ are indeed correlation functions on $\mathbb{S}^d$, one can proceed similarly as in (4.49).

The coefficient λ is determined by the moments of N and U, which we have not specified yet. If $\lambda \in (0, 1]$ the validity of the correlation functions (4.77) and (4.78) would also follow from Lemma 4.1.2. Therefore it is somewhat surprising that we can pick λ even from the interval $(0, 2)$, which we show now. If N is $\mathrm{Geo}_{\mathbb{N}}(p)$ distributed and if the marginals $U_{i,j}$, $i, j \in \mathbb{N}$, follow a $\mathcal{N}(\mu, \sigma^2)$ distribution, the factor λ becomes

$$\lambda = \frac{p\,(\mu^2 + \sigma^2)}{p\,\sigma^2 + \mu^2/2}, \qquad \mu \in \mathbb{R}, \sigma > 0, p \in (0, 1).$$

This λ can attain any value in $(0, 2)$, as can be seen from the parametrization

$$\mu = 1, \quad \sigma(\lambda) = \sqrt{\frac{2-\lambda}{\lambda}}, \quad p(\lambda) = \frac{\lambda^2}{2(\lambda-1)^2 + 2}, \quad \lambda \in (0, 2). \tag{4.79}$$

This representation of the parametrization is also useful for picking the corresponding parameter in the simulation. Figure 4.10 visualizes the functions $\lambda \mapsto \sigma(\lambda)$ and $\lambda \mapsto p(\lambda)$.

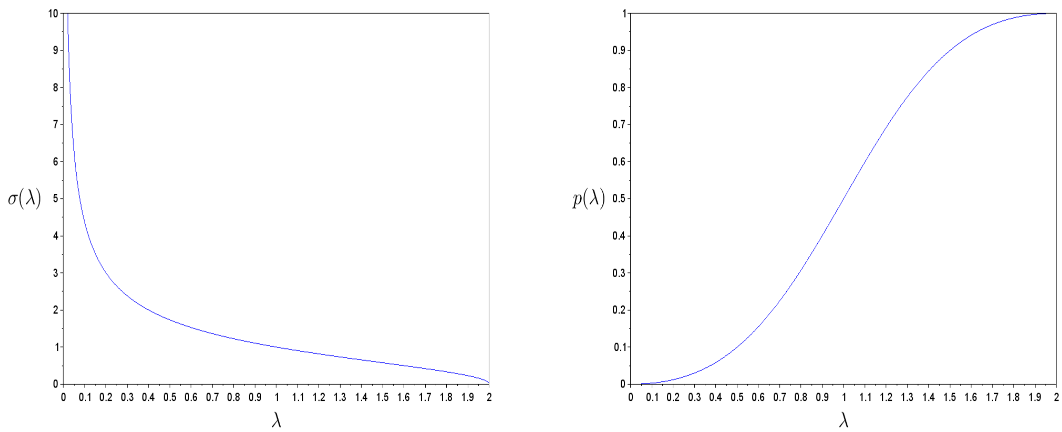


Figure 4.10.: The functions $\lambda \mapsto \sigma(\lambda)$ and $\lambda \mapsto p(\lambda)$.

The correlation function (4.77) on $\mathbb{S}^2$ (recall that $c_3 = 1/2$) is depicted in Figure 4.11a. Realizations of the associated random token field on $\mathbb{S}^2$ are given in Figures 4.11b and 4.11c. The big difference in the number of realized random sets and in the range of the random fields in Figures 4.11b and 4.11c is due to the particular dependence of the success probability p and the variance σ^2 on λ, cf. Figure 4.10.

Concerning the simulation of an approximate Gaussian random field, some computations show that the absolute third moment of a $\mathcal{N}(\mu, \sigma^2)$ distributed random variable Y is given by

$$\mathbb{E}(|Y|^3) = \mu(\mu^2 + 3\sigma^2)\Big(1 + 2\Phi\Big(-\frac{\mu}{\sigma}\Big)\Big) + \sigma(\mu^2 + 2\sigma^2)\frac{2}{\sqrt{2\pi}}e^{-\mu^2/(2\sigma^2)}.$$

Plugging in the corresponding values in (4.55) we obtain the bounds $127.72/\sqrt{n}$ in case $\lambda = 1/2$ and $73.07/\sqrt{n}$ if $\lambda = 3/2$. However our simulations in Figures 4.11d and 4.11e show, that already $n = 10000$ yields acceptable results in the sense that none of the simulated balls on $\mathbb{S}^2$ is visible anymore. This is not surprising as our bound in (4.39) is not sharp.

Figure 4.12 displays the analogous simulations for the random token field with the radius $\tilde{R}$ and the correlation function (4.78).

More correlation functions on $\mathbb{S}^d$ for which there are associated mosaic random fields are given in Tables 4.3 and 4.4. Most of these correlation functions did already appear in Tables 4.1 and 4.2, except that some of the constants are different on the sphere. We refer the reader to Appendix B.1 for details.

4.7. Cylinder and Torus

In this section we give a short excursion to two more exotic spaces, cylinder and torus. Let $O = \mathbb{S}^1 \times [0,h]$ be an open cylinder with radius 1 and height $h > 0$. Let $d_{\mathbb{S}^1}(x_1, y_1) = \arccos\big(\langle x_1, y_1\rangle\big)$, $x_1, y_1 \in \mathbb{S}^1$, be the great circle metric on $\mathbb{S}^1$. Then

$$d_O\big((x_1,x_2),(y_1,y_2)\big) = \sqrt{d^2_{\mathbb{S}^1}(x_1,y_1) + |x_2 - y_2|^2}, \quad x_1, y_1 \in \mathbb{S}^1,\ x_2, y_2 \in [0,h],$$

defines a metric on O. Fix $a \in (0,\pi]$, and let $(D_n, n \in \mathbb{N})$ be a sequence of $[0,a]$-valued random variables, let $(U_n, n \in \mathbb{N})$ be a sequence of uniformly distributed random variables on $[0, 2\pi)$, and let $(V_n, n \in \mathbb{N})$ be a sequence of uniformly distributed random variables on $[-a/2, h + a/2]$. Suppose all random variables above are independent. Let $F(u,v) = \big(\cos u, \sin u, v\big)$, $u \in [0,2\pi)$, $v \in [-a/2, h+a/2]$, and let us define $(X_n, n \in \mathbb{N})$ by $X_n = F(U_n, V_n)$, $n \in \mathbb{N}$. Then

$$B_n = B_{D_n/2}(X_n) = \Big\{ z \in O \,\Big|\, d_O(z, X_n) \le \frac{D_n}{2} \Big\}$$

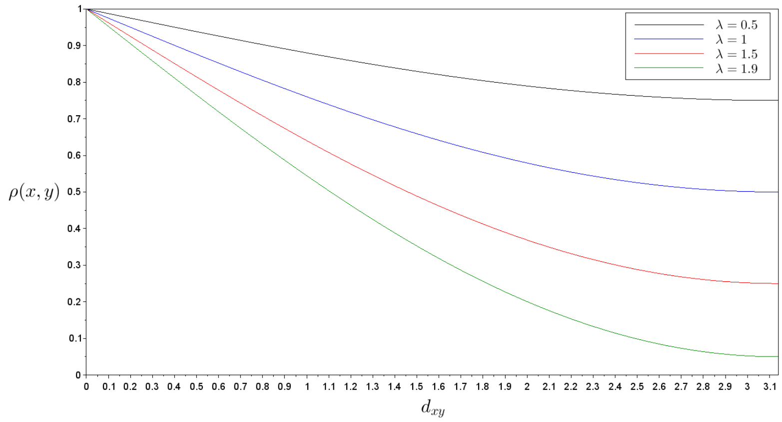
defines an i.i.d. sequence $(B_n, n \in \mathbb{N})$ of random closed balls on O. Let D be equal to some constant $t \in (0,a]$, such that a single ball does not intersect itself. Then

$$P(x, y \in B) = P\big(X \in B_{t/2}(x) \cap B_{t/2}(y)\big) = \frac{\lambda^2\big(F^{-1}\big(B_{t/2}(x) \cap B_{t/2}(y)\big)\big)}{2\pi(h+a)}$$

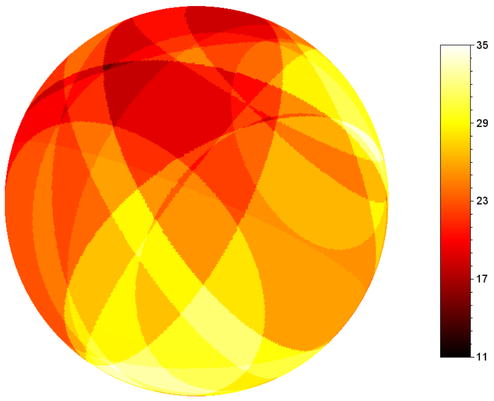
holds for all $x, y \in O$. The set $F^{-1}\big(B_{t/2}(x) \cap B_{t/2}(y)\big)$ is the intersection of two balls $B_{t/2}(\tilde{x})$ and $B_{t/2}(\tilde{y})$ in $\mathbb{R}^2$ with $\tilde{x}, \tilde{y} \in [-t/2, 2\pi + t/2) \times [0,h]$ and $\|\tilde{x} - \tilde{y}\| = d_O(x,y)$, where a part of this intersection which possibly exceeds the left or right boundary of $[0,2\pi) \times [-a/2, h + a/2]$ is reflected to the opposite side. The volume of the intersection of the balls does not change by this reflection and hence we can use (4.42) and (4.43) to get

$$P(x,y \in B) = \frac{1}{4\pi(h+a)}\Big(t^2 \arccos\Big(\frac{d_O(x,y)}{t}\Big) - d_O(x,y)\sqrt{t^2 - d_O^2(x,y)}\Big)\mathbb{1}_{d_O(x,y) \le t}. \tag{4.80}$$

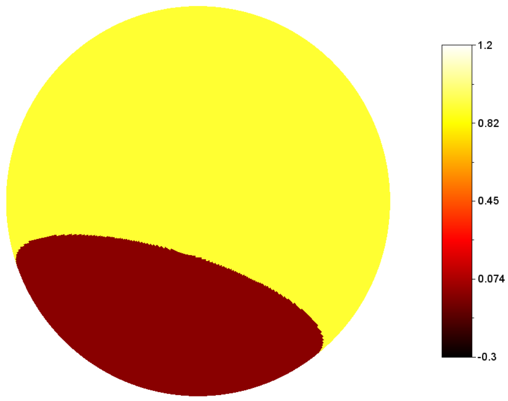
Just as in Section 4.3 this formula can be integrated with respect to the distribution of D in order to obtain $P(x, y \in B)$ for a random diameter of the ball B and the resulting expressions are up to the normalization constant equal to (4.45) and (4.47) with $d_{xy} = d_O(x,y)$. Using for example the distribution function (4.44) of Sironvalle



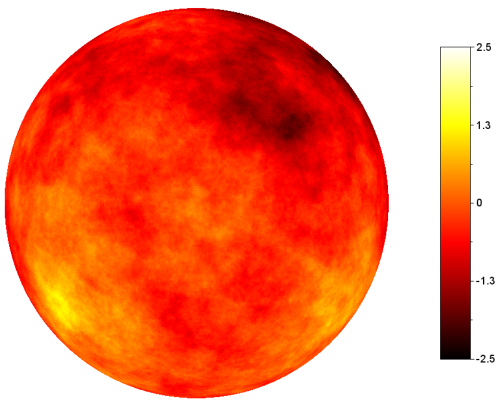
(a) Correlation function (4.77) for different λ.



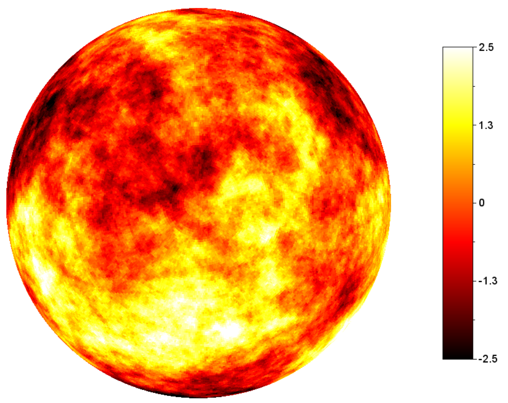
(b) Random token field with $\lambda = 1/2$.



(c) Random token field with $\lambda = 3/2$.

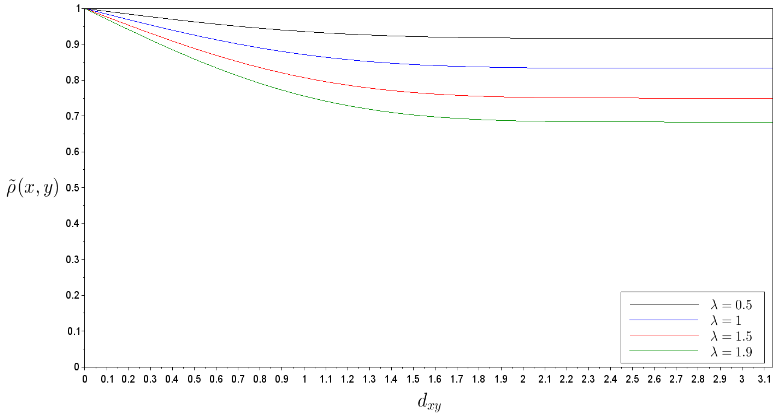


(d) Approximate Gaussian random field with $\lambda = 1/2$, $n = 10000$.

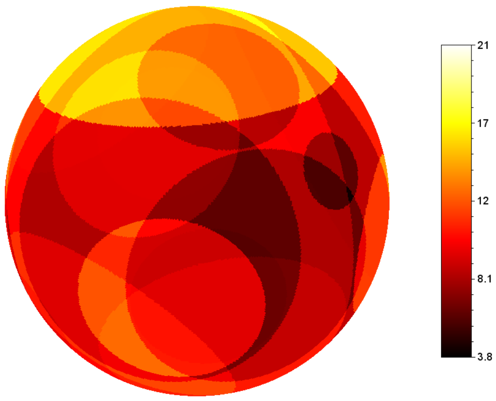


(e) Approximate Gaussian random field with $\lambda = 3/2$, $n = 10000$.

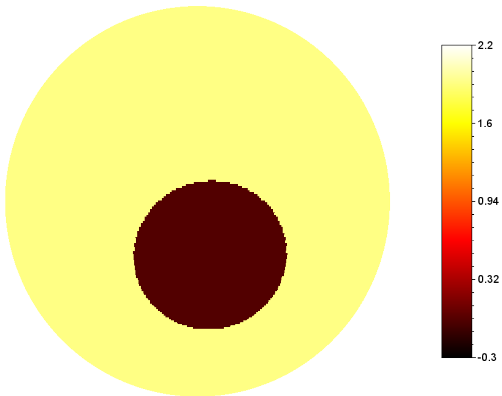
Figure 4.11.: Random token field with the correlation function (4.77) and the radius R.



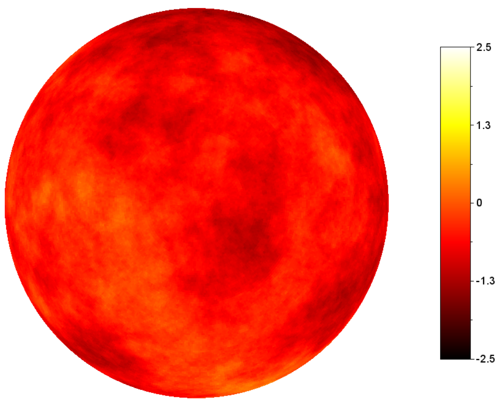
(a) Correlation function (4.78) for different λ.



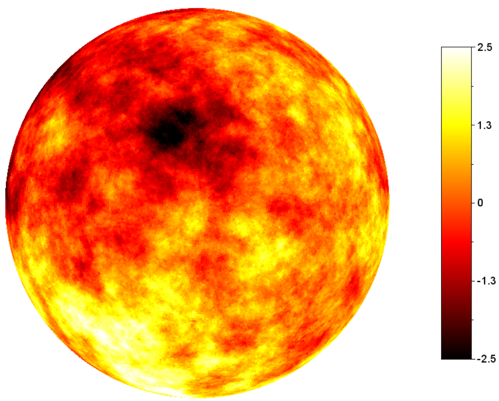
(b) Random token field with $\lambda = 1/2$.



(c) Random token field with $\lambda = 3/2$.



(d) Approximate Gaussian random field with $\lambda = 1/2$, $n = 10000$.



(e) Approximate Gaussian random field with $\lambda = 3/2$, $n = 10000$.

Figure 4.12.: Random token field with the correlation function (4.78) and the radius $\tilde{R}$.

	Correlation function	Parameter	Lit.
1.	$\rho(x,y) = \big(1 + \big(\frac{d_{xy}}{c}\big)^{\alpha}\big)^{-\beta/\alpha}$	$\alpha \in (0,1]$, $\beta, c > 0$	[33]
2.	$\rho(x,y) = e^{-(d_{xy}/c)^{\alpha}}$	$\alpha \in (0,1]$, $c > 0$	[33]
3.	$\rho(x,y) = 1 - \big(\frac{(d_{xy}/c)^{\alpha}}{1+(d_{xy}/c)^{\alpha}}\big)^{\beta}$	$\alpha, \beta \in (0,1]$, $c > 0$	[33]
4.	$\rho(x,y) = 1 - \big(\frac{(1+(d_{xy}/c)^{\alpha})^{\gamma}-1}{(1+(d_{xy}/c)^{\alpha})^{\gamma}}\big)^{\beta}$	$\alpha, \beta \in (0,1]$, $\gamma, c > 0$	
5.	$\rho(x,y) = \big(1 - \big(\frac{d_{xy}}{c}\big)^{\alpha}\big)^{n}$	$\alpha \in (0,1]$, $c \geq \pi, n \in \mathbb{N}_0$	[40]*
6.	$\rho(x,y) = \big(1 - \frac{d_{xy}}{\pi}\big)^{n} e^{-(d_{xy}/c)^{\alpha}}$	$\alpha \in (0,1]$, $c > 0$, $n \in \mathbb{N}_0$	[33]*
7.	$\rho(x,y) = \frac{1-(d_{xy}/\pi)^{\alpha}}{1-d_{xy}/\pi}$	$\alpha \in (0,1]$	
8.	$\rho(x,y) = \frac{1-e^{-(d_{xy}/c)^{\alpha}}}{(d_{xy}/c)^{\alpha}}$	$\alpha \in (0,1]$, $c > 0$	
9.	$\rho(x,y) = \frac{\log(1+(d_{xy}/c)^{\alpha})}{(d_{xy}/c)^{\alpha}}$	$\alpha \in (0,1]$, $c > 0$	
10.	$\rho(x,y) = \big(1 + \beta - \beta e^{-(d_{xy}/c)^{\alpha}}\big)^{-\gamma}$	$\alpha \in (0,1]$, $\beta, \gamma, c > 0$	
11.	$\rho(x,y) = e^{-\beta(d_{xy}/c)^{\alpha}/(1+(d_{xy}/c)^{\alpha})}$	$\alpha \in (0,1], \beta, c > 0$	
12.	$\rho(x,y) = \frac{1+(d_{xy}/c)^{\alpha}}{1+\beta(d_{xy}/c)^{\alpha}}$	$\alpha \in (0,1]$, $\beta \geq 1$, $c > 0$	
13.	$\rho(x,y) = e^{-\beta(1-e^{-(d_{xy}/c)^{\alpha}})}$	$\alpha \in (0,1]$, $\beta, c > 0$	
14.	$\rho(x,y) = 1 - \big(\frac{\gamma(d_{xy}/c)^{\alpha}}{1+(1+\gamma)(d_{xy}/c)^{\alpha}}\big)^{\beta}$	$\alpha, \beta \in (0,1], \gamma \geq 0, c > 0$	
15.	$\rho(x,y) = e^{-(\sin(d_{xy}/2)/c)^{\alpha}}$	$\alpha \in (0,1]$, $c > 0$	[33, 86]
16.	$\rho(x,y) = \big(1 + \big(\frac{1}{c}\sin\big(\frac{d_{xy}}{2}\big)\big)^{\alpha}\big)^{-\beta/\alpha}$	$\alpha \in (0,1]$, $\beta, c > 0$	[33, 86]
17.	$\rho(x,y) = 1 - \big(c_{d+1}\sin\big(\frac{d_{xy}}{2}\big)\big)^{\alpha}$	$\alpha \in (0,1]$	[33, 86]

Table 4.3.: Correlation functions of simple mosaic random fields on $\mathbb{S}^d$. A '*' at the reference indicates that the given correlation function is new, but can be obtained as convex combinations or products of known correlation functions.

	Correlation function	Parameter	Lit.
1.	$\rho(x,y) = \lambda\big(1 - \frac{d_{xy}}{\pi}\big) + 1 - \lambda$	$\lambda \in (0,2)$	[33]
2.	$\rho_2(x,y) = \mathbb{1}_{d_{xy}=0} + \frac{1}{\pi(1-\cos(r))}\Big(\arccos\big(\frac{\cos^2(r)-\cos(d_{xy})}{\sin^2(r)}\big)$		
	$-2\cos(r)\arccos\big(\frac{\cos(r)(1-\cos(d_{xy}))}{\sin(r)\sin(d_{xy})}\big)\Big)\mathbb{1}_{0<d_{xy}\leq 2r}$	$r \in (0, \frac{\pi}{2}]$	[85]
3.	$\rho(x,y) = \frac{1-d_{xy}/\pi}{1+d_{xy}/c}$	$c > \pi$	
4.	$\rho(x,y) = \lambda\big(1 - \frac{d_{xy}}{\pi}\big)e^{-d_{xy}/c} + (1-\lambda)\big(1 - \frac{d_{xy}}{\pi}\big)$	$c > 0,\ \lambda \in (0,1)$	[33]*
5.	$\rho(x,y) = 1 - 2^{1-\alpha}\frac{d_{xy}/\pi}{(1+d_{xy}/\pi)^{1-\alpha}}$	$\alpha \in (0,1]$	
6.	$\rho(x,y) = \lambda\big(1 - c_{d+1}\sin\big(\frac{d_{xy}}{2}\big)\big) + 1 - \lambda$	$\lambda \in (0,2)$	[33]
7.	$\rho(x,y) = 1 - c_{d+1}\frac{3}{d+2}\sin\big(\frac{d_{xy}}{2}\big)\cos^2\big(\frac{d_{xy}}{2}\big)$		
	$-c_{d+1}\frac{2}{d+2}\sin^3\big(\frac{d_{xy}}{2}\big)$		

Table 4.4.: Correlation functions of random token, dead leaves, and mixture random fields on $\mathbb{S}^d$. Correlation function 2. is a correlation functions on $\mathbb{S}^2$. A '*' at the reference indicates that the given correlation function is new, but can be obtained as convex combinations or products of known correlation functions.

for the diameter of the balls, we get for the corresponding random token field (4.21) with Poisson distributed number of balls the correlation function

$$\rho(x,y) = \Bigg(1 - \frac{3\sqrt{d^2_{\mathbb{S}^1}(x_1,y_1) + |x_2-y_2|^2}}{2a} + \frac{(d^2_{\mathbb{S}^1}(x_1,y_1) + |x_2-y_2|^2)^{3/2}}{2a^3}\Bigg)\mathbb{1}_{\sqrt{d^2_{\mathbb{S}^1}(x_1,y_1)+|x_2-y_2|^2}\le a}.$$

The two-dimensional torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ can be treated in a similar way. A convenient choice for the random closed ball is

$$B = B_{D/2}(X) = \Big\{ z \in \mathbb{T}^2 \,\Big|\, d_{\mathbb{T}^2}(z,X) \le \frac{D}{2} \Big\}$$

with $d_{\mathbb{T}^2}(x,y) = \sqrt{d^2_{\mathbb{S}^1}(x_1,y_1) + d^2_{\mathbb{S}^1}(x_2,y_2)}$, $x,y \in \mathbb{T}^2$. Here, we let $X = F(U,V)$ with the parametrization $F(u,v) = \big(\cos u, \sin u, \cos v, \sin v\big)$, $u,v \in [0,2\pi)$, the random variables U and V are uniformly distributed on $[0,2\pi)$, the diameter D is a $[0,a]$-valued random variable for a cutoff $a \in [0,\pi]$, and all random variables are assumed to be independent. For example, if the diameter D is equal to some constant $t \in (0,a]$, then the probability $P(x,y \in B)$ for $x,y \in \mathbb{T}^2$ is given by

$$P(x,y\in B) = \frac{1}{8\pi^2}\Big(t^2 \arccos\Big(\frac{d_{\mathbb{T}^2}(x,y)}{t}\Big) - d_{\mathbb{T}^2}(x,y)\sqrt{t^2 - d_{\mathbb{T}^2}(x,y)^2}\Big)\mathbb{1}_{d_{\mathbb{T}^2}(x,y)\le t}.$$

As an example, a simple mosaic random field Z with $\mathrm{Poi}(\lambda)$ distributed N and this random closed sets has then by virtue of Corollary 4.2.4 the correlation function

$$\rho(x,y) = e^{-\lambda\left(t^2/(8\pi) - 1/(4\pi^2)\left(t^2\arccos(d_{\mathbb{T}^2}(x,y)/t) - d_{\mathbb{T}^2}(x,y)\sqrt{t^2-d_{\mathbb{T}^2}(x,y)^2}\right)\mathbb{1}_{d_{\mathbb{T}^2}(x,y)\le t}\right)} \tag{4.81}$$

for all $x,y \in \mathbb{T}^2$. This correlation function and simulations of the associated simple mosaic random field are depicted in Figure 4.13. For the simulations we have chosen standard normally distributed marginals.

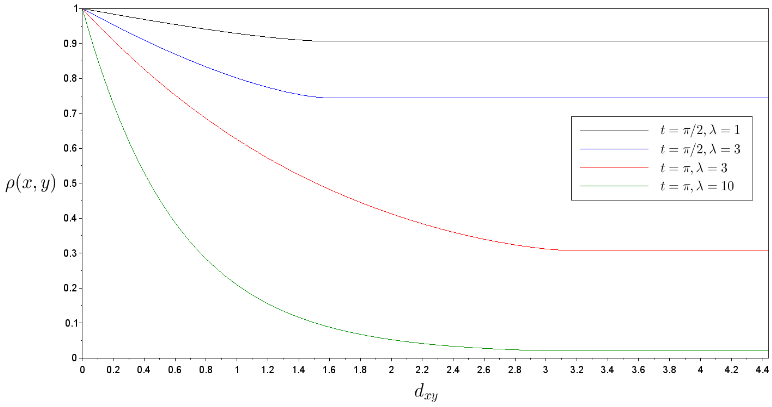
4.8. Simple Mosaic Random Fields Revisited

If ρ is a correlation function on some set M and ψ defined by

$$\psi(t) = \sum_{n\in\mathbb{N}_0} p_n t^n, \quad p_n \ge 0, \sum_{n\in\mathbb{N}_0} p_n = 1, t \in [-1,1],$$

is a probability generating function, then it follows from Lemma 4.1.2 on page 47, that $\psi \circ \rho$ is a correlation function on M. Lemma 4.8.1 gives an alternative proof of this well-known result, which reveals a possibility to simulate a random field having the correlation function $\psi \circ \rho$ given that we can simulate a random field with the correlation function ρ.

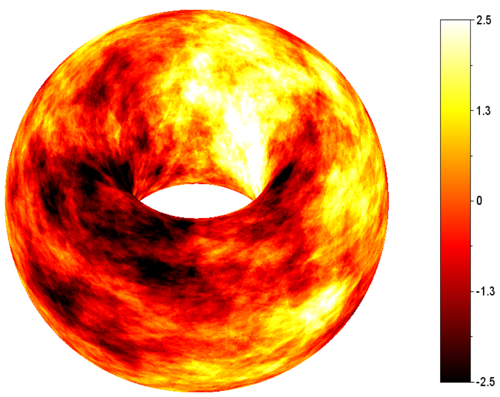
Lemma 4.8.1. *Suppose ρ is a real-valued correlation function on M and ψ is a probability generating function. Then $\psi \circ \rho$ is a correlation function on M.*



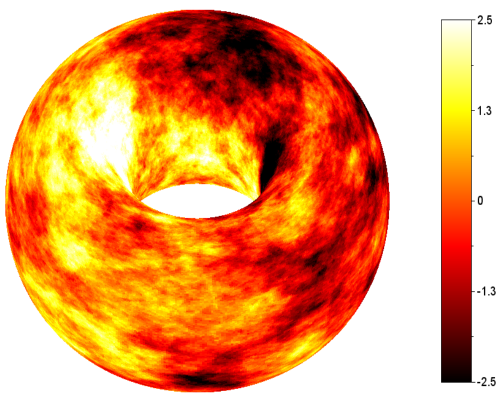
(a) Correlation function (4.81) for different t and λ.

(b) Simple mosaic random field with $t = \pi/2$, $\lambda = 10$.

(c) Simple mosaic random field with $t = \pi$, $\lambda = 10$.



(d) Approximate Gaussian random field with $t = \pi$, $\lambda = 10$, $n = 300$.



(e) Approximate Gaussian random field with $t = \pi$, $\lambda = 10$, $n = 1000$.

Figure 4.13.: Simple mosaic random field on $\mathbb{T}^2$ with the correlation function (4.81).

Proof. Since ρ is a correlation function, its range is contained in $[-1,1]$ which in turn is contained in the region of convergence of any probability generating function. Hence the composition is well-defined. The assumptions imply the existence of a probability space $(\Omega, \mathscr{A}, P)$ on which there are defined an $\mathbb{N}_0$-valued random variable N having the probability generating function ψ, a real-valued random variable U with $\mathbb{E}(U) = 0$ and $\mathbb{E}(U^2) = 1$, and centered random fields $(Z_n, n \in \mathbb{N})$, $Z_n = (Z_n(x), x \in M)$, such that $\operatorname{Cov}(Z_n(x), Z_n(y)) = \rho(x,y)$ for all $n \in \mathbb{N}$. Furthermore, we may assume that N, U, and the random fields Z_n, $n \in \mathbb{N}$, are independent. More precisely, the family formed by N, U, and the sequence of random vectors $\big((Z_n(x_1), \ldots, Z_n(x_m)), n \in \mathbb{N}\big)$ is independent for any choice of $m \in \mathbb{N}$ and $\{x_1, \ldots x_m\} \subseteq M$. Consider the random field

$$Y(x) = U \prod_{n=1}^{N} Z_n(x), \quad x \in M. \tag{4.82}$$

By independence

$$\mathbb{E}\big(Y(x)\big) = \sum_{n \in \mathbb{N}_0} P(N = n) \mathbb{E}(U) \prod_{k=1}^{n} \mathbb{E}\big(Z_k(x)\big) = 0$$

and since furthermore $\mathbb{E}(U^2) = 1$ and $\mathbb{E}\big(Z_n(x) Z_n(y)\big) = \rho(x,y)$ for all $n \in \mathbb{N}$,

$$\begin{aligned}
\operatorname{Cov}\big(Y(x), Y(y)\big) &= \mathbb{E}\big(Y(x)Y(y)\big) \\
&= \sum_{n \in \mathbb{N}_0} P(N = n) \mathbb{E}(U^2) \prod_{k=1}^{n} \mathbb{E}\big(Z_k(x) Z_k(y)\big) \\
&= \sum_{n \in \mathbb{N}_0} P(N = n) \big(\rho(x,y)\big)^n = (\psi \circ \rho)(x,y)
\end{aligned} \tag{4.83}$$

for all $x, y \in M$. Setting $y = x$ in (4.83) we get $\operatorname{Var}\big(Y(x)\big) = 1$ for all $x \in M$ and the claim follows because $\psi \circ \rho$ is the correlation function of the random field (4.82). $\square$

Remark 4.8.2. A construction of the form (4.82) with Poisson distributed N has been used by Matheron in [64] in order to show that $e^{-t\gamma}$ is positive definite for any $t > 0$ if γ is a conditionally negative definite kernel.

Let us now assume that M is as in Section 4.2 and let Z be a simple mosaic random field with $N \equiv 1$ random sets, such that Z may be written as

$$Z(x) = V\, \mathbb{1}_{x \in B} + W\, \mathbb{1}_{x \notin B}, \quad x \in M, \tag{4.84}$$

for a random set B in M and real-valued random variables V and W. We assume that B, V, and W are independent and furthermore, that the random variables V and W are centered and normalized, i.e.

$$\mathbb{E}(V) = \mathbb{E}(W) = 0 \quad \text{and} \quad \mathbb{E}(V^2) = \mathbb{E}(W^2) = 1.$$

Then $\mathbb{E}\big(Z(x)\big) = 0$ and $\operatorname{Var}\big(Z(x)\big) = 1$ and we get either directly or with (4.29)

$$\rho_Z(x,y) = P(x, y \in B) + P(x, y \notin B) = 1 + 2p_{xy} - p_x - p_y, \quad x, y \in M,$$

for the correlation function ρ_Z of Z. Applying Lemma 4.8.1 to this correlation function and a probability generating function ψ_N of some $\mathbb{N}_0$-valued random variable N, we get for the correlation function of the random field Y in (4.82)

$$\rho(x,y) = \psi_N(1 + 2p_{xy} - p_x - p_y), \quad x, y \in M,$$

which is exactly the correlation function (4.29) of the simple mosaic random field in section 4.2. To clarify the connection between the simple mosaic random field and Y, we use the following formula.

Lemma 4.8.3. *Let $(a_n, \in \mathbb{N}), (b_n, n \in \mathbb{N})$ be sequences in $\mathbb{R}$, then for any $n \in \mathbb{N}_0$*

$$\prod_{i=1}^{n}(a_i + b_i) = \sum_{I \in \mathcal{P}_n} \Big(\prod_{i \in I} a_i \prod_{j \in \{1,\dots,n\} \setminus I} b_j \Big). \tag{4.85}$$

Proof. Formula (4.85) is true for $n = 0$, since both sides equal 1 by definition of the empty product and because $\mathcal{P}_0 = \{\emptyset\}$. Suppose (4.85) holds true for some $n \in \mathbb{N}$, then

$$\begin{aligned}
\prod_{i=1}^{n+1}(a_i + b_i) &= \Big(\sum_{I \in \mathcal{P}_n} \Big(\prod_{i \in I} a_i \prod_{j \in \{1,\dots,n\} \setminus I} b_j \Big) \Big)(a_{n+1} + b_{n+1}) \\
&= \sum_{I \in \mathcal{P}_n} \Big(\prod_{i \in I \cup \{n+1\}} a_i \prod_{j \in \{1,\dots,n\} \setminus I} b_j \Big) + \sum_{I \in \mathcal{P}_n} \Big(\prod_{i \in I} a_i \prod_{j \in \{1,\dots,n+1\} \setminus I} b_j \Big) \\
&= \sum_{J \in \{I \cup \{n+1\} \mid I \in \mathcal{P}_n\}} \Big(\prod_{i \in J} a_i \prod_{j \in \{1,\dots,n+1\} \setminus J} b_j \Big) + \sum_{I \in \mathcal{P}_n} \Big(\prod_{i \in I} a_i \prod_{j \in \{1,\dots,n+1\} \setminus I} b_j \Big).
\end{aligned}$$

Because the summands of the two sums in the last line are equal and because $\mathcal{P}_{n+1}$ may be written as

$$\mathcal{P}_{n+1} = \{ I \cup \{n+1\} \mid I \in \mathcal{P}_n \} \uplus \mathcal{P}_n,$$

the assertion in the lemma follows by induction. □

Since the random field Z in (4.84) is centered and normalized, the associated random field Y in (4.84) may be taken to be

$$Y(x) = U \prod_{i=1}^{N} \big(V_i \, \mathbb{1}_{x \in B_i} + W_i \, \mathbb{1}_{x \notin B_i} \big), \quad x \in M, \tag{4.86}$$

with identically distributed sequences $(V_i, i \in \mathbb{N})$ and $(W_i, i \in \mathbb{N})$ of centered and normalized random variables, and an identically distributed sequence $(B_i, i \in \mathbb{N})$ of random closed sets, such that the family formed by N, $(V_i, i \in \mathbb{N})$, $(W_i, i \in \mathbb{N})$, and $(B_i, i \in \mathbb{N})$ is independent. The simple mosaic random field Z_M in (4.20) is of the form

$$Z_M(x) = \sum_{I \in \mathcal{P}_N} U_I \, \mathbb{1}_{x \in C_I}, \quad x \in M,$$

with

$$C_I = \Big(\bigcap_{i \in I} B_i\Big) \cap \Big(\bigcap_{j \in \{1,\dots,n\} \setminus I} B_j^c\Big), \quad I \in \mathcal{P}_n, n \in \mathbb{N}_0,$$

and identically distributed sequences $(U_I, I \in \mathcal{P}_n)$, $n \in \mathbb{N}$, and $(B_i, i \in \mathbb{N})$, such that the family formed by N, $(U_I, I \in \mathcal{P}^*(\mathbb{N}))$, and $(B_i, i \in \mathbb{N})$ is independent.

Because of

$$\prod_{i \in I} \mathbb{1}_{x \in B_i} \prod_{j \in \{1,\dots,n\} \setminus I} \mathbb{1}_{x \notin B_j} = \mathbb{1}_{x \in (\bigcap_{i \in I} B_i) \cap (\bigcap_{j \in \{1,\dots,n\} \setminus I} B_j^c)} = \mathbb{1}_{x \in C_I}$$

for all $x \in M$, $I \in \mathcal{P}_n$, and $n \in \mathbb{N}_0$, an application of (4.85) yields for every $x \in M$ the pointwise equality

$$Y(x) = \sum_{I \in \mathcal{P}_N} \tilde{U}_I \mathbb{1}_{x \in C_I} \quad \text{with} \quad \tilde{U}_I = U \prod_{i \in I} V_i \prod_{j \in \{1,\dots,n\} \setminus I} W_j, \quad I \in \mathcal{P}_n, n \in \mathbb{N}_0.$$

Hence Y in (4.86) and Z_M are of the same form, except that the random variables $(\tilde{U}_I, I \in \mathcal{P}_n)$, $n \in \mathbb{N}_0$, are not independent but uncorrelated: If $n \in \mathbb{N}_0$ and $I, J \in \mathcal{P}_n$, then the assumptions on the random variables U, V_i, and W_j imply

$$\begin{aligned}
\mathbb{E}\big(\tilde{U}_I \tilde{U}_J\big) &= \mathbb{E}(U^2)\, \mathbb{E}(V^2)^{|I \cap J|}\, \mathbb{E}(V)^{|I \triangle J|}\, \mathbb{E}(W^2)^{|\{1,\dots,n\} \setminus (I \cup J)|}\, \mathbb{E}(W)^{|\{1,\dots,n\} \cap (I \triangle J)|} \\
&= \begin{cases} 1, & I = J, \\ 0, & I \neq J. \end{cases}
\end{aligned}$$

It is of course irrelevant for the correlation function of a mosaic random field whether the marginals are independent as for Z_M or uncorrelated as for Y. In fact, we could have also assumed uncorrelated marginals in Theorem 4.2.2 and the proof would have followed the same lines. We may therefore think of Y in (4.86) as a simplified representation of the simple mosaic random field.

As an application of this new representation, we now impose a modest correlation structure on the marginals by assuming

$$\mathbb{E}(VW) = 1 - \kappa, \quad \kappa \in [0, 2],$$

instead of independence for the random variables V and W in (4.84). Lemma 4.8.1 is still applicable, so that we obtain

$$\rho(x, y) = (\psi_N \circ \rho_\kappa)(x, y), \quad x, y \in M, \tag{4.87}$$

as the correlation function of Y in (4.86) and ρ_κ is the correlation function of the random field Z in (4.84) under correlated marginals V and W. A direct calculation in the spirit of Theorem 4.2.2 shows

$$\begin{aligned}
\rho_\kappa(x, y) &= \mathbb{E}(V^2) P(x, y \in B) + \mathbb{E}(VW)\big(P(x \in B, y \notin B) + P(x \notin B, y \in B)\big) \\
&\qquad + \mathbb{E}(W^2) P(x, y \notin B) \\
&= 1 - \kappa(p_x + p_y - 2p_{xy}), \quad x, y \in M. \qquad (4.88)
\end{aligned}$$

If M is a bounded subset of $\mathbb{R}^d$ and if we take random half-spaces $H(X,R)$ as random sets for our mosaic random field, the combination of (4.87) and (4.88) result in the correlation function

$$\rho(x,y) = \psi_N\Big(1 - \kappa\, c_d\, \frac{d_{xy}}{2C_M}\Big), \quad x,y \in M, \kappa \in [0,2].$$

For $\kappa = 1$ we recover our former correlation function. Through the new parameter κ, we may now for example weaken the constraint $c \geq 2C_M/c_d$ for the scale parameter c of the correlation function 5. in Table 4.1. Taking $\kappa = 2$ and a compound distribution for N, consisting of a $\mathrm{Bin}\big(n, (C_M/c_d\, c)^\alpha\big)$ distribution and a $\mathrm{Sibuya}(\alpha)$ distribution, we obtain the correlation function

$$\rho(x,y) = \Big(1 - \Big(\frac{d_{xy}}{c}\Big)^\alpha\Big)^n, \quad \alpha \in (0,1], c \geq \frac{C_M}{c_d}, n \in \mathbb{N}_0. \tag{5'}$$

The same can be done for the analogous correlation function on the sphere, as is shown in the next section.

Note that it is not possible to get rid of the dimension-depending constant c_d in the bound on the scale factor c in (5') with the help of probability generating functions. An application of the identity theorem for power series as in Remark 4.6.2 shows that there can not exist a probability generating function ψ_N such that

$$\psi_N\Big(1 - \kappa\, c_d\, \frac{d_{xy}}{2C_M}\Big) = \Big(1 - \Big(\frac{d_{xy}}{c}\Big)^\alpha\Big)^n$$

holds true for all $x,y \in M$ if $c < C_M/c_d$.

4.9. Power Covariance Function on $\mathbb{S}^d$

The power covariance function on $\mathbb{S}^2$

$$C(x,y) = c_0 - \Big(\frac{d_{\mathbb{S}^2}(x,y)}{c}\Big)^\alpha, \quad x,y \in \mathbb{S}^2, \tag{4.89}$$

was introduced in [40]. It was shown there, that for the associated coefficients $a_{n,2}$ in (4.11) the inequalities

$$a_{n,2} \geq 0 \quad \text{and} \sum_{n \in \mathbb{N}_0} a_{n,2} C_n^{1/2}(1) < \infty \tag{4.90}$$

hold true, which by Theorem 4.1.5 is equivalent to (4.89) being a continuous and isotropic covariance function on $\mathbb{S}^2$. The parameter constraints that were given in [40] are $\alpha \in (0,1]$, $c > 0$, and

$$c_0 \geq \int_0^\pi \Big(\frac{\theta}{c}\Big)^\alpha \sin(\theta)\, d\theta. \tag{4.91}$$

To the best of the authors knowledge, the validity of (4.89) on spheres of arbitrary dimension has not been shown so far. A generalization of the proof in [40] is not straightforward as the proof relies on sine expansions of Legendre polynomials, which are the Gegenbauer polynomials $C_n^{(d-1)/2}$ in case $d = 2$ (see section 4.1.2).

In what follows we given a proof of the validity of (4.89) using the simplified representation of a mosaic random field with correlated marginals in Section 4.8. Our proof shows the validity of (4.89) on spheres $\mathbb{S}^d$ of arbitrary dimension $d \in \mathbb{N}$. In doing so, we obtain a lower bound for c_0, which is smaller than the lower bound given in [40] in case $d = 2$ (cf. Remark 4.9.2 below). In case $\alpha = 1$ our bound is also necessary for the validity of (4.89).

Proposition 4.9.1. *The power covariance function* (4.89) *is a covariance function on* $\mathbb{S}^d$ *if* $\alpha \in (0, 1]$, $c > 0$, *and*

$$c_0 \geq \left(\frac{\pi}{2c}\right)^{\alpha}. \tag{4.92}$$

In case $\alpha = 1$, *the condition* (4.92) *is also necessary in order for* (4.89) *being a covariance function on* $\mathbb{S}^d$ *for all* $c > 0$.

Proof. Consider a mosaic random field $\bar{Z}$ on $\mathbb{S}^d$ of the form

$$\bar{Z}(x) = U\,\mathbb{1}_{x \in B} + V\,\mathbb{1}_{x \notin B}, \quad x \in \mathbb{S}^d,$$

with centered and normalized random variables U and V such that $\mathbb{E}(UV) = -1$. Let

$$B = B_{\pi/2}(X) = \left\{ z \in \mathbb{S}^d \,\middle|\, d_{\mathbb{S}^d}(z, X) \leq \frac{\pi}{2} \right\}, \quad X \sim \mathcal{U}(\mathbb{S}^d),$$

be a random hemisphere on $\mathbb{S}^d$. We may assume that X is independent of U and V. From (4.88) and (4.64) we get for the correlation function $\bar{\rho}$ of $\bar{Z}$

$$\bar{\rho}(x, y) = 1 - 2(p_x + p_y - 2p_{xy}) = 1 - \frac{d_{\mathbb{S}^d}(x, y)}{\pi/2}, \quad x, y \in \mathbb{S}^d. \tag{4.93}$$

From Lemma 4.8.1 it follows that

$$\rho(x, y) = (\psi \circ \bar{\rho})(x, y), \quad x, y \in \mathbb{S}^d, \tag{4.94}$$

is a correlation function on $\mathbb{S}^d$ for any probability generating function ψ. Let $\psi = \psi_A \circ \psi_S$, where ψ_A is the probability generating function of a Bernoulli distribution with success probability p and ψ_S is the probability generating function of a Sibuya(α) distribution, so that

$$\psi(t) = 1 - p\,(1 - t)^{\alpha}, \quad \alpha \in (0, 1], p \in [0, 1], t \in [-1, 1]. \tag{4.95}$$

Combining (4.94) and (4.95) we have that

$$\rho(x, y) = 1 - p\,\frac{d_{\mathbb{S}^d}(x, y)}{\pi/2}, \quad x, y \in \mathbb{S}^d,$$

is a correlation function on $\mathbb{S}^d$. Let us now take

$$p = \frac{1}{c_0}\left(\frac{\pi}{2c}\right)^{\alpha}, \tag{4.96}$$

then our assumption on c_0 implies that $p \in [0,1]$ for all $\alpha \in (0,1]$ and all $c > 0$. With this p, the correlation function ρ becomes

$$\rho(x,y) = 1 - \frac{1}{c_0}\Big(\frac{d_{\mathbb{S}^d}(x,y)}{c}\Big)^{\alpha}, \quad x,y \in \mathbb{S}^d, \tag{4.97}$$

from which it follows that

$$C(x,y) = c_0 - \Big(\frac{d_{\mathbb{S}^d}(x,y)}{c}\Big)^{\alpha}, \quad x,y \in \mathbb{S}^d,$$

is a covariance function on $\mathbb{S}^d$.

To prove necessity in case $\alpha = 1$, we consider the first Schoenberg coefficient $b_{0,d}$ of (4.97). If $d = 1$, we have by (4.15)

$$b_{0,1} = \frac{1}{\pi}\int_0^{\pi}\Big(1 - \frac{1}{c_0}\frac{\theta}{c}\Big)\,d\theta = 1 - \frac{1}{c_0}\frac{\pi}{2c},$$

which is negative if $c_0 < \pi/(2c)$. If $d \geq 2$, equation (4.14) yields

$$b_{0,d} = \frac{d-1}{2^{3-d}\pi}\frac{\Gamma((d-1)/2)^2}{\Gamma(d-1)}\int_0^{\pi}\Big(1 - \frac{1}{c_0}\frac{\theta}{c}\Big)C_0^{(d-1)/2}\big(\cos(\theta)\big)\sin^{d-1}(\theta)\,d\theta.$$

Since $C_0^{\lambda} \equiv 1$ for all $\lambda \geq 0$, the assumption $c_0 < \pi/(2c)$ and integration by substitution lead to

$$b_{0,d} < -\frac{d-1}{2^{2-d}\pi^2}\frac{\Gamma((d-1)/2)^2}{\Gamma(d-1)}\int_{-\pi/2}^{\pi/2}\theta\cos^{d-1}(\theta)\,d\theta = 0.$$

Therefore Theorem 4.1.5 implies that $c_0 \geq \pi/(2c)$ is also necessary in case $\alpha = 1$ and $c > 0$. □

Simulations of the mosaic random field that is constructed in the proof of Proposition 4.9.1 are given in Example 5.3.6 in Chapter 5.

Remark 4.9.2. From

$$\begin{aligned}\int_0^{\pi}\Big(\frac{\theta}{c}\Big)^{\alpha}\sin(\theta)\,d\theta &= \int_0^{\pi/2}\Big(\frac{\theta}{c}\Big)^{\alpha}\sin(\theta)\,d\theta + \int_{\pi/2}^{\pi}\Big(\frac{\theta}{c}\Big)^{\alpha}\sin(\theta)\,d\theta\\ &\geq \Big(\frac{\pi}{2c}\Big)^{\alpha} + \int_{\pi/2}^{\pi}\Big(\frac{\theta}{c}\Big)^{\alpha}\sin(\theta)\,d\theta\\ &> \Big(\frac{\pi}{2c}\Big)^{\alpha}\end{aligned}$$

for all $\alpha \in (0,1]$ and all $c > 0$, it follows that the bound on c_0 given in Proposition 4.9.1 for $d = 2$ is lower than the bound given in [40]. However, a closer look at the proof given in [40] suggests that it might be possible to work with the constraint

$$c_0 \geq \frac{1}{2}\int_0^{\pi}\Big(\frac{\theta}{c}\Big)^{\alpha}\sin(\theta)\,d\theta \tag{4.98}$$

there. Figure 4.14 compares this new bound with the bound given in Proposition 4.9.1 and indicates, that the bound in (4.98) is slightly smaller (the integration for the plot of $\alpha \mapsto 2^{-1}\int_0^{\pi}(\theta/c)^{\alpha}\sin(\theta)\,d\theta$ was done numerically). Note that the

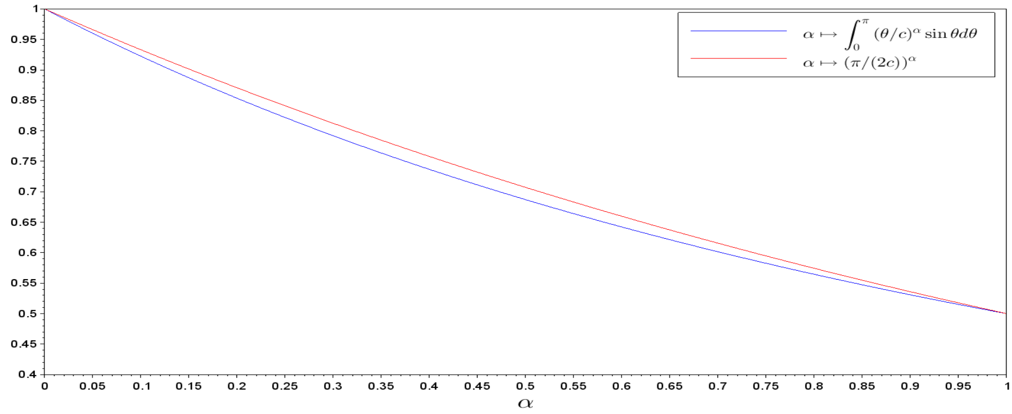


Figure 4.14.: Comparison of the bounds in Remark 4.9.2 for $c = \pi$.

condition (4.98) is not compatible with the proof of Proposition 4.9.1, because for any c_0 smaller than the bound given in Proposition 4.9.1 the probability p of the Bernoulli distribution in (4.96) is not well-defined. However, the validity of (4.98) is not guaranteed, as the details of a modified version of the proof given in [40] still need to be worked out. Also, as mentioned before, the proof in [40] is valid only in $d = 2$.

5. Spectral Turning Bands on the Sphere

5.1. The Spectral Turning Bands Method

The spectral turning bands method is a simulation method for random fields on $\mathbb{R}^d$ which is based on the Schoenberg's spectral representation (Theorem 4.1.4 on page 49) of covariance functions. In this section we motivate the method in a way that allows us to apply it to the sphere in the next section.

We begin with Karhunen's theorem on the spectral representation of second-order random fields (the classical reference is [48]), in its general form given in [10].

Let $(\Omega, \mathscr{A}, P)$ be a probability space, $(W, \mathscr{W})$ a measurable space, and σ a σ-finite measure on $(W, \mathscr{W})$. The set consisting of all elements $B \in \mathscr{W}$ such that $\sigma(B) < \infty$ is denoted by $\mathscr{W}_0$. A mapping $\xi : \mathscr{W}_0 \to L^2(\Omega, \mathscr{A}, P; \mathbb{C})$ is called *random orthogonal measure with structure function* σ if

1. $\xi(A \cup B) = \xi(A) + \xi(B)$ for all disjoint $A, B \in \mathscr{W}_0$,
2. $\mathbb{E}\big(\xi(A)\overline{\xi(B)}\big) = \sigma(A \cap B)$ for all $A, B \in \mathscr{W}_0$.

Theorem 5.1.1 (Karhunen, 1946; Berschneider, Sasvári, 2012)**.** *Let M be a set and suppose that $Z : M \to L^2(\Omega, \mathscr{A}, P; \mathbb{C})$ is a second-order random field on M. Let furthermore $C(x, y) = \mathbb{E}\big(Z(x)\overline{Z(y)}\big)$, $x, y \in M$, and assume that C can be represented as*

$$C(x, y) = \int_W f(x, z)\overline{f(y, z)}\, d\sigma(z), \quad x, y \in M, \tag{5.1}$$

where

1. *σ is a σ-finite meaure on the measurable space $(W, \mathscr{W})$,*
2. *$f : M \times W \to \mathbb{C}$ is such that $f(x, \cdot) \in L^2(W, \mathscr{W}, \sigma; \mathbb{C})$ for all $x \in M$,*
3. *$\operatorname{span}\{f(x, \cdot) \,|\, x \in M\}$ is dense in $L^2(W, \mathscr{W}, \sigma; \mathbb{C})$.*

Then there exists a uniquely determined random orthogonal measure ξ with structure function σ such that

$$Z(x) = \int_W f(x, z)\, d\xi(z), \quad x \in M. \tag{5.2}$$

Additionally, setting $H(Z) = \overline{\operatorname{span}}\{Z(x) \,|\, x \in M\}$ and $H(\xi) = \overline{\operatorname{span}}\{\xi(A) \,|\, A \in \mathscr{W}_0\}$, it is true that $H(Z) = H(\xi)$.

Let a function C with a representation of the form (5.1) be given. Note that (5.1) implies the positive definiteness of C (cf. Remark 5.1.3 below). Then theoretically, Theorem 5.1.1 yields a random field Z such that $\mathbb{E}\big(Z(x)\overline{Z(y)}\big) = C(x,y)$, if the sufficient conditions can be shown to be satisfied. However, if we wish to simulate Z, there are two drawbacks concerning this approach. First, Condition 3 is quite technical and hard to verify in general (see [10] for a discussion of this condition). Second, the random orthogonal measure ξ is obtained as a limit and might not be given explicitly, implying that the representation (5.2) is not explicit.

On the other hand the statement of Theorem 5.2 is quite strong, as it gives a representation of the random field Z in $L^2(\Omega, \mathscr{A}, P; \mathbb{C})$. If our goal is only to find a random field such that $\mathbb{E}\big(Z(x)\overline{Z(y)}\big) = C(x,y)$ holds true, we can hope for a statement with less restrictive assumptions.

If the measure σ of Theorem 5.1.1 is finite and if $S : \Omega \to W$ is a measurable mapping such that $P_S = \sigma/\sigma(W)$, then a very simple orthogonal measure with structure function σ is given by $\xi(B) = \varepsilon_S(B)$, $B \in \mathscr{W}_0$. Equation (5.2) with this orthogonal measure becomes

$$Z(x) = \int_W f(x,z)\, d\xi(z) = f(x,S), \quad x \in M.$$

Passing over to a real-valued, centered, and normalized version of the random field Z, we arrive at the following weaker variant of Theorem 5.1.1.

Lemma 5.1.2. *Let M be a set, let $(\Omega, \mathscr{A}, P)$ be a probability space, suppose that $(W, \mathscr{W}, \sigma)$ is a finite measure space, and assume that $f : M \times W \to \mathbb{R}$ is such that $f(x,\cdot) \in L^2(W, \mathscr{W}, \sigma; \mathbb{R})$ for all $x \in M$. Suppose that a function $C : M \times M \to \mathbb{R}$ can be represented as*

$$C(x,y) = \int_W f(x,z) f(y,z)\, d\sigma(z), \quad x,y \in M. \tag{5.3}$$

If $S : \Omega \to W$ and $U : \Omega \to \mathbb{R}$ are independent measurable mappings such that $P_S = \sigma/\sigma(W)$, $\mathbb{E}(U) = 0$, and $\mathbb{E}(U^2) = 1$, then the random field Z defined by

$$Z(x) = \sqrt{\sigma(W)}\, U f(x,S), \quad x \in M,$$

has C as its covariance function.

Proof. By independence, we have

$$\mathbb{E}\big(Z(x)\big) = \sqrt{\sigma(W)}\, \mathbb{E}(U)\, \mathbb{E}\big(f(x,S)\big) = 0, \quad x \in M,$$

and therefore

$$\begin{aligned} \operatorname{Cov}\big(Z(x), Z(y)\big) &= \mathbb{E}\big(Z(x)Z(y)\big) \\ &= \sigma(W)\, \mathbb{E}(U^2)\, \mathbb{E}\big(f(x,S)f(y,S)\big) \\ &= \int_W f(x,z) f(y,z)\, d\sigma(z) \\ &= C(x,y), \quad x,y \in M. \end{aligned}$$

□

Remark 5.1.3. Neither the statement of Lemma 5.1.2 nor the implicated positive definiteness of functions C which can be represented in the form (5.3) are new. In fact, for any Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle_{\mathscr{H}})$ and any function $f : M \to \mathscr{H}$ it follows from the properties of an inner product, that $C(x,y) = \langle f(x), f(y)\rangle_{\mathscr{H}}$, $x, y \in M$, is positive definite (for instance [75, Section 2.2]). For our purposes, it suffices to consider the special case $\mathscr{H} = L^2(\Omega, \mathscr{A}, P; \mathbb{R})$.

Let us now turn to the spectral turning bands method on $M = \mathbb{R}^d$ (see [58, 75]). By Theorem 4.1.4 on page 49, any continuous, stationary, and isotropic covariance function C can be represented as

$$C(x,y) = \int_{[0,\infty)} \Omega_d(r\|x-y\|)\, d\mu(r), \quad x, y \in \mathbb{R}^d, \tag{5.4}$$

with a uniquely determined finite measure μ on $\big([0,\infty), \mathscr{B}^1\big([0,\infty)\big)\big)$. A random field Z having the covariance function (5.4) is given by

$$Z(x) = \sqrt{2\mu\big([0,\infty)\big)} \cos\big(R\langle x, X\rangle + V\big), \quad x \in \mathbb{R}^d, \tag{5.5}$$

where R, X, and V are independent random variables such that $R \sim \mu/\mu([0,\infty))$, $X \sim \mathscr{U}(\mathbb{S}^{d-1})$, and $V \sim \mathscr{U}([0, 2\pi))$. This follows from Lemma 5.1.2, as we will show now. Recall that

$$\Omega_d(r) = \Gamma\Big(\frac{d}{2}\Big)\Big(\frac{2}{r}\Big)^{(d-2)/2} J_{(d-2)/2}(r), \quad r > 0,$$

where J_ν is the Bessel function of the first kind (4.7). By [35, Formula 8.411.5] and the definition of the normalized surface measure $\bar{\sigma}_{d-1}$ on $\mathbb{S}^{d-1}$ (cf. equations (1.2) and (1.3) on page 20), we have

$$\begin{aligned}\Omega_d(r\|x-y\|) &= \frac{\Gamma\big(d/2\big)}{\sqrt{\pi}\,\Gamma\big((d-1)/2\big)} \int_0^\pi \cos\big(r\|x-y\|\cos(\theta)\big) \sin^{d-2}(\theta)\, d\theta \\ &= \int_{\mathbb{S}^{d-1}} \cos\big(r\|x-y\|\langle z, e_d\rangle\big)\, d\bar{\sigma}_{d-1}(z).\end{aligned}$$

The surface measure is invariant with respect to rotations, hence we may replace the unit vector e_d in the integrand by any other element of $\mathbb{S}^{d-1}$, in particular by $(x-y)/\|x-y\|$ if $x \neq y$, $x, y \in \mathbb{R}^d$. This yields for the integrand above with the addition formula for the cosine function

$$\cos\Big(r\|x-y\|\Big\langle z, \frac{x-y}{\|x-y\|}\Big\rangle\Big) = \cos\big(r\langle z, x\rangle\big)\cos\big(r\langle z, y\rangle\big) + \sin\big(r\langle z, x\rangle\big)\sin\big(r\langle z, y\rangle\big),$$

which can be written as

$$\frac{1}{\pi}\int_0^{2\pi} \cos\big(r\langle z, x\rangle + \theta\big)\cos\big(r\langle z, y\rangle + \theta\big)\, d\theta. \tag{5.6}$$

Therefore the representation (5.4) becomes

$$C(x,y) = \frac{1}{\pi}\int_{[0,\infty)}\int_{\mathbb{S}^{d-1}}\int_0^{2\pi} \cos\big(r\langle z, x\rangle + \theta\big)\cos\big(r\langle z, y\rangle + \theta\big)\, d\theta\, d\bar{\sigma}_{d-1}(z)\, d\mu(r) \tag{5.7}$$

for all $x, y \in \mathbb{R}^d$, $x \neq y$. For $x = y$ (5.6) equals 1 and therefore (5.7) is also true in that case. Letting $\bar{\nu}$ denote the uniform probability measure on $[0, 2\pi)$ and $\bar{\mu} = \mu/\mu([0,\infty))$, we have from (5.7)

$$C(x,y) = \int_W f(x,(r,z,\theta)) f(y,(r,z,\theta))\, d\sigma(r,z,\theta), \quad x,y \in \mathbb{R}^d,$$

with

$$\begin{aligned} W &= [0,\infty) \times \mathbb{S}^{d-1} \times [0,2\pi), \\ \sigma &= \bar{\mu} \otimes \bar{\sigma}_{d-1} \otimes \bar{\nu}, \\ \text{and} \quad f(x,(r,z,\theta)) &= \sqrt{2\mu([0,\infty))} \cos\big(r\langle z,x\rangle + \theta\big), \quad x \in \mathbb{R}^d, (r,z,\theta) \in W. \end{aligned}$$

By Lemma 5.1.2, a random field having the covariance function (5.4) is therefore given by

$$Z(x) = U\sqrt{2\mu\big([0,\infty)\big)} \cos\big(R\langle x,X\rangle + V\big), \quad x \in \mathbb{R}^d,$$

with independent $R \sim \bar{\mu}$, $X \sim \mathcal{U}(\mathbb{S}^{d-1})$, $V \sim \mathcal{U}([0,2\pi])$, and a real-valued random variable U such that $\mathbb{E}(U) = 0$, and $\mathbb{E}(U^2) = 1$. Since the random variable U is merely responsible for centering the random field and because

$$Z(x) = \sqrt{2\mu\big([0,\infty)\big)} \cos\big(R\langle x,X\rangle + V\big)$$

is already centered, U can be omitted and we arrive at (5.5). We illustrate the method with an example.

Example 5.1.4. For $d = 2$, Schoenberg's representation (5.4) of a continuous, stationary, and isotropic correlation function ρ becomes

$$\rho(x,y) = \int_{[0,\infty)} J_0\big(r\|x-y\|\big)\, d\mu(r),$$

where μ is a probability measure because ρ is a correlation function. Suppose we want to simulate a random field having a Gaussian correlation function

$$\rho(x,y) = e^{-(\|x-y\|/c)^2}, \quad c > 0, x, y \in \mathbb{R}^2. \tag{5.8}$$

From [35, Formula 6.631.4] and integration by substitution we have

$$2e^{-(\|x-y\|/c)^2} = \int_0^\infty J_0\big(s\|x-y\|\big) c^2\, r\, e^{-(cr)^2/4}\, dr,$$

and hence the spectral measure μ of the Gaussian correlation function is

$$\mu(A) = \int_A \frac{r}{2/c^2} e^{-\frac{r^2}{2\cdot 2/c^2}}\, dr, \quad A \in \mathscr{B}^1([0,\infty)),$$

i.e. a Rayleigh distribution with scale parameter $\sigma = \sqrt{2}/c$. A random variable R having this distribution can be simulated by means of the inverse transformation method:

$$R \overset{d}{=} \frac{2}{c}\sqrt{-\log(U)}, \quad U \sim \mathcal{U}((0,1)).$$

For the simulation of $X \sim \mathcal{U}(\mathbb{S}^1)$ we can simply set $X = \big(\cos(\tilde{U}), \sin(\tilde{U})\big)$, where $\tilde{U}$ is uniformly distributed on the interval $[0, 2\pi)$. With $V \sim \mathcal{U}([0, 2\pi))$, the random field (5.5) becomes

$$Z(x) = \sqrt{2}\cos\big(R\langle x, X\rangle + V\big), \quad x \in \mathbb{R}^2.$$

The Gaussian correlation function is depicted in Figure 5.1a. Simulations of the spectral turning bands field on $[-1,1]^2$ are given in Figures 5.1b and 5.1c. These Figures illustrate why the method is called *turning bands*. Figures 5.1d and 5.1e show simulations of the corresponding approximate Gaussian random fields.

5.2. Spectral Turning Bands on $\mathbb{S}^d$

In this section, we derive a spectral turning bands method on the sphere $\mathbb{S}^d$ by the application of Lemma 5.1.2. Recall that Schoenberg's representation of a continuous and isotropic correlation function ρ on $\mathbb{S}^d$ is given by

$$\rho(x,y) = \sum_{n \in \mathbb{N}_0} b_{n,d} \frac{C_n^{(d-1)/2}\big(\cos\big(d_{\mathbb{S}^d}(x,y)\big)\big)}{C_n^{(d-1)/2}(1)}, \quad x, y \in \mathbb{S}^d, \tag{5.9}$$

with

$$b_{n,d} = \frac{2n+d-1}{2^{3-d}\pi} \frac{\Gamma\big((d-1)/2\big)^2}{\Gamma(d-1)} \int_0^\pi C_n^{(d-1)/2}\big(\cos(\theta)\big) \sin^{d-1}(\theta) \tilde{\rho}(\theta)\, d\theta \tag{5.10}$$

for $d \geq 2$ and $n \in \mathbb{N}_0$, and in case $d = 1$

$$b_{n,1} = \frac{\min\{n+1, 2\}}{\pi} \int_0^\pi \cos(n\theta)\tilde{\rho}(\theta)\, d\theta, \quad n \in \mathbb{N}_0. \tag{5.11}$$

Here $\tilde{\rho} : [0,\pi] \to [-1,1]$ is the function associated to the isotropic function ρ by $\rho(x,y) = \tilde{\rho}(d_{\mathbb{S}^d}(x,y))$.

Before we state the actual result, let us mention a more apparent approach and its drawbacks. Building on Schoenberg's representation of continuous and isotropic correlation functions on $\mathbb{S}^d$, Ziegel [88] showed that any such function has a *spherical convolution root*.

Theorem 5.2.1 (Ziegel, 2014)**.** *For any continuous and isotropic correlation function ρ there exists an isotropic function $g \in L^2\big(\mathbb{S}^d \times \mathbb{S}^d, \mathscr{B}(\mathbb{S}^d \times \mathbb{S}^d), \sigma_d \otimes \sigma_d; \mathbb{R}\big)$ such that*

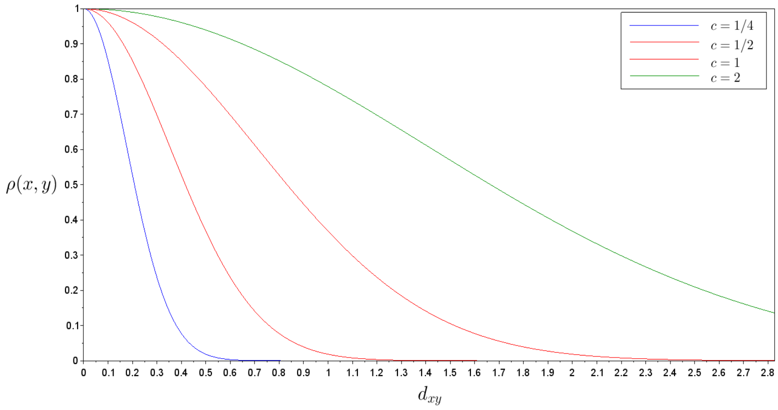
$$\rho(x,y) = \int_{\mathbb{S}^d} g(x,z)g(y,z)\, d\sigma_d(z), \quad x, y \in \mathbb{S}^d. \tag{5.12}$$

The representation (5.12) of the correlation function is already in the form of Lemma 5.1.2, so that the following corollary is evident.

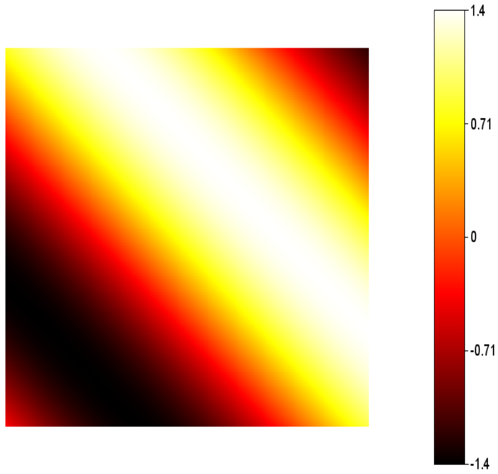
Corollary 5.2.2. *Let ρ be a continuous and isotropic correlation function on $\mathbb{S}^d$ and let U and X be independent random variables such that $X \sim \mathcal{U}(\mathbb{S}^d)$, $\mathbb{E}(U) = 0$, and $\mathbb{E}(U^2) = 1$. Then there is a function $g : \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{R}$ such that the random field*

$$Z(x) = \sqrt{\sigma_d(\mathbb{S}^d)}\, U g(x, X), \quad x \in \mathbb{S}^d, \tag{5.13}$$

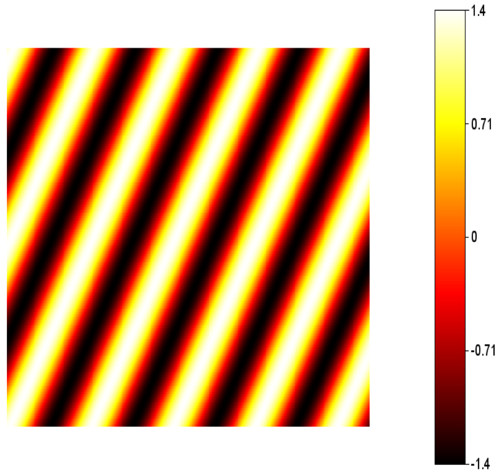
has the correlation function ρ.



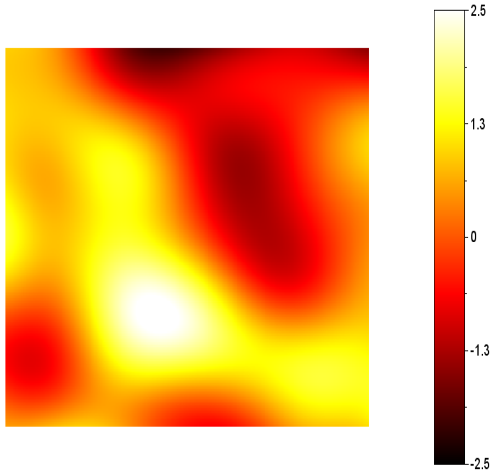
(a) Gaussian correlation function (5.8) with different scales c.



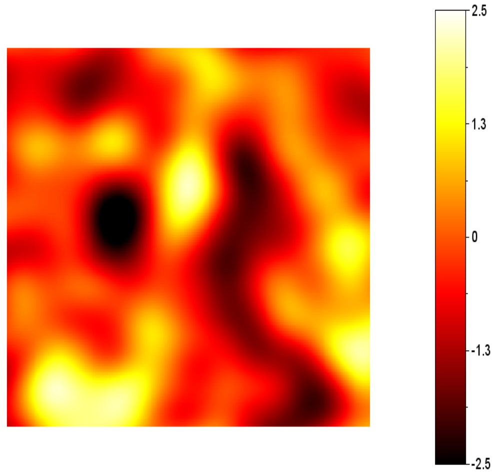
(b) Spectral turning bands field with $c = 1/2$.



(c) Spectral turning bands field with $c = 1/4$.



(d) Approximate Gaussian random field with $c = 1/2$, $n = 5000$.



(e) Approximate Gaussian random field with $c = 1/4$, $n = 5000$.

Figure 5.1.: Spectral turning bands random field with Gaussian correlation function.

Corollary 5.2.2 raises the question of how to determine a convolution root g for a given correlation function ρ. In general, there can be more than one convolution root leading to the same correlation function, as has been shown in [88]. However, an explicit construction of g in the form of an infinite linear combination of Gegenbauer polynomials involving the Schoenberg coefficients (5.10) of ρ has been given in [88] in order to prove Theorem 5.2.1. This function g seems to be a natural candidate for the application of Corollary 5.2.2. When it comes to the applicability in practice, there are two drawbacks of this approach. First, we have to know the coefficients (5.10) of a given correlation function ρ – in other words, we have to know the spectral measure of ρ – and much like in the $\mathbb{R}^d$ case the determination of the spectral measure of a given correlation function is a demanding task (see for instance [31] for solutions in case of $\mathbb{R}^2$). The second drawback is that in any computer implementation, an infinite linear combination must be truncated if no closed form is available, leading to an approximate simulation method only.

In what follows we use an orthogonality relation for Gegenbauer polynomials in order to show that Lemma 5.1.2 can be applied to Schoenberg's spectral representation (5.9) directly, resulting in a simulation method that avoids the infinite linear combination problem. The mentioned orthogonality relation is as follows (see [74, Corollary 4.9]): For all $d \geq 2$, $n, m \in \mathbb{N}_0$, and $x, y \in \mathbb{S}^d$ it is true that

$$\int_{\mathbb{S}^d} C_n^{(d-1)/2}\big(\langle x, z\rangle\big) C_m^{(d-1)/2}\big(\langle y, z\rangle\big)\, d\sigma_d(z) = \delta_{n,m}\sigma_d(\mathbb{S}^d)\frac{d-1}{2n+d-1} C_n^{(d-1)/2}\big(\langle x, y\rangle\big), \tag{5.14}$$

for $d = 1$, $n, m \in \mathbb{N}_0$, and $x, y \in \mathbb{S}^1$ the relation is

$$\int_{\mathbb{S}^1} C_n^0\big(\langle x, z\rangle\big) C_m^0\big(\langle y, z\rangle\big)\, d\sigma_1(z) = \delta_{n,m} \max\{2-n, 1\}\, \pi\, C_n^0(\langle x, y\rangle). \tag{5.15}$$

Theorem 5.2.3. *Let ρ be a continuous and isotropic correlation function on $\mathbb{S}^d$ and let $(b_{n,d}, n \in \mathbb{N}_0)$ denote the coefficients of ρ in* (5.10) *or* (5.11)*. Suppose R, X, and U are independent random variables such that $P(R = n) = b_{n,d}$ for all $n \in \mathbb{N}_0$, $X \sim \mathcal{U}(\mathbb{S}^d)$, $\mathbb{E}(U) = 0$, and $\mathbb{E}(U^2) = 1$. Then the random field Z on $\mathbb{S}^d$ defined by*

$$Z(x) = \begin{cases} \sqrt{\min\{R+1, 2\}}\, U \cos\big(R\langle x, X\rangle\big), & d = 1, \\ \sqrt{\frac{2R+d-1}{(d-1)C_R^{(d-1)/2}(1)}}\, U\, C_R^{(d-1)/2}\big(\langle x, X\rangle\big), & d \geq 2, \end{cases} \qquad x \in \mathbb{S}^d, \tag{5.16}$$

has the correlation function ρ.

Proof. Let $d \geq 2$ and suppose μ is the distribution of the random variable R. Since $\cos\big(d_{\mathbb{S}^d}(x, y)\big) = \langle x, y\rangle$, Schoenberg's representation (5.9) of ρ can be written as

$$\rho(x, y) = \int_{[0,\infty)} \frac{C_n^{(d-1)/2}\big(\langle x, y\rangle\big)}{C_n^{(d-1)/2}(1)}\, d\mu(n), \quad x, y \in \mathbb{S}^d.$$

Using (5.14) this integral becomes

$$\int_{[0,\infty)} \frac{2n+d-1}{(d-1)C_n^{(d-1)/2}(1)} \frac{1}{\sigma_d(\mathbb{S}^d)} \int_{\mathbb{S}^d} C_n^{(d-1)/2}\big(\langle x, z\rangle\big) C_n^{(d-1)/2}\big(\langle y, z\rangle\big)\, d\sigma_d(z)\, d\mu(n).$$

Rearranging we obtain

$$\rho(x,y) = \int_W f(x,s) f(y,s)\, d\sigma(s),$$

with

$$\begin{aligned} W &= [0,\infty) \times \mathbb{S}^d, \\ \sigma &= \mu \otimes \bar{\sigma}_d, \\ \text{and} \quad f(x,(n,z)) &= \sqrt{\frac{2n+d-1}{(d-1)C_n^{(d-1)/2}(1)}} C_n^{(d-1)/2}\big(\langle x,z\rangle\big), \quad x \in \mathbb{S}^d, (n,z) \in W. \end{aligned}$$

Thus Lemma 5.1.2 yields the claim for $d \geq 2$. In case $d = 1$, we have because of $C_n^0(1) = \cos(n \cdot 0) = 1$ for all $n \in \mathbb{N}_0$

$$\rho(x,y) = \int_{[0,\infty)} C_n^0\big(\langle x,y\rangle\big)\, d\mu(n), \quad x,y \in \mathbb{S}^1,$$

if μ denotes the distribution of R. Using (5.15) we obtain

$$\rho(x,y) = \int_{[0,\infty)} \int_{\mathbb{S}^1} \frac{2}{\max\{2-n,1\}} C_n^0\big(\langle x,z\rangle\big) C_n^0\big(\langle y,z\rangle\big)\, d\bar{\sigma}_1(z) d\mu(n), \quad x,y \in \mathbb{S}^1,$$

and the assertion follows also in case $d = 1$ with Lemma 5.1.2 and the identity $2/\max\{2-n,1\} = \min\{n+1,2\}$ for all $n \geq 0$. □

In view of the simulations in the next section, we investigate the third absolute moment of the spectral turning bands random field (5.16).

Proposition 5.2.4. *Let Z be the spectral turning bands random field* (5.16) *on $\mathbb{S}^d$, $d \geq 1$, and let $(c_d, d \in \mathbb{N})$ be the sequence of constants defined in Proposition 4.3.1 on page 63. Then we have in case $d = 1$*

$$\mathbb{E}(|Z(x)|^3) = \mathbb{E}(|U|^3)\Big(\frac{8\sqrt{2}}{3\pi} + \Big(1 - \frac{8\sqrt{2}}{3\pi}\Big) b_{0,1}\Big), \quad x \in \mathbb{S}^1,$$

and in case $d \geq 2$

$$\begin{aligned} \mathbb{E}(|Z(x)|^3) = \frac{1}{\pi c_d} \mathbb{E}(|U|^3) \sum_{n\in\mathbb{N}_0} b_{n,d} \Big(\frac{(2n+d-1)\, n!\, \Gamma(d-1)}{(d-1)\Gamma(n+d-1)}\Big)^{3/2} \\ \times \int_0^\pi \big|C_n^{(d-1)/2}\big(\cos(\theta)\big)\big|^3 \sin^{d-1}(\theta)\, d\theta, \quad x \in \mathbb{S}^d. \end{aligned} \tag{5.17}$$

Additionally, in case $d = 2$ the following inequality holds true

$$\mathbb{E}(|Z(x)|^3) \leq \mathbb{E}(|U|^3)\Big(b_{0,2} + \Big(\frac{\Gamma(1/4)}{\pi}\Big)^2 \sum_{n\in\mathbb{N}} b_{n,2} \Big(\frac{2n+1}{n}\Big)^{3/2}\Big), \quad x \in \mathbb{S}^2. \tag{5.18}$$

Proof. Let $d \geq 2$ and $x \in \mathbb{S}^d$, then by independence

$$\begin{aligned} \mathbb{E}(|Z(x)|^3) &= \sum_{n\in\mathbb{N}_0} b_{n,d} \mathbb{E}\Big(\Big|\sqrt{\frac{2n+d-1}{(d-1)C_n^{(d-1)/2}(1)}}\, U\, C_n^{(d-1)/2}\big(\langle x,X\rangle\big)\Big|^3\Big) \\ &= \mathbb{E}(|U|^3) \sum_{n\in\mathbb{N}_0} b_{n,d} \Big(\frac{2n+d-1}{(d-1)C_n^{(d-1)/2}(1)}\Big)^{3/2} \mathbb{E}\big(\big|C_n^{(d-1)/2}\big(\langle x,X\rangle\big)\big|^3\big). \end{aligned}$$

For all $n \in \mathbb{N}_0$ we find by the rotational invariance of the uniform distribution on $\mathbb{S}^d$

$$\mathbb{E}\big(\big|C_n^{(d-1)/2}\big(\langle x, X\rangle\big)\big|^3\big) = \mathbb{E}\big(\big|C_n^{(d-1)/2}\big(\langle e_{d+1}, X\rangle\big)\big|^3\big).$$

Passing to spherical coordinates this expectation becomes

$$\frac{\sigma_{d-1}(\mathbb{S}^{d-1})}{\sigma_d(\mathbb{S}^d)} \int_0^\pi \big|C_n^{(d-1)/2}\big(\cos(\theta)\big)\big|^3 \sin^{d-1}(\theta)\, d\theta.$$

With $\sigma_d(\mathbb{S}^d) = 2\pi^{(d+1)/2}/\Gamma((d+1)/2)$ (cf. equation (1.3) on page 20), $C_n^{(d-1)/2}(1) = \Gamma(n+d-1)/(n!\Gamma(d-1))$ (cf. (4.13) on page 51), and $c_d = \Gamma(d/2)/(\sqrt{\pi}\Gamma((d+1)/2))$ (cf. (4.46) on page 63), equation (5.17) follows by collecting terms.

For $d = 1$, the representation $X = \big(\sin(V), \cos(V)\big) \sim \mathcal{U}(\mathbb{S}^1)$ with $V \sim \mathcal{U}([0, 2\pi))$ and the definition of C_n^0 (cf. (4.10) on page 50) lead to

$$\mathbb{E}\big(\big|C_n^0\big(\langle e_2, X\rangle\big)\big|^3\big) = \mathbb{E}\big(|C_n^0\big(\cos(V)\big)|^3\big) = \mathbb{E}(|\cos(nV)|^3)$$

for all $n \in \mathbb{N}_0$, so that by the rotational invariance of the uniform distribution on $\mathbb{S}^1$ we have for all $x \in \mathbb{S}^1$ and all $n \geq 1$

$$\begin{aligned}
\mathbb{E}\big(\big|C_n^0\big(\langle x, X\rangle\big)\big|^3\big) &= \mathbb{E}(|\cos(nV)|^3) \\
&= \frac{1}{2\pi}\int_0^{2\pi} |\cos(nv)|^3 dv \\
&= \frac{1}{2\pi}\int_0^{2\pi} |\cos(v)|^3 dv = \frac{4}{3\pi}.
\end{aligned}$$

For $n = 0$ it follows from $C_0^0 \equiv 1$ that $\mathbb{E}(|C_0^0(\langle x, X\rangle)|^3) = 1$. Therefore in case $d = 1$

$$\begin{aligned}
\mathbb{E}(|Z(x)|^3) &= \mathbb{E}(|U|^3) \sum_{n\in\mathbb{N}_0} b_{n,1}\big(\min\{n+1, 2\}\big)^{3/2}\mathbb{E}\big(\big|C_n^0\big(\langle x, X\rangle\big)\big|^3\big) \\
&= \mathbb{E}(|U|^3)\Big(b_{0,1} + \sum_{n\in\mathbb{N}} b_{n,1} 2^{3/2}\frac{4}{3\pi}\Big) \\
&= \mathbb{E}(|U|^3)\Big(\frac{8\sqrt{2}}{3\pi} + \Big(1 - \frac{8\sqrt{2}}{3\pi}\Big)b_{0,1}\Big), \quad x \in \mathbb{S}^1.
\end{aligned}$$

Let us now turn to the case $d = 2$. Since $C_n^{1/2} = P_n$ and $c_2 = 2/\pi$, we have from (5.17) for all $x \in \mathbb{S}^2$

$$\mathbb{E}(|Z(x)|^3) = \frac{1}{2}\,\mathbb{E}(|U|^3) \sum_{n\in\mathbb{N}_0} b_{n,2}(2n+1)^{3/2} \int_0^\pi \big|P_n\big(\cos(\theta)\big)\big|^3 \sin(\theta)\, d\theta.$$

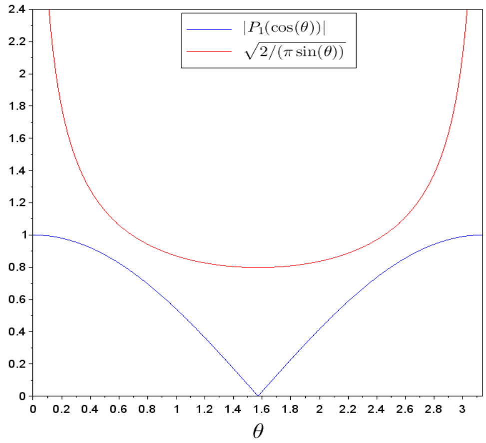
By [1, Formula 22.14.3] we have the bound

$$\big|P_n\big(\cos(\theta)\big)\big| \leq \sqrt{\frac{2}{\pi\, n \sin(\theta)}}, \quad n \in \mathbb{N}, 0 < \theta < \pi. \tag{5.19}$$

(Figure 5.2 presents a visualization of this bound.) Thus we have by [35, Formula 3.621.1] for all $n \in \mathbb{N}$

$$\int_0^\pi \big|P_n\big(\cos(\theta)\big)\big|^3 \sin(\theta)\, d\theta \leq \Big(\frac{2}{n\pi}\Big)^{3/2} \int_0^\pi \frac{1}{\sqrt{\sin(\theta)}}\, d\theta = \frac{2\,\Gamma(1/4)^2}{\pi^2} n^{-3/2}.$$

For $n = 0$ the integral is equal to 2 and (5.18) follows. $\square$



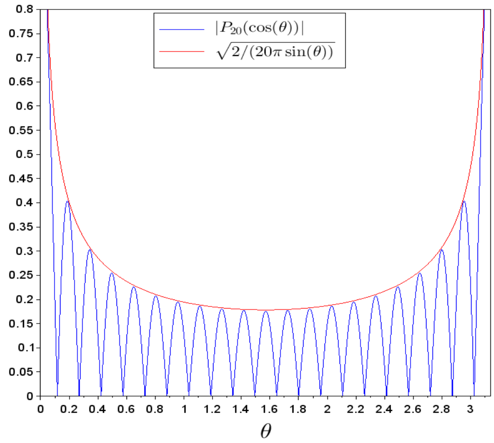


Figure 5.2.: Visualization of the bound (5.19).

5.3. Examples and Simulations on $\mathbb{S}^2$

In this section we consider the practically most relevant case of two-dimensional unit sphere $\mathbb{S}^2$. We give various examples of spectral turning bands random fields and their associated correlation functions and provide simulations.

Since $C_n^{1/2} = P_n$ and $P_n(1) = 1$, Schoenberg's representation (5.9) on $\mathbb{S}^2$ becomes

$$\rho(x,y) = \sum_{n\in\mathbb{N}_0} b_{n,2} P_n\big(\cos\big(d_{\mathbb{S}^2}(x,y)\big)\big), \quad x,y \in \mathbb{S}^2, \tag{5.20}$$

$$\text{with} \quad b_{n,2} = \frac{2n+1}{2} \int_0^\pi \tilde{\rho}(\theta) P_n\big(\cos(\theta)\big)\sin(\theta)\,d\theta, \quad n \in \mathbb{N}_0. \tag{5.21}$$

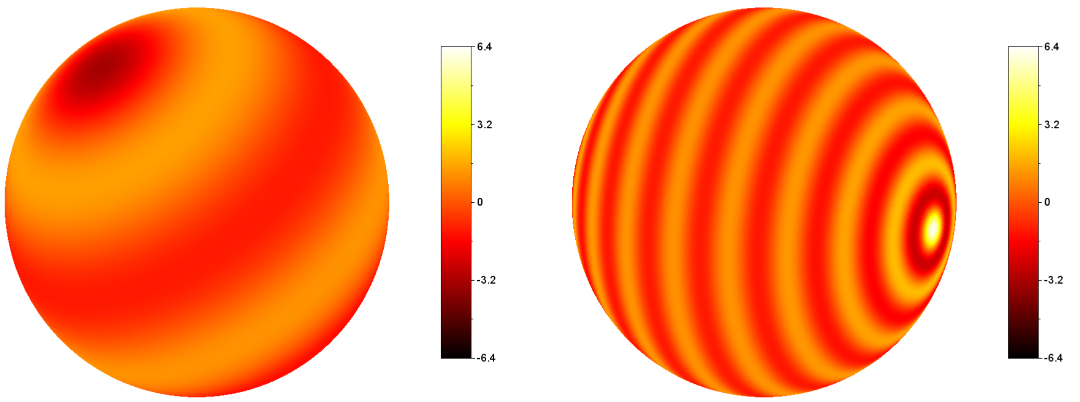
The corresponding spectral turning bands random field (5.16) is given by

$$Z(x) = \sqrt{2R+1}\,U P_R\big(\langle x, X\rangle\big), \quad x \in \mathbb{S}^2, \tag{5.22}$$

$$\text{with independent } R \sim \sum_{n\in\mathbb{N}_0} b_{n,2}\varepsilon_n, X \sim \mathcal{U}(\mathbb{S}^2), U \text{ such that } \mathbb{E}(U) = 0, \mathbb{E}(U^2) = 1.$$

In order to simulate the random field (5.22) we have to simulate a centered and normalized random variable U, a random vector X which is uniformly distributed on $\mathbb{S}^2$, and an $\mathbb{N}_0$-valued random variable R with $P(R = n) = b_{n,2}$ for all $n \in \mathbb{N}_0$. The simulation of U and X poses no problem, so the actual questions are how to simulate R and what kind of correlation functions are obtained. In theory, this question is answered by equations (5.20) and (5.21). However, in practice closed form representations of (5.20) might be unknown for a given discrete distribution $\sum_{n\in\mathbb{N}_0} b_{n,2}\varepsilon_n$. If on the other hand we are given a correlation function, solutions of the integrals (5.21) might be unknown or given in a way, such that an exact simulation procedure for the random variable R might be unknown, as the example in (4.17) of [40] on page 51 demonstrates.

Before we start with the examples, let us take a closer look at the random field Z in (5.22). Figure 5.3 displays two realizations of Z with random X, U fixed to 1, $R = 5$ in Figure 5.3a and $R = 20$ in Figure 5.3b. The random variable $X \sim \mathcal{U}(\mathbb{S}^2)$ in (5.22) may be thought of as a random North Pole, defining a random spherical coordinate system on $\mathbb{S}^2$. In a spherical coordinate system, a circle of latitude can be seen as the set consisting of all points in $\mathbb{S}^2$ which have a certain great circle distance from the North Pole. From $\langle x, y\rangle = \cos(d_{\mathbb{S}^2}(x, y))$ for all $x, y \in \mathbb{S}^2$ it is therefore evident that the realizations of Z are constant on all circles of latitude in the random spherical coordinate system. These are the *bands* that are visible in Figure 5.3. On the circles of longitude, the random field Z behaves like the function $\varphi \mapsto \sqrt{2R+1}\, U P_R(\cos(\varphi))$. Figure 5.4 displays this function for $U = 1$ and different values of R. Apart from the scaling, the random variable R determines the index of the Legendre polynomial and is therefore responsible for the oscillation of Z on each circle of longitude. In consequence, there a few *bands* visible in a realization of Z if the simulated value of R is small and we get many *bands* if R is large. Finally, the random variable U makes sure that the mean of Z is 0.



(a) $U = 1,\ R = 5$. (b) $U = 1,\ R = 20$.

Figure 5.3.: Realizations of $Z(x) = \sqrt{2R+1}\, U P_R(\langle x, X\rangle)$.

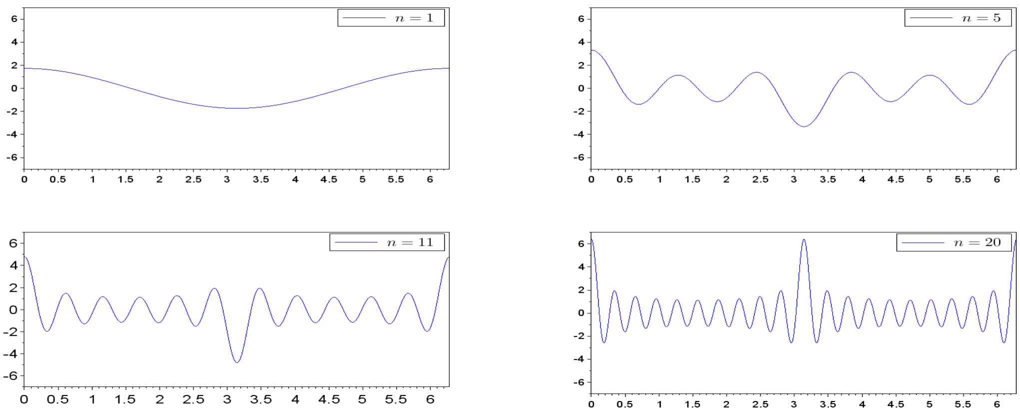


Figure 5.4.: The function $\varphi \mapsto \sqrt{2n+1} P_n(\cos(\varphi))$ for $n = 1, 5, 11$, and 20.

In what follows, we provide examples of spectral turning bands random fields Z, for which the correspondence between the correlation function and the coefficients $(b_{n,2}, n \in \mathbb{N}_0)$ is explicit and we illustrate these examples with simulations. Note that the spectral turning bands random field Z in (5.22) is centered and normalized.

Therefore, the sequence of random fields $(S_n, n \in \mathbb{N})$ defined by

$$S_n(x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i(x), \quad x \in \mathbb{S}^2,$$

converges to a Gaussian random field as n tends to infinity (see Lemma 4.4.1). In all simulations, we choose a standard normal distribution for the centering random variable U. It follows therefore from (4.55) in Proposition 5.2.4, $\mathbb{E}(|U|^3) = 4/\sqrt{2\pi}$, and $C_{\mathrm{BE}} = 1/2$, that for all $x \in \mathbb{S}^2$ and all $n \in \mathbb{N}$ the inequality

$$\sup_{y\in\mathbb{R}} |P(S_n(x) \le y) - \Phi(y)| \le \Big(b_{0,2} + \Big(\frac{\Gamma(1/4)}{\pi}\Big)^2 \sum_{k\in\mathbb{N}} b_{k,2}\Big(\frac{2k+1}{k}\Big)^{3/2}\Big)\sqrt{\frac{2}{\pi n}}, \quad (5.23)$$

holds true, where Φ is the distribution function of the standard normal distribution. Just as in Section 4.4 we choose for the simulation of the approximate Gaussian random field S_n a number of superpositions n such that the bound in (5.23) becomes smaller than 0.05.

Example 5.3.1. Of course the simplest example is to let $R \equiv n$ with probability one for some $n \in \mathbb{N}_0$. If we take for example $n = 3$, then (5.20) becomes [35, Formula 8.912.4]

$$\rho(x,y) = P_3\big(\cos\big(d_{\mathbb{S}^2}(x,y)\big)\big) = \frac{1}{8}\big(5\cos\big(3\,d_{\mathbb{S}^2}(x,y)\big) + 3\cos\big(d_{\mathbb{S}^2}(x,y)\big)\big).$$

Figure 5.5a displays this correlation function on $\mathbb{S}^2$. It admits a full range of $[-1,1]$ and is not monotone. Note that taking a single Legendre polynomial $\rho(x,y) = P_n\big(\cos\big(d_{\mathbb{S}^2}(x,y)\big)\big)$ results in a correlation function which is symmetric with respect to the point $(\pi/2, 0)$ if n is odd and symmetric with respect to the y-axis shifted by $\pi/2$ to the right if n is even. Figure 5.5b displays a realization of the spectral turning bands random field with $R \equiv 3$. Figures 5.5c, 5.5d, and 5.5e show simulations of an approximate Gaussian random field obtained from this spectral turning bands field. From $b_{n,2} = \delta_{3,n}$, $n \in \mathbb{N}_0$, we get a bound in (5.23) which is lower than $3.79/\sqrt{n}$, and we choose $n = 10$, $n = 500$ and $n = 6000$ superpositions for the approximations in Figures 5.5c, 5.5d, and 5.5e.

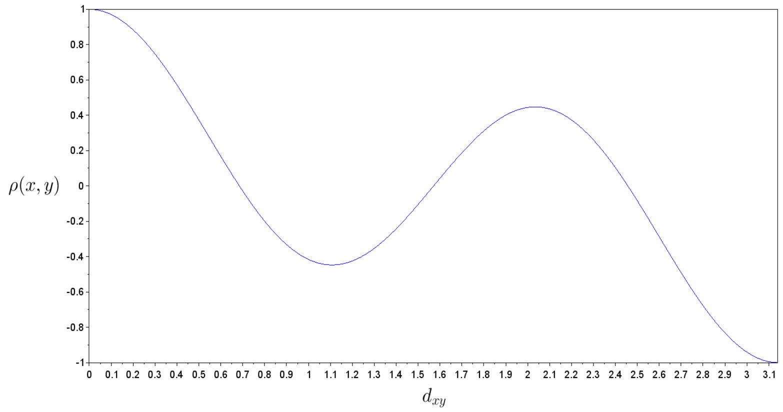
Example 5.3.2. The next step is to take finite convex combinations of Legendre polynomials. Let us take for example

$$b_{1,2} = \frac{1}{3}, b_{2,2} = \frac{1}{6}, b_{4,2} = \frac{1}{3}, \quad \text{and} \quad b_{16,2} = \frac{1}{6},$$

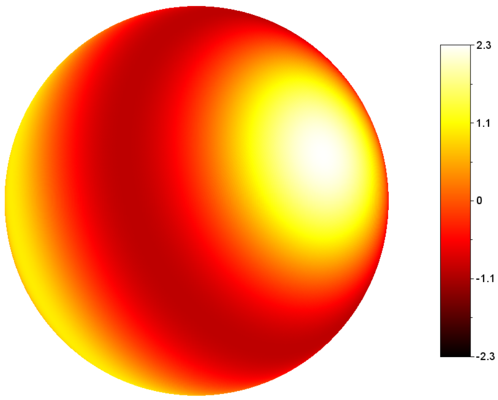
so that (5.20) becomes

$$\begin{aligned}\rho(x,y) = \frac{1}{3}P_1\big(\cos\big(d_{\mathbb{S}^2}(x,y)\big)\big) + \frac{1}{6}P_2\big(\cos\big(d_{\mathbb{S}^2}(x,y)\big)\big) + \frac{1}{3}P_4\big(\cos\big(d_{\mathbb{S}^2}(x,y)\big)\big)\\ + \frac{1}{6}P_{16}\big(\cos\big(d_{\mathbb{S}^2}(x,y)\big)\big), \quad x,y \in \mathbb{S}^2. \qquad (5.24)\end{aligned}$$

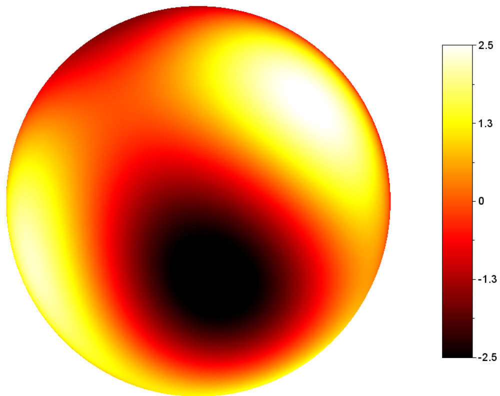
The graph of ρ is depicted in Figure 5.6a. It follows a global trend while oscillating locally. Realizations of the corresponding spectral turning bands random field are given in Figures 5.6b and 5.6c. The bound in (5.23) is approximately $4.27/\sqrt{n}$. Figures 5.6d and 5.6e display approximations to a Gaussian random field with this



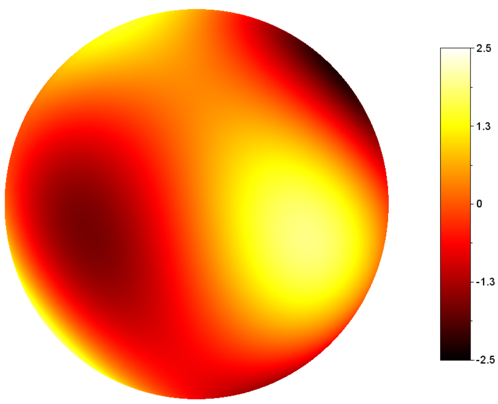
(a) Correlation function $\rho(x, y) = P_3\big(\cos(d_{\mathbb{S}^2}(x, y))\big)$.



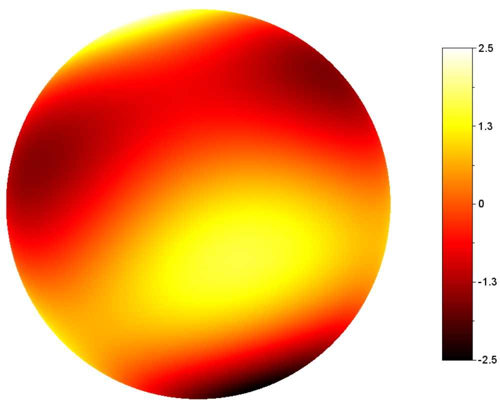
(b) Spectral turning bands random field with $R \equiv 3$.



(c) Approximate Gaussian random field with $n = 10$.

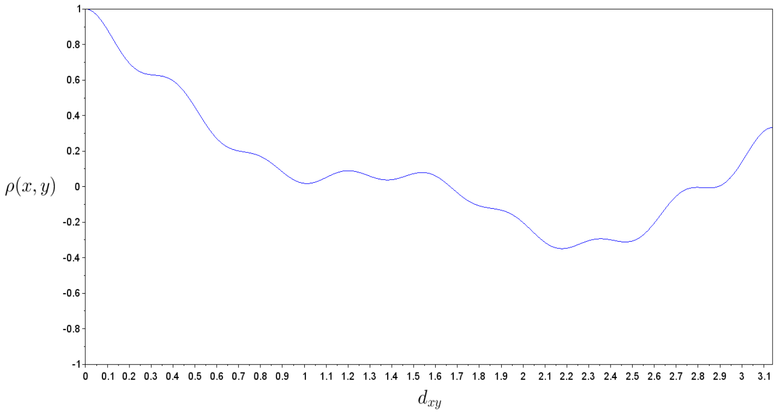


(d) Approximate Gaussian random field with $n = 500$.

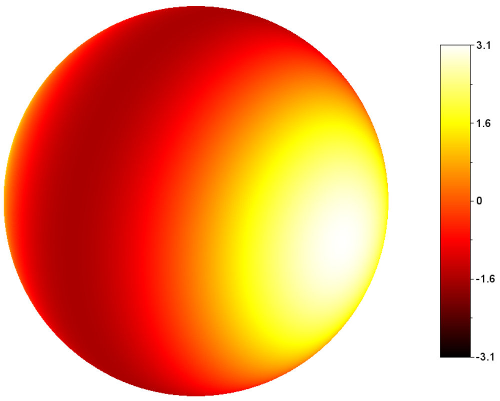


(e) Approximate Gaussian random field with $n = 6000$.

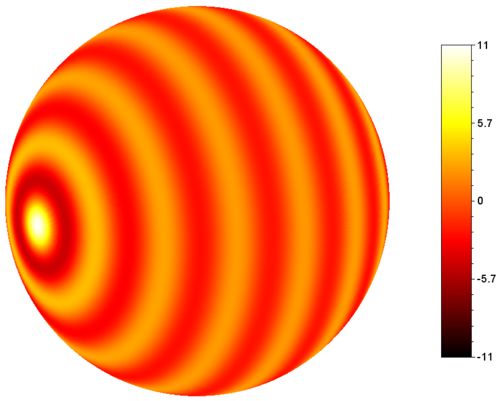
Figure 5.5.: Spectral turning bands random field with $R \equiv 3$.



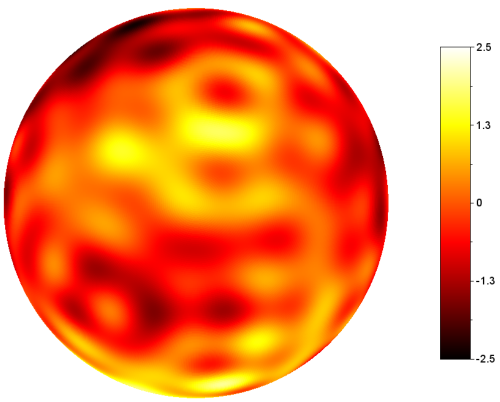
(a) Correlation function (5.24).



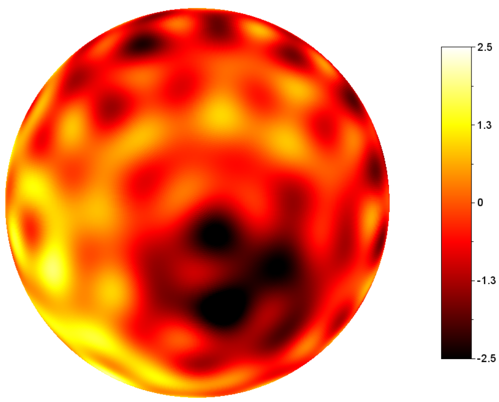
(b) Spectral turning bands random field.



(c) Spectral turning bands random field.



(d) Approximate Gaussian random field with $n = 500$.



(e) Approximate Gaussian random field with $n = 7500$.

Figure 5.6.: Spectral turning bands random field with correlation function (5.24).

correlation function and $n = 500$ and $n = 7500$ superpositions of single realizations of the spectral turning bands random field. Due to the combination of Legendre polynomials with small and large indices, this random field exhibits just as its correlation function a global behaviour paired with a local and quite regular oscillation.

Let us examine what happens if we increase the local oscillation while giving it less weight. We take

$$b_{1,2} = \frac{1}{3}, b_{2,2} = \frac{1}{2}, b_{20,2} = \frac{1}{12}, \quad \text{and} \quad b_{50,2} = \frac{1}{12},$$

so that the correlation function is now

$$\begin{aligned}\rho(x,y) = \frac{1}{3}P_1\big(\cos\big(d_{\mathbb{S}^2}(x,y)\big)\big) + \frac{1}{2}P_2\big(\cos\big(d_{\mathbb{S}^2}(x,y)\big)\big) + \frac{1}{12}P_{20}\big(\cos\big(d_{\mathbb{S}^2}(x,y)\big)\big) \\ + \frac{1}{12}P_{50}\big(\cos\big(d_{\mathbb{S}^2}(x,y)\big)\big), \quad x,y \in \mathbb{S}^2. \qquad (5.25)\end{aligned}$$

Figure 5.7a displays this correlation function and Figures 5.7b depicts a possible realization of the corresponding spectral turning bands random field. Figure 5.7c displays the associated approximate Gaussian random field with $n = 500$ superpositions. The realization looks somewhat artificial and the local oscillation is reminiscent of the surface of a golf ball. The bound on the Kolmogorov distance is $4.71/\sqrt{n}$, hence only marginally larger than the bound in the example before. Figures 5.7d and 5.7e show, that the structure of the realizations stays the same if we increase n.

Example 5.3.3. In the next example we fix $N \in \mathbb{N}$ and let

$$b_{n,2} = \frac{2n+1}{(N+1)^2}, \quad n = 0,\dots,N, \quad b_{k,2} = 0, \quad k \geq N+1.$$

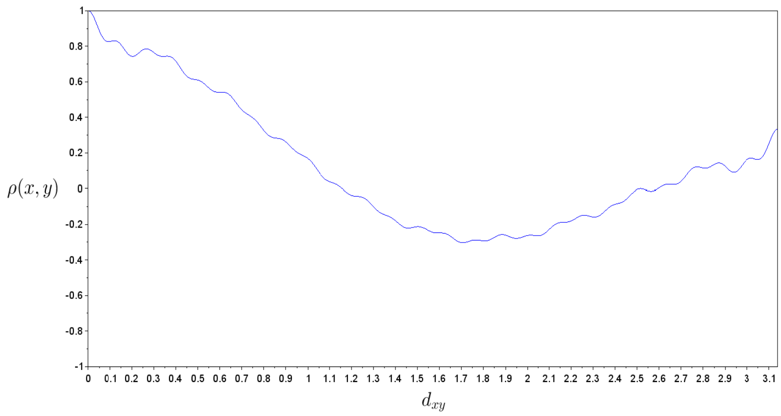
Then this defines a probability distribution on $\mathbb{N}_0$. By the Christoffel–Darboux formula (for example [35, Formula 8.915.1]) and since $P_k(1) = 1$ for all $k \in \mathbb{N}_0$, Schoenberg's representation (5.20) becomes

$$\begin{aligned}\rho(x,y) &= \frac{1}{(N+1)^2}\sum_{n=0}^{N}(2n+1)P_n\big(\cos\big(d_{\mathbb{S}^2}(x,y)\big)\big) \\ &= \frac{P_N\big(\cos\big(d_{\mathbb{S}^2}(x,y)\big)\big) - P_{N+1}\big(\cos\big(d_{\mathbb{S}^2}(x,y)\big)\big)}{(N+1)\big(1-\cos\big(d_{\mathbb{S}^2}(x,y)\big)\big)}, \quad N \in \mathbb{N}. \qquad (5.26)\end{aligned}$$

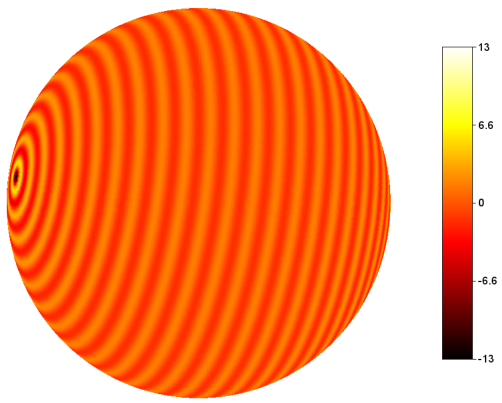
Figure 5.8a displays this correlation function for different values of N. For large N this correlation function drops quickly towards 0 and swings off. Simulations of the spectral turning bands random field for $N = 5$ and $N = 50$ are given in Figures 5.8b and 5.8c. Concerning the simulation of an approximate Gaussian random field and the Kolmogorov distance of the marginal distributions, we have found the bounds $3.76/\sqrt{n}$ in case $N = 5$ and $3.1/\sqrt{n}$ in case $N = 50$. Figures 5.8d and 5.8e display simulations with $n = 6000$.

Example 5.3.4. Let us take a negative binomial distribution for the random variable R, i.e.

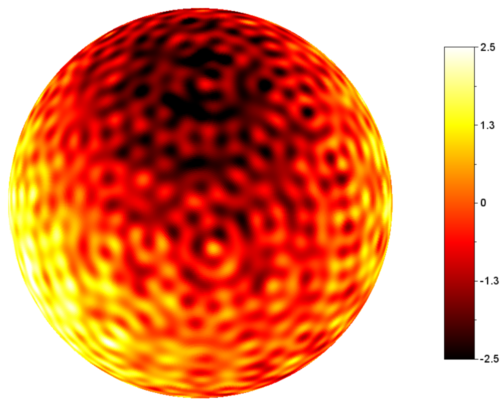
$$b_{n,2} = P(R = n) = \frac{\Gamma(n+r)}{\Gamma(r)\,n!}p^r(1-p)^n, \quad r > 0, p \in (0,1), n \in \mathbb{N}_0.$$



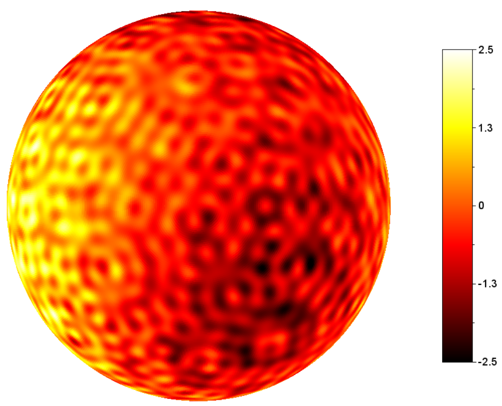
(a) Correlation function (5.25).



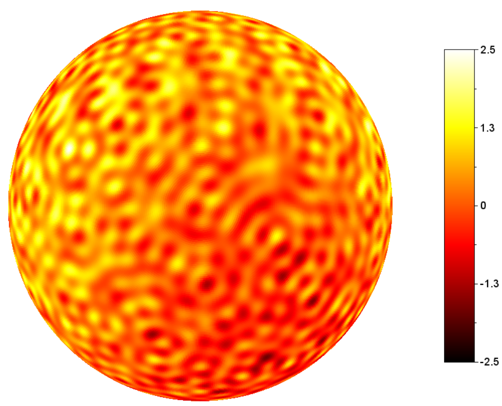
(b) Spectral turning bands random field.



(c) Approximate Gaussian random field with $n = 500$.

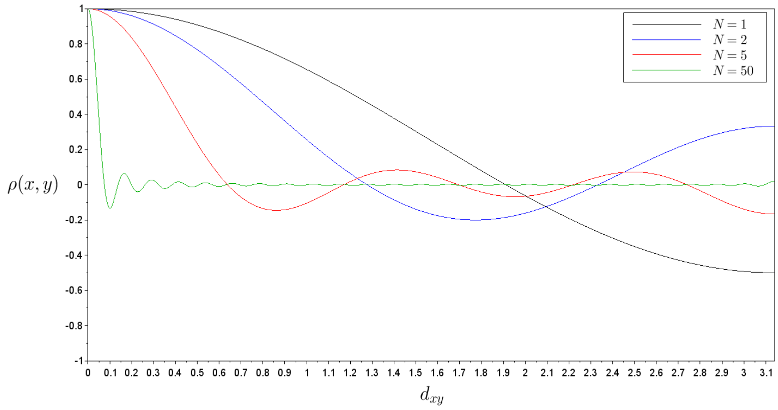


(d) Approximate Gaussian random field with $n = 9000$.

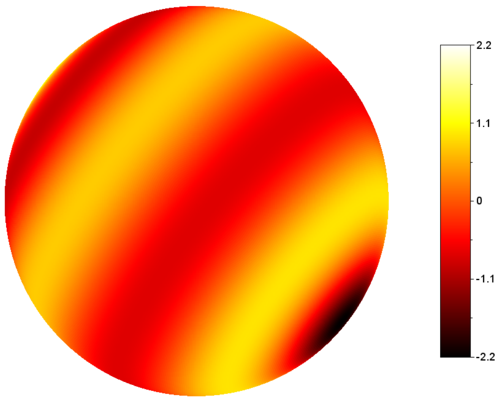


(e) Approximate Gaussian random field with $n = 20000$.

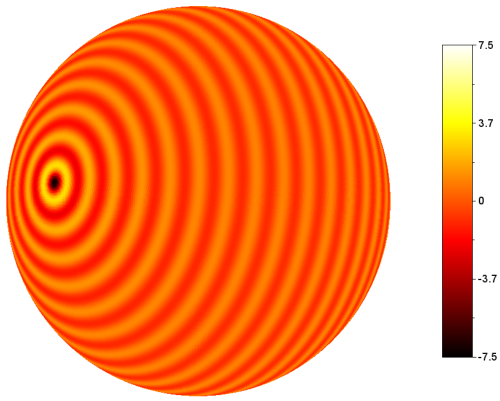
Figure 5.7.: Spectral turning bands random field with correlation function (5.25).



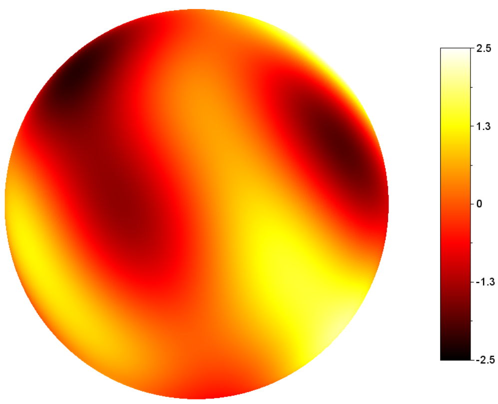
(a) Correlation function (5.26).



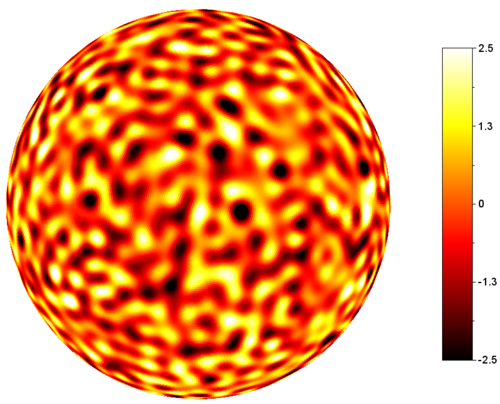
(b) Spectral turning bands random field with $N = 5$.



(c) Spectral turning bands random field with $N = 50$.



(d) Approximate Gaussian random field with $N = 5$ and $n = 6000$.



(e) Approximate Gaussian random field with $N = 50$ and $n = 6000$.

Figure 5.8.: Spectral turning bands random field with correlation function (5.26).

Using [72, Formula 5.10.1.16] Schoenberg's representation (5.20) becomes

$$\begin{aligned}\rho(x,y) &= p^r \sum_{n\in\mathbb{N}_0} \frac{\Gamma(n+r)}{\Gamma(r)\, n!}(1-p)^n P_n\big(\cos\big(d_{\mathbb{S}^2}(x,y)\big)\big)\\ &= p^r\big(1+2(1-p)\cos\big(d_{\mathbb{S}^2}(x,y)\big)+(1-p)^2\big)^{-r/2}\\ &\quad\times {}_2F_1\left(r,1-r;1;\frac{1}{2}-\frac{1-(1-p)\cos\big(d_{\mathbb{S}^2}(x,y)\big)}{2\sqrt{1-2(1-p)\cos\big(d_{\mathbb{S}^2}(x,y)\big)+(1-p)^2}}\right).\end{aligned} \tag{5.27}$$

For the first factor in (5.27) we have

$$\begin{aligned}&p^r\big(1+2(1-p)\cos\big(d_{\mathbb{S}^2}(x,y)\big)+(1-p)^2\big)^{-r/2}\\ &\qquad= \Big(\frac{1}{p^2}-2\frac{1-p}{p^2}\cos\big(d_{\mathbb{S}^2}(x,y)\big)+\frac{(1-p)^2}{p^2}\Big)^{-r/2}\\ &\qquad= \Big(1+2\frac{1-p}{p^2}\big(1-\cos\big(d_{\mathbb{S}^2}(x,y)\big)\big)\Big)^{-r/2}\\ &\qquad= \left(1+\left(\frac{\sin\big(d_{\mathbb{S}^2}(x,y)/2\big)}{p/(2\sqrt{1-p})}\right)^2\right)^{-r/2}.\end{aligned}$$

Let $c>0$, then $p=2\,c\,(\sqrt{c^2+1}-c)\in(0,1)$ and thus p is a valid choice for the second parameter of the negative binomial distribution. For this p, the last term in the equality above becomes

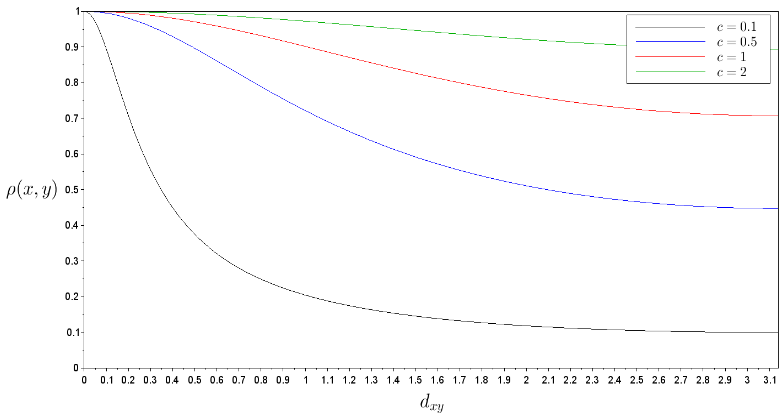
$$\left(1+\left(\frac{\sin\big(d_{\mathbb{S}^2}(x,y)/2\big)}{c}\right)^2\right)^{-r/2},$$

i.e. the first factor in (5.27) is equal to the composition of a generalized Cauchy model with parameter $\alpha=2$ and $\beta=r>0$ (see [34]) and the function $\theta\mapsto\sin(\theta/2)$ (see [33, 86]). If $r=1$, i.e. if $R\sim\mathrm{Geo}_{\mathbb{N}_0}$, the hypergeometric function in (5.27) equals 1 from the definition of the hypergeometric function (e.g. [35, Formula 9.100]) and we have the correlation function

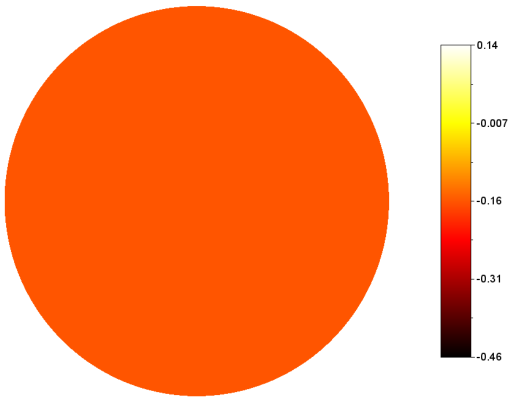
$$\rho(x,y)=\left(1+\left(\frac{\sin\big(d_{\mathbb{S}^2}(x,y)/2\big)}{c}\right)^2\right)^{-1/2},\qquad x,y\in\mathbb{S}^2. \tag{5.28}$$

Figure 5.9a displays this correlation function for different parameter c. This correlation function only admits positive correlations. Furthermore, large values of c result in correlation functions which are close to $\rho\equiv 1$. Realizations of the corresponding spectral turning bands field on $\mathbb{S}^2$ are depicted in Figures 5.9b and 5.9c. Concerning the simulation of a Gaussian random field we have for the bound in (5.23) the following inequality in terms of $p=2c(\sqrt{c^2+1}-c)$:

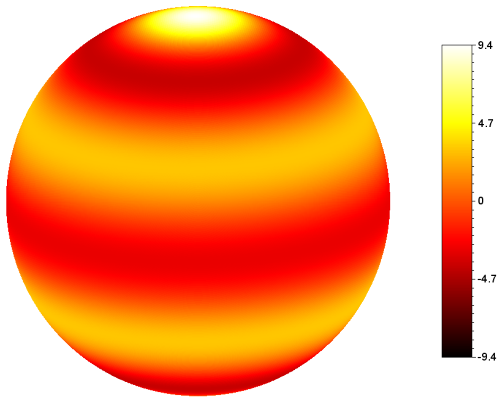
$$\begin{aligned}&\Big(b_{0,2}+\Big(\frac{\Gamma(1/4)}{\pi}\Big)^2\sum_{k\in\mathbb{N}} b_{k,2}\Big(\frac{2k+1}{k}\Big)^{3/2}\Big)\sqrt{\frac{2}{\pi n}}\\ &\leq \Big(p+\Big(\frac{\Gamma(1/4)}{\pi}\Big)^2\Big(p\sum_{k=1}^{m}(1-p)^k\Big(\frac{2k+1}{k}\Big)^{3/2}+\frac{(1-p)^{m+1}(2m+3)^{3/2}}{(m+1)^{3/2}}\Big)\Big)\sqrt{\frac{2}{\pi n}}\end{aligned}$$



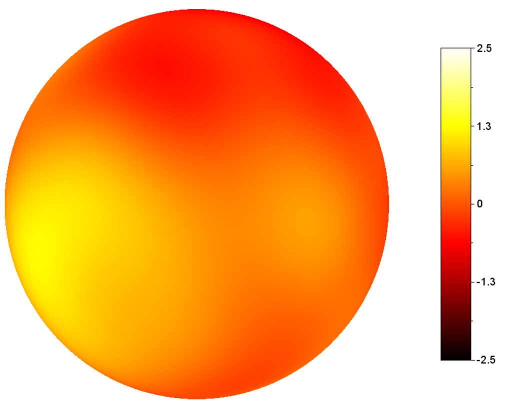
(a) Correlation function (5.28).



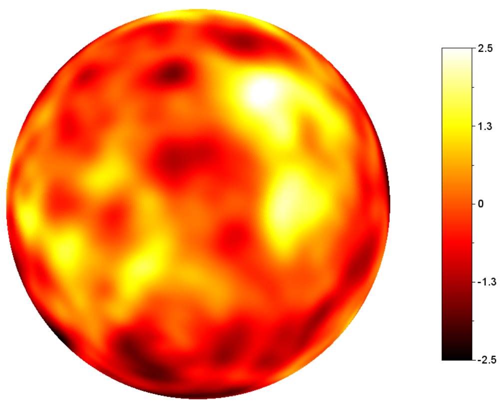
(b) Spectral turning bands random field with $c = 0.5$.



(c) Spectral turning bands random field with $c = 0.1$.



(d) Approximate Gaussian random field with $c = 0.5$ and $n = 9000$.



(e) Approximate Gaussian random field with $c = 0.1$ and $n = 9000$.

Figure 5.9.: Spectral turning bands random field with correlation function (5.28).

for all $m \in \mathbb{N}$. For $m = 100$ we get a bound of approximately $3.36/\sqrt{n}$ on the Kolmogorov distance in (5.23) for $c = 0.01$ and if $c = 0.5$ the bound is lower than $2.39/\sqrt{n}$. Simulations of the approximate Gaussian random fields for $c = 0.1$, $c = 0.5$, and $n = 9000$ superpositions are given in Figures 5.9d and 5.9e.

Example 5.3.5. By means of [72, Formula 5.10.1.13] a Poi(c) distributed random variable R for the spectral turning bands random field (5.22) results in a correlation function of the form

$$\begin{aligned}\rho(x,y) &= e^{-c}\sum_{n\in\mathbb{N}_0}\frac{c^n}{n!}P_n\big(\cos\big(d_{\mathbb{S}^2}(x,y)\big)\big)\\ &= e^{-2c\sin^2(d_{\mathbb{S}^2}(x,y)/2)}J_0\big(c\sin\big(d_{\mathbb{S}^2}(x,y)\big)\big), \quad c>0, x,y\in\mathbb{S}^2. \end{aligned} \tag{5.29}$$

Figure 5.10a displays this correlation function for different values of the parameter c. This correlation function looks similar to (5.26). The parameter c controls the speed at which the correlation function levels off at 0 and is similar to a scale factor. Simulations of the corresponding spectral turning bands field are given in Figures 5.10b and 5.10c. With a similar estimation as in Example 5.3.4, we get in (5.23) the bounds $3.6/\sqrt{n}$ if $c = 5$ and $3.27/\sqrt{n}$ if $c = 10$ and we have chosen $n = 9000$ for the simulations in Figures 5.10d and 5.10e. Figure 5.11 presents simulations with larger values of c. The bounds on the Kolmogorov distance are $3.13/\sqrt{n}$ for $c = 20$ and $3.06/\sqrt{n}$ in case $c = 50$.

Example 5.3.6. Our last example exhibits the practical limitations of the spectral turning bands method. Let us consider the correlation function corresponding to the power covariance function of [40], which we have already discussed in Section 4.9:

$$\rho(x,y) = 1-\Big(\frac{d_{\mathbb{S}^2}(x,y)}{c}\Big)^{\alpha}, \quad x,y\in\mathbb{S}^2. \tag{5.30}$$

Proposition 4.9.1 shows, that (5.30) is a correlation function for all $\alpha \in (0,1]$ and all $c \geq \pi/2$. Let us first consider the case $\alpha = 1$:

$$\rho(x,y) = 1-\frac{d_{\mathbb{S}^2}(x,y)}{c}, \quad c\geq\frac{\pi}{2}, x,y\in\mathbb{S}^2. \tag{5.31}$$

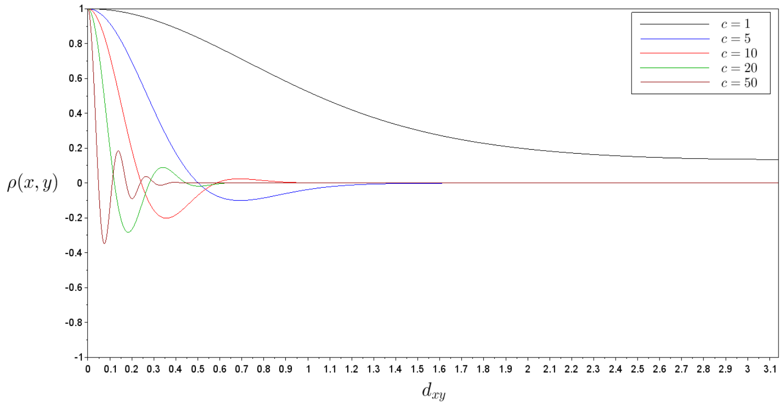
By [40] the coefficients $b_{n,2}$ of this correlation function are given by

$$b_{n,2} = \begin{cases} 1-\frac{\pi}{2c}, & n=0,\\ 0, & n \text{ even},\\ \frac{(2n+1)\pi}{2c}\Big(\frac{(n-2)!!}{2^{(n+1)/2}((n+1)/2)!}\Big)^2, & n \text{ odd}. \end{cases} \tag{5.32}$$

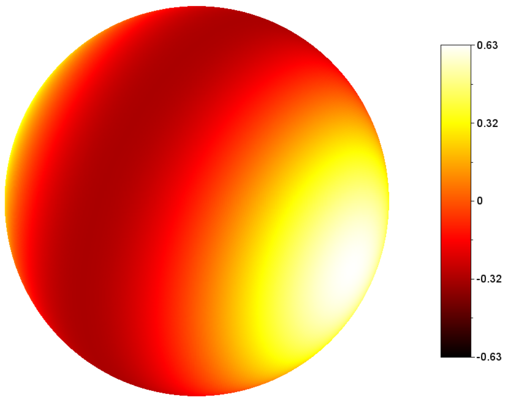
It is easy to see, that for all odd $n \geq 1$ the recursion

$$b_{n+2,2} = b_{n,2}\frac{2n+5}{2n+1}\Big(\frac{n}{n+3}\Big)^2$$

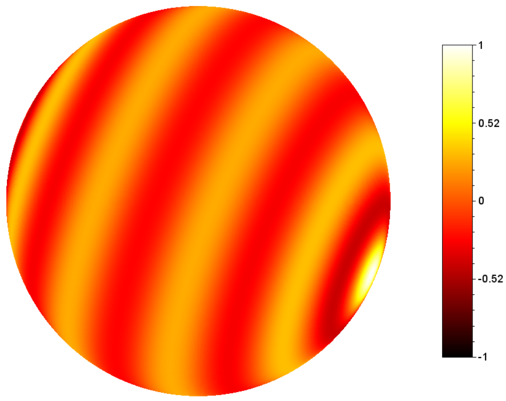
holds true, therefore a random variable R having the distribution $\sum_{n\in\mathbb{N}_0} b_{n,2}\varepsilon_n$ can conveniently be simulated with the inverse transformation method. Consequently, we can simulate a spectral turning bands random field (5.22) which has the correlation function (5.31). Figure 5.13a presents the correlation function (5.31).



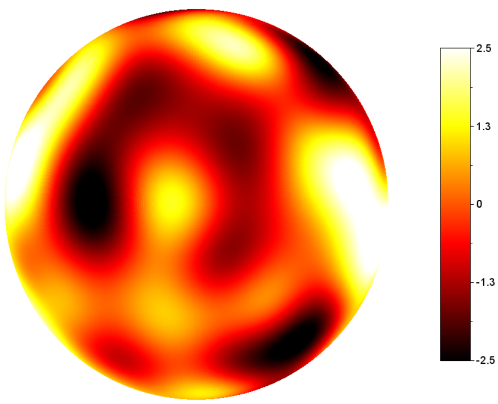
(a) Correlation function (5.29).



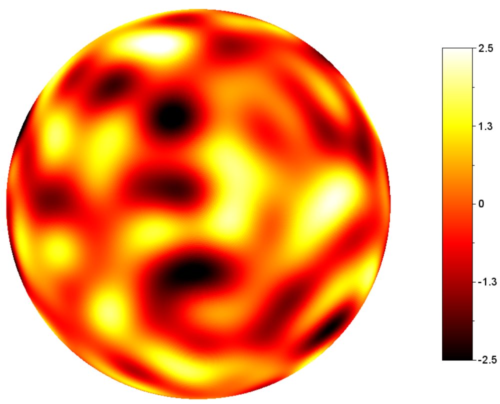
(b) Spectral turning bands random field with $c = 5$.



(c) Spectral turning bands random field with $c = 10$.

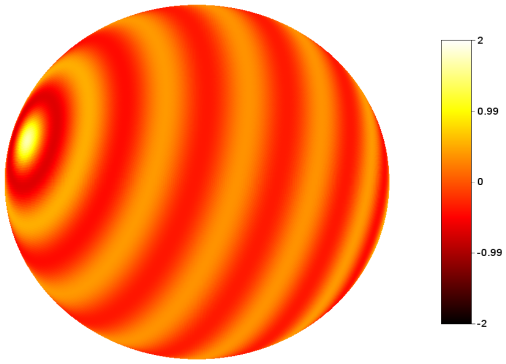


(d) Approximate Gaussian random field with $c = 5$ and $n = 9000$.

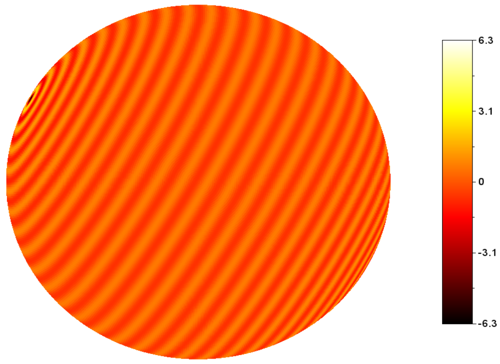


(e) Approximate Gaussian random field with $c = 10$ and $n = 9000$.

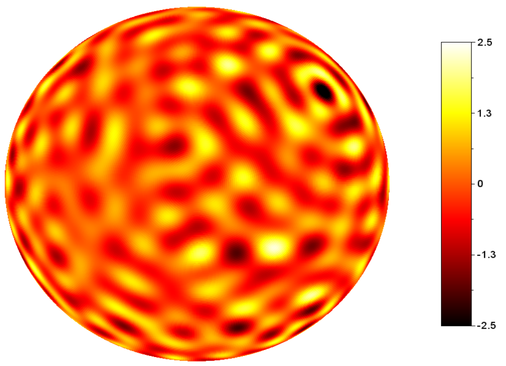
Figure 5.10.: Spectral turning bands random field with correlation function (5.29) and $c = 5$ or $c = 10$.



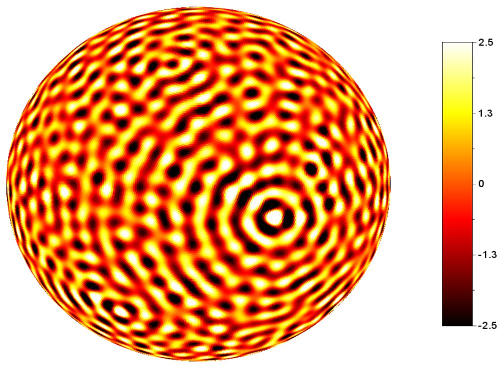
(a) Spectral turning bands random field with $c = 20$.



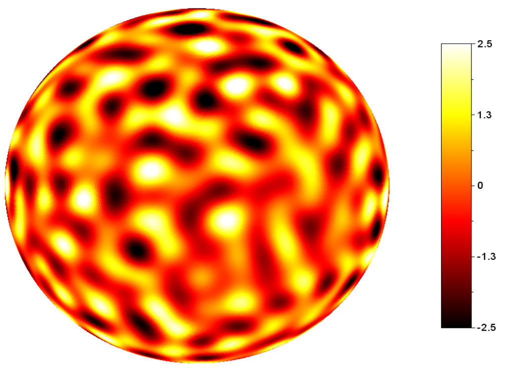
(b) Spectral turning bands random field with $c = 50$.



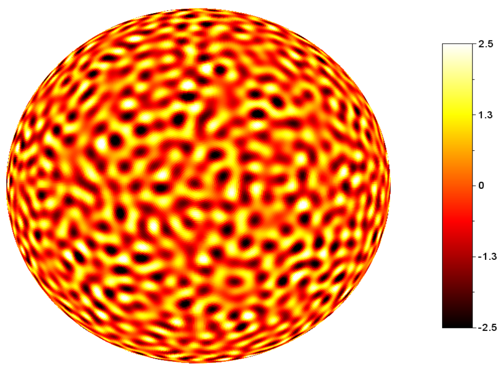
(c) Approximate Gaussian random field with $c = 20$ and $n = 20$.



(d) Approximate Gaussian random field with $c = 50$ and $n = 20$.

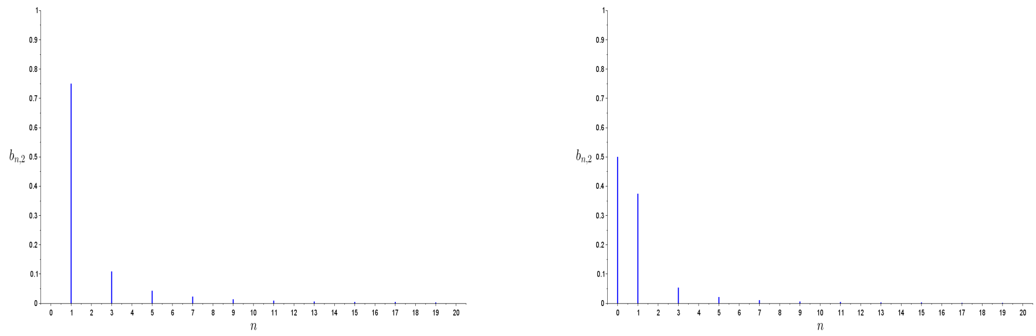


(e) Approximate Gaussian random field with $c = 20$ and $n = 9000$.



(f) Approximate Gaussian random field with $c = 50$ and $n = 9000$.

Figure 5.11.: Spectral turning bands random field with correlation function (5.29) and $c = 20$ or $c = 50$.



(a) $c = \pi/2$. (b) $c = \pi$.

Figure 5.12.: The coefficients $b_{0,2}, \ldots, b_{20,2}$.

In order to explain the outcomes of the simulations, we have to take a closer look at the coefficients $(b_{n,2}, n \in \mathbb{N}_0)$ in (5.32). Figure 5.12 displays the first few coefficients and shows, that the random variable R is likely to attain a small value in the simulations, which results in a spectral turning bands random field that looks like the one in Figure 5.13b (the simulated value of R is 1 in this figure). However, if we simulate repeatedly, occasionally a very large value of R and hence a random field that looks like the one in 5.13c occurs (the simulated value of R in Figure 5.13c is 151). The reason for this is the slow decay of the coefficients $(b_{n,2}, n \in \mathbb{N}_0)$. This can be seen as follows. For every odd index $2m+1$, $m \in \mathbb{N}_0$, we have from (5.32)

$$\begin{aligned} b_{2m+1,2} &= \frac{(4m+3)\pi}{2c}\Big(\frac{(2m)!}{2^{2m+1}m!(m+1)!}\Big)^2 \\ &= \frac{(4m+3)\pi}{2c}\Big(\frac{\Gamma(2m+1)}{2^{2m+1}\Gamma(m+1)\Gamma(m+2)}\Big)^2. \end{aligned} \tag{5.33}$$

Writing $\Gamma(2m+1) = \Gamma\big(2(m+1/2)\big)$, we obtain with the product theorem for the gamma function (e.g. [35, Formula 8.335])

$$b_{2m+1,2} = \frac{1}{8c}\frac{4m+3}{(m+2)^2}\Big(\frac{\Gamma\big(m+1/2\big)}{\Gamma(m+1)}\Big)^2, \quad m \in \mathbb{N}_0. \tag{5.34}$$

An application of Gautschi's inequality (for instance [26, Formula 5.6.4]) for the quotient of gamma functions yields

$$\frac{1}{2c}\frac{m+3/4}{(m+2)^2(m+1)} < b_{2m+1,2} < \frac{1}{2c}\frac{m+3/4}{(m+2)^2 m}, \quad m \in \mathbb{N}, \tag{5.35}$$

from which it follows that

$$\mathbb{E}(R) \geq \frac{3\pi}{32c} + \frac{7}{72c}\sum_{m\in\mathbb{N}}\frac{1}{m},$$

and we conclude that the spectral measure of (5.31) does not have a finite mean.

Figure 5.13d displays a simulation of an approximate Gaussian random field which is build from $n = 500$ simulations of spectral turning bands random fields with the correlation function (5.31). Out of this 500 simulations, there were 7 simulations for which the value of R was close to 90 or even exceeded this value, and these 7 simulations stand out from the rest. Because the correlation function (5.31) is isotropic and generally quite simple, we suspect that this outcome is not characteristic for the correlation function but can be explained with a rather slow convergence of the spectral turning bands random field to a Gaussian random field. Therefore, let us examine the bound on the Kolmogorov distance in (5.23) for this example. Using (5.33), the bound in (5.23) becomes

$$\left(\left(1-\frac{\pi}{2c}\right)+\frac{1}{8c}\left(\frac{\Gamma(1/4)}{\pi}\right)^2\sum_{m\in\mathbb{N}_0}\frac{(4m+3)^{5/2}}{(m+2)^2(2m+1)^{3/2}}\left(\frac{\Gamma(m+1/2)}{\Gamma(m+1)}\right)^2\right)\sqrt{\frac{2}{\pi n}}.$$

The quotient of gamma functions can be estimated with Gautschi's inequality. The other quotient in the sum can be estimated as follows:

$$\frac{(4m+3)^{5/2}}{(m+2)^2(2m+1)^{3/2}}\leq\frac{\big(4(m+1)\big)^{5/2}}{(m+1)^2\big(3/2(m+1)\big)^{3/2}}=\sqrt{\frac{2^{13}}{3^3}}(m+1)^{-1},\quad m\in\mathbb{N}.$$

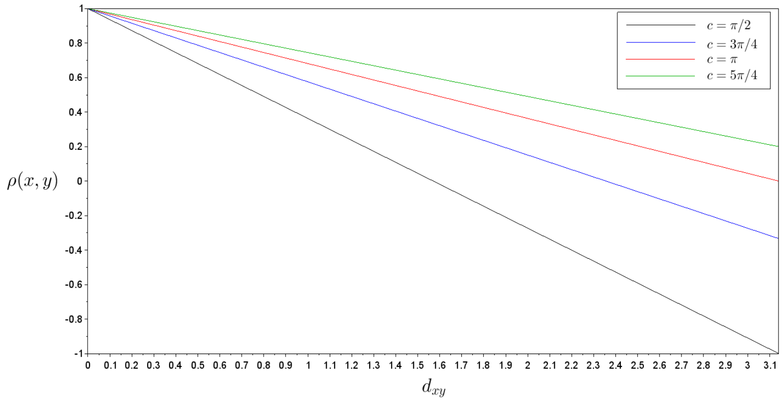
Therefore, the Kolmogorov distance of the marginal distributions is smaller than

$$\begin{aligned}&\left(\left(1-\frac{\pi}{2c}\right)+\frac{1}{8c}\left(\frac{\Gamma(1/4)}{\pi}\right)^2\left(\frac{3^{5/2}\pi}{4}+\sqrt{\frac{2^{13}}{3^3}}\sum_{m\in\mathbb{N}}\frac{1}{m(m+1)}\right)\right)\sqrt{\frac{2}{\pi n}}\\&\qquad=\left(\left(1-\frac{\pi}{2c}\right)+\frac{81\pi+2^{17/2}}{32\cdot 3^{3/2}c}\left(\frac{\Gamma(1/4)}{\pi}\right)^2\right)\sqrt{\frac{2}{\pi n}},\end{aligned}$$

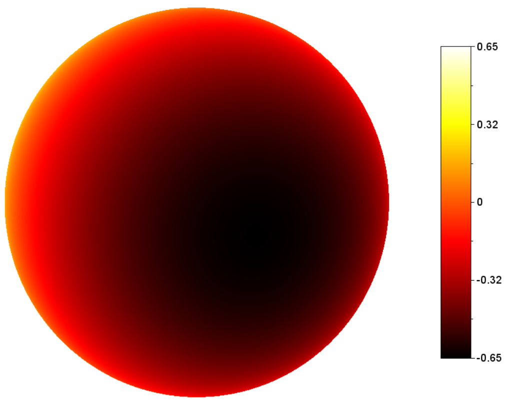
where the equality follows because the series in the first line is a telescope series from which it can be seen that the value of the series equals 1. This way, we get moderate bounds of approximately $2.51/\sqrt{n}$ in case $c=\pi/2$ and $1.66/\sqrt{n}$ if $c=\pi$. But the simulation of an approximate Gaussian random field with $n=10000$ superpositions in Figure 5.13e is not what we would expect from this bounds. Although the simulation looks better than the one in Figure 5.13d, it is still somewhat blurred and single simulations with large simulated value of R continue to stand out.

The practical explanation for this is the heavy-tailed spectral measure (5.35) of the correlation function (5.31), which results in many simulated values of R that are small and few simulated values which are very large. For instance, in the simulation depicted in Figure 5.13e we had 9029 simulated values of R out of 10000, which were lower or equal to 5, and 60 realizations of R exceeded 100. Another explanation could be a slow convergence to a Gaussian random field for this particular correlation function and the spectral turning bands method. Because we have seen that the bound on the Kolmogorov distance in (5.23) is comparably small here, we assume that the convergence of the multivariate distributions to that of a Gaussian random field must be disproportionally slow.

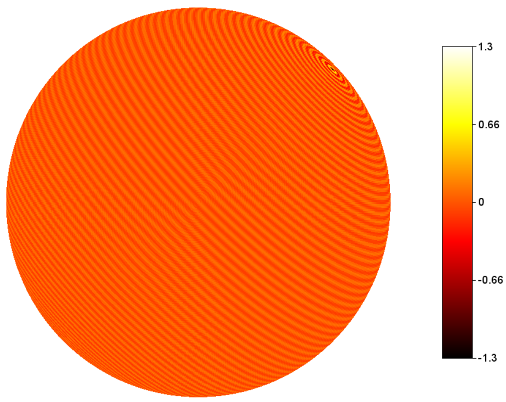
Note that the correlation function (5.31) can also be simulated with mosaic random fields. Indeed, we have shown the validity of (5.30) on any unit sphere $\mathbb{S}^d$ and for all $\alpha\in(0,1]$ in Proposition 4.9.1 with an explicit construction of a corresponding simple mosaic random field. Although the mosaic random field in the proof of Proposition 4.9.1 does also involve a heavy–tailed distribution in the Sibuya(α) distribution, the case $\alpha=1$ is unproblematic because the Sibuya(1) distribution



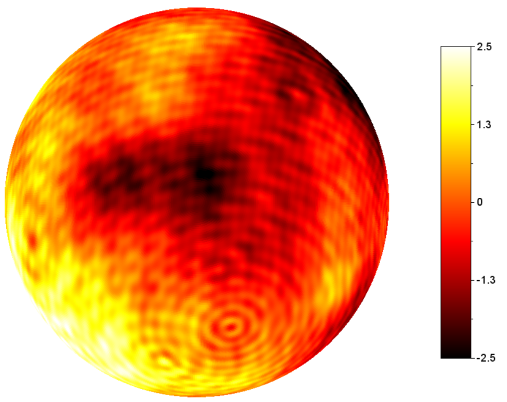
(a) Correlation function (5.31).



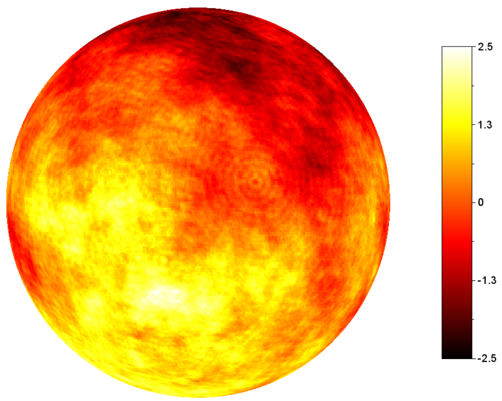
(b) Spectral turning bands random field with $c = \pi/2$.



(c) Spectral turning bands random field with $c = \pi/2$.



(d) Approximate Gaussian random field with $c = \pi/2$ and $n = 500$.



(e) Approximate Gaussian random field with $c = \pi/2$ and $n = 10000$.

Figure 5.13.: Spectral turning bands random field with correlation function (5.31).

is just the Dirac measure in 1. Figure 5.14 compares simulations based on mosaic random fields and spectral turning bands random field. Figures 5.14e and 5.14f confirm our expectation that simulations based on simple mosaic random fields are better suited for this particular correlation function than the spectral turning bands method.

Now that we know how to simulate the case $\alpha = 1$, we can for the spectral turning bands random field also use Lemma 4.8.1 to simulate the general case $\alpha \in (0, 1]$. In detail, if $(Z_i, i \in \mathbb{N})$ are independent copies of a spectral turning bands random field with the correlation function ρ in (5.31), U is a centered and normalized random variable, and N is Sibuya(α) distributed, then by Lemma 4.8.1 the correlation function ρ_α of the random field

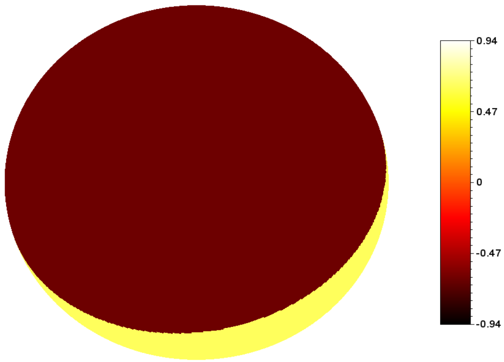
$$Z(x) = U \prod_{i=1}^{N} Z_i(x), \quad x \in \mathbb{S}^2,$$

is given by

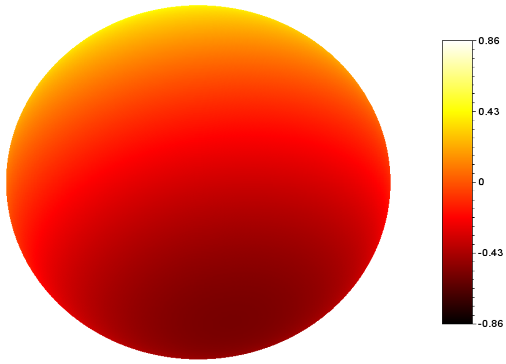
$$\rho_\alpha(x, y) = (\psi_N \circ \rho)(x, y) = 1 - \left(\frac{d_{\mathbb{S}^2}(x, y)}{c}\right)^\alpha, \quad \alpha \in (0, 1], c \geq \frac{\pi}{2}, x, y \in \mathbb{S}^2, \quad (5.36)$$

provided that U, N, and $(Z_i, i \in \mathbb{N})$ are independent (see the proof of Lemma 4.8.1). Figure 5.15a displays the correlation function (5.36) for $c = \pi/2$ and different α. Figure 5.15b depicts a spectral turning bands random field with this correlation function and $\alpha = 0.8$. Figures 5.15c, 5.15d, and 5.15e display the corresponding approximate Gaussian random fields. Because we have added another heavy-tailed distribution into our simulation, we have an even slower convergence to a Gaussian random field than in the $\alpha = 1$ example.

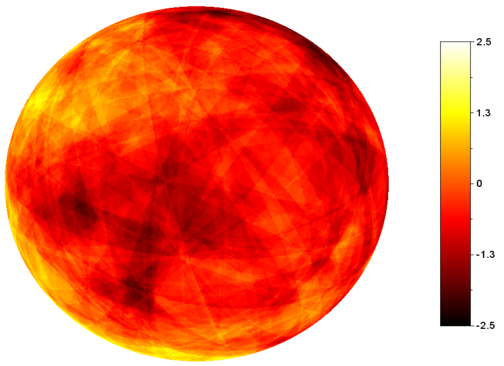
Figure 5.16 compares the spectral turning bands method with the mosaic random field model in the case $\alpha = 0.8$. Now that both methods include the simulation of slowly decaying distributions, there seems to be no difference in their performance.



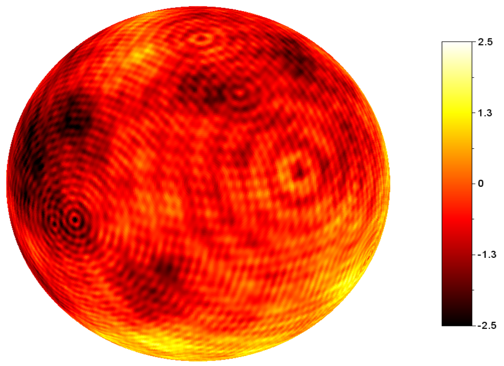
(a) Single realization of a simple mosaic random field with the correlation function (5.31).



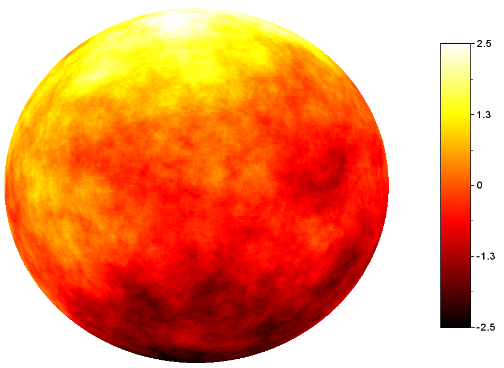
(b) Single realization of a spectral turning bands random field with the correlation function (5.31).



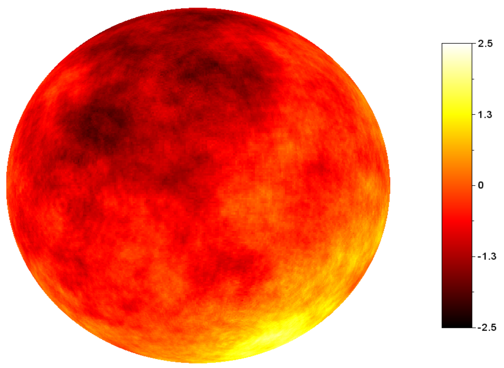
(c) Approximate Gaussian random field build from simple mosaic random fields, $n = 500$.



(d) Approximate Gaussian random field build from spectral turning bands random fields, $n = 500$.

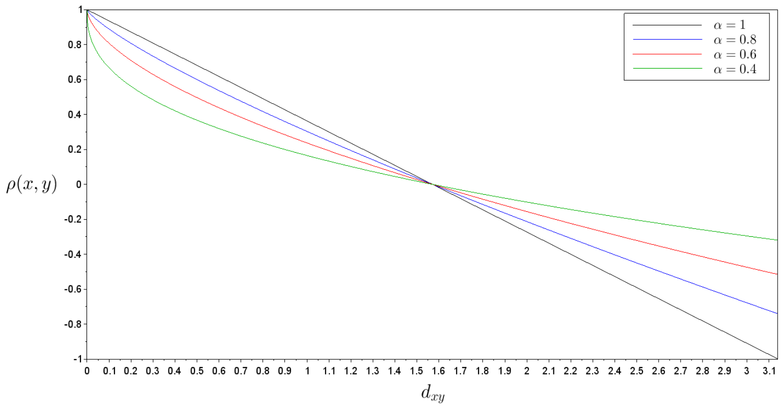


(e) Approximate Gaussian random field build from simple mosaic random fields, $n = 20000$.

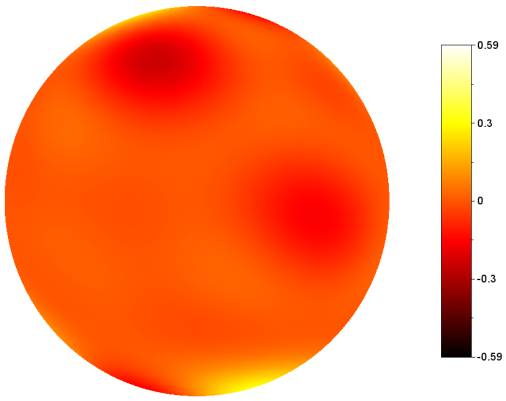


(f) Approximate Gaussian random field build from spectral turning bands random fields, $n = 20000$.

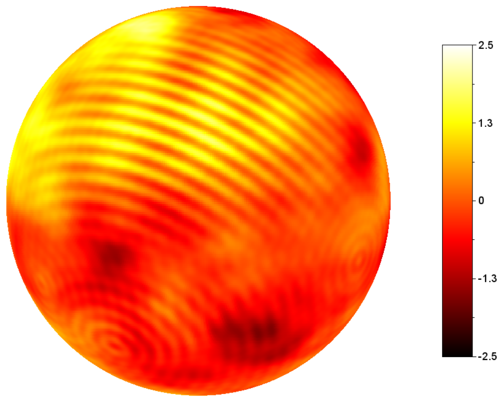
Figure 5.14.: Comparison of the mosaic random field model and the spectral turning bands method for the correlation function (5.31).



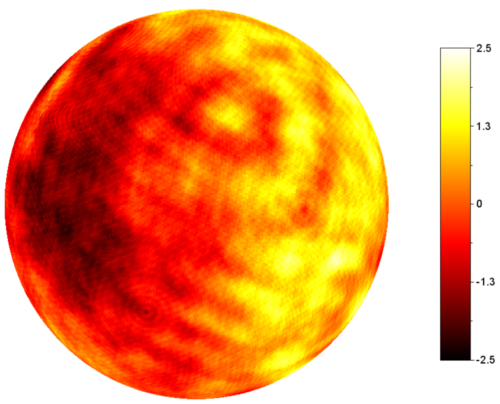
(a) Correlation function (5.36) with $c = \pi/2$.



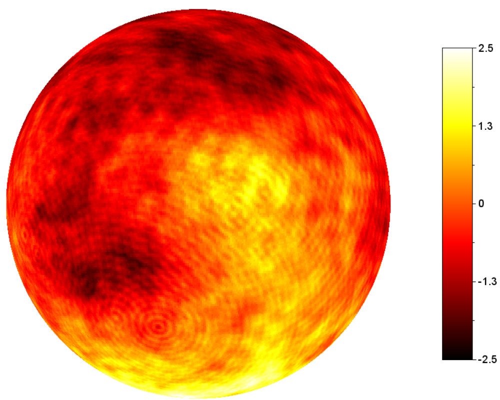
(b) Spectral turning bands random field with $\alpha = 0.8$ and $c = \pi/2$.



(c) Approximate Gaussian random field with $\alpha = 0.8$, $c = \pi/2$, and $n = 500$.

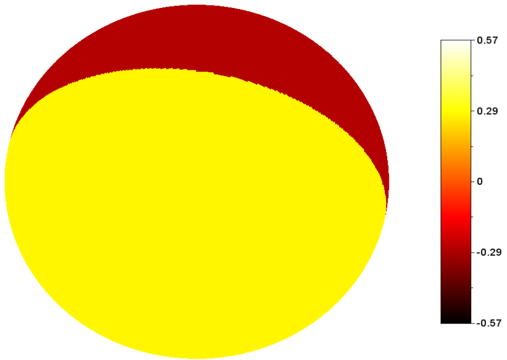


(d) Approximate Gaussian random field with $\alpha = 0.8$, $c = \pi/2$, and $n = 5000$.

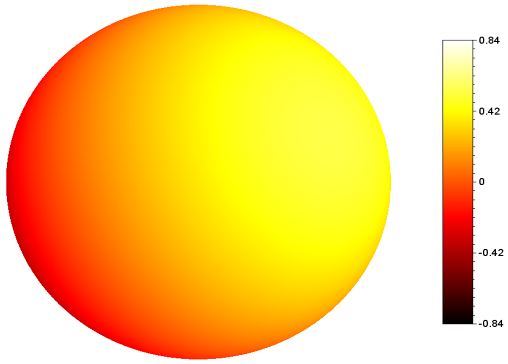


(e) Approximate Gaussian random field with $\alpha = 0.8$, $c = \pi/2$, and $n = 10000$.

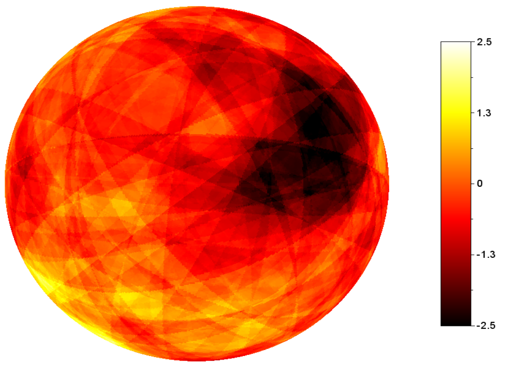
Figure 5.15.: Spectral turning bands random field with correlation function (5.36).



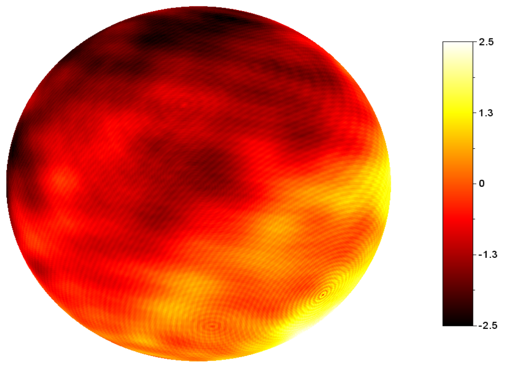
(a) Single realization of a simple mosaic random field with the correlation function (5.36) with $\alpha = 0.8$ and $c = \pi/2$.



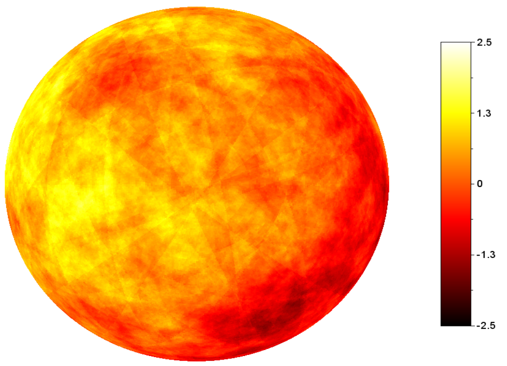
(b) Single realization of a spectral turning bands random field with the correlation function (5.36) with $\alpha = 0.8$ and $c = \pi/2$.



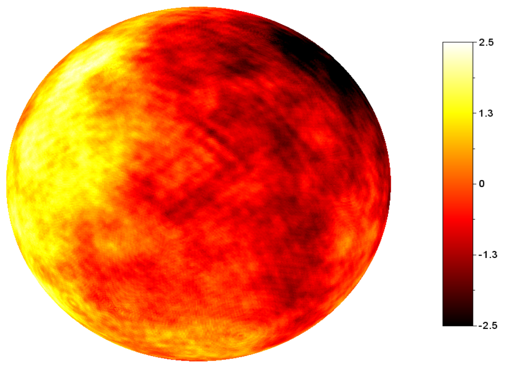
(c) Approximate Gaussian random field build from simple mosaic random fields, $n = 500$.



(d) Approximate Gaussian random field build from spectral turning bands random fields, $n = 500$.



(e) Approximate Gaussian random field build from simple mosaic random fields, $n = 20000$.



(f) Approximate Gaussian random field build from spectral turning bands random fields, $n = 20000$.

Figure 5.16.: Comparison of the mosaic random field and the spectral turning bands method for the correlation function (5.36) with $\alpha = 0.8$ and $c = \pi/2$.

Part III.

Appendices

A. Kolmogorov-Chentsov on $\mathbb{S}^2$ - Auxiliary Results

A.1. Distances in the Grids $(D_n, n \in \mathbb{N})$ on $\mathbb{S}^2$

In this appendix we present the proofs of auxiliary results that were presented in Sections 1.2 and 3.1. We develop the the techniques that allow us to make quantitative and qualitative statements about the distances of grid points in the spherical grid on $\mathbb{S}^2$.

Consider the metric space $(\mathbb{S}^2, d_{\mathbb{S}^2})$, where the great circle metric $d_{\mathbb{S}^2}$ is defined by

$$d_{\mathbb{S}^2}(x,y) = \arccos\big(\langle x,y\rangle\big), \quad x,y \in \mathbb{S}^2.$$

The spherical coordinate map ϕ_2 is defined as

$$\phi_2 : \begin{cases} [0,\pi]\times[0,2\pi) & \longrightarrow \quad \mathbb{S}^2, \\ (\theta,\varphi) & \longmapsto \quad \phi_2(\theta,\varphi) = \big(\cos(\varphi)\sin(\theta), \sin(\varphi)\sin(\theta), \cos(\theta)\big). \end{cases}$$

On $(\mathbb{S}^2, d_{\mathbb{S}^2})$ it is often convenient to express the great circle distance $d_{\mathbb{S}^2}(x,y)$ in terms of spherical coordinates. Using the addition theorem for the cosine function, we have

$$d_{\mathbb{S}^2}(x,y) = \arccos\big(g(\theta_x, \theta_y, |\varphi_x - \varphi_y|)\big) \tag{A.1}$$

with the function

$$g : \begin{cases} \mathbb{R}^3 & \longrightarrow \quad \mathbb{R}, \\ (x,y,z) & \longmapsto \quad g(x,y,z) = \sin(x)\sin(y)\cos(z) + \cos(x)\cos(y). \end{cases} \tag{A.2}$$

Let us now take the spherical grid on $\mathbb{S}^2$ from Section 1.2:

$$D_n = \Big\{\phi_2\Big(k\frac{\pi}{2^{n-1}}, l\frac{\pi}{2^{n-1}}\Big)\Big| k = 0,1,\dots,2^{n-1}, l = 0,1,\dots,2^n-1\Big\}, \quad n \in \mathbb{N}.$$

For any $n \in \mathbb{N}$, the number δ_n^0 is defined as the minimal distance of distinct grid points in the n-grid, i.e.

$$\delta_n^0 = \min_{x,y\in D_n, x\neq y} d_{\mathbb{S}^2}(x,y), \quad n \in \mathbb{N}.$$

It is intuitively clear, that the minimal distance of grid points is attained somewhere at the poles. The following lemma justifies this intuition.

Lemma A.1.1. *For all $n \geq 2$ we have*

$$\delta_n^0 = \arccos\Big(\sin^2\Big(\frac{\pi}{2^{n-1}}\Big)\cos\Big(\frac{\pi}{2^{n-1}}\Big) + \cos^2\Big(\frac{\pi}{2^{n-1}}\Big)\Big). \tag{A.3}$$

Proof. Let $n \geq 2$. If $x, y \in D_n$, there are $k_x, k_y \in \{0, 1, \ldots, 2^{n-1}\}$ and $l_x, l_y \in \{0, 1, \ldots, 2^n - 1\}$ such that X can be written as $x = \phi_2\big(k_x\pi/2^{n-1}, l_x\pi/2^{n-1}\big)$ and for y we have $y = \phi_2\big(k_y\pi/2^{n-1}, l_y\pi/2^{n-1}\big)$. In view of (A.1) we have

$$d_{\mathbb{S}^2}(x, y) = \arccos\Big(g\Big(k_x\frac{\pi}{2^{n-1}}, k_y\frac{\pi}{2^{n-1}}, |l_x - l_y|\frac{\pi}{2^{n-1}}\Big)\Big).$$

In order to minimize $d_{\mathbb{S}^2}(x, y)$ we have to maximize

$$\begin{aligned} &g\Big(k_x\frac{\pi}{2^{n-1}}, k_y\frac{\pi}{2^{n-1}}, |l_x - l_y|\frac{\pi}{2^{n-1}}\Big) \\ &\quad = \sin\Big(k_x\frac{\pi}{2^{n-1}}\Big)\sin\Big(k_y\frac{\pi}{2^{n-1}}\Big)\cos\Big(|l_x - l_y|\frac{\pi}{2^{n-1}}\Big) + \cos\Big(k_x\frac{\pi}{2^{n-1}}\Big)\cos\Big(k_y\frac{\pi}{2^{n-1}}\Big). \end{aligned} \tag{A.4}$$

If $k_x \neq k_y$, it is optimal to choose $l_x = l_y$ so that

$$g\Big(k_x\frac{\pi}{2^{n-1}}, k_y\frac{\pi}{2^{n-1}}, 0\Big) = \cos\Big(|k_x - k_y|\frac{\pi}{2^{n-1}}\Big).$$

Consequently, we maximize g in the case $k_x \neq k_y$ by picking $k \in \{0, 1, \ldots, 2^{n-1} - 1\}$ and $l \in \{0, 1, \ldots, 2^n - 1\}$ and choosing $x = \phi_2\big(k\pi/2^{n-1}, l\pi/2^{n-1}\big)$ and furthermore $y = \phi_2\big((k+1)\pi/2^{n-1}, l\pi/2^{n-1}\big)$. This leaves us with

$$g\Big(k\frac{\pi}{2^{n-1}}, (k+1)\frac{\pi}{2^{n-1}}, 0\Big) = \cos\Big(\frac{\pi}{2^{n-1}}\Big)$$

as a candidate for the maximum in (A.4). In the other case, it follows from the monotonicity of the cosine function that it is optimal to take any $l_x, l_y \in \{0, 1, \ldots, 2^n - 1\}$ such that $|l_x - l_y| \bmod (2^n - 2) = 1$, e.g. $l_x = 0, l_y = 1$. Because we require $x \neq y$, neither $k_x = k_y = 0$ nor $k_x = k_y = 2^{n-1}$ are permissible choices, so that we have to find $k \in \{1, \ldots, 2^{n-1} - 1\}$ for which

$$g\Big(k\frac{\pi}{2^{n-1}}, k\frac{\pi}{2^{n-1}}, \frac{\pi}{2^{n-1}}\Big) = \sin^2\Big(k\frac{\pi}{2^{n-1}}\Big)\cos\Big(\frac{\pi}{2^{n-1}}\Big) + \cos^2\Big(k\frac{\pi}{2^{n-1}}\Big) \tag{A.5}$$

becomes maximal. A straightforward optimization shows that (A.5) is maximal for the boundary values $k = 1$ or $k = 2^{n-1} - 1$. We have therefore the candidates

$$\begin{aligned} g\Big(\frac{\pi}{2^{n-1}}, \frac{\pi}{2^{n-1}}, \frac{\pi}{2^{n-1}}\Big) &= \sin^2\Big(\frac{\pi}{2^{n-1}}\Big)\cos\Big(\frac{\pi}{2^{n-1}}\Big) + \cos^2\Big(\frac{\pi}{2^{n-1}}\Big) \\ \text{and} \quad g\Big(0, \frac{\pi}{2^{n-1}}, 0\Big) &= \cos\Big(\frac{\pi}{2^{n-1}}\Big) \end{aligned}$$

for the maximum in (A.4). The assertion of the lemma then follows from

$$\begin{aligned} \sin^2\Big(\frac{\pi}{2^{n-1}}\Big)\cos\Big(\frac{\pi}{2^{n-1}}\Big) + \cos^2\Big(\frac{\pi}{2^{n-1}}\Big) - \cos\Big(\frac{\pi}{2^{n-1}}\Big) &= -\cos\Big(\frac{\pi}{2^{n-1}}\Big)^3 + \cos\Big(\frac{\pi}{2^{n-1}}\Big)^2 \\ &= \cos\Big(\frac{\pi}{2^{n-1}}\Big)^2\Big(1 - \cos\Big(\frac{\pi}{2^{n-1}}\Big)\Big), \end{aligned}$$

since the last quantity is greater than or equal to 0 for all $n \in \mathbb{N}$. ☐

As explained in Section 3.1, we think of the sets $(\pi_n, n \in \mathbb{N})$ as the sets which define the notion of neighborhood in the grids. Let us define $(\pi_n, n \in \mathbb{N})$ as in

Section 3.1 with the help of the spherical maximum metric d_∞, i.e. for $n \in \mathbb{N}$ we define the set π_n by

$$\pi_n = \left\{ \langle x, y \rangle_{up} \,\middle|\, x, y \in D_n, d_\infty(x, y) \leq \frac{\pi}{2^{n-1}} \right\} \tag{A.6}$$

where

$$\begin{aligned} d_\infty(x, y) &= \min\{d^1_\infty(x, y), d^2_\infty(x, y)\} \\ \text{with} \quad d^1_\infty(x, y) &= \max\bigl\{|\theta_x - \theta_y|, \min\{|\varphi_x - \varphi_y|, 2\pi - |\varphi_x - \varphi_y|\}\bigr\} \\ \text{and} \quad d^2_\infty(x, y) &= \min\{\theta_x + \theta_y, 2\pi - (\theta_x + \theta_y)\}, \end{aligned} \tag{A.7}$$

for all $x, y \in \mathbb{S}^2$. Defining π_n this way, for any grid point $x \in D_n$ away from the poles the grid points $y \in D_n$ which satisfy $\langle x, y \rangle_{up} \in \pi_n$ are exactly the 8 adjacent grid points and x itself. It was mentioned in Section 1.2, that we need to identify the surrounding grid points of a point $x \in D_n$ as neighbors of x in order for Condition (c) of Theorem 1.2.1 to be satisfied. Let Δ_n denote the maximal distance of neighboring grid points in the n-grid with this notion of neighborhood, i.e.

$$\Delta_n = \max_{\langle x,y \rangle_{up} \in \pi_n} d_{\mathbb{S}^2}(x, y), \quad n \in \mathbb{N}.$$

We have the following formula for Δ_n:

Lemma A.1.2. *For all $n \geq 2$ it is true that*

$$\Delta_n = \arccos\Bigl(\cos^2\Bigl(\frac{\pi}{2^{n-1}}\Bigr)\Bigr).$$

Proof. Let $n \geq 2$ and $x, y \in D_n$, i.e. there are some $k_x, k_y \in \{0, 1, \ldots, 2^{n-1}\}$ and $l_x, l_y \in \{0, 1, \ldots, 2^n - 1\}$ such that $x = \phi_2\bigl(k_x\pi/2^{n-1}, l_x\pi/2^{n-1}\bigr)$ and $y = \phi_2\bigl(k_y\pi/2^{n-1}, l_y\pi/2^{n-1}\bigr)$. It follows from the definition of d_∞ in (A.7) that the assertion $\langle x, y \rangle_{up} \in \pi_n$ is equivalent to

$$\Bigl(|k_x - k_y| \leq 1 \text{ and } |l_x - l_y| \bmod (2^n - 1) \leq 1\Bigr) \text{ or } \Bigl((k_x + k_y) \bmod (2^n - 1) \leq 1\Bigr). \tag{A.8}$$

Let us assume, that the first statement in (A.8) holds true. Then in particular $|l_x - l_y| \in \{0, 1, 2^n - 1\}$ since $l_x, l_y \in \{0, 1, \ldots, 2^n - 1\}$. Hence $|l_x - l_y|\pi/2^{n-1}$ is an element of $\bigl\{0, \pi/2^{n-1}, 2\pi - \pi/2^{n-1}\bigr\}$ and consequently $\cos\bigl(|l_x - l_y|\pi/2^{n-1}\bigr) \in \bigl\{1, \cos\bigl(\pi/2^{n-1}\bigr)\bigr\}$ by the symmetry of the cosine function. For any k_x, k_y it is therefore optimal to choose l_x, l_y such that $|l_x - l_y| \bmod (2^n - 2) = 1$ in order to minimize (A.4). Concerning k_x, k_y we are free to make any choice that satisfies $|k_x - k_y| \leq 1$. We have by the addition theorem for the cosine function for all $x, y, z \in \mathbb{R}$

$$\begin{aligned} g(x, y, z) &= \bigl(\cos(|x - y|) - \cos(x)\cos(y)\bigr)\cos(z) + \cos(x)\cos(y) \\ &= \cos(|x - y|)\cos(z) + \cos(x)\cos(y)\bigl(1 - \cos(z)\bigr), \end{aligned}$$

so that we are left with the minimization of

$$\begin{aligned} &g\Bigl(k_x\frac{\pi}{2^{n-1}}, k_y\frac{\pi}{2^{n-1}}, \frac{\pi}{2^{n-1}}\Bigr) \\ &\quad = \cos\Bigl(|k_x - k_y|\frac{\pi}{2^{n-1}}\Bigr)\cos\Bigl(\frac{\pi}{2^{n-1}}\Bigr) + \cos\Bigl(k_x\frac{\pi}{2^{n-1}}\Bigr)\cos\Bigl(k_y\frac{\pi}{2^{n-1}}\Bigr)\Bigl(1 - \cos\Bigl(\frac{\pi}{2^{n-1}}\Bigr)\Bigr) \end{aligned} \tag{A.9}$$

subject to the condition $|k_x - k_y| \leq 1$. Because the second summand in (A.9) is non-negative for any choice of $k_x, k_y \in \{0, 1, \dots, 2^{n-1}\}$ with $|k_x - k_y| \leq 1$, it is easy to see that (A.9) can not become smaller than

$$g\Big(\frac{\pi}{2}, \frac{\pi}{2} + \frac{\pi}{2^{n-1}}, \frac{\pi}{2^{n-1}}\Big) = \cos^2\Big(\frac{\pi}{2^{n-1}}\Big), \tag{A.10}$$

corresponding to $x = \phi_2\big(2^{n-2}\pi/2^{n-1}, \pi/2^{n-1}\big)$ and $y = \phi_2\big((2^{n-2}+1)\pi/2^{n-1}, \pi/2^{n-1}\big)$.

In the case $(k_x + k_y) \bmod (2^n - 1) \leq 1$ at least one of the grid points x, y must be a pole and the other can be located on the adjacent circle of latitude or be a pole as well. The function g may therefore attain the following values:

$$g\Big(k_x\frac{\pi}{2^{n-1}}, k_y\frac{\pi}{2^{n-1}}, |l_x - l_y|\frac{\pi}{2^{n-1}}\Big) = \begin{cases} 1, & x, y \text{ poles}, \\ \cos\big(\frac{\pi}{2^{n-1}}\big), & \text{otherwise}. \end{cases}$$

Both of these outcomes are however larger than (A.10), which shows that

$$\Delta_n = \arccos\Big(g\Big(\frac{\pi}{2}, \frac{\pi}{2} + \frac{\pi}{2^{n-1}}, \frac{\pi}{2^{n-1}}\Big)\Big) = \arccos\Big(\cos^2\Big(\frac{\pi}{2^{n-1}}\Big)\Big)$$

for all $n \geq 2$. □

In view of Condition (b) in Theorem 1.2.1 we are interested in the behavior of the quotients Δ_n/δ_n^0, $n \in \mathbb{N}$. Since $\Delta_n = \arccos(a_n)$ and $\delta_n^0 = \arccos(b_n)$ for two sequences $(a_n, n \in \mathbb{N})$ and $(b_n, n \in \mathbb{N})$ which increase to 1 and because $\arccos(1) = 0$, the behavior of the sequence $(\Delta_n/\delta_n^0, n \in \mathbb{N})$ depends on the behavior of the sequences $(a_n, n \in \mathbb{N})$ and $(b_n, n \in \mathbb{N})$. The following result allows us to reduce the question of the convergence of $(\arccos(a_n)/\arccos(b_n), n \in \mathbb{N})$ to that of $(b_n\sqrt{1 - a_n^2}/(a_n/\sqrt{1 - b_n^2}), n \in \mathbb{N})$. This will enable us to show that $(\Delta_n/\delta_n^0, n \in \mathbb{N})$ diverges to ∞, and hence that Condition (b) in Theorem 1.2.1 does not hold true for the grids (A.3) and the metric space $(\mathbb{S}^2, d_{\mathbb{S}^2})$.

We assume that an inequality of the form (A.11) is well-known to specialists, but since we could not find it in the literature, we give a proof here.

Lemma A.1.3. *For any $\varepsilon > 0$ there exists $C \in (0, 1)$ such that*

$$(1 - \varepsilon)\frac{y\sqrt{1 - x^2}}{x\sqrt{1 - y^2}} < \frac{\arccos(x)}{\arccos(y)} < (1 + \varepsilon)\frac{y\sqrt{1 - x^2}}{x\sqrt{1 - y^2}} \tag{A.11}$$

for all $x, y \in (C, 1)$.

Proof. By [26, Formula 4.23.3] the inverse tangent function may be represented as

$$\arctan(x) = \int_0^x \frac{1}{1 + t^2}\, dt, \quad x > 0.$$

A substitution yields

$$\arctan(x) = x\int_0^1 \frac{1}{1 + x^2t^2}\, dt. \tag{A.12}$$

By dominated convergence we find that

$$\lim_{x \searrow 0} \int_0^1 \frac{1}{1 + x^2t^2}\, dt = 1,$$

hence the quotient $\int_0^1 1/(1+x^2t^2)\,dt / \int_0^1 1/(1+y^2t^2)\,dt$ converges to 1 as x, y fall to 0 and consequently

$$\frac{\int_0^1 1/(1+x^2t^2)\,dt}{\int_0^1 1/(1+y^2t^2)\,dt} \in (1-\varepsilon, 1+\varepsilon) \tag{A.13}$$

for all $\varepsilon > 0$ and all x, y which are sufficiently small. By [35, Formula 1.624.5] we have

$$\frac{\arccos(x)}{\arccos(y)} = \frac{\arctan\bigl(\sqrt{1-x^2}/x\bigr)}{\arctan\bigl(\sqrt{1-y^2}/y\bigr)} \tag{A.14}$$

for all $x, y \in (0,1)$. Because $\sqrt{1-x^2}/x$ falls to 0 as x increases to 1, for all $\varepsilon > 0$ there is a constant $C \in (0,1)$ such that for all $x, y \in (C, 1)$ we can combine (A.14), (A.12), and (A.13) to get (A.11). □

Corollary A.1.4. *The sequence $(\Delta_n/\delta_n^0, n \in \mathbb{N})$ diverges to ∞ as $n \to \infty$.*

Proof. By Lemmas A.1.1 and A.1.2 we have

$$\frac{\Delta_n}{\delta_n^0} = \frac{\arccos\bigl(\cos^2(\pi/2^{n-1})\bigr)}{\arccos\Bigl(\sin^2(\pi/2^{n-1})\cos(\pi/2^{n-1}) + \cos^2(\pi/2^{n-1})\Bigr)}.$$

As $n \to \infty$, both arguments of the inverse cosine functions increase to 1. In view of Lemma A.1.3, for any $\varepsilon \in (0,1)$ there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\frac{\Delta_n}{\delta_n^0} > (1-\varepsilon)\frac{\sqrt{1-\cos^4(\pi/2^{n-1})}\Bigl(\sin^2(\pi/2^{n-1})\cos(\pi/2^{n-1}) + \cos^2(\pi/2^{n-1})\Bigr)}{\sqrt{1-\Bigl(\sin^2(\pi/2^{n-1})\cos(\pi/2^{n-1}) + \cos^2(\pi/2^{n-1})\Bigr)^2}\cos^2(\pi/2^{n-1})}. \tag{A.15}$$

Squaring inequality (A.15) and rearranging gives

$$\begin{aligned}
\Bigl(\frac{\Delta_n}{\delta_n^0}\Bigr)^2 &> (1-\varepsilon)^2(1 + a_n + 2\,b_n) \\
\text{with}\quad a_n &= \frac{\sin^4(\pi/2^{n-1})}{c_n}, \\
b_n &= \frac{\sin^2(\pi/2^{n-1})\cos(\pi/2^{n-1})}{c_n}, \\
\text{and}\quad c_n &= \cos^2(\pi/2^{n-1}) - \sin^4(\pi/2^{n-1})\cos^4(\pi/2^{n-1}) \\
&\qquad - 2\sin^2(\pi/2^{n-1})\cos^5(\pi/2^{n-1}) - \cos^6(\pi/2^{n-1}).
\end{aligned}$$

By the Pythagorean trigonometric identity, the double-angle formula and the triple-angle formula we find

$$a_n = \frac{4\,\bigl(1+\cos(\pi/2^{n-1})\bigr)}{\cos^2(\pi/2^{n-1})\Bigl(\cos(\pi/2^{n-1}) + 2\cos(\pi/2^{n-2}) - \cos(3\pi/2^{n-1}) + 6\Bigr)},$$

so that $a_n \geq 0$ for all $n \geq 3$. Similarly,

$$b_n = \frac{2}{\sin^2(\pi/2^n)\cos(\pi/2^{n-1})\Bigl(\cos(\pi/2^{n-1}) + 2\cos(\pi/2^{n-2}) - \cos(3\pi/2^{n-1}) + 6\Bigr)},$$

from which it is seen that b_n and hence also Δ_n/δ_n^0 diverge to ∞ as $n \to \infty$. □

A.2. Application of Theorem 2.4.1 to $\mathbb{S}^2$

The next lemma shows that Condition (NC) in Theorem 2.4.1 is true if we would have chosen the metric d_∞ instead of $d_{\mathbb{S}^2}$. In combination with the following lemma we then obtain the validity of Condition (NC) in Theorem 2.4.1 for the metric space $(\mathbb{S}^2, d_{\mathbb{S}^2})$ and the choice of $(\pi_n, n \in \mathbb{N})$ given in (A.6). The following proof is motivated by the arguments given in Section 4 of [70].

Lemma A.2.1. *Let $n \geq 3$, $m \geq n$, and $x, y \in D_m$. Then $d_\infty(x,y) \leq \pi/2^{n-1}$ implies the existence of $x_1, y_1 \in D_{m-1}, \dots, x_{m-n}, y_{m-n} \in D_n$ such that*

$$\langle x, x_1\rangle_{up}, \langle y, y_1\rangle_{up} \in \pi_m, \dots, \langle x_{m-n-1}, x_{m-n}\rangle_{up}, \langle y_{m-n-1}, y_{m-n}\rangle_{up} \in \pi_{n+1},$$
$$\text{and} \quad \langle x_{m-n}, y_{m-n}\rangle_{up} \in \pi_n.$$

Proof. We show the assertion by induction over $m \geq n$. In case $m = n$, we have to show that $\langle x, y\rangle_{up} \in \pi_n$, but this follows from the definition of π_n and because we have $d_\infty(x,y) \leq \pi/2^{n-1}$. Now let us assume that the statement of the lemma is true for some $m \geq n$ and suppose $x, y \in D_{m+1}$ such that $d_\infty(x,y) \leq \pi/2^{n-1}$. With $k_x, k_y \in \{0, 1, \dots, 2^m\}$ and $l_x, l_y \in \{0, 1, \dots, 2^{m+1}-1\}$ the grid points $x, y \in D_{m+1}$ can be represented as $x = \phi_2(k_x\pi/2^m, l_y\pi/2^m)$ and $y = \phi_2(k_y\pi/2^m, l_y\pi/2^m)$. From the definition of d_∞ it follows that the statement $d_\infty(x,y) \leq \pi/2^{n-1}$ can be divided into three cases:

Case 1: $k_x + k_y \leq 2^{m-n+1}$,
Case 2: $k_x + k_y \geq 2^{m+1} - 2^{m-n+1}$,
Case 3: $k_x + k_y \in \{2^{m-n+1}+1, \dots, 2^{m+1} - 2^{m-n+1} - 1\}$ and $|k_x - k_y| \leq 2^{m-n+1}$ and $|l_x - l_y| \bmod (2^{m+1} - 2^{m-n+1}) \leq 2^{m-n+1}$.

In each case we will now define grid points $x' = \phi_2(k'_x\pi/2^{m-1}, l'_x\pi/2^{m-1}) \in D_m$ and $y' = \phi_2(k'_y\pi/2^{m-1}, l'_y\pi/2^{m-1}) \in D_m$ such that $\langle x, x'\rangle_{up}, \langle y, y'\rangle_{up} \in \pi_{m+1}$ and $d_\infty(x', y') \leq \pi/2^{n-1}$. Then, in view of the induction hypothesis, the lemma is proven.

Let us begin with the case of x, y being near the North Pole, i.e. $k_x + k_y \leq 2^{m-n+1}$. In that case we simply move north in the polar coordinate to the nearest grid points $x', y' \in D_m$. For the azimuthal coordinate the direction is irrelevant as long as we arrive at a grid point in D_m. This is because d_∞ has a different behavior for points at the poles. If x or y are already elements of $D_{m+1} \cap D_m$, we do not have to move at all. In detail, choose

$$j'_z = \max\Big\{v \in \mathbb{N}_0 \,\Big|\, v \leq \frac{j_z}{2}\Big\}, \quad j \in \{k, l\}, z \in \{x, y\},$$

then $k'_z \in \{0, 1, \dots, 2^{m-1}\}$, $l'_z \in \{0, 1, \dots, 2^m - 1\}$ for $z \in \{x, y\}$, so that we have $x', y' \in D_m$. Because $k_x - 2k'_x, l_x - 2l'_x \in \{0, 1\}$ it follows from the inequality $d_\infty(x, x') \leq d^1_\infty(x, x')$ that $d_\infty(x, x') \in \{0, \pi/2^m\}$, so that $\langle x, x'\rangle_{up} \in \pi_{m+1}$. Analogously, we have $\langle y, y'\rangle_{up} \in \pi_{m+1}$. The statement $d_\infty(x', y') \leq \pi/2^{n-1}$ can be obtained from $k_x + k_y \leq 2^{m-n+1}$ and $2k'_x + 2k'_y \leq k_x + k_y$ so that

$$d_\infty(x', y') \leq d^2_\infty(x', y') \leq (k'_x + k'_y)\frac{\pi}{2^{m-1}} \leq \frac{\pi}{2^{n-1}}.$$

For the second case we can proceed analogously with the obvious difference that we have to move south. Thus let us come to the third case. In this case the grid points

x and y are located away from the poles and we can proceed similarly as if we had a grid in $\mathbb{R}^2$. To reduce the distance between the polar coordinates we define

$$k'_x = \begin{cases} \max\{v \in \mathbb{N}_0 \,|\, v \le \frac{k_x}{2}\}, & k_x \ge k_y, \\ \min\{v \in \mathbb{N}_0 \,|\, v \ge \frac{k_x}{2}\}, & k_x < k_y, \end{cases}$$
$$\text{and} \quad k'_y = \begin{cases} \min\{v \in \mathbb{N}_0 \,|\, v \ge \frac{k_y}{2}\}, & k_x > k_y, \\ \max\{v \in \mathbb{N}_0 \,|\, v \le \frac{k_y}{2}\}, & k_x \le k_y. \end{cases}$$

Concerning the azimuthal coordinates, we have to be careful with the identification of the boundary values. In order to decrease the distance of the azimuthal coordinates we define

$$l'_x = \begin{cases} \max\{v \in \mathbb{N}_0 \,|\, v \le \frac{l_x}{2}\}, & |l_x - l_y| \le 2^{m-n+1}, l_x \ge l_y, \\ \min\{v \in \mathbb{N}_0 \,|\, v \ge \frac{l_x}{2}\}, & |l_x - l_y| \le 2^{m-n+1}, l_x < l_y, \\ \min\{v \in \mathbb{N}_0 \,|\, v \ge \frac{l_x}{2}\} \bmod 2^m, & |l_x - l_y| \ge 2^m - 2^{m-n+1}, l_x > l_y, \\ \max\{v \in \mathbb{N}_0 \,|\, v \le \frac{l_x}{2}\}, & |l_x - l_y| \ge 2^m - 2^{m-n+1}, l_x < l_y, \end{cases}$$
$$l'_y = \begin{cases} \min\{v \in \mathbb{N}_0 \,|\, v \ge \frac{l_y}{2}\}, & |l_x - l_y| \le 2^{m-n+1}, l_x > l_y, \\ \max\{v \in \mathbb{N}_0 \,|\, v \le \frac{l_y}{2}\}, & |l_x - l_y| \le 2^{m-n+1}, l_x \le l_y, \\ \max\{v \in \mathbb{N}_0 \,|\, v \le \frac{l_y}{2}\}, & |l_x - l_y| \ge 2^m - 2^{m-n+1}, l_x > l_y, \\ \min\{v \in \mathbb{N}_0 \,|\, v \ge \frac{l_y}{2}\} \bmod 2^m, & |l_x - l_y| \ge 2^m - 2^{m-n+1}, l_x < l_y. \end{cases}$$

It follows as in the first case that the so defined grid points x', y' are elements of D_m and furthermore, we have $\langle x, x' \rangle_{up}, \langle y, y' \rangle_{up} \in \pi_{m+1}$ because

$$|k_x - 2k'_x|, |k_y - 2k'_y| \in \{0, 1\} \quad \text{and} \quad |l_x - 2l'_x|, |l_y - 2l'_y| \in \{0, 1, 2^{m+1} - 1\}.$$

To show that $d_\infty(x', y') \le \pi/2^{n-1}$, we note that from $|k_x - k_y| \le 2^{m-n-1}$ and $|2k'_x - 2k'_y| \le |k_x - k_y|$ it follows that

$$|k'_x - k'_y| \le 2^{m-n}. \tag{A.16}$$

Furthermore, we have in case $|l_x - l_y| \le 2^{m-n+1}$ the inequality $|2l'_x - 2l'_y| \le |l_x - l_y|$ and if $|l_x - l_y| \ge 2^m - 2^{m-n+1}$ then $|2l'_x - 2l'_y| \ge |l_x - l_y|$ by construction, so that

$$|l'_x - l'_y| \bmod (2^m - 2^{m-n+1}) \le 2^{m-n}. \tag{A.17}$$

It follows therefore from (A.16) and (A.17) that

$$d_\infty(x', y') \le d^1_\infty(x', y') \le \frac{\pi}{2^{n-1}}.$$

This concludes the proof. □

Now let us define

$$\tilde{\delta}_n = \arccos\Big(\sin^2\Big(\frac{\pi}{2^n}\Big)\cos\Big(\frac{\pi}{2^{n-1}}\Big) + \cos^2\Big(\frac{\pi}{2^n}\Big)\Big), \quad n \in \mathbb{N}.$$

Then we can show the following:

Lemma A.2.2. *Let $n \ge 2$, $m \ge n$, and $x, y \in D_m$. Then $d_{\mathbb{S}^2}(x, y) \le \tilde{\delta}_n$ implies $d_\infty(x, y) \le \pi/2^{n-1}$.*

Proof. Let n, m be as in the lemma and $x, y \in D_m$. We show the reverse statement, i.e. we show that $d_\infty(x,y) > \pi/2^{n-1}$ implies $d_{\mathbb{S}^2}(x,y) > \tilde{\delta}_n$. Hence let $x = \phi_2(k_x\pi/2^{m-1}, l_x\pi/2^{m-1}) \in D_m$ and $y = \phi_2(k_y\pi/2^{m-1}, l_y\pi/2^{m-1}) \in D_m$, then $d_\infty(x,y) > \pi/2^{n-1}$ is equivalent to

$$\begin{aligned} &\big(|k_x - k_y| > 2^{m-n} \text{ and } 2^{m-n} < k_x + k_y < 2^m - 2^{m-n}\big) \\ \text{or} \quad &\big(2^{m-n} < |l_x - l_y| < 2^m - 2^{m-n} \text{ and } 2^{m-n} < k_x + k_y < 2^m - 2^{m-n}\big). \end{aligned}$$

In particular, the points x, y can not be too close to the poles. In the first case we have in particular $k_x \neq k_y$, and we have already seen in the proof of Lemma A.1.1 that in this case we have for the function g defined in (A.2)

$$g\Big(k_x\frac{\pi}{2^{m-1}}, k_y\frac{\pi}{2^{m-1}}, |l_x - l_y|\frac{\pi}{2^{m-1}}\Big) \le g\Big(k_x\frac{\pi}{2^{m-1}}, k_y\frac{\pi}{2^{m-1}}, 0\Big) = \cos\Big(|k_x - k_y|\frac{\pi}{2^{m-1}}\Big).$$

Since $|k_x - k_y| > 2^{m-n}$ we have therefore

$$g\Big(k_x\frac{\pi}{2^{m-1}}, k_y\frac{\pi}{2^{m-1}}, |l_x - l_y|\frac{\pi}{2^{m-1}}\Big) < \cos\Big(2^{m-n}\frac{\pi}{2^{m-1}}\Big) = \cos\Big(\frac{\pi}{2^{n-1}}\Big)$$

and consequently $d_{\mathbb{S}^2}(x,y) > \pi/2^{n-1}$ in the first case. In the second case, the restriction $2^{m-n} < |l_x - l_y| < 2^m - 2^{m-n}$ implies

$$g\Big(k_x\frac{\pi}{2^{m-1}}, k_y\frac{\pi}{2^{m-1}}, |l_x - l_y|\frac{\pi}{2^{n-1}}\Big) \le g\Big(k_x\frac{\pi}{2^{m-1}}, k_y\frac{\pi}{2^{m-1}}, \frac{\pi}{2^{n-1}}\Big). \tag{A.18}$$

For any $k_x, k_y \in \{0, 1, \dots, 2^{m-1}\}$ we have from the addition theorems for the sine and the cosine function the relation

$$\begin{aligned} g\Big(\frac{k_x + k_y}{2}\frac{\pi}{2^{m-1}}, \frac{k_x + k_y}{2}\frac{\pi}{2^{m-1}}, \frac{\pi}{2^{n-1}}\Big) - g\Big(k_x\frac{\pi}{2^{m-1}}, k_y\frac{\pi}{2^{m-1}}, \frac{\pi}{2^{n-1}}\Big) \\ = \sin^2\Big(\frac{k_x - k_y}{2}\frac{\pi}{2^{m-1}}\Big)\Big(1 + \cos\Big(\frac{\pi}{2^{n-1}}\Big)\Big). \end{aligned}$$

Because the last term is non-negative, we get with (A.18)

$$g\Big(k_x\frac{\pi}{2^{m-1}}, k_y\frac{\pi}{2^{m-1}}, |l_x - l_y|\frac{\pi}{2^{n-1}}\Big) \le g\Big(\frac{k_x + k_y}{2}\frac{\pi}{2^{m-1}}, \frac{k_x + k_y}{2}\frac{\pi}{2^{m-1}}, \frac{\pi}{2^{n-1}}\Big).$$

The restriction $2^{m-n} < k_x + k_y < 2^m - 2^{m-n}$ entails that $2^{m-n-1} < (k_x + k_y)/2 < 2^{m-1} - 2^{m-n-1}$, and it is easy to see that

$$\begin{aligned} g\Big(\frac{k_x + k_y}{2}\frac{\pi}{2^{m-1}}, \frac{k_x + k_y}{2}\frac{\pi}{2^{m-1}}, \frac{\pi}{2^{n-1}}\Big) &< g\Big(\frac{\pi}{2^n}, \frac{\pi}{2^n}, \frac{\pi}{2^{n-1}}\Big) \\ &= \sin^2\Big(\frac{\pi}{2^n}\Big)\cos\Big(\frac{\pi}{2^{n-1}}\Big) + \cos^2\Big(\frac{\pi}{2^n}\Big) \end{aligned}$$

for all k_x, k_y such that $2^{m-n} < k_x + k_y < 2^m - 2^{m-n}$. Altogether we have in the second case

$$g\Big(k_x\frac{\pi}{2^{m-1}}, k_y\frac{\pi}{2^{m-1}}, |l_x - l_y|\frac{\pi}{2^{n-1}}\Big) < \sin^2\Big(\frac{\pi}{2^n}\Big)\cos\Big(\frac{\pi}{2^{n-1}}\Big) + \cos^2\Big(\frac{\pi}{2^n}\Big)$$

and hence $d_{\mathbb{S}^2}(x,y) > \tilde{\delta}_n$ also in the second case. This concludes the proof. $\square$

B. Isotropic Functions on $\mathbb{S}^d$ and Correlation Functions of Mosaic Random Fields

B.1. Mosaic Random Fields Corresponding to the Correlation Functions in Tables 4.1, 4.2, 4.3, and 4.4

In this appendix we present for each correlation function ρ of Tables 4.1, 4.2, 4.3, and 4.4 a mosaic random field

$$Z(x) = \sum_{I \in \mathcal{P}_N} \left(\sum_{i \in \mathbb{I}_I} U_{g(I),i} \right) \mathbb{1}_{x \in C_I}, \quad x \in M, \tag{B.1}$$

$$C_I = \left(\bigcap_{i \in I} B_i \right) \cap \left(\bigcap_{j \in \{1,\dots,N\} \setminus I} B_j^c \right), \quad I \in \mathcal{P}_N, \tag{B.2}$$

which has ρ as its correlation function. The mosaic random field (B.1) is determined by the following objects:

- An $\mathbb{N}_0$-valued random variable N,
- an independent and identically distributed sequence of random closed sets $(B_n, n \in \mathbb{N})$ defining the cells $C_I, I \in \mathcal{P}_n$, $n \in \mathbb{N}_0$, via (B.2),
- an independent and identically distributed sequence of real-valued random variables $(U_{i,j}, i, j \in \mathbb{N})$,
- a function $g : \mathcal{P}^*(\mathbb{N}) \to \mathbb{N}$,
- families $(\mathbb{I}_I, I \in \mathcal{P}_n)$, $n \in \mathbb{N}_0$, of finite subsets of $\mathbb{N}$.

Random variables with different characters are assumed to be independent.

Each choice of g and $(\mathbb{I}_I, I \in \mathcal{P}_n)$, $n \in \mathbb{N}_0$, determines a submodel of the mosaic random field (B.1). We make the following conventions: the random field (B.1) is called

- *simple mosaic random field* if g is an injection and $\mathbb{I}_I = \{1\}$ for all I,
- *random token field* if $g \equiv 1$ and $\mathbb{I}_I = I$ for all I,
- *mixture random field* if g is an injection and $\mathbb{I}_I = I$ for all I,
- *dead leaves random field* if $g(I) = \mathbb{1}_{I \neq \emptyset} \max I$ and $\mathbb{I}_I = \{1\}$ for all I.

The abbreviations of the random sets are as follows:

- random half-spaces $H = H(X, R)$ defined by

$$H(X, R) = \{z \in \mathbb{R}^d \,|\, \langle z, X\rangle \geq R\} \cap M$$

  in case M is a bounded subset of $\mathbb{R}^d$,

- random closed balls $B = B_{D/2}(X)$ defined by

$$B_{D/2}(X) = \left\{z \in \mathbb{R}^d \,\middle|\, \|x - y\| \leq \frac{D}{2}\right\} \cap M$$

  in case M is a bounded subset of $\mathbb{R}^d$ and

$$B_{D/2}(X) = \left\{z \in \mathbb{S}^d \,\middle|\, d_{\mathbb{S}^d}(x, X) \leq \frac{D}{2}\right\}$$

  in case $M = \mathbb{S}^d$,

- random rectangles $E = E(Z)$, $Z = (Z^1, \ldots, Z^d)$, defined by

$$E(Z) = \{z \in \mathbb{R}^d \,|\, |z_1 - Z^1| \leq a_1, \ldots, |z_d - Z^d| \leq a_d\} \cap M$$

  in case M is a bounded subset of $\mathbb{R}^d$ and $a_1, \ldots, a_d > 0$.

B.1.1. Table 4.1, M Bounded Subset of $\mathbb{R}^d$

For the correlation functions in Table 4.1 the respective random fields are simple mosaic random fields. From (4.29) it follows, that the distribution of U has no impact on the correlation function of the simple mosaic random field, as long as $0 < \mathrm{Var}(U) < \infty$. Furthermore, all mosaic random fields corresponding to the correlation functions of Table 4.1 are build from random half-spaces $H(X, R)$ with $X \sim \mathcal{U}(\mathbb{S}^{d-1})$ and $R \sim \mathcal{U}\big([-C_M, C_M]\big)$. Therefore, we only specify the distribution of the number of random sets N in this subsection.

1. Generalized Cauchy correlation function

$$\rho(x, y) = \left(1 + \left(\frac{d_{xy}}{c}\right)^{\alpha}\right)^{-\beta/\alpha}, \quad \alpha \in (0, 1], \beta, c > 0.$$

   - $N = \sum_{l=1}^{L} K_l$, $L \sim \mathrm{NegBin}\left(\beta/\alpha, (1+(2C_M/(c_d\, c))^{\alpha})^{-1}\right)$, $K_l \sim \mathrm{Sibuya}(\alpha)$.

2. Powered exponential correlation function

$$\rho(x, y) = e^{-(d_{xy}/c)^{\alpha}}, \quad \alpha \in (0, 1], c > 0.$$

   - $N = \sum_{l=1}^{L} K_l$, $L \sim \mathrm{Poi}\left((2C_M/(c_d\, c))^{\alpha}\right)$, this compound Poisson distribution is also called *discrete stable distribution* (see [83]), $K_l \sim \mathrm{Sibuya}(\alpha)$.

3. Dagum correlation function

$$\rho(x, y) = 1 - \left(\frac{(d_{xy}/c)^{\alpha}}{1 + (d_{xy}/c)^{\alpha}}\right)^{\beta}, \quad \alpha, \beta \in (0, 1], c > 0.$$

- $N = \sum_{m=1}^{M} \sum_{l=1}^{L_m} K_{l,m}$, $M \sim \text{Sibuya}(\beta)$, $L_m \sim \text{Geo}_{\mathbb{N}_0}\big((1 + (2C_M/(c_d\, c))^\alpha)^{-1}\big)$, $K_{l,m} \sim \text{Sibuya}(\alpha)$.

4.

$$\rho(x,y) = 1 - \Big(\frac{(1 + (d_{xy}/c)^\alpha)^\gamma - 1}{(1 + (d_{xy}/c)^\alpha)^\gamma}\Big)^\beta, \quad \alpha, \beta \in (0,1], \gamma, c > 0.$$

- $N = \sum_{m=1}^{M} \sum_{l=1}^{L_m} K_{l,m}$, $M \sim \text{Sibuya}(\beta)$, $L_m \sim \text{NegBin}\big(\gamma, (1 + (2C_M/(c_d\, c))^\alpha)^{-1}\big)$, $K_{l,m} \sim \text{Sibuya}(\alpha)$.

5.

$$\rho(x,y) = \Big(1 - \Big(\frac{d_{xy}}{c}\Big)^\alpha\Big)^n, \quad \alpha \in (0,1], c \geq \frac{2C_M}{c_d}, n \in \mathbb{N}_0.$$

- $N = \sum_{l=1}^{L} K_l$, $L \sim \text{Bin}\big(n, (2C_M/(c_d\, c))^\alpha\big)$, $K_l \sim \text{Sibuya}(\alpha)$.

6.

$$\rho(x,y) = \Big(1 - c_d \frac{d_{xy}}{2C_M}\Big)^n e^{-(d_{xy}/c)^\alpha}, \quad \alpha \in (0,1], c > 0, n \in \mathbb{N}_0.$$

- $N = n + \sum_{l=1}^{L} K_l$, $n \in \mathbb{N}_0$, $L \sim \text{Poi}\big((2C_M/(c_d\, c))^\alpha\big)$, $K_l \sim \text{Sibuya}(\alpha)$.

7.

$$\rho(x,y) = \frac{1 - \big(c_d\, d_{xy}/(2C_M)\big)^\alpha}{1 - c_d\, d_{xy}/(2C_M)}, \quad \alpha \in (0,1].$$

- $N = K - 1$, $K \sim \text{Sibuya}(\alpha)$.

8.

$$\rho(x,y) = \frac{1 - e^{-(d_{xy}/c)^\alpha}}{(d_{xy}/c)^\alpha}, \quad \alpha \in (0,1], c > 0.$$

- $N = \sum_{l=1}^{L} K_l$, $L \sim \mathcal{U}\big(\{0,\dots,M\}\big)$, $M \sim \text{Poi}\big((2C_M/(c_d\, c))^\alpha\big)$, $K_l \sim \text{Sibuya}(\alpha)$.

9.

$$\rho(x,y) = \frac{\log(1 + (d_{xy}/c)^\alpha)}{(d_{xy}/c)^\alpha}, \quad \alpha \in (0,1], c > 0.$$

- $N = \sum_{l=1}^{L} K_l$, $L \sim \mathcal{U}\big(\{0,\dots,M\}\big)$, $M \sim \text{Geo}_{\mathbb{N}_0}\big((1 + (2C_M/(c_d\, c))^\alpha)^{-1}\big)$, $K_l \sim \text{Sibuya}(\alpha)$.

10.

$$\rho(x,y) = \big(1 + \beta - \beta e^{-(d_{xy}/c)^\alpha}\big)^{-\gamma}, \quad \alpha \in (0,1], \beta, \gamma, c > 0.$$

- $N = \sum_{m=1}^{M} \sum_{l=1}^{L_m} K_{l,m}$, $M \sim \text{NegBin}\big(\gamma, (1+\beta)^{-1}\big)$, $L_m \sim \text{Poi}\big((2C_M/(c_d\, c))^\alpha\big)$, $K_{l,m} \sim \text{Sibuya}(\alpha)$.

11.

$$\rho(x,y) = e^{-\beta(d_{xy}/c)^\alpha/(1+(d_{xy}/c)^\alpha)}, \quad \alpha \in (0,1], \beta, c > 0.$$

- $N = \sum_{m=1}^{M} \sum_{l=1}^{L_m} K_{l,m}$, $M \sim \mathrm{Poi}(\beta)$, $L_m \sim \mathrm{Geo}_{\mathbb{N}_0}\big((1+(2C_M/(c_d\, c))^\alpha)^{-1}\big)$, $K_{l,m} \sim \mathrm{Sibuya}(\alpha)$.

12.

$$\rho(x,y) = \frac{1+(d_{xy}/c)^\alpha}{1+\beta(d_{xy}/c)^\alpha}, \quad \alpha \in (0,1], \beta \geq 1, c > 0.$$

- $N = \sum_{m=1}^{M} \sum_{l=1}^{L_m} K_{l,m}$, $M \sim \mathrm{Geo}_{\mathbb{N}_0}\big(\beta^{-1}\big)$, $L_m \sim \mathrm{Geo}_{\mathbb{N}_0}\big((1+(2C_M/(c_d c))^\alpha)^{-1}\big)$, $K_{l,m} \sim \mathrm{Sibuya}(\alpha)$.

13.

$$\rho(x,y) = e^{-\beta(1-e^{-(d_{xy}/c)^\alpha})}, \quad \alpha \in (0,1], \beta, c > 0.$$

- $N = \sum_{m=1}^{M} \sum_{l=1}^{L_m} K_{l,m}$, $M \sim \mathrm{Poi}(\beta)$, $L_m \sim \mathrm{Poi}\big((2C_M/(c_d\, c))^\alpha\big)$, $K_{l,m} \sim \mathrm{Sibuya}(\alpha)$.

14.

$$\rho(x,y) = 1 - \Big(\frac{\gamma\,(d_{xy}/c)^\alpha}{1+(1+\gamma)(d_{xy}/c)^\alpha}\Big)^\beta, \quad \alpha, \beta \in (0,1], \gamma \geq 0, c > 0.$$

- $N = \sum_{q=1}^{Q} \sum_{m=1}^{M_q} \sum_{l=1}^{L_{m,q}} K_{l,m,q}$, $Q \sim \mathrm{Sibuya}(\beta)$, $M_q \sim \mathrm{Geo}_{\mathbb{N}_0}\big((1+\gamma)^{-1}\big)$, $L_{m,q} \sim \mathrm{Geo}_{\mathbb{N}_0}\big((1+(2C_M/(c_d\, c))^\alpha)^{-1}\big)$, $K_{l,m,q} \sim \mathrm{Sibuya}(\alpha)$.

B.1.2. Table 4.2, M Bounded Subset of $\mathbb{R}^d$

1.

$$\rho(x,y) = \lambda\Big(1 - c_d \frac{d_{xy}}{2C_M}\Big) + 1 - \lambda, \quad \lambda \in (0,2).$$

- Random token field,
- random half-spaces $H(X,R)$, $X \sim \mathcal{U}(\mathbb{S}^{d-1})$, $R \sim \mathcal{U}\big([-C_M, C_M]\big)$,
- $N \sim \mathrm{Geo}_{\mathbb{N}}\big(\lambda^2/(2(1-\lambda)^2+2)\big)$,
- $U_{i,j} \sim \mathcal{N}\big(1, (2-\lambda)/\lambda\big)$.

2.

$$\rho(x,y) = \frac{B_{1-d_{xy}^2/a^2}((d+1)/2, 1/2)}{B((d+1)/2, 1/2)} \mathbb{1}_{d_{xy} \leq a}, \quad a > 0.$$

- Random token field
- random closed balls $B_{D/2}(X)$, $X \sim \mathcal{U}(\mathbb{S}^{d-1})$, $D = a$,
- $N \sim \mathrm{Poi}$ (the parameter of the Poisson distribution is arbitrary),
- the distribution of U is arbitrary as long as $0 < \mathrm{Var}(U) < \infty$.

3.

$$\rho(x,y) = \Big(\frac{2}{\pi}\arccos\Big(\frac{d_{xy}}{a}\Big) - \frac{2}{\pi a^2} d_{xy}\sqrt{a^2 - d_{xy}^2}\Big)\mathbb{1}_{d_{xy}\le a}, \quad a > 0.$$

- Here M is a subset of $\mathbb{R}^2$,
- random token field,
- random closed balls $B_{D/2}(X)$, $X \sim \mathcal{U}(\mathbb{S}^1)$, $D = a$,
- $N \sim$ Poi (the parameter of the Poisson distribution is arbitrary),
- the distribution of U is arbitrary as long as $0 < \mathrm{Var}(U) < \infty$.

4. Spherical correlation function

$$\rho(x,y) = \Big(1 - \frac{3d_{xy}}{2a} + \frac{d_{xy}^3}{2a^3}\Big)\mathbb{1}_{d_{xy}\le a}, \quad a > 0.$$

- Here M is a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$,
- random token field,
- random closed balls $B_{D/2}(X)$,
  - $d = 2$: $X \sim \mathcal{U}(\mathbb{S}^1)$, D is continuously distributed with the distribution function $F(x) = \frac{1}{a}\big(a - \sqrt{a^2 - x^2}\big)\mathbb{1}_{0\le x\le a} + \mathbb{1}_{x>a}$, $x \in \mathbb{R}$,
  - $d = 3$: $X \sim \mathcal{U}(\mathbb{S}^2)$, $D = a$,
- $N \sim$ Poi (the parameter of the Poisson distribution is arbitrary),
- the distribution of U is arbitrary as long as $0 < \mathrm{Var}(U) < \infty$.

5.

$$\rho(x,y) = \Big(\frac{B_{1-d_{xy}^2/a^2}((d+1)/2, 1/2)}{B((d+1)/2, 1/2)} - \frac{d_{xy}^{d+1}}{a^{d+1}}\frac{B_{1-d_{xy}^2/a^2}((d+1)/2, -d/2)}{B((d+1)/2, 1/2)}\Big)\mathbb{1}_{d_{xy}\le a},$$
$$a > 0.$$

- Random token field,
- random closed balls $B_{D/2}(X)$, $X \sim \mathcal{U}(\mathbb{S}^{d-1})$, $D \sim \mathcal{U}\big([0,a]\big)$,
- $N \sim$ Poi (the parameter of the Poisson distribution is arbitrary),
- the distribution of U is arbitrary as long as $0 < \mathrm{Var}(U) < \infty$.

6.

$$\rho(x,y) = \Big(\frac{2}{\pi}\arccos\Big(\frac{d_{xy}}{a}\Big) - \frac{4}{\pi a^2} d_{xy}\sqrt{a^2 - d_{xy}^2} + \frac{2}{\pi a^3} d_{xy}^3 \operatorname{artanh}\Big(\Big(1 - \frac{d_{xy}^2}{a^2}\Big)^{1/2}\Big)\Big)\mathbb{1}_{d_{xy}\le a}, \quad a > 0.$$

- Here M is a subset of $\mathbb{R}^2$,
- random token field,
- random closed balls $B_{D/2}(X)$, $X \sim \mathcal{U}(\mathbb{S}^1)$, $D \sim \mathcal{U}\big([0,a]\big)$,

- $N \sim$ Poi (the parameter of the Poisson distribution is arbitrary),
- the distribution of U is arbitrary as long as $0 < \mathrm{Var}(U) < \infty$.

7.

$$\rho(x,y) = \frac{1 - c_d\, d_{xy}/(2C_M)}{1 + d_{xy}/c}, \quad c > \frac{2C_M}{c_d}.$$

- Dead leaves random field,
- random half-spaces $H(X,R)$, $X \sim \mathcal{U}(\mathbb{S}^{d-1})$, $R \sim \mathcal{U}\big([-C_M, C_M]\big)$,
- $N \sim \mathrm{Geo}_{\mathbb{N}}\left((c\,c_d - 2C_M)/(c\,c_d + 2C_M)\right)$,
- the distribution of U is arbitrary as long as $0 < \mathrm{Var}(U) < \infty$.

8.

$$\rho(x,y) = \lambda\Big(1 - c_d\frac{d_{xy}}{2C_M}\Big)e^{-d_{xy}/c} + (1-\lambda)\Big(1 - c_d\frac{d_{xy}}{2C_M}\Big), \quad c > 0, \lambda \in (0,1).$$

- Mixture random field,
- random half-spaces $H(X,R)$, $X \sim \mathcal{U}(\mathbb{S}^{d-1})$, $R \sim \mathcal{U}\big([-C_M, C_M]\big)$,
- $N \sim \mathrm{Poi}\left(2C_M/(c_d\, c)\right)$,
- $U_{i,j} \sim \mathcal{N}\big(1, \lambda/(1-\lambda)\big)$.

9.

$$\rho(x,y) = 1 - 2^{1-\alpha}\frac{c_d\, d_{xy}/(2C_M)}{(1 + c_d\, d_{xy}/(2C_M))^{1-\alpha}}, \quad \alpha \in (0,1].$$

- Dead leaves random field
- random half-spaces $H(X,R)$, $X \sim \mathcal{U}(\mathbb{S}^{d-1})$, $R \sim \mathcal{U}\big([-C_M, C_M]\big)$,
- $N \sim \mathrm{Sibuya}(\alpha)$,
- the distribution of U is arbitrary as long as $0 < \mathrm{Var}(U) < \infty$.

10.

$$\rho(x,y) = \frac{\prod_{k=1}^{n}(2a_k - |x_k - y_k|)_+}{2^d \prod_{k=1}^{d} a_k}, \quad a_1, \dots, a_d > 0.$$

- Random token field,
- random closed rectangles $E(Z)$,
  $Z = (Z^1, \dots, Z^d) \sim \mathcal{U}\big(\prod_{k=1}^{d}[-(R_k + a_k), R_k + a_k]\big)$,
- $N \sim$ Poi (the parameter of the Poisson distribution is arbitrary),
- the distribution of U is arbitrary as long as $0 < \mathrm{Var}(U) < \infty$.

B.1.3. Table 4.3, $M = \mathbb{S}^d$

In case $M = \mathbb{S}^d$ we restrict ourselves to random closed balls $B_R(X)$ as random sets. The center of the ball X is always uniformly distributed over $\mathbb{S}^d$. Hence we must only specify the distribution of the radius R in order to specify the random closed set in $\mathbb{S}^d$. The random fields corresponding to the correlation function in Table 4.3 are simple mosaic random fields and so are the random fields described in this subsection. As before, the distribution of U is irrelevant for the correlation function of a simple mosaic random field.

1. Generalized Cauchy correlation function

$$\rho(x,y) = \Big(1 + \Big(\frac{d_{xy}}{c}\Big)^{\alpha}\Big)^{-\beta/\alpha}, \quad \alpha \in (0,1], \beta, c > 0.$$

- $R \equiv \pi/2$,
- $N = \sum_{l=1}^{L} K_l$, $L \sim \text{NegBin}\big(\beta/\alpha, (1 + (\pi/c)^{\alpha})^{-1}\big)$, $K_l \sim \text{Sibuya}(\alpha)$.

2. Powered exponential correlation function

$$\rho(x,y) = e^{-(d_{xy}/c)^{\alpha}}, \quad \alpha \in (0,1], c > 0.$$

- $R \equiv \pi/2$,
- $N = \sum_{l=1}^{L} K_l$, $L \sim \text{Poi}\big((\pi/c)^{\alpha}\big)$, $K_l \sim \text{Sibuya}(\alpha)$.

3. Dagum correlation function

$$\rho(x,y) = 1 - \Big(\frac{(d_{xy}/c)^{\alpha}}{1 + (d_{xy}/c)^{\alpha}}\Big)^{\beta}, \quad \alpha, \beta \in (0,1], c > 0.$$

- $R \equiv \pi/2$,
- $N = \sum_{m=1}^{M} \sum_{l=1}^{L_m} K_{l,m}$, $M \sim \text{Sibuya}(\beta)$, $L_m \sim \text{Geo}_{\mathbb{N}_0}\big((1 + (\pi/c)^{\alpha})^{-1}\big)$, $K_{l,m} \sim \text{Sibuya}(\alpha)$.

4.

$$\rho(x,y) = 1 - \Big(\frac{(1 + (d_{xy}/c)^{\alpha})^{\gamma} - 1}{(1 + (d_{xy}/c)^{\alpha})^{\gamma}}\Big)^{\beta}, \quad \alpha, \beta \in (0,1], \gamma, c > 0.$$

- $R \equiv \pi/2$,
- $N = \sum_{m=1}^{M} \sum_{l=1}^{L_m} K_{l,m}$, $M \sim \text{Sibuya}(\beta)$, $L_m \sim \text{NegBin}\big(\gamma, (1+(\pi/c)^{\alpha})^{-1}\big)$, $K_{l,m} \sim \text{Sibuya}(\alpha)$.

5.

$$\rho(x,y) = \Big(1 - \Big(\frac{d_{xy}}{c}\Big)^{\alpha}\Big)^{n}, \quad \alpha \in (0,1], c \geq \pi, n \in \mathbb{N}_0.$$

- $R \equiv \pi/2$,
- $N = \sum_{l=1}^{L} K_l$, $L \sim \text{Bin}\big(n, (\pi/c)^{\alpha}\big)$, $K_l \sim \text{Sibuya}(\alpha)$.

6.

$$\rho(x,y) = \Big(1 - \frac{d_{xy}}{\pi}\Big)^n e^{-(d_{xy}/c)^\alpha}, \quad \alpha \in (0,1], c > 0, n \in \mathbb{N}_0.$$

- $R \equiv \pi/2$,
- $N = n + \sum_{l=1}^{L} K_l$, $n \in \mathbb{N}_0$, $L \sim \mathrm{Poi}\left((\pi/c)^\alpha\right)$, $K_l \sim \mathrm{Sibuya}(\alpha)$.

7.

$$\rho(x,y) = \frac{1-(d_{xy}/\pi)^\alpha}{1-d_{xy}/\pi}, \quad \alpha \in (0,1].$$

- $R \equiv \pi/2$,
- $N = K - 1$, $K \sim \mathrm{Sibuya}(\alpha)$.

8.

$$\rho(x,y) = \frac{1-e^{-(d_{xy}/c)^\alpha}}{(d_{xy}/c)^\alpha}, \quad \alpha \in (0,1], c > 0.$$

- $R \equiv \pi/2$,
- $N = \sum_{l=1}^{L} K_l$, $L \sim \mathcal{U}\big(\{0,\dots,M\}\big)$, $M \sim \mathrm{Poi}\left((\pi/c)^\alpha\right)$, $K_l \sim \mathrm{Sibuya}(\alpha)$.

9.

$$\rho(x,y) = \frac{\log(1+(d_{xy}/c)^\alpha)}{(d_{xy}/c)^\alpha}, \quad \alpha \in (0,1], c > 0.$$

- $R \equiv \pi/2$,
- $N = \sum_{l=1}^{L} K_l$, $L \sim \mathcal{U}(\{0,\dots,M\})$, $M \sim \mathrm{Geo}_{\mathbb{N}_0}\left((1+(\pi/c)^\alpha)^{-1}\right)$, $K_l \sim \mathrm{Sibuya}(\alpha)$.

10.

$$\rho(x,y) = \left(1+\beta-\beta e^{-(d_{xy}/c)^\alpha}\right)^{-\gamma}, \quad \alpha \in (0,1], \beta, \gamma, c > 0.$$

- $R \equiv \pi/2$,
- $N = \sum_{m=1}^{M} \sum_{l=1}^{L_m} K_{l,m}$, $M \sim \mathrm{NegBin}\left(\gamma, (1+\beta)^{-1}\right)$, $L_m \sim \mathrm{Poi}\left((\pi/c)^\alpha\right)$, $K_{l,m} \sim \mathrm{Sibuya}(\alpha)$.

11.

$$\rho(x,y) = e^{-\beta(d_{xy}/c)^\alpha/(1+(d_{xy}/c)^\alpha)}, \quad \alpha \in (0,1], \beta, c > 0.$$

- $R \equiv \pi/2$,
- $N = \sum_{m=1}^{M} \sum_{l=1}^{L_m} K_{l,m}$, $M \sim \mathrm{Poi}(\beta)$, $L_m \sim \mathrm{Geo}_{\mathbb{N}_0}\left((1+(\pi/c)^\alpha)^{-1}\right)$, $K_{l,m} \sim \mathrm{Sibuya}(\alpha)$.

12.

$$\rho(x,y) = \frac{1+(d_{xy}/c)^\alpha}{1+\beta(d_{xy}/c)^\alpha}, \quad \alpha \in (0,1], \beta \geq 1, c > 0.$$

- $R \equiv \pi/2$,
- $N = \sum_{m=1}^{M} \sum_{l=1}^{L_m} K_{l,m}$, $M \sim \mathrm{Geo}_{\mathbb{N}_0}\left(\beta^{-1}\right)$, $L_m \sim \mathrm{Geo}_{\mathbb{N}_0}\left((1+(\pi/c)^{\alpha})^{-1}\right)$, $K_{l,m} \sim \mathrm{Sibuya}(\alpha)$.

13.

$$\rho(x,y) = e^{-\beta(1-e^{-(d_{xy}/c)^{\alpha}})}, \quad \alpha \in (0,1], \beta, c > 0.$$

- $R \equiv \pi/2$,
- $N = \sum_{m=1}^{M} \sum_{l=1}^{L_m} K_{l,m}$, $M \sim \mathrm{Poi}(\beta)$, $L_m \sim \mathrm{Poi}\left((\pi/c)^{\alpha}\right)$, $K_{l,m} \sim \mathrm{Sibuya}(\alpha)$.

14.

$$\rho(x,y) = 1 - \left(\frac{\gamma\,(d_{xy}/c)^{\alpha}}{1+(1+\gamma)(d_{xy}/c)^{\alpha}}\right)^{\beta}, \quad \alpha, \beta \in (0,1], \gamma \geq 0, c > 0.$$

- $R \equiv \pi/2$,
- $N = \sum_{q=1}^{Q} \sum_{m=1}^{M_q} \sum_{l=1}^{L_{m,q}} K_{l,m,q}$, $Q \sim \mathrm{Sibuya}(\beta)$, $M_q \sim \mathrm{Geo}_{\mathbb{N}_0}\left((1+\gamma)^{-1}\right)$, $L_{m,q} \sim \mathrm{Geo}_{\mathbb{N}_0}\left((1+(\pi/c)^{\alpha})^{-1}\right)$, $K_{l,m,q} \sim \mathrm{Sibuya}(\alpha)$.

15.

$$\rho(x,y) = e^{-(\sin(d_{xy}/2)/c)^{\alpha}}, \quad \alpha \in (0,1], c > 0.$$

- $\cos(R) \sim \mathcal{U}\big([-1,1]\big)$,
- $N = \sum_{l=1}^{L} K_l$, $L \sim \mathrm{Poi}\left((c_{d+1}c)^{-\alpha}\right)$, $K_l \sim \mathrm{Sibuya}(\alpha)$.

16.

$$\rho(x,y) = \left(1 + \left(\frac{1}{c}\sin\left(\frac{d_{xy}}{2}\right)\right)^{\alpha}\right)^{-\beta/\alpha}, \quad \alpha \in (0,1], \beta, c > 0.$$

- $\cos(R) \sim \mathcal{U}\big([-1,1]\big)$,
- $N = \sum_{l=1}^{L} K_l$, $L \sim \mathrm{NegBin}\left(\beta/\alpha, (1+(c_{d+1}c)^{-\alpha})^{-1}\right)$, $K_l \sim \mathrm{Sibuya}(\alpha)$.

17.

$$\rho(x,y) = 1 - \left(c_{d+1}\sin\Big(\frac{d_{xy}}{2}\Big)\right)^{\alpha}, \quad \alpha \in (0,1].$$

- $\cos(R) \sim \mathcal{U}\big([-1,1]\big)$,
- $N \sim \mathrm{Sibuya}(\alpha)$.

B.1.4. Table 4.4, $M = \mathbb{S}^d$

1.

$$\rho(x,y) = \lambda\Big(1 - \frac{d_{xy}}{\pi}\Big) + 1 - \lambda, \quad \lambda \in (0,2).$$

- Random token field,

- $R \equiv \pi/2$,
- $N \sim \mathrm{Geo}_{\mathbb{N}}\left(\lambda^2/(2(1-\lambda)^2+2)\right)$,
- $U_{i,j} \sim \mathcal{N}\big(1, (2-\lambda)/\lambda\big)$.

2.

$$\rho(x,y) = \mathbb{1}_{d_{xy}=0} + \frac{1}{\pi\big(1-\cos(r)\big)}\Bigg(\arccos\Big(\frac{\cos^2(r)-\cos(d_{xy})}{\sin^2(r)}\Big) - 2\cos(r)\arccos\Big(\frac{\cos(r)\big(1-\cos(d_{xy})\big)}{\sin(r)\sin(d_{xy})}\Big)\Bigg)\mathbb{1}_{0<d_{xy}\leq 2r}, \quad r \in \Big(0, \frac{\pi}{2}\Big].$$

- Here M is the two-dimensional sphere $\mathbb{S}^2$,
- random token field,
- $R \equiv r$,
- $N \sim \mathrm{Poi}$ (the parameter of the Poisson distribution is arbitrary),
- the distribution of U is arbitrary as long as $0 < \mathrm{Var}(U) < \infty$.

3.

$$\rho(x,y) = \frac{1-d_{xy}/\pi}{1+d_{xy}/c}, \quad c > \pi.$$

- Random token field,
- $R \equiv \pi/2$,
- $N \sim \mathrm{Geo}_{\mathbb{N}}\big((c-\pi)/(c+\pi)\big)$,
- the distribution of U is arbitrary as long as $0 < \mathrm{Var}(U) < \infty$.

4.

$$\rho(x,y) = \lambda\Big(1 - \frac{d_{xy}}{\pi}\Big)e^{-d_{xy}/c} + (1-\lambda)\Big(1-\frac{d_{xy}}{\pi}\Big), \quad c > 0, \lambda \in (0,1).$$

- Mixture random field,
- $R \equiv \pi/2$,
- $N \sim \mathrm{Poi}\,(\pi/c)$,
- $U_{i,j} \sim \mathcal{N}\big(1, \lambda/(1-\lambda)\big)$.

5.

$$\rho(x,y) = 1 - 2^{1-\alpha}\frac{d_{xy}/\pi}{(1+d_{xy}/\pi)^{1-\alpha}}, \quad \alpha \in (0,1].$$

- Dead leaves random field
- $R \equiv \pi/2$,
- $N \sim \mathrm{Sibuya}(\alpha)$,
- the distribution of U is arbitrary as long as $0 < \mathrm{Var}(U) < \infty$.

6.
$$\rho(x,y) = \lambda\Big(1 - c_{d+1}\sin\left(\frac{d_{xy}}{2}\right)\Big) + 1 - \lambda, \quad \lambda \in (0,2).$$

- Random token field,
- $\cos(R) \sim \mathcal{U}\big([-1,1]\big)$,
- $N \sim \mathrm{Geo}_{\mathbb{N}}\left(\lambda^2/(2(1-\lambda)^2+2)\right)$,
- $U_{i,j} \sim \mathcal{N}\big(1,(2-\lambda)/\lambda\big)$.

7.
$$\rho(x,y) = 1 - c_{d+1}\frac{3}{d+2}\sin\Big(\frac{d_{xy}}{2}\Big)\cos^2\Big(\frac{d_{xy}}{2}\Big) - c_{d+1}\frac{2}{d+2}\sin^3\Big(\frac{d_{xy}}{2}\Big).$$

- Random token field,
- $\cos(R) \sim \mathcal{U}\big([-1,1]\big)$,
- $N \sim \mathrm{Poi}$ (the parameter of the Poisson distribution is arbitrary),
- the distribution of U is arbitrary as long as $0 < \mathrm{Var}(U) < \infty$.

B.2. Isotropic Functions on $\mathbb{S}^d$

A function $C : \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{R}$ is called *isotropic* if for all $x, y \in \mathbb{S}^d$ and all $\mathcal{R} \in \mathrm{SO}(d+1)$ it is true that

$$C(x,y) = C(\mathcal{R}x, \mathcal{R}y).$$

In this appendix we give a proof of a statement which is well-known but which we could not find in the literature.

Lemma B.2.1. *A function $C : \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{R}$ is isotropic if and only if there is $\tilde{C} : [0,\pi] \to \mathbb{R}$ such that*

$$C(x,y) = \tilde{C}\big(d_{\mathbb{S}^d}(x,y)\big), \quad x, y \in \mathbb{S}^d. \tag{B.3}$$

Proof. Suppose there is a function $\tilde{C} : [0,\pi] \to \mathbb{R}$ such that (B.3) is true. Because any rotation $\mathcal{R} \in \mathrm{SO}(d+1)$ is an orthogonal matrix, we have for all $x, y \in \mathbb{S}^d$

$$C(\mathcal{R}x, \mathcal{R}y) = \tilde{C}\big(d_{\mathbb{S}^d}(\mathcal{R}x, \mathcal{R}y)\big) = \tilde{C}\big(\arccos\big(\langle \mathcal{R}x, \mathcal{R}y\rangle\big)\big) = C(x,y),$$

which shows the reverse implication. To see that the isotropy of C also implies the existence of a function $\tilde{C} : [0,\pi] \to \mathbb{R}$ such that (B.3) holds true, let $x, \tilde{x}, y, \tilde{y}$ be some points in $\mathbb{S}^d$ such that $d_{\mathbb{S}^d}(x,y) = d_{\mathbb{S}^d}(\tilde{x},\tilde{y})$. We aim to show $C(x,y) = C(\tilde{x},\tilde{y})$.

Because $\mathrm{SO}(d+1)$ acts transitively on $\mathbb{S}^d$ there are rotations $\mathcal{R}_x, \mathcal{R}_{\tilde{x}} \in \mathrm{SO}(d+1)$ such that

$$\mathcal{R}_x x = \mathcal{R}_{\tilde{x}}\tilde{x} = e_{d+1} \text{ with } e_{d+1} = (0,\dots,0,1) \in \mathbb{S}^d.$$

Because of $d_{\mathbb{S}^d}(x,y) = d_{\mathbb{S}^d}(\tilde{x},\tilde{y})$ we have

$$\langle e_{d+1}, \mathcal{R}_x y\rangle = \langle x, y\rangle = \cos\big(d_{\mathbb{S}^d}(x,y)\big) = \langle e_{d+1}, \mathcal{R}_{\tilde{x}}\tilde{y}\rangle,$$

so that we may write $\mathcal{R}_x y$ and $\mathcal{R}_{\tilde{x}}\tilde{y}$ in block matrix notation as

$$\mathcal{R}_x y = \big(u, \cos\big(d_{\mathbb{S}^d}(x,y)\big)\big), \quad \mathcal{R}_{\tilde{x}}\tilde{y} = \big(v, \cos\big(d_{\mathbb{S}^d}(x,y)\big)\big) \tag{B.4}$$

with some vectors $u, v \in \mathbb{R}^d$. Since $\|\mathcal{R}_x y\| = \|\mathcal{R}_{\tilde{x}}\tilde{y}\| = 1$, the length of u and v is given by

$$\|u\| = \|v\| = \sin\big(d_{\mathbb{S}^d}(x,y)\big). \tag{B.5}$$

Suppose that $d_{\mathbb{S}^d}(x,y) \in \{0,\pi\}$, so that $u = v = 0$ by (B.5). Then we have $\mathcal{R}_x y = \mathcal{R}_{\tilde{x}}\tilde{y}$ from (B.4) and the isotropy of C implies

$$C(x,y) = C(e_{d+1}, \mathcal{R}_x y) = C(\tilde{x},\tilde{y}).$$

If $d_{\mathbb{S}^d}(x,y) \in (0,\pi)$, then $u \neq 0$ and $v \neq 0$ and the vectors $u' = u/\sin\big(d_{\mathbb{S}^d}(x,y)\big)$ and $v' = v/\sin\big(d_{\mathbb{S}^d}(x,y)\big)$ are elements of $\mathbb{S}^{d-1}$. Using the transitive action of $\mathrm{SO}(d)$ on $\mathbb{S}^{d-1}$ there is a rotation $\mathcal{R}' \in \mathrm{SO}(d)$ such that $\mathcal{R}'u' = v'$. It is easy to see, that the rotation (given in block matrix notation)

$$\mathcal{R}^* = \begin{pmatrix} \mathcal{R}' & 0 \\ 0 & 1 \end{pmatrix}$$

is an element of $\mathrm{SO}(d+1)$, and furthermore, that e_{d+1} is a fixed point of $\mathcal{R}^*$. Hence in case $d_{\mathbb{S}^d}(x,y) \in (0,\pi)$ we have

$$\begin{aligned} C(x,y) &= C(e_{d+1}, \mathcal{R}_x y) \\ &= C(e_{d+1}, \mathcal{R}^*\mathcal{R}_x y) \\ &= C\Big(e_{d+1}, \mathcal{R}^*\Big(\sin\big(d_{\mathbb{S}^d}(x,y)\big)u', \cos\big(d_{\mathbb{S}^d}(x,y)\big)\Big)\Big) \\ &= C\Big(e_{d+1}, \Big(\sin\big(d_{\mathbb{S}^d}(x,y)\big)v', \cos\big(d_{\mathbb{S}^d}(x,y)\big)\Big)\Big) \\ &= C(e_{d+1}, \mathcal{R}_{\tilde{x}}\tilde{y}) \\ &= C(\tilde{x},\tilde{y}). \end{aligned}$$

Altogether we have shown that the isotropy of C implies

$$C(x,y) = C(\tilde{x},\tilde{y}) \quad \text{for all } x, \tilde{x}, y, \tilde{y} \in \mathbb{S}^d \text{ such that } d_{\mathbb{S}^d}(x,y) = d_{\mathbb{S}^d}(\tilde{x},\tilde{y}). \tag{B.6}$$

For $\theta \in [0,\pi]$ we may now pick any $x, y \in \mathbb{S}^d$ with $d_{\mathbb{S}^d}(x,y) = \theta$ and define $\tilde{C}(\theta)$ by

$$\tilde{C}(\theta) = C(x,y).$$

Then this defines a function $\tilde{C} : [0,\pi] \to \mathbb{R}$ and (B.6) shows that the condition (B.3) holds true for this $\tilde{C}$. □

Bibliography

[1] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards – Applied Mathematics Series*. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[2] R. J. Adler. *The geometry of random fields*. John Wiley & Sons, Ltd., Chichester, 1981. Wiley Series in Probability and Mathematical Statistics.

[3] R. J. Adler and J. E. Taylor. *Random fields and geometry*. Springer Monographs in Mathematics. Springer, New York, 2007.

[4] R. Allan and T. Ansell. A new globally complete monthly historical gridded mean sea level pressure dataset (hadslp2): 1850–2004. *J. Climate*, 19(22):5816–5842, 2006.

[5] R. Andreev and A. Lang. Kolmogorov–Chentsov theorem and differentiability of random fields on manifolds. *Potential Anal.*, 41(3):761–769, 2014.

[6] H. Bauer. *Wahrscheinlichkeitstheorie*. de Gruyter Lehrbuch. [de Gruyter Textbook]. Walter de Gruyter & Co., Berlin, fourth edition, 1991.

[7] C. Berg, J. P. R. Christensen, and P. Ressel. *Harmonic analysis on semigroups*, volume 100 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1984. Theory of positive definite and related functions.

[8] C. Berg, J. Mateu, and E. Porcu. The Dagum family of isotropic correlation functions. *Bernoulli*, 14(4):1134–1149, 2008.

[9] A. C. Berry. The accuracy of the Gaussian approximation to the sum of independent variates. *Trans. Amer. Math. Soc.*, 49:122–136, 1941.

[10] G. Berschneider and Z. Sasvári. On a theorem of Karhunen and related moment problems and quadrature formulae. In *Spectral theory, mathematical system theory, evolution equations, differential and difference equations*, volume 221 of *Oper. Theory Adv. Appl.*, pages 173–187. Birkhäuser/Springer Basel AG, Basel, 2012.

[11] R. L. Bishop and S. I. Goldberg. *Tensor analysis on manifolds*. The Macmillan Co., New York; Collier-Macmillan Ltd., London, 1968.

[12] S. Bochner. Monotone Funktionen, Stieltjessche Integrale und harmonische Analyse. *Math. Ann.*, 108(1):378–410, 1933.

[13] D. Bolin and F. Lindgren. Spatial models generated by nested stochastic partial differential equations, with an application to global ozone mapping. *Ann. Appl. Stat.*, 5(1):523–550, 2011.

[14] M. Bottomley, C. Folland, J. Hsiung, R. Newell, and D. Parker. Global ocean surface temperature atlas (gosta), joint meteorological office and massachusetts institute of technology project, uk depts. of the environment and energy, 1990.

[15] P. Cabella and D. Marinucci. Statistical challenges in the analysis of cosmic microwave background radiation. *Ann. Appl. Stat.*, 3(1):61–95, 2009.

[16] D. Chen, V. A. Menegatto, and X. Sun. A necessary and sufficient condition for strictly positive definite functions on spheres. *Proc. Amer. Math. Soc.*, 131(9):2733–2740, 2003.

[17] N. N. Chentsov. Weak convergence of stochastic processes whose trajectories have no discontinuities of the second kind and the "heuristic" approach to the Kolmogorov-Smirnov tests. *Theory Probab. Appl.*, 1(1):140–144, 1956.

[18] A. Chessa. On the object-based method for simulating sandstone deposits. In *ECMOR III-3rd European Conference on the Mathematics of Oil Recovery*, pages 67–77, 1992.

[19] A. Chessa and A. Martinus. Object-based modelling of the spatial distribution of fluvial sandstone deposits. In *ECMOR III-3rd European Conference on the Mathematics of Oil Recovery*, pages 5–14, 1992.

[20] J.-P. Chilès and P. Delfiner. *Geostatistics: modeling spatial uncertainty.* John Wiley & Sons, Inc., Hoboken, NJ, 2012.

[21] H. Cramér and M. R. Leadbetter. *Stationary and related stochastic processes. Sample function properties and their applications.* John Wiley & Sons, Inc., New York-London-Sydney, 1967.

[22] P. E. Creasey and A. Lang. Fast generation of isotropic Gaussian random fields on the sphere. *Monte Carlo Methods Appl.*, 24(1):1–11, 2018.

[23] D. Daley. The non-existence of stationary infinite Newtonian universes and a multi-dimensional model of shot noise. *Nature*, 227(5261):935, 1970.

[24] D. J. Daley and E. Porcu. Dimension walks and Schoenberg spectral measures. *Proc. Amer. Math. Soc.*, 142(5):1813–1824, 2014.

[25] L. Devroye. A triptych of discrete distributions related to the stable law. *Statist. Probab. Lett.*, 18(5):349–351, 1993.

[26] *NIST Digital Library of Mathematical Functions.* http://dlmf.nist.gov/, Release 1.0.15 of 2017-06-01. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller and B. V. Saunders, eds.

[27] C.-G. Esseen. On the Liapounoff limit of error in the theory of probability. *Ark. Mat. Astr. Fys.*, 28A(9):19, 1942.

[28] A. Estrade and J. Istas. Ball throwing on spheres. *Bernoulli*, 16(4):953–970, 2010.

[29] J. Feinauer, T. Brereton, A. Spettl, M. Weber, I. Manke, and V. Schmidt. Stochastic 3D modeling of the microstructure of lithium-ion battery anodes via Gaussian random fields on the sphere. *Comput. Mater. Sci.*, 109:137–146, 2015.

[30] E. Freitag and R. Busam. *Funktionentheorie.* Springer-Lehrbuch. [Springer Textbook]. Springer-Verlag, Berlin, 1993.

[31] T. Gneiting. Closed form solutions of the two-dimensional turning bands equation. *Math. Geol.*, 30(4):379–390, 1998.

[32] T. Gneiting. Power-law correlations, related models for long-range dependence and their simulation. *J. Appl. Probab.*, 37(4):1104–1109, 2000.

[33] T. Gneiting. Strictly and non-strictly positive definite functions on spheres. *Bernoulli*, 19(4):1327–1349, 2013.

[34] T. Gneiting and M. Schlather. Stochastic models that separate fractal dimension and the Hurst effect. *SIAM Rev.*, 46(2):269–282, 2004.

[35] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products.* Fourth edition prepared by Ju. V. Geronimus and M. Ju. Ceĭtlin. Translated from the Russian by Scripta Technica, Inc. Translation edited by Alan Jeffrey. Academic Press, New York-London, 1965.

[36] J. Guinness and M. Fuentes. Isotropic covariance functions on spheres: some properties and modeling considerations. *J. Multivar. Anal.*, 143:143–152, 2016.

[37] L. V. Hansen, T. L. Thorarinsdottir, E. Ovcharov, T. Gneiting, and D. Richards. Gaussian random particles with flexible Hausdorff dimension. *Adv. Appl. Prob.*, 47(2):307–327, 2015.

[38] S. Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 80 of *Pure and Applied Mathematics.* Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.

[39] J. Hoffmann-Jørgensen. *Stochastic processes on Polish spaces*, volume 39 of *Various Publications Series (Aarhus).* Aarhus Universitet, Matematisk Institut, Aarhus, 1991.

[40] C. Huang, H. Zhang, and S. M. Robeson. On the validity of commonly used covariance and variogram functions on the sphere. *Math. Geosci.*, 43(6):721–733, 2011.

[41] J. Jeong and M. Jun. A class of Matérn-like covariance functions for smooth processes on a sphere. *Spat. Stat.*, 11:1–18, 2015.

[42] D. Jeulin and P. Jeulin. Synthesis of rough surfaces by random morphological models. *Stereol. Iugosl*, 3(1):239–246, 1981.

[43] J. Jost. *Riemannian geometry and geometric analysis.* Universitext. Springer, Heidelberg, sixth edition, 2011.

[44] M. Jun and M. L. Stein. An approach to producing space-time covariance functions on spheres. *Technometrics*, 49(4):468–479, 2007.

[45] M. Jun and M. L. Stein. Nonstationary covariance models for global data. *Ann. Appl. Stat.*, 2(4):1271–1289, 2008.

[46] O. Kallenberg. *Foundations of modern probability.* Probability and its applications. Springer, New York ; Heidelberg [u.a.], 1997.

[47] A. Kaplan, M. A. Cane, Y. Kushnir, A. C. Clement, M. B. Blumenthal, and B. Rajagopalan. Analyses of global sea surface temperature 1856–1991. *J. Geophys. Res.*, 103(C9):18567–18589, 1998.

[48] K. Karhunen. Zur Spektraltheorie stochastischer Prozesse. *Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys.*, 1946(34):7, 1946.

[49] V. Korolev and I. Shevtsova. An improvement of the Berry–Esseen inequality with applications to Poisson and mixed Poisson random sums. *Scand. Actuar. J.*, 2012(2):81 – 105, 2012.

[50] A. Laforgia and P. Natalini. Exponential, gamma and polygamma functions: Simple proofs of classical and new inequalities. *J. Math. Anal. Appl.*, 407(2):495–504, 2013.

[51] A. Lang, J. Potthoff, M. Schlather, and D. Schwab. Continuity of random fields on Riemannian manifolds. *Commun. Stoch. Anal.*, 10(2):185–193, 2016.

[52] A. Lang and C. Schwab. Isotropic Gaussian random fields on the sphere: regularity, fast simulation and stochastic partial differential equations. *Ann. Appl. Prob.*, 25(6):3047–3094, 2015.

[53] C. Lantuejoul. *Geostatistical simulation: models and algorithms.* Springer-Verlag, Berlin, Heidelberg, 2002.

[54] J. M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics.* Springer, New York, second edition, 2013.

[55] S. Li. Concise formulas for the area and volume of a hyperspherical cap. *Asian J. Math. Stat.*, 4(1):66–70, 2011.

[56] F. Lindgren, H. v. Rue, and J. Lindström. An explicit link between Gaussian fields and Gaussian Markov random fields: the stochastic partial differential equation approach. *J. R. Stat. Soc. Ser. B Stat. Methodol.*, 73(4):423–498, 2011.

[57] A. Malyarenko. *Invariant random fields on spaces with a group action.* Probability and its Applications (New York). Springer, Heidelberg, 2013.

[58] A. Mantoglou. Digital simulation of multivariate two-and three-dimensional stochastic processes with a spectral turning bands method. *Math. Geol.*, 19(2):129–149, 1987.

[59] D. Marinucci and G. Peccati. *Random fields on the sphere*, volume 389 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 2011. Representation, limit theorems and cosmological applications.

[60] G. Marsaglia. Choosing a point from the surface of a sphere. *Ann. Math. Statist.*, 43(2):645–646, 04 1972.

[61] G. Matheron. Modele séquentiel de partition aléatoire. Technical report, Centre de Morphologie Mathématique, Fontainebleau, 1968.

[62] G. Matheron. Ensembles fermés aléatoires, ensembles semi-markoviens et polyèdres poissoniens. *Adv. Appl. Prob.*, 4:508–541, 1972.

[63] G. Matheron. *Random sets and integral geometry.* John Wiley & Sons, New York-London-Sydney, 1975.

[64] G. Matheron. Leçons sur les fonctions aléatoires d'ordre 2. Technical report c-53, Ecole nationale supérieure des Mines de Paris, March 1972.

[65] R. D. McPeters, P. Bhartia, A. J. Krueger, J. R. Herman, B. M. Schlesinger, C. G. Wellemeyer, C. J. Seftor, G. Jaross, S. L. Taylor, T. Swissler, et al. Nimbus-7 total ozone mapping spectrometer (toms) data products user's guide. *NASA Reference Publication 1384*, 1996.

[66] R. E. Miles. Poisson flats in Euclidean spaces. I. A finite number of random uniform flats. *Adv. Appl. Prob.*, 1:211–237, 1969.

[67] M. E. Muller. Some continuous Monte Carlo methods for the Dirichlet problem. *Ann. Math. Statist.*, 27:569–589, 1956.

[68] P. Petersen. *Riemannian geometry.* Graduate texts in mathematics; 171. Springer, New York, NY, 2. ed. edition, 2006.

[69] J. Potthoff. Sample properties of random fields. I. Separability and measurability. *Commun. Stoch. Anal.*, 3(1):143–153, 2009.

[70] J. Potthoff. Sample properties of random fields. II. Continuity. *Commun. Stoch. Anal.*, 3(3):331–348, 2009.

[71] J. Potthoff. Sample properties of random fields III: differentiability. *Commun. Stoch. Anal.*, 4(3):335–353, 2010.

[72] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev. *Integrals and series. Vol. 2.* Gordon & Breach Science Publishers, New York, second edition, 1988. Special functions, Translated from the Russian by N. M. Queen.

[73] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev. *Integrals and series. Vol. 3.* Gordon and Breach Science Publishers, New York, 1990. More special functions, Translated from the Russian by G. G. Gould.

[74] M. Reimer. *Multivariate polynomial approximation.* International series of numerical mathematics; 144. Birkhäuser, Basel ; Berlin [u.a.], 2003.

[75] M. Schlather. Construction of covariance functions and unconditional simulation of random fields. In *Advances and challenges in space-time modelling of natural events*, pages 25–54. Springer-Verlag, Berlin, Heidelberg,, 2012.

[76] I. J. Schoenberg. Metric spaces and completely monotone functions. *Ann. of Math. (2)*, 39(4):811–841, 1938.

[77] I. J. Schoenberg. Positive definite functions on spheres. *Duke Math. J.*, 9:96–108, 1942.

[78] D. Schwab, M. Schlather, and J. Potthoff. A general class of mosaic random fields. *ArXiv e-prints*, 2017.

[79] Scilab Enterprises. *Scilab: free and open source software for numerical computation.* Scilab Enterprises, Orsay, France, 2012.

[80] M. Sibuya. Generalized hypergeometric, digamma and trigamma distributions. *Ann. Inst. Statist. Math.*, 31(3):373–390, 1979.

[81] M. A. Sironvalle. The random coin method: solution of the problem of the simulation of a random function in the plane. *J. Internat. Assoc. Math. Geol.*, 12(1):25–32, 1980.

[82] E. Slutsky. Qualche proposizione relativa alla teoria delle funzioni aleatorie. *Giorn. Ist. Ital. Attuari*, 8:183–199, 1937.

[83] F. W. Steutel and K. van Harn. Discrete analogues of self-decomposability and stability. *Ann. Probab.*, 7(5):893–899, 1979.

[84] S. Torquato. *Random heterogeneous materials: microstructure and macroscopic properties*, volume 16. Springer Science & Business Media, 2013.

[85] A. Tovchigrechko and I. A. Vakser. How common is the funnel-like energy landscape in protein-protein interactions? *Protein Sci.*, 10(8):1572–1583, 2001.

[86] M. u. I. Yadrenko. *Spectral theory of random fields.* Translation Series in Mathematics and Engineering. Optimization Software, Inc., Publications Division, New York, 1983. Translated from the Russian.

[87] A. M. Yaglom. Second-order homogeneous random fields. In *Proc. 4th Berkeley Sympos. Math. Statist. and Prob., Vol. II*, pages 593–622. Univ. California Press, Berkeley, Calif., 1961.

[88] J. Ziegel. Convolution roots and differentiability of isotropic positive definite functions on spheres. *Proc. Amer. Math. Soc.*, 142(6):2063–2077, 2014.