Period maps and Torelli results for marked hypersurface singularities

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ABSTRACT

The moduli space of marked singularities was introduced by Claus Hertling in 2010 and parameterizes μ-homotopic isolated hypersurface singularities equipped with certain markings. This moduli space can be understood either as a global μ-constant stratum or as a Teichmüller space of singularities. The additional marking allows one to formulate the conjecture on the analytic behavior of singularities within a distinguished μ-homotopy class in terms of Torelli(-type) problems in a very efficient way. In the monograph at hand these Torelli problems are solved for many different types of singularities. The proofs use such diverse concepts as $\mathbb{Z}$-lattice computations, Fuchsian groups and the Gauss-Manin connection. In particular, the monograph contains proofs of the Torelli problems for all singularities of modality zero, one and two. This closes some long-standing gaps.

ZUSAMMENFASSUNG

Der Modulraum markierter Singularitäten, der im Jahr 2010 erstmals von Claus Hertling betrachtet wurde, parametrisiert μ-homotope isolierte Hyperflächensingularitäten, die mit bestimmten Markierungen versehen sind. Dieser Modulraum kann einerseits als globales μ-konstant-Stratum, andererseits als Teichmüller-Raum für Singularitäten aufgefasst werden. Die zusätzliche Markierung erlaubt es, die Frage nach dem analytischen Verhalten der Singularitäten innerhalb einer bestimmten μ-Homotopieklasse, effizient als Torelli Problem zu formulieren. In der vorliegenden Monographie werden diese Torelli Probleme für eine Vielzahl unterschiedlicher Typen von Singularitäten gelöst. Dabei werden so unterschiedliche Konzepte wie ganzzahlige Gitterberechnungen, Fuchssche Gruppen und der Gauss-Manin Zusammenhang verwendet. Insbesondere enthält die Monographie Torelli Resultate für alle Singularitäten von Modalität null, eins und zwei. Dies schließt einige seit langem bekannte Lücken.

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LIST OF IMPORTANT SYMBOLS

BL	Period map for marked singularities
D_{BL}	Classifying space for Brieskorn lattices
$\deg_w$	Weighted degree of some monomial
f	Isolated hypersurface singularity
$G^{mar}(f_0)$	Subgroup of $G_{\mathbb{Z}}(f_0)$ which acts on $M_{\mu}^{mar}(f_0)^0$
$G_{\mathbb{Z}}(f_0)$	Group of automorphisms on $Ml(f_0)$ that respect the Seifert form
$\mathrm{GL}(n;R)$	General linear group of degree n over ring R
$H_0''(f)$	Brieskorn lattice of f
I	Intersection form
$\mathcal{I}$	Period matrix (of Varchenko)
L	Seifert form
LBL	Period map
M_h	(Picard-Lefschetz) monodromy
M_T	Transversal monodromy
$M_{\mu}(f_0)$	Moduli space of singularities
$M_{\mu}^{mar}(f_0)$	Moduli space of marked singularities w/ reference singularity f_0
$Ml(f)$	Milnor lattice of the singularity f
N	Nilpotent part of the monodromy M_h
$\mathrm{PGL}(n;R)$	Projective general linear group of degree n over ring R
$\mathrm{PSL}(n;R)$	Projective special linear group of degree n over ring R
Φ_n	n-th cyclotomic polynomial
$\mathcal{R}$	Group of biholomorphic map germs
S^1	Unit circle
S_{μ}	μ-constant stratum
$\mathrm{SL}(n;R)$	Special linear group of degree n over ring R
$\mathrm{Sym}(k)$	Symmetric group on a finite set of k elements
$\mathbb{Z}_p$	Multiplicative group of integers modulo p

1

INTRODUCTION

Isolated hypersurface singularities (short: singularities) are holomorphic function germs $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with a finite Milnor number μ. The Milnor number is a topological invariant of a singularity[1]. So loosely speaking in a family of singularities with constant Milnor number the singularities are "topologically the same".

Now in the seminal two-volumed book [AGV85]+[AGV88] of Arnold, Gusein-Zade and Varchenko the question is raised which properties in a family of singularities with constant Milnor number can still vary. They conjectured that the singularities in all such families are determined by certain analytic invariants, the so-called Picard-Fuchs singularities $PFS(f)$. However Claus Hertling gave an example in his doctoral thesis [He93] which shows that Picard-Fuchs singularities in general do not globally classify all singularities in a family of singularities with constant Milnor number. This phenomenon can appear as soon as singularities of modality > 0 are involved. Therefore, he introduced new analytic invariants which use all of the information on the position of the Brieskorn lattice $H_0''(f)$ in the Gauss-Manin connection and showed in [He93], [He95] and [He98] that they classify all simple singularities, all unimodal singularities, all semiquasihomogeneous singularities with weights $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, all Brieskorn-Pham singularities with coprime exponents and almost all bimodal singularities. This classification works if and only if some period map LBL, like the one in the classical Torelli Theorem for Riemann surfaces, is injective. So this question can be referred to as the Torelli Conjecture for hypersurface singularities (see Section 2.4 for the precise statement). Up to the present day no type of singularity is known where this conjecture fails.

To prove the Torelli Conjecture for hypersurface singularities for certain types of singularities Hertling made an enormous effort. He did very detailed computations with eigenspace parts of certain differential forms inside the Gauss-Manin connection. Some people consider these computations as "very cumbersome and difficult" (see [Ku98, p. 180]). Moreover for the bimodal series singularities they were published only partly at first (cf. [He95, p. 391]). So one aim of this thesis is it to make these old computations more accessible and visible in the literature.

Later a more conceptual approach to the aforementioned question was carried out in [He11]. There the singularities come equipped with markings and a sort of Teichmüller space for singularities is established. This space M_μ^{mar} is then locally isomorphic to the μ-constant stratum S_μ and has an analogue $G_{\mathbb{Z}}$ of a mapping class group acting properly discontinously on it. Now the problem can be formulated in terms of the period maps as a Torelli problem more cleverly (see Section 3.2). This yields a stronger classification conjecture, namely the strong global Torelli Conjecture for hypersurface singularities. It

1 Note that all the objects mentioned in this introduction will be defined properly in the upcoming chapters of the thesis.

was proved in [He11] for the simple singularities, the Brieskorn-Pham singularities with coprime exponents and some exceptional singularities.

Now in the monograph at hand we look at the old computations concerning the Gauss-Manin connection and use this information to prove the strong global Torelli Conjecture for all types of singularities of modality ≤ 2. The main new ingredient to achieve this is precise knowledge about the group $G_{\mathbb{Z}}$. The group $G_{\mathbb{Z}}$ contains $\mathbb{Z}$-lattice automorphisms on the Milnor lattice which respect a certain bilinear form. So a priori it is not easy to compute the group $G_{\mathbb{Z}}$. Here, we can bypass this obstacle in the following way. At first we split up the Milnor lattice into smaller parts in the sense of Orlik's Conjecture from 1972 (see Section A.2), then we compute groups of automorphism on these smaller parts and later we put everything together again to determine $G_{\mathbb{Z}}$ or a sufficiently large subgroup of $G_{\mathbb{Z}}$. All necessary computations are included.

All in all, the main results of the monograph at hand are:

- We verify the strong global Torelli Conjecture for hypersurface singularities for all the simple elliptic, hyperbolic, the remaining exceptional uni- and bimodal, the quadrangle and the bimodal series singularities (see Sections 4.2, 4.3, 4.4 and 4.5).
- We show (as a by-product) that also for singularities of types $Z_{1,14k}$, $S_{1,10k}$ and $S^{\sharp}_{1,10k}$ with $k \in \mathbb{N}$ the Torelli Conjecture for hypersurface singularities is true. This was unclear for a long time (cf. Remark 4.4.6).
- We falsify Conjecture 3.2 (a) from [He11], i.e. we show that the moduli space M_{μ}^{mar} is not connected in general (see Section 4.4).
- We verify Orlik's Conjecture for several types of singularities where it applies (e.g. see Section 4.6).
- We compare M_{μ}^{mar} with the classical Teichmüller space for Riemann surfaces and argue that they do not have so much in common (see Section 3.3).
- We summarize the state of the art of the Torelli Conjectures mentioned above and discuss some ideas for future research in Sections 4.6 and 4.7.

1.1 PLAN OF THESIS

This thesis consists of four chapters and an extensive appendix with several sections each. Chapter 1 provides a very rough idea of what the thesis is about and names the most important supporters of the thesis project. Chapter 2 introduces the basic notions, such as the Milnor lattice, the Brieskorn lattice, the Gauss-Manin connection and μ-constant families of singularities. Moreover it states the Torelli Conjecture for hypersurface singularities mentioned above. Chapter 3 explains the moduli space of marked singularities M_{μ}^{mar} and the strong global Torelli Conjecture. Finally, Chapter 4 elucidates the proofs of the strong global Torelli Conjecture for all types of singularities up to modality two. A more detailed

description of its contents is given at the beginning of each chapter.
The Appendix A consists of three sections. The first one discusses Newton nondegenerate and semiquasihomogeneous singularities. This is important as many of the uni- and bimodal singularities are quasihomogeneous, semiquasihomogeneous and/or Newton nondegenerate. The second one discusses Orlik's Conjecture and contains a collection of techniques which are useful to control the automorphisms on some smaller parts of the Milnor lattice. The last one discusses infinite Fuchsian groups which arise as subgroups of groups $\mathrm{GL}(2; \mathbb{Z}[\zeta])$ where ζ is a root of unity. This is needed for the quadrangle singularities in particular. So a good understanding of the material in the appendix is essential for the understanding of the results of the thesis.

Note that most of the results of this thesis appeared beforehand in the articles [GH17] and [GH18]. Those articles were written during my time as a doctoral student at the University of Mannheim. All of them are already published or currently in press. More specifically, the book chapter [GH17] contains material of the Sections 4.2, 4.3 and A.2 and the article [GH18] contains material of the Sections 4.4, 4.5, A.1, A.2 and A.3.

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2

BASIC NOTIONS ON ISOLATED HYPERSURFACE SINGULARITIES

In this chapter we recall many important definitions on isolated hypersurface singularities. We start in Section 2.1 with the classical definition of the Milnor fibration of a singularity and present several results on the topology of isolated hypersurface singularities. The material presented in Section 2.1 is standard textbook material and can be found in almost all books on hypersurface singularities. We recommend [Eb07]. In Section 2.2 we briefly discuss the Gauss-Manin connection and Brieskorn lattice for singularities. In Section 2.3 we generalize all objects studied before to μ-constant families of singularities. We end up this chapter by stating the Torelli Conjectures for singularities of Claus Hertling in Section 2.4. The definitions and further details on these notions can be found in more specialized textbooks, like [Ku98] or [He02-2].

2.1 THE TOPOLOGY OF ISOLATED HYPERSURFACE SINGULARITIES

2.1.1 *Milnor fibration*

The notion "singularity" is ubiquitous throughout mathematics. In this thesis by the term *singularity* or more precisely by *isolated hypersurface singularity* we refer to a holomorphic function germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at zero. This means that for every representative $f : U \to \mathbb{C}$ of the germ, where $U \subseteq \mathbb{C}^{n+1}$ is a suitable neighborhood of zero, zero is an isolated point in the set $\{x \in U \mid \frac{\partial f}{\partial x_0}(x) = \ldots = \frac{\partial f}{\partial x_n}(x) = 0\}$. A classical equivalent formulation for a singularity being isolated is that its *Milnor number*

$$\mu := \mu(f) := \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^{n+1},0} / \left\langle \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle \tag{1}$$

is finite. The ideal $\left\langle \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle$ is called *Jacobi ideal* and the complex vector space $\mathcal{O}_{\mathbb{C}^{n+1},0} / \left\langle \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle$ is called *Jacobi algebra*. The Milnor number is a topological invariant of an isolated hypersurface singularity. Aside from the definition as the complex vector space dimension of the Jacobi algebra, it can be also defined purely topologically via the Milnor fibration.

A Milnor fibration for f can be constructed as described in the following. The situation is visualized in Figure 1. We choose $\epsilon > 0$ in such a way that f is defined on the ball $B_\epsilon^{2n+2} := \{x \in \mathbb{C}^{n+1} \mid |x| < \epsilon\}$ around zero and $f^{-1}(0)$ is transversal to $\partial B_{\widetilde{\epsilon}}^{2n+2}$ for all $\widetilde{\epsilon} \leq \epsilon$. Moreover, we choose $\delta > 0$ such that $f^{-1}(\tau)$ is transversal to $\partial B_\epsilon^{2n+2}$ for all $\tau \in T_\delta := \{\tau \in \mathbb{C} \mid |\tau| < \delta\}$. We define the punctured disk $T'_\delta := T_\delta \setminus \{0\}$ and the sets $X(\epsilon, \delta) := B_\epsilon^{2n+2} \cap f^{-1}(T_\delta)$ as well as $X'(\epsilon, \delta) := X(\epsilon, \delta) \setminus f^{-1}(0)$. Then $f : X'(\epsilon, \delta) \to$

T'_δ is a locally trivial $\mathcal{C}^\infty$-fibration with fibers $X_\tau := f^{-1}(\tau)$ for $\tau \in T'_\delta$. This is the *Milnor fibration*, which can be understood as a good representative of the germ. The fibers X_τ

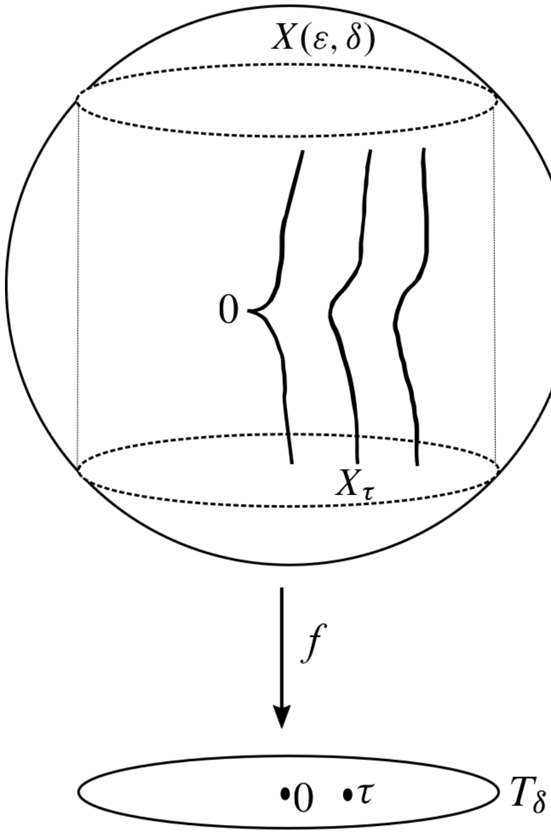

Figure 1: Milnor fibration for a singularity

of the Milnor fibration are homotopy equivalent to a bouquet of μ n-spheres (see [Mi68, Theorem 7.2]). If $u : T_\infty \to T'_\delta$ is the universal covering and $X_\infty := X'(\epsilon, \delta) \times_{T'_\delta} T_\infty$, then for all $\tau \in T_\infty$ the natural inclusion $X_{u(\tau)} \hookrightarrow X_\infty$ is a homotopy equivalence. So by taking the middle homology[1] with integer coefficients we obtain a $\mathbb{Z}$-lattice

$$Ml(f) := H_n(X_\infty, \mathbb{Z}) \cong H_n(X_\tau, \mathbb{Z}) \cong \mathbb{Z}^\mu \tag{2}$$

for any $\tau \in T'_\delta$. The lattice $Ml(f)$ is called *Milnor lattice* and will be of great importance for the rest of this thesis. Besides the classical Milnor lattice also the space $Ml(f)_\mathbb{C} := Ml(f) \otimes_\mathbb{Z} \mathbb{C}$ will appear in what follows.

The Milnor fibration defines a (geometric) monodromy diffeomorphism via parallel translation along the path $\gamma : [0,1] \to T_\delta, t \mapsto \delta e^{2\pi i t}$. The induced monodromy on the homology then gives a quasiunipotent monodromy automorphism $M_h : Ml(f) \to Ml(f)$. This is the *(Picard-Lefschetz) monodromy*. In the following the semisimple part of the monodromy M_h is denoted as M_s, the unipotent part as M_u and the nilpotent part as $N := \log M_u$.

The most significant properties of the Picard-Lefschetz monodromy follow from the Monodromy Theorem of Brieskorn and Grothendieck stated below. Here we present the Monodromy Theorem together with some supplement of van Doorn and Steenbrink.

1 For $n = 0$ we have to take the reduced (!) homology.

Theorem 2.1.1 (Monodromy Theorem, see [Br70] for parts (a)-(c) and [vDS89] for part (d))**.** *For the monodromy M_h of a singularity as above the following is true.*

(a) The eigenvalues of M_h are roots of unity.

(b) The size of the blocks in the Jordan normal form (= Jordan blocks*) of M_h is at most $(n+1)\times(n+1)$.*

(c) The size of the Jordan blocks of M_h for the eigenvalue 1 is at most $n\times n$.

(d) If there exists a Jordan block of (maximal) size $(n+1)\times(n+1)$, then there exists also a Jordan block of size $n\times n$ for the eigenvalue 1.

Often it is important to study certain "parts" of the Milnor lattice. Here, for some element $\lambda \in Ml\,(f)$ we will write

$$Ml\,(f)_\lambda := \ker\left(M_h - \lambda\,\mathrm{id}\right)^\mu : Ml\,(f)_{\mathbb{C}} \to Ml\,(f)_{\mathbb{C}}\,, \tag{3}$$
$$Ml\,(f)_{\neq\lambda} := \bigoplus_{\lambda'\neq\lambda} Ml\,(f)_{\lambda'}\,. \tag{4}$$

The Milnor lattice comes equipped with two important bilinear forms. The first one is the $(-1)^n$-symmetric *intersection form* $I : Ml\,(f)\times Ml\,(f)\to\mathbb{Z}$, which is given by the intersection numbers of the homology. The second one is the *Seifert form* $L : Ml\,(f)\times Ml\,(f)\to\mathbb{Z}$. It is unimodular and its precise construction can be found in [AGV88, Section I.2.3]. The three objects M_h, I and L are related via the following formulas from [AGV88, Section I.2.3]

$$L\,(M_h a, b) = (-1)^{n+1}\,L\,(b,a)\,, \tag{5}$$
$$I\,(a,b) = -\,L\,(a,b) + (-1)^{n+1}\,L\,(b,a)\,. \tag{6}$$

The two formulas above imply $I\,(a,b) = L\,((M_h - \mathrm{id})a, b)$. Hence, the eigenspace with eigenvalue 1 of M_h is the *radical* $\mathrm{Rad}(I)\subseteq Ml\,(f)$ of I. Note that for $n=1$, i.e. $f:(\mathbb{C}^2,0)\to(\mathbb{C},0)$, the radical $\mathrm{Rad}(I)$ is a $\mathbb{Z}$-lattice of rank $r-1$ where r is the number of branches of $f=\prod_{j=1}^r f_j$ (see [Ka96]). Generators of this lattice are given by any subset of cardinality $r-1$ of the classes $\{l_1,\dots,l_r\}$, where $l_j\in Ml\,(f)$ is obtained by pushing the (correctly oriented) cycle $\partial B_\epsilon^{2n+2}\cap f_j^{-1}\,(0)$ from the boundary of the fiber $f^{-1}\,(0)$ to the boundary of the fiber X_τ.

Since we want to achieve results which are independent of coordinate changes we will formulate many statements up to so-called right equivalence. It is the finest meaningful equivalence relation among isolated hypersurface singularities. Two singularities $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ and $g:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ are *right equivalent* (write $f\sim_{\mathcal{R}} g$) if there exists a biholomorphic map germ $\varphi:(\mathbb{C}^{n+1},0)\to(\mathbb{C}^{n+1},0)$ with $f=\varphi\circ g$. For sake of brevity we denote the group of biholomorphic map germs by $\mathcal{R}$.

Since the early days of singularity theory singularities are related to deformation processes. This is in particular related to the view of René Thom on the subject. He also coined the term unfolding of a singularity. For us an *unfolding* of a singularity f is a holomorphic function germ $F : (\mathbb{C}^{n+1} \times M, 0) \to (\mathbb{C}, 0)$ such that $F|_{(\mathbb{C}^{n+1}\times\{0\},0)} = f$. Here $(M, 0)$ denotes the germ of a manifold. There are certain special unfoldings which give rise to all other unfoldings. This will be subject of the next definition.

Definition 2.1.2. *An unfolding $F : (\mathbb{C}^{n+1} \times M, 0) \to (\mathbb{C}, 0)$ is* versal *if it induces any unfolding $G : (\mathbb{C}^{n+1} \times M', 0) \to (\mathbb{C}, 0)$, that means morphisms $\varphi : (M', 0) \to (M, 0)$ and $\Phi : (\mathbb{C}^{n+1} \times M', 0) \to (\mathbb{C}^{n+1} \times M, 0)$ with $\mathrm{pr}_M \circ\Phi = \varphi \circ \mathrm{pr}_{M'}$ and $\Phi|_{(\mathbb{C}^{n+1}\times\{0\},0)} = \mathrm{id}$ exist. Furthermore, a versal unfolding is called* semiuniversal *if* $\dim(M, 0)$ *is minimal.*

Note that semiuniversal unfoldings exist (see [AGV85] or [Ma68]).
Finally we define the *modality* of a singularity. It is the maximum dimension of the set of orbits of the group $\mathcal{R}$ of germs of biholomorphic coordinate transformations in a neighborhood of the orbit of the singularity. Singularities of modality zero, one and two are called *simple*, *unimodal* and *bimodal* singularities, respectively. We write $\mod(f)$ for the modality of f. There exists a famous list of polynomial representatives, so-called *normal forms*, of all singularities (modulo right equivalence) of modality at most two. This list [Ar76] is due to Vladimir Arnold (see also [Ar74] and [AGV85]) and will be of significant importance for what follows. Namely large parts of this monograph are concerned with working along that list and showing conjectures for all singularities on it. The easiest so-called *type* of a singularity is the type A_1. It consists of simple singularities of Milnor number $\mu = 1$ that are right equivalent to the normal form x_0^2.

2.1.2 *Distinguished bases and Coxeter-Dynkin diagrams*

Another classical notion associated to isolated hypersurface singularities is that of distinguished bases. To give the definition we choose a "good" representative F of the semiuniversal unfolding of f and a generic parameter $t \in M$. Then $F_t : X_t \to T_\delta$ with $T_\delta \subseteq \mathbb{C}$ as above and some $X_t \subseteq \mathbb{C}^{n+1}$ is a *morsification* of f. It has μ singularities of type A_1 and μ pairwise different corresponding critical values $u_1, \ldots, u_\mu \in T_\delta$. Now we fix a value $\tau \in (T_\delta \cap \mathbb{R}_{>0}) \setminus \{u_1, \ldots, u_\mu\}$ and choose a path γ_i from u_i to τ for each $i = 1, \ldots, \mu$, such that the paths intersect only at τ and arrive at τ in clockwise order. Lastly, shift from the singularity above each value u_i along γ_i to the Milnor fiber and call the image δ_i. Then the $(\delta_1, \ldots, \delta_\mu)$ are special $\mathbb{Z}$-bases of the Milnor lattice, the so-called *distinguished bases*. The elements of such a basis are called *vanishing cycles*. A distinguished basis is usually encoded in terms of *Stokes matrices* and *Coxeter-Dynkin diagrams*. A Stokes matrix associated to a distinguished basis $(\delta_1, \ldots, \delta_\mu)$ is an upper triangular $\mu \times \mu$ matrix with ones on the diagonal and entries $S_{i,j} := (-1)^{n(n+1)/2} \cdot I(\delta_i, \delta_j)$ for $i < j$. The Coxeter-Dynkin diagram of a distinguished basis (resp. its Stokes matrix $S = (S_{i,j})_{i,j=1,\ldots,\mu}$) is a graph with vertices $1, \ldots, \mu$, where two vertices i and j are connected via

$$\begin{cases} \text{no edge,} & \text{if } S_{i,j} = 0, \\ |S_{i,j}| \text{ edges,} & \text{if } S_{i,j} < 0, \\ S_{i,j} \text{ dotted edges,} & \text{if } S_{i,j} > 0. \end{cases} \tag{7}$$

Coxeter-Dynkin diagrams of several singularities had been computed explicitly. Most notably by Norbert A'Campo, Wolfgang Ebeling, Andrei Gabrielov and Sabir Gusein-Zade (e.g. see [Eb83], [Eb07] and [Ga74-2]). Below some first examples of Coxeter-Dynkin diagrams are given.

Example 2.1.3. *(i) The Coxeter-Dynkin diagram of a singularity of type A_1 is a graph consisting of just one vertex.*

(ii) Singularities of type $E_{3,0}$ have Milnor number $\mu = 16$ and modality two. The graph shown in Figure 2 is a Coxeter-Dynkin diagram for them (cf. [He93, Kapitel 3 b)]).

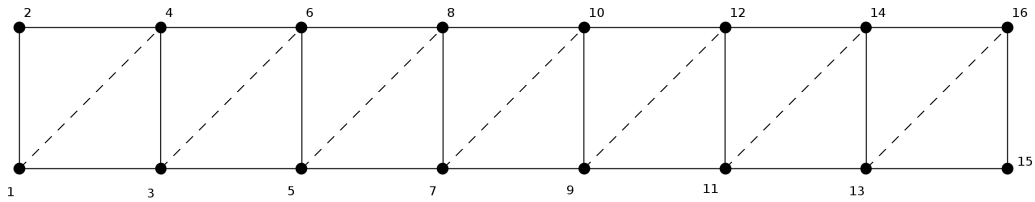

Figure 2: Coxeter-Dynkin diagram for (singularities of type) $E_{3,0}$

Many more Coxeter-Dynkin diagrams can be found in Chapter 4 of the monograph at hand. We close this subsection by describing how to compute the mondromy, the intersection form and the Seifert form from a given Coxeter-Dynkin diagram resp. Stokes matrix. First of all, the *Picard-Lefschetz transformation* on $Ml(f)$ of a vanishing cycle δ is $s_\delta(b) := b - (-1)^{n(n+1)/2} \cdot I(\delta, b) \cdot \delta$. Then the monodromy M_h is

$$M_h = s_{\delta_1} \circ \ldots \circ s_{\delta_\mu} \tag{8}$$

for any distinguished basis $\underline{\delta} = (\delta_1, \ldots, \delta_\mu)$. The matrices of the monodromy, Seifert form and intersection form with respect to a distinguished basis are given by formulas from [AGV88, Section I.2.3], namely by

$$M_h(\underline{\delta}) = \underline{\delta} \cdot (-1)^{n+1} \cdot S^{-1} S^T, \tag{9}$$

$$I\left(\underline{\delta}^T, \underline{\delta}\right) = (-1)^{n(n+1)/2} \cdot \left(S + (-1)^n \cdot S^T\right), \tag{10}$$

$$L\left(\underline{\delta}^T, \underline{\delta}\right) = (-1)^{(n+1)(n+2)/2} \cdot S^T. \tag{11}$$

These formulas show that the Seifert form L determines the intersection form I and the monodromy M_h. Hence every automorphism on the Milnor lattice, which respects the

Seifert form, also respects the monodromy and the intersection form. The group of all automorphisms which respect the Seifert form will be denoted as $G_{\mathbb{Z}}(f) := \mathrm{Aut}(Ml(f), L)$. It is of particular importance in the main part of this thesis.
Finally, there is a nondegenerate bilinear form $S : H^n(X_\infty, \mathbb{Q}) \times H^n(X_\infty, \mathbb{Q}) \to \mathbb{Q}$ which is invariant under the monodromy and determined by the Stokes matrix and the monodromy (see [He02-2, Section 10.6] for a precise definition). It is called the *polarizing form* and it will be important in the next section.

2.1.3 *Stabilization and the Thom-Sebastiani result*

In the following we often prove results only for a fixed number of variables $n+1$ — for example for *curve singularities* ($n=1$) or *surface singularities* ($n=2$). The general results follow then via the process of stabilization. This will be elucidated in this subsection.
For a given function germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ in the variables $x_0, \dots, x_n$ the function germ $f + x_{n+1}^2$ for an extra variable x_{n+1} is called the *stabilization*[2] of f. A function germ and its stabilization have very similar topological properties. In particular, the Stokes matrix remains the same under stabilization and it holds

$$G_{\mathbb{Z}}(f) = G_{\mathbb{Z}}\left(f + x_{n+1}^2\right). \tag{12}$$

Those properties follow from the more general framework of Thom and Sebastiani below (see [AGV88, Section I.2.7]). Namely, aside from just adding one singularity of type A_1, it is also interesting to add other singularities. In particular, for singularities $f = f(x_0, \dots, x_n) : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ and $g = g(y_0, \dots, y_m) : (\mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)$ there is an isomorphism

$$\Phi : Ml(f+g) \to Ml(f) \otimes Ml(g) \tag{13}$$

with $M_h(f+g) \cong M_h(f) \otimes M_h(g)$ for $f + g := f(x_0, \dots, x_n) + g(x_{n+1}, \dots, x_{m+n+1})$. The result of Thom and Sebastiani had been extended to Seifert forms by Deligne and to distinguished bases by Gabrielov (see [AGV88, Section I.2.7]). It holds

$$\begin{aligned} &L(f+g) \cong (-1)^{(n+1)(m+1)} \cdot L(f) \otimes L(g), \text{ and} \\ &S(f+g) \cong S(f) \otimes S(g) \end{aligned} \tag{14}$$

for appropriate distinguished bases. In the following we will refer to all those kind of statements as the *Thom-Sebastiani result.*
Finally, we have the famous Splitting Lemma (cf. [He11, Section 2]). Note that the *multiplicity* of a singularity f, write $\mathrm{mult}(f)$, is $\mathrm{mult}(f) := \max\left(k \mid f \in \mathfrak{m}^k\right)$ where $\mathfrak{m} \subseteq \mathcal{O}_{\mathbb{C}^{n+1},0}$ is the maximal ideal.

2 Sometimes authors prefer to call this *suspension* instead, e.g. see [He11].

Theorem 2.1.4 (Splitting Lemma). *Let $f_1, f_2 \in \mathfrak{m}^2 \subseteq \mathcal{O}_{\mathbb{C}^{n+1},0}$ be isolated hypersurface singularities. Then it is*

$$\operatorname{mult}(f_1) = 2 \Leftrightarrow f_1 \sim_{\mathcal{R}} g_1(x_0, \ldots, x_{n-1}) + x_n^2 \text{ for some } g_1 \in \mathfrak{m}^2 \subseteq \mathcal{O}_{\mathbb{C}^n,0}, \tag{15}$$
$$f_1 \sim_{\mathcal{R}} f_2 \Leftrightarrow f_1 + x_{n+1}^2 \sim_{\mathcal{R}} f_2 + x_{n+1}^2. \tag{16}$$

2.2 GAUSS-MANIN CONNECTION AND BRIESKORN LATTICE

The Gauss-Manin connection for isolated hypersurface singularities was described first in [Br70]. Inside the Gauss-Manin connection we have the so-called Brieskorn lattice, which is the central building block of the invariants we will define later. So a good understanding of the Gauss-Manin connection is essential for what follows. Our presentation here is similar to [He99, Section 4] and [HS99, Section 2]. It builds on results of Greuel, Hertling, Malgrange, K. Saito, M. Saito, Varchenko and others.

We consider the cohomology bundle $H^n := \bigcup_{\tau \in T'_\delta} H^n(X_\tau, \mathbb{C})$. It is a flat complex vector bundle. The μ-dimensional vector space of global flat multivalued sections in H^n can be identified with $H^n(X_\infty, \mathbb{C})$. It comes equipped with a $\mathbb{Z}$-lattice $H^n(X_\infty, \mathbb{Z}) \subseteq H^n(X_\infty, \mathbb{C})$ and a monodromy which we denote (by abuse of notation) also by M_h. Let $A \in H^n(X_\infty, \mathbb{C})_\lambda$ and $\alpha \in \mathbb{Q}$ with $e^{-2\pi i \alpha} = \lambda$, where $H^n(X_\infty, \mathbb{C})_\lambda$ is defined analogous to (3). Then

$$s(A, \alpha)(\tau) := \tau^\alpha \cdot \exp\left(\log \tau \cdot \frac{-N}{2\pi i}\right) A(\tau) \tag{17}$$

is a unique holomorphic section in H^n. Now let $\mathcal{H}^n$ be the sheaf of germs of holomorphic sections in H^n. Then there is a differential operator $\partial_\tau : (i_*\mathcal{H}^n)_0 \to (i_*\mathcal{H}^n)_0$ induced by the covariant derivative, where $i : T'_\delta \to T_\delta$ denotes an inclusion and the lower index 0 denotes stalks resp. germs at 0. Furthermore, there is a $\mathbb{C}\{\tau\}[\tau^{-1}]$-vector space $\mathcal{G}_0$ of dimension μ which is invariant under ∂_τ. It is called the *Gauss-Manin connection*. The Gauss-Manin connection is solely determined through the complex monodromy. The Gauss-Manin connection is

$$\mathcal{G}_0 = \bigoplus_{-1<\alpha\leq 0} \mathbb{C}\{\tau\}[\tau^{-1}]C^\alpha, \tag{18}$$

where $C^\alpha := \ker(\tau\partial_\tau - \alpha)^{n+1} \subseteq \mathcal{G}_0$ is a complex vector space. It admits a decreasing $V^\bullet$-filtration, the *Kashiwara-Malgrange filtration*, with

$$V^\alpha := V^\alpha \mathcal{G}_0 := \sum_{\alpha\leq\beta} \mathbb{C}\{\tau\}C^\beta = \bigoplus_{\alpha\leq\beta<\alpha+1} \mathbb{C}\{\tau\}C^\beta, \tag{19}$$
$$V^{>\alpha} := V^{>\alpha}\mathcal{G}_0 := \sum_{\alpha<\beta} \mathbb{C}\{\tau\}C^\beta = \bigoplus_{\alpha<\beta\leq\alpha+1} \mathbb{C}\{\tau\}C^\beta. \tag{20}$$

Note that V^{α} and $V^{>\alpha}$ are free $\mathbb{C}\{\tau\}$-modules of rank μ with $V^{\alpha}/V^{>\alpha} \cong C^{\alpha}$. Lastly, there is an isomorphism $\psi_{\alpha} : H^{n}\left(X_{\infty}, \mathbb{C}\right)_{\lambda} \to C^{\alpha}, A \mapsto s\left(A, \alpha\right)_{0}$.

Let $\omega \in \Omega^{n+1}_{\mathbb{C}^{n+1}}$ be a holomorphic $(n+1)$-form. Then the Gelfand-Leray form $\frac{\omega}{df}|_{X_{\tau}}$ gives a holomorphic section $s\left[\omega\right]\left(\tau\right)$ in the cohomology bundle H^{n} via

$$s\left[\omega\right]\left(\tau\right) := \left[\frac{\omega}{df}|_{X_{\tau}}\right] \in H^{n}\left(X_{\tau}, \mathbb{C}\right) \text{ for } \tau \in T'_{\delta}. \tag{21}$$

The germ $s\left[\omega\right]_{0} \in \left(i_{*}\mathcal{H}^{n}\right)_{0}$ of such a holomorphic section is in $\mathcal{G}_{0}$. Thus in the sense of (18) each germ $s\left[\omega\right]_{0}$ is an infinite sum of uniquely determined eigenspace parts $s\left(\omega, \alpha\right) \in C^{\alpha}$ with respect to $\tau\partial_{\tau}$, i.e.

$$s\left[\omega\right]_{0} = \sum_{\alpha > -1} s\left(\omega, \alpha\right). \tag{22}$$

The *order* of $\omega \in \Omega^{n+1}_{\mathbb{C}^{n+1},0}$ is $\alpha\left(s\left[\omega\right]\right) := \alpha\left(\omega\right) := \min\left(\alpha \,|\, s\left(\omega, \alpha\right) \neq 0\right)$ and the eigenspace part $s\left(\omega, \alpha\left(\omega\right)\right)$ is the principal part of $s\left[\omega\right]_{0}$.

According to the work of Malgrange [Ma74, Lemma 4.5], the growth towards 0 of the sections $s\left[\omega\right]\left(\tau\right)$ is bounded, i.e. we even have $s\left[\omega\right]_{0} \in V^{>-1}$. So there is a well-defined map

$$\Omega^{n+1}_{\mathbb{C}^{n+1},0} \to V^{>-1}, \omega \mapsto s\left[\omega\right]_{0}. \tag{23}$$

The kernel of this map is $df \wedge d\Omega^{n-1}_{\mathbb{C}^{n+1},0}$ (see [Ma74]).

Definition 2.2.1. *We define the* Brieskorn lattice *as*

$$H''_{0}\left(f\right) := \Omega^{n+1}_{\mathbb{C}^{n+1},0}/df \wedge d\Omega^{n-1}_{\mathbb{C}^{n+1},0}. \tag{24}$$

The Brieskorn lattice and the fine analytic information which it carries were studied first in [Br70]. Here we will think of it as a subset of $V^{>-1}$ (via the map (23)). By the Leray Residue Theorem (cf. [Br70]) the inverse elements of $H''_{0}\left(f\right) \subseteq V^{>-1}$ satisfy

$$\partial_{\tau}^{-1} s\left[d\eta\right]_{0} = s\left[df \wedge \eta\right]_{0} \text{ for } \eta \in \Omega^{n+1}_{\mathbb{C}^{n+1},0}. \tag{25}$$

Now to justify the name Brieskorn "lattice" (cf. [SaM89]) we have to look at the equations

$$\tau \cdot s\left[\omega\right]_{0} = s\left[f \cdot \omega\right]_{0}, \tag{26}$$

$$\mathbb{C}\{\tau\}\left[\tau^{-1}\right] \cdot H''_{0}\left(f\right) = \mathcal{G}_{0}. \tag{27}$$

They show that $H''_{0}\left(f\right)$ is a free $\mathbb{C}\{\tau\}$-module of rank μ. To justify the use of the symbol $H''_{0}\left(f\right)$ it is necessary to know that Brieskorn also studied the object

$$H'_{0}\left(f\right) := df \wedge d\Omega^{n}_{\mathbb{C}^{n+1},0}/df \wedge \Omega^{n-1}_{\mathbb{C}^{n+1},0}. \tag{28}$$

They are connected via the formulas (cf. (25))

$$\partial_\tau^{-1} H_0''(f) = H_0'(f)\,, \tag{29}$$

$$H_0''(f)\,/\,H_0'(f) = \Omega^{n+1}_{\mathbb{C}^{n+1},0}/df \wedge \Omega^{n}_{\mathbb{C}^{n+1},0}, \tag{30}$$

$$\dim H_0''(f)\,/\,H_0'(f) = \mu. \tag{31}$$

So the Brieskorn lattice $H_0''(f)$ can be also considered as a free $\mathbb{C}\{\{\partial_\tau^{-1}\}\}$-module of rank μ. The same holds for $V^{>-1}$.

We can define a ∂_τ^{-1}-sesquilinear pairing K_f on $V^{>-1}$ via the polarizing form S (cf. Subsection 2.1.2). Restricted to the Brieskorn lattice it coincides with the restriction of K. Saito's higher residue pairing to the Brieskorn lattice. The properties of K_f are summarized in the following proposition.

Proposition 2.2.2. *There is a unique pairing*

$$K_f : V^{>-1} \times V^{>-1} \to \mathbb{C}\{\{\partial_\tau^{-1}\}\} \tag{32}$$

with the properties in (i)–(iv). Let $A \in H^n(X_\infty, \mathbb{C})_{e^{-2\pi i\alpha}}$ and $B \in H^n(X_\infty, \mathbb{C})_{e^{-2\pi i\beta}}$.

(i) For α, $\beta \in (-1, 0)$ with $\alpha + \beta = -1$ it is

$$K_f(s(A,\alpha), s(B,\beta)) = \frac{1}{(2\pi i)^n} S(A,B) \cdot \partial_\tau^{-1}. \tag{33}$$

(ii) For $\alpha = \beta = 0$ it is

$$K_f(s(A,\alpha), s(B,\beta)) = \frac{-1}{(2\pi i)^{n+1}} S(A,B) \cdot \partial_\tau^{-2}. \tag{34}$$

(iii) For $\alpha, \beta \in \mathbb{R}_{>-1}$ with $\alpha + \beta \notin \mathbb{Z}$ it is

$$K_f : C^\alpha \times C^\beta \to 0. \tag{35}$$

(iv) For $a, b \in V^{>-1}$ it is

$$\partial_\tau^{-1} \cdot K_f(a,b) = K_f(\partial_\tau^{-1} a, b) = K_f(a, -\partial_\tau^{-1} b). \tag{36}$$

It satisfies also (for $\alpha, \beta \in \mathbb{R}_{>-1}$)

$$K_f : C^\alpha \times C^\beta \to \mathbb{C} \cdot \partial_\tau^{-\alpha-\beta-2} \qquad \text{if } \alpha+\beta \in \mathbb{Z}, \tag{37}$$

$$K_f(\tau a, b) - K_f(a, \tau b) = [\tau, K_f(a,b)] \qquad \text{for } a, b \in V^{>-1}, \tag{38}$$

where $[\tau, \partial_\tau^{-k}] = k\partial_\tau^{-k-1}$. If one writes $K_f(a,b) = \sum_{k\geq 1} K_f^{(-k)}(a,b) \cdot \partial_\tau^{-k}$ with $K_f^{(k)}(a,b) \in \mathbb{C}$, then $K_f^{(-k)}$ is $(-1)^{k+n+1}$-symmetric.

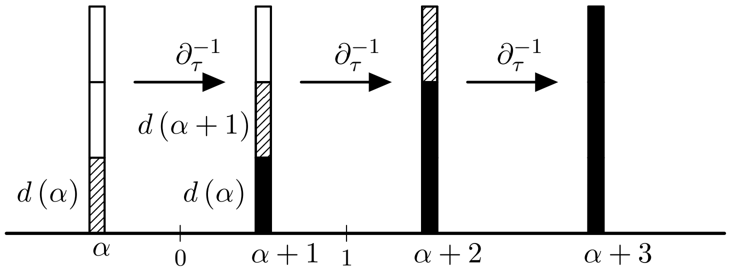

Figure 3: Spectral data (schematic)

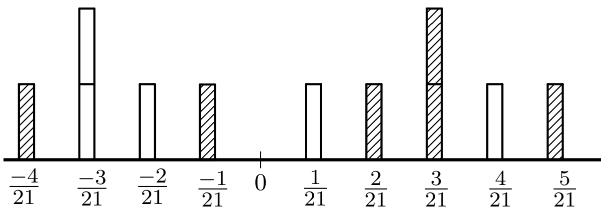

Figure 4: Spectral data of U_{24}

In what follows information on the position of the Brieskorn lattice inside the Gauss-Manin connection will be crucial. Some of this information is captured by the so-called *spectrum*. It consists of a tuple of μ rational numbers with multiplicities $d(\alpha)$, where

$$d(\alpha) := \dim \underbrace{V^\alpha \cap H_0''(f)/V^{>\alpha}}_{=:\mathbf{Gr}_V^\alpha H_0''(f)} - \dim V^\alpha \cap \partial_\tau^{-1} H_0''(f)/V^{>\alpha}. \tag{39}$$

Those rational numbers are called *spectral numbers*. For spectral numbers $(\alpha_1, \ldots, \alpha_\mu)$ the eigenvalues of the monodromy of the respective singularity are $e^{-2\pi i \alpha_i}$ for $i = 1, \ldots \mu$. Moreover, the spectral numbers satisfy the symmetry relation $d(\alpha) = d(n-1-\alpha)$. The structure of the spectral numbers and the spaces C^α can be visualized in pictures like Figure 3.

Here the total number of bars above a value α, $\alpha+1$, $\alpha+2$ or $\alpha+3$ gives the dimension of the eigenspace space C^α, $C^{\alpha+1}$, $C^{\alpha+2}$ or $C^{\alpha+3}$, respectively. The number of shaded bars above a value α, $\alpha+1$, $\alpha+2$ or $\alpha+3$ is the respective multiplicity $d(\alpha)$, $d(\alpha+1)$, $d(\alpha+2)$ or $d(\alpha+3)$. Finally, the number of shaded and black bars above a value α, $\alpha+1$, $\alpha+2$ or $\alpha+3$ is the dimension of the space $\mathbf{Gr}_V^\alpha H_0''(f)$, $\mathbf{Gr}_V^{\alpha+1} H_0''(f)$, $\mathbf{Gr}_V^{\alpha+2} H_0''(f)$ or $\mathbf{Gr}_V^{\alpha+3} H_0''(f)$, respectively. An example of such a picture associated to a true isolated hypersurface singularity is given in the example below.

Example 2.2.3. *The Brieskorn-Pham singularity*[3] $x^3+y^3+z^7$ *has the spectral numbers* $(\alpha_1,\ldots,\alpha_{24}) = \left(-\frac{4}{21}, -\frac{1}{21}, \frac{2}{21}, \frac{1}{7}, \frac{1}{7}, \frac{5}{21}, \frac{2}{7}, \frac{2}{7}, \frac{8}{21}, \frac{3}{7}, \frac{3}{7}, \frac{10}{21}, \frac{11}{21}, \frac{4}{7}, \frac{4}{7}, \frac{13}{21}, \frac{5}{7}, \frac{5}{7}, \frac{16}{21}, \frac{6}{7}, \frac{6}{7}, \frac{19}{21}, \frac{22}{21}, \frac{25}{21}\right)$. *Those numbers can be computed easily via the library* gmssing.lib *of the computer algebra system* SINGULAR, *see [Sch01] and [DGPS16]. Knowing the spectral numbers we can draw the picture in Figure 4. It contains all eigenspaces C^α between $\frac{-4}{21}$ and $\frac{5}{21}$. The singularity $x^3+y^3+z^7$ will be studied in more detail in Section 4.6.*

3 A *Brieskorn-Pham singularity* is a singularity of the form $x_0^{p_0} + x_1^{p_1} + \ldots + x_n^{p_n}$.

Note that *Steenbrink's Hodge filtration* is given via the relation

$$F^p_{St}H^n\left(X_\infty,\mathbb{C}\right)_\lambda \quad = \quad \psi_\alpha^{-1}\left(\partial_\tau^{n-p}\,\mathbf{Gr}_V^{n-p+\alpha}\,H_0''(f)\right) \tag{40}$$

with $\alpha\in(-1,0]$. Originally, Steenbrink defined the Hodge filtration $F^\bullet_{St}$ via the Hironaka resolution of singularities (see [St77]). Later, Varchenko [Va80] constructed a closely related Hodge filtration $F^\bullet_{Va}$ from the Brieskorn lattice $H_0''(f)$. Scherk and Steenbrink and also M. Saito modified this construction to obtain $F^\bullet_{St}$ (see [SchS85] and [SaM89]).

Now let $V^{>\infty}(f)$ be the space of germs of sections that have moderate growth at 0. Then in [He99] a classifying space $D_{PMHS}\left(f\right)$ for polarized mixed Hodge structures and a classifying space $D_{BL}\left(f\right)$ for $\mathbb{C}\{\tau\}$-lattices in $V^{>\infty}(f)$ was constructed. More precisely, $D_{PMHS}\left(f\right)$ is a classifying space for M_s-invariant Hodge filtrations $F^\bullet$ on $H^n\left(X_\infty,\mathbb{C}\right)$ with weight filtration $W_\bullet$ on $H^n\left(X_\infty,\mathbb{Q}\right)$ such that $(H^\infty_{\neq 1},H^\infty_{\neq 1,\mathbb{Z}},F^\bullet,W_\bullet,-N,S)$ and $(H^\infty_1,H^\infty_{1,\mathbb{Z}},F^\bullet,W_\bullet,-N,S)$ are polarized mixed Hodge structures of weight n and $n+1$, respectively, with the same Hodge numbers as $F^\bullet_{St}$. $D_{BL}\left(f\right)$ is the *classifying space for Brieskorn lattices* given by

$$\begin{aligned}D_{BL}\left(f\right) := \{\mathcal{L}_0\subseteq V^{>-1}\,|\,&\mathcal{L}_0 \text{ is a free } \mathbb{C}\{\tau\}\text{-module of rank } \mu,\\ &\mathcal{L}_0 \text{ is a free } \mathbb{C}\{\{\partial_\tau^{-1}\}\}\text{-module of rank } \mu,\\ &\text{The filtration } F^\bullet \text{ on } H^\infty_{\mathbb{C}}(f_0) \text{ defined by (40)}\\ &\text{with } \mathcal{L}_0 \text{ instead of } H_0''(f_0) \text{ is in } D_{PMHS}\left(f\right),\\ &K_f(\mathcal{L}_0,\mathcal{L}_0)\subseteq \partial_\tau^{-n-1}\cdot\mathbb{C}\{\{\partial_\tau^{-1}\}\}\}\end{aligned} \tag{41}$$

According to the conditions imposed on the elements of $D_{BL}\left(f\right)$ they are *marked Brieskorn(-like) lattices*. The structure of the spaces $D_{PMHS}\left(f\right)$ and $D_{BL}\left(f\right)$ is described in the next proposition.

Proposition 2.2.4 ([He99, Sections 2 and 5])**.** *Fix a singularity $f(x_0,\ldots,x_n)$.*

(a) $D_{PMHS}\left(f\right)$ is a real homogeneous space and a complex manifold. The group $G_{\mathbb{Z}}(f)$ acts properly discontinuously[4] on $D_{PMHS}\left(f\right)$.

(b) $D_{BL}\left(f\right)$ is a complex manifold and a locally trivial bundle over $D_{PMHS}\left(f\right)$. The fibers have a natural $\mathbb{C}^*$-action with negative weights and are affine algebraic manifolds. Moreover they are isomorphic to $\mathbb{C}^{N_{BL}}$ for some $N_{BL}\in\mathbb{Z}_{\geq 0}$. The group $G_{\mathbb{Z}}\left(f\right)$ acts properly discontinuously on $D_{BL}\left(f\right)$.

(c) The two spaces $D_{PMHS}(f)$ and $D_{PMHS}(f+x_{n+1}^2)$ and the two spaces $D_{BL}(f)$ and $D_{BL}(f+x_{n+1}^2)$ are canonically isomorphic.

Note that part (c) of Proposition 2.2.4 does not come from [He99], but it can be deduced easily from [BH18, Theorem 4.7]. In the following we will employ $D_{BL}\left(f\right)$ as a sort of Siegel upper half-space for singularities.

4 Loosely speaking, this means that each point in $D_{PMHS}\left(f\right)$ has a neighborhood with the property that all "non-trivial" elements of $G_{\mathbb{Z}}\left(f\right)$ move this neighborhood outside itself.

2.3 μ-CONSTANT FAMILIES OF ISOLATED HYPERSURFACE SINGULARITIES

In this thesis we usually do not consider a single singularity, but rather families of singularities. One such family is the semiuniversal unfolding, which already appeared in Section 2.1. Additionally the following "smaller" family is of particular interest.

Definition 2.3.1. *A holomorphic μ-constant family of singularities consists of a number $\mu \in \mathbb{Z}_{\geq 1}$, a complex manifold T, an open neighborhood $X \subseteq \mathbb{C}^{n+1} \times T$ of $\{0\} \times T$ and a holomorphic function $F : X \to \mathbb{C}$ such that $F_t := F|_{X_t}$ with $X_t := X \cap \mathbb{C}^{n+1} \times \{t\}$ for any $t \in T$ has an isolated singularity at 0 with Milnor number μ.*

The properties of a μ-constant family of singularities are discussed now. Loosely speaking, the topological properties of singularities inside a μ-constant family stay the same while the analytic properties vary. This is made precise in Theorem 2.3.2. Details on the proof of this rather classical theorem can be found in [AGV88], [Ku98], [He93, Kapitel 2], [He11], [Va80-2] and [Va82]. In particular, parts (a) and (b) of Theorem 2.3.2 can be found in [He11, Theorem 2.2], part (c) can be found in [Ku98] and part (d) originates from [Va82]. Yet a proof of this theorem will be omitted here.

Theorem 2.3.2. *Consider a holomorphic μ-constant family as in Definition 2.3.1. Then the following statements are true.*

(a) The Milnor lattices $(Ml(F_t), L)$ with Seifert forms for parameters $t \in T$ are locally canonically isomorphic. They glue to a local system $\bigcup_{t \in T} Ml(F_t)$ of free $\mathbb{Z}$-modules of rank μ.

(b) Therefore also the spaces $C^\alpha(F_t)$, $V_\tau^{>\infty}(F_t)$, $V_\tau^\alpha(F_t)$ are locally canonically isomorphic and glue to local systems.

(c) But the Brieskorn lattices $H_0''(F_t) \subseteq V_\tau^{>-1}(F_t)$ vary holomorphically. For $\omega \in \Omega^{n+1}_{X/T}$ it is $s[\omega]_0(t) := s[\omega|_{X_t}]_0 \in H_0''(F_t)$. Let ξ be a holomorphic vector field on T. Its canonical lifts to $\mathbb{C} \times T$ (with coordinate τ on $\mathbb{C}$) and X are also denoted ξ. The covariant derivative of $s[\omega]_0(t)$ by ξ is

$$\xi \; s[\omega]_0(t) = s[\mathrm{Lie}_\xi \, \omega]_0(t) + (-\partial_\tau) s[\xi(F) \cdot \omega]_0(t). \tag{42}$$

(d) All germs F_t have the same spectrum.

2.4 TORELLI PROBLEMS FOR SINGULARITIES

The question on *how singularities inside a μ-constant family can vary modulo right equivalence* is very classical. It appears already in [AGV88, Part III] and is sometimes referred to as *Varchenko's conjecture* (cf. [Ku98, p. 174]). Varchenko tried to answer this question via introducing certain invariants, the so-called Picard-Fuchs singularities, that vary continuously with the right equivalence class inside a μ-constant family. However it turned out that the Picard-Fuchs singularities do not determine the right equivalence class in the whole

μ-constant family if exceptional unimodal singularities are involved. An example where the Picard-Fuchs singularities fail for a singularity of type E_{12} can be found in [He93, Satz 3.9 c)] or [Ku98, Part III]. That is why Claus Hertling introduced refined versions[5] of the Picard-Fuchs singularities. The refined invariants are constructed along the same lines as the invariants in the famous Torelli Theorem for compact Riemann surfaces, see Theorem 2.4.1. This is what we will explain next.

Let f be an isolated hypersurface singularity. Furthermore, let $\omega_1, \ldots, \omega_\mu$ be a $\mathbb{C}\{\tau\}$-basis of $H_0''(f)$ and $\delta_1, \ldots, \delta_\mu$ be a basis of $Ml(f)$. Then a period matrix $\mathcal{I} := (\mathcal{I}_{k,j})_{k,j=1,\ldots,\mu}$ of f is given by $\mathcal{I}_{k,j} := \left(\int_{\delta_k(\tau)} \omega_j / df\right)$ where $\int_{\delta_k(\tau)} \omega_j / df = \int_{\partial(\delta_k)} \frac{w_j}{f-\tau}$. The *Picard-Fuchs singularity* $PFS(f)$ of f is the equivalence class of period matrices of f modulo base change. This means modulo the equivalence relation

$$\mathcal{I}' \sim \mathcal{I} :\Leftrightarrow \exists B \in \mathrm{GL}(\mu; \mathbb{C}\{\tau\}), U \in \mathrm{GL}(\mu; \mathbb{Z}) \text{ with } \mathcal{I}' = B\mathcal{I}U \tag{43}$$

for period matrices $\mathcal{I}$ and $\mathcal{I}'$ of f. The invariant $PFS(f)$ has the weakness mentioned above. Therefore we want to refine it now. We use the Riemannian pair of matrices in the classical Torelli theorem for compact Riemann surfaces as a model for proper invariants.

Theorem 2.4.1 (Torelli Theorem, see [To13] and [He95, Section (3.1)]). *Let X be a Riemann surface of genus g. Moreover let $\omega_1, \ldots, \omega_g$ be a basis of $H^0(X, \Omega^1)$ and $\delta_1, \ldots, \delta_{2g}$ be a basis of $H_1(X, \mathbb{Z})$. Then the Riemann surface X is uniquely determined by the equivalence class modulo base change of the Riemannian pair of matrices (A, C) where $C := \left(\int_{\delta_j} \omega_i\right)$ and $A := (\delta_i \circ \delta_j)$.*

So in analogy to the classical Torelli Theorem 2.4.1 we consider pairs of matrices in the case of singularities as well. The first refined invariant $SBG(f)$ is then the equivalence class $(\mathcal{I}, I)$ of the period matrix $\mathcal{I}$ and the intersection form I modulo the equivalence relation

$$\begin{aligned}(\mathcal{I}, I) \sim (\mathcal{I}', I') :\Leftrightarrow \exists B \in \mathrm{GL}(\mu; \mathbb{C}\{\tau\}), U \in \mathrm{GL}(\mu; \mathbb{Z}) \text{ with } \mathcal{I}' = B\mathcal{I}U \\ \text{and } I' = U^T I U.\end{aligned} \tag{44}$$

The second refined invariant $LBL(f)$ is the equivalence class $(\mathcal{I}, L)$ of the period matrix $\mathcal{I}$ and the Seifert form L modulo the equivalence relation

$$\begin{aligned}(\mathcal{I}, L) \sim (\mathcal{I}', L') :\Leftrightarrow \exists B \in \mathrm{GL}(\mu; \mathbb{C}\{\tau\}), U \in \mathrm{GL}(\mu; \mathbb{Z}) \text{ with } \mathcal{I}' = B\mathcal{I}U \\ \text{and } L' = U^T L U.\end{aligned} \tag{45}$$

The Seifert form determines the intersection form as explained earlier in Subsection 2.1.2. Thus, the invariant $SBG(f)$ is finer than $PFS(f)$ but possibly coarser than $LBL(f)$. In contrast to $PFS(f)$ the invariants $SBG(f)$ and $LBL(f)$ are both fine enough to deal with all exceptional unimodal singularities. Currently there is no tangible example known where the invariant $SBG(f)$ fails in the same way as $PFS(f)$ does for singularities of

5 Note that some authors call these refined invariants the *Hertling invariants*, e.g. see [Ku98].

type E_{12}. Nevertheless we avoid working with $SBG(f)$ here and consider the invariant $LBL(f)$ right away.
There is a second way to construct $LBL(f)$. Namely, as the isomorphism class of the triple $(Ml(f), L, H_0''(f))$. Both constructions yield equivalent invariants by [He95, (3.4) Proposition]. Since $D_{BL}/G_{\mathbb{Z}}$ can be also thought of as the set of isomorphism classes for such tuples (e.g. cf. [He98, p. 180]), it is $LBL(f) \in D_{BL}/G_{\mathbb{Z}}$.

Conjecture 2.4.2 (Torelli Conjecture (for hypersurface singularities))**.** *The invariant $LBL(f)$ of a hypersurface singularity f determines the right equivalence class of f.*

This conjecture was formulated first in [He93, Kapitel 2 d)]. It holds for all simple, unimodal and bimodal singularities.

In algebraic geometry one usually wants to study and classify objects in terms of a moduli space. Such a moduli space of singularities $M_\mu(f)$ is constructed in [He02-2, Part 2]. Set-theoretically it is

$$M_\mu(f) := \{g \text{ in the } \mu\text{-homotopy class of } f\}/\sim_{\mathcal{R}} . \tag{46}$$

Moreover it is locally isomorphic to the μ-constant stratum modulo some finite group and from the μ-constant stratum it also inherits a canonical complex structure (see [He02-2, Section 13.3]). Recall that the *μ-constant stratum* of a singularity is the analytic variety

$$S_\mu := \{t \in M \mid F_t \text{ has only one singularity } x_0 \text{ and } F_t(x_0) = 0\}, \tag{47}$$

where M is the base space of a semiuniversal unfolding. The dimension of the μ-constant stratum S_μ is the modality by [Ga74]. Now we can reformulate the Torelli Conjecture 2.4.2 in a more abstract, but equivalent (!), version.

Conjecture 2.4.3 (Torelli Conjecture (for hypersurface singularities))**.** *The period map*

$$LBL : M_\mu(f) \to D_{BL}/G_{\mathbb{Z}}, [g] \mapsto LBL(g) \tag{48}$$

is injective.

We will see a sharpening (with the help of markings of singularities) of this conjecture in the next chapter.

3 THE MODULI SPACE OF MARKED SINGULARITIES

As mentioned earlier the moduli space of singularities $M_\mu(f)$ is locally isomorphic to the μ-constant stratum modulo some finite group. However it is desirable to get rid of this finite group and to construct a moduli space which is locally isomorphic to the μ-constant stratum only. That is why Claus Hertling introduced in [He11] the notion of a *marking* of a singularity. This allows him to construct a moduli space $M_\mu^{mar}(f_0)$ which is everywhere locally isomorphic to the μ-constant stratum and, hence, can be seen as a global μ-constant stratum (see Definition 3.1.3). Now in Section 3.1 we recall the construction of this moduli space $M_\mu^{mar}(f_0)$. Note that the idea of the construction of this moduli space is closely related to that of Teichmüller space for compact Riemann surfaces. In Section 3.2 we state the Torelli Conjectures for $M_\mu^{mar}(f_0)$ from [He11]. Finally in Section 3.3 we compare $M_\mu^{mar}(f_0)$ with the Teichmüller space.

3.1 MARKINGS OF SINGULARITIES

The construction of the moduli space of marked singularities is inspired by Teichmüller theory for (marked) compact Riemann surfaces. So it is crucial to assign markings also to singularities (see Definition 3.1.1). Note that as in classical Teichmüller theory it is necessary to choose some sort of base point. Here this is the reference singularity.

Definition 3.1.1 ([He11, Definition 4.1])**.** *Let f_0 be an arbitrary but fixed singularity, the* reference singularity*.*

(a) A strongly marked singularity *is a couple (f, ρ) where f is a singularity, such that there is a μ-constant family which contains f and f_0, and*

$$\rho : (Ml(f), L) \to (Ml(f_0), L)$$

is an isomorphism.

(b) Two strongly marked singularities (f, ρ) and (g, τ) are called right equivalent *(write: $(f, \rho) \sim_R (g, \tau)$), if f and g are right equivalent in the unmarked sense with $f = g \circ \varphi$ and $\rho = \tau \circ \varphi_{hom}$ where $\varphi_{hom} : (Ml(f), L) \to (Ml(g), L)$ is the induced isomorphism.*

(c) A marked singularity *is a couple $(f, \pm\rho)$ where f is a singularity, such that there is a μ-constant family which contains f and f_0, and*

$$\rho : (Ml(f), L) \to (Ml(f_0), L)$$

is an isomorphism. In particular, we refer to the morphism ρ as the marking *of the marked singularity $(f, \pm\rho)$.*

(d) Two marked singularities $(f, \pm\rho)$ and $(g, \pm\tau)$ are called right equivalent *(write: $(f, \pm\rho) \sim_R (g, \pm\tau)$), if f and g are right equivalent in the unmarked sense with $f = g \circ \varphi$ and $\rho = \pm\tau \circ \varphi_{hom}$ where $\varphi_{hom} : (Ml(f), L) \to (Ml(g), L)$ is the induced isomorphism.*

At first sight the sign ambiguity of the marking in the notion of a marked singularity seems to be very peculiar. However, it is not known in general (but conjectured (!)) whether all μ-homotopy families of singularities satisfy one of the following two properties:

$$\text{Any singularity in the } \mu\text{-homotopy class of } f_0 \text{ has multiplicity} \geq 3, \tag{49}$$

$$\text{Any singularity in the } \mu\text{-homotopy class of } f_0 \text{ has multiplicity } 2. \tag{50}$$

For μ-homotopy families where neither of the two properties (resp. assumptions) hold, the notion of strongly marked singularities behaves badly (see [He11, Remarks 4.2] for further details). Luckily, for all families of singularities studied in this thesis one of the assumptions is correct. If assumption (49) is true we have the following conjecture in addition.

Conjecture 3.1.2 ([He11, Conjecture 3.2 (b)])**.** *If the μ-homotopy class of f_0 satisfies assumption (49), then $-\mathrm{id} \notin G^{smar}(f_0)$.*

Now we start with defining the moduli space of marked singularities and giving some of its most important properties.

Definition 3.1.3 ([He11, Theorem 4.3])**.** *Let f_0 be a fixed reference singularity. We define the sets:*

$$M_\mu^{smar}(f_0) := \{\textit{strongly marked } (f, \rho) \mid f \textit{ in the } \mu\textit{-homotopy class of } f_0\} / \sim_R, \tag{51}$$

$$M_\mu^{mar}(f_0) := \{\textit{marked } (f, \pm\rho) \mid f \textit{ in the } \mu\textit{-homotopy class of } f_0\} / \sim_R . \tag{52}$$

(a) $M_\mu^{mar}(f_0)$ carries a natural canonical complex structure. It can be constructed with the underlying reduced complex structure as an analytic geometric quotient[1].

(b) The germ $\left(M_\mu^{mar}(f_0), [(f, \pm\rho)]\right)$ with its canonical complex structure is isomorphic to the μ-constant stratum of f with its canonical complex structure (see [He02-2, Chapter 12] for the definition of that).

(c) For any $\psi \in G_{\mathbb{Z}}(f_0)$ we obtain an automorphism

$$\psi_{mar} : M_\mu^{mar}(f_0) \to M_\mu^{mar}(f_0), [(f, \pm\rho)] \mapsto [(f, \pm\psi \circ \rho)]. \tag{53}$$

There is a group action from the left

$$G_{\mathbb{Z}}(f_0) \times M_\mu^{mar}(f_0) \to M_\mu^{mar}(f_0), (\psi, [(f, \pm\rho)]) \mapsto \psi_{mar}([(f, \pm\rho)]). \tag{54}$$

(d) The action of $G_{\mathbb{Z}}(f_0)$ on $M_\mu^{mar}(f_0)$ is properly discontinuous. The quotient $M_\mu^{mar}(f_0) / G_{\mathbb{Z}}(f_0)$ is the moduli space M_μ for right equivalence classes in the μ-

1 See [Pu07, Section 1.2] for the definition of an *analytic geometric quotient*.

homotopy class of f_0 with its canonical complex structure. In particular, $[(f_1, \pm\rho_1)]$ and $[(f_2, \pm\rho_2)]$ are in one $G_{\mathbb{Z}}(f_0)$-orbit if and only if f_1 and f_2 are right equivalent.

In [He11] it was conjectured that the moduli $M_\mu^{mar}(f_0)$ space is connected (see [He11, Conjecture 3.2]). However the results in this thesis show that this is not true for certain subseries of bimodal series singularities (see Section 4.4). So in general $M_\mu^{mar}(f_0)$ consists of different connected components. We denote the connected component of $M_\mu^{mar}(f_0)$ and $M_\mu^{smar}(f_0)$ that contains the reference singularity as $M_\mu^{mar}(f_0)^0$ and $M_\mu^{smar}(f_0)^0$, respectively. The following theorem gives a good characterization of the connected components of the moduli space of marked singularities.

Theorem 3.1.4 ([He11, Theorem 4.4 (a),(b),(e)]). *We are in the situation as above.*

(a) The map

$$\begin{aligned} G_{\mathbb{Z}}(f_0)/G^{mar}(f_0) &\to \{\textit{topological components of } M_\mu^{mar}\} \\ \psi \cdot G^{mar}(f_0) &\mapsto (\textit{the component } \psi_{mar}((M_\mu^{mar})^0)) \end{aligned}$$

is a bijection.

(b) If assumption (49) or (50) holds then (a) is also true for M_μ^{smar} and $G^{smar}(f_0)$.

(c) $-\mathrm{id} \in G_{\mathbb{Z}}(f_0)$ acts trivially on $M_\mu^{mar}(f_0)$. Suppose that assumption (50) holds and that $f_0 = g_0(x_0, \dots, x_{n-1}) + x_n^2$. Then $-\mathrm{id}$ acts trivially on $M_\mu^{smar}(f_0)$ and

$$M_\mu^{smar}(f_0) = M_\mu^{mar}(f_0) = M_\mu^{mar}(g_0), \tag{55}$$
$$G^{smar}(f_0) = G^{mar}(f_0) = G^{mar}(g_0). \tag{56}$$

Suppose additionally that assumption (49) holds for g_0 (instead of f_0). Then $\{\pm\mathrm{id}\}$ acts freely on $M_\mu^{smar}(g_0)$, and the quotient map

$$M_\mu^{smar}(g_0) \xrightarrow{/\{\pm \mathrm{id}\}} M_\mu^{mar}(g_0), \quad [(f,\rho)] \mapsto [(f, \pm\rho)]$$

is a double covering.

Besides the moduli space $M_\mu^{mar}(f_0)$ Claus Hertling also introduced in [He11] certain subgroups of $G_{\mathbb{Z}}(f_0)$. Namely, the groups

$$G^{smar}(f_0) := \{\psi \in G_{\mathbb{Z}}(f_0) \mid \psi \text{ maps } M_\mu^{smar}(f_0)^0 \text{ to itself}\}, \tag{57}$$
$$G^{mar}(f_0) := \{\psi \in G_{\mathbb{Z}}(f_0) \mid \psi \text{ maps } M_\mu^{mar}(f_0)^0 \text{ to itself}\}, \tag{58}$$
$$G_{\mathcal{R}}^{smar}(f_0) := \{\varphi_{hom} \mid \varphi \in \mathcal{R} \text{ with } f_0 \circ \varphi = f_0\}, \tag{59}$$
$$G_{\mathcal{R}}^{mar}(f_0) := \{\pm\psi \mid \psi \in G_{\mathcal{R}}^{smar}(f_0)\}. \tag{60}$$

The group $G^{mar}(f_0)$ is called the *μ-constant monodromy group* of f. It holds $G_{\mathcal{R}}^{smar}(f_0) \subseteq G^{smar}(f_0) \subseteq G^{mar}(f_0)$. In fact, $G^{mar}(f_0)$ was originally constructed in [He11] as the subgroup of elements of $G_{\mathbb{Z}}(f_0)$ that can be realized as transversal monodromies of some

μ-constant families modulo (± 1) (cf. [GH17, Section 5]). This characterization of $G^{mar}(f_0)$ will be important for us later on. Finally we will also need the group

$$G_{\mathcal{R}}^{smar,gen}(f_0) := \bigcap_{[(f,\rho)] \in M_\mu^{smar}} \rho^{-1} \circ G_{\mathcal{R}}^{smar}(f) \circ \rho. \tag{61}$$

Aside from the Picard-Lefschetz monodromy a second form of monodromy will play an important role in the study of μ-constant families of isolated hypersurface singularities. It is the monodromy M_T of the homology bundle. For reasons of clarity we refer to it as *transversal monodromy.* Note that in the literature this is also sometimes called horizontal monodromy (e.g. see [Ku98, Section III.5.2]). The next lemma makes precise what we mean by that.

Lemma 3.1.5. *Let T be the parameter space of a μ-constant family as in Definition 2.3.1. The transversal monodromy M_T of it is the representation $\pi_1(T, t^0) \to G_{\mathbb{Z}}(F_{t^0})$ which comes from the local system $\bigcup_{t \in T} Ml(F_t)$.*
If its image is in $G_{\mathcal{R}}^{smar,gen}(F_{t^0})$, then there is a natural map $T \to M_\mu^{smar}(F_{t^0})$.

Proof. The trivial strong marking $+\mathrm{id}$ for F_{t^0} induces strong markings of other singularities F_t along any path. Now two paths, which meet at some point t, might not induce the same strong marking of F_t. However the two markings differ only by an element of $G_{\mathcal{R}}^{smar}(F_t)$. Hence, they induce the same right equivalence class of a marked singularity. □

The following theorem amasses known results from [He11] on the μ-constant monodromy groups and so-called jets. Recall that the *1-jet* of a function germ $f \in \mathcal{O}_{\mathbb{C}^{n+1},0}$ is the class $j_1 f \in \mathcal{O}_{\mathbb{C}^{n+1},0}/\mathfrak{m}^2$ and the *1-jet* of a coordinate change $\varphi = (\varphi_0, \ldots, \varphi_n) \in \mathcal{R}$ is $j_1\varphi = (j_1\varphi_0, \ldots, j_1\varphi_n)$.

Proposition 3.1.6. *Consider the data in Definition 3.1.3. We define the group $\mathcal{R}^f := \{\varphi \in \mathcal{R} \,|\, f \circ \varphi = f\}$ and the quotient of jet spaces $R_f := j_1\mathcal{R}^f / \left(j_1\mathcal{R}^f\right)^0$. Then the following statements hold.*

(a) If $\mathrm{mult}(f) \geq 3$ then $j_1\mathcal{R}^f = R_f$.

(b) The homomorphism $()_{hom} : \mathcal{R}^f \to G_{\mathbb{Z}}(f)$ factors through R_f. Its image is $(R_f)_{hom} = G_{\mathcal{R}}^{smar}(f) \subseteq G_{\mathbb{Z}}(f)$.

(c) The homomorphism $()_{hom} : R_f \to G_{\mathcal{R}}^{smar}(f)$ is an isomorphism.

(d) We have the equivalence

$$-\mathrm{id} \notin G_{\mathcal{R}}^{smar}(f) \Leftrightarrow \mathrm{mult}(f) \geq 3. \tag{62}$$

This is equivalent to the following statement: $G_R^{mar}(f) = G_R^{smar}(f)$ if $\mathrm{mult}(f) = 2$, and $G_{\mathcal{R}}^{mar}(f) = G_{\mathcal{R}}^{smar}(f) \times \{\pm\mathrm{id}\}$ if $\mathrm{mult}(f) \geq 3$.

(e) It holds $G_{\mathcal{R}}^{mar}(f) = G_{\mathcal{R}}^{mar}(f + x_{n+1}^2)$.

(f) $M_h \in G^{smar}(f)$. If f is quasihomogeneous, then $M_h \in G_{\mathcal{R}}^{smar}(f)$.

(g) For any $[(f,\rho)] \in M_\mu^{smar}$ *we have*

$$\mathrm{Stab}_{G_{\mathbb{Z}}}\left([(f,\rho)]\right) = \rho \circ G_{\mathcal{R}}^{smar}(f) \circ \rho^{-1}, \tag{63}$$
$$\mathrm{Stab}_{G_{\mathbb{Z}}}\left([(f,\pm\rho)]\right) = \rho \circ G_{\mathcal{R}}^{mar}(f) \circ \rho^{-1}. \tag{64}$$

For reasons of brevity we omit (f_0) in $M_\mu^{mar}(f_0)$, $G_{\mathbb{Z}}(f_0)$, $G^{mar}(f_0)$, $D_{BL}(f_0)$, etc., when the reference singularity is clear from the context.

3.2 TORELLI PROBLEMS FOR MARKED SINGULARITIES

Classically, there already exists a Torelli result for marked singularities. Namely, the infinitesimal (!) Torelli result which is due to M. Saito [SaM91]. We present it here in a strengthened version which is due to Hertling [He02-2]. Note that it is formulated originally for the μ-constant stratum. However the results of the previous section make it possible to state it immediately also for the moduli space of marked singularities $M_\mu^{mar}(f_0)$.

Theorem 3.2.1 ([He02-2, Theorem 12.8])**.** *Let f_0 be an arbitrary but fixed reference singularity. Then the following statements are true.*

(a) There is a natural holomorphic period map

$$BL : M_\mu^{mar}(f_0) \to D_{BL}(f_0). \tag{65}$$

It is $G_{\mathbb{Z}}$-equivariant.

(b) It is an immersion, here the reduced complex structure on $M_\mu^{mar}(f_0)$ is considered.

Now we want to have also a global marked Torelli result analogous to the classical Torelli Theorem. The following conjecture makes this precise. It collects the conjectures [He11, Conjecture 5.3], [He02-2, Conjecture 13.12] and [He93, Kapitel 2 d)].

Conjecture 3.2.2. *Fix a reference singularity f_0.*

(a) The period map $BL : M_\mu^{mar} \to D_{BL}$ is injective.

(b) The period map $LBL : M_\mu := M_\mu^{mar}/G_{\mathbb{Z}} \to D_{BL}/G_{\mathbb{Z}}$ is injective.

(c) For any singularity f in the μ-homotopy class of f_0 and any marking ρ it holds

$$\mathrm{Stab}_{G_{\mathbb{Z}}}\left([(f,\pm\rho)]\right) = \mathrm{Stab}_{G_{\mathbb{Z}}}\left(BL\left([(f,\pm\rho)]\right)\right). \tag{66}$$

Note that the inclusion $\mathrm{Stab}_{G_{\mathbb{Z}}}\left([(f,\pm\rho)]\right) \subseteq \mathrm{Stab}_{G_{\mathbb{Z}}}\left(BL\left([(f,\pm\rho)]\right)\right)$ *and the finiteness of both groups are clear.*

Note that part (b) of Conjecture 3.2.2 is equivalent to Conjecture 2.4.2 resp. 2.4.3 in Chapter 2. Moreover by [He11, Lemma 5.5] part (a) of Conjecture 3.2.2 holds if and only if (b) and (c) hold. So Conjecture 2.4.2 (a) is a strengthened version of the "old" Torelli

Conjecture 2.4.2 resp. 2.4.3. Note that we will call Conjecture 2.4.2 (a) the *strong global Torelli Conjecture 2.4.2 (a)* in what follows. Finally, it is shown below that the Torelli Conjectures are also stable under stabilization. So it is sufficient to prove them only for a fixed number of variables n.

Proposition 3.2.3. *Consider the μ-homotopy class of some arbitrary but fixed reference singularity $f_0(x_0,\dots,x_n)$ which satisfies assumption (49) and such that for any $m \geq 1$ the μ-homotopy class of $f_0 + \sum_{j=n+1}^{n+m} x_j^2$ satisfies assumption (50). Let $m \geq 1$. Then the strong global Torelli Conjecture 3.2.2 (a) holds for f_0 if any only if it holds for the reference singularity $f_0 + \sum_{j=n+1}^{n+m} x_j^2$.*

Proof. According to Theorem 3.1.4 (c) the moduli spaces of marked singularities $M_\mu^{mar}(f_0)$ and $M_\mu^{mar}\left(f_0 + \sum_{j=n+1}^{n+m} x_j^2\right)$ are canonically isomorphic. Moreover by Proposition 2.2.4 (c) the classifying spaces $D_{BL}(f_0)$ and $D_{BL}\left(f_0 + \sum_{j=n+1}^{n+m} x_j^2\right)$ are canonically isomorphic, too. So it remains to see that these isomorphisms are compatible with the period maps BL for f_0 and $f_0 + \sum_{j=n+1}^{n+m} x_j^2$. This follows via applying [BH18, Theorem 4.7] for the TEZP-structure of a stabilization. □

In order to prove the Torelli conjectures in the Sections 4.4 and 4.5 we make use of Proposition 3.2.3. In the other sections of Chapter 4 we sometimes make additional calculations for curve singularities. In particular, in cases where this seems easier than using abstract arguments.

3.3 COMPARISON OF MODULI SPACE OF MARKED SINGULARITIES AND TEICHMÜLLER SPACE

The previously studied analogies between the moduli space of marked singularities and the Teichmüller space for compact Riemann surfaces lead naturally to the question, if there are more similarities between both objects. In this section we look at several properties of the Teichmüller space and compare them with the moduli space of marked singularities. Here our basic reference for Teichmüller theory is the excellent book [FM12]. All in all, it turns out that many properties of the Teichmüller space and the mapping class group do not translate to M_μ^{mar} and $G_{\mathbb{Z}}$. Some of this follows from older results in [He11] and some of this relates to new results in Chapter 4 of this thesis. In any case, it was never stated explicitly before.

First of all, we note that the Teichmüller space is contractible (see [FM12, Theorem 10.6]). However, the moduli space of marked singularities M_μ^{mar} is not contractible in general. This was presumed for quite some time and now verified as $M_\mu^{mar} \cong (\mathbb{H} \setminus \{\text{discrete set}\}) \times \mathbb{C}$ for singularities of type $E_{3,0}$ where $\mathbb{H}$ denotes the complex upper half-plane (see [He11, Remarks 8.5. (iii)] and Section 4.5 for the details). The next natural thing to ask is now, if the moduli space of marked singularities M_μ^{mar} is at least connected. This is also not true, because for singularities of type $E_{3,18}$ the space M_μ^{mar} consists of infinitely many copies of $\mathbb{C}^* \times \mathbb{C}$. This is shown in Section 4.4.
Now we will consider the mapping class group of the Teichmüller space and its counterpart

the group $G_{\mathbb{Z}}$. We know already from Definition 3.1.3 (d) that $G_{\mathbb{Z}}$ acts properly discontinuously on the space M_μ^{mar}. This fits exactly to the Teichmüller theory picture (see [FM12, Theorem 12.2] for the Theorem of Fricke). Moreover, the mapping class group has trivial center except for Riemann surfaces of genus 0 and 1 (see [FM12, Theorem 3.10]). Recall that the center of a group is the subgroup of it consisting of those elements that commute with every element of the group. Now by [He11, Theorem 8.3] the group $G_{\mathbb{Z}}$ is a finite cyclic group for singularities of type A_l. All its elements commute with one another. This means in particular that the center of $G_{\mathbb{Z}}$ is the cyclic group itself and becomes arbitrarily large as l grows.

Lastly, the mapping class group of the Teichmüller space is also the group of biholomorphisms on the Teichmüller space (except for Riemann surfaces of genus 0, 1 and 2). Classically this is a corollary of Royden's Theorem (see [FM12, Chapter 12]). For singularities this is false, because the group of biholomorphisms on $\mathbb{C}$ is $\mathrm{Aut}(\mathbb{C}) = \{z \mapsto az + b \,|\, a \in \mathbb{C}^*, b \in \mathbb{C}\}$ and the automorphism group for singularities of type Z_{12} is a finite cyclic group acting on $M_\mu^{mar} \cong \mathbb{C}$ (see Section 4.3).

The situation is summarized in Table 1.

Property of the Teichmüller space	Analogous property for M_μ^{mar}
Contractibility	Not true
Connectedness	Not true
Mapping class group acts properly discontinuously	Yes, true
Mapping class group has (almost always) trivial center	Not true
Royden's Theorem holds	Not true

Table 1: Moduli space of marked singularities vs. Teichmüller space

4

TORELLI RESULTS FOR μ-CONSTANT FAMILIES OF SINGULARITIES

In this chapter, which is the longest chapter of this monograph, we will see proofs of the Conjectures 3.2.2, 3.1.2 and A.2.2 for many types of isolated hypersurface singularities. Above all we will prove the strong global Torelli Conjecture 3.2.2 (a) for all types of singularities on Arnold's famous list [Ar76] of singularities of modality ≤ 2. The chapter comprises the seven Sections 4.1–4.7. Each of the Sections 4.1–4.6 deals with a certain class of types of isolated hypersurface singularities. Furthermore, each of the Sections 4.2–4.5 is divided into two smaller subsections. The first one computes (a large enough subgroup of) $G_{\mathbb{Z}}$ for singularities of the respective types. The second one puts this new piece of information together with the computations inside the Gauss-Manin connection and proves the Torelli Conjectures for the respective types. Finally, Section 4.7 summarizes the results of this chapter and the current state of the art of the Torelli results for μ-constant families of isolated hypersurface singularities.

All this relies heavily on the proofs of the Torelli Conjecture 2.4.2 from [He93] (resp. [He95]) and the already existing proofs of the strong global Torelli Conjecture 3.2.2 for the simple and some of the exceptional unimodal and bimodal singularities from [He11]. Of course the Torelli Conjectures are formulated in terms of period maps. So a good understanding of the (often multivalued) period map LBL and $T \to D_{BL}(f_0)$, where T is the parameter space of a well chosen family of normal forms, is crucial. Most of the computations involving the Gauss-Manin connection and the period maps LBL had been carried out already in [He93] (resp. [He95]). The most important new ingredient to obtain the marked Torelli results from the older results is the precise calculation of the automorphism group $G_{\mathbb{Z}}$ on the Milnor lattice. Here, in order to compute $G_{\mathbb{Z}}$ our strategy is as follows. At first we split up the Milnor lattice into certain parts B_1, B_2 and B_3 in the sense of Orlik's Conjecture A.2.2, then we compute groups of automorphisms that respect the Seifert form on these smaller parts and, lastly, we put all the parts together again. In several cases we will not gain a complete description of $G_{\mathbb{Z}}$, but at least a good control of its action on $M_\mu^{mar}(f_0)$ and $D_{BL}(f_0)$. This is sufficient to prove Torelli results. Note that important details on $\mathbb{Z}$-lattices in general and Orlik's Conjecture A.2.2 in particular can be found in Section A.2.

Unfortunately, almost all types have their own characteristic difficulties to deal with. For example for some specific types of bimodal series singularities we have to consider Fuchsian groups. This makes the proofs quite lengthy. That is also why we decided to shift some of the most technical parts of those proofs to the Appendix (see in particular Sections A.2 and A.3).

4.1 SIMPLE SINGULARITIES

The simple singularities are the singularities of modality zero. They are right equivalent to one element of the two infinite series of types A_k for $k \geq 1$ or D_k for $k \geq 4$ or to the three exceptional types E_6, E_7 or E_8. The types together with the respective Milnor number μ and spectral numbers are given in Table 2. All this data is classical material and can be found everywhere in the literature (e.g. see [AGV85, Section 15.1] for the normal forms and [Ku98, (8.8.1) Table] for the spectral numbers).

Type	normal form	μ	spectral numbers
A_k for $k \geq 1$	x^{k+1}	k	$\left(-1+\frac{i}{k+1}\right)$ with $i = 1, 2, \dots, k$
D_k for $k \geq 4$	$x^2y + y^{k-1}$	k	0 and $\left(\frac{2i-k+2}{2(k-1)}\right)$ with $i = 0, \dots, k-2$
E_6	$x^3 + y^4$	6	$\left(\frac{i}{12}\right)$ with $i = -5, -2, -1, 1, 2, 5$
E_7	$x^3 + xy^3$	7	$\left(\frac{i}{18}\right)$ with $i = -8, -4, -2, 0, 2, 4, 8$
E_8	$x^3 + y^5$	8	$\left(\frac{i}{30}\right)$ with $i = -14, -8, -4, -2, 2, 4, 8, 14$

Table 2: Facts about simple singularities

The Torelli problems for the simple singularities are rather easy, but not as simple as one might excpect. They were solved in [He11, Section 8] for marked simple singularities. For the sake of completeness we summarize these results here.

First of all, we notice that the difference of the largest and the smallest spectral number is less than one in all of the cases above. Thus the Brieskorn lattice is just $H_0'' = V^{\alpha_1}$ where α_1 is the smallest spectral number. That means the Brieskorn lattice is completely determined by the spectral numbers and, hence, the classifying space of Brieskorn lattices is just a point $D_{BL} = \{pt\}$ for any type of simple singularity. Now with some further effort, one obtains the following theorem.

Theorem 4.1.1. *We consider singularities of multiplicity at least three. Then for simple singularities of type D_4 the group $G_{\mathbb{Z}}$ is $\{\pm M_h^k \,|\, k \in \mathbb{Z}\} \times \mathrm{Sym}\,(3)$, for singularities of types D_{2k} with $k \geq 3$ the group $G_{\mathbb{Z}}$ is $\{\pm M_h^k \,|\, k \in \mathbb{Z}\} \times \mathrm{Sym}\,(2)$ and for all other types of simple singularities it is $\{\pm M_h^k \,|\, k \in \mathbb{Z}\}$. In all cases the period map $BL : M_\mu^{mar} \to D_{BL} = \{pt\}$ is an isomorphism, i.e. the strong global Torelli Conjecture 3.2.2 (a) is true.*

Proof. See [He11, Theorem 8.3+Theorem 8.4]. □

4.2 SIMPLE ELLIPTIC AND HYPERBOLIC SINGULARITIES

Aside from the simple singularities the simple elliptic and hyperbolic singularities are the most popular types of isolated hypersurface singularities. The simple elliptic and the hyperbolic singularities are both 1-parameter families of singularities. For each triple $(p, q, r) \in \mathbb{Z}^3_{\geq 2}$ with $p \geq q \geq r$ and $\kappa := \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$ we have a certain type of singularity, denoted as T_{pqr}. Normal forms for $t \in (\mathbb{C} \setminus \{0\})$ with the respective Milnor number are given in Table 3 (e.g. cf. [AGV85, Section 15.1]).

Type	normal form	μ
T_{pqr}	$x^p + y^q + z^r + t \cdot xyz$	$p+q+r-1$

Table 3: Facts about simple elliptic and hyperbolic singularities

The hyperbolic types are those with $\kappa < 1$ and the simple elliptic are those with $\kappa = 1$. Note that the only triples $(p,q,r) \in \mathbb{Z}^3_{\geq 2}$ with $p \geq q \geq r$ and $\kappa = 1$ are $(3,3,3)$, $(4,4,2)$ and $(6,3,2)$. So there are only three different types of simple elliptic singularities, namely $\widetilde{E}_6 := T_{333}$, $\widetilde{E}_7 := T_{442}$ and $\widetilde{E}_8 := T_{632}$.
The simple elliptic types have some deep connections with elliptic curves. For example there is also a Legendre normal form with parameter $t \in (\mathbb{C} \setminus \{0,1\})$ for each of the three types of simple elliptic singularities (see Table 4).

Type	normal form	μ
$\widetilde{E}_6$	$y\,(y-x)\,(y-tx) - xz^2$	8
$\widetilde{E}_7$	$yx\,(y-x)\,(y-tx) + z^2$	9
$\widetilde{E}_8$	$y\,(y-x^2)\,(y-tx^2) + z^2$	10

Table 4: Legendre normal forms of the simple elliptic singularities

These Legendre normal forms have the advantage that they do contain representatives of all right equivalence classes. They originate from [SaK74, Satz 1.9 iii)]. For the normal forms in Table 3 this is not the case for $\widetilde{E}_7 := T_{442}$ (see [SaK74, 1.11, Bemerkung ii)]). We will work with both normal forms of the simple elliptic singularities in the following. We set $T := \mathbb{C} \setminus \{0\}$ for $\kappa < 1$ and $T := \mathbb{C} \setminus \{0,1\}$ for $\kappa = 1$.

4.2.1 *The group $G_{\mathbb{Z}}$ for simple elliptic and hyperbolic singularities*

A Coxeter-Dynkin diagram for the singularities of type T_{pqr} (with $\kappa \leq 1$) is displayed in Figure 5. This Coxeter-Dynkin diagram originates from [Ga74-2].

We choose a distinguished basis $\underline{\delta} = (\delta_1, \ldots, \delta_\mu)$ with this Coxeter-Dynkin diagram. Then the monodromy matrix M_M with $M_h(\underline{\delta}) = \underline{\delta} \cdot M_M$ can be calculated via the formulas in Chapter 2 — either via (8) or via (9). In [He93, Kapitel 3a)] M_M was calculated via (8). It is (here all not specified entries are 0)

$$M_M = \begin{pmatrix} M_1 & & & M_8 \\ & M_2 & & M_9 \\ & & M_3 & M_{10} \\ M_5 & M_6 & M_7 & M_4 \end{pmatrix} \tag{67}$$

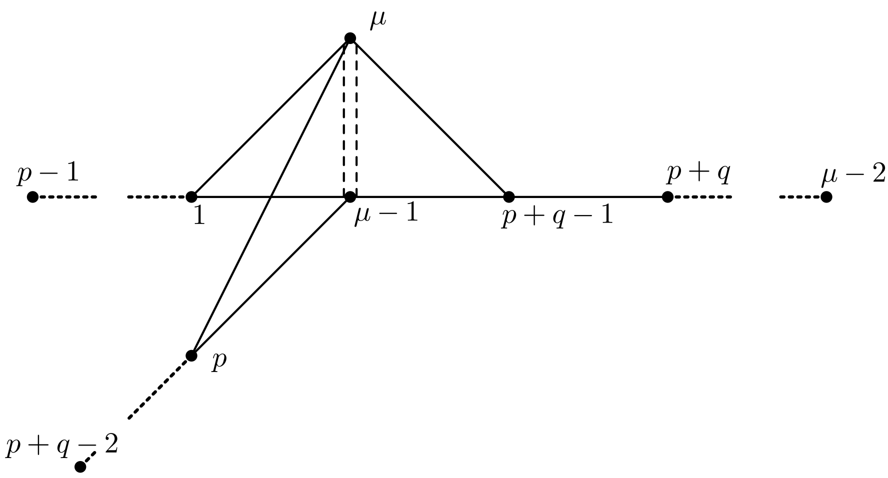

Figure 5: Coxeter-Dynkin diagram for T_{pqr}

with the following blocks,

$$\begin{aligned}
M_1 &= \begin{pmatrix} 0 & & & -1 \\ 1 & \ddots & & -1 \\ & \ddots & 0 & -1 \\ & & 1 & -1 \end{pmatrix} \in M((p-1)\times(p-1),\mathbb{Z}), \\
M_2 &\in M((q-1)\times(q-1),\mathbb{Z}) \text{ and} \\
M_3 &\in M((r-1)\times(r-1),\mathbb{Z}) \text{ are defined analogously,} \\
M_4 &= \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix},
\end{aligned}$$

and M_5, M_6, M_7, M_8, M_9, M_{10} are of suitable sizes with all entries except the following being 0,

$$\begin{aligned}
&(M_5)_{1,1} = (M_6)_{1,1} = (M_7)_{1,1} = -1, \quad (M_5)_{2,1} = (M_6)_{2,1} = (M_7)_{2,1} = 1, \\
&(M_8)_{1,1} = (M_8)_{1,2} = (M_9)_{1,1} = (M_9)_{1,2} = (M_{10})_{1,1} = (M_{10})_{1,2} = 1.
\end{aligned}$$

The characteristic polynomial of the monodromy is

$$p_{ch} = \frac{t^p-1}{t-1} \cdot \frac{t^q-1}{t-1} \cdot \frac{t^r-1}{t-1} \, (t-1)^2 \,. \tag{68}$$

Now we will use this data to determine the group $G_{\mathbb{Z}}$. This computation results in an explicit characterization of $G_{\mathbb{Z}}$ in the case of $\kappa < 1$ and an almost explicit characterization in the case of $\kappa = 1$. Here the crucial step is to gain a good control of the automorphisms on each of the two eigenspaces $Ml\,(f)_1 = \ker\left((M_h - \mathrm{id})^\mu\right)$ and $Ml\,(f)_{\neq 1} = \bigoplus_{\lambda\neq 1} Ml\,(f)_\lambda$ with $Ml\,(f)_\lambda = \ker\left((M_h - \lambda\,\mathrm{id})^\mu\right)$ resp. lattices $Ml\,(f)_{1,\mathbb{Z}} := Ml\,(f)_1 \cap Ml\,(f)$ and $Ml\,(f)_{\neq 1,\mathbb{Z}} := Ml\,(f)_{\neq 1} \cap Ml\,(f)$.

Theorem 4.2.1. *Consider a surface singularity f of type T_{pqr} with Milnor lattice $Ml(f)$, monodromy M_h, intersection form I and Seifert form L. We set $\chi := \mathrm{lcm}\,(p,q,r) \in \mathbb{Z}$.*

(a) Then $\dim Ml(f)_1 = 2$, $\operatorname{rank}\operatorname{Rad}(I) = 1$ if $\kappa < 1$ and $= 2$ if $\kappa = 1$. Choose a $\mathbb{Z}$-basis b_1, b_2 of $Ml(f)_{1,\mathbb{Z}}$ with $b_1 \in \operatorname{Rad}(I)$ and $L(b_1,b_2) \le 0$. Then

$$L(b_1,b_2) = -\chi \quad \textit{and} \quad M_h b_2 = b_2 + \chi(\kappa-1)\cdot b_1. \tag{69}$$

(b) The restriction map $G_{\mathbb{Z}} \to \operatorname{Aut}(Ml(f)_{1,\mathbb{Z}}, L)$ is surjective. Moreover it holds

$$\begin{aligned} \operatorname{Aut}(Ml(f)_{1,\mathbb{Z}}, L) &= \{\underline{b} \mapsto \underline{b}\cdot A \mid A \in \mathrm{SL}(2;\mathbb{Z})\} && (70)\\ &\cong \mathrm{SL}(2;\mathbb{Z}) \quad \textit{if} \quad \kappa = 1, \\ \operatorname{Aut}(Ml(f)_{1,\mathbb{Z}}, L) &= \{\pm T_{aut}^k \mid k \in \mathbb{Z}\} \quad \textit{if} \quad \kappa < 1, && (71) \end{aligned}$$

where $\underline{b} := (b_1,b_2)$ and $T_{aut} \in \operatorname{Aut}(Ml(f)_{1,\mathbb{Z}})$ is the automorphism with $T_{aut}(b_1) = b_1$ and $T_{aut}(b_2) = b_2 + b_1$.

(c) The group $G_{\mathbb{Z}}$ for $\kappa < 1$ and the subgroup $\{g \in G_{\mathbb{Z}} \mid g(b_1) = \pm b_1\} \subseteq G_{\mathbb{Z}}$ for $\kappa = 1$ will be described explicitly below — except for the part U_2. There is a monodromy invariant decomposition

$$Ml(f)_{\neq 1} = Ml_{\mathbb{C}}^{(1)} \oplus Ml_{\mathbb{C}}^{(2)} \oplus Ml_{\mathbb{C}}^{(3)} \tag{72}$$

such that the characteristic polynomial of $M_h|_{Ml^{(j)}}$ is

$$\frac{t^p-1}{t-1},\ \frac{t^q-1}{t-1},\ \frac{t^r-1}{t-1} \qquad \textit{for } j = 1,2,3 \tag{73}$$

and such that the following holds.

$$\left.\begin{array}{rl} G_{\mathbb{Z}} & \textit{for } \kappa < 1 \\ \{g \in G_{\mathbb{Z}} \mid g(b_1) = \pm b_1\} & \textit{for } \kappa = 1 \end{array}\right\} = (U_1 \rtimes U_2) \times \{\pm \mathrm{id}\}, \tag{74}$$

where U_1 is the infinite subgroup of $G_{\mathbb{Z}}$

$$\begin{aligned} U_1 \;=\; & \{T_{aut}^{\delta} \times (M_h|_{Ml_{\mathbb{C}}^{(1)}})^{\alpha} \times (M_h|_{Ml_{\mathbb{C}}^{(2)}})^{\beta} \times (M_h|_{Ml_{\mathbb{C}}^{(3)}})^{\gamma} \mid && (75) \\ & (\delta,\alpha,\beta,\gamma) \in \mathbb{Z} \times \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r \textit{ with } \frac{\alpha}{p} + \frac{\beta}{q} + \frac{\gamma}{r} \equiv \frac{\delta}{\chi} \mod 1\} \end{aligned}$$

and where U_2 is a finite subgroup of $G_{\mathbb{Z}}$ with

$$U_2 \begin{cases} = \{\mathrm{id}\} & \textit{if } p > q > r, \\ \cong \mathrm{Sym}\,(2) & \textit{if } p = q > r \textit{ or } p > q = r, \\ \cong \mathrm{Sym}\,(3) & \textit{if } p = q = r. \end{cases} \tag{76}$$

which consists of certain automorphisms which act trivially on $Ml(f)_1$ and which permute those of the subspaces $Ml_{\mathbb{C}}^{(j)}$ which have equal dimension.

Proof. (a) At first, we set

$$\widetilde{b}_1 := \delta_{\mu-1} - \delta_\mu, \tag{77}$$

$$\widetilde{b}_2 := \chi \cdot \left(\sum_{i=1}^{p-1} \frac{p-i}{p}\delta_i + \sum_{i=1}^{q-1} \frac{q-i}{q}\delta_{p-1+i} + \sum_{i=1}^{r-1} \frac{r-i}{r}\delta_{p+q-2+i} + \delta_{\mu-1} \right). \tag{78}$$

Then we compute

$$M_h(\widetilde{b}_1) = \widetilde{b}_1, \quad M_h(\widetilde{b}_2) = \widetilde{b}_2 + \chi(\kappa-1)\cdot \widetilde{b}_1. \tag{79}$$

With (9) we obtain

$$\begin{pmatrix} L(\widetilde{b}_1,\widetilde{b}_1) & L(\widetilde{b}_1,\widetilde{b}_2) \\ L(\widetilde{b}_2,\widetilde{b}_1) & L(\widetilde{b}_2,\widetilde{b}_2) \end{pmatrix} = \begin{pmatrix} 0 & -\chi \\ \chi & \frac{\chi^2}{2}(\kappa-1) \end{pmatrix}. \tag{80}$$

By (79), $\widetilde{b}_1, \widetilde{b}_2$ is a $\mathbb{Q}$-basis of $Ml(f)_{1,\mathbb{Q}}$ and M_h is on $Ml(f)_1$ semisimple if $\kappa = 1$ and it has a 2×2 Jordan block if $\kappa < 1$. Moreover from the coefficients we see that $\widetilde{b}_1$, $\widetilde{b}_2$ is also a $\mathbb{Z}$-basis of $Ml(f)_{1,\mathbb{Z}}$. Here it is important that the coefficients of $\widetilde{b}_2$ have greatest common divisor 1. As the equations (69) hold for $\widetilde{b}_1$, $\widetilde{b}_2$, they hold for any basis b_1, b_2 as in (a).

(b) For $\kappa = 1$ the Seifert form L on $Ml(f)_{1,\mathbb{Z}}$ is up to the factor χ the standard symplectic form (see (80)). Therefore (70) holds. If $\kappa < 1$ then (71) holds because of (80).

The restriction map $G_\mathbb{Z} \to \mathrm{Aut}(Ml(f)_{1,\mathbb{Z}})$ contains T_{aut}. This follows from (74) (whose proof below does not use this fact), because obviously there are $(\delta,\alpha,\beta,\gamma)$ as in (75) with $\delta = 1$. This shows (b) in the case $\kappa < 1$.

For $\kappa = 1$ the surjectivity of the map $G_\mathbb{Z} \to \mathrm{Aut}(Ml(f)_{1,\mathbb{Z}}, L)$ follows from [Ku98, III.2.6].

(c) We will prove (c) for the special choice $\widetilde{b}_1, \widetilde{b}_2$. Then (c) holds for any b_1, b_2 as in (a) because by the surjectivity of the map $G_\mathbb{Z} \to \mathrm{Aut}(Ml(f)_{1,\mathbb{Z}}, L)$, an element $g \in G_\mathbb{Z}$ with $g(\widetilde{b}_1) = b_1, g(\widetilde{b}_2) = b_2$ exists. We define

$$Ml_\mathbb{Z}^{[1]} := \mathbb{Z}\cdot\widetilde{b}_1 \oplus \bigoplus_{i=1}^{p-1} \mathbb{Z}\cdot\delta_i, \ Ml_\mathbb{C}^{[1]} := Ml_\mathbb{Z}^{[1]} \otimes_\mathbb{Z} \mathbb{C}, \tag{81}$$

$$Ml_\mathbb{C}^{(1)} := Ml_\mathbb{C}^{[1]} \cap Ml(f)_{\neq 1}, \tag{82}$$

and analogously $Ml_\mathbb{Z}^{[2]}, Ml_\mathbb{C}^{[2]}, Ml_\mathbb{C}^{(2)}$ and $Ml_\mathbb{Z}^{[3]}, Ml_\mathbb{C}^{[3]}, Ml_\mathbb{C}^{(3)}$.

Looking carefully at the matrix M_M yields

$$M_h : \delta_1 + \widetilde{b}_1 \mapsto \delta_2 \mapsto \ldots \mapsto \delta_{p-1} \mapsto -(\delta_1 + \ldots + \delta_{p-1}) \mapsto \delta_1 + \widetilde{b}_1. \tag{83}$$

Therefore $Ml_\mathbb{Z}^{[1]}$ is a cyclic M_h-module with characteristic polynomial $t^p - 1$, and $Ml_\mathbb{C}^{[1]} = \mathbb{C}\widetilde{b}_1 \oplus Ml_\mathbb{C}^{(1)}$, and M_h on $Ml_\mathbb{C}^{(1)}$ has the characteristic polynomial $(t^p-1)/(t-1)$. So Lemma A.2.3 applies and shows

$$\mathrm{Aut}(Ml_\mathbb{Z}^{[1]}, L) = \{\pm (M_h|_{Ml_\mathbb{Z}^{[1]}})^\alpha \,|\, \alpha \in \{0,1,\ldots,p-1\}\}. \tag{84}$$

Finally, M_h, I and L are well-defined on the quotient lattice $Ml_{\mathbb{Z}}^{[1]}/\mathbb{Z}\cdot\widetilde{b}_1$, and $(Ml_{\mathbb{Z}}^{[1]}/\mathbb{Z}\cdot\widetilde{b}_1, -I)$ is a root lattice of type A_{p-1}. The last statement follows immediately from the part of the Coxeter-Dynkin diagram which corresponds to $\delta_1,\dots,\delta_{p-1}$.

$Ml_{\mathbb{Z}}^{[2]}$ and $Ml_{\mathbb{Z}}^{[3]}$ have the same properties as $Ml_{\mathbb{Z}}^{[1]}$, with q respectively r instead of p.

Now it is clear that

$$Ml(f)_{\neq 1} = Ml_{\mathbb{C}}^{(1)} \oplus Ml_{\mathbb{C}}^{(2)} \oplus Ml_{\mathbb{C}}^{(3)}.$$

The $\mathbb{Z}$-lattice

$$Ml_{\mathbb{Z}}^{[1]} + Ml_{\mathbb{Z}}^{[2]} + Ml_{\mathbb{Z}}^{[3]} = \mathbb{Z}\cdot\widetilde{b}_1 \oplus \bigoplus_{i=1}^{\mu-2}\mathbb{Z}\cdot\delta_i = (\mathbb{C}\cdot\widetilde{b}_1 \oplus Ml(f)_{\neq 1}) \cap Ml(f)$$

is a primitive sublattice of $Ml(f)$ of rank $\mu-1$. Any $g\in G_{\mathbb{Z}}$ with $g(\widetilde{b}_1)=\pm\widetilde{b}_1$ maps it to itself, because it maps $\mathbb{C}\widetilde{b}_1$ and $Ml(f)_{\neq 1}$ and $Ml(f)$ to themselves. g maps also the quotient lattice

$$(Ml_{\mathbb{Z}}^{[1]} + Ml_{\mathbb{Z}}^{[2]} + Ml_{\mathbb{Z}}^{[3]})/\mathbb{Z}\cdot\widetilde{b}_1 = Ml_{\mathbb{Z}}^{[1]}/\mathbb{Z}\cdot\widetilde{b}_1 \oplus Ml_{\mathbb{Z}}^{[2]}/\mathbb{Z}\cdot\widetilde{b}_1 \oplus Ml_{\mathbb{Z}}^{[3]}/\mathbb{Z}\cdot\widetilde{b}_1$$

to itself. But this is together with $-I$ an orthogonal sum of lattices of types $A_{p-1},\ A_{q-1}$ and A_{r-1}. Therefore g can only permute the summands, and only those summands of equal rank.

If $p=q$, a special element $\sigma_{12}\in G_{\mathbb{Z}}$ is given by

$$\begin{aligned}\sigma_{12}(\delta_i) &= \delta_{p-1+i},\quad \sigma_{12}(\delta_{p-1+i})=\delta_i \quad \text{for } 1\le i\le p-1,\\ \sigma_{12}(\delta_j) &= \delta_j \quad \text{for } p+q-2\le j\le \mu.\end{aligned}$$

That $\sigma_{12}\in G_{\mathbb{Z}}$ follows immediately from the symmetry of the Coxeter-Dynkin diagram. Similarly $\sigma_{23}\in G_{\mathbb{Z}}$ is defined if $q=r$. In any case, these elements generate a subgroup $U_2\subseteq G_{\mathbb{Z}}$ with the properties in (c).

Therefore, starting with an arbitrary element $\widetilde{g}\in G_{\mathbb{Z}}$ if $\kappa<1$ respectively $\widetilde{g}\in\{g\in G_{\mathbb{Z}}\,|\,g(\widetilde{b}_1)=\pm\widetilde{b}_1\}$ if $\kappa=1$, we can compose it with $\pm\,\mathrm{id}$ and an element of U_2, and we obtain an element $g\in G_{\mathbb{Z}}$ with $g(\widetilde{b}_1)=\widetilde{b}_1$ and $g(Ml_{\mathbb{Z}}^{[j]})=Ml_{\mathbb{Z}}^{[j]}$ for $j=1,2,3$. Then it is $g|_{Ml_{\mathbb{Z}}^{[1]}}=(M_h|_{Ml_{\mathbb{Z}}^{[1]}})^{\alpha}$ for a unique $\alpha\in\{0,1,\dots,p-1\}$, and similarly with $\beta\in\{0,1,\dots,q-1\}$ and $\gamma\in\{0,1,\dots,r-1\}$ for $Ml_{\mathbb{Z}}^{[2]}$ and $Ml_{\mathbb{Z}}^{[3]}$. We also have $g(\widetilde{b}_2)=\widetilde{b}_2+\delta\widetilde{b}_1$ for some $\delta\in\mathbb{Z}$. Now we calculate, while observing (83),

$$M_h\left(\sum_{i=1}^{p-1}\frac{p-i}{p}\delta_i\right) = \left(\sum_{i=1}^{p-1}\frac{p-i}{p}\delta_i\right) - (\delta_1+\widetilde{b}_1) + \frac{1}{p}\widetilde{b}_1, \tag{85}$$

$$M_h^{\alpha}\left(\sum_{i=1}^{p-1}\frac{p-i}{p}\delta_i\right) = \left(\sum_{i=1}^{p-1}\frac{p-i}{p}\delta_i\right) - \left(\widetilde{b}_1+\sum_{i=1}^{\alpha}\delta_i\right) + \frac{\alpha}{p}\widetilde{b}_1. \tag{86}$$

The definition (78) of $\widetilde{b}_2$ shows

$$-\delta_{\mu-1} = -\frac{1}{\chi}\widetilde{b}_2 + \sum_{i=1}^{p-1}\frac{p-i}{p}\delta_i + \sum_{i=1}^{q-1}\frac{q-i}{q}\delta_{p-1+i} + \sum_{i=1}^{r-1}\frac{r-i}{r}\delta_{p+q-2+i} \tag{87}$$

and (86) gives then

$$\begin{aligned} g(-\delta_{\mu-1}) &= -\delta_{\mu-1} + \left(\frac{-\delta}{\chi} + \frac{\alpha}{p} + \frac{\beta}{q} + \frac{\gamma}{r}\right) \cdot \widetilde{b}_1 \\ &- \left(\widetilde{b}_1 + \sum_{i=1}^{\alpha} \delta_i\right) - \left(\widetilde{b}_1 + \sum_{i=p}^{p-1+\beta} \delta_i\right) - \left(\widetilde{b}_1 + \sum_{i=p+q-1}^{p+q-2+\gamma} \delta_i\right). \end{aligned} \tag{88}$$

Therefore it is

$$\begin{aligned} \frac{\alpha}{p} + \frac{\beta}{q} + \frac{\gamma}{r} &\equiv \frac{\delta}{\chi} \mod 1 \\ \text{and} \quad g &= T_{aut}^{\delta} \times (M_h|_{Ml_{\mathbb{C}}^{(1)}})^{\alpha} \times (M_h|_{Ml_{\mathbb{C}}^{(2)}})^{\beta} \times (M_h|_{Ml_{\mathbb{C}}^{(3)}})^{\gamma}. \end{aligned} \tag{89}$$

Thus $g \in U_1$, so $G_{\mathbb{Z}} \subseteq (U_1 \rtimes U_2) \times \{\pm \mathrm{id}\}$.

Vice versa, we have to show $U_1 \subseteq G_{\mathbb{Z}}$. Fix a $g \in U_1$. It respects the decomposition

$$Ml(f)_{\mathbb{C}} = Ml(f)_1 \oplus Ml_{\mathbb{C}}^{(1)} \oplus Ml_{\mathbb{C}}^{(2)} \oplus Ml_{\mathbb{C}}^{(3)}.$$

This is a left and right orthogonal decomposition with respect to the Seifert form L. The restriction of g to each of the four blocks respects L there, so $g \in \mathrm{Aut}(Ml(f)_{\mathbb{C}}, L)$. It restricts on $Ml_{\mathbb{C}}^{[1]}$ to M_h^{α}, so it maps the lattice $Ml_{\mathbb{Z}}^{[1]}$ to itself, and analogously the lattices $Ml_{\mathbb{Z}}^{[2]}$ and $Ml_{\mathbb{Z}}^{[3]}$, thus also the sum $Ml_{\mathbb{Z}}^{[1]} + Ml_{\mathbb{Z}}^{[2]} + Ml_{\mathbb{Z}}^{[3]}$. This sum is a primitive sublattice of $Ml(f)$ of rank $\mu - 1$ with

$$Ml(f) = \left(Ml_{\mathbb{Z}}^{[1]} + Ml_{\mathbb{Z}}^{[2]} + Ml_{\mathbb{Z}}^{[3]}\right) \oplus \mathbb{Z} \cdot \delta_{\mu-1}.$$

The calculation above of $g(-\delta_{\mu-1})$ shows $g(\delta_{\mu-1}) \in Ml(f)$ and $g(\delta_{\mu-1}) \equiv \delta_{\mu-1}$ modulo the sublattice. Therefore it is $g \in G_{\mathbb{Z}}$. □

Now we are ready to prove the strong global Torelli Conjecture. This will be done in the next section. An application of Theorem 4.2.1, aside from Torelli problems, can be found in [GH17, Section 8].

4.2.2 *Torelli Conjectures for simple elliptic and hyperbolic singularities*

For almost all simple, unimodal and bimodal singularities there exist older Torelli results from [He93] and [He95]. So we can make use of them here. In particular for simple elliptic and hyperbolic singularities there are two Torelli results from [He93], which we will apply in order to prove the next theorem.

Theorem 4.2.2. *(a) For the simple elliptic singularities and the hyperbolic singularities in any number of variables, the space M_{μ}^{mar} of right equivalence classes of marked singularities is*

$$M_{\mu}^{mar} \cong T^{univ} := \begin{cases} \mathbb{C} & \text{if } \kappa < 1 \\ \mathbb{H} & \text{if } \kappa = 1 \end{cases}, \tag{90}$$

so it is connected, and thus $G^{mar} = G_{\mathbb{Z}}$. The period map $BL : M_\mu^{mar} \to D_{BL}$ is an isomorphism, so the strong global Torelli Conjecture 3.2.2 (a) is true.

(b) Now consider the singularities of type T_{pqr} as curve singularities if $r = 2$ and as surface singularities if $r \geq 3$. Then

$$G_{\mathbb{Z}} = G^{mar} = G^{smar} \times \{\pm \operatorname{id}\}, \quad \text{equivalently: } -\operatorname{id} \notin G^{smar}. \tag{91}$$

The subgroup of G^{smar}, which acts trivially on M_μ^{mar}, is the kernel of the surjective map

$$G^{smar} \to \operatorname{Aut}(Ml(f_0)_{1,\mathbb{Z}}, L)/\{\pm \operatorname{id}\}. \tag{92}$$

It is equal to $\rho \circ (\mathcal{R}^f)_{hom} \circ \rho^{-1}$ for a generic $[(f, \pm\rho)] \in M_\mu^{mar}$. Its size is 54, 16 and 6 for T_{333}, T_{442} and T_{632}.

Proof. We start and choose a marked reference singularity $[(f_0, \pm\rho)]$ with parameter in T^{univ}. Then immediately all singularities with a parameter in T^{univ} become marked singularities, because T^{univ} is simply connected. Hence, we have a period map $T^{univ} \to D_{BL}$ that is well-defined. The Torelli result from [He93, Kapitel 3] tells us that this map is an isomorphism. In particular this means that the marked Brieskorn lattices of the marked singularities in T^{univ} are all different. Thus the marked singularities with parameters in T^{univ} are all not right equivalent and we have an embedding $T^{univ} \hookrightarrow M_\mu^{mar}(f_0)^0$.
On the other hand, we have the immersion $BL : M_\mu^{mar}(f_0)^0 \to D_{BL}$ (see Theorem 3.2.1). It restricts to the isomorphism $T^{univ} \to D_{BL}$ and, hence, it must hold $T^{univ} = M_\mu^{mar}(f_0)^0$. So in order to prove part (a) it remains to show that $M_\mu^{mar}(f_0)$ is connected, i.e. $G_{\mathbb{Z}} = G^{mar}$. This will be done via the characterization of $G_{\mathbb{Z}}$ coming from Theorem 4.2.1. Therefore, we have to consider D_{BL} and the action of $G_{\mathbb{Z}}$ on it. It is

$$D_{BL} \cong \begin{cases} \{V \subseteq Ml(f_0)_1 \mid \dim V = 1, V \neq \ker(M_h - \operatorname{id})\} & , \text{ if } \kappa < 1, \\ \text{one component of } \{V \subseteq Ml(f_0)_1 \mid \dim V = 1, V \neq \overline{V}\} & , \text{ if } \kappa = 1. \end{cases} \tag{93}$$

In both cases the group $\operatorname{Aut}\left(Ml(f_0)_{1,\mathbb{Z}}, L\right)/\{\pm \operatorname{id}\}$ acts faithfully on D_{BL}. Furthermore, in both cases we know that the period map

$$T^{univ}/\sim_{\mathcal{R}} \to D_{BL}/\operatorname{Aut}\left(Ml(f_0)_{1,\mathbb{Z}}, L\right) \tag{94}$$

is an isomorphism by [He93, Kapitel 3]. Using both isomorphism we obtain

$$\left(M_\mu^{mar}(f_0)\right)^0/G^{mar} \cong T^{univ}/\sim_{\mathcal{R}} \to D_{BL}/\operatorname{Aut}\left(Ml(f_0)_{1,\mathbb{Z}}, L\right), \tag{95}$$

and the surjectivity of $G^{mar} \to \operatorname{Aut}\left(Ml(f_0)_{1,\mathbb{Z}}, L\right)$. Note that this shows part (b) of Theorem 4.2.1, too.
By the isomorphism theorem it is now sufficient to show that the kernels of the surjective maps $G_{\mathbb{Z}} \to \operatorname{Aut}\left(Ml(f_0)_{1,\mathbb{Z}}, L\right)$ and $G^{mar} \to \operatorname{Aut}\left(Ml(f_0)_{1,\mathbb{Z}}, L\right)$ coincide. Then also

the groups G^{mar} and $G_{\mathbb{Z}}$ coincide and part (a) of the theorem follows. The kernel of $G_{\mathbb{Z}} \to \operatorname{Aut}\left(Ml\left(f_0\right)_{1,\mathbb{Z}}, L\right)$ was already determined in Theorem 4.2.1. It contains exactly those elements from $(U_1 \rtimes U_2) \times \{\pm \operatorname{id}\}$ in (74) with $\delta = 0$. So it holds

$$\begin{aligned} &\ker\left(G_{\mathbb{Z}} \to \operatorname{Aut}\left(Ml\left(f_0\right)_{1,\mathbb{Z}}, L\right)\right) \\ &\cong \left(\{(\alpha,\beta,\gamma) \in \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r \mid \frac{\alpha}{p} + \frac{\beta}{q} + \frac{\gamma}{r} \equiv 0 \mod 1\} \rtimes U_2\right) \times \{\pm \operatorname{id}\}. \end{aligned} \tag{96}$$

On the other hand the kernel of the map $G^{mar} \to \operatorname{Aut}\left(Ml\left(f_0\right)_{1,\mathbb{Z}}, L\right)$ can be determined as the subgroup of G^{mar} which acts trivially on $M_\mu^{mar}\left(f_0\right)^0$. In other words, it is the isotropy group in G^{mar} of a generic point $[(f, \pm\rho)] \in M_\mu^{mar}\left(f_0\right)^0$. By Proposition 3.1.6 (g) this means it is

$$\rho \circ G_{\mathcal{R}}^{mar}(f) \circ \rho^{-1} = \rho \circ \{\pm \varphi_{hom} \mid \varphi \in \mathcal{R}^f\} \circ \rho^{-1}. \tag{97}$$

For generic f we can now use the normal forms from Table 3 (also for $\kappa = 1$). So it is $f = x^p + y^q + z^r + t \cdot xyz$. Now we define coordinate changes that generate a finite subgroup of $S \subseteq \mathcal{R}^f$. Those coordinate changes are

$$\begin{aligned} \varphi^{\alpha,\beta,\gamma} : (x,y,z) &\mapsto (e^{2\pi i\alpha/p}x, e^{2\pi i\beta/q}y, e^{2\pi i\gamma/r}z) \\ &\quad\ \text{with } \frac{\alpha}{p} + \frac{\beta}{q} + \frac{\gamma}{r} \equiv 0 \mod 1, \\ \varphi^{1,2} : (x,y,z) &\mapsto (y,x,z) \quad \text{if } p = q, \\ \varphi^{2,3} : (x,y,z) &\mapsto (x,z,y) \quad \text{if } q = r, \\ \varphi^{minus} : (x,y,z) &\mapsto (x,y,-z-txy) \quad \text{if } r = 2. \end{aligned} \tag{98}$$

The map φ^{minus} has order 2 and commutes with the other coordinate changes. Note here that $q = r = 2$ is impossible, because of $\kappa \le 1$. The group S is then (in an abstract sense) isomorphic to $U_1^0 \rtimes U_2$ if $r \ge 3$ and to $(U_1^0 \rtimes U_2) \times \{\pm \operatorname{id}\}$ if $r = 2$. The map to 1-jets of coordinate changes is injective,

$$S \xrightarrow{\cong} j_1 S \subseteq j_1 \mathcal{R}^f \subseteq j_1 \mathcal{R}. \tag{99}$$

Now we have to treat the cases $r \ge 3$ and $r = 2$ separately.

The case $r \ge 3$: Then $j_1\mathcal{R}^f$ is finite and isomorphic to R_f, the map

$$()_{hom} : R_f \to G_{\mathbb{Z}}(f) = \rho^{-1} \circ G_{\mathbb{Z}} \circ \rho$$

is injective, and the image $G_{\mathcal{R}}^{smar}(f)$ does not contain $-\operatorname{id}$ by Theorem 3.1.6. Therefore then $S \cong (S)_{hom} \subseteq G_{\mathbb{Z}}(f)$ and $-\operatorname{id} \notin (S)_{hom}$. Thus the group $(S)_{hom} \times \{\pm \operatorname{id}\}$ is isomorphic to $(U_1^0 \rtimes U_2) \times \{\pm \operatorname{id}\}$. Now it is clear that the group in (97) is at least as big as the group in (96). But it cannot be bigger. So they are of equal size. This implies $G^{mar} = G_{\mathbb{Z}}$.

The case $r = 2$: We claim that the map $S \to (S)_{hom}$ is injective. If this is true then $(S)_{hom} \cong (U_1^0 \rtimes U_2) \times \{\pm \operatorname{id}\}$, and this is of equal size as the group in (96). Then again the group in (97) is at least as big as the group in (96), but it cannot be bigger. So they are of equal size. This implies $G^{mar} = G_{\mathbb{Z}}$.

It is enough to prove the claim. For this we consider the curve singularity

$$g := x^p + y^q - \frac{1}{4}tx^2y^2. \tag{100}$$

Then

$$\begin{array}{rcl} g+z^2 &=& f\circ\psi \quad \text{with } \psi(x,y,z)=(x,y,z-\frac{1}{2}txy),\\ \mathcal{R}^{g+z^2} &=& \psi^{-1}\circ\mathcal{R}^f\circ\psi,\\ \psi^{-1}\circ\varphi^{\alpha,\beta,\gamma}\circ\psi &=& \varphi^{\alpha,\beta,\gamma},\\ \psi^{-1}\circ\varphi^{1,2}\circ\psi &=& \varphi^{1,2}, \quad \text{if } p=q,\\ \psi^{-1}\circ\varphi^{minus}\circ\psi &=& ((x,y,z)\mapsto(x,y,-z)). \end{array} \tag{101}$$

($q=r=2$ is impossible because of $\kappa\leq 1$). The subgroup

$$S^{curve} := \{\varphi^{\alpha,\beta,\gamma}\circ(\varphi^{minus})^{-\gamma}\,|\,(\alpha,\beta,\gamma)\in U_1^0\}\rtimes U_2 \tag{102}$$

has index 2 in S, its conjugate $\psi^{-1}\circ S^{curve}\circ\psi$ restricts to $\mathcal{R}^g$, and it maps injectively to $j_1\mathcal{R}^g\cong R_g$. According to Theorem 3.1.6 (c) the map $S^{curve}\to(S^{curve})_{hom}$ is injective, and $-\mathrm{id}$ is not in the image. But $(\varphi^{minus})_{hom}=-\mathrm{id}$. This proves the claim.

(b) According to Theorem 3.1.4 (c), the projection $M_\mu^{smar}\to M_\mu^{mar}$ is a twofold covering and $-\mathrm{id}$ exchanges the two sheets of this covering. Because of $M_\mu^{mar}=\mathbb{C}$ if $\kappa<1$ and $M_\mu^{mar}=\mathbb{H}$ if $\kappa=1$, M_μ^{smar} has two components. Therefore $-\mathrm{id}\notin G^{smar}$ and $G_{\mathbb{Z}}=G^{mar}=G^{smar}\times\{\pm\mathrm{id}\}$. The statements right before and after (92) were already proved and used in the proof of part (a) above.

The group $\rho\circ(\mathcal{R}^f)_{hom}\circ\rho^{-1}$ for a generic $[(f,\rho)]\in M_\mu^{mar}$ has size 54, 16 and 6 for T_{333}, T_{442} and T_{632}, because it is isomorphic to an index 2 subgroup of the group in (96), and that group has 108, 32 and 12 elements in the cases T_{333}, T_{442} and T_{632}. □

4.3 EXCEPTIONAL UNIMODAL AND BIMODAL SINGULARITIES

The exceptional unimodal and bimodal singularities are far less exceptional than their name suggests. In many respects they can be seen as rather generic (especially when compared to the simple elliptic singularities). Normal forms and the Milnor numbers μ of the 14+14 families of exceptional unimodal and bimodal singularities are given in Table 5. Here it is $t\in\mathbb{C}$ and $(t_1,t_2)\in\mathbb{C}^2$. Those normal forms can be found for example in [AGV85, Section 15.1].

Here we will treat just 6 out of the 28 quasihomogeneous exceptional unimodal and bimodal singularities. Namely we will consider the families of singularities of types Z_{12}, Q_{12}, U_{12},

Type	normal form	μ
E_{12}	$x^3+y^7+txy^5$	12
E_{13}	$x^3+xy^5+ty^8$	13
E_{14}	$x^3+y^8+txy^6$	14
E_{18}	$x^3+y^{10}+(t_1+t_2y)\,xy^7$	18
E_{19}	$x^3+xy^7+(t_1+t_2y)\,y^{11}$	19
E_{20}	$x^3+y^{11}+(t_1+t_2y)\,xy^8$	20
Z_{11}	$x^3y+y^5+txy^4$	11
Z_{12}	$x^3y+xy^4+tx^2y^3$	12
Z_{13}	$x^3y+y^6+txy^5$	13
Z_{17}	$x^3y+y^8+(t_1+t_2y)\,xy^6$	17
Z_{18}	$x^3y+xy^6+(t_1+t_2y)\,y^9$	18
Z_{19}	$x^3y+y^9+(t_1+t_2y)\,xy^7$	19
Q_{10}	$x^3+y^4+yz^2+txy^3$	10
Q_{11}	$x^3+y^2z+xz^3+tz^5$	11
Q_{12}	$x^3+y^5+yz^2+txy^4$	12
Q_{16}	$x^3+y^7+yz^2+(t_1+t_2y)\,xy^5$	16
Q_{17}	$x^3+yz^2+xy^5+(t_1+t_2y)\,y^8$	17
Q_{18}	$x^3+yz^2+y^8+(t_1+t_2y)\,xy^6$	18
W_{12}	$x^4+y^5+tx^2y^3$	12
W_{13}	$x^4+xy^4+ty^6$	13
W_{17}	$x^4+xy^5+(t_1+t_2y)\,y^7$	17
W_{18}	$x^4+y^7+(t_1+t_2y)\,x^2y^4$	18
S_{11}	$x^4+y^2z+xz^2+tx^3z$	11
S_{12}	$x^2y+y^2z+xz^3+tz^5$	12
S_{16}	$x^2z+yz^2+xy^4+(t_1+t_2y)\,y^6$	16
S_{17}	$x^2z+yz^2+y^6+(t_1+t_2y)\,y^4z$	17
U_{12}	$x^3y+y^3+z^4+txyz^2$	12
U_{16}	$x^3y+xz^2+y^5+(t_1+t_2y)\,x^2y^2$	16

Table 5: Exceptional singularities

Z_{18}, Q_{16}, U_{16}. For the other 22 types the automorphism group $G_{\mathbb{Z}}$ is just $\{\pm M_h^k \mid k \in \mathbb{Z}\}$ and the proof of the Torelli result in the marked case was already given in 2011 by Claus Hertling (see [He11, Section 8]). The automorphism group $G_{\mathbb{Z}}$ for the types Z_{12}, Q_{12}, U_{12}, Z_{18}, Q_{16}, U_{16} will be computed in the upcoming subsection and will turn out to be slightly more complicated.

4.3.1 *The group $G_{\mathbb{Z}}$ for exceptional singularities*

The Coxeter-Dynkin diagrams of the exceptional singularities at stake are displayed in Figure 6, Figure 7 and Figure 8. Figure 6 shows the Coxeter-Dynkin diagram for singu-

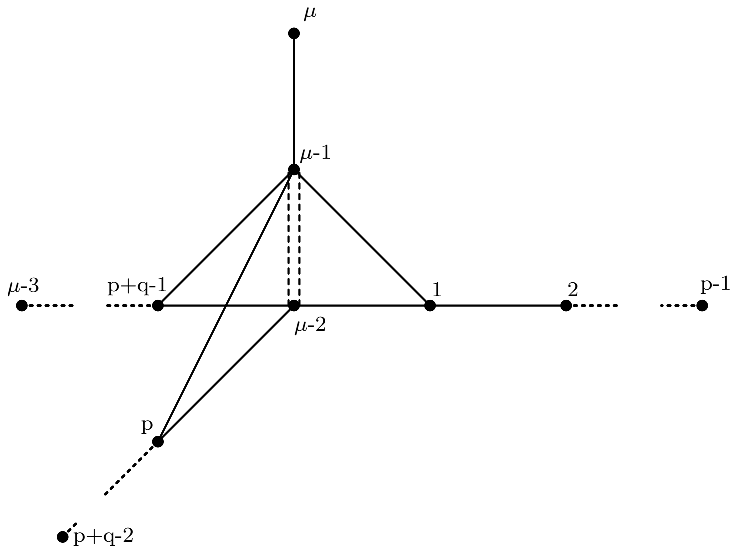

Figure 6: Coxeter-Dynkin diagram for Z_{12}, Q_{12} resp. U_{12}

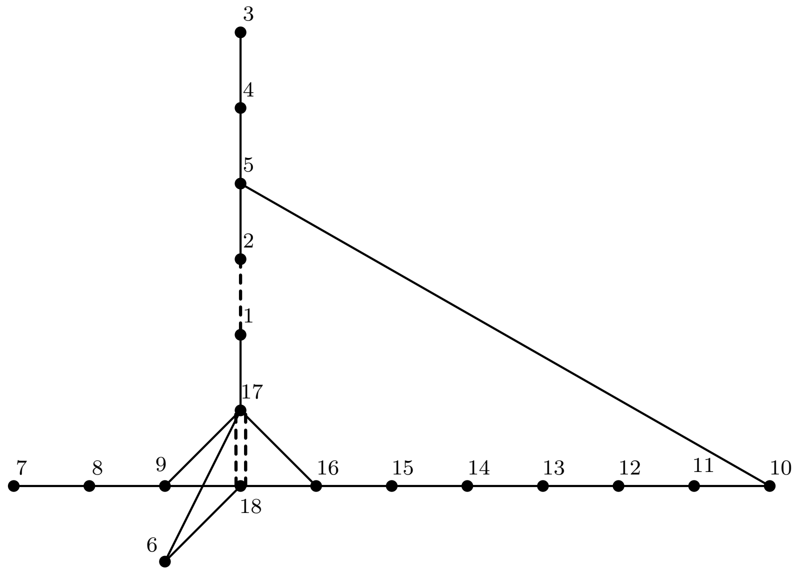

Figure 7: Coxeter-Dynkin diagram for Z_{18}

larities of type Z_{12}, Q_{12} and U_{12} where the triple (p, q, r) is $(2, 4, 6)$, $(3, 3, 6)$ and $(4, 4, 4)$, respectively.

The Coxeter-Dynkin diagram for singularities of type Z_{18} is shown in Figure 7 and the Coxeter-Dynkin diagram for singularities of type Q_{16} in Figure 8.

With the Coxeter-Dynkin diagrams we can easily compute the Seifert form L, the intersection form I and the monodromy M_h via the formulas in Chapter 2. On top of that for the (Brieskorn-Pham) singularity U_{16} we can write down immediately the monodromy, which is

$$M_h = S_{\Phi_3} \otimes S_{\Phi_3} \otimes S_{\Phi_5} \tag{103}$$

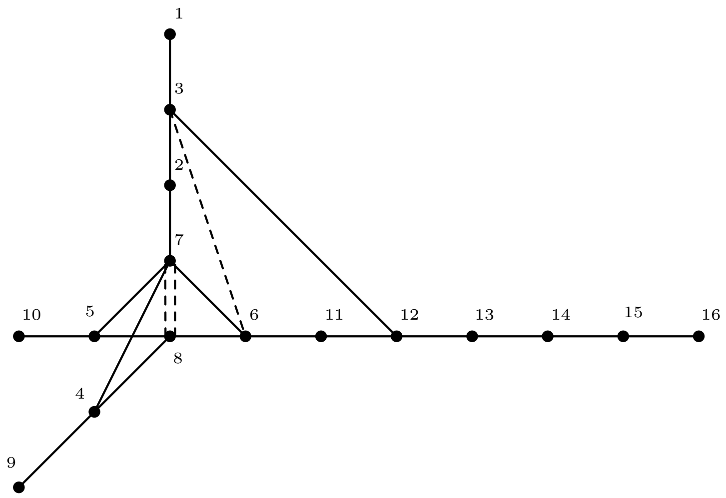

Figure 8: Coxeter-Dynkin diagram for Q_{16}

(cf. [He93, Kapitel 3 a)]). The n-th cyclotomic polynomial is denoted as Φ_n and S_{Φ_3} and S_{Φ_5} are the companion matrices[1] for the respective cyclotomic polynomials. The characteristic polynomial of the monodromy will be denoted as p_{ch} again.

Moreover we can decompose Milnor lattices of the six types of singularities into the sum of cyclic monodromy modules, the Orlik blocks (see Section A.2 for details). We obtain

$$Ml\left(f\right)=\bigoplus_{i=0}^{\deg p_1-1}\mathbb{Z}\cdot M_h^i\left(a_1\right)\oplus\bigoplus_{i=0}^{\deg p_2-1}\mathbb{Z}\cdot M_h^i\left(a_2\right) \tag{104}$$

where a_1 and a_2 are generators and p_1 and p_2 are polynomials with $p_{ch}=p_1\cdot p_2$ as specified in Table 6. This means that Orlik's Conjecture A.2.2 is true for all those types of singularities. Note that the Coxeter-Dynkin diagrams above are taken from [He93, Kapitel 3] — except the one for Z_{18}, which comes from [Eb83, Table 2 + Figure 4]. Furthermore, also the data in Table 6 is taken from [He93, Kapitel 3]. Yet in [He93] no explicit generators for the types Z_{12} and Z_{18} are given. There the author deduces the existence of the Orlik blocks from an abstract result in [MW86] (cf. Section A.2).

Now with the decomposition of the Milnor lattice into Orlik blocks we can compute the group $G_{\mathbb{Z}}$ via applying Lemma A.2.3 to the blocks. We prove the main theorem of this subsection.

1 For us the *companion matrix* of a polynomial $p(x)=a_gx^g+a_{g-1}x^{g-1}+\ldots+a_1x+a_0$ is the quadratic matrix $S_p:=\begin{pmatrix}0&0&\ldots&0&-a_0\\1&0&\ldots&0&-a_1\\0&1&\ddots&\vdots&-a_2\\\vdots&\ddots&\ddots&0&\vdots\\0&\ldots&0&1&-a_{g-1}\end{pmatrix}$.

Type	p_{ch}	p_1	p_2	a_1	a_2
Z_{12}	$\Phi_{22}\Phi_2^2$	$\Phi_{22}\Phi_2$	Φ_2	δ_2	$\delta_1-\delta_2-\delta_4$
Q_{12}	$\Phi_{15}\Phi_3^2$	$\Phi_{15}\Phi_3$	Φ_3	δ_6	$\delta_5+\delta_8-\delta_1$
U_{12}	$\Phi_{12}\Phi_6\Phi_4^2\Phi_2^2$	$\Phi_{12}\Phi_6\Phi_4\Phi_2$	$\Phi_4\Phi_2$	δ_8	$\delta_4-\delta_7$
Z_{18}	$\Phi_{34}\Phi_2^2$	$\Phi_{34}\Phi_2$	Φ_2	δ_7	$\delta_6-\delta_7-\delta_9$
Q_{16}	$\Phi_{21}\Phi_3^2$	$\Phi_{21}\Phi_3$	Φ_3	δ_4	$\delta_4-\delta_5$
U_{16}	$\Phi_{15}\Phi_5^2$	$\Phi_{15}\Phi_5$	Φ_5	δ_1	$\delta_1+\delta_5+\delta_{13}$

Table 6: Characteristic polynomials of the exceptional singularities

Theorem 4.3.1. *For the three 1-parameter families of exceptional unimodal singularities of types Z_{12}, Q_{12}, U_{12} and the three 2-parameter families of exceptional bimodal singularities of types Z_{18}, Q_{16}, U_{16} the automorphism group is $G_{\mathbb{Z}} = \{\pm M_h^k \mid k \in \mathbb{Z}\} \times U$ with*

$$\begin{array}{c|c|c|c|c|c|c} & Z_{12} & Q_{12} & U_{12} & Z_{18} & Q_{16} & U_{16} \\ \hline U \cong & \{\mathrm{id}\} & \mathrm{Sym}\,(2) & \mathrm{Sym}\,(3) & \{\mathrm{id}\} & \mathrm{Sym}\,(2) & \mathrm{Sym}\,(3) \end{array}. \tag{105}$$

Proof. We set

$$Ml\,(f) = \bigoplus_{i=0}^{\deg p_1 - 1} \mathbb{Z} \cdot M_h^i\,(a_1) \oplus \bigoplus_{i=0}^{\deg p_2 - 1} \mathbb{Z} \cdot M_h^i\,(a_2) =: B_1 \oplus B_2 \tag{106}$$

and

$$B_3 := \ker\left(p_2\,(M_h) : Ml\,(f)_{\mathbb{C}} \to Ml\,(f)_{\mathbb{C}}\right) \cap Ml\,(f)\,. \tag{107}$$

The space B_3 is then a primitive rank $2\deg p_2$ sublattice of the Milnor lattice. It is generated by the primitive sublattices $B_1 \cap B_3$ and B_2 of rank $\deg p_2$ of B_3. Now consider some $g \in G_{\mathbb{Z}}$ with $g|_{B_3} = \pm\,(M_h|_{B_3})^k$ for some $k \in \mathbb{Z}$. This restricts to an automorphism of B_1, because of $B_2 \subseteq B_3$ and

$$(B_1)_{\mathbb{C}} = \ker\left(\frac{p_1}{p_2}\,(M_h)\right) \oplus (B_1 \cap B_3)_{\mathbb{C}}\,. \tag{108}$$

So we are able to apply Lemma A.2.3 and obtain $g|_{B_1} = \pm\,(M_h|_{B_1})^l$ for some $l \in \mathbb{Z}$. Now $g|_{B_3} = \pm\,(M_h|_{B_3})^k$ implies $k \equiv l \mod \mathrm{lcm}(m : \Phi_m | p_2)$ and, thus, $g = \pm\,(M_h)^l$. It holds

$$\{g \in G_{\mathbb{Z}} \mid g|_{B_3} = \pm\,(M_h|_{B_3})^k \text{ for some } k \in \mathbb{Z}\} = \{\pm\,(M_h)^k \mid k \in \mathbb{Z}\}. \tag{109}$$

Additionally we can use Lemma A.2.7. In part (c) of this lemma all possible automorphisms on B_3 were computed. It holds

$$\mathrm{Aut}\,(B_3, L) = \{\pm\,(M_h|_{B_3})^k\,|k \in \mathbb{Z}\} \times U, \tag{110}$$

where U is as in (105). In part (d) it was shown that the map $G_{\mathbb{Z}} \to \mathrm{Aut}\,(B_3, L) \cong \{\pm\,(M_h|_{B_3})^k\,|k \in \mathbb{Z}\} \times U$ is surjective. Now with the information about the kernel of this surjective map from (109) we can deduce the claim via the fundamental homomorphism theorem. □

4.3.2 *Torelli Conjectures for exceptional singularities*

The Torelli results for the exceptional singularities are in a sense the "litmus test" for the period maps LBL resp. BL. For the types of singularities studied in Sections 4.1 and 4.2 already the Picard-Fuchs singularities are sufficient to determine the right equivalence class inside a μ-constant family. So this is the first time where we actually need the period maps LBL resp. BL.

We will prove now the injectivity of BL for the six families of exceptional unimodal and bimodal singularities that we considered in Subsection 4.3.1. In so doing we again build on the Torelli results for LBL in [He93]. However in contrast to the proof of Theorem 4.2.2 we only need one Torelli result from [He93] here.

Theorem 4.3.2. *(a) For the 6 families of exceptional unimodal and bimodal singularities of types $Z_{12}, Q_{12}, U_{12}, Z_{18}, Q_{16}$ and U_{16} in any number of variables, the space M_μ^{mar} is $M_\mu^{mar} \cong T := \mathbb{C}^{\mathrm{mod}(f_0)}$, so it is connected, and thus $G^{mar} = G_{\mathbb{Z}}$. Moreover, the period map $BL : M_\mu^{mar} \to D_{BL}$ is an isomorphism and, thus, the strong global Torelli Conjecture 3.2.2 (a) is true.*

(b) Now consider the singularities of type Z_{12} and Z_{18} as curve singularities and the singularities of types Q_{12}, U_{12}, Q_{16} and U_{16} as surface singularities. Then their multiplicities are ≥ 3. It holds

$$G_{\mathbb{Z}} = G^{mar} = G^{smar} \times \{\pm \mathrm{id}\}, \quad \textit{equivalently: } -\mathrm{id} \notin G^{smar}. \tag{111}$$

Proof. (a) We choose as a reference singularity the quasihomogeneous singularity in T with the marking $\pm\,\mathrm{id}$, i.e. $[(f_0, \pm\,\mathrm{id})]$. Then all elements elements in T become marked singularities, because T is simply connected. So we have a well-defined period map $T \to D_{BL}$. Now the Torelli result from [He93, Kapitel 3+Kapitel 4] says that this period map is an isomorphism. So all marked Brieskorn lattices of the marked singularities with parameter in T are different. Moreover, this means that all the marked singularities with parameter in T are different. We have an embedding $T \hookrightarrow M_\mu^{mar}(f_0)^0$.

Furthermore, there is the immersive period map $BL : M_\mu^{mar}(f_0)^0 \to D_{BL}$ according to Theorem 3.2.1. This restricts to the isomorphism $T \to D_{BL}$ and we have $T = M_\mu^{mar}(f_0)^0$. So it rests to show that $G^{mar} = G_{\mathbb{Z}}$ in order to prove part (a). Since if this is true, $T = M_\mu^{mar}(f_0)^0 = M_\mu^{mar}$ is connected and $BL : M_\mu^{mar} \to D_{BL}$ is an isomorphism.

Therefore we consider the weight(s) of the parameter(s) t for singularities of modality 1 and t_1 and t_2 for singularities of modality 2. They equip $T = M_\mu^{mar}$ with a good $\mathbb{C}^*$-action. It commutes with the action of $G_{\mathbb{Z}}$. This yields

$$G^{mar} = \mathrm{Stab}_{G_{\mathbb{Z}}}([(f_0, \pm\,\mathrm{id})]) =_{\text{Prop. 3.1.6 (g)}} G_{\mathcal{R}}^{mar}(f_0)\,. \tag{112}$$

Moreover, we know that $G^{smar}_{\mathcal{R}}(f_0) \times \{\pm \mathrm{id}\} = G^{mar}_{\mathcal{R}}(f_0)$ by Proposition 3.1.6 (d). Luckily, as f_0 is a quasihomogeneous singularity of degree ≥ 3, the group $G^{smar}_{\mathcal{R}}(f_0)$ can be calculated easily via $\mathrm{Stab}_{G_w}(f_0)$. Namely, by Theorem 3.1.6 (c) and Theorem A.1.5 it is $\mathrm{Stab}_{G_w}(f_0) \xrightarrow{\cong} G^{smar}_{\mathcal{R}}(f_0)$. It suffices to show that $\mathrm{Stab}_{G_w}(f_0)$ has half as many elements as the group $G_{\mathbb{Z}}$. We postpone its proof for a moment. If it holds, then

$$G_{\mathbb{Z}} = G^{mar} = G^{mar}_{\mathcal{R}}(f_0) = G^{smar}_{\mathcal{R}}(f_0) \times \{\pm \mathrm{id}\} \tag{113}$$

follows, and part (a) of the theorem holds. For part (b), the same arguments as in the proof of Theorem 4.2.2 (b) apply: M^{smar}_{μ} is a twofold covering of M^{mar}_{μ}, and the two sheets are exchanged by the action of $-\mathrm{id}$. Since $T = \mathbb{C}^{\mathrm{mod}(f_0)}$, we can conclude that M^{smar}_{μ} has two components and $-\mathrm{id} \notin M^{smar}_{\mu}$.

In Theorem 4.3.1 it was shown that $G_{\mathbb{Z}}$ is $G_{\mathbb{Z}} = \{\pm M_h^k \mid k \in \mathbb{Z}\} \times U$ with U as in Table (105). Now we compare it with $\mathrm{Stab}_{G_w}(f_0)$. It is sufficient to find enough elements such that the resulting group has half as many elements as $G_{\mathbb{Z}}$.

The cases Z_{12} and Z_{18}: Then

$$\varphi_1 : (x, y) \mapsto (e^{2\pi i w_x} x, e^{2\pi i w_y} y) \quad \text{satisfies} \quad (\varphi_1)_{hom} = M_h. \tag{114}$$

Here the quasihomogeneous weights (w_x, w_y) of the coordinates x and y are $\left(\frac{3}{11}, \frac{2}{11}\right)$ for Z_{12} and $\left(\frac{5}{17}, \frac{2}{17}\right)$ for Z_{18}. This is already sufficient. Here it is $G^{smar} = G^{smar}_{\mathcal{R}} = \{M_h^k \mid k \in \mathbb{Z}\}$.

The cases Q_{12}, Q_{16}, U_{12}, U_{16}: Here it is convenient to make use of the decomposition of the singularity f_0 into a sum of an A_l singularity g_0 in one variable and a D_{2m} singularity h_0 in two variables. In all four cases the weight system w' of the A_l singularity and the weight system w'' of the D_{2m} singularity have denominators $l+1$ and $2m-1$ with $\gcd(l+1, 2m-1) = 1$. Therefore it is

$$\begin{aligned} \mathrm{Stab}_{G_w}(f_0) &= \mathrm{Stab}_{G_{w'}}(g_0) \times \mathrm{Stab}_{G_{w''}}(h_0) \\ &\cong \mathbb{Z}_{l+1} \times \begin{cases} \mathbb{Z}_{2m-1} \times \mathrm{Sym}(2) & \text{if } m \geq 3 \\ \mathbb{Z}_3 \times \mathrm{Sym}(3) & \text{if } m = 2. \end{cases} \end{aligned} \tag{115}$$

In all four cases this group has half as many elements as $G_{\mathbb{Z}}$. □

Those were the hard cases among the exceptional unimodal and bimodal singularities. All in all, we can state the following.

Theorem 4.3.3. *For all families of exceptional unimodal and bimodal singularities it is $G^{mar} = G_{\mathbb{Z}}$. So $M^{mar}_{\mu} \cong \mathbb{C}^{\mathrm{mod}(f_0)}$ is connected. Moreover, the period map $BL : M^{mar}_{\mu} \to D_{BL}$ is injective and the strong global Torelli Conjecture 3.2.2 (a) is true.*

Proof. Combining [He11, Theorem 8.3] and Theorem 4.3.2 yields the claim. □

4.4 BIMODAL SERIES SINGULARITIES

The eight infinite series of bimodal singularities $E_{3,p}$, $Z_{1,p}$, $Q_{2,p}$, $U_{1,p}$, $W_{1,p}$, $W^{\sharp}_{1,p}$, $S_{1,p}$ and $S^{\sharp}_{1,p}$ from Arnold's list (see [AGV85, Section 15.1.2]), i.e. the singularities with Kodaira types $I_p^*(2,-,-,-)$ for $p \in \mathbb{Z}_{>0}$, are mysterious ever since. Nevertheless they also have to be considered in order to get Torelli results for all bimodal singularities. We have collected information on the types of the bimodal series singularities. In fact, Table 7 shows certain normal forms of the singularities with $(t_1, t_2) \in T := (\mathbb{C} \setminus \{0\}) \times \mathbb{C}$ and the Milnor number μ. For some series we have to distinguish between the terms of odd number p and the terms of even number p. Therefore we use the number $q \in \mathbb{Z}_{\geq 0}$. The data as it appears here is taken from [He95, Table 6.2].

Type	normal form	μ
$E_{3,p}$	$x^3 + x^2y^3 + (t_1 + t_2y)\, y^{9+p} + z^2$	$16 + p$
$Z_{1,p}$	$x^3y + x^2y^3 + (t_1 + t_2y)\, y^{7+p} + z^2$	$15 + p$
$Q_{2,p}$	$x^3 + yz^2 + x^2y^2 + (t_1 + t_2y)\, y^{6+p}$	$14 + p$
$U_{1,2q}$	$x^3 + xz^2 + xy^3 + (t_1 + t_2y)\, y^{3+q}z$	$14 + 2q$
$U_{1,2q-1}$	$x^3 + xz^2 + xy^3 + (t_1 + t_2y)\, y^{1+q}z^2$	$14 + 2q - 1$
$W_{1,p}$	$x^4 + x^2y^3 + (t_1 + t_2y)\, y^{6+p} + z^2$	$15 + p$
$W^\sharp_{1,2q}$	$(x^2 + y^3)^2 + (t_1 + t_2y)\, x^2y^{3+q} + z^2$	$15 + 2q$
$W^\sharp_{1,2q-1}$	$(x^2 + y^3)^2 + (t_1 + t_2y)\, xy^{4+q} + z^2$	$15 + 2q - 1$
$S_{1,p}$	$x^2z + yz^2 + x^2y^2 + (t_1 + t_2y)\, y^{5+p}$	$14 + p$
$S^\sharp_{1,2q}$	$x^2z + yz^2 + y^3z + (t_1 + t_2y)\, x^2y^{2+q}$	$14 + 2q$
$S^\sharp_{1,2q-1}$	$x^2z + yz^2 + y^3z + (t_1 + t_2y)\, xy^{3+q}$	$14 + 2q - 1$

Table 7: Facts about the bimodal series singularities

Note that all normal forms in Table 7 except the normal forms for $W^\sharp_{1,p}$ are Newton nondegenerate (cf. Definition A.1.1). Luckily, the normal form $f_p(x, y, \widetilde{z})$ for $W^\sharp_{1,p}$ can be easily made Newton nondegenerate as well. The respective coordinate change is $\widetilde{z} = z + i(x^2 + y^3)$ and this then yields

$$f_p(x, y, z + i(x^2 + y^3)) = z^2 + 2ix^2z + 2iy^3z + \begin{cases} (t_1 + t_2y)xy^{4+q} & \text{if } p = 2q - 1, \\ (t_1 + t_2y)x^2y^{3+q} & \text{if } p = 2q. \end{cases} \tag{116}$$

In the following we will work with the normal forms in Table 7, unless stated otherwise. Moreover for each 2-parameter family of singularities in Table 7 we choose $f_0 := f_{(1,0)}$ as reference singularity.

4.4.1 *The group $G_\mathbb{Z}$ for bimodal series singularities*

Now we list the Coxeter-Dynkin diagrams for all of the eight bimodal series. For each of the eight series of surface singularities, a distinguished basis $\delta_1, \ldots, \delta_\mu$ with the Coxeter-

Dynkin diagram in the corresponding figures is given. The distinguished basis is the one in [Eb81, Tabelle 6 + Abbildung 16], with a small change in the cases $W_{1,1}$ and $S_{1,1}$[2].

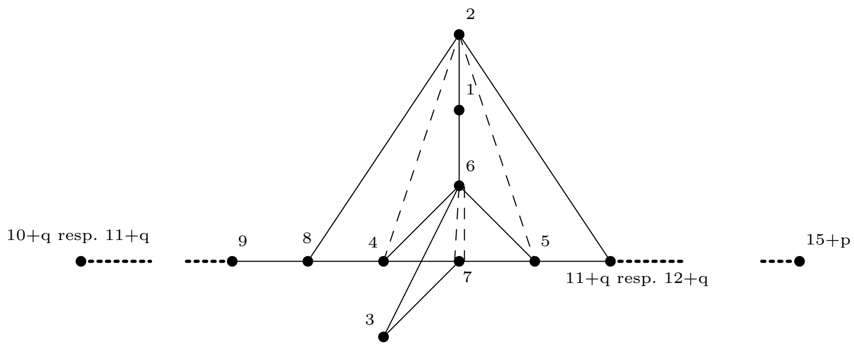

Figure 9: Coxeter-Dynkin diagram for $W^{\sharp}_{1,2q-1}$ resp. $W^{\sharp}_{1,2q}$

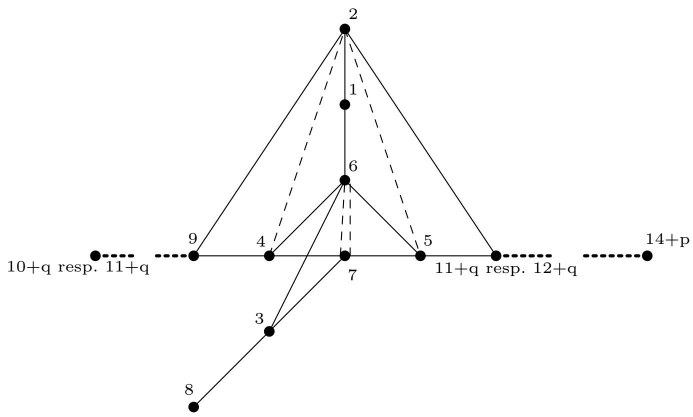

Figure 10: Coxeter-Dynkin diagram for $S^{\sharp}_{1,2q-1}$ resp. $S^{\sharp}_{1,2q}$

In contrast to the previous section there is in general no decomposition of the full Milnor lattice of bimodal series singularities into Orlik blocks. Counterexamples are for example given by the Milnor lattices of singularities of types $E_{3,p}$ where $p \equiv 0 \mod 6$ (cf. [He93, p. 172]). However we can find a decomposition of a certain index $r_I = [Ml(f) : \bigoplus_{j\geq 1} B_j]$, where B_j are Orlik blocks. This will be done in the next theorem. Note that the proof of part (c) of this theorem makes use of Lemma A.2.5 and results in Section A.3. Hence, it makes sense to recall this first.

2 In [Eb81] another numbering of the Coxeter-Dynkin diagram for $W_{1,1}$ and $S_{1,1}$ is suggested. I am grateful to Wolfgang Ebeling who provided information on how to transform the numbering in [Eb81] to the numbering we chose in Figure 15 and Figure 16 for $W_{1,1}$ and $S_{1,1}$.

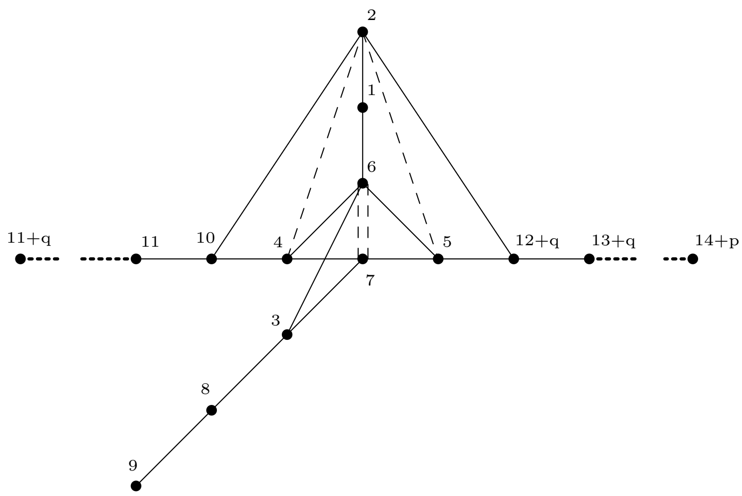

Figure 11: Coxeter-Dynkin diagram for $U_{1,p}$

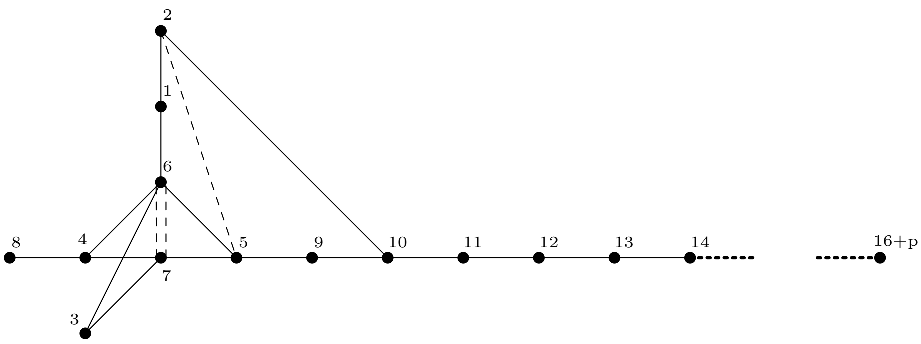

Figure 12: Coxeter-Dynkin diagram for $E_{3,p}$

Theorem 4.4.1. *For any surface singularity f in any of the eight bimodal series, the following holds.*

(a) For all series except $Z_{1,p}$, there are Orlik blocks $B_1, B_2 \subseteq Ml(f)$, and for the series $Z_{1,p}$, there are Orlik blocks $B_1, B_2, B_3 \subseteq Ml(f)$ with the following properties. The characteristic polynomial p_{B_j} of the monodromy on B_j is b_j. The sum $\sum_{j\geq 1} B_j$ is a direct sum $\bigoplus_{j\geq 1} B_j$, and it is a sublattice of $Ml(f)$ of full rank μ and of index r_I. Define

$$\widetilde{B_1} := \begin{cases} B_1 & \text{for all series except } Z_{1,p}, \\ B_1 \oplus B_3 & \text{for the series } Z_{1,p}. \end{cases} \tag{117}$$

Then it holds

$$\begin{aligned} L(\widetilde{B_1}, B_2) &= 0 = L(B_2, \widetilde{B_1}) && \text{for all series,} && (118) \\ G_{\mathbb{Z}} &= \mathrm{Aut}(\bigoplus_{j\geq 1} B_j, L) && \text{for all series except } S_{1,10}. && (119) \end{aligned}$$

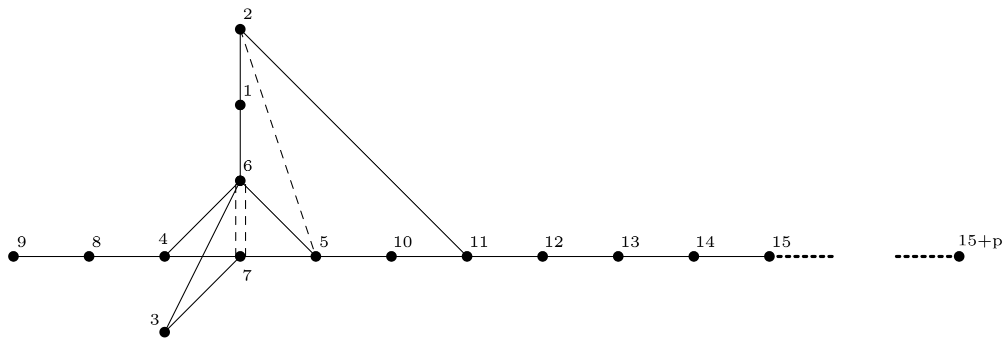

Figure 13: Coxeter-Dynkin diagram for $Z_{1,p}$

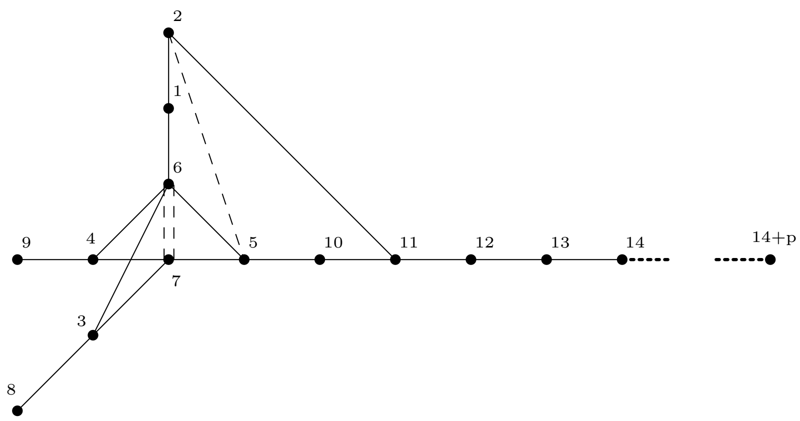

Figure 14: Coxeter-Dynkin diagram for $Q_{2,p}$

In the case $S_{1,10}$, a substitute for (119) *is*

$$g \in G_{\mathbb{Z}} \text{ with } g((B_1)_{\Phi_{10}}) = (B_1)_{\Phi_{10}} \Rightarrow g(B_j) = B_j \text{ for } j = 1, 2. \tag{120}$$

(b) It is $\Phi_m \not| b_2 \iff m \not| p$. In that case it holds

$$G_{\mathbb{Z}} = \{(\pm M_h^{k_1}|_{\widetilde{B_1}}) \times (\pm M_h^{k_2}|_{B_2}) \,|\, k_1, k_2 \in \mathbb{Z}\}. \tag{121}$$

(c) In the case of the subseries with $m|p$, the eigenspace $Ml(f)_\zeta \subseteq Ml(f)_{\mathbb{C}}$ is 2-dimensional. The Hermitian form h_ζ on it according to Lemma A.2.6 (a) with $h_\zeta(a,b) := \sqrt{-\zeta} \cdot L(a, \overline{b})$ for $a, b \in Ml(f)_\zeta$ is nondegenerate and indefinite, so $\mathrm{Proj}(Ml(f)_\zeta) \cong P^1\mathbb{C}$ *contains the half-plane*

$$\mathcal{H}_\zeta := \{\mathbb{C} \cdot a \,|\, a \in Ml(f)_\zeta \text{ with } h_\zeta(a,a) < 0\} \subseteq \mathrm{Proj}(Ml(f)_\zeta). \tag{122}$$

Therefore the group $\mathrm{Aut}(Ml(f)_\zeta, h_\zeta)/S^1 \cdot \mathrm{id}$ *is isomorphic to* $\mathrm{PSL}(\mathbb{R}; 2)$. *The homomorphism*

$$\Psi : G_{\mathbb{Z}} \to \mathrm{Aut}(Ml(f)_\zeta, h_\zeta)/S^1 \cdot \mathrm{id}, \quad g \mapsto g|_{Ml(f)_\zeta} \mod S^1 \cdot \mathrm{id}, \tag{123}$$

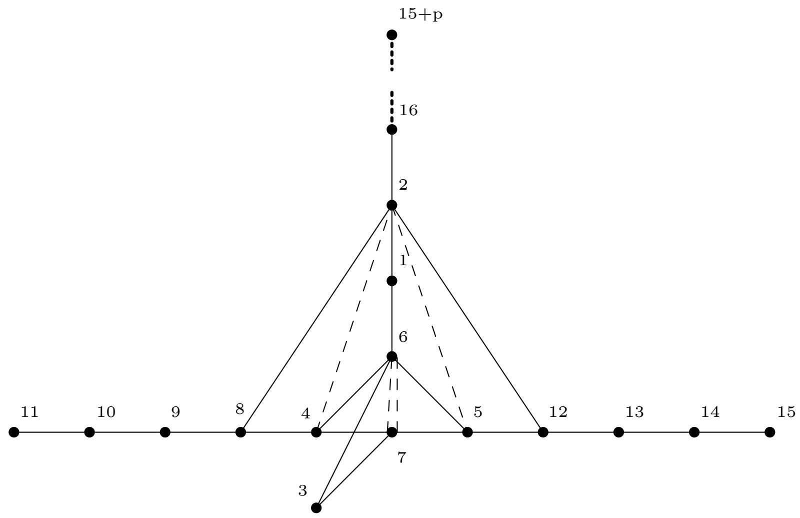

Figure 15: Coxeter-Dynkin diagram for $W_{1,p}$

is well-defined. The image of the homomorphism $\Psi(G_{\mathbb{Z}})$ *is an infinite Fuchsian group acting on the half-plane* $\mathcal{H}_\zeta$*. The kernel is*

$$\ker \Psi \quad = \quad \{\pm M_h^k \,|\, k \in \mathbb{Z}\}. \tag{124}$$

Before we start with the actual (very long) proof, we summarize the key idea of its longest part, i.e. the proof of part (a). So in this section (and Section 4.5) we work most times with two Orlik blocks B_1 and B_2 such that $B_1 + B_2 = B_1 \oplus B_2$. This direct sum is then either equal to $Ml\,(f)$ or has index 2 in $Ml\,(f)$ and it holds $L(B_1, B_2) = L(B_2, B_1) = 0$. For all bimodal series singularities (especially for those with $[Ml\,(f) : B_1 \oplus B_2] = 2$) except $S_{1,10}$ and $Z_{1,0}$, we will show

$$\mathrm{Aut}(Ml\,(f)\,, L) = \mathrm{Aut}(B_1 \oplus B_2, L). \tag{125}$$

This (often) works as follows.
In several cases, there exists an element $\gamma_5 \in (B_1)_{\Phi_2} \setminus \{0\}$ which is mapped to itself (modulo sign) by all elements in $\mathrm{Aut}(Ml\,(f)\,, L) \cup \mathrm{Aut}(B_1 \oplus B_2, L)$ and which fulfills

$$B_1 \oplus B_2 = \{a \in \mathrm{Aut}(Ml\,(f) \mid L(a, \gamma_5) \in 2\mathbb{Z}\}. \tag{126}$$

Then any element $g \in \mathrm{Aut}(Ml\,(f)\,, L)$ maps $B_1 \oplus B_2$ to itself. Thus we obtain the first inclusion $\mathrm{Aut}(Ml\,(f)\,, L) \subseteq \mathrm{Aut}(B_1 \oplus B_2, L)$. Now if this inclusion "$\supseteq$" holds, we can conclude that $\mathrm{Aut}(Ml\,(f)\,, L)$ is either equal to $\mathrm{Aut}(B_1 \oplus B_2, L)$ or to a subgroup of index 2 in $\mathrm{Aut}(B_1 \oplus B_2, L)$ via the argument in [GH18, Remarks 2.6 (iv)]. Note that, in general it is difficult to see that $\mathrm{Aut}(Ml\,(f)\,, L)$ is really equal to $\mathrm{Aut}(B_1 \oplus B_2, L)$ and not just a subgroup of index 2 in $\mathrm{Aut}(B_1 \oplus B_2, L)$. Hence, in the following, we need a different strategy to show equality in (125) (cf. Proof of Theorem 4.4.1).

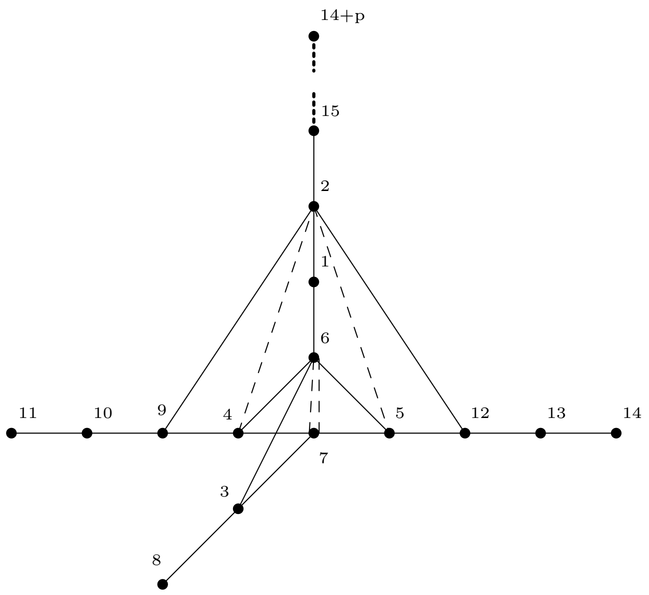

Figure 16: Coxeter-Dynkin diagram for $S_{1,p}$

Proof of Theorem 4.4.1. The Orlik blocks $B_j \subseteq Ml(f)$ are chosen by choosing the generating lattice vectors β_j with

$$B_j := \sum_{i \geq 0} \mathbb{Z} \cdot M_h^i(\beta_j). \tag{127}$$

The generators β_j are given in (128).

Type	β_1	β_2	β_3
$W_{1,p}^\sharp$	δ_3	δ_8	$-$
$S_{1,p}^\sharp$	δ_8	δ_9	$-$
$U_{1,p}$	δ_8	δ_{10}	$-$
$E_{3,p}$	δ_3	δ_{10}	$-$
$Z_{1,p}$	δ_8	δ_{11}	$\delta_3 - \delta_4 - \delta_9$
$Q_{2,p}$	δ_8	δ_{11}	$-$
$W_{1,p}$	$\delta_3 + \delta_9 + \delta_{11}$	δ_{16}	$-$
$S_{1,p}$	$-\delta_8 + \delta_{13}$	δ_{15}	$-$

(128)

Now the action of the powers of the monodromy on the generators

$$\beta_j \mapsto M_h(\beta_j) \mapsto M_h^2(\beta_j) \mapsto \ldots \mapsto M_h^{\deg b_j}(\beta_j). \tag{129}$$

is computed and the characteristic polynomials b_j of M_h on the B_j are determined. Moreover we show that B_j is a primitive sublattice of $Ml(f)$, that $\sum_{j\geq 1} B_j = \bigoplus_{j\geq 1} B_j$ is a direct sum and that it is a sublattice of full rank and of index r_I in $Ml(f)$. This data is summarized in Table 8.

Type	b_1	b_2	b_3	m	r_I
$W^\sharp_{1,p}$	Φ_{12}	$(t^{12+p}-1)/\Phi_1$	-	12	1
$S^\sharp_{1,p}$	$\Phi_{10}\Phi_2$	$(t^{10+p}-1)/\Phi_1$	-	10	1
$U_{1,p}$	Φ_9	$(t^{9+p}-1)/\Phi_1$	-	9	1
$E_{3,p}$	$\Phi_{18}\Phi_2$	$t^{9+p}+1$	-	18	2
$Z_{1,p}$	$\Phi_{14}\Phi_2$	$t^{7+p}+1$	Φ_2	14	2
$Q_{2,p}$	$\Phi_{12}\Phi_4\Phi_3$	$t^{6+p}+1$	-	12	2
$W_{1,p}$	$\Phi_{12}\Phi_6\Phi_3\Phi_2$	$t^{6+p}+1$	-	12	2
$S_{1,p}$	$\Phi_{10}\Phi_5\Phi_2$	$t^{5+p}+1$	-	10	2

Table 8: Characteristic polynomials of the bimodal series singularities

For the series $W^\sharp_{1,p}$ we obtain the monodromy action on the respective generators β_j as follows:

$$\begin{aligned}
\delta_3 &\mapsto -\delta_1-\delta_3-\delta_6+\delta_7 \mapsto \delta_1+\delta_2-\delta_4-\delta_5-\delta_6 \\
&\mapsto -\delta_1 \mapsto \delta_1+\delta_2-\delta_3-\delta_4-\delta_5-\delta_6, \qquad (130)\\
\delta_8 &\mapsto \delta_9 \mapsto \ldots \mapsto \delta_{11+q} \mapsto -\delta_4-\delta_8-\delta_9-\ldots-\delta_{11+q} \\
&\mapsto -\delta_2+\delta_4+\delta_6-\delta_7 \mapsto -\delta_{12+q} \mapsto -\delta_{13+q} \mapsto \ldots \mapsto -\delta_{15+p} \\
&\mapsto \delta_5+\delta_{12+q}+\ldots+\delta_{15+p} \mapsto \delta_2-\delta_5-\delta_6+\delta_7 \mapsto \delta_8. \qquad (131)
\end{aligned}$$

Thus the characteristic polynomial of M_h on B_j is b_j, and the blocks are

$$B_1 = \langle \delta_3, \delta_1, \delta_6-\delta_7, \delta_2-\delta_4-\delta_5-\delta_6 \rangle, \qquad (132)$$

$$B_2 = \langle \delta_8, \delta_9, \ldots, \delta_{15+p}; \delta_4, \delta_5, -\delta_2+\delta_6-\delta_7 \rangle. \qquad (133)$$

This shows that B_1 and B_2 are primitive sublattices with $B_1+B_2 = B_1 \oplus B_2 = Ml(f)$, i.e. $r_I = 1$.

For the series $S^\sharp_{1,p}$ the monodromy acts on the generators as follows:

$$\begin{aligned}
\delta_8 &\mapsto -\delta_3-\delta_8 \mapsto \delta_1+\delta_3+\delta_6-\delta_7 \\
&\mapsto -\delta_1-\delta_2+\delta_3+\delta_4+\delta_5+\delta_6+\delta_8 \\
&\mapsto -\delta_3-\delta_6+\delta_7 \mapsto -\delta_8, \qquad (134)\\
\delta_9 &\mapsto \delta_{10} \mapsto \ldots \mapsto \delta_{11+q} \mapsto -\delta_4-\delta_9-\delta_{10}-\ldots-\delta_{11+q} \\
&\mapsto -\delta_2+\delta_4+\delta_6-\delta_7 \mapsto -\delta_{12+q} \mapsto -\delta_{13+q} \mapsto \ldots \mapsto -\delta_{14+p} \\
&\mapsto \delta_5+\delta_{12+q}+\ldots+\delta_{14+p} \mapsto \delta_2-\delta_5-\delta_6+\delta_7 \mapsto \delta_9. \qquad (135)
\end{aligned}$$

Thus the characteristic polynomial of M_h on B_j is b_j, and the blocks are

$$\begin{aligned} B_1 &= \langle \delta_8, \delta_3, \delta_6 - \delta_7, \delta_1, -\delta_2 + \delta_4 + \delta_5 + \delta_6 \rangle, &(136)\\ B_2 &= \langle \delta_9, \delta_{10}, \ldots, \delta_{14+p}; \delta_4, \delta_5, -\delta_2 + \delta_6 - \delta_7 \rangle. &(137)\end{aligned}$$

This shows that B_1 and B_2 are primitive sublattices with $B_1 + B_2 = B_1 \oplus B_2 = Ml\,(f)$ and $r_I = 1$.

For the series $U_{1,p}$ the monodromy action on the generators is:

$$\begin{aligned} \delta_8 &\mapsto \delta_9 \mapsto -\delta_3 - \delta_8 - \delta_9 \mapsto \delta_1 + \delta_3 + \delta_6 - \delta_7 \\ &\mapsto -\delta_1 - \delta_2 + \delta_3 + \delta_4 + \delta_5 + \delta_6 + \delta_8 \\ &\mapsto -\delta_6 + \delta_7 + \delta_8 + \delta_9 \mapsto -\delta_1 - \delta_3 - \delta_6 + \delta_7 - \delta_8, &(138)\\ \delta_{10} &\mapsto \delta_{11} \mapsto \ldots \mapsto \delta_{11+q} \mapsto -\delta_4 - \delta_{10} - \delta_{11} - \ldots - \delta_{11+q} \\ &\mapsto -\delta_2 + \delta_4 + \delta_6 - \delta_7 \mapsto -\delta_{12+q} \mapsto -\delta_{13+q} \mapsto \ldots \mapsto -\delta_{14+p} \\ &\mapsto \delta_5 + \delta_{12+q} + \ldots + \delta_{14+p} \mapsto \delta_2 - \delta_5 - \delta_6 + \delta_7 \mapsto \delta_{10}. &(139)\end{aligned}$$

Thus the characteristic polynomial of M_h on B_j is b_j, and the blocks are

$$\begin{aligned} B_1 &= \langle \delta_1, \delta_3, \delta_8, \delta_9, \delta_6 - \delta_7, -\delta_2 + \delta_4 + \delta_5 + \delta_6 \rangle, &(140)\\ B_2 &= \langle \delta_{10}, \delta_{11}, \ldots, \delta_{14+p}; \delta_4, \delta_5, -\delta_2 + \delta_6 - \delta_7 \rangle. &(141)\end{aligned}$$

Again B_1 and B_2 are primitive sublattices with $B_1 + B_2 = B_1 \oplus B_2 = Ml\,(f)$ and $r_I = 1$.

For the series $E_{3,p}$ the monodromy acts as follows:

$$\begin{aligned} \delta_3 &\mapsto -\delta_1 - \delta_3 - \delta_6 + \delta_7 \mapsto -\delta_4 - \delta_5 - \delta_6 \\ &\mapsto \delta_2 - \delta_3 - \delta_4 - \delta_5 - \delta_6 - \delta_8 - \delta_9 \mapsto -\delta_5 - \delta_7 \\ &\mapsto \delta_2 - \delta_3 - \delta_4 - \delta_5 - \delta_6 - \delta_9 \mapsto -\delta_4 - \delta_5 - \delta_7 - \delta_8 \\ &\mapsto \delta_1 + \delta_2 - \delta_3 - \delta_5 - \delta_7 - \delta_9 \mapsto \delta_3 + \delta_6 - \delta_7 \\ &\mapsto -\delta_3, &(142)\\ \delta_{10} &\mapsto \delta_{11} \mapsto \ldots \mapsto \delta_{16+p} \mapsto -\delta_5 - \sum_{i=9}^{16+p} \delta_i \\ &\mapsto -\delta_2 + \delta_5 + \delta_6 - \delta_7 \mapsto -\delta_{10}. &(143)\end{aligned}$$

Thus the characteristic polynomial of M_h on B_j is b_j, and the blocks are

$$\begin{aligned} B_1 &= \langle \delta_1, \delta_3, \delta_4, \delta_8, \delta_6 - \delta_7, \delta_5 + \delta_6, \delta_2 - \delta_9 \rangle, &(144)\\ B_2 &= \langle \delta_{10}, \delta_{11}, \ldots, \delta_{16+p}, \delta_5 + \delta_9, \delta_2 - \delta_6 + \delta_7 + \delta_9 \rangle. &(145)\end{aligned}$$

This shows that B_1 and B_2 are primitive sublattices with $B_1 + B_2 = B_1 \oplus B_2$. Furthermore $B_1 \oplus B_2 \supseteq \{2\delta_2\}$ and $B_1 + B_2 + \mathbb{Z} \cdot \delta_2 = Ml(f)$. This shows $[Ml(f) : B_1 \oplus B_2] = 2 = r_I$.

For the series $Z_{1,p}$ the monodromy action is as follows:

$$\begin{aligned}
\delta_8 &\mapsto \delta_9 \mapsto -\delta_4-\delta_8-\delta_9 \mapsto \delta_1+\delta_4+\delta_6-\delta_7 \\
&\mapsto \delta_3+\delta_4+\delta_5+\delta_6+\delta_8 \\
&\mapsto -\delta_1-\delta_2+\delta_4+\delta_5+\delta_7+\delta_8+\delta_9+\delta_{10} \\
&\mapsto -\delta_4-\delta_6+\delta_7 \mapsto -\delta_8, \qquad (146) \\
\delta_{11} &\mapsto \delta_{12} \mapsto \ldots \mapsto \delta_{15+p} \mapsto -\delta_5 - \sum_{i=10}^{15+p} \delta_i \\
&\mapsto -\delta_2+\delta_5+\delta_6-\delta_7 \mapsto -\delta_{11}, \qquad (147) \\
\delta_3-\delta_4-\delta_9 &\mapsto -\delta_3+\delta_4+\delta_9. \qquad (148)
\end{aligned}$$

Thus the characteristic polynomial of M_h on B_j is b_j, and the blocks are

$$\begin{aligned}
B_1 &= \langle \delta_8, \delta_9, \delta_4, \delta_1, \delta_6-\delta_7, \delta_3+\delta_5+\delta_6, \\
&\quad -\delta_2+\delta_5+\delta_7+\delta_{10}\rangle, \qquad (149) \\
B_2 &= \langle \delta_{11}, \delta_{12}, \ldots, \delta_{15+p}; \delta_5+\delta_{10}, -\delta_2+\delta_5+\delta_6-\delta_7\rangle, \qquad (150) \\
B_3 &= \langle \delta_3-\delta_4-\delta_9\rangle. \qquad (151)
\end{aligned}$$

This shows that B_1, B_2 and B_3 are primitive sublattices with $B_1+B_2+B_3 = B_1\oplus B_2\oplus B_3$. Furthermore $B_1\oplus B_2\oplus B_3 \supseteq \{2\delta_5\}$ and $B_1+B_2+B_3+\mathbb{Z}\cdot\delta_5 = Ml(f)$. This shows $[Ml(f) : B_1\oplus B_2\oplus B_3] = 2 = r_I$.

For the series $Q_{2,p}$ the monodromy action on the generators is as follows:

$$\begin{aligned}
\delta_8 &\mapsto -\delta_3-\delta_8 \mapsto \delta_1+\delta_3+\delta_6-\delta_7 \mapsto \delta_3+\delta_4+\delta_5+\delta_6+\delta_8 \\
&\mapsto -\delta_1-\delta_2+\delta_4+\delta_5+\delta_7+\delta_9+\delta_{10} \mapsto -\delta_4-\delta_6+\delta_7 \\
&\mapsto -\delta_9 \mapsto \delta_4+\delta_9 \mapsto -\delta_1-\delta_4-\delta_6+\delta_7, \qquad (152) \\
\delta_{11} &\mapsto \delta_{12} \mapsto \ldots \mapsto \delta_{14+p} \mapsto -\delta_5 - \sum_{i=10}^{14+p} \delta_i \\
&\mapsto -\delta_2+\delta_5+\delta_6-\delta_7 \mapsto -\delta_{11}. \qquad (153)
\end{aligned}$$

Thus the characteristic polynomial of M_h on B_j is b_j, and the blocks are

$$\begin{aligned}
B_1 &= \langle \delta_8, \delta_3, \delta_9, \delta_4, \delta_1, \delta_6-\delta_7, \\
&\quad \delta_5+\delta_6, -\delta_2+\delta_5+\delta_7+\delta_{10}\rangle, \qquad (154) \\
B_2 &= \langle \delta_{11}, \delta_{12}, \ldots, \delta_{14+p}; \delta_5+\delta_{10}, -\delta_2+\delta_5+\delta_6-\delta_7\rangle. \qquad (155)
\end{aligned}$$

This shows that B_1 and B_2 are primitive sublattices with $B_1+B_2 = B_1\oplus B_2$. Furthermore $B_1\oplus B_2 \supseteq \{2\delta_5\}$ and $B_1+B_2+\mathbb{Z}\cdot\delta_5 = Ml(f)$. This shows $[Ml(f) : B_1\oplus B_2] = 2 = r_I$.

For the series $W_{1,p}$ the monodromy action on the generators is as follows:

$$\begin{aligned}
\delta_3+\delta_9+\delta_{11} &\mapsto -\delta_1-\delta_3-\delta_4-\delta_6+\delta_7-\delta_8-\delta_9-\delta_{11}\\
&\mapsto \delta_1-\delta_5-\delta_7+\delta_{11}\\
&\mapsto -\delta_1-\delta_4-\delta_8-\delta_9-\delta_{10}-\delta_{11}-\delta_{12}\\
&\mapsto \delta_1-\delta_3-\delta_5-\delta_7-\delta_{13}\\
&\mapsto \delta_3+\delta_6-\delta_7-\delta_{12}-\delta_{14}\\
&\mapsto -\delta_3-\delta_{13}-\delta_{15}\\
&\mapsto \delta_1+\delta_3+\delta_5+\delta_6-\delta_7+\delta_{12}+\delta_{13}+\delta_{15}\\
&\mapsto -\delta_1+\delta_4+\delta_7-\delta_{15}\\
&\mapsto \delta_1+\delta_5+\delta_8+\delta_{12}+\delta_{13}+\delta_{14}+\delta_{15} \qquad (156)\\
&\mapsto -\delta_1+\delta_3+\delta_4+\delta_7+\delta_9\\
&\mapsto -\delta_3-\delta_6+\delta_7+\delta_8+\delta_{10}\\
&\mapsto \delta_3+\delta_9+\delta_{11},
\end{aligned}$$

$$\begin{aligned}
\delta_{16} &\mapsto \delta_{17}\mapsto\ldots\mapsto\delta_{14+p}\mapsto\delta_{15+p}\\
&\mapsto -\delta_1-\delta_2-\sum_{i=16}^{15+p}\delta_i\\
&\mapsto -\delta_3-\delta_4-\delta_5-\delta_6-\delta_8-\delta_{12}\\
&\mapsto -\delta_4-\delta_5-\delta_7-\delta_8-\delta_9-\delta_{12}-\delta_{13}\\
&\mapsto -\delta_3-\delta_4-\delta_5-\delta_7-\delta_8-\delta_9-\delta_{10}-\delta_{12}-\delta_{13}-\delta_{14}\\
&\mapsto \delta_1-\delta_4-\delta_5+\delta_6-2\delta_7-\sum_{i=8}^{15}\delta_i\\
&\mapsto -\delta_2+\delta_4+\delta_5+2\delta_6-2\delta_7 \qquad (157)\\
&\mapsto -\delta_{16}.
\end{aligned}$$

Thus the characteristic polynomial of M_h on B_j is b_j. Here the blocks B_1 and B_2 are generated by the first $\deg b_1$ respectively $\deg b_2$ of the elements above. Here $B_1+B_2=B_1\oplus B_2$ and $[Ml(f):B_1\oplus B_2]=2=r_I$ follow by the calculation of the determinant which expresses these generators of B_1 and B_2 in the distinguished basis $\delta_1,\ldots,\delta_\mu$. Then it also follows that B_1 and B_2 are primitive sublattices.

For the series $S_{1,p}$ the monodromy action is as follows:

$$\begin{aligned}
-\delta_8+\delta_{13} &\mapsto \delta_3+\delta_8+\delta_{14} \\
&\mapsto -\delta_1-\delta_3-\delta_5-\delta_6+\delta_7-\delta_{12}-\delta_{13}-\delta_{14} \\
&\mapsto \delta_1-\delta_3-\delta_4-\delta_7-\delta_8 \\
&\mapsto \delta_3+\delta_6-\delta_7-\delta_9 \\
&\mapsto \delta_8-\delta_{10} \\
&\mapsto -\delta_3-\delta_8-\delta_{11} \\
&\mapsto \delta_1+\delta_3+\delta_4+\delta_6-\delta_7+\delta_9+\delta_{10}+\delta_{11} \\
&\mapsto -\delta_1+\delta_3+\delta_5+\delta_7+\delta_8 \\
&\mapsto -\delta_3-\delta_6+\delta_7+\delta_{12} \\
&\mapsto -\delta_8+\delta_{13},
\end{aligned} \tag{158}$$

$$\begin{aligned}
\delta_{15} &\mapsto \delta_{16}\mapsto\ldots\mapsto\delta_{14+p}\mapsto -\delta_1-\delta_2-\sum_{i=15}^{14+p}\delta_i \\
&\mapsto -\delta_3-\delta_4-\delta_5-\delta_6-\delta_9-\delta_{12} \\
&\mapsto -\delta_3-\delta_4-\delta_5-\delta_7-\delta_8-\delta_9-\delta_{10}-\delta_{12}-\delta_{13} \\
&\mapsto \delta_1-\delta_4-\delta_5+\delta_6-2\delta_7-\sum_{j\in\{9,10,11,12,13,14\}}\delta_j \\
&\mapsto -\delta_2+\delta_4+\delta_5+2\delta_6-2\delta_7\mapsto -\delta_{15}.
\end{aligned} \tag{159}$$

Thus the characteristic polynomial of M_h on B_j is b_j. Here the blocks B_1 and B_2 are generated by the first $\deg b_1$ respectively $\deg b_2$ of the elements above. Here $B_1+B_2=B_1\oplus B_2$ and $[Ml(f):B_1\oplus B_2]=2=r_I$ follow by the calculation of the determinant which expresses these generators of B_1 and B_2 in the distinguished basis $\delta_1,\ldots,\delta_\mu$. Then it also follows that B_1 and B_2 are primitive sublattices.
Now with the nice generators for $\widetilde{B_1}$ and B_2 as given above, we can show the left and right L-orthogonality of $\widetilde{B_1}$ and B_2 in (118). First of all, we see that δ_μ is a cyclic generator for B_2 in all eight series. The generators of $\widetilde{B_1}$ yield $\widetilde{B_1}\subseteq\bigoplus_{j=1}^{\mu-2}\mathbb{Z}\cdot\delta_j$ for all cases except $W_{1,1}$ and $S_{1,1}$. This together with $L(\delta_i,\delta_\mu)=0$ for $i<\mu$ show $L(\widetilde{B_1},\delta_\mu)=0$ and, thus, $L(\widetilde{B_1},B_2)=0$. Looking carefully at the Coxeter-Dynkin diagrams we see that $L(\delta_\mu,\delta_i)=0$ for $i\le\mu-2$ for all cases except $W_{1,1}$ and $S_{1,1}$, thus $L(\delta_\mu,\widetilde{B_1})=0$ and $L(B_2,\widetilde{B_1})=0$. However, in the cases $W_{1,1}$ and $S_{1,1}$ $L(B_1,\delta_\mu)=0=L(\delta_\mu,B_1)$ and, hence, $L(B_1,B_2)=0=L(B_2,B_1)$ hold also.

The substitute of (119) for $S_{1,10}$ is (120). It will be shown at the very end of this proof as some sort of a special case. So in order to prove part (a), it remains to show (119). This is trivial for the three series with $r_I=1$. It will be shown for the series $Q_{2,p}$ and for the subseries $W_{1,6s-3}$ with $s\in\mathbb{Z}_{\ge1}$) of the series $W_{1,p}$ also at the very end of this proof.

For the other cases we have to study smaller Orlik blocks. In particular, for the series $S^{\sharp}_{1,p}$, $E_{3,p}$, $Z_{1,p}$, $W_{1,p}$ and $S_{1,p}$ it is $\Phi_2|b_1$. In these cases we can define

$$\gamma_1 := v(\beta_1, -1) := \frac{b_1}{\Phi_2}(M_h)(\beta_1) \tag{160}$$

and compute $L(\gamma_1, \gamma_1) = \frac{b_1}{\Phi_2}(-1) \cdot L(\gamma_1, \beta_1)$ via (478). The results are listed in (161).

Type	γ_1	$L(\gamma_1, \gamma_1)$
$S^{\sharp}_{1,p}$	$\Phi_{10}(M_h)(\delta_8) = 2\delta_1 + \delta_2 - \delta_4 - \delta_5 - \delta_6 + \delta_8$	5
$E_{3,p}$	$\Phi_{18}(M_h)(\delta_3) = -\delta_2 + 2\delta_3 + \delta_6 - \delta_7 + \delta_9$	6
$Z_{1,p}$	$\Phi_{14}(M_h)(\delta_8)$	
	$= \delta_2 + \delta_3 - 3\delta_4 - \delta_6 + \delta_7 - 3\delta_9 - \delta_{10}$	21
$W_{1,p}$	$(\Phi_{12}\Phi_6\Phi_3)(M_h)(\delta_3 + \delta_9 + \delta_{11})$	
	$= \delta_4 - \delta_5 + \delta_9 + \delta_{11} - \delta_{13} - \delta_{15}$	6
$S_{1,p}$	$(\Phi_{10}\Phi_5)(M_h)(-\delta_8 + \delta_{13})$	
	$= -2\delta_1 + \delta_7 - \delta_8 - \delta_9 - \delta_{11} - \delta_{12} - \delta_{14}$	10

(161)

In the case of the series $Z_{1,p}$ we set additionally $\gamma_3 := \beta_3$ and compute

$$L(\gamma_3, \gamma_3) = 3, \quad L(\gamma_1, \gamma_3) = L(\gamma_3, \gamma_1) = 7. \tag{162}$$

Now in certain subseries of the series $S^{\sharp}_{1,p}$, $E_{3,p}$, $Z_{1,p}$, $W_{1,p}$ and $S_{1,p}$ it is also $\Phi_2|b_2$ (see (163) for details).

Type	Condition for $\Phi_2|b_2$	$L(\gamma_2, \gamma_2)$
$S^{\sharp}_{1,p}$	$p \equiv 0(2)$	$5 + \frac{p}{2}$
$E_{3,p}$	$p \equiv 0(2)$	$18 + 2p$
$Z_{1,p}$	$p \equiv 0(2)$	$14 + 2p$
$W_{1,p}$	$p \equiv 1(2)$	$12 + 2p$
$S_{1,p}$	$p \equiv 0(2)$	$10 + 2p$

(163)

Hence, in these cases we can also define

$$\gamma_2 := v(\beta_2, -1) := \frac{b_2}{\Phi_2}(M_h)(\beta_2) \tag{164}$$

and compute $L(\gamma_2,\gamma_2)=\frac{b_2}{\Phi_2}(-1)\cdot L(\gamma_2,\beta_2)$ (cf. (478)). The results are listed in (165).

$$\begin{array}{l|l} \text{Type} & \gamma_2 \\ \hline S^\sharp_{1,p} & -\delta_2+\delta_4+\delta_5+\delta_6-\delta_7+\sum_{j=1}^{2+p/4}(\delta_{7+2j}+\delta_{10+\frac{p}{2}+2j}) \\ & -\delta_4+\delta_5+\sum_{j=1}^{(6+p)/4}(-\delta_{8+2j}+\delta_{11+\frac{p}{2}+2j}) \\ E_{3,p} & -\delta_2+2\delta_5+\delta_6-\delta_7+\delta_9+2\sum_{j=1}^{4+p/2}\delta_{8+2j} \\ Z_{1,p} & -\delta_2+2\delta_5+\delta_6-\delta_7+\delta_{10}+2\sum_{j=1}^{3+p/2}\delta_{9+2j} \\ W_{1,p} & -2\delta_3+\delta_4+\delta_5+\delta_9+\delta_{11}+\delta_{13}+\delta_{15}+2\sum_{j=1}^{(1+p)/2}\delta_{14+2j} \\ S_{1,p} & 2(-\delta_1-\delta_2+\delta_4+\delta_5+\delta_6)-\delta_7-\delta_8 \\ & \qquad +\delta_9+\delta_{11}+\delta_{12}+\delta_{14}-2\sum_{j=1}^{p/2}\delta_{14+2j} \end{array} \tag{165}$$

Note that in (165) the first line for $S^\sharp_{1,p}$ is the case $p\equiv 0(4)$, the second line is the case $p\equiv 2(4)$.

In the subseries of $E_{3,p}, W_{1,p}$ and $S_{1,p}$ with $\Phi_2|b_2$ we see

$$\widetilde{\gamma_2}:=\frac{1}{2}(\gamma_1+\gamma_2)\overset{!}{\in} Ml(f). \tag{166}$$

In the subseries of $Z_{1,p}$ with $\Phi_2|b_2$ we see

$$\widetilde{\gamma_2}:=\frac{1}{2}(\gamma_1+\gamma_2-3\gamma_3)\overset{!}{\in} Ml(f). \tag{167}$$

Together with $[Ml(f):B_1\oplus B_2]=2$ for these subseries, this shows

$$Ml(f)_{\Phi_2} = \mathbb{Z}\gamma_1\oplus\mathbb{Z}\widetilde{\gamma_2} \qquad \text{for } E_{3,2q}, W_{1,2q-1}, S_{1,2q}, \tag{168}$$

$$Ml(f)_{\Phi_2} = \mathbb{Z}(\gamma_1-2\gamma_3)\oplus\mathbb{Z}\widetilde{\gamma_2}\oplus\mathbb{Z}\gamma_3 \qquad \text{for } Z_{1,2q}. \tag{169}$$

For $S^\sharp_{1,2q}$ the Orlik decomposition $Ml(f)=B_1\oplus B_2$ gives a decomposition $Ml(f)_{\Phi_2}=\mathbb{Z}\gamma_1\oplus\mathbb{Z}\gamma_2$. The matrices of L for these bases of $Ml(f)_{\Phi_2}$ in the aforementioned cases are:

$$\begin{array}{ccc} S^\sharp_{1,2q} & E_{3,2q} & Z_{1,2q} \\ \begin{pmatrix} 5 & 0 \\ 0 & 5+q \end{pmatrix} & \begin{pmatrix} 6 & 3 \\ 3 & 6+q \end{pmatrix} & \begin{pmatrix} 5 & 2 & 1 \\ 2 & 5+q & -1 \\ 1 & -1 & 3 \end{pmatrix} \\ W_{1,2q-1} & S_{1,2q} & \\ \begin{pmatrix} 6 & 3 \\ 3 & 4+q \end{pmatrix} & \begin{pmatrix} 10 & 5 \\ 5 & 5+q \end{pmatrix} & \end{array} \tag{170}$$

All these matrices are positive definite. Moreover, the corresponding quadratic forms $(x_1\, x_2)(\text{matrix})\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ respectively $(x_1\, x_2\, x_3)(\text{matrix})\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ are

$$\begin{array}{rl} 5x_1^2+(5+q)x_2^2 & \text{for } S^\sharp_{1,2q} \\ 3x_1^2+3(x_1+x_2)^2+(3+q)x_2^2 & \text{for } E_{3,2q} \\ (2x_1+x_2)^2+(x_1+x_3)^2+(x_2-x_3)^2+(3+q)x_2^2+x_3^2 & \text{for } Z_{1,2q} \\ 3x_1^2+3(x_1+x_2)^2+(1+q)x_2^2 & \text{for } W_{1,2q-1} \\ 5x_1^2+5(x_1+x_2)^2+qx_2^2 & \text{for } S_{1,2q} \end{array} \tag{171}$$

This shows

$$\{a\in Ml(f)_{\Phi_2}\,|\,L(a,a)=L(\gamma_1,\gamma_1)\}=\{\pm\gamma_1\}. \tag{172}$$

for $W_{1,2q-1}$ with $q\neq 2$, for $S_{1,2q}$ with $q\neq 5$, and for all $S^\sharp_{1,2q}$ and $E_{3,2q}$. It shows for $Z_{1,2q}$

$$\{a\in Ml(f)_{\Phi_2}\,|\,L(a,a)=3\} \quad = \quad \{\pm\gamma_3\}, \tag{173}$$
$$\{a\in Ml(f)_{\Phi_2}\,|\,L(a,a)=5\} \quad = \quad \{\pm(\gamma_1-2\gamma_3)\}. \tag{174}$$

All this implies

$$\begin{array}{rcl} \operatorname{Aut}(Ml(f)_{\Phi_2},L) & = & \{\pm\operatorname{id}|_{\mathbb{Z}\gamma_1}\}\times\{\pm\operatorname{id}|_{\mathbb{Z}\gamma_2}\} \qquad \text{for } S^\sharp_{1,2q}, \\ & & \text{for } E_{3,2q},\ \text{for } S_{1,2q} \text{ with } q\neq 5, \\ & & \text{and for } W_{1,2q-1} \text{ with } q\neq 2, \end{array} \tag{175}$$
$$\operatorname{Aut}(Ml(f)_{\Phi_2},L) \quad = \quad \{\pm\operatorname{id}|_{\mathbb{Z}\gamma_1\oplus\mathbb{Z}\gamma_3}\}\times\{\pm\operatorname{id}|_{\mathbb{Z}\gamma_2}\} \text{ for } Z_{1,2q}. \tag{176}$$

Now we consider the cases $S^\sharp_{1,2q-1}$, $E_{3,2q-1}$, $Z_{1,2q-1}$, $W_{1,2q}$ and $S_{1,2q-1}$ with $\Phi_2 \not| b_2$. Here it is

$$Ml(f)_{\Phi_2}=(\widetilde{B_1})_{\Phi_2} \quad \text{and} \quad \operatorname{Aut}(Ml(f)_{\Phi_2},L)=\{\pm\operatorname{id}\}. \tag{177}$$

We define

$$\gamma_4:=\begin{cases} \gamma_1 & \text{for } E_{3,p},W_{1,p},S_{1,p} \\ \gamma_1-3\gamma_3 & \text{for } Z_{1,p}. \end{cases} \tag{178}$$

Then for $E_{3,p}$, $W_{1,p}$ with $p\neq 3$, $S_{1,p}$ with $p\neq 10$ and $Z_{1,p}$ it holds

$$g(\gamma_4)=\pm\gamma_4 \qquad \text{for } g\in G_{\mathbb{Z}}, \tag{179}$$

and for $E_{3,p}$, $W_{1,p}$ (including $p=3$), $S_{1,p}$ (including $p=10$) and $Z_{1,p}$ it holds

$$\widetilde{B_1}\oplus B_2=\{a\in Ml(f)\,|\,L(a,\gamma_4)\equiv 0(2)\}. \tag{180}$$

Here $\subseteq$ (180) follows from $L(B_2,\gamma_4)=0$ and $L(\beta_1,\gamma_4)\equiv 0(2)$ and in the case of $Z_{1,p}$ $L(\beta_3,\gamma_4)=4$. Now $=$ in (180) follows from $L(Ml(f),\gamma_4)=\mathbb{Z}$ and $[Ml(f):\widetilde{B_1}\oplus B_2]=2$. Together (179) and (180) show that any $g\in G_{\mathbb{Z}}$ respects $\widetilde{B_1}\oplus B_2$, so

$$G_{\mathbb{Z}}\subseteq \mathrm{Aut}(\widetilde{B_1}\oplus B_2,L) \tag{181}$$

for $E_{3,p}$, $W_{1,p}$ with $p\neq 3$, $S_{1,p}$ with $p\neq 10$ and $Z_{1,p}$. We claim that (179) and thus (181) hold also for $W_{1,3}$. That will be proved at the very end of this proof as a special case.

Now it remains to show $\mathrm{Aut}(\widetilde{B_1}\oplus B_2,L)\subseteq G_{\mathbb{Z}}$ for the series $E_{3,p}$, $Z_{1,p}$, $W_{1,p}$, $S_{1,p}$. Therefore we will extend the definition of $\widetilde{\gamma_2}$ in such a way to the cases with $\Phi_2\not| b_2$ that $(\widetilde{B_1}\oplus B_2)+\mathbb{Z}\cdot\widetilde{\gamma_2}=Ml(f)$. Furthermore, we will show that $g(\widetilde{\gamma_2})\in Ml(f)$ for any $g\in\mathrm{Aut}(\widetilde{B_1}\oplus B_2,L)$. This then implies $\mathrm{Aut}(\widetilde{B_1}\oplus B_2,L)\subseteq G_{\mathbb{Z}}$. But the proof of $g(\widetilde{\gamma_2})\in Ml(f)$ requires a better control of $\mathrm{Aut}(\widetilde{B_1}\oplus B_2,L)$.

Consider again all eight series and define

$$b_4:=\frac{\gcd(b_1,b_2)}{\gcd(b_1,b_2,\Phi_m)}=\gcd(\frac{b_1}{\Phi_m},b_2)\in\mathbb{Z}[t]. \tag{182}$$

Then it is

$$b_4=\begin{cases}1 & \text{for } W^\sharp_{1,p}, S^\sharp_{1,2q-1}, U_{1,p}, E_{3,2q-1}, Z_{1,2q-1},\\ & Q_{2,p} \text{ with } p\not\equiv 0(4), W_{1,2q}, S_{1,2q-1},\\ \Phi_2 & \text{for } S^\sharp_{1,2q}, E_{3,2q}, Z_{1,2q}, W_{1,2q-1} \text{ with } q\not\equiv 2(3), S_{1,2q},\\ \Phi_4 & \text{for } Q_{2,4s},\\ \Phi_6\Phi_2 & \text{for } W_{1,6s-3}.\end{cases} \tag{183}$$

We claim that in all cases except $S_{1,10}$, any $g\in G_{\mathbb{Z}}\cup\mathrm{Aut}(\widetilde{B_1}\oplus B_2,L)$ maps $(\widetilde{B_1})_{b_4}$ to $(\widetilde{B_1})_{b_4}$ and $(B_2)_{b_4}$ to $(B_2)_{b_4}$. In the cases with $b_4=1$ there is nothing to show as then $(\widetilde{B_1})_{b_4}=\{0\}=(B_2)_{b_4}$. In the cases $Q_{2,p}$ with $p\equiv 0(4)$ and $W_{1,6s-3}$, this will be shown at the very end of the proof as some special case.

In all other cases $b_4=\Phi_2$ and $(B_2)_{b_4}=\mathbb{Z}\cdot\gamma_2$ and

$$(\widetilde{B_1})_{b_4}=\begin{cases}\mathbb{Z}\cdot(\gamma_1-2\gamma_3)\oplus\mathbb{Z}\cdot\gamma_3 & \text{for } Z_{1,2q},\\ \mathbb{Z}\cdot\gamma_1 & \text{else.}\end{cases} \tag{184}$$

Since $(\widetilde{B_1}\oplus B_2)_{\Phi_2}\subseteq Ml(f)_{\Phi_2}$, the equations (172), (173) and (174) hold also with $(\widetilde{B_1}\oplus B_2)_{\Phi_2}$ instead of $Ml(f)_{\Phi_2}$. They characterize $(\widetilde{B_1})_{\Phi_2}$ within $Ml(f)_{\Phi_2}$ and within $(\widetilde{B_1}\oplus B_2)_{\Phi_2}$. Thus any $g\in G_{\mathbb{Z}}\cup\mathrm{Aut}(\widetilde{B_1}\oplus B_2,L)$ maps $(\widetilde{B_1})_{\Phi_2}$ to itself, and then it maps also the L-orthogonal sublattice $(B_2)_{\Phi_2}$ to itself.

For all eight series except $S_{1,10}$, this implies the following. For any automorphism $g \in G_{\mathbb{Z}} \cup \mathrm{Aut}(\widetilde{B_1} \oplus B_2, L)$ it is

$$g : \widetilde{B_1} \to \widetilde{B_1} \text{ and } B_2 \to B_2 \qquad \text{if } m \not| p, \tag{185}$$

$$\left.\begin{array}{l} g : (\widetilde{B_1})_{b_1/\Phi_m} \to (\widetilde{B_1})_{b_1/\Phi_m} \\ g : (B_2)_{b_2/\Phi_m} \to (B_2)_{b_2/\Phi_m} \end{array}\right\} \quad \begin{array}{c} \text{if } m|p \text{ and the} \\ \text{type is not } S_{1,10}. \end{array} \tag{186}$$

Now we want to apply Lemma A.2.3 to the Orlik blocks at stake. One checks easily that all assumptions of the lemma are satisfied. Therefore

$$\begin{aligned} & \mathrm{Aut}(\widetilde{B_1} \oplus B_2, L) \qquad (187) \\ = \;& \{\pm M_h^k|_{\widetilde{B_1}} \,|\, k \in \mathbb{Z}\} \times \{\pm M_h^k|_{B_2} \,|\, k \in \mathbb{Z}\} \qquad \text{if } m \not| p, \end{aligned}$$

and if $m|\,p$ and the type is not $S_{1,10}$, then $\mathrm{Aut}(\widetilde{B_1} \oplus B_2, L)$ projects to a subgroup of

$$\begin{aligned} & \mathrm{Aut}((\widetilde{B_1})_{b_1/\Phi_m}, L) \times \mathrm{Aut}((B_2)_{b_2/\Phi_m}, L) \qquad (188) \\ = \;& \{\pm M_h^k|_{(\widetilde{B_1})_{b_1/\Phi_m}} \,|\, k \in \mathbb{Z}\} \times \{\pm M_h^k|_{(B_2)_{b_2/\Phi_m}} \,|\, k \in \mathbb{Z}\}. \end{aligned}$$

The group $\mathrm{Aut}(\widetilde{B_1} \oplus B_2, L)$ for $m \not| p$ is generated by the automorphisms M_h, $-\mathrm{id}$, $M_h|_{\widetilde{B_1}} \times \mathrm{id}\,|_{B_2}$ and $(-\mathrm{id}\,|_{\widetilde{B_1}}) \times \mathrm{id}\,|_{B_2}$, and analogously for the group in (188) if $m|\,p$.

Now we extend the definition of γ_2. For $E_{3,2q-1}$, $Z_{1,2q-1}$ and $S_{1,2q-1}$ define it as follows:

$$\begin{aligned} \gamma_2 &:= \delta_2 - \delta_6 + \delta_7 + \delta_9 \quad \text{for } E_{3,2q-1}, \qquad (189) \\ \gamma_2 &:= \delta_2 - \delta_6 + \delta_7 + \delta_{10} \quad \text{for } Z_{1,2q-1}, \\ \gamma_2 &:= 2(-\delta_1 - \delta_2 + \sum_{j\in\{4,5,6\}} \delta_j) - \delta_7 - \delta_8 + \sum_{j\in\{9,11,12,14\}} \delta_j \quad \text{for } S_{1,2q-1}. \end{aligned}$$

The equations (144), (149) and (159) show that $\gamma_2 \in B_2$. For $W_{1,2q}$ (so $p = 2q$) define

$$\begin{aligned} \gamma_2 &:= (t^p(t+1)\Phi_{12} + \sum_{j=0}^{p-1} t^j)(M_h)(\delta_{16}) \qquad (190) \\ &= (t^p(1+t-t^2-t^3+t^4+t^5) + \sum_{j=0}^{p-1} t^j)(M_h)(\delta_{16}) \\ &= -2\delta_2 + 2\delta_6 - 2\delta_7 + \delta_4 + \delta_5 + \delta_9 - \delta_{11} + \delta_{13} - \delta_{15}. \end{aligned}$$

We observe that in the case $12|p$, Φ_{12} divides $\sum_{j=0}^{p-1} t^j$ so that then $\gamma_2 \in \Phi_{12}(M_h)(B_2) = (B_2)_{b_2/\Phi_{12}}$. In all four cases it holds $\frac{1}{2}(\gamma_4 + \gamma_2) \in Ml(f)$.

Now for the series $E_{3,p}$, $Z_{1,p}$, $W_{1,p}$ and $S_{1,p}$ it is

$$\gamma_4 \in (B_1)_{\Phi_2}, \qquad \begin{cases} \gamma_2 \in B_2 & \text{if } m \not| p, \\ \gamma_2 \in (B_2)_{b_2/\Phi_m} & \text{if } m | p, \end{cases} \tag{191}$$

$$\widetilde{\gamma_2} := \frac{1}{2}(\gamma_4 + \gamma_2) \overset{!}{\in} Ml(f), \tag{192}$$

$$Ml(f) = (\widetilde{B_1} \oplus B_2) + \mathbb{Z}\widetilde{\gamma_2}, \tag{193}$$

$$\begin{aligned}(M_h|_{\widetilde{B_1}} \times \mathrm{id}\,|_{B_2})(\widetilde{\gamma_2}) &= ((-\mathrm{id}\,|_{\widetilde{B_1}}) \times \mathrm{id}\,|_{B_2})(\widetilde{\gamma_2}) \\ &= \frac{1}{2}(-\gamma_4 + \gamma_2) = -\gamma_4 + \widetilde{\gamma_2} \in Ml(f).\end{aligned} \tag{194}$$

Therefore any $g \in \mathrm{Aut}(\widetilde{B_1} \oplus B_2, L)$ maps $\widetilde{\gamma_2}$ to an element of $Ml(f)$. Thus it maps $Ml(f)$ to $Ml(f)$, thus $g \in G_{\mathbb{Z}}$. This finishes the proof of (119) and of part (a) for all series except $Q_{2,p}$ and $W_{1,6s-3}$ and $S_{1,10}$. All those cases are treated at the very end of the proof as special cases.

(b) This part follows immediately from (119) and (187). Note that at the very end of the proof of part (c) we will show (119) and (187) also for the series $Q_{2,p}$ and $W_{1,6s-3}$.

(c) In this part we consider the eight subseries with $m|p$. We write $p = m \cdot r$ with $r \in \mathbb{Z}_{\geq 1}$. Recall that for $\zeta = e^{2\pi i/m}$ the ring $\mathbb{Z}[\zeta]$ is a principal ideal domain (cf. [He93, Kapitel 3]). Here ξ denotes any primitive m-th unit root.

We can apply formula (483) from Lemma A.2.5 (b) with $\Lambda = Ml(f)$, $\Lambda^{(1)} = \widetilde{B}_1 \oplus B_2$ and $p = \Phi_m$. This gives

$$Ml(f)_{\Phi_m} = (\widetilde{B}_1 \oplus B_2)_{\Phi_m} = (B_1 \oplus B_2)_{\Phi_m} = (B_1)_{\Phi_m} \oplus (B_2)_{\Phi_m}. \tag{195}$$

Therefore the space

$$Ml(f)_{\xi,\mathbb{Z}[\zeta]} \quad := \quad Ml(f)_\xi \cap Ml(f)_{\mathbb{Z}[\zeta]} \tag{196}$$

is a free $\mathbb{Z}[\zeta]$-module of rank 2 with basis $v_{1,\xi}, v_{2,\xi}$ with

$$v_{j,\xi} := v(\beta_j, \xi) = \frac{b_j}{t - \xi}(M_h)(\beta_j) \qquad \text{for } j = 1, 2 \tag{197}$$

(see (477) for the notion $v(\beta_j, \xi)$). It holds $v_{j,\overline{\xi}} = \overline{v_{j,\xi}}$.

Now we start the proof. It will comprise four steps. Step 1 calculates the values of the Hermitian form h_ξ according to Lemma A.2.6 on a suitable $\mathbb{Z}[\zeta]$-basis of $Ml(f)_{\xi,\mathbb{Z}[\zeta]}$. Step 2 analyzes what this implies for automorphisms of the pair $(Ml(f)_{\xi,\mathbb{Z}[\zeta]}, L)$ and thus gives a first approximation of $\Psi(G_{\mathbb{Z}})$. Step 3 uses (120) for $S_{1,10}$ and (186) for all other singularities and the Orlik block structure of the blocks B_j to control the action of $g \in G_{\mathbb{Z}}$ on all eigenspaces simultaneously. It will prove (124). Step 4 combines the Steps 2 and 3 with results from Section A.3 and shows that $\Psi(G_{\mathbb{Z}})$ is some infinite Fuchsian group.

Step 1: The form

$$h_\xi : Ml(f)_\xi \times Ml(f)_\xi \to \mathbb{C}, \quad (a,b) \mapsto \sqrt{-\xi} \cdot L(a, \overline{b})$$

is Hermitian according to Lemma A.2.6. In this step it will be calculated with respect to the $\mathbb{Z}[\zeta]$-basis $v_{1,\xi}, v_{2,\xi}$ of $Ml(f)_{\xi,\mathbb{Z}[\zeta]}$. For $i \neq j$

$$h_\xi(v_{i,\xi}, v_{j,\xi}) = \sqrt{-\xi} \cdot L(v_{i,\xi}, v_{j,\overline{\xi}}) = 0 \tag{198}$$

because of (118). $L(v_{j,\xi}, v_{j,\overline{\xi}})$ will be calculated with (478),

$$L(v_{j,\xi}, v_{j,\overline{\xi}}) = \frac{b_j}{t-\overline{\xi}}(\overline{\xi}) \cdot L(\frac{b_j}{t-\xi}(M_h)(\beta_j), \beta_j), \tag{199}$$

first for $j=2$, then for $j=1$.

We calculate for all eight subseries:

k	0	1	2	$\cdots$	$\deg b_2 - 1$	$\deg b_2$
$L(M_h^k(\beta_2), \beta_2)$	1	-1	0	$\cdots$	0	0 if $r_I = 1$, -1 if $r_I \geq 2$

For the three subseries with $r_I = 1$, i.e. $W^\sharp_{1,12r}$, $S^\sharp_{1,10r}$, $U_{1,9r}$, it is

$$\frac{b_2}{t-\xi} = \frac{t^{m+p}-1}{(t-\xi)\cdot\Phi_1} = \Phi_1^{-1} \cdot \sum_{j=0}^{m+p-1} \xi^{m+p-1-j} \cdot t^j, \tag{200}$$

$$\frac{b_2}{t-\overline{\xi}}(\overline{\xi}) = (\overline{\xi}-1)^{-1} \cdot (m+p) \cdot \xi = m(1+r)(\overline{\xi}-1)^{-1} \cdot \xi, \tag{201}$$

$$L(\frac{b_2}{t-\xi}(M_h)(\beta_2), \beta_2) = (\xi-1)^{-1} \cdot \overline{\xi} \cdot (1-\overline{\xi}) = \overline{\xi}^2, \tag{202}$$

$$h_\xi(v_{2,\xi}, v_{2,\xi}) = m(1+r) \cdot (1-\xi)^{-1} \cdot \sqrt{-\xi} > 0. \tag{203}$$

For the five subseries with $r_I = 2$ it is

$$\frac{b_2}{t-\xi} = \frac{t^{m/2+p}+1}{t-\xi} = \sum_{j=0}^{m/2+p-1} \xi^{m/2+p-1-j} \cdot t^j, \tag{204}$$

$$\frac{b_2}{t-\overline{\xi}}(\overline{\xi}) = (\frac{m}{2}+p)(-\xi) = \frac{m}{2}(1+2r)(-\xi), \tag{205}$$

$$L(\frac{b_2}{t-\xi}(M_h)(\beta_2), \beta_2) = -\overline{\xi}(1-\overline{\xi}), \tag{206}$$

$$h_\xi(v_{2,\xi}, v_{2,\xi}) = \frac{m}{2}(1+2r) \cdot (1-\overline{\xi}) \cdot \sqrt{-\xi} > 0. \tag{207}$$

Now we turn to $h_\xi(v_{1,\xi}, v_{1,\xi})$. We compute for all eight series

k	0	1	2	3	4	5	6	7	8	9	10	11
$L(M_h^k(\beta_1),\beta_1)$ for $W_{1,p}^\sharp$	1	-1	1	0	0	1						
$L(M_h^k(\beta_1),\beta_1)$ for $S_{1,p}^\sharp$	1	-1	0	1	0							
$L(M_h^k(\beta_1),\beta_1)$ for $U_{1,p}$	1	-1	0	0	1	0	-1	0	0			
$L(M_h^k(\beta_1),\beta_1)$ for $E_{3,p}$	1	-1	1	0	1	0	1	0	1			
$L(M_h^k(\beta_1),\beta_1)$ for $Z_{1,p}$	1	-1	0	0	1	0	0					
$L(M_h^k(\beta_1),\beta_1)$ for $Q_{2,p}$	1	-1	0	1	0	0	0	0	0	0	-1	0
$L(M_h^k(\beta_1),\beta_1)$ for $W_{1,p}$	3	-3	2	-1	0	1	-1	1	-1	0	1	-2
$L(M_h^k(\beta_1),\beta_1)$ for $S_{1,p}$	2	-2	0	1	0	-1	1	0	-1	0		

and

$$\text{for } W_{1,p}^\sharp \qquad \frac{b_1}{t-\xi} = \frac{\Phi_{12}}{t-\xi} = t^3+\xi t^2+(\xi^2-1)t+(\xi^3-\xi),$$

$$\text{for } S_{1,p}^\sharp \qquad \frac{b_1}{t-\xi} = \frac{\Phi_{10}\Phi_2}{t-\xi} = \frac{t^5+1}{t-\xi} = t^4+\xi t^3+\xi^2t^2+\xi^3 t+\xi^4,$$

$$\begin{aligned}\text{for } U_{1,p} \qquad & \frac{b_1}{t-\xi} = \frac{\Phi_9}{t-\xi} = \frac{t^6+t^3+1}{t-\xi} \\ & = t^5+\xi t^4+\xi^2t^3+(\xi^3+1)t^2+(\xi^4+\xi)t+(\xi^5+\xi^2),\end{aligned}$$

$$\begin{aligned}\text{for } E_{3,p} \qquad & \frac{b_1}{t-\xi} = \frac{\Phi_{18}\Phi_2}{t-\xi} = \frac{t^7+t^6-t^4-t^3+t+1}{t-\xi} = t^6+(\xi+1)t^5 \\ & +(\xi^2+\xi)t^4+(\xi^6+\xi^2)t^3+(\xi^7+\xi^6)t^2+(\xi^8+\xi^7)t+\xi^8,\end{aligned}$$

$$\text{for } Z_{1,p} \qquad \frac{b_1}{t-\xi} = \frac{t^7+1}{t-\xi} = t^6+\xi t^5+\xi^2t^4+\xi^3t^3+\xi^4t^2+\xi^5t+\xi^6,$$

$$\begin{aligned}\text{for } Q_{2,p} \qquad & \frac{b_1}{t-\xi} = \frac{\Phi_{12}\Phi_4\Phi_3}{t-\xi} = \frac{t^8+t^7+t^6+t^2+t+1}{t-\xi} \\ & = t^7+(\xi+1)t^6+(\xi^2+\xi+1)t^5+(\xi^3+\xi^2+\xi)t^4 \\ & +(\xi^4+\xi^3+\xi^2)t^3+(\xi^5+\xi^4+\xi^3)t^2+(\xi^5+\xi^4)t+\xi^5,\end{aligned}$$

$$\begin{aligned}\text{for } W_{1,p} \qquad & \frac{b_1}{t-\xi} = \frac{\Phi_{12}\Phi_6\Phi_3\Phi_2}{t-\xi} = \frac{t^9+t^8+t^5+t^4+t+1}{t-\xi} \\ & = t^8+(\xi+1)t^7+(\xi^2+\xi)t^6+(\xi^3+\xi^2)t^5+(\xi^3+\xi^2)t^4 \\ & +(\xi^3+\xi^2)t^3+(\xi^4+\xi^3)t^2+(\xi^5+\xi^4)t+\xi^5,\end{aligned}$$

$$\begin{aligned}\text{for } S_{1,p} \qquad & \frac{b_1}{t-\xi} = \frac{\Phi_{10}\Phi_5\Phi_2}{t-\xi} = \frac{\sum_{j=0}^9 t^j}{t-\xi} \\ & = t^8+(\xi+1)t^7+(\xi^2+\xi+1)t^6+(\xi^3+\xi^2+\xi+1)t^5 \\ & +(\xi^4+\xi^3+\xi^2+\xi+1)t^4+(\xi^4+\xi^3+\xi^2+\xi)t^3 \\ & +(\xi^4+\xi^3+\xi^2)t^2+(\xi^4+\xi^3)t+\xi^4.\end{aligned}$$

This table and this list give the following values.

	$\frac{b_1}{t-\xi}(\overline{\xi})$	$L(\frac{b_1}{t-\xi}(M_h)(\beta_1), \beta_1)$
$W^\sharp_{1,p}$	$4\overline{\xi}^3 - 2\overline{\xi} = -2(\xi+\overline{\xi})\xi^2$	$\xi^3(1-\xi)$
$S^\sharp_{1,p}$	$5\overline{\xi}^4 = -5\xi$	$-\xi(\xi^2+\overline{\xi}^2-1)$
$U_{1,p}$	$6\overline{\xi}^5 + 3\overline{\xi}^2 = 3\xi(\xi^3-1)$	$-\xi^6(\xi^2+\overline{\xi}^2)$
$E_{3,p}$	$3(\overline{\xi}^6+\overline{\xi}^5+\overline{\xi}^9+\overline{\xi}^8) = -3(\xi+1)(\xi^3+1)$	$\xi^2(\xi+\overline{\xi})(\xi^2+\overline{\xi}^2)$
$Z_{1,p}$	$7\overline{\xi}^6 = -7\xi$	$\xi^2(\xi^4+\overline{\xi}^4+1)$
$Q_{2,p}$	$6(\overline{\xi}^7+\overline{\xi}^6+\overline{\xi}^5) = -6(\xi+\overline{\xi}+1)$	$\xi^2(\xi+1) = (1-\xi)^{-1}$
$W_{1,p}$	$4(\overline{\xi}^8+\overline{\xi}^7+\overline{\xi}^6+\overline{\xi}^5) = 4\overline{\xi}^7(1+\xi)(\xi+\overline{xi})$	$\xi^3(\xi-1)(\overline{\xi}-1)$
$S_{1,p}$	$5(\overline{\xi}^8+\overline{\xi}^7+\overline{\xi}^6+\overline{\xi}^5+\overline{\xi}^4)$	$-1+\xi+\xi^2-2\xi^3+\xi^4$

According to $h_\xi(v_{1,\xi}, v_{1,\xi}) = \sqrt{-\xi}\cdot L(v_{1,\xi}, v_{1,\overline{\xi}})$ and (199) we obtain the following values (see (208)). Note that in order to compute the values in (208) by hand it may be helpful to have some basic information on the rings $\mathbb{Z}[\zeta]$ available. For example such information can be found in the proof of [He93, Lemma 3.8] for $\mathbb{Z}[\zeta] = \mathbb{Z}[e^{\frac{2\pi i}{18}}]$ and in [GH18, Lemma 2.11] for all cases at stake.

	$h_\xi(v_{1,\xi}, v_{1,\xi})$
$W^\sharp_{1,p}$	$(-2)(\xi+\overline{\xi})\cdot(1-\overline{\xi})\sqrt{-\xi}$
$S^\sharp_{1,p}$	$5(\xi^2+\overline{\xi}^2)(\xi^2+\overline{\xi}^2-1)\cdot(1-\xi)^{-1}\sqrt{-\xi}$
$U_{1,p}$	$3(\xi^4+\overline{\xi}^4+1)\cdot(1-\overline{\xi})\sqrt{-\xi}$
$E_{3,p}$	$(-3)(1+\xi)(1+\overline{\xi})(\xi+\overline{\xi}-1)\cdot(1-\xi)^{-1}\sqrt{-\xi}$
$Z_{1,p}$	$(-7)(\xi^2+\overline{\xi}^2)\cdot(1-\overline{\xi})\sqrt{-\xi}$
$Q_{2,p}$	$(-6)(\xi+\overline{\xi}+1)\cdot(1-\xi)^{-1}\sqrt{-\xi}$
$W_{1,p}$	$(-4)(\xi+\overline{\xi})\cdot(1-\overline{\xi})\sqrt{-\xi}$
$S_{1,p}$	$(-10)(\xi^2+\overline{\xi}^2)\cdot(1-\overline{\xi})\sqrt{-\xi}$

(208)

Here we observe that as in (203) and (207) it holds $(1-\overline{\xi})\sqrt{-\xi} > 0$ and $(1-\xi)^{-1}\sqrt{-\xi} > 0$. In each of the eight cases we find

$$h_\xi(v_{1,\xi}, v_{1,\xi}) > 0 \quad \text{for } \xi \notin \{\zeta, \overline{\zeta}\}, \tag{209}$$

$$h_\xi(v_{1,\xi}, v_{1,\xi}) < 0 \quad \text{for } \xi \in \{\zeta, \overline{\zeta}\}, \tag{210}$$

and

$$L(v_{1,\xi}, \beta_1) = L(\frac{b_1}{t-\xi}(M_h)(\beta_1), \beta_1) \in \mathbb{Z}[\zeta]^*. \tag{211}$$

Step 2: We define for each of the eight series

$$b_5 := \frac{b_1}{\Phi_m} \in \mathbb{Z}[t] \qquad \text{unitary.} \tag{212}$$

Then we obtain

series	$W^\sharp_{1,p}$	$S^\sharp_{1,p}$	$U_{1,p}$	$E_{3,p}$	$Z_{1,p}$	$Q_{2,p}$	$W_{1,p}$	$S_{1,p}$
b_5	1	Φ_2	1	Φ_2	Φ_2	$\Phi_4\Phi_3$	$\Phi_6\Phi_3\Phi_2$	$\Phi_5\Phi_2$

and

$$b_5(\xi)/b_5(\overline{\xi}) \in \{\pm\xi^k \mid k \in \mathbb{Z}\}. \tag{213}$$

Define for each of the eight subseries with $m|p$

$$b_6 := \frac{b_2}{\Phi_m} \in \mathbb{Z}[t] \qquad \text{unitary} \tag{214}$$

and

$$\begin{aligned} w(\xi) &:= -\frac{h_\xi(v_{2,\xi}, v_{2,\xi})}{h_\xi(v_{1,\xi}, v_{1,\xi})} = -\frac{\frac{b_2}{t-\overline{\xi}}(\overline{\xi}) \cdot L(v_{2,\xi}, \beta_2)}{\frac{b_1}{t-\overline{\xi}}(\overline{\xi}) \cdot L(v_{1,\xi}, \beta_1)} \\ &= -\frac{b_6}{b_5}(\overline{\xi}) \cdot \frac{L(v_{2,\xi}, \beta_2)}{L(v_{1,\xi}, \beta_1)}. \end{aligned} \tag{215}$$

Then

$$b_5(\overline{\xi})w(\xi) = b_6(\overline{\xi}) \cdot \frac{L(v_{2,\xi}, \beta_2)}{L(v_{1,\xi}, \beta_1)} \in \mathbb{Z}[\zeta]. \tag{216}$$

It is in $\mathbb{Z}[\zeta]$ because of (211). The $w(\xi)$ are as follows.

	$w(\xi)$
$W^\sharp_{1,p}$	$(1+r)(+6)[(1-\xi)(1-\overline{\xi})(\xi+\overline{\xi})]^{-1}$
$S^\sharp_{1,p}$	$(1+r)(-2)[(\xi^2+\overline{\xi}^2)(\xi^2+\overline{\xi}^2-1)]^{-1}$
$U_{1,p}$	$(1+r)(-3)[(1-\xi)(1-\overline{\xi})(\xi^4+\overline{\xi}^4+1)]^{-1}$
$E_{3,p}$	$(1+2r)(+3)(1-\xi)(1-\overline{\xi})[(1+\xi)(1+\overline{\xi})(\xi+\overline{\xi}-1)]^{-1}$
$Z_{1,p}$	$(1+2r)(+1)[\xi^2+\overline{\xi}^2]^{-1}$
$Q_{2,p}$	$(1+2r)(+1)(1-\xi)(1-\overline{\xi})[\xi+\overline{\xi}+1]^{-1}$
$W_{1,p}$	$(1+2r)(+\frac{3}{2})[\xi+\overline{\xi}]^{-1}$
$S_{1,p}$	$(1+2r)(+\frac{1}{2})[\xi^2+\overline{\xi}^2]^{-1}$

(217)

The inequalities (203), (207), (209) and (210) give

$$w(\xi) \begin{cases} < 0 & \text{for } \xi \notin \{\zeta, \overline{\zeta}\}, \\ > 0 & \text{for } \xi \in \{\zeta, \overline{\zeta}\}. \end{cases} \tag{218}$$

Using the basis $v_{1,\xi}$,$v_{2,\xi}$ of $Ml(f)_{\xi,\mathbb{Z}[\zeta]}$, the automorphism group $\mathrm{Aut}(Ml((f)_{\xi,\mathbb{Z}[\zeta]}, h_\xi)$ can be identified via the isomorphism $A(\xi) \mapsto g$ given by

$$g(v_{1,\xi}, v_{2,\xi}) \quad = \quad (v_{1,\xi}, v_{2,\xi}) \cdot A(\xi). \tag{219}$$

with the matrix group

$$\{A(\xi) \in \mathrm{GL}(2; \mathbb{Z}[\zeta]) \mid \begin{pmatrix} -1 & 0 \\ 0 & w(\xi) \end{pmatrix} = A(\xi)^T \cdot \begin{pmatrix} -1 & 0 \\ 0 & w(\xi) \end{pmatrix} \cdot \overline{A(\xi)}\}. \tag{220}$$

The inequalities (218) and Theorem A.3.4 say that the matrix group in the case of $\xi = \zeta$ projects to an infinite Fuchsian group. Furthermore, Theorem A.3.4 tells us that the elements of the matrix group for any ξ can be represented by triples $(a(\xi), c(\xi), \delta(\xi)) \in \mathbb{Z}[\zeta]^2 \times \{\pm\zeta^k \mid k \in \mathbb{Z}\}$ with

$$a(\xi)a(\overline{\xi}) - 1 = w(\xi) \cdot c(\xi)c(\overline{\xi}), \tag{221}$$

where

$$A(\xi) = \begin{pmatrix} a(\xi) & w(\xi) \cdot c(\overline{\xi}) \cdot \delta(\xi) \\ c(\xi) & a(\overline{\xi}) \cdot \delta(\xi) \end{pmatrix}. \tag{222}$$

This gives a first approximation of $\Psi(G_{\mathbb{Z}})$. It took into account only the eigenspace $Ml(f)_{\xi,\mathbb{Z}[\zeta]}$ and the pairing h_ξ which L and complex conjugation induce on it.

Step 3: Now (124) will be shown. Therefore, we will employ Lemma A.2.3 and (120) for $S_{1,10}$ and (188) for all other singularities.

We start and show that the assumptions of Lemma A.2.3 are fulfilled. Let $g \in \ker \Psi \subseteq G_{\mathbb{Z}}$, i.e. $g|_{Ml(f)_\zeta} \in \mathbb{C}^* \cdot \mathrm{id}$. Then $g|_{Ml(f)_\xi} \in \mathbb{C}^* \cdot \mathrm{id}$ for all ξ with $\Phi_m(\xi) = 0$. Thus we have

$$g((B_j)_{\Phi_m}) = (B_j)_{\Phi_m} \qquad \text{for } j = 1, 2. \tag{223}$$

For $S_{1,10}$ the equalities $g(B_j) = B_j$ with $j = 1, 2$ follow from (120). For all other singularities $g(B_j) = B_j$ with $j = 1, 2$ follows with (188) (and (177) for B_3 in the case $Z_{1,14r}$).

Now we are ready to apply Lemma A.2.3 to the Orlik blocks B_1 and B_2. One checks easily that all hypotheses are satisfied. In the case $Z_{1,14r}$ B_3 is glued to B_1 by (177). Therefore in all cases we obtain

$$g = (\varepsilon_1 \cdot M_h^{k_1})|_{B_1} \times (\varepsilon_2 \cdot M_h^{k_2})|_{B_2} \tag{224}$$

for some $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ and $k_1, k_2 \in \mathbb{Z}$. Now we consider

$$\widetilde{g} \quad := \quad \varepsilon_2 \cdot M_h^{-k_2} \circ g. \tag{225}$$

It satisfies

$$\begin{aligned} \widetilde{g}|_{B_1} = \varepsilon_1\varepsilon_2 \cdot M_h^{k_1-k_2}|_{B_1}, \quad \widetilde{g}|_{B_2} = \mathrm{id}, \quad \widetilde{g}|_{Ml(f)_\xi} \in \mathbb{C}^* \cdot \mathrm{id}, \\ \text{thus} \quad \widetilde{g}|_{Ml(f)_\xi} = \mathrm{id}, \quad \widetilde{g}|_{Ml(f)_{\Phi_m}} = \mathrm{id}\,. \end{aligned} \tag{226}$$

Comparing this with Table 7 shows

$$\begin{aligned} \widetilde{g} &= \mathrm{id} \text{ or } \widetilde{g} = -M_h^{\frac{m}{2}(1+2r)} \quad \text{for the series } Q_{2,p},\, W_{1,p},\, S_{1,p}, \\ \widetilde{g} &= \mathrm{id} \quad \text{for the other series in Table 7.} \end{aligned}$$

In any case, $\widetilde{g}$ and g are in $\{\pm M_h^k \,|\, k \in \mathbb{Z}\}$ and, thus, it is $\ker \Psi = \{\pm M_h^k \,|\, k \in \mathbb{Z}\}$.

Step 4: By Step 2, the image $\Psi(G_\mathbb{Z})$ is a subgroup of an infinite Fuchsian group and therefore itself a Fuchsian group. It rests to show that it is an infinite group. By Step 3, the kernel of $\Psi : G_\mathbb{Z} \to \Psi(G_\mathbb{Z})$ is $\{\pm M_h^k \,|\, k \in \mathbb{Z}\}$, so it is finite. Therefore it rests to show that $G_\mathbb{Z}$ is infinite. We will see that the subgroup of elements $g \in G_\mathbb{Z}$ with

$$\begin{aligned} g &= \mathrm{id} \text{ on any eigenspace } Ml(f)_\lambda \text{ with } \Phi_m(\lambda) \neq 0, \\ \text{i.e. } g &= \mathrm{id} \text{ on } (\widetilde{B_1})_{b_5} \text{ and on } (B_2)_{b_6}. \end{aligned} \tag{227}$$

is infinite.

Consider an element $g \in G_\mathbb{Z}$ with (227). For all singularities except $S_{1,10}$ (119) holds. For $S_{1,10}$ (227) implies $g(\gamma_4) = \pm\gamma_4$, and then (181) gives $g \in \mathrm{Aut}(B_1 \oplus B_2, L)$. In the case of the series $Z_{1,14r}$, the element g maps $B_1 \oplus B_2$ to itself because $(B_1 \oplus B_2)_\mathbb{C}$ contains $\ker \Phi_m(M_h)$. In any case, Lemma A.2.4 applies with $k = 2$, $\Lambda^{(1)} = B_1$, $\Lambda^{(2)} = B_2$, $e^{(1)} = \beta_1$, $e^{(2)} = \beta_2$ and $p_0 = \Phi_m$. By (481) there are unique polynomials $p_{ij} \in \mathbb{Z}[t]_{<\deg b_i}$ for $i = 1, 2$ with

$$g(\beta_j) = p_{1j}(M_h)(\beta_1) + p_{2j}(M_h)(\beta_2) \tag{228}$$

and

$$\begin{aligned} p_{11} &= 1 + b_5 \cdot q_{11}, \quad p_{12} = b_5 \cdot q_{12}, \\ p_{21} &= b_6 \cdot q_{21}, \qquad\; p_{22} = 1 + b_6 \cdot q_{22} \end{aligned} \tag{229}$$

for suitable polynomials $q_{ij} \in \mathbb{Z}[t]_{<\varphi(m)}$.

We know that g restricts to an automorphism of the pair $((B_1 \oplus B_2)_{\Phi_m}, L)$. By (482), the matrix $A(\xi)$ from (219) in Step 2 takes the form

$$A(\xi) = \begin{pmatrix} 1 + b_5(\xi)q_{11}(\xi) & b_6(\xi)q_{12}(\xi) \\ b_5(\xi)q_{21}(\xi) & 1 + b_6(\xi)q_{22}(\xi) \end{pmatrix}. \tag{230}$$

By Step 2, this matrix $A(\xi)$ satisfies (221) and (222).

Vice versa, all polynomials $q_{ij} \in \mathbb{Z}[t]_{<\varphi(m)}$ for $i = 1, 2$ such that the matrix in (230) satisfies (221) and (222), give rise to an element $g \in G_\mathbb{Z}$ with (227) via (229) and (228).

So it remains to show that there are infinitely many polynomials $q_{ij} \in \mathbb{Z}[t]_{<\varphi(m)}$ such that the matrix in (230) satisfies (221) and (222) and that $q_{12}(\xi) \neq 0$ and $q_{21}(\xi) \neq 0$. We start by defining

$$w_0(\xi) := w(\xi)b_5(\xi)b_5(\overline{\xi}) \in \mathbb{Z}[\zeta] \cap \mathbb{R} \tag{231}$$

and asking for infinitely many solutions $a(\xi), f(\xi) \in \mathbb{Z}[\zeta] \cap \mathbb{R}$ of the Pell equation

$$a(\xi)^2 - 1 = w_0(\xi) \cdot f(\xi)^2 \tag{232}$$

with the additional condition

$$w_0(\xi) \quad | \quad a(\xi) - 1. \tag{233}$$

Such solutions must exist according to Lemma A.3.5. They give rise to the elements

$$q_{11}(\xi) := \frac{a(\xi) - 1}{b_5(\xi)}, \qquad q_{12}(\xi) := f(\xi) \cdot \frac{w(\xi)b_5(\overline{\xi})}{b_6(\xi)}, \tag{234}$$

$$q_{21}(\xi) := f(\xi), \qquad q_{22}(\xi) := \frac{a(\xi) - 1}{b_6(\xi)}. \tag{235}$$

In the situation above it is

$$b_6(\xi) \,|\, w(\xi)b_5(\overline{\xi}) \,|\, w_0(\xi) \,|\, a(\xi) - 1,$$

(cf. (211), (213) and (216)). These elements come from unique polynomials $q_{ij} \in \mathbb{Z}[t]_{<\varphi(m)}$, whereas the polynomials satisfy all the desired properties. So there are infinitely many polynomials and, hence, $G_\mathbb{Z}$ contains infinitely many elements. This finishes the proof of the theorem except for the parts, which we explicitly omitted. They will be proved now.

First of all, the proof of (119) for $Q_{2,p}$ was postponed and has to be given now. Recall the definition (182) of b_4 and recall that $b_4 = \Phi_4$ for $Q_{2,4s}$ and $b_4 = 1$ for the other $Q_{2,p}$. We proceed by proving the following three claims:

(i) In the case $Q_{2,4s}$ and for any $g \in G_\mathbb{Z} \cup \mathrm{Aut}(B_1 \oplus B_2, L)$

$$g : (B_1)_{b_4} \to (B_1)_{b_4} \text{ and } (B_2)_{b_4} \to (B_2)_{b_4}. \tag{236}$$

(ii) In the case $Q_{2,p}$ there is an element $\gamma_4 \in (B_1)_{\Phi_4}$ with

$$\begin{aligned} B_1 \oplus B_2 &= \{a \in Ml(f) \,|\, L(a, \gamma_4) \equiv 0(2)\} \qquad (237)\\ &= \{a \in Ml(f) \,|\, L(a, M_h(\gamma_4)) \equiv 0(2)\}, \\ g(\gamma_4) &\in \{\pm\gamma_4, \pm M_h(\gamma_4)\} \qquad \text{for any } g \in G_\mathbb{Z}. \qquad (238) \end{aligned}$$

(iii) In the case $Q_{2,p}$ there is an element $\gamma_5 \in Ml(f)$ with

$$B_1 + B_2 + \mathbb{Z}\cdot\gamma_5 = Ml(f) \tag{239}$$

$$\text{and} \quad g(\gamma_5) \in Ml(f) \text{ for any } g \in \mathrm{Aut}(B_1 \oplus B_2, L). \tag{240}$$

For all $Q_{2,p}$ we set

$$\begin{aligned}\gamma_1 &:= \frac{b_1}{\Phi_4}(M_h)(\beta_1) = (\Phi_{12}\Phi_3)(M_h)(\delta_8)\\ &= (t^6 + t^5 - t^3 + t + 1)(M_h)(\delta_8)\\ &= -2\delta_3 - 2\delta_4 - \delta_5 - 2\delta_6 + \delta_7 - \delta_8 - \delta_9.\end{aligned} \tag{241}$$

We easily see that it is $M_h^2(\gamma_1) = -\gamma_1$. Now $(B_1)_{\Phi_4}$ is an Orlik block with cyclic generator γ_1 (cf. Section A.2). Hence, it is $(B_1)_{\Phi_4} = \mathbb{Z}\cdot\gamma_1 \oplus \mathbb{Z}\cdot M_h(\gamma_1)$. We compute

$$M_h(\gamma_1) = 2\delta_1 + \delta_2 - \delta_5 + \delta_6 - 2\delta_7 - \delta_8 - \delta_9 - \delta_{10}. \tag{242}$$

For $Q_{2,4s}$ we define

$$\begin{aligned}\gamma_2 &:= \frac{b_2}{\Phi_4}(M_h)(\beta_2) = \frac{t^{6+4s}+1}{t^2+1}(M_h)(\delta_{11})\\ &= (t^{4+4s} - t^{2+4s} + t^{4s} - \ldots - t^2 + 1)(M_h)(\delta_{11})\\ &= -\delta_5 - \delta_{10} + (-1)\sum_{j=1}^{2+2s}\delta_{10+2j} + (-2)\sum_{j=1}^{1+s}\delta_{9+4j}.\end{aligned} \tag{243}$$

It holds $M_h^2(\gamma_2) = -\gamma_2$. Now $(B_2)_{\Phi_4}$ is an Orlik block with cyclic generator γ_2 (cf. Section A.2). Thus we obtain $(B_2)_{\Phi_4} = \mathbb{Z}\cdot\gamma_2 \oplus \mathbb{Z}\cdot M_h(\gamma_2)$. We can compute

$$M_h(\gamma_2) = -\delta_2 + \delta_5 + \delta_6 - \delta_7 + \sum_{j=1}^{2+2s}(-1)^{j+1}\delta_{10+2j}. \tag{244}$$

For $Q_{2,4s}$ we define

$$\gamma_3 := \frac{1}{2}(\gamma_1 + M_h(\gamma_1) + \gamma_2 + M_h(\gamma_2)) \tag{245}$$

and we conclude that

$$\gamma_3 = \delta_1 - \sum_{j\in\{3,4,5,7,8,9,10\}}\delta_j - \sum_{j=1}^{1+s}(\delta_{9+4j} + \delta_{10+4j}) \overset{!}{\in} Ml(f). \tag{246}$$

Together with $[Ml(f) : B_1 \oplus B_2] = 2$ this shows (239). In particular, γ_1, $M_h(\gamma_1)$, γ_3, $M_h(\gamma_3)$ is a $\mathbb{Z}$-basis of $Ml(f)_{\Phi_4}$. Now we want to calculate the matrices of L with respect to the basis $\gamma_1, M_h(\gamma_1), \gamma_2, M_h(\gamma_2)$ of $(B_1 \oplus B_2)_{\Phi_4}$ and the basis $\gamma_1, M_h(\gamma_1), \gamma_3, M_h(\gamma_3)$ of

$Ml(f)_{\Phi_4}$. Note that we need to calculate only the values $L(\gamma_1, \gamma_1)$ and $L(\gamma_2, \gamma_2)$, because of (118) and the following identities for any $a \in Ml(f)_{\Phi_4}$:

$$\begin{aligned} L(a, M_h(a)) &= L(M_h(a), M_h^2(a)) = -L(M_h(a), a) \\ &= L(a, a) = L(M_h(a), M_h(a)). \end{aligned} \tag{247}$$

Using $M_h^2(\gamma_j) = -\gamma_j$ and calculations similar to (478), we obtain

$$L(\gamma_1, \gamma_1) = L(\frac{b_1}{\Phi_4}(-M_h^{-1})(\gamma_1), \delta_8) = 3 \cdot L(M_h(\gamma_1), \delta_8) = 3, \tag{248}$$

$$\begin{aligned} L(\gamma_2, \gamma_2) &= L(\frac{b_2}{\Phi_4}(M_h^{-1})(\gamma_2), \delta_{11}) \\ &= (3 + 2s) \cdot L(\gamma_2, \delta_{11}) = 3 + 2s, \end{aligned} \tag{249}$$

All in all, we obtain

$$L(\begin{pmatrix} \gamma_1 \\ M_h(\gamma_1) \\ \gamma_2 \\ M_h(\gamma_2) \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ M_h(\gamma_1) \\ \gamma_2 \\ M_h(\gamma_2) \end{pmatrix}^T) = \begin{pmatrix} 3 & 3 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & 3+2s & 3+2s \\ 0 & 0 & -(3+2s) & 3+2s \end{pmatrix} \tag{250}$$

and

$$L(\begin{pmatrix} \gamma_1 \\ M_h(\gamma_1) \\ \gamma_3 \\ M_h(\gamma_3) \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ M_h(\gamma_1) \\ \gamma_3 \\ M_h(\gamma_3) \end{pmatrix}^T) = \begin{pmatrix} 3 & 3 & 3 & 0 \\ -3 & 3 & 0 & 3 \\ 0 & 3 & 3+s & 3+s \\ -3 & 0 & -(3+s) & 3+s \end{pmatrix}. \tag{251}$$

The quadratic form associated to the last matrix is

$$\frac{3}{2} \cdot \Big[(x_1 + x_3)^2 + (x_1 - x_4)^2 + (x_2 + x_3)^2 + (x_2 + x_4)^2 \Big] + s \cdot (x_3^2 + x_4^2). \tag{252}$$

This shows (most importantly for $Q_{2,4s}$ — but in fact for all $Q_{2,p}$)

$$\{a \in Ml(f)_{\Phi_4} \,|\, L(a, a) = 3\} = \{\pm\gamma_1, \pm M_h(\gamma_1)\}, \tag{253}$$

and because of $(B_1 \oplus B_2)_{\Phi_4} \subseteq Ml(f)_{\Phi_4}$

$$\{a \in (B_1 \oplus B_2)_{\Phi_4} \,|\, L(a, a) = 3\} = \{\pm\gamma_1, \pm M_h(\gamma_1)\}, \tag{254}$$

This implies that any $g \in G_{\mathbb{Z}} \cup \mathrm{Aut}(B_1 \oplus B_2, L)$ maps the set $\{\pm\gamma_1, \pm M_h(\gamma_1)\}$ to itself and thus $(B_1)_{\Phi_4}$ to itself and thus the L-orthogonal sublattice $(B_2)_{\Phi_4}$ to itself. This shows (236) and gives (i).

Now define for all $Q_{2,p}$

$$\begin{aligned} \gamma_4 &:= \gamma_1 + M_h(\gamma_1) \\ &= 2\delta_1 + \delta_2 - 2\delta_3 - 2\delta_4 - 2\delta_5 - \delta_6 - \delta_7 - 2\delta_8 - 2\delta_9 - \delta_{10}. \end{aligned} \tag{255}$$

We can observe that

$$M_h(\gamma_4) = -\gamma_1 + M_h(\gamma_1) \tag{256}$$
$$= -2\gamma_1 + \gamma_4. \tag{257}$$

(253) and (256) imply (238). (257) implies the second equality in (237). We compute

$$L(\delta_8, \gamma_4) = 0. \tag{258}$$

This shows $L(\delta_8, M_h(\gamma_4)) \equiv 0(2)$ (in fact, it is $= -2$). The M_h-invariance of L and the fact that δ_8 is a cyclic generator of the Orlik block B_1 give $B_1 \subseteq \{a \in Ml(f) \,|\, L(a, \gamma_4) \equiv 0(2)\}$. As (118) implies $L(B_2, \gamma_4) = 0$, so $B_1 \oplus B_2 \subseteq \{a \in Ml(f) \,|\, L(a, \gamma_4) \equiv 0(2)\}$. Now $r_I = 2$ and for example $L(\delta_2, \gamma_4) = -1 \not\equiv 0(2)$ show (237) and (ii). Now (ii) implies $G_{\mathbb{Z}} \subseteq \mathrm{Aut}(B_1 \oplus B_2, L)$.

Finally, (iii) implies now $\mathrm{Aut}(B_1 \oplus B_2, L) \subseteq G_{\mathbb{Z}}$, but (iii) has still to be proved.

We continue as in the final part of the proof of part (a) for the other series. The claim in (i) holds and Lemma A.2.3 can be applied. Therefore (187) and (188) hold for $Q_{2,p}$. The group $\mathrm{Aut}(B_1 \oplus B_2, L)$ for $12 \not| p$ is generated by $M_h, -\mathrm{id}, M_h|_{B_1} \times \mathrm{id}|_{B_2}$ and $(-\mathrm{id})|_{B_1} \times \mathrm{id}|_{B_2}$, and analogously for the group in (188) if $12|p$.

For $Q_{2,4s}$ we define $\gamma_5 := \gamma_3$. It satisfies (239). If $12|4s$, it is in $(B_1)_{b_1/\Phi_m} + (B_3)_{b_2/\Phi_m}$. Thus we can work with the group in (188). If $12 \not| 4s$, we work with the group in (187). In both cases γ_5 satisfies (240), because of

$$(M_h|_{B_1} \times \mathrm{id}|_{B_2})(\gamma_5) = \gamma_5 - M_h(\gamma_1) \in Ml(f), \tag{259}$$
$$((-\mathrm{id})|_{B_1} \times \mathrm{id}|_{B_2})(\gamma_5) = \gamma_5 - (\gamma_1 + M_h(\gamma_1)) \in Ml(f). \tag{260}$$

For the other $Q_{2,p}$, we choose the following rather simple γ_5, namely

$$\begin{aligned} \gamma_5 &:= \delta_{10} = \frac{1}{2}(-\delta_2 + \delta_6 - \delta_7 + \delta_{10}) - \frac{1}{2}(-\delta_2 + \delta_6 - \delta_7 - \delta_{10}), \\ \text{with} \quad & -\delta_2 + \delta_6 - \delta_7 + \delta_{10} \in B_1, \quad -\delta_2 + \delta_6 - \delta_7 - \delta_{10} \in B_2. \end{aligned} \tag{261}$$

Then (239) holds. Moreover it is

$$(M_h|_{B_1} \times \mathrm{id}|_{B_2})(\gamma_5) = \delta_1 + \delta_2 \in Ml(f), \tag{262}$$
$$((-\mathrm{id})|_{B_1} \times \mathrm{id}|_{B_2})(\gamma_5) = \delta_2 - \delta_6 + \delta_7 \in Ml(f). \tag{263}$$

In any case (239), (240) and (iii) hold. Thus $\mathrm{Aut}(B_1 \oplus B_2, L) \subseteq G_{\mathbb{Z}}$, and (119) follows for $Q_{2,p}$.

Secondly, the proof of (119) for $W_{1,6s-3}$ was postponed and has to be given now. Luckily, the majority of the arguments was already given in the proof of part (a). It remains to show the following two statements:

(i) Equation (179) holds for $W_{1,3}$.

(ii) In the case $W_{1,6s-3}$, any $g \in G_{\mathbb{Z}} \cup \mathrm{Aut}(B_1 \oplus B_2, L)$ maps $(B_1)_{b_4}$ and $(B_2)_{b_4}$ to itself.

For the rest of this proof we consider W_{6s-3} only. Here it holds $b_4 = \Phi_6\Phi_2$. We set

$$\begin{aligned}
\varrho_1 &:= \frac{b_1}{\Phi_6\Phi_2}(M_h)(\beta_1) = (\Phi_{12}\Phi_3)(M_h)(\delta_3+\delta_9+\delta_{11}) && (264)\\
&= \Phi_3(M_h)(\delta_9-\delta_{13}) = \delta_9+\delta_{10}+\delta_{11}-\delta_{13}-\delta_{14}-\delta_{15},\\
\varrho_2 &:= \frac{b_2}{\Phi_6\Phi_2}(M_h)(\beta_2) = \frac{t^{6+p}+1}{t^3+1}(M_h)(\delta_{16}) && (265)\\
&= (t^{3+p}-t^p+\ldots-t^3+1)(M_h)(\delta_{16})\\
&= \delta_1+\delta_2-\sum_{j\in\{3,4,5,7,8,9,10,12,13,14\}}\delta_j+\sum_{j=1}^{p}\delta_{15+j}+\sum_{j=0}^{p/3-1}(-1)^j\delta_{16+3j}.
\end{aligned}$$

We have cyclic generators ϱ_1 and ϱ_2 of the Orlik blocks $(B_1)_{\Phi_6\Phi_2}$ and $(B_2)_{\Phi_6\Phi_2}$ (cf. Section A.2). Thus ϱ_i, $M_h(\varrho_i)$ and $M_h^2(\varrho_i)$ give a $\mathbb{Z}$-basis of $(B_i)_{\Phi_6\Phi_2}$. We compute

$$\begin{aligned}
M_h(\varrho_1) &= -\delta_4-\delta_8-\delta_9+\delta_5+\delta_{12}+\delta_{13}, && (266)\\
M_h^2(\varrho_1) &= -\delta_8-\delta_9-\delta_{10}+\delta_{12}+\delta_{13}+\delta_{14}, && (267)\\
M_h(\varrho_2) &= \delta_1+\delta_3+2\delta_6-2\delta_7-\sum_{j=9,10,11,13,14,15}\delta_j+\sum_{j=0}^{p/3-1}(-1)^j\delta_{17+3j}, && (268)\\
M_h^2(\varrho_2) &= -\delta_2+2\delta_4+2\delta_5+2\delta_6-\delta_7+\delta_8+\delta_9+\delta_{12}+\delta_{13}+\sum_{j=0}^{p/3-1}(-1)^j\delta_{18+3j}. && (269)
\end{aligned}$$

Furthermore, we have to calculate the 6×6 matrix of values of L for the $\mathbb{Z}$-basis ϱ_1, $M_h(\varrho_1)$, $M_h^2(\varrho_1)$, ϱ_2, $M_h(\varrho_2)$, $M_h^2(\varrho_2)$ of $(B_1\oplus B_2)_{\Phi_6\Phi_2}$. Since (118) holds, it is block diagonal with two 3×3 blocks. This makes life easy. Since L is M_h-invariant and because of the identities for any $a \in Ml(f)_{\Phi_6\Phi_2}$ we obtain

$$\begin{gathered}
L(M_h(a),a) = -L(a,a), \quad L(M_h^2(a),a) = -L(a,M_h(a)),\\
L(a,M_h^2(a)) = L(M_h(a),M_h^3(a)) = -L(M_h(a),a) = L(a,a),
\end{gathered}$$

and each 3×3 matrix is determined by two values. The matrices are

$$L(M_h^i(\varrho_1), M_h^j(\varrho_1))_{i,j=0,1,2} = \begin{pmatrix} 2 & 2 & 2 \\ -2 & 2 & 2 \\ -2 & -2 & 2 \end{pmatrix}, \tag{270}$$

$$L(M_h^i(\varrho_2), M_h^j(\varrho_2))_{i,j=0,1,2} = \begin{pmatrix} 1+2s & 0 & 1+2s \\ -1-2s & 1+2s & 0 \\ 0 & -1-2s & 1+2s \end{pmatrix}. \tag{271}$$

Now recall the definition $\widetilde{\gamma_2} := \frac{1}{2}(\gamma_1+\gamma_2)$ in (166) and recall that

$$Ml(f)_{\Phi_2} = \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\widetilde{\gamma_2} \overset{2:1}{\supseteq} \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2 = (B_1 \oplus B_2)_{\Phi_2}. \tag{272}$$

Thus it holds also

$$Ml(f)_{\Phi_6\Phi_2} = \langle \varrho_1, M_h(\varrho_1), M_h^2(\varrho_1), \varrho_2, M_h(\varrho_2), \widetilde{\gamma_2}\rangle \overset{2:1}{\supseteq} (B_1 \oplus B_2)_{\Phi_6\Phi_2}, \tag{273}$$

where

$$\widetilde{\gamma_2} = \frac{1}{2}(\gamma_1+\gamma_2) = \frac{1}{2}(\varrho_1 - M_h(\varrho_1) + M_h^2(\varrho_1) + \varrho_2 - M_h(\varrho_2) + M_h^2(\varrho_2)).$$

The matrix of L for the $\mathbb{Z}$-basis ϱ_1, $M_h(\varrho_1)$, $M_h^2(\varrho_1)$, ϱ_2, $M_h(\varrho_2)$, $\widetilde{\gamma_2}$ of $Ml(f)_{\Phi_6\Phi_2}$ is

$$\begin{pmatrix} 2 & 2 & 2 & 0 & 0 & 1 \\ -2 & 2 & 2 & 0 & 0 & -1 \\ -2 & -2 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1+2s & 0 & 1+2s \\ 0 & 0 & 0 & -1-2s & 1+2s & -1-2s \\ 1 & -1 & 1 & 1+2s & -1-2s & 3+3s \end{pmatrix}. \tag{274}$$

The respective quadratic form $(x_1 \dots x_6)(\text{matrix})\begin{pmatrix} x_1 \\ \vdots \\ x_6 \end{pmatrix}$ is

$$\begin{aligned} & \frac{1}{2}\left[(2x_1+x_6)^2 + (2x_2-x_6)^2 + (2x_3+x_6)^2\right] \\ + \quad & \frac{1}{2}(1+2s)\left[(x_4-x_5+x_6)^2 + (x_4+x_6)^2 + (x_5-x_6)^2\right]. \end{aligned} \tag{275}$$

We see that

$$\{a \in Ml(f)_{\Phi_6\Phi_2} \mid L(a,a) = 2\} = \{\pm M_h^j(\varrho_1) \mid j = 0,1,2\}, \tag{276}$$

and also that

$$\{a \in (B_1 \oplus B_2)_{\Phi_6\Phi_2} \,|\, L(a,a) = 2\} = \{\pm M_h^j(\varrho_1) \,|\, j = 0,1,2\}. \tag{277}$$

Thus any $g \in G_{\mathbb{Z}} \cup \mathrm{Aut}(B_1 \oplus B_2, L)$ maps ϱ_1 to an element of $\{\pm M_h^j(\varrho_1) \,|\, j = 0,1,2\}$. These are cyclic generators of the Orlik block $(B_1)_{\Phi_6\Phi_2}$. Thus any $g \in G_{\mathbb{Z}} \cup \mathrm{Aut}(B_1 \oplus B_2, L)$ maps $(B_1)_{\Phi_6\Phi_2}$ to itself. As the Orlik block $(B_2)_{\Phi_6\Phi_2}$ is the L-orthogonal sublattice within $Ml(f)_{\Phi_6\Phi_2}$, such a g maps also $(B_2)_{\Phi_6\Phi_2}$ to itself. This shows (ii) above. Especially such a g maps $(B_1)_{\Phi_2}$ to itself and its generator $\gamma_4 = \gamma_1$ to $\pm\gamma_4$. This shows (i) above.

Lastly, the proof of (120) for $S_{1,10}$ was postponed and has to be given now. From now on only $S_{1,10}$ is considered. (170) shows that $(Ml(f)_{\Phi_2}, L)$ is an A_2-lattice with roots $\{\pm\gamma_1, \pm\widetilde{\gamma_2}, \pm(\widetilde{\gamma_2}-\gamma_1)\}$ (see [BH16, Section 2] for some details on such lattices). Note that here γ_1 generates $(B_1)_{\Phi_2}$. We will show that $(B_1)_{\Phi_{10}}$ and $\pm\gamma_1$ satisfy the following special relationship:

$$\Big[((B_1)_{\Phi_{10}} + \mathbb{Z}\cdot a)_{\mathbb{Q}} \cap Ml(f) : ((B_1)_{\Phi_{10}} + \mathbb{Z}\cdot a)\Big] = \begin{cases} 5 & \text{if } a = \pm\gamma_1, \\ 1 & \text{if } a \in \{\pm\widetilde{\gamma_2}, \pm(\widetilde{\gamma_2}-\gamma_1)\}. \end{cases} \tag{278}$$

If $a = \pm\gamma_1$, then we have

$$\begin{aligned} ((B_1)_{\Phi_{10}} + \mathbb{Z}\cdot a)_{\mathbb{Q}} \cap Ml(f) &= (B_1)_{\Phi_{10}\Phi_2} = \bigoplus_{j=0}^{4} \mathbb{Z}\cdot (t^j\Phi_5)(M_h)(\beta_1), \\ (B_1)_{\Phi_{10}} + \mathbb{Z}\cdot a &= (B_1)_{\Phi_{10}} + (B_1)_{\Phi_2} \\ &= \bigoplus_{j=0}^{3} \mathbb{Z}\cdot (t^j\Phi_2\Phi_5)(M_h)(\beta_1) \oplus \mathbb{Z}\cdot (\Phi_{10}\Phi_5)(M_h)(\beta_1), \end{aligned}$$

so the respective index is

$$\left[\bigoplus_{j=0}^{4} \mathbb{Z}\cdot t^j \;:\; \bigoplus_{j=0}^{3} \mathbb{Z}\cdot t^j\Phi_2 \oplus \mathbb{Z}\cdot \Phi_{10}\right] = 5.$$

Now recall that $(B_1)_{\Phi_{10}}$ is a primitive sublattice of $Ml(f)$ and that

$$B_1 \subseteq \bigoplus_{j=1}^{14} \mathbb{Z}\cdot\delta_j, \quad \text{so } (B_1)_{\Phi_{10}} \subseteq \bigoplus_{j=1}^{14} \mathbb{Z}\cdot\delta_j.$$

We observe that

$$\widetilde{\gamma_2} \equiv \widetilde{\gamma_2} - \gamma_1 \equiv -\sum_{j=15}^{24} \delta_j \mod \sum_{j=1}^{14} \mathbb{Z}\cdot\delta_j.$$

Since the sum $-\sum_{j=15}^{24}\delta_j$ in $\widetilde{\gamma_2}$ and in $\widetilde{\gamma_2}-\gamma_1$, the sublattices $(B_1)_{\Phi_{10}} \oplus \mathbb{Z}\cdot\widetilde{\gamma_2}$ and $(B_1)_{\Phi_{10}} \oplus \mathbb{Z}\cdot(\widetilde{\gamma_2}-\gamma_1)$ are primitive in $Ml(f)$. This means their index above is 1. So this shows (278). Now (120) follows easily. In fact, consider an element $g \in G_{\mathbb{Z}}$ with $g((B_1)_{\Phi_{10}}) = (B_1)_{\Phi_{10}}$. It must map γ_1 to some root of the A_2-lattice $(Ml(f)_{\Phi_{10}}, L)$. By

(278) the image must be $\pm\gamma_1$ and, hence, so $g((B_1)_{\Phi_2}) = (B_1)_{\Phi_2}$. Therefore $g((B_1)_{\Phi_{10}\Phi_2}) = (B_1)_{\Phi_{10}\Phi_2}$ and by its L-orthogonality it is also $g((B_2)_{\Phi_{10}\Phi_2}) = (B_2)_{\Phi_{10}\Phi_2}$.

For $S_{1,10}$ it is $b_1 = \Phi_{10}\Phi_5\Phi_2$ and $b_2 = \Phi_{30}\Phi_{10}\Phi_6\Phi_2$. Thus, the eigenspaces with eigenvalues different from the roots of $\Phi_{10}\Phi_2$ are one-dimensional and are either in $(B_1)_\mathbb{C}$ or in $(B_2)_\mathbb{C}$. This implies (120) for $S_{1,10}$.

Now everything is proved. □

4.4.2 *Torelli Conjectures for bimodal series singularities*

Now we want to prove the strong global Torelli Conjecture 3.2.2 (a) for the bimodal series singularities. Therefore we have to make a couple of preparations. In particular, we need the covering map

$$c_T : T^{cov} \to T, \qquad (\tau_1, t_2) \mapsto (\tau_1^{m_2}, t_2). \tag{279}$$

with covering space $T^{cov} := (\mathbb{C} \setminus \{0\}) \times \mathbb{C}$, where

$$m_2 := m + r_I \cdot p \tag{280}$$

with m, p and r_I as in Table 8. Moreover, we need to compute the classifying space D_{BL} and the period map $T \to D_{BL}$. This was done partially already in [He93]. However we will compute it here again for practical reasons.

We set

$$\alpha_1 := \frac{-1}{m} < \beta_1 := \frac{-1}{m_2} < 0 < \alpha_2 := \frac{1}{m_2} < \beta_2 := \frac{1}{m}. \tag{281}$$

Recall that here $\zeta := e^{2\pi i/m}$ and $\psi_\alpha : H^n(X_\infty, \mathbb{C}) \to C^\alpha$ is an isomorphism, where for each 2-parameter family in Table 7 M_μ^{mar}, $(M_\mu^{mar})^0$, $G_\mathbb{Z}$, G^{mar}, Ml, $H^n(X_\infty, \mathbb{C})$ and C^α denotes the respective object for the reference singularity f_0. By Table 8 it is

$$\begin{aligned} \dim C^{\beta_1} &= \dim C^{\alpha_2} = 1, \\ \dim C^{\alpha_1} &= \dim C^{\beta_2} = \begin{cases} 1 & \text{if } m \nmid p, \\ 2 & \text{if } m | p. \end{cases} \end{aligned} \tag{282}$$

Thus we have to distinguish between the two cases $m \nmid p$ and $m|p$. Now we define a space $D_{BL}^{sub} \subseteq D_{BL}$, which will contain the image of T under the period map. In the case $m \nmid p$ the space is the 2-dimensional space

$$\begin{aligned} D_{BL}^{sub} &:= \{\mathbb{C} \cdot (v_1 + v_2 + v_4) \,|\, v_1 \in C^{\alpha_1} \setminus \{0\}, v_2 \in C^{\beta_1} \setminus \{0\}, v_4 \in C^{\beta_2}\} \\ &= \{\mathbb{C} \cdot (v_1^0 + \rho_1 v_2^0 + \rho_2 v_4^0) \,|\, (\rho_1, \rho_2) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{C}\} \\ &\quad \text{for some generators } v_1^0, v_2^0, v_4^0 \text{ of } C^{\alpha_1}, C^{\beta_1}, C^{\beta_2} \\ &\cong (\mathbb{C} \setminus \{0\}) \times \mathbb{C}. \end{aligned} \tag{283}$$

In the case $m|p$ the polarizing form S defines an indefinite Hermitian form $((a,b) \mapsto S(a,\overline{b}))$ on $H^n(X_\infty, \mathbb{C})$. This follows from the respective statement for h_ζ on Ml_ζ in Theorem 4.4.1, Lemma A.2.5 (b) and from the relation between the Seifert form L and polarizing form S (see Subsection (2.1.2)). Thus we get the half-plane

$$\mathcal{H}(C^{\alpha_1}) \;:=\; \{\mathbb{C}\cdot v \,|\, v \in C^{\alpha_1} \text{ with } S(\psi_{\alpha_1}^{-1}(v), \overline{\psi_{\alpha_1}^{-1}(v)}) < 0\} \subseteq \mathrm{Proj}(C^{\alpha}). \tag{284}$$

In the case with $m|p$ we have the 3-dimensional space

$$\begin{aligned} D_{BL}^{sub} \;:=\;& \{\mathbb{C}\cdot(v_1+v_2+v_4) \,|\, v_1 \in C^{\alpha_1}\setminus\{0\} \text{ with } [\mathbb{C}\cdot v_1] \in \mathcal{H}(C^{\alpha_1}), \\ & v_2 \in C^{\beta_1}\setminus\{0\}, v_4 \in \mathbb{C}\cdot\psi_{\beta_2}(\overline{\psi_{\alpha_1}^{-1}(v_1)}) \subseteq C^{\beta_2}\} \\ \cong\;& \mathcal{H}(C^{\alpha_1}) \times (\mathbb{C}\setminus\{0\}) \times \mathbb{C}. \end{aligned} \tag{285}$$

Finally, we denote by $M_T \in G_{\mathbb{Z}}$ the transversal monodromy, i.e. the monodromy of the homology bundle $\bigcup_{(t_1,t_2)\in T} Ml(f_{(t_1,t_2)}) \to T$ along the cycle $\{(e^{2\pi i s}, 0) \,|\, s \in [0,1]\}$ (cf. Lemma 3.1.5). The properties of the spaces D_{BL}^{sub} and the transversal monodromy are described in the next theorem.

Theorem 4.4.2. *(a) D_{BL}^{sub} embeds canonically into D_{BL}.*

(b) For suitable generators $v_1^0 \in C^{\alpha_1}\setminus\{0\}$, $v_2^0 \in C^{\beta_1}\setminus\{0\}$ and $v_4^0 := \psi_{\beta_2}(\overline{\psi_{\alpha_1}^{-1}(v_1^0)}) \in C^{\beta_2}\setminus\{0\}$, the multivalued period map $BL_T : T \to D_{BL}$ has its image in D_{BL}^{sub} and takes the form

$$(t_1,t_2) \mapsto \mathbb{C}\cdot\left(v_1^0 + t_1^{1/m_2}\cdot v_2^0 + \left(\frac{t_2}{t_1} + r(t_1)\right) v_4^0\right) \tag{286}$$

with

$$r(t_1) = \begin{cases} 0 & \textit{in the cases } (r_I = 1 \;\&\; p \geq 3), \\ & \textit{the cases } (r_I = 2 \;\&\; p \geq 2) \\ & \textit{and the case } U_{1,2}, \\ c_T \cdot t_1 & \textit{in the cases } (r_I = 2 \;\&\; p = 1) \\ & \textit{and the cases } W_{1,2}^\sharp \textit{ and } S_{1,2}^\sharp, \\ c_T \cdot t_1^2 & \textit{in the cases } (r_I = 1 \;\&\; p = 1), \end{cases} \tag{287}$$

for a suitable constant $c_T \in \mathbb{C}$. In the cases with $m|p$, the transversal monodromy M_T has on C^{α_1} the eigenvalues 1 and $\overline{\zeta}$, and $\mathbb{C}\cdot v_1^0$ is the eigenspace with eigenvalue 1. The class $[\mathbb{C}\cdot v_1^0]$ is in $\mathcal{H}(C^{\alpha_1})$.

(c) The induced period map $BL_{T^{cov}} : T^{cov} \to D_{BL}^{sub}$ is an isomorphism if $m \not| p$ and an isomorphism to the fiber above $[\mathbb{C}\cdot v_1^0] \in \mathcal{H}(C^{\alpha_1})$ of the projection $D_{BL}^{sub} \to \mathcal{H}(C^{\alpha_1})$ if $m|p$.

(d) In the case of the subseries $U_{1,9r}$, G^{mar} contains an element g_3 such that $\Psi(g_3)$ is elliptic of order 18 (for all subseries with $p = m\cdot r$, $\Psi(M_T)$ is elliptic of order m, for $U_{1,9r}$ $m = 9$).

(e) $f_{(t_1,t_2)}$ *and* $f_{(\widetilde{t_1},\widetilde{t_2})}$ *are right equivalent*

$$\iff \begin{cases} \exists\, k\in\mathbb{Z} \textit{ with } (\widetilde{t_1},\widetilde{t_2}) = (\zeta^{r_I p k}\cdot t_1, \zeta^{(r_I p+2)k}\cdot t_2) \\ \qquad \textit{for all 8 series except } U_{1,2q}, \\ \exists\, k\in\mathbb{Z} \textit{ and } \varepsilon\in\{\pm 1\} \textit{ with} \\ \qquad (\widetilde{t_1},\widetilde{t_2}) = (\varepsilon\zeta^{r_I p k}\cdot t_1, \varepsilon\zeta^{(r_I p+2)k}\cdot t_2) \textit{ for } U_{1,2q}. \end{cases} \tag{288}$$

Note that the parts (a), (b) and (d) of Theorem 4.4.2 will be proved after Theorem 4.4.4.

Proof of Theorem 4.4.2 (c) and (e): (c) This follows immediately from (286).

(e) First we prove "$\Leftarrow$". Therefore we give explicit coordinate changes here. Looking at the normal forms in Table 7 one case after the other we see that

$$f_{(t_1,t_2)}(x\cdot\zeta^{\varsigma_1\cdot k}, y\cdot\zeta^{\varsigma_2\cdot k}, z\cdot\zeta^{\varsigma_3\cdot k}) = f_{(t_1\cdot\zeta^{r_I p k}, t_2\cdot\zeta^{(r_I p+2)k})}(x,y,z). \tag{289}$$

Here $k\in\mathbb{Z}$ and $(\varsigma_1,\varsigma_2,\varsigma_3)$ are as in (290).

	ς_1	ς_2	ς_3
$W_{1,p}^\sharp$ and $W_{1,p}$	3	2	0
$S_{1,p}^\sharp$ and $S_{1,p}$	3	2	4
$U_{1,p}$	3	2	3
$E_{3,p}$	6	2	0
$Z_{1,0}$	4	2	0
$Q_{2,p}$	4	2	5

(290)

In the case $U_{1,2q}$ we have additionally

$$f_{(t_1,t_2)}(x,y,-z) \quad = \quad f_{(-t_1,-t_2)}(x,y,z). \tag{291}$$

Altogether this shows "$\Leftarrow$".

Now we show "$\Rightarrow$". In order to do so, let $f_{(t_1,t_2)}$ and $f_{(\widetilde{t_1},\widetilde{t_2})}$ be right equivalent. Then $BL_T(t_1,t_2)$ and $BL_T(\widetilde{t_1},\widetilde{t_2})$ are isomorphic, so a $g\in G_{\mathbb{Z}}$ with $g(BL_T(t_1,t_2)) = BL_T(\widetilde{t_1},\widetilde{t_2})$ exists. We claim that v_1^0, v_2^0 and v_4^0 are eigenvectors of g with some eigenvalues λ_1,λ_2 and $\overline{\lambda_1}$. For v_2^0 this is trivial as $\dim C^{\beta_1} = 1$, for v_1^0 in the case $m \not| p$ also. In the case $m|p$, it follows for v_1^0 from (286). For v_4^0 use $v_4^0 = \psi_{\beta_2}(\overline{\psi_{\alpha_1}^{-1}(v_1^0)})$. We claim also that

$$\lambda_1 \in Eiw(\zeta), \quad \lambda_2\in Eiw(e^{2\pi i/m_2}). \tag{292}$$

Recall that $Eiw(\zeta) = \{\pm\zeta^k \,|\, k\in\mathbb{Z}\}$ (cf. Section A.3). For λ_2 (292) is a consequence of the following three facts and of [He93, Lemma 3.4 a)+c)].

(i) The 1-dimensional eigenspace $Ml_{e^{2\pi i/m_2}}$ is already defined over $\mathbb{Q}(e^{2\pi i/m_2})$. Therefore $\lambda_2\in\mathbb{Q}(e^{2\pi i/m_2})$.

(ii) $|\lambda_2| = 1$ because L pairs $Ml_{e^{2\pi i/m_2}}$ and $Ml_{e^{-2\pi i/m_2}}$.

(iii) λ_2 is an algebraic integer because $g \in G_{\mathbb{Z}}$.

If $m \not| p$, the same reasoning applies also to λ_1. Suppose for a moment $m|p$.

By part (b), the transversal monodromy M_T acts on C^{α_1} and on $H^n(X_\infty, \mathbb{C})_\zeta$ with eigenvalues 1 and ζ, and the 1-dimensional eigenspaces with eigenvalue 1 are $\mathbb{C} \cdot v_1^0$ and $\mathbb{C} \cdot \psi_{\alpha_1}^{-1}(v_1^0)$. Therefore $\mathbb{C} \cdot \psi_{\alpha_1}^{-1}(v_1^0)$ is already defined over $\mathbb{Q}(\zeta)$, i.e. $\mathbb{C} \cdot \psi_{\alpha_1}^{-1}(v_1^0) \cap H^n(X_\infty, \mathbb{Q})$ is a 1-dimensional $\mathbb{Q}(\zeta)$-vector space. This implies (i) $\lambda_1 \in \mathbb{Q}(\zeta)$. Moreover, (ii) holds as $v_1^0 \in \mathcal{H}(C^{\alpha_1})$. Finally, (iii) is trivial. So by [He93, Lemma 3.4 a)+c)] we can conclude that $\lambda_1 \in Eiw(\zeta)$. Now (292) follows in all cases.

The equality $g(BL_T(t_1, t_2)) = BL_T(\widetilde{t_1}, \widetilde{t_2})$ becomes

$$\begin{aligned}
& \mathbb{C} \cdot \left(\lambda_1 \cdot v_1^0 + \lambda_2 \cdot t_1^{1/m_2} \cdot v_2^0 + \overline{\lambda_1} \left(\frac{t_2}{t_1} + r(t_1) \right) \cdot v_4^0 \right) \\
= \; & \mathbb{C} \cdot \left(v_1^0 + \widetilde{t_1}^{1/m_2} \cdot v_2^0 + \left(\frac{\widetilde{t_2}}{\widetilde{t_1}} + r(\widetilde{t_1}) \right) \cdot v_4^0 \right), \\
\text{so} \quad & \widetilde{t_1}^{1/m_2} = \lambda_2 \overline{\lambda_1} \cdot t_1^{1/m_2}, \quad \frac{\widetilde{t_2}}{\widetilde{t_1}} + r(\widetilde{t_1}) = \overline{\lambda_1}^2 \left(\frac{t_2}{t_1} + r(t_1) \right), \\
\text{so} \quad & \widetilde{t_1} = \lambda_2^{m_2} \overline{\lambda_1}^{m_2} \cdot t_1, \\
\text{and} \quad & \widetilde{t_2} = \overline{\lambda_1}^2 \cdot \frac{\widetilde{t_1}}{t_1} \cdot t_2 + \widetilde{t_1} \cdot (\overline{\lambda_1}^2 \cdot r(t_1) - r(\widetilde{t_1})).
\end{aligned} \tag{293}$$

Because of (292), we can write λ_1 and λ_2 as follows:

	λ_1	λ_2
All cases with $m \equiv 0(2),\ m_2 \equiv 0(2)$	$\overline{\zeta}^k$	$e^{2\pi i l/m_2}$
The cases $W^\sharp_{1,2q-1}$ and $S^\sharp_{1,2q-1}$	$\varepsilon_2 \cdot \overline{\zeta}^k$	$\varepsilon_2 \cdot e^{2\pi i l/m_2}$
The cases $U_{1,2q-1}$	$\varepsilon_1 \cdot \overline{\zeta}^k$	$e^{2\pi i l/m_2}$
The cases $U_{1,2q}$	$\varepsilon_1 \cdot \overline{\zeta}^k$	$\varepsilon_2 \cdot e^{2\pi i l/m_2}$

(294)

Here $k, l \in \mathbb{Z}$ and $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$. It is esay to check that (293) boils down to

$$\widetilde{t_1} = \zeta^{r_I p k} \cdot t_1, \quad \widetilde{t_2} = \zeta^{(r_I p + 2)k} \cdot t_2, \tag{295}$$

in all cases except $U_{1,2q}$. In the cases $U_{1,2q}$, it boils down to

$$\widetilde{t_1} = \varepsilon_1 \varepsilon_2 \cdot \zeta^{pk} \cdot t_1, \widetilde{t_2} = \varepsilon_1 \varepsilon_2 \cdot \zeta^{(p+2)k}. \tag{296}$$

This settles "$\Rightarrow$" and finishes the proof of Theorem 4.4.2 (e). $\square$

Now we describe the space D_{BL} for the bimodal series singularities.

Lemma 4.4.3. *Consider a family of bimodal series singularities.*

(a) The spectral numbers $\alpha_1, \ldots, \alpha_\mu$ with $\alpha_1 \leq \ldots \leq \alpha_\mu$ satisfy

$$\alpha_1 = \frac{-1}{m} < \alpha_2 = \frac{1}{m_2} < \alpha_3 \leq \ldots \leq \alpha_{\mu-2} < \alpha_{\mu-1} = 1 - \frac{1}{m_2} < \alpha_\mu = 1 + \frac{1}{m} \tag{297}$$

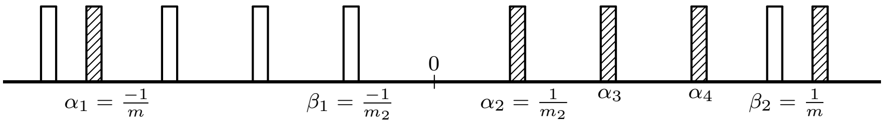

Figure 17: Spectral data for the bimodal series singularities ($r_I = 1$)

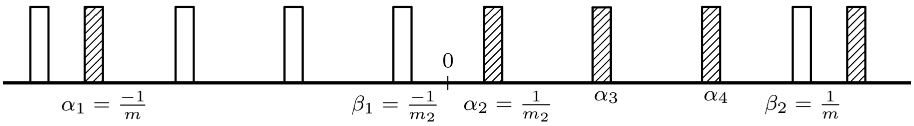

Figure 18: Spectral data for the bimodal series singularities ($r_I = 2$)

and are uniquely determined by this and the characteristic polynomial $\prod_{j\geq 1} b_j$ *of the monodromy with* b_j *as in given Table 8.*

(b) Recall from (281) that $\beta_1 = \frac{-1}{m_2} = -\alpha_2$ *and* $\beta_2 = \frac{1}{m} = -\alpha_1$*. Then*

$$\dim C^{\alpha_1} = \begin{cases} 1 & \textit{if } m \not| p, \\ 2 & \textit{if } m|p, \end{cases} \tag{298}$$

$$\dim C^{\beta} = \begin{cases} 1 & \textit{for } \beta \in (\alpha_1, \beta_2) \cap \frac{1}{m_2}(\mathbb{Z} \setminus \{0\}) \textit{ if } r_I = 1, \\ & \textit{and for } \beta \in (\alpha_1, \beta_2) \cap (\frac{1}{m_2} + \frac{2}{m_2}\mathbb{Z}) \textit{ if } r_I = 2, \\ 0 & \textit{for other } \beta \in (\alpha_1, \beta_2). \end{cases} \tag{299}$$

The following two pictures illustrate this for $2m < p < 3m$*. The first one shows the situation for* $r_I = 1$*, the second one for* $r_I = 2$*.*

(c) Denote by $(*)$ *the condition*

$$(*) : \qquad \beta \in (\alpha_1, 0) \textit{ with } C^\beta \neq \{0\} \quad (\textit{then } \dim C^\beta = 1).$$

If $m \not| p$ *the classfying space* D_{BL} *in [He99] is*

$$\begin{aligned} D_{BL} &= \{\mathbb{C} \cdot (v_1 + \sum_{\beta:(*)} v_{(\beta)} + v_2) \,|\, v_1 \in C^{\alpha_1} - \{0\}, v_{(\beta)} \in C^\beta, v_2 \in C^{\beta_2}\} \\ &\cong \mathbb{C}^{N_{BL}} \qquad \textit{with } N_{BL} := |\{\beta : (*)\}| + 1. \end{aligned} \tag{300}$$

In (284) the half-plane $\mathcal{H}(C^{\alpha_1})$ *was defined for* $m|p$*. If* $m|p$ *then* D_{BL} *is*

$$\begin{aligned} D_{BL} &= \{\mathbb{C} \cdot (v_1 + \sum_{\beta:(*)} v_{(\beta)} + v_2) \,| \\ &\qquad v_1 \in C^{\alpha_1} - \{0\} \textit{ with } [\mathbb{C} \cdot v_1] \in \mathcal{H}(C^{\alpha_1}), \\ &\qquad v_{(\beta)} \in C^\beta, v_2 \in \mathbb{C} \cdot \psi_{\beta_2}(\overline{\psi_{\alpha_1}^{-1}(v_1)}) \subseteq C^{\beta_2}\} \\ &\cong \mathcal{H}(C^{\alpha_1}) \times \mathbb{C}^{N_{BL}} \qquad \textit{with } N_{BL} := |\{\beta : (*)\}| + 1. \end{aligned} \tag{301}$$

Proof. (a) The spectral numbers of the bimodal series singularities can be found in [AGV88, Subsection 13.3.4]. The rest of part (a) is clear.

(b) Here we have to show the two equations (298) and (299). We start with (298). It can be already found in [He95, p. 391]. But it also follows easily from $\dim C^{\alpha_1} = \dim Ml_\zeta$ and $\Phi_m \not| b_2 \iff m \not| p$. Equation (299) can be deduced from the values of b_j in Table 8.

(c) We want to compute the spaces D_{BL}. So let $\mathcal{L}_0 \in D_{BL}$. Then by the spectral data given in part (a) we obtain

$$\mathcal{L}_0 = \mathbb{C} \cdot \sigma_1 \oplus \mathcal{L}_0 \cap \bigoplus_{\beta:\, \alpha_2 \leq \beta \leq \beta_2} C^\beta \oplus V^{>\beta_2} \tag{302}$$

where β with $C^\beta \neq \{0\}$ and

$$\alpha(\sigma_1) = \alpha_1, \quad \sigma_1 \in C^{\alpha_1} \oplus \bigoplus_{\beta:(*)} C^\beta \oplus C^{\beta_2}. \tag{303}$$

Since the spaces C^β are one-dimensional for β with $\alpha_2 \leq \beta < \beta_2$ and $C^\beta \neq \{0\}$, they are generated by the principal part of a section in $\mathcal{L}_0$. The space C^{β_2} has dimension $\dim C^{\beta_2} = 1$ and is not generated by the principal part of a section in $\mathcal{L}_0$ in the case $m \not| p$. If $m|p$ then $\dim C^{\beta_2} = 2$ and we have a one-dimensional subspace $\{v \in C^{\beta_2} \,|\, K_f^{(-2)}(v, s(\sigma_1, \alpha_1)) = 0\} \subseteq C^{\beta_2}$. It is in $\mathcal{L}_0$, because then β_2 is a spectral number with multiplicity 1. The principal part $s(\sigma_1, \alpha_1)$ must be compatible with a polarized Hodge structure of weight 2 on $H^n(X_\infty, \mathbb{C})_\zeta \oplus H^n(X_\infty, \mathbb{C})_{\overline{\zeta}}$ (cf. Section 2.2). This results in $[\mathbb{C} \cdot s(\sigma_1, \alpha_1)] \in \mathcal{H}(C^{\alpha_1})$. In particular, it holds

$$C^{\beta_2} = \mathbb{C} \cdot \psi_{\beta_2}(\overline{\psi_{\alpha_1}^{-1} s(\sigma_1, \alpha_1)}) \oplus \{v \in C^{\beta_2} \,|\, K_f^{(-2)}(v, s(\sigma_1, \alpha_1)) = 0\}, \tag{304}$$

and σ_1 can be chosen with

$$\alpha(\sigma_1) = \alpha_1, \quad \sigma_1 \in C^{\alpha_1} \oplus \bigoplus_{\beta:(*)} C^\beta \oplus \mathbb{C} \cdot \psi_{\beta_2}(\overline{\psi_{\alpha_1}^{-1} s(\sigma_1, \alpha_1)}). \tag{305}$$

Now for all m we have unique characterization (up to rescaling) of σ_1. For $m \not| p$ it is uniquely determined by (303) and for $m|p$ it is uniquely determined by (305). It can be chosen freely with (303) respectively with (305) and $[\mathbb{C} \cdot s(\sigma_1, \alpha_1)] \in \mathcal{H}(C^{\alpha_1})$. The condition $K_f^{(-2)}(\mathcal{L}_0, \mathcal{L}_0) = 0$ on D_{BL} implies that $\mathcal{L}_0 \cap \bigoplus_{\alpha_2 \leq \beta \leq \beta_2} C^\beta$ is uniquely determined by σ_1. So also the whole of $\mathcal{L}_0$ is uniquely determined by σ_1. All in all, D_{BL} is as stated in (300) and (301). $\square$

Theorem 4.4.4. *Consider the normal form in (116) for $W^\sharp_{1,p}$ and the normal forms in Table 7 for the other seven series. Once again it is $\omega_0 := dxdydz$. Define*

$$\begin{aligned}
b_1 &:= s(\omega_0,\alpha_1)(1,0)\in C^{\alpha_1},\\
b_2 &:= s(\omega_0,\beta_1)(1,0)\in C^{\beta_1},\\
b_3 &:= s(y\omega_0,\alpha_2)(1,0)\in C^{\alpha_2},\\
b_4 &:= s(y\omega_0,\beta_2)(1,0)\in C^{\beta_2}.
\end{aligned}$$

If $m|p$, choose $b_5\in C^{\beta_2}$ with $\mathbb{C}\cdot b_5=\{v\in C^{\beta_2}\,|\,K_f^{(-2)}(b_1,v)=0\}$.

(a) *All b_j are nonzero and $K_f^{(-2)}(b_1+b_2,b_3+b_4)=0$. If $m|p$ then it is $C^{\beta_2}=\mathbb{C}\cdot b_4\oplus\mathbb{C}\cdot b_5$.*

(b) *We write $t=(t_1,t_2)$. Then it is*

$$\alpha(s[\omega_0]_0(t)) = \alpha_1, \tag{306}$$
$$s(\omega_0,\alpha_1)(t) = b_1, \tag{307}$$
$$s(\omega_0,\beta)(t) = 0 \quad \text{for } \alpha_1<\beta<\beta_1, \tag{308}$$
$$s(\omega_0,\beta_1)(t) = t_1^{1/m_2}\cdot b_2, \tag{309}$$
$$s(\omega_0,\alpha_2)(t) = \frac{t_2}{t_1}\cdot\frac{-1}{m_2}\cdot t_1^{-1/m_2}\cdot b_3+s(\omega,\alpha_2)(t_1,0), \tag{310}$$
$$s(\omega_0,\beta_2)(t)\ \begin{cases} = s(\omega_0,\beta_2)(t_1,0) & \text{if } m\not|p,\\ \in s(\omega_0,\beta_2)(t_1,0)+\mathbb{C}\cdot b_5 & \text{if } m|p,\end{cases} \tag{311}$$

with

	$s(\omega_0,\alpha_2)(t_1,0)$	$s(\omega_0,\beta_2)(t_1,0)$
$(r_I=2\,\&\,p\geq 2)$ *or* $(r_I=1\,\&\,p\geq 3)$ *or* $U_{1,2}$	0	0
$W^\sharp_{1,1},S^\sharp_{1,1},U_{1,1}$	$c_1\cdot t_1^{2-1/m_2}\cdot b_3$	$c_2\cdot t_1^2\cdot b_4$
$W^\sharp_{1,2},S^\sharp_{1,2},E_{3,1},Z_{1,1},Q_{2,1},W_{1,1},S_{1,1}$	$c_1\cdot t_1^{1-1/m_2}\cdot b_3$	$c_2\cdot t_1\cdot b_4$

(312)

for some values $c_1,c_2\in\mathbb{C}$. Here $\alpha(s[\omega]_0(t))=\min(\alpha\,|\,s(\omega,\alpha)(t)\neq 0)$ is again the order (see Section 2.2). Finally it is

$$\alpha(s[y\omega_0]_0(t) = \alpha_2, \tag{313}$$
$$s(y\omega_0,\alpha_2)(t) = t_1^{-1/m_2}\cdot b_3, \tag{314}$$
$$s(y\omega_0,\beta_2)(t)\ \begin{cases} = b_4 & \text{if } m\not|p \text{ or } t_2=0,\\ \in b_4+\mathbb{C}\cdot b_5 & \text{if } m|p,\end{cases} \tag{315}$$

$$s(\sigma,\beta_2)(t)\begin{cases} =0 & \text{if } m\not|p\\ \in\mathbb{C}\cdot b_5 & \text{if } m|p\end{cases} \tag{316}$$

for $\sigma\in H_0''(f_t)$ with $\alpha(\sigma)>\alpha_2$.

(c) For the five series with $r_I = 2$ (cf. Table 8) and for $b \in \mathbb{Z}_{\geq 0}$ it holds

$$\alpha(s[y^{b+1}\omega_0]_0(t)) = \alpha_2 + \frac{2b}{m_2} = \frac{2b+1}{m_2}, \tag{317}$$

$$s(y^{b+1}\omega_0, \frac{2b+1}{m_2})(t) = t_1^{-(2b+1)/m_2} \cdot s(y^{b+1}\omega_0, \frac{2b+1}{m_2})(1,0). \tag{318}$$

In particular, if $p = mr$ then it is $\frac{2r+1}{m_2} = \frac{1}{m} = \beta_2$, b_5 can be chosen as $b_5 = s(y^{r+1}\omega_0, \beta_2)(1,0)$, and

$$s(y^{r+1}\omega_0, \beta_2)(t) = t_1^{-1/m} \cdot b_5. \tag{319}$$

(d) For the three subseries $W^\sharp_{1,12r}, S^\sharp_{1,10r}, U_{1,9r}$ (i.e. the subseries with $r_I = 1$ and $m|p$), b_5 can be chosen such that b_5 and ω in (322) satisfy

$$\alpha(s[\omega]_0(t)) = \beta_2 = \frac{1}{m}, \tag{320}$$

$$s(\omega, \beta_2 + 1)(t) = t_1^{-1/m} \cdot b_5. \tag{321}$$

(322)

	ω
$W^\sharp_{1,12+24r}, S^\sharp_{1,10+20r}$	$xy^r\omega_0$
$U_{1,9+18r}$	$y^r z\omega_0$
$W^\sharp_{1,24r}, S^\sharp_{1,20r}, U_{1,18r}$	$y^{r+1}\omega_0$

Proof. (a) First of all, we notice that $\nu(\omega_0) - 1 = \alpha_1 < s(f)$ and $\nu(y\omega_0) - 1 = \alpha_2 < s(f)$ (cf. Section A.1). With Corollary A.1.3 this already shows (306) and (307). Hence, it holds $b_1 \neq 0$ and $b_3 \neq 0$. The proof that $b_2 \neq 0$ will be postponed. For $b_4 \neq 0$ we need (308), which will be shown below. With (308) and $K_f^{-2}(H_0''(f_t), H_0''(f_t)) = 0$ we can conclude that

$$0 = K_f^{-2}\left(s[\omega_0]_0(1,0), s[y\omega_0]_0(1,0)\right) = K_f^{-2}(b_1 + b_2, b_3 + b_4). \tag{323}$$

Since it is $K_f^{-2}(b_2, b_3) \neq 0$, equation (323) yields $K_f^{-2}(b_1, b_4) \neq 0$ and $b_4 \neq 0$. Furthermore, this shows $C^{\beta_2} = \mathbb{C} \cdot b_4 \oplus \mathbb{C} \cdot b_5$ if $m|p$.

(b)–(d) We do the proof here only for the series $E_{3,p}$. The other cases are similar[3]. Note that, for the series $W^\sharp_{1,p}$, $S^\sharp_{1,p}$ and $U_{1,p}$ more case discussions are necessary. For $E_{3,p}$ we have two compact faces σ_1 and σ_2 of the Newton polyhedron (cf. [He93, Kapitel 4 d)]). We have corresponding linear forms l_{σ_1} and l_{σ_2} in the sense of Section A.1. Such linear forms are encoded by the triples $(l_{\sigma_j}(x), l_{\sigma_j}(y), l_{\sigma_j}(z))$ with $j = 1, 2$. For $E_{3,p}$ the triples

3 Some helpful data for the other series can be found in [GH18, (9.37)]

are $\frac{1}{18}(6,2,9)$ and $\frac{1}{2(9+p)}(6+p,2,9+p)$. The value $s(f)$ of Corollary A.1.3 is $\frac{4}{9}$. This gives us the following two relations:

$$f = \frac{1}{3}xf_x + \frac{1}{9}yf_y + \frac{1}{2}zf_z - \frac{p}{9}t_1y^{9+p} - \frac{p+1}{9}t_2y^{10+p}, \quad (324)$$
$$f = \frac{6+p}{2(9+p)}xf_x + \frac{2}{2(9+p)}yf_y + \frac{1}{2}zf_z - \frac{p}{2(9+p)}x^3 - \frac{1}{9+p}t_2y^{10+p}. \quad (325)$$

Via these two relations and (25) we obtain two values for $\partial_\tau \tau s[x^ay^b\omega_0]_0(t)$, namely

$$\begin{aligned} &\partial_\tau \tau s[x^ay^b\omega_0]_0(t) \\ = \ & l_{\sigma_1}(a+1,b+1,1)\cdot s[x^ay^b\omega_0]_0(t) && (326)\\ &-\frac{p}{9}t_1\partial_\tau s[x^ay^{b+9+p}\omega_0]_0(t) - \frac{p+1}{9}t_2\partial_\tau s[x^ay^{b+10+p}\omega_0]_0(t), \\ = \ & l_{\sigma_2}(a+1,b+1,1)\cdot s[x^ay^b\omega_0]_0(t) && (327)\\ &-\frac{p}{2(9+p)}\partial_\tau s[x^{a+3}y^b\omega_0]_0(t) \ - \frac{1}{9+p}t_2\partial_\tau s[x^ay^{b+10+p}\omega_0]_0(t). \end{aligned}$$

Now for any β with $\dim C^\beta \neq 0$ this yields

$$\begin{aligned} &(\beta+1-l_{\sigma_1}(a+1,b+1,1))s(x^ay^b\omega_0,\beta)(t) \\ = \ & -\frac{p}{9}t_1\partial_\tau s(x^ay^{b+9+p}\omega_0,\beta+1)(t) \\ & -\frac{p+1}{9}t_2\partial_\tau s(x^ay^{b+10+p}\omega_0,\beta+1)(t), && (328)\\ &(\beta+1-l_{\sigma_2}(a+1,b+1,1))s(x^ay^b\omega_0,\beta)(t) \\ = \ & -\frac{p}{2(9+p)}\partial_\tau s(x^{a+3}y^b\omega_0,\beta+1)(t \\ & -\frac{1}{9+p}t_2\partial_\tau s(x^ay^{b+10+p}\omega_0,\beta+1)(t). && (329) \end{aligned}$$

Furthermore, by Theorem 2.3.2 (c) we can “replace” τ with t_1 and t_2 as follows:

$$\begin{aligned} \partial_{t_1}s[x^ay^b\omega_0]_0(t) &= (-\partial_\tau)s[x^ay^{b+9+p}\omega_0]_0(t), && (330)\\ \partial_{t_2}s[x^ay^b\omega_0]_0(t) &= (-\partial_\tau)s[x^ay^{b+10+p}\omega_0]_0(t) \\ &= \partial_{t_1}s[x^ay^{b+1}\omega_0]_0(t). && (331) \end{aligned}$$

All in all, the equations (328)–(331) give

$$\left(\frac{p}{9}t_1\partial_{t_1} + \frac{p+1}{9}t_2\partial_{t_2} - (\beta+1) + l_{\sigma_1}(a+1,b+1,1)\right)s(x^ay^b\omega_0,\beta)(t) = 0, \quad (332)$$
$$\begin{aligned} &\left(\frac{1}{9+p}t_2\partial_{t_2} - (\beta+1) + l_{\sigma_2}(a+1,b+1,1)\right)s(x^ay^b\omega_0,\beta)(t) \\ &= \frac{p}{2(9+p)}\partial_\tau s(x^{a+3}y^b\omega_0,\beta+1)(t). \quad (333) \end{aligned}$$

Plugging in $t_2 = 0$ in (332) yields

$$s(x^ay^b\omega_0,\beta)(t_1,0) = t_1^{\frac{9}{p}(\beta+1-l_{\sigma_1}(a+1,b+1,1))}\cdot s(x^ay^b\omega_0,\beta)(1,0). \quad (334)$$

The following eight equations are special cases of (334). They are:

$$
\begin{aligned}
s(\omega_0,\alpha_1)(t_1,0) &= b_1, && (335)\\
s(\omega_0,\beta_1)(t_1,0) &= t_1^{1/m_2}\cdot b_2, && (336)\\
s(\omega_0,\alpha_2)(t_1,0) &= t_1^{-1/m_2+1/p}\cdot s(\omega_0,\alpha_2)(1,0), && (337)\\
s(\omega_0,\beta_2)(t_1,0) &= t_1^{1/p}\cdot s(\omega_0,\beta_2)(1,0), && (338)
\end{aligned}
$$

$$
\begin{aligned}
s(y^{b+1}\omega_0,\frac{2b+1}{m_2})(t_1,0) &= t_1^{-(2b+1)/m_2}\cdot s(y^{b+1}\omega_0,\frac{2b+1}{m_2})(1,0), && (339)\\
s(y\omega_0,\alpha_2)(t_1,0) &= t_1^{-\alpha_2}\cdot b_3 = t_1^{-1/m_2}\cdot b_3, && (340)\\
s(y\omega_0,\beta_2)(t_1,0) &= b_4, && (341)\\
s(y^{r+1}\omega_0,\beta_2)(t_1,0) &= t_1^{-1/m_2}\cdot s(y^{r+1}\omega_0,\beta_2)(1,0) \quad \text{if } p=18r. && (342)
\end{aligned}
$$

Now we formulate a claim. It will be useful later on.

Claim: Fix some $b\in\mathbb{Z}_{\geq 0}$. Then the following three statements on the Newton order are claimed to be true.

(i) $\nu(y^{b+1}\omega_0)=\alpha_2+\frac{b}{9+p}=\frac{2b+1}{m_2}$.

(ii) Any $(n+1)$-form $df\wedge d\eta$ which contains $y^{b+1}\omega_0$ as a summand, contains a summand $g\cdot\omega_0$, where g is a monomial (times a nonzero scalar) with $\nu(g\cdot\omega_0)\leq \nu(y^{b+1}\omega_0)$.

(iii) $\overline{\nu}(y^{b+1}\omega_0)=\frac{2b+1}{m_2}$.

Proof of the Claim: Statement (i) is trivial and statement (iii) follows from (i) and (ii). So it is enough to prove (ii).

(ii) The only monomial differential $(n-1)$-forms η such that $df\wedge d\eta$ contains $f_y\cdot y^c\cdot\omega_0$ are $\eta_1=-xy^c dz$ and $\eta_2=y^c z dx$. For them it is

$$
\begin{aligned}
df\wedge d\eta_1 &= f_y\cdot y^c\cdot\omega_0 - f_x\cdot c\cdot xy^{c-1}\cdot\omega_0,\\
df\wedge d\eta_2 &= f_y\cdot y^c\cdot\omega_0 - f_z\cdot c\cdot y^{c-1}z\cdot\omega_0.
\end{aligned}
$$

These $(n+1)$-forms contain $(3-2c)x^2y^{c+2}\omega_0$ respectively $3x^2y^{c+2}\omega_0$, and

$$\nu(x^2y^{c+2}\omega_0)\leq\nu(y^{c+8+p}\omega_0).$$

This finishes the proof of the claim.

With the claim at hand and Theorem A.1.2 it follows

$$
\begin{aligned}
\alpha(s[y^{b+1}\omega_0]_0(t)) &= \frac{2b+1}{m_2}, && (343)\\
s(y^{b+1}\omega_0,\frac{2b+1}{m_2})(t) &\neq 0. && (344)
\end{aligned}
$$

In particular, it holds $b_3 \neq 0$. For $p = 18r$ it is also $s(y^{r+1}\omega_0, \beta_2)(t) \neq 0$. In this case the equality

$$K_f^{(-2)}(s[\omega_0]_0(1,0), s[y^{r+1}\omega_0]_0(1,0)) = 0$$

then gives $K_f^{(-2)}(b_1, s(y^{r+1}\omega_0, \beta_2)(1,0)) = 0$. For that reason we can choose b_5 as $b_5 = s(y^{r+1}\omega_0, \beta_2)(1,0)$ if $p = 18r$.

Finally, the sections $s(y^{b+1}\omega_0, \frac{2b+1}{m_2})(t)$ are independent of t_2 because (331) gives

$$\partial_{t_2} s(y^{b+1}\omega_0, \frac{2b+1}{m_2})(t) = \partial_{t_1} s(y^{b+2}\omega_0, \frac{2b+1}{m_2})(t) = 0.$$

So part (c) is proved, i.e. (314) and (317)–(319) are true.

Another thing that follows from (331) is

$$\begin{array}{rcll} \partial_{t_2} s[\omega_0]_0(t) & = & \partial_{t_1} s[y\omega_0]_0(t), & \\ \text{so } s(\omega_0,\beta)(t) & = & s(\omega_0,\beta)(t_1,0) & \text{for } \alpha_1 \leq \beta < \alpha_2. \end{array} \tag{345}$$

With (335) and (328) and (344) we can conclude

$$s(\omega_0,\beta)(t_1,0) = \begin{cases} b_1 & \text{if } \beta = \alpha_1, \\ \frac{-p}{9(\beta-\alpha_1)} t_1 \partial_\tau s(y^{9+p}\omega_0, \beta+1)(t_1,0) = 0 & \text{if } \alpha_1 < \beta < \beta_1, \\ \frac{-p}{9(\beta_1-\alpha_1)} t_1 \partial_\tau s(y^{9+p}\omega_0, \beta_1+1)(t_1,0) \neq 0 & \text{if } \beta = \beta_1. \end{cases}$$

This gives $b_2 \neq 0$ and, thus, the finishes the proof of part (a). Moreover, via (335) and (336) we obtain (307)–(309).

By part (a) we know already that $b_4 \neq 0$. This now yields (315) and (316). It rests to show (310)–(312). We start with (310) and (311). With (345), (314) and (315) we obtain

$$\begin{array}{rcl} \partial_{t_2} s(\omega_0,\alpha_2)(t) & = & \partial_{t_1} s(y\omega_0,\alpha_2)(t) = \partial_{t_1}(t_1^{-1/m_2}\cdot b_3), \\ \partial_{t_2} s(\omega_0,\beta_2)(t) & = & \partial_{t_1} s(y\omega_0,\beta_2)(t) \begin{cases} = 0 & \text{if } m \not| p, \\ \in \mathbb{C}\cdot b_5 & \text{if } m|p, \end{cases} \end{array}$$

which gives the desired two statements (310) and (311).

For (312) we note that the sections

$$\begin{array}{rrcl} & s(y\omega_0,\alpha_2)(t_1,0) & = & t_1^{-1/m_2}\cdot b_3, \\ & s(y\omega_0,\beta_2)(t_1,0) & = & b_4, \\ \text{and in the case } m|p & s(y^{r+1}\omega_0,\beta_2)(t_1,0) & = & t_1^{-1/m}\cdot b_5 \end{array}$$

are univalued nowhere vanishing sections generating the two bundles $\bigcup_{t_1\in T} C^{\alpha_2}(t_1,0)$ and $\bigcup_{t_1\in T} C^{\beta_2}(t_1,0)$. But also $s(\omega_0,\alpha_2)(t_1,0)$ and $s(\omega_0,\beta_2)(t_1,0)$ are univalued sections in these bundles. In contrast to the sections above they are everywhere vanishing for $p \geq 2$ by (337) and (338). For $p = 1$ they give the statement for $E_{3,1}$ in the last line of (312). This settles the parts (b) and (c) for the series $E_{3,p}$. □

Finally, we are able to formulate the main theorem of this subsection. Besides the Torelli Conjectures, part (e) is the most surprising part of this theorem.

Theorem 4.4.5. *Consider a family of bimodal series singularities.*

(a) The transversal monodromy satisfies $M_T^{m_2} = \mathrm{id}$. Therefore the pull back to T^{cov} with c_T of the family of singularities over T has trivial transversal monodromy. Thus the strong marking $+\mathrm{id}$ for $f_{(1,0)}$ induces a well defined strong marking for each singularity of this family over T^{cov}. This gives a map $T^{cov} \to (M_\mu^{smar})^0$ and a map $T^{cov} \to (M_\mu^{mar})^0$.

(b) The maps from (a) are isomorphisms and $-\mathrm{id} \notin G^{smar}$, where G^{smar} is the group for the singularities of multiplicity ≥ 3, namely the curve singularities $W_{1,p}^\sharp$, $E_{3,p}$, $Z_{1,p}$, $W_{1,p}$ and the surface singularities $S_{1,p}^\sharp$, $U_{1,p}$, $Q_{2,p}$, $S_{1,p}$. In particular, this means Conjecture 3.1.2 is true.

(c) The period map $BL : M_\mu^{mar} \to D_{BL}$ is an embedding, i.e. the strong global Torelli Conjecture 3.2.2 (a) is true.

(d) If $p \not\equiv 0 \mod m$ then $G^{mar} = G_\mathbb{Z}$.

(e) If $p \equiv 0 \mod m$ then $G_\mathbb{Z} \supsetneqq G^{mar}$ with $|G^{mar}| = |\{g \in G_\mathbb{Z} \,|\, g(Ml_{\zeta,1}) = Ml_{\zeta,1}\}| < \infty$ and $|G_\mathbb{Z}| = \infty$. Here M_T has on the 2-dimensional $\mathbb{C}$-vector space Ml_ζ the eigenvalues 1 and $\overline{\zeta}$ and $Ml_{\zeta,1}$ is the 1-dimensional eigenspace of M_T on Ml_ζ with eigenvalue 1. $\Psi(G_\mathbb{Z})$ is an infinite Fuchsian group by Theorem 4.4.1 (c). $\Psi(G^{mar})$ is the finite subgroup of elliptic elements which fix the point $[Ml_{\zeta,1}] \in \mathcal{H}_\zeta$ (cf. Theorem 4.4.1 (c)). Finally, M_μ^{mar} consists of infinitely many copies of T^{cov}.

Proof. (a) Everything is clear except the statements on the transversal monodromy. This will be proved later.

(b) We consider the following maps:

$$\begin{tikzcd} T^{cov} \arrow[r] \arrow[d, "\cong"'] & (M_\mu^{smar})^0 \arrow[d, "BL"] \\ D_{BL}^{sub} \arrow[r, hook] & D_{BL} \end{tikzcd} \tag{346}$$

Since $T^{cov} \hookrightarrow D_{BL}^{sub} \hookrightarrow D_{BL}$ is an embedding, $T^{cov} \to (M_\mu^{smar})^0$ is an embedding as well.

Note that both spaces T^{cov} and $(M_\mu^{smar})^0$ are locally μ-constant strata of semiuniversal unfoldings and, hence, they are smooth of dimension 2. The space D_{BL}^{sub} is almost closed in D_{BL}. The closure of D_{BL}^{sub} consists of itself and the space

$$\{\mathbb{C} \cdot (v_1 + v_4) \,|\, v_1 \text{ and } v_4 \text{ as in (283) resp. (285)}\}. \tag{347}$$

In any case it is $v_2 = 0$. No element $g \in G_\mathbb{Z}$ maps a point of the space in (347) to a point of D_{BL}^{sub}. The covering space T^{cov} contains representatives of any right equivalence class in the μ-homotopy family. Therefore the image of $(M_\mu^{smar})^0$ in D_{BL} cannot be bigger than D_{BL}^{sub}. Thus it is $T^{cov} \cong (M_\mu^{smar})^0$.

In the case of singularities of multiplicity 2, $M_\mu^{smar} \cong M_\mu^{mar}$ holds anyway by Theorem 3.1.4 (c). Then also $(M_\mu^{smar})^0 \cong (M_\mu^{mar})^0$ holds.

Consider the case of singularities of multiplicity ≥ 3. Then $-\mathrm{id} \in G_\mathbb{Z}$ acts nontrivially on M_μ^{smar} by Theorem 3.1.4 (c). Furthermore, it acts trivially on D_{BL}. The map $(M_\mu^{smar})^0 \to D_{BL}$ is an embedding. So $-\mathrm{id} \in G_\mathbb{Z}$ does not act on $(M_\mu^{smar})^0$ and, hence, $-\mathrm{id} \notin G^{smar}$. Finally, this shows that $(M_\mu^{smar})^0 \to (M_\mu^{mar})^0$ is an isomorphism by Theorem 3.1.4 (c).

(c) for $m \not| p$ and (d): By the discussion above we have the map $(M_\mu^{mar})^0 \xrightarrow{\cong} T^{cov} \xrightarrow{\cong} D_{BL}^{sub} \hookrightarrow D_{BL}$. This map is an embedding. Now the fact $G_\mathbb{Z} = G^{mar}$ would imply $M_\mu^{mar} = (M_\mu^{mar})^0$ and, hence, would verify the strong global Torelli Conjecture 3.2.2. So it remains to prove $G_\mathbb{Z} = G^{mar}$.

Let $g_1 \in G_\mathbb{Z}$. This element acts on D_{BL}^{sub}. According to the proof of Theorem 4.4.2 (e) the map

$$(M_\mu^{mar})^0/G^{mar} \to D_{BL}^{sub}/G_\mathbb{Z} \tag{348}$$

is an isomorphism. Thus there is an element $g_2 \in G^{mar}$ which acts in the same way on D_{BL}^{sub} as g_1. We set $g_3 := g_1 \circ g_2^{-1}$. This element acts trivially on D_{BL}^{sub}. It has eigenvalues λ_1, λ_2 and $\overline{\lambda_1}$ on $C^{\alpha_1}, C^{\beta_1}$ and C^{β_2}. Therefore

$$\begin{aligned}
&\mathbb{C}(v_1+v_2+v_4) = \mathbb{C}(\lambda_1 \cdot v_1 + \lambda_2 \cdot v_2 + \overline{\lambda_1} \cdot v_4) \text{ for any } \mathbb{C}(v_1+v_2+v_4) \in D_{BL}^{sub},\\
&\text{thus } \lambda_2\overline{\lambda_1} = 1, \quad \overline{\lambda_1}^2 = \mathrm{id}, \quad \text{so } \lambda_1 = \lambda_2 \in \{\pm 1\},\\
&\text{and } g_3 = \lambda_1 \cdot \mathrm{id} \text{ on } Ml_\zeta \oplus Ml_{e^{2\pi i/m_2}}.
\end{aligned} \tag{349}$$

By Theorem 4.4.1 (b) we know how the group $G_\mathbb{Z}$ looks like. It contains very few automorphisms g_3 with (349). By (121) and Table 8 show that the group $\{g \in G_\mathbb{Z} \,|\, g = \pm\mathrm{id} \text{ on } Ml_\zeta \oplus Ml_{e^{2\pi i/m_2}}\}$ is as follows:

$$\begin{aligned}
&\{\pm\mathrm{id}\} \qquad \text{in the cases } W^\sharp_{1,2q-1}, S^\sharp_{1,2q-1}, U_{1,2q}, E_{3,p}, Z_{1,p}, \\
&\{\pm\mathrm{id}, \pm(\mathrm{id}\,|_{B_1} \times (-M_h^{m_2/2})|_{B_2})\} \quad \text{in the cases } W^\sharp_{1,2q}, S^\sharp_{1,2q}, U_{1,2q-1},\\
&\{\pm\mathrm{id}, \pm((-M_h^{m/2})|_{B_1} \times \mathrm{id}\,|_{B_2})\} \qquad \text{in the cases } Q_{2,p}, W_{1,p}, S_{1,p}.
\end{aligned} \tag{350}$$

Now we can formulate a claim. This claim then shows $g_3 \in G_\mathcal{R}^{mar}$ and $g_1 \in G^{mar}$ and, hence, that $G_\mathbb{Z} = G^{mar}$.

Claim:

$$\{g \in G_\mathbb{Z} \,|\, g = \pm\mathrm{id} \text{ on } Ml_\zeta \oplus Ml_{e^{2\pi i/m_2}}\} = G_\mathcal{R}^{mar}. \tag{351}$$

The inclusion "$\supseteq$" in (351) holds, because any element of $G_\mathcal{R}^{mar} = G_\mathcal{R}^{mar}(f_{(1,0)})$ acts on D_{BL}^{sub} with $BL_T(1,0)$ as fixed point and the proof of Theorem 4.4.2 (e) shows that it acts then trivially on D_{BL}^{sub}.

We know that the group $G_{\mathcal{R}}^{mar}$ contains the two elements $\pm\,\mathrm{id}$. In order to prove equality in (351) for the cases in the second and third line of (350), it is sufficient to show that $G_{\mathcal{R}}^{mar}$ contains more elements than those two. It is equivalent to show that $G_{\mathcal{R}}^{smar}(f)$ for a generic singularity f with multiplicity ≥ 3 contains one other element than $+\,\mathrm{id}$. The following table lists coordinate changes which give such an element.

$$\begin{array}{ll} W^{\sharp}_{1,2q} & (x,y)\mapsto(-x,y) \\ S^{\sharp}_{1,2q} & (x,y,z)\mapsto(-x,y,z) \\ U_{1,2q-1} & (x,y,z)\mapsto(x,y,-z) \\ Q_{2,p} & (x,y,z)\mapsto(x,y,-z) \\ W_{1,p} & (x,y)\mapsto(-x,y) \\ S_{1,p} & (x,y,z)\mapsto(-x,y,z) \end{array} \tag{352}$$

This proves the claim and finishes the proof of part (c) for $m \not| p$ and part (d).

(c) for $m|p$ and (e): First we will show the characterization

$$G^{mar} = \{g \in G_{\mathbb{Z}} \,|\, g(Ml_{\zeta,1}) = Ml_{\zeta,1}\}. \tag{353}$$

We know that $\Psi(M_T)$ is an elliptic element with fixed point $[Ml_{\zeta,1}] \in \mathcal{H}_{\zeta}$ and angle $\frac{2\pi}{m} = \arg(\frac{\zeta}{1})$. Moreover, all elements of G^{mar} (including M_T) act on $\mathcal{H}(C^{\alpha_1})$ as elliptic elements with fixed point $[\mathbb{C}\cdot v_1^0]$, because all elements in G^{mar} act on $(M_\mu^{mar})^0$ and on its image $BL_{T^{cov}}((M_\mu^{mar})^0) \subseteq D_{BL}^{sub}$. Therefore all elements of G^{mar} act on $\mathcal{H}_{\zeta}$ as elliptic elements with fixed point $[Ml_{\zeta,1}]$. This shows "$\subseteq$" in (353).

Now we will show "$\supseteq$" in (353). So let $g_1 \in \{g \in G_{\mathbb{Z}} \,|\, g(Ml_{\zeta,1}) = Ml_{\zeta,1}\}$. It has an eigenvalue λ_1 on $Ml_{\zeta,1}$ and an eigenvalue λ_2 on the other eigenspace within Ml_{ζ} (which is the h_{ζ}-orthogonal subspace of Ml_{ζ}). By (292) it is λ_1 and $\lambda_2 \in Eiw(\zeta)$. Thus $\Psi(g_1)$ is an elliptic element with fixed point $[Ml_{\zeta,1}] \in \mathcal{H}_{\zeta}$ and angle $\arg\frac{\lambda_2}{\lambda_1}$.

In all cases except possibly $U_{1,9r}$, the product $g_2 = g_1 \circ M_T^k$ for a suitable $k \in \mathbb{Z}$ acts trivially on $\mathcal{H}_{\zeta}$. In the cases $U_{1,9r}$, the product $g_2 = g_1 \circ g_3^k$ for $g_3 \in G^{mar}$ as in Theorem 4.4.2 (d) acts trivially on $\mathcal{H}_{\zeta}$ as well.

Now (124) from Theorem 4.4.1 (c) applies to g_2 and shows $g_2 \in \{\pm M_h^k \,|\, k \in \mathbb{Z}\}$. Hence it holds $g_2 \in G^{mar}$ and $g_1 \in G^{mar}$. This shows "$\supseteq$" in (353) and in total we have equality in (353).

Hence, $\Psi(G^{mar})$ and G^{mar} are finite. Moreover, by Theorem 4.4.1 (c) the groups $\Psi(G_{\mathbb{Z}})$ and $G_{\mathbb{Z}}$ are infinite. Therefore we have the surprising fact that $G_{\mathbb{Z}} \supsetneqq G^{mar}$. Furthermore, now by Theorem 3.1.4 (a) M_μ^{mar} must consist of infinitely many copies of $(M_\mu^{mar})^0$.

Note that, if two different copies would have intersecting images in D_{BL} under the period map BL, the images would coincide and there would be a copy different from $(M_\mu^{mar})^0$ with the same image in D_{BL} as $(M_\mu^{mar})^0$. An element $g_3 \in G_{\mathbb{Z}}$ which maps $(M_\mu^{mar})^0$ to this copy would be in $\{g \in G_{\mathbb{Z}} \,|\, g(Ml_{\zeta,1}) = Ml_{\zeta,1}\} \setminus G^{mar} = \emptyset$. This is a contradiction. All in all, then $BL : M_\mu^{mar} \to D_{BL}^{sub}$ is an embedding. The proof of the remaining statement on M_T in part (e) will be postponend to the very end of this proof.

Now we want to proof the properties of the transversal monodromy M_T that we omitted above. According to Theorem 4.4.4, the following sections in the bundles $\bigcup_{t_1\in T} C^{\beta}(t_1,0)$ for β as in (354) are univalued nowhere vanishing sections that generate the respective bundle (in the case $\beta=\alpha_1$ only if $m \not| p$).

section	b_1	$t_1^{1/m_2}\cdot b_2$	$t_1^{-1/m_2}\cdot b_3$	b_4	$t_1^{-1/m}\cdot b_5$ if $m\vert p$
β	α_1	β_1	α_2	β_2	β_2
eigenvalue of M_T on $\mathbb{C}\cdot b_j$	1	$e^{-2\pi i/m_2}$	$e^{2\pi i/m_2}$	1	$e^{2\pi i/m}$

(354)

Therefore b_1 and b_4 are univalued, and b_2 and b_3 (and b_5 if $m|p$) are multivalued flat sections with eigenvalues of M_T as specified above. Hence, we can already see that $M_T^{m_2}$ is the identity on all the spaces C^{α_1}, C^{β_1}, C^{α_2}, C^{β_2}, Ml_ζ and $Ml_{e^{2\pi i/m_2}}$. We will show that it is the identity on all of Ml.

Firstly, we consider the case $m\not| p$. Then by (350) $M_T^{m_2}$ is in

$$\begin{aligned}
&\{\mathrm{id}\} \qquad \text{in the cases } W^\sharp_{1,2q-1}, S^\sharp_{1,2q-1}, U_{1,2q}, E_{3,p}, Z_{1,p},\\
&\{\mathrm{id}, \mathrm{id}|_{B_1}\times(-M_h^{m_2/2})|_{B_2}\} \quad \text{in the cases } W^\sharp_{1,2q}, S^\sharp_{1,2q}, U_{1,2q-1},\\
&\{\mathrm{id}, (-M_h^{m/2})|_{B_1}\times \mathrm{id}|_{B_2}\} \qquad \text{in the cases } Q_{2,p}, W_{1,p}, S_{1,p}.
\end{aligned} \tag{355}$$

On the other hand, in the cases in the second and third line of (355), $m_2=m+r_Ip$ is even, and M_T itself is in $G_{\mathbb{Z}}$ which is given by (121) in Theorem 4.4.1. So it holds $M_T^{m_2}=\mathrm{id}$ also in the second and third line of (355).

Secondly, we consider the case $m|p$, i.e. $p=mr$. By Theorem 4.4.1 and more precisely by (124) $M_T^{m_2}=\varepsilon\cdot M_h^k$ for some $\varepsilon\in\{\pm1\}$ and some $k\in\mathbb{Z}$. Then $\varepsilon\cdot\zeta^k=1$ and $\varepsilon\cdot e^{2\pi ik/m_2}=1$. If $\varepsilon=1$, then the two conditions boil down to $m|k$ and $m_2|k$. Thus is is $m_2|k$ and $M_T^{m_2}=\mathrm{id}$. If $\varepsilon=-1$, we will arrive at a contradiction as specified below. So the two conditions require that m and m_2 are both even.

For each eigenvalue λ of M_h on Ml with $\dim Ml_\lambda=1$, an eigenvector in $Ml_{\lambda,\mathbb{Z}[\lambda]}$ exists. Then M_T has an eigenvalue in $Eiw(\lambda)$ on this eigenvector, and $M_T^{m_2}$ has the eigenvalue 1 on this eigenvector. Here m_2 even is used. Therefore $M_T^{m_2}=\mathrm{id}$ on Ml_λ for each

$$\lambda\in\{\zeta, e^{2\pi i/m_2}\}\cup\{\widetilde{\lambda}\,|\,\dim Ml_{\widetilde{\lambda}}=1\}.$$

Comparing this with Table 8 we see that some $k\in\mathbb{Z}$ with $-\lambda^k=1$ for all these λ exists. This gives a contradiction. The case $\varepsilon=-1$ is impossible. Thus $M_T^{m_2}=\mathrm{id}$ is proved in all cases (cf. Theorem 4.4.5 (a)).

Finally, we will prove that M_T has the eigenvalues 1 and $\overline{\zeta}$ on Ml_ζ and on C^{α_1} to finish part the proof of part (e). A priori by (354) M_T has on C^{β_2} and on $H^n(X_\infty,\mathbb{C})_{e^{-2\pi i\beta_2}}=H^n(X_\infty,\mathbb{C})_{\overline{\zeta}}$ the eigenvalues 1 and ζ. As Ml_ζ is dual to $H^n(X_\infty,\mathbb{C})_{\overline{\zeta}}$ and $H^n(X_\infty,\mathbb{C})_\zeta$ is complex conjugate to $H^n(X_\infty,\mathbb{C})_{\overline{\zeta}}$, M_t has on Ml_ζ, $H^n(X_\infty,\mathbb{C})_\zeta=H^n(X_\infty,\mathbb{C})_{e^{-2\pi i\alpha_1}}$ and C^{α_1} the eigenvalues 1 and $\overline{\zeta}$.

□

Remark 4.4.6. *Theorem 4.4.5 shows that the Torelli Conjecture 3.2.2 (b) holds also for the three subseries $Z_{1,14k}$, $S_{1,10k}$ and $S^{\sharp}_{1,10k}$ with $k \in \mathbb{N}$. Hence the limitation in [He95, Theorem on p. 360] can be dropped.*

With Theorem 4.4.4 and the latest results on M_T we can now prove the remaining parts of Theorem 4.4.2.

Proof of Theorem 4.4.2 (a), (b) and (d): (a) This follows immediately from the respective definition of D_{BL}^{sub} (see (283) or (285)) together with Lemma 4.4.3 (c).

(b) All of this follows by carefully looking at the results and the proof of Theorem 4.4.4. Note that the part with the eigenvalues of M_T on C^{α_1} was already shown above in the proof of Theorem 4.4.5. Here let us consider suitable $v_1^0 = b_1$, $v_2^0 = b_2$ and $v_4^0 \in \mathbb{C}^* \cdot b_4$. Then the section in the brackets on the right hand side of (286) is

$$s[\omega_0]_0(t) + \left(\frac{1}{m} \frac{t_2}{t_1} + \left\{ \begin{matrix} 0 \\ -c_1 \cdot t_1^2 \\ -c_1 \cdot t_1 \end{matrix} \right\} \right) \cdot s[y\omega_0]_0(t) \mod \bigoplus_{\alpha_2 < \beta < \beta_2} C^{\beta} \oplus \mathbb{C} \cdot b_5 \oplus V^{>\beta_2}. \quad (356)$$

Here we have three different cases in $\{\ldots\}$. They correspond to the three lines in (312). Note that, the linear combination is chosen such that it has no part in C^{α_2}. This section together with the fact $K_f^{(-2)}(H_0''(f_t), H_0''(f_t)) = 0$ determine $H_0''(f_t)$. The remaining parts of part (b) follow by the values in (354). In fact, M_T has on $v_1^0 = b_4$ the eigenvalue 1.

(c) We have to consider the subseries of type $U_{1,9r}$. Here we will treat the cases $U_{1,9+18r}$ and $U_{1,18r}$ separately. Let

$$\varphi : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0), \quad (x, y, z) \mapsto (x, y, -z). \quad (357)$$

be a coordinate change.

Case $U_{1,9+18r}$: It is $\varphi \in G_{\mathcal{R}}^{smar,gen} \subseteq G^{smar}$ and

$$\varphi^*(\omega_0) = -\omega_0, \quad \varphi^*(y^r z\omega_0) = y^r z\omega_0. \quad (358)$$

Now compare (307) and (322). The coordinate change φ induces an automorphism $(\varphi)_{coh}$ on C^{α_1} and C^{β_2} with

$$(\varphi)_{coh}(b_1) = -b_1, \quad (\varphi)_{coh}(b_4) = -b_4, \quad (\varphi)_{coh}(b_5) = b_5. \quad (359)$$

Thus $g_3 = -M_T \circ (\varphi)_{hom} \in G^{mar}$ is the desired element.

Case $U_{1,18r}$: By (322) and (354) we obtain (cf. (358)) the identities

$$\varphi^*(\omega_0) = -\omega_0, \quad , \varphi^*(y^{r+1}\omega_0) = -y^{r+1}\omega_0. \quad (360)$$

Now by (291) $(\varphi)_{coh}$ is an isomorphism

$$H_0''(f_{(t_1,0)}) \to H_0''(f_{(-t_1,0)}), \quad C^{\beta_2}(t_1,0) \to C^{\beta_2}(-t_1,0).$$

The composition

$$(-\,\mathrm{id}) \circ (\text{math. pos. flat shift from } C^{\beta_2}(-t_1,0) \text{ to } C^{\beta_2}(t_1,0)) \circ (\varphi)_{coh}$$

acts on $C^{\beta_2}(t_1,0)$ and has because of (345) the eigenvectors b_4 and b_5 with the eigenvalues 1 and $e^{\pi i/9}$. The situation looks as follows:

$$\begin{array}{ccccc}
b_4 & t_1^{-1/9} b_5 & & C^{\beta_2}(t_1,0) \\
\downarrow & \downarrow & (\varphi)_{coh} & \downarrow \\
-b_4 & -(e^{-\pi i} t_1)^{-1/9} b_5 & & C^{\beta_2}(-t_1,0) \\
\downarrow & \downarrow & \text{shift} & \downarrow \\
-b_4 & -e^{\pi i/9} t_1^{-1/9} b_5 & & C^{\beta_2}(t_1,0) \\
\downarrow & \downarrow & -\,\mathrm{id} & \downarrow \\
b_4 & e^{\pi i/9} t_1^{-1/9} b_5 & & C^{\beta_2}(t_1,0)
\end{array}$$

The corresponding composition

$$(-\,\mathrm{id}) \circ (\text{math. pos. flat shift from } Ml(f_{(-t_1,0)}) \text{ to } Ml(f_{(t_1,0)})) \circ (\varphi)_{hom}$$

is in G^{mar} and can be chosen as the desired element g_3. □

4.5 QUADRANGLE SINGULARITIES

In this section we consider the singularities of types $E_{3,0}$, $Z_{1,0}$, $Q_{2,0}$, $U_{1,0}$, $W_{1,0}$ and $S_{1,0}$ from [AGV85, Section 15.1.2], i.e. the singularities with Kodaira types I_0^* $(2,-,-,-)$. They are commonly referred to as quadrangle singularities, because they correspond to certain quadrangles in the hyperbolic plane just as the simple elliptic singularities correspond to triangles. Note that some authors also use the term quadrilateral singularities instead of quadrangle singularities (e.g. see [Ur93]).
The names of the types $E_{3,0}$, $Z_{1,0}$, $Q_{2,0}$, $U_{1,0}$, $W_{1,0}$ and $S_{1,0}$ are not given accidentally. In fact, the quadrangle singularities can be considered as the special 0-th members of the eight bimodal series from Section 4.4, with the two series $W_{1,p}$ and $W_{1,p}^\sharp$ for $W_{1,0}$ and the two series $S_{1,p}$ and $S_{1,p}^\sharp$ for $S_{1,0}$.
Here we work with different normal forms than the ones in [AGV85, Section 15.1.2]. They are given in Table 9 with $(t_1,t_2) \in T^{(5)} := (\mathbb{C} \setminus \{0,1\}) \times \mathbb{C}$ and originate from [GH18, Section 10].

Those normal forms $f_{(t_1,t_2)}$ contain representatives of each right equivalence class. Additionally Table 10 lists weights (w_x, w_y, w_z) such that $f_{(t_1,0)}$ with the normal form as above

Type	normal form	μ
$E_{3,0}$	$x(x-y^3)(x-t_1y^3)+t_2x^2y^4+z^2$	16
$Z_{1,0}$	$xy(x-y^2)(x-t_1y^2)+t_2x^2y^4+z^2$	15
$Q_{2,0}$	$x(x-y^2)(x-t_1y^2)+yz^2+t_2xz^2$	14
$U_{1,0}$	$xz(x-z)+y^3(x-t_1z)+t_2y^4z$	14
$W_{1,0}$	$x^4+(4t_1-2)x^2y^3+y^6+t_2x^2y^4+z^2$	15
$S_{1,0}$	$x^2z+y^3z+yz^2+t_1x^2y^2+t_2x^2y^3$	14

Table 9: Facts about the quadrangle singularities

is quasihomogeneous of weighted degree 1 and two numbers m_0 and m_∞. In this sense the quadrangle singularities have semiquasihomogeneous normal forms. We set $m_1 := m_0$. Note that it holds $w_y = \frac{2}{m} < w_x \leq w_z$.

	(w_x, w_y, w_z)	m_0	m_∞
$W_{1,0}$	$(\frac{1}{4}, \frac{1}{6}, \frac{1}{2})$	12	6
$S_{1,0}$	$(\frac{3}{10}, \frac{2}{10}, \frac{4}{10})$	10	5
$U_{1,0}$	$(\frac{1}{3}, \frac{2}{9}, \frac{1}{3})$	9	9
$E_{3,0}$	$(\frac{1}{3}, \frac{1}{9}, \frac{1}{2})$	9	9
$Z_{1,0}$	$(\frac{2}{7}, \frac{1}{7}, \frac{1}{2})$	7	7
$Q_{2,0}$	$(\frac{1}{3}, \frac{1}{6}, \frac{5}{12})$	6	6

Table 10: Facts about the quadrangle singularities, part 2

Besides the parameter space $T^{(5)}$ we need seven more spaces $T^{(1)}$, $T^{(2)}$, $T^{(3)}$, $T^{(4)}$, $T^{(6)}$, $T^{(7)}$ and $T^{(8)}$ here. The situation is summarized in (361) and (362), where $c^{(2)}$ and $c^{(6)}$ are branched and $c^{(1)}$ and $c^{(5)}$ unbranched coverings.

$$\begin{tikzcd} T^{(3)} \arrow[d, "c^{(1)}"'] & \subseteq & T^{(4)} := \mathbb{H} \arrow[d, "c^{(2)}"] \\ T^{(1)} := \mathbb{C} \setminus \{0,1\} & \subseteq & T^{(2)} := P^1\mathbb{C} \end{tikzcd} \tag{361}$$

$$\begin{tikzcd} T^{(7)} := T^{(3)} \times \mathbb{C} \arrow[d, "c^{(5)}:=c^{(1)}\times\mathrm{id}"'] & \subseteq & T^{(8)} := T^{(4)} \times \mathbb{C} \arrow[d, "c^{(6)}:=c^{(2)}\times\mathrm{id}"] \\ T^{(5)} = T^{(1)} \times \mathbb{C} & \subseteq & T^{(6)} := T^{(2)} \times \mathbb{C} \end{tikzcd} \tag{362}$$

To define $c^{(2)}$, $c^{(2)}$ and $T^{(3)}$ in the situation above, we have to consider a triangle group $\Gamma \subseteq \mathrm{PGL}(2;\mathbb{R})$ of type $(\frac{1}{m_0}, \frac{1}{m_1}, \frac{1}{m_\infty})$. The quotient $\mathbb{H}/\Gamma$ is then an orbifold with three orbifold points of orders m_0, m_1 and m_∞. They are the images of the elliptic fixed points of Γ on $T^{(4)}$ of orders m_0, m_1 and m_∞. It is $T^{(4)}/\Gamma \cong P^1\mathbb{C}$ in the sense of manifolds. Now we choose coordinates on $T^{(4)}/\Gamma$ such that 0 and 1 are orbifold points of order $m_0 = m_1$ and ∞ is an orbifold point of order m_∞. Then we denote by

$$c^{(2)} : T^{(4)} \to T^{(2)} \tag{363}$$

the quotient map. It is a branched covering. Finally, we define $T^{(3)}$ and $c^{(1)}$ as

$$\begin{aligned} T^{(3)} &:= T^{(4)} \setminus (c^{(2)})^{-1}(\{0,1,\infty\}), \\ \text{and} \quad c^{(1)} &:= c^{(2)}|_{T^{(4)}} : T^{(3)} \to T^{(1)}. \end{aligned} \tag{364}$$

Note that the map $c^{(1)}$ is again a covering.

We denote by G_3 and $G_2 \subseteq G_3$ the following groups of automorphisms of $T^{(2)} = P^1\mathbb{C}$:

$$\begin{aligned} G_3 &:= \{t_1 \mapsto t_1, 1-t_1, \tfrac{1}{t_1}, \tfrac{t_1}{t_1-1}, \tfrac{1}{1-t_1}, \tfrac{t_1-1}{t_1}\} &&\cong \mathrm{Sym}\,(3) \text{ as a group,} \\ G_2 &:= \{t_1 \mapsto t_1, 1-t_1\} &&\cong \mathrm{Sym}\,(2) \text{ as a group.} \end{aligned} \tag{365}$$

They also act on $T^{(1)} = \mathbb{C} \setminus \{0,1\}$ and give symmetries on the quasihomogeneous part of the parameter space $T^{(5)}$ of the normal forms. This will be made explicit in the next proposition.

Proposition 4.5.1. *Consider a bimodal family of quadrangle surface singularities as in Table 9. A function*

$$\begin{aligned} \kappa : G_2 \times T^{(1)} \to \mathbb{C}^* \qquad &\textit{for } W_{1,0}, S_{1,0}, \\ \kappa : G_3 \times T^{(1)} \to \mathbb{C}^* \qquad &\textit{for } U_{1,0}, E_{3,0}, Z_{1,0}, Q_{2,0}, \end{aligned} \tag{366}$$

with the following properties exists.

$$f_{(t_1,t_2)} \sim_{\mathcal{R}} f_{(\widetilde{t_1},\widetilde{t_2})} \iff \exists\, g \in \begin{cases} G_2 & \textit{for } W_{1,0}, S_{1,0}, \\ G_3 & \textit{for } U_{1,0}, E_{3,0}, Z_{1,0}, Q_{2,0}, \end{cases}$$

$$\textit{with } \widetilde{t_1} = g(t_1), \widetilde{t_2}^{m_\infty} = \kappa(g,t_1)\cdot t_2^{m_\infty}, \tag{367}$$

$$\kappa(\mathrm{id}, t_1) = 1, \tag{368}$$

$$\kappa(g_2 g_1, t_1) = \kappa(g_1, t_1)\cdot \kappa(g_2, g_1(t_1)). \tag{369}$$

The values $\kappa(g,t_1)$ for generators g of the group are listed in (370).

(370)

	$W_{1,0}$	$S_{1,0}$	$U_{1,0}$	$E_{3,0}$	$Z_{1,0}$	$Q_{2,0}$
$t_1 \mapsto 1-t_1$	1	-1	1	$\left(\frac{1-t_1}{t_1}\right)^{18}$	$\left(\frac{1-t_1}{t_1}\right)^{14}$	-1
$t_1 \mapsto t_1^{-1}$	–	–	$-t_1^{-3}$	t_1^{-12}	t_1^{-10}	t_1^{3}

Proof. The equations (368)–(370) are consistent (to check this is nontrivial only for $E_{3,0}$ and $Z_{1,0}$) and define a unique function κ as in (366).

We will show now that this function satisfies “$\Leftarrow$” in (367). We postpone the proof of “$\Rightarrow$” in (367) to the very end of this section.

"$\Leftarrow$": The equality

$$f_{(t_1,t_2)}(x\cdot e^{2\pi i w_x}, y\cdot e^{2\pi i w_y}, z\cdot e^{2\pi i w_z}) = f_{(t_1,t_2\cdot e^{2\pi i 2/m})} \tag{371}$$

gives "$\Leftarrow$" in (367) for $g = \mathrm{id}$ and $\kappa(\mathrm{id}, t_1) = 1$ (for $U_{1,0}$ $m = m_\infty = 9$, in the other cases $m_\infty = \frac{m}{2}$). Now we list coordinate changes $(x,y,z) \mapsto \varphi^{(1)}(x,y,z)$ and $(x,y,z) \mapsto \varphi^{(2)}(x,y,z)$ with

$$\begin{aligned} f_{(t_1,t_2)}(\varphi^{(1)}(x,y,z)) &= f_{(1-t_1,0)} + t_2\cdot p^{(1)}(t_1,x,y,z) \\ &\quad \text{for all 6 cases,} \qquad (372) \\ f_{(t_1,t_2)}(\varphi^{(2)}(x,y,z)) &= f_{(t_1^{-1},0)}(x,y,z) + t_2\cdot p^{(2)}(t_1,x,y,z) \\ &\quad \text{for } U_{1,0}, E_{3,0}, Z_{1,0}, Q_{2,0} \qquad (373) \end{aligned}$$

for certain quasihomogeneous polynomials $p^{(1)}$ and $p^{(2)}$ in the three variables x, y, z with $\deg_w p^{(1)} = \deg_w p^{(2)} = 1 + \frac{2}{m}$.

	$\varphi^{(1)}(x,y,z)$	$\varphi^{(2)}(x,y,z)$
$W_{1,0}$	$(x,-y,z)$	–
$S_{1,0}$	$(ix, y, -z-y^2)$	–
$U_{1,0}$	$(-x+z, -y, z)$	$(-z, t_1^{-1/3}y, -x)$
$E_{3,0}$	$(x-y^3, -y, z)$	$(x, t_1^{-1/3}y, z)$
$Z_{1,0}$	$(e^{-2\pi i/14}(x-y^2), i\cdot e^{-2\pi i/28}y, z)$	$(t_1^{1/7}x, t_1^{-3/7}y, z)$
$Q_{2,0}$	$(x-y^2, iy, e^{-2\pi i/8}z)$	$(x, t_1^{-1/2}y, t_1^{1/4}z)$

(374)

One can calculate $p^{(1)}$ and $p^{(2)}$ easily. We set $p_{>1} := \frac{\partial f_{(t_1,t_2)}}{\partial t_2}$. The proof of [AGV85, 12.6 Lemma] implies here

$$\begin{aligned} & f_{(\widetilde{t_1},0)} + t_2\cdot\widetilde{p} \sim_{\mathcal{R}} f_{(\widetilde{t_1},\widetilde{t_2})} \\ & \text{where } t_2\cdot\widetilde{p} \equiv \widetilde{t_2}\cdot p_{>1} \mod (\text{Jacobi ideal of } f_{(\widetilde{t_1},0)}). \end{aligned} \tag{375}$$

Here $\widetilde{p}$ is a monomial of weighted degree $1 + \frac{2}{m}$. All such monomials are given in (376).

	$W_{1,0}$	$S_{1,0}$	$U_{1,0}$	$E_{3,0}$	$Z_{1,0}$	$Q_{2,0}$
$\widetilde{p}$	x^2y^4	x^2yz	x^2yz	x^3y	x^3y^2	x^2y^3
	y^7	y^4z	xyz^2	xy^7	xy^6	x^3y
	x^2yz	y^2z^2	xy^4	y^{10}	y^8	xy^5
	y^4z	y^6	x^3y	yz^2	x^4	y^7
	yz^2	x^4	yz^3		yz^2	y^2z^2
		z^3				

(376)

Now with (376) we find $\widetilde{t_2}$ with (375) for $\widetilde{p} = p^{(1)}$ and for $\widetilde{p} = p^{(2)}$. Then we can easily verify (370). $\square$

4.5.1 *The group $G_{\mathbb{Z}}$ for quadrangle singularities*

The Coxeter-Dynkin diagrams of the quadrangle singularities can be obtained from the ones in Subsection 4.4.1 (so from the ones in [Eb81, Tabelle 6 + Abbildung 16]). Note that the diagrams for $W^\sharp_{1,p}$ and $W_{1,p}$ specialize both to the same diagram for $W_{1,0}$. However only the description of the action of the monodromy on the distinguished basis for $W^\sharp_{1,p}$ in Subsection 4.4.1 specializes to $W_{1,0}$. The description of the monodromy action for $W_{1,p}$ does not, because in this case δ_2 maps to $2\delta_1+2\delta_2+\delta_8+\delta_{12}+\delta_{16}$ under M_h and δ_{16} does not exist for $W_{1,0}$. Hence, we work with the specialization to $p=0$ of the formulas for $W^\sharp_{1,p}$ in Subsection 4.4.1.
The same argument applies to $S_{1,0}$, too. Thus we work with the specialization to $p=0$ of the formulas for $S^\sharp_{1,p}$ in Subsection 4.4.1.
In particular, Table 11 below specializes Table 8 to the case $p=0$.

Type	p_1	p_2	p_3	m	r_I
$E_{3,0}$	$\Phi_{18}\Phi_2$	$\Phi_{18}\Phi_6\Phi_2$	-	18	2
$Z_{1,0}$	$\Phi_{14}\Phi_2$	$\Phi_{14}\Phi_2$	Φ_2	14	2
$Q_{2,0}$	$\Phi_{12}\Phi_4\Phi_3$	$\Phi_{12}\Phi_4$	-	12	2
$U_{1,0}$	Φ_9	$\Phi_9\Phi_3$	-	9	1
$W_{1,0}$	Φ_{12}	$\Phi_{12}\Phi_6\Phi_4\Phi_3\Phi_2$	-	12	1
$S_{1,0}$	$\Phi_{10}\Phi_2$	$\Phi_{10}\Phi_5\Phi_2$	-	10	1

Table 11: Characteristic polynomials of the quadrangle singularities

Theorem 4.5.2. *For any surface singularity f in any of the six families of quadrangle singularities, the following holds.*

(a) For all families except $Z_{1,0}$, there are Orlik blocks $B_1, B_2 \subseteq Ml(f)$, and for $Z_{1,0}$, there are Orlik blocks $B_1, B_2, B_3 \subseteq Ml(f)$ with the following properties. The characteristic polynomial p_{B_j} of the monodromy on B_j is b_j. The sum $\sum_{j\geq 1} B_j$ is a direct sum $\bigoplus_{j\geq 1} B_j$, and it is a sublattice of $Ml(f)$ of full rank μ and of index r_I. Define

$$\widetilde{B_1} := \begin{cases} B_1 & \text{for all cases except } Z_{1,0}, \\ B_1\oplus B_3 & \text{for } Z_{1,0}. \end{cases} \tag{377}$$

Then it holds

$$L(\widetilde{B_1}, B_2) \;=\; 0 = L(B_2, \widetilde{B_1}), \tag{378}$$

$$g \in G_{\mathbb{Z}} \text{ with } g((B_1)_{\Phi_m}) = (B_1)_{\Phi_m} \;\Rightarrow\; g(B_j) = B_j \text{ for } j \geq 1. \tag{379}$$

(b) The eigenspace $Ml(f)_\zeta \subseteq Ml(f)_\mathbb{C}$ is 2-dimensional. The Hermitian form h_ζ on it from Lemma A.2.5 (a) with $h_\zeta(a,b) := \sqrt{-\zeta}\cdot L(a,\overline{b})$ for $a,b\in Ml(f)_\zeta$ is nondegenerate and indefinite, so $\mathrm{Proj}\,(Ml(f)_\zeta)\cong P^1\mathbb{C}$ *contains the half-plane*

$$\mathcal{H}_\zeta := \{\mathbb{C}\cdot a \,|\, a\in Ml(f)_\zeta \text{ with } h_\zeta(a,a)<0\}\subseteq \mathrm{Proj}(Ml(f)_\zeta). \tag{380}$$

Therefore the group $\mathrm{Aut}(Ml(f)_\zeta,h_\zeta)/S^1\cdot\mathrm{id}$ *is isomorphic to* $\mathrm{PSL}(2;\mathbb{R})$*. The homomorphism*

$$\Psi: G_\mathbb{Z}\to \mathrm{Aut}(Ml(f)_\zeta,h_\zeta)/S^1\cdot\mathrm{id},\quad g\mapsto g|_{Ml(f)_\zeta} \mod S^1\cdot\mathrm{id}, \tag{381}$$

is well-defined. The image of the homomorphism $\Psi(G_\mathbb{Z})$ is an infinite Fuchsian group acting on the half-plane $\mathcal{H}_\zeta$. It is a triangle group of the same type as in Theorem A.3.6, i.e. of the types:

$W_{1,0}$	$S_{1,0}$	$E_{3,0}$ & $U_{1,0}$	$Z_{1,0}$	$Q_{2,0}$
$(2,12,12)$	$(2,10,10)$	$(2,3,18)$	$(2,3,14)$	$(2,3,12)$

(382)

The kernel of the homomorphism is

$$\ker\Psi = \{\pm M_h^k\,|\,k\in\mathbb{Z}\}. \tag{383}$$

Proof. (a) We define the Orlik blocks B_1 and B_2 (and B_3 for $Z_{1,0}$) as in the proof of Theorem 4.4.1. Of course now it is $p=0$. By the arguments of the proof of Theorem 4.4.1, the sum $\sum_{j\geq 1}B_j$ is again a direct sum $\bigoplus_{j\geq 1}B_j$. Moreover it is again a sublattice of $Ml(f)$ of full rank μ and index r_I and equation (378) holds.

So for part (a), it remains to show (379). In the cases $W_{1,0}$ and $U_{1,0}$ this is trivial as $r_I=1$ and $b_1=\Phi_m$ and B_1 and B_2 are L-orthogonal. In the other cases, i.e. $S_{1,0}$, $E_{3,0}$, $Z_{1,0}$ and $Q_{2,0}$, the proof works along the same lines as the proof of (120) for $S_{1,10}$. This will be made explicit now. We treat $S_{1,0}$, $E_{3,0}$ and $Z_{1,0}$ first and then we consider $Q_{2,0}$.

Note that many formulas in the proof of part (a) of Theorem 4.4.1 specialize to the cases $S_{1,0}$, $E_{3,0}$ and $Z_{1,0}$. Most notably, (127)–(129), (160)–(171), (173), (178) and (180) translate naturally. The quadratic forms in (171) give the following sets of "small" vectors (cf. (172) and (174)):

$$\begin{aligned}
\{a\in Ml(f)_{\Phi_2}\,|\,L(a,a)=5\} &= \{\pm\gamma_1,\pm\gamma_2\} \quad \text{for } S_{1,0}, \\
\{a\in Ml(f)_{\Phi_2}\,|\,L(a,a)=6\} &= \{\pm\gamma_1,\pm\widetilde{\gamma_2},\pm(\widetilde{\gamma_2}-\gamma_1)\} \quad \text{for } E_{3,0}, \\
\{a\in Ml(f)_{\Phi_2}\,|\,L(a,a)=5\} &= \{\pm(\gamma_1-3\gamma_2),\pm\widetilde{\gamma_2},\pm(\widetilde{\gamma_2}-\gamma_2)\} \quad \text{for } Z_{1,0}.
\end{aligned} \tag{384}$$

In each of the cases the first element (up to sign) of each of these three sets generates $(B_1)_{\Phi_2}$. We claim that $(B_1)_{\Phi_m}$ and this first element satisfy the following special relationship. For a in any of these three sets define

$$r(a) := [((B_1)_{\Phi_m}+\mathbb{Z}\cdot a)_\mathbb{Q}\cap Ml(f) : ((B_1)_{\Phi_m}+\mathbb{Z}\cdot a)]\in\mathbb{Z}_{\geq 1}. \tag{385}$$

Then we claim:

$$\begin{array}{cc|cc|cc|cc}
 & & S_{1,0} & & E_{3,0} & & Z_{1,0} & \\ \hline
a & r(a) & \pm\gamma_1 & 5 & \pm\gamma_1 & 3 & \pm(\gamma_1-2\gamma_3) & 7 \\
a & r(a) & \pm\gamma_2 & 1 & \pm\widetilde{\gamma_2}, \pm(\widetilde{\gamma_2}-\gamma_1) & 1 & \pm\widetilde{\gamma_2}, \pm(\widetilde{\gamma_2}-\gamma_2) & 1
\end{array} \tag{386}$$

The proof is the same as the proof of (278) for $S_{1,10}$. Here we use that for any unitary polynomial $p(t) \in \mathbb{Z}[t]$ it holds

$$\left[\bigoplus_{j=0}^{\deg p} \mathbb{Z}\cdot t^j : \bigoplus_{j=0}^{\deg p-1} \mathbb{Z}\cdot t^j \Phi_2 \oplus \mathbb{Z}\cdot p(t)\right] = |p(-1)|, \tag{387}$$

and by [He93, Lemma 3.5] it holds

$$\Phi_{10}(-1) = 5, \quad \Phi_{18}(-1) = 3, \quad \Phi_{14}(-1) = 7. \tag{388}$$

Furthermore, we use

$$B_1 \subseteq \sum_{j=1}^{m_1} \mathbb{Z}\cdot\delta_j \qquad \text{with } m_1 := 8, 9, 10 \text{ for } S_{1,0}, E_{3,0}, Z_{1,0} \tag{389}$$

and that the elements in the second line of (386) are modulo $\sum_{j=1}^{m_1}\mathbb{Z}\cdot\delta_j$ of the forms:

$$\begin{aligned}
S_{1,0} &: \ \gamma_2 \equiv \delta_9+\delta_{11}+\delta_{12}+\delta_{14}, \\
E_{3,0} &: \ \widetilde{\gamma_2} \equiv \delta_{10}+\delta_{12}+\delta_{14}+\delta_{16}, \quad \widetilde{\gamma_2}-\gamma_1 \equiv \widetilde{\gamma_2}, \\
Z_{1,0} &: \ \widetilde{\gamma_2} \equiv \delta_{11}+\delta_{13}+\delta_{15}, \quad \widetilde{\gamma_2}-\gamma_2 \equiv -\widetilde{\gamma_2}.
\end{aligned} \tag{390}$$

Therefore $(B_1)_{\Phi_m} + \mathbb{Z}\cdot a$ is primitive in $Ml(f)$ for those elements a and, thus, $r(a) = 1$.

The derivation of (379) from (386) and (384) for $S_{1,0}$, $E_{3,0}$ and $Z_{1,0}$ works along the same lines as the derivation of (120) from (278) for $S_{1,10}$.

The only additional argument concerns $B_3 = \mathbb{Z}\cdot\gamma_3$ in the case $Z_{1,0}$. By (173) any $g \in G_{\mathbb{Z}}$ maps B_3 to itself. Since $L(\gamma_1 - 2\gamma_3, \gamma_3) = 1 \neq 0$, B_3 and $(B_1)_{\Phi_2}$ are glued together. That means, if $g = \varepsilon\cdot\mathrm{id}$ on $(B_1)_{\Phi_2}$ for some $\varepsilon \in \{\pm 1\}$, then $g = \varepsilon\cdot\mathrm{id}$ on B_3.

Now we come to $Q_{2,0}$. The formulas (152)–(155), (237)–(238), (241)–(252), (255)–(258) are also valid for $p = 0$ respectively $s = 0$. The quadratic form in (252) now gives the following variant of (253):

$$A := \{\gamma_1, \gamma_3, \gamma_1 - \gamma_3 + M_h(\gamma_3), \gamma_1 - M_h(\gamma_1) + M_h(\gamma_3)\}, \tag{391}$$

$$\{b \in Ml(f)_{\Phi_4} \mid L(b,b) = 3\} = \bigcup_{a\in A} \{\pm a, \pm M_h(a)\}. \tag{392}$$

There are 16 elements which come in 4 sets of 4 elements such that each set is M_h-invariant. Recall that $M_h^2 = -\mathrm{id}$ on $Ml(f)_{\Phi_4}$. The set $\{\pm\gamma_1, \pm M_h(\gamma_1)\}$ generates $(B_1)_{\Phi_4}$. We claim that $(B_1)_{\Phi_{12}}$ and this set satisfy the following special relationship. For $a \in A$ define the index

$$\begin{aligned} r(a) \; := \; & \Big[((B_1)_{\Phi_{12}} + \mathbb{Z}\cdot a + \mathbb{Z}\cdot M_h(a))_{\mathbb{Q}} \cap Ml(f) \\ & : \; ((B_1)_{\Phi_{12}} + \mathbb{Z}\cdot a + \mathbb{Z}\cdot M_h(a))\Big] \in \mathbb{Z}_{\geq 1}. \end{aligned} \tag{393}$$

Then we claim that the index is

$$r(a) = \begin{cases} 9 & \text{for } a = \gamma_1, \\ 1 & \text{for } a \in \{\gamma_3, \gamma_1 - M_h(\gamma_1) + M_h(\gamma_3)\}, \\ 1 \text{ or } 2 & \text{for } a = \gamma_1 - \gamma_3 + M_h(\gamma_3). \end{cases} \tag{394}$$

So $r(\gamma_1) = 9$ holds because of

$$\begin{aligned} & ((B_1)_{\Phi_{12}} + \mathbb{Z}\cdot\gamma_1 + \mathbb{Z}\cdot M_h(\gamma_1))_{\mathbb{Q}} \cap Ml(f) \qquad (395) \\ = \; & (B_1)_{\Phi_{12}\Phi_4} = \bigoplus_{j=0}^{5} \mathbb{Z}\cdot (t^j\Phi_3)(M_h)(\beta_1), \\ & (B_1)_{\Phi_{12}} + \mathbb{Z}\cdot\gamma_1 + \mathbb{Z}\cdot M_h(\gamma_1) \qquad (396) \\ = \; & \bigoplus_{j=0}^{3} \mathbb{Z}\cdot (t^j\Phi_4\Phi_3)(M_h)(\beta_1) \oplus \bigoplus_{j=0}^{1} \mathbb{Z}\cdot (t^j\Phi_{12}\Phi_3)(M_h)(\beta_1), \end{aligned}$$

and, thus, it is

$$r(\gamma_1) = \left[\bigoplus_{j=0}^{5} \mathbb{Z}\cdot t^j : \bigoplus_{j=0}^{3} \mathbb{Z}\cdot t^j\Phi_4 \oplus \bigoplus_{j=0}^{1} \mathbb{Z}\cdot t^j\Phi_{12}\right] = 3\cdot 3. \tag{397}$$

For $a \in A\setminus\{\gamma_1\}$, $r(a) \in \{1,2\}$ holds because of

$$B_1 \subseteq \sum_{j=1}^{10} \mathbb{Z}\cdot\delta_j, \tag{398}$$

and because the elements a and $M_h(a)$ for $a \in A\setminus\{\gamma_1\}$ are modulo $\sum_{j=1}^{10} \mathbb{Z}\cdot\delta_j$ of the form

$$\begin{aligned} \gamma_1 &\equiv -\delta_{13} - \delta_{14}, \\ M_h(\gamma_1) &\equiv \delta_{12} + \delta_{13}, \\ \gamma_1 - \gamma_3 + M_h(\gamma_3) &\equiv \delta_{12} + 2\delta_{13} + \delta_{14}, \\ M_h(\gamma_1 - \gamma_3 + M_h(\gamma_3)) &\equiv -\delta_{12} + \delta_{14}, \\ \gamma_1 - M_h(\gamma_1) + M_h(\gamma_3) &\equiv \delta_{12} + \delta_{13}, \\ M_h(\gamma_1 - M_h(\gamma_1) + M_h(\gamma_3)) &\equiv \delta_{13} + \delta_{14}. \end{aligned} \tag{399}$$

The derivation of (379) for $Q_{2,0}$ from (392) and (394) is a simple variant of the derivation of (120) from (278) for $S_{1,10}$ at the very end of the proof of Theorem 4.4.1. Namely, consider an element $g \in G_{\mathbb{Z}}$ with $g((B_1)_{\Phi_{12}}) = (B_1)_{\Phi_{12}}$. By (392) it maps the set $\{\pm\gamma_1, \pm M_h(\gamma_1)\}$ to one of the four sets on the right hand side of (392). Moreover by (394) the image must

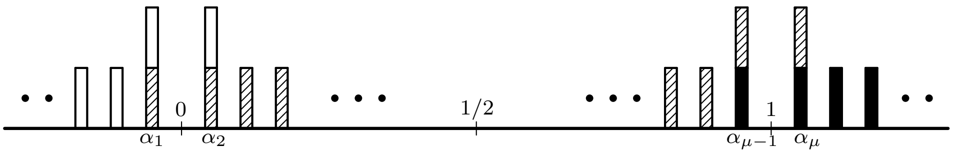

Figure 19: Spectral data for the quadrangle singularities

be the set $\{\pm\gamma_1, \pm M_h(\gamma_1)\}$ itself. As this set generates $(B_1)_{\Phi_4}$, g maps $(B_1)_{\Phi_4}$ to itself. Then g maps the sets $(B_1)_{\Phi_{12}\Phi_4}$, $B_1 = (B_1)_{\Phi_{12}\Phi_4\Phi_3}$ and $B_2 = (B_2)_{\Phi_{12}\Phi_4}$ to themselves. This finishes the proof of part (a).

(b) All the formulas and arguments in the proof of part (c) of Theorem 4.4.1 for the cases $W^\sharp_{1,12r}, S^\sharp_{1,10r}, U_{1,9r}, E_{3,18r}, Z_{1,14r}$ and $Q_{2,12r}$ translate to $r = 0$.

In Step 3 (379) is used instead of (119), just as (120) for $S_{1,10}$. Therefore $\Psi(G_\mathbb{Z})$ is an infinite Fuchsian group of type as in (382). By (217), (525) and Theorem A.3.6, $\Psi(G_\mathbb{Z})$ is a subgroup of a triangle group of the same type as in Theorem A.3.6, for each case. Lastly, the proof of Theorem 4.5.6 will show that it is the full triangle group. □

4.5.2 *Torelli Conjectures for quadrangle singularities*

Here we choose $f_0 := f_{(i,0)}$ as reference singularity for each 2-parameter family in Table 9. The spectral data for the singularities is given below. Moreover the classifying space D_{BL} for the quadrangle singularities is computed.

Lemma 4.5.3. *Consider a bimodal family of quadrangle surface singularities with normal form as in Table 9.*

(a) The spectral numbers $\alpha_1, \ldots, \alpha_\mu$ with $\alpha_1 \leq \ldots \leq \alpha_\mu$ satisfy

$$\alpha_1 = \frac{-1}{m} < \alpha_2 = \frac{1}{m} < \alpha_3 \leq \ldots \leq \alpha_{\mu-2} < \alpha_{\mu-1} = 1 - \frac{1}{m} < \alpha_\mu = 1 + \frac{1}{m}, \quad (400)$$

$$\dim C^{\alpha_1} = \dim C^{\alpha_2} = 2. \quad (401)$$

Figure 19 illustrates the spectral data.

We also have

$$\begin{aligned} V^{\alpha_1}(f_{(t_1,t_2)}) &\supseteq H_0''(f_{(t_1,t_2)} \supseteq V^{>\alpha_2}(f_{(t_1,t_2)}), & (402)\\ H_0''(f_{(t_1,t_2)}) &= \mathbb{C}\cdot(s(\omega_0,\alpha_1)(t_1,t_2) + s(\omega_0,\alpha_2)(t_1,t_2)) \\ &+ \mathbb{C}\cdot s(y\omega_0,\alpha_2)(t_1,t_2) + V^{>\alpha_2}(f_{(t_1,t_2)}). & (403)\end{aligned}$$

(b) The polarizing form S defines an indefinite form $((a,b) \mapsto S(a,\overline{b}))$ on the space $H^n(X_\infty, \mathbb{C})_\zeta$. We get the half-plane

$$\mathcal{H}(C^{\alpha_1}) := \{\mathbb{C}\cdot v \mid v \in C^{\alpha_1} \textit{ with } S(\psi^{-1}_{\alpha_1}(v), \overline{\psi^{-1}_{\alpha_1}(v)}) < 0\} \subseteq \mathrm{Proj}(C^{\alpha_1}). \quad (404)$$

(c) The classifying space for Brieskorn lattices is

$$\begin{aligned} D_{BL} &= \{\mathbb{C}\cdot(v_1+v_2)\,|\,v_1\in C^{\alpha_1}-\{0\} \text{ with } [\mathbb{C}\cdot v_1]\in\mathcal{H}(C^{\alpha_1}), \\ &\qquad\qquad v_2\in\mathbb{C}\cdot\psi_{\alpha_2}(\overline{\psi_{\alpha_1}^{-1}(v_1)})\subseteq C^{\alpha_2}\} \\ &\cong \mathcal{H}(C^{\alpha_1})\times\mathbb{C}. \end{aligned} \tag{405}$$

Proof. (a) The spectral numbers can be found in [AGV88, 13.3.4, p. 389]. With the spectral numbers at hand (401) and (402) are obvious. Moreover, the form of the Brieskorn lattice given in (403) follows from Proposition A.1.4 and $\deg_w(\omega_0)=\alpha_1+1, \deg_w(y\omega_0)=\alpha_2+1$ and $\deg_w(x^iy^jz^k\omega_0)>\alpha_2+1$ for any other monomial $x^iy^jz^k$, because $w_y<w_x\leq w_z$.

(b) This follows as in Subsection 4.4.2. In particular, look right before Theorem 4.4.2.

(c) This follows as in Lemma 4.4.3 (c) in the case $m|p$. □

Now we can compute the period map $T^{(5)}\to D_{BL}$. Note that this period map was already computed in [He93, Kapitel 4 b)]. For reasons of self-consistency of this writing we do this computation again here.

Theorem 4.5.4. *Consider a bimodal family of quadrangle surface singularities with normal form as in Table 9.*

(a) The section $s(\omega_0,\alpha_1)(t_1,t_2)=s(\omega_0,\alpha_1)(t_1,0)=s[\omega_0](t_1,0)$ is independent of t_2 and satisfies the hypergeometric differential equation

$$0=\Big(t_1(1-t_1)\partial_{t_1}^2+(c-(a+b+1)t_1)\partial_{t_1}-ab\Big)\,s[\omega_0](t_1,0) \tag{406}$$

with $(1-c,c-a-b,a-b)=(\frac{1}{m_0},\frac{1}{m_1},\frac{1}{m_\infty})$.

(b) The multivalued period map

$$BL_{T^{(1)}}:T^{(1)}\to\mathcal{H}(C^{\alpha_1}),\quad t_1\mapsto\mathbb{C}\cdot s[\omega_0](t_1,0), \tag{407}$$

lifts to a univalued period map

$$BL_{T^{(3)}}:T^{(3)}\to\mathcal{H}(C^{\alpha_1}) \tag{408}$$

which is an open embedding and extends to an isomorphism

$$BL_{T^{(4)}}:T^{(4)}\to\mathcal{H}(C^{\alpha_1}). \tag{409}$$

(c) We have

$$s(\omega_0,\alpha_2)(t_1,t_2)=t_2\cdot(-\partial_\tau)s[p_{>1}\omega_0](t_1,0), \tag{410}$$

$$C^{\alpha_2}=\mathbb{C}\cdot s[y\omega_0](t_1,0)\oplus\mathbb{C}\cdot\partial_\tau s[p_{>1}\omega_0](t_1,0). \tag{411}$$

(d) The multivalued period map

$$BL_{T^{(5)}} : T^{(5)} \to D_{BL} \tag{412}$$

is locally in $T^{(1)}$ and $\mathcal{H}(C^{\alpha_1})$ an isomorphism of line bundles and lifts to an open embedding of line bundles

$$BL_{T^{(7)}} : T^{(7)} \to D_{BL}. \tag{413}$$

(e) In the case of $S_{1,0}$ it holds

$$\partial_{t_1} s[x\omega_0](t_1, 0) = \frac{2t_1 - 1}{5t_1(1-t_1)} \cdot s[x\omega_0](t_1, 0). \tag{414}$$

Proof. (a) Here we just describe the calculations that are necessary to prove (406). Note that, $f_{(t_1,0)}$ and $\partial_{t_1} f_{(t_1,0)}$ are quasihomogeneous of weighted degree 1. Now list all monomials $d_1, \ldots, d_l$ in x, y, z which turn up in $f^2_{(t_1,0)}$, $f_{(t_1,0)} \cdot \partial_{t_1} f_{(t_1,0)}$ and $(\partial_{t_1} f_{(t_1,0)})^2$, find $l-2$ independent linear combinations of $d_1\omega_0, \ldots, d_l\omega_0$ in $df_{(t_1,0)} \wedge d\Omega^1_{\mathbb{C}^3}$ and determine an equation

$$\begin{aligned} &p_1 \cdot (\partial_{t_1} f_{(t_1,0)})^2 \cdot \omega_0 + p_2 \cdot f_{(t_1,0)} \cdot \partial_{t_1} f_{(t_1,0)} \cdot \omega_0 + p_3 \cdot f^2_{(t_1,0)} \cdot \omega_0 \\ &\equiv 0 \mod df_{(t_1,0)} \wedge d\Omega^1_{\mathbb{C}^3} \end{aligned} \tag{415}$$

with $p_1, p_2, p_3 \in \mathbb{Q}[t_1]$. Then we obtain the hypergeometric equation

$$\Big(p_1 \partial^2_{t_1} - (\alpha_1 + 2)p_2 \partial_{t_1} + (\alpha_1+2)(\alpha_1+1)p_3\Big) s[\omega_0](t_1, 0). \tag{416}$$

Note that by stabilization it is sufficient to work in the cases $W_{1,0}$, $E_{3,0}$, $Z_{1,0}$ with the curve singularities (see Proposition 3.2.3). There the number l of monomials is $l = 5$. In the other cases we need surfaces singularities and there it is $l = 9$.

(b) The period map $BL_{T^{(1)}}$ is not constant because $s[\omega_0](t_1, 0)$ and $\partial_{t_1} s[\omega_0](t_1, 0) = (-\partial_\tau) s[\partial_{t_1} f_{(t_1,0)} \cdot \omega_0](t_1, 0)$ are linearly independent. This follows from the fact that the derivative $\partial_{t_1} f_{(t_1,0)}$ is not in the Jacobi ideal. Therefore the multivalued coefficient functions $f_1(t_1)$ and $f_2(t_1)$ with

$$s[\omega_0](t_1, 0) = f_1(t_1) \cdot v^0_1 + f_2(t_1) \cdot v^0_2 \tag{417}$$

for an arbitrary basis v^0_1, v^0_2 of C^{α_1} are linearly independent scalar solutions of the same hypergeometric differential equation. Their quotient $\left(t_1 \mapsto \frac{f_1(t_1)}{f_2(t_1)}\right)$ is a Schwarzian function [Fo51, Sections 113+114], which maps the closure of the upper half-plane to a hyperbolic triangle with angles $\frac{\pi}{m_0}, \frac{\pi}{m_1}, \frac{\pi}{m_\infty}$. The vertices are the images of 0, 1, ∞. Hence, the multivalued map $BL_{T^{(1)}} : T^{(1)} \to \mathcal{H}(C^{\alpha_1})$ is an inverse of the quotient map $c^{(1)} : T^{(3)} \to T^{(1)}$. This shows (408) and (409).

(c) It is $s(\omega_0, \alpha_2)(t_1, 0) = 0$ because of Proposition A.1.4 (a). Furthermore, it holds

$$\begin{aligned}
\partial_{t_2} s(\omega_0, \alpha_2)(t_1, t_2) &= (-\partial_\tau) s(p_{>1}\omega_0, \alpha_2 + 1)(t_1, t_2) \\
&= (-\partial_\tau) s[p_{>1}\omega_0](t_1, 0) \\
\text{thus } s(\omega_0, \alpha_2)(t_1, t_2) &= t_2 \cdot (-\partial_\tau) s[p_{>1}\omega_0](t_1, 0) \\
&\equiv t_2 \cdot v_2 \mod \mathbb{C} \cdot s[y\omega_0](t_1, 0) \qquad (418) \\
\text{with a suitable } v_2 &\in \psi_{\alpha_2}^{-1}(\overline{\psi_{\alpha_1}(s[\omega_0](t_1, 0))}) \setminus \{0\}.
\end{aligned}$$

Here $v_2 \neq 0$ follows from (411) which is a consequence of the fact that $p_{>1}$ is not in the Jacobi ideal of $f_{(t_1,0)}$.

(d) This follows from (408) and part (c).

(e) To prove this part we have to do calculations which are similar to the ones in part (a) — but simpler. In fact, it is

$$\begin{aligned}
\partial_{t_1} s[x\omega_0](t_1, 0) &= (-\partial_\tau) s[\partial_{t_1} f_{(t_1,0)} \cdot x\omega_0](t_1, 0) = (-\partial_\tau) s[x^3 y^2 \omega_0](t_1, 0) \\
&\overset{(*)}{=} \frac{2t_1 - 1}{6t_1(t_1 - 1)} (-\partial_\tau) s[f_{(t_1,0)} \cdot x\omega_0](t_1, 0) \\
&= \frac{2t_1 - 1}{6t_1(t_1 - 1)} (-\partial_\tau \tau) s[x\omega_0](t_1, 0) \\
&= \frac{2t_1 - 1}{6t_1(t_1 - 1)} (-\frac{6}{5}) s[x\omega_0](t_1, 0) \\
&= \frac{2t_1 - 1}{5t_1(1 - t_1)} s[x\omega_0](t_1, 0).
\end{aligned}$$

For $\overset{(*)}{=}$ one has to find 3 relations in $df_{(t_1,0)} \wedge d\Omega^1_{\mathbb{C}^3}$ between the monomial differential forms $x^3y^2\omega_0, xy^3z\omega_0, xyz^2\omega_0$ and $x^3z\omega_0$ in $f_{(t_1,0)} \cdot x\omega_0$ and $x^3y^2\omega_0$. □

The last step before we come to the main theorem of this subsection is the following result on the transversal monodromy. In order to proof this result we will use (383) from Theorem 4.5.2.

Theorem 4.5.5. *Consider a bimodal family of quadrangle surface singularities with normal form as given in Table 9. The pull back to $T^{(3)}$ with $c^{(1)}$ of the homology group $\bigcup_{t_1 \in T^{(1)}} Ml(f_{(t_1,0)}) \to T^{(1)}$ comes equipped with a monodromy representation of the form $\pi^{(3)} : \pi_1(T^{(3)}, \tau^{(3)}) \to G_{\mathbb{Z}}$ (with $c^{(1)}(\tau^{(3)}) = i$). It is called the* transversal monodromy group.

(a) In (419) the local monodromies around the elliptic fixed points in $(c^{(2)})^{-1}(0)$, $(c^{(2)})^{-1}(1)$ and $(c^{(2)})^{-1}(\infty)$ are given.

	$W_{1,0}$	$S_{1,0}$	$U_{1,0}$	$E_{3,0}$	$Z_{1,0}$	$Q_{2,0}$
$(c^{(2)})^{-1}(\{0,1\})$	id	id	id	id	id	id
$(c^{(2)})^{-1}(\infty)$	id	M_h^5	id	id	id	M_h^6

(419)

Therefore $\operatorname{im}(\pi^{(3)}) = \{\mathrm{id}\}$ *for* $W_{1,0}$, $U_{1,0}$, $E_{3,0}$, $Z_{1,0}$, *and* $\operatorname{im}(\pi^{(3)}) = \{\mathrm{id}, M_h^{m_\infty}\}$ *for* $S_{1,0}$ *and* $Q_{2,0}$.

(b) We have:

$$\begin{aligned} \{g\in G_{\mathbb{Z}}\,|\, g \text{ acts trivially on } D_{BL}\} &= \{g\in G_{\mathbb{Z}}\,|\, g=\pm\,\mathrm{id} \text{ on } Ml_\zeta\} \\ &= \{\pm\,\mathrm{id}, \pm M_h^{m_\infty}\} \\ &= \begin{cases} \{\pm\,\mathrm{id}\} & \text{for } U_{1,0}, E_{3,0}, Z_{1,0} \\ \{\pm\,\mathrm{id}, \pm M_h^{m_\infty}\} & \text{for } W_{1,0}, S_{1,0}, Q_{2,0}. \end{cases} \end{aligned} \tag{420}$$

(c) We consider the group $G_{\mathcal{R}}^{smar,gen}$ *(cf. (61)) for the singularities of multiplicity* ≥ 3, *i.e. the curve singularities* $W_{1,0}$, $E_{3,0}$, $Z_{1,0}$ *and the surface singularities* $S_{1,0}$, $U_{1,0}$, $Q_{2,0}$. *It is*

$$G_{\mathcal{R}}^{smar,gen} = \begin{cases} \{\mathrm{id}\} & \text{for } U_{1,0}, E_{3,0}, Z_{1,0}, \\ \{\mathrm{id}, M_h^{m_\infty}\} & \text{for } W_{1,0}, S_{1,0}, Q_{2,0}. \end{cases} \tag{421}$$

Proof. We start with the proof of part (b).
(b) Suppose that $g \in G_{\mathbb{Z}}$ acts trivially on D_{BL}. Then it acts trivially also on $\mathcal{H}(C^{\alpha_1})$, i.e. $g = \lambda\cdot\mathrm{id}$ on Ml_ζ for some $\lambda\in\mathbb{C}^*$. It is $\mathbb{C}\cdot(v_1+v_2) = \mathbb{C}\cdot(\lambda v_1+\overline{\lambda}v_2)$ and, thus, $\lambda=\overline{\lambda}\in\{\pm 1\}$. This together with formula (383) and the set of eigenvalues of M_h gives (420).

(a) The (Papperitz-)Riemann symbol

$$\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{matrix}\right\} \tag{422}$$

encodes the local behaviour near 0, 1 and ∞ of scalar solutions of the hypergeometric equation. Locally suitable solutions have the following form (h.o.t. = higher order terms):

$$\begin{aligned} &\text{near } 0: && t_1^0 + \text{h.o.t.} && \text{and} && t_1^{1-c} + \text{h.o.t.}, \\ &\text{near } 1: && (t_1-1)^0 + \text{h.o.t.} && \text{and} && (t_1-1)^{c-a-b} + \text{h.o.t.}, \\ &\text{near } \infty: && t_1^{-a} + \text{h.o.t.} && \text{and} && t_1^{-b} + \text{h.o.t.} \end{aligned} \tag{423}$$

Especially, the local monodromy of the space of solutions has the eigenvalues

$$\begin{aligned} &\text{around } 0: && 1 && \text{and} && e^{2\pi i(1-c)}, \\ &\text{around } 1: && 1 && \text{and} && e^{2\pi i(c-a-b)}, \\ &\text{around } \infty: && e^{-2\pi i a} && \text{and} && e^{-2\pi i b}. \end{aligned} \tag{424}$$

In our situation, i.e. $(1-c, c-a-b, a-b) = (\frac{1}{m_0}, \frac{1}{m_1}, \frac{1}{m_\infty})$, we obtain

$$\begin{array}{c|cccccc} & W_{1,0} & S_{1,0} & U_{1,0} & E_{3,0} & Z_{1,0} & Q_{2,0} \\ \hline a & \frac{1}{2} & \frac{1}{2} & \frac{4}{9} & \frac{4}{9} & \frac{3}{7} & \frac{5}{12} \\ b & \frac{1}{3} & \frac{3}{10} & \frac{1}{3} & \frac{1}{3} & \frac{2}{7} & \frac{1}{4} \\ c & \frac{11}{12} & \frac{9}{10} & \frac{8}{9} & \frac{8}{9} & \frac{6}{7} & \frac{5}{6} \end{array} \tag{425}$$

The branched covering $c^{(2)} : T^{(4)} \to T^{(2)}$ has at the elliptic fixed points the orders m_0, m_1, m_∞. Therefore the local monodromies of the pull back to $T^{(3)}$ of the solutions on $T^{(1)} = \mathbb{C} \setminus \{0, 1\} \subseteq T^{(2)} = P^1\mathbb{C}$ become $+\mathrm{id}$ except around the elliptic fixed points in $(c^{(2)})^{-1}(\infty)$ in the cases $S_{1,0}$ and $Q_{2,0}$ where they become $-\mathrm{id}$.

The same holds for the restrictions to Ml_ζ of the local monodromies in $\pi^{(3)}$.

With (383) we obtain (419) for $U_{1,0}$, $E_{3,0}$, $Z_{1,0}$ and the following approximation of (419) for $W_{1,0}, S_{1,0}, Q_{2,0}$.

$$\begin{array}{c|ccc} & W_{1,0} & S_{1,0} & Q_{2,0} \\ \hline (c^{(2)})^{-1}(\{0,1\}) & \mathrm{id} \text{ or } -M_h^6 & \mathrm{id} \text{ or } -M_h^5 & \mathrm{id} \text{ or } -M_h^6 \\ (c^{(2)})^{-1}(\infty) & \mathrm{id} \text{ or } -M_h^6 & -\mathrm{id} \text{ or } M_h^5 & -\mathrm{id} \text{ or } M_h^6 \end{array} \tag{426}$$

Now we show that several entries in (426) cannot occur by case-by-case analysis to end up with (419).

The case $W_{1,0}$: Here, the sublattice $Ml_{-1,\mathbb{Z}}$ has rank 1. Hence, the local transversal monodromies of the homology bundle on $T^{(1)}$ around 0, 1 and ∞ have on $Ml_{-1,\mathbb{Z}}$ eigenvalues in $\{\pm 1\}$. The branched covering $T^{(4)} \to T^{(2)}$ is at the elliptic fixed points of even order. Thus the map $\pi^{(3)}$ restricts to the trivial monodromy on $Ml_{-1,\mathbb{Z}}$. This excludes $-M_h^6$ in (426).

The case $S_{1,0}$: The local transversal monodromies of the homology bundle on $T^{(1)}$ around 0, 1 and ∞ have on $Ml_{e^{-2\pi i/5}}$ eigenvalues in $Eiw(\zeta)$. The branched covering is at the elliptic fixed points in $(c^{(2)})^{-1}(\{0,1\})$ of order 10. Thus the local monodromies of $\pi^{(3)}$ around points in $(c^{(2)})^{-1}(\{0,1\})$ are trivial on $Ml_{e^{-2\pi i/5}}$. This excludes $-M_h^5$ in the first line of (426). The branched covering is at the elliptic fixed points in $(c^{(2)})^{-1}(\infty)$ of order 5. Now via Theorem 4.5.4 (e) we obtain near ∞

$$s[x\omega_0](t_1, 0) = (t_1^{-2/5} + \text{h.o.t.}) \cdot (\text{a flat multivalued section}). \tag{427}$$

Therefore also the local monodromy of $\pi^{(3)}$ around points in $(c^{(2)})^{-1}(\infty)$ is trivial. This excludes $-\mathrm{id}$ in the second line of (426).

The case $Q_{2,0}$: The local transversal monodromies of the homology bundle on $T^{(1)}$ around 0, 1 and ∞ have on $Ml_{e^{-2\pi i/3}}$ eigenvalues in $Eiw(e^{2\pi i/6})$. The branched covering $T^{(4)} \to T^{(2)}$ is at the elliptic fixed points of order 6. Thus $\pi^{(3)}$ restricts to the trivial monodromy on $Ml_{e^{-2\pi i/3}}$. This excludes $-M_h^6$ in the first line and $-\mathrm{id}$ in the second line of (426).

(c) By Proposition 3.1.6 (d) it is $-\mathrm{id} \notin G_{\mathcal{R}}^{smar,gen}$. Furthermore, the group $G_{\mathcal{R}}^{smar,gen}$ fixes $BL([f, \pm\rho])$ for any $[f, \pm\rho] \in M_\mu^{mar}$. Since $T^{(7)} \to D_{BL}$ is an open embedding, $G_{\mathcal{R}}^{smar,gen}$ fixes D_{BL}. By part (b) $G_{\mathcal{R}}^{smar,gen} = \{\mathrm{id}\}$ for $U_{1,0}$, $E_{3,0}$, $Z_{1,0}$, and $G_{\mathcal{R}}^{smar,gen} = \{\mathrm{id}\}$ or $\{\mathrm{id}, M_h^{m_\infty}\}$ or $\{\mathrm{id}, -M_h^{m_\infty}\}$ for $W_{1,0}, S_{1,0}, Q_{2,0}$. The coordinate changes φ of the curve singularities $W_{1,0}$ and the surface singularities $S_{1,0}$ and $Q_{2,0}$ in (428) give a nontrivial element of $G_{\mathcal{R}}^{smar,gen}$.

$$\begin{array}{lll} W_{1,0} & S_{1,0} & Q_{2,0} \\ (x,y) \mapsto (-x,y) & (x,y,z) \mapsto (-x,y,z) & (x,y,z) \mapsto (x,y,-z) \end{array} \tag{428}$$

The coordinate change φ maps ω_0 to $-\omega_0$ and $s[\omega_0](t_1,0)$ to $-s[\omega_0](t_1,0)$. Therefore $(\varphi)_{hom}|_{Ml_\zeta} = -\mathrm{id}$ and $(\varphi)_{hom} = M_h^{m_\infty}$ (and not $-M_h^{m_\infty}$). This shows (420) for $W_{1,0}$, $S_{1,0}$ and $Q_{2,0}$. □

Now we are ready to prove the main theorem of this subsection. We will need many results from all over this chapter. Note that the proof will also close the gaps that are left in the proof of Theorem 4.5.2.

Theorem 4.5.6. *Consider a bimodal family of quadrangle surface singularities with normal form as given in Table 9.*

(a) There are canonical isomorphisms

$$T^{(7)} \to (M_\mu^{smar})^0 \to (M_\mu^{mar})^0. \tag{429}$$

(b) $-\mathrm{id} \notin G^{smar}$, where G^{smar} is the group for the singularities of multiplicity ≥ 3, namely the curve singularities $W_{1,0}, E_{3,0}, Z_{1,0}$ and the surface singularities $S_{1,0}$, $U_{1,0}$, $Q_{2,0}$. So Conjecture 3.1.2 is true.

(c) It is $G_{\mathbb{Z}} = G^{mar}$, i.e. $M_\mu^{mar} = (M_\mu^{mar})^0$.

(d) The period map $BL : M_\mu^{mar} \to D_{BL}$ is an embedding. So the strong global Torelli Conjecture 3.2.2 (a) is true.

Proof. According to Theorem 4.5.5 (a)+(c) the transversal monodromy representation $\pi^{(7)}$ of the pull back to $T^{(7)}$ with $c^{(5)}$ of the homology bundle $\bigcup_{(t_1,t_2)\in T^{(5)}} Ml(f_{(t_1,t_2)}) \to T^{(5)}$ is trivial in the cases $W_{1,0}$, $U_{1,0}$, $E_{3,0}$, $Z_{1,0}$ and has image in $G_{\mathcal{R}}^{smar,gen} = \{\mathrm{id}, M_h^{m_\infty}\}$ in the cases $S_{1,0}$ and $Q_{2,0}$. Thus, the strong marking $+\mathrm{id}$ on $f_{(i,0)}$ induces for each $f_{(t_1,t_2)}$ two strong markings in the same right equivalence class in the cases $S_{1,0}$ and $Q_{2,0}$ and one strong marking in the other cases. In any case, we obtain a map $T^{(7)} \to (M_\mu^{smar})^0$.

Composed with the period map we get an open embedding $T^{(7)} \to (M_\mu^{smar})^0 \to D_{BL}$ by Theorem 4.5.4. Moreover, we know that $(M_\mu^{smar})^0 \to D_{BL}$ is even an immersion (cf. Theorem 3.2.1) and that all three spaces above are 2-dimensional manifolds. So $T^{(7)} \to$

$(M_\mu^{smar})^0$ and $(M_\mu^{smar})^0 \to D_{BL}$ are open embeddings, too. For the moment we postpone the proof that the map $T^{(7)} \to (M_\mu^{smar})^0$ is an isomorphism.

Part (b) follows now easily. We consider the case of singularities of multiplicity ≥ 3. For singularities of multiplicity 2, $M_\mu^{smar} = M_\mu^{mar}$ and $(M_\mu^{smar})^0 = (M_\mu^{mar})^0$ hold anyways. For singularities of multiplicity ≥ 3, $-\mathrm{id} \in G_{\mathbb{Z}}$ acts trivially on D_{BL}. But it acts non-trivially on M_μ^{smar} by Theorem 3.1.4 (c). The map $(M_\mu^{smar})^0 \to D_{BL}$ is an embedding. Therefore $-\mathrm{id} \in G_{\mathbb{Z}}$ does not act on $(M_\mu^{smar})^0$. Hence, it is $-\mathrm{id} \notin G^{smar}$. This shows part (b). In this case $(M_\mu^{smar})^0 \cong (M_\mu^{mar})^0$ according to Theorem 3.1.4 (c).

The map $c^{(2)} : T^{(4)} = \mathbb{H} \to T^{(2)} = P^1\mathbb{C}$ is the branched covering from an action of a triangle group Γ of type $(\frac{1}{m_0}, \frac{1}{m_1}, \frac{1}{m_\infty})$ on $\mathbb{H}$. The triangle group Γ is a normal subgroup of index 2 of a triangle group Γ^{qh} of type $(2, 2m, 2m)$ with $\Gamma^{qh}/\Gamma = G_2$ for $W_{1,0}$ and $S_{1,0}$. It is a normal subgroup of index 6 of a triangle group Γ^{qh} of type $(2, 3, 2m)$ with $\Gamma^{qh}/\Gamma = G_3$ for $U_{1,0}$, $E_{3,0}$, $Z_{1,0}$ and $Q_{2,0}$. Figure 20 shows the two hyperbolic triangles associated to Γ and Γ^{qh}. The symbols $[0], [1], [\infty], [\frac{1}{2}], [2], [-1], [e^{2\pi i/6}]$ at special points indicate the images of these points under $c^{(2)}$.

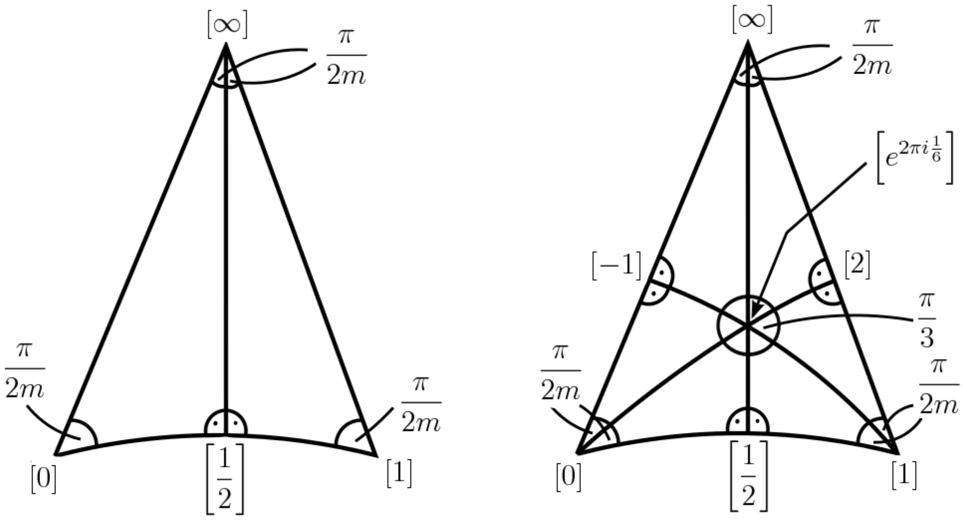

Figure 20: Hyperbolic triangles

Since the group Γ^{qh} maps the set of elliptic fixed points $(c^{(2)})^{-1}(\{0, 1, \infty\}) = T^{(4)} \setminus T^{(3)}$ of Γ to itself, it acts on $T^{(3)}$. The orbits of Γ^{qh} in $T^{(3)}$ are contained in the right equivalence classes of quasihomogeneous singularities by the (already proved) implication "$\Leftarrow$" in (367) in Proposition 4.5.1. By the embedding $T^{(3)} \to \mathcal{H}(C^{\alpha_1})$ in Theorem 4.5.4, Γ^{qh} acts also on $\mathcal{H}(C^{\alpha_1})$, and the orbits are contained in the orbits of $\Psi(G^{mar})$, because the orbits of G^{mar} on $(M_\mu^{mar})^0$ are the right equivalence classes in $(M_\mu^{mar})^0$.

Now we compare the actions of Γ^{qh} and $\Psi(G^{mar})$ on $\mathcal{H}(C^{\alpha_1})$. Γ^{qh} acts as a triangle group of type $(2, 2m, 2m)$ respectively $(2, 3, 2m)$, and $\Psi(G^{mar})$ acts by Theorem 4.5.2 (b) as a subgroup of a triangle group of the same type. Moreover, the orbits of Γ^{qh} are contained in the orbits of $\Psi(G^{mar})$. Therefore the actions must coincide and $\Psi(G^{mar}) = \Psi(G_{\mathbb{Z}})$ is a triangle group of the types as given in (382). This gives the surjectivity in Theorem 4.5.2 and finishes the proof of Theorem 4.5.2.

It also shows that G^{mar} acts on $T^{(3)}$. Since $T^{(3)}$ contains representatives of the right equivalence classes of all quasihomogeneous singularities in the given μ-homotopy family, the marked quasihomogeneous singularities in $(M_\mu^{mar})^0$ must all be in $T^{(3)}$. This shows that the open embedding $T^{(7)} \to (M_\mu^{mar})^0$ is an isomorphism.

Lastly, we prove $G_{\mathbb{Z}} = G^{mar}$. In order to do so we consider an element $g_1 \in G_{\mathbb{Z}}$. Since it holds $\Psi(G^{mar}) = \Psi(G_{\mathbb{Z}})$, we can multiply it with an element $g_2 \in G^{mar}$ such that $g_3 = g_1 g_2$ satisfies $\Psi(g_3) = \mathrm{id}$. Now by (383) it is $g_3 \in \{\pm M_h^k \,|\, k \in \mathbb{Z}\} \subseteq G^{mar}$. This proves $G_{\mathbb{Z}} = G^{mar}$ and, hence, $M_\mu^{mar} = (M_\mu^{mar})^0$ holds. Now as $BL : (M_\mu^{mar})^0 \to D_{BL}$ is an embedding, $BL : M_\mu^{mar} \to D_{BL}$ is an embedding, too. This completes the proof of Theorem 4.5.6. □

Lastly, we have to finish the proof of Proposition 4.5.1.

Proof of "⇒" in (367). By the discussion above the group $G_{\mathbb{Z}}$ acts as Γ^{qh} on $\mathcal{H}(C^{\alpha_1})$ and, hence, as G_2 respectively G_3 on $T^{(1)}$. This already shows "⇒" in (367) for the quasihomogeneous singularities. Note that this is proved also in [Bi92, Satz 1.5.2] with a different sort of argument.

Now an element $g \in G_{\mathbb{Z}}$ which acts trivially on $T^{(3)}$ is in $\{\pm M_h^k \,|\, k \in \mathbb{Z}\}$ and restricts to $\lambda \cdot \mathrm{id}$ on Ml_ζ for some $\lambda \in Eiw(\zeta)$. We observe that

$$g: \ \mathbb{C} \cdot (v_1 + v_2) \mapsto \mathbb{C}(\lambda \cdot v_1 + \overline{\lambda} \cdot v_2) = \mathbb{C} \cdot (v_1 + \overline{\lambda}^2 \cdot v_2).$$

Thus it acts on the fibers of the projection $D_{BL} \to \mathcal{H}(C^{\alpha_1})$ by multiplication with $\overline{\lambda}^2$, and it acts in the same way on the fibers of the projection $T^{(7)} \to T^{(3)}$. But it is $(\overline{\lambda}^2)^{m_\infty} = 1$. This then shows "⇒" in (367) for all quadrangle singularities. □

4.6 BEYOND SINGULARITIES OF MODALITY 2 — THE TYPE U_{24}

After we proved the Torelli Conjectures for all simple, unimodal and bimodal singularities in the previous sections, it is natural to ask what happens for singularities of higher modality. Here some partial results are available. They were achieved by concrete calculations, just as the results for the singularities of modality less than or equal to two. In particular, we know that Brieskorn-Pham singularities with pairwise coprime exponents and semiquasihomogeneous singularities with weights $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ fulfill the Torelli Conjectures (see [He98]). The current state of the art is summarized in Table 14.

Until now there is no general method to attack these problems. But even without a complete general understanding it is possible to consider a certain type of singularity and prove the conjectures by hand. Here we do this exemplarily for the quasihomogeneous polynomial $f = x^3 + y^3 + z^7$, which defines an isolated hypersurface singularity of Milnor number $\mu = 24$ and multiplicity three. This singularity is of type U_{24}, of modality five and quasihomogeneous of weighted degree one with weights $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{7}\right)$. Especially, it is not of any of the types studied earlier in this thesis. It turns up as some kind of standard example of a quasihomogeneous singularity in several articles, e.g. see [Ba03, Example 7] or [GHP97, Example 2.8]. This is also the reason why we choose it here.

The Coxeter-Dynkin diagram for this singularity can be found in [Ga74-2, Figure 8] and looks as shown in Figure 21.

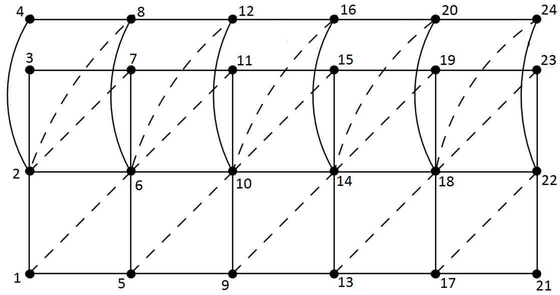

Figure 21: Coxeter-Dynkin diagram for $x^3 + y^3 + z^7$

This encodes its Seifert form. From this Seifert form we can compute the monodromy operator M_h on the Milnor lattice via the formulas in Chapter 2. This then yields

$$M_h = \left(\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix} \right)^T .$$

The characteristic polynomial of this monodromy matrix is $p_{ch} = \Phi_7^2 \Phi_{21}$. We set $p_2 := \Phi_7$ and $p_{\min} := \Phi_7 \Phi_{21}$. Then we can compute that $Ml\,(f) = \bigoplus_{i=0}^{17} \mathbb{Z} \cdot M_h^i\,(\delta_{12}) \oplus \bigoplus_{i=0}^{5} \mathbb{Z} \cdot M_h^i\,(\delta_{12} - \delta_{18}) =: B_1 \oplus B_2$ and that the monodromy with respect to this new basis has the form

$$M_M = \begin{pmatrix} S_{p_{\min}} & \\ & S_{p_2} \end{pmatrix} .$$

Recall that S_{p_2} and $S_{p_{\min}}$ are the companion matrices for the polynomials p_2 and $p_{\min}$. In other words that means that Orlik's Conjecture A.2.2 is true for this singularity.
Now as Orlik's Conjecture is true here we can employ the lattice theory from Appendix A.2 to compute the group $G_{\mathbb{Z}}$. Note that this proof is similar to the case of singularities of type U_{16}.

Proposition 4.6.1. *For $f_0 = x^3 + y^3 + z^7$ we have the following two equalities*

$$\{g \in G_{\mathbb{Z}} | g|_{B_3} = \pm (M_h|_{B_3})^k\} = \{\pm M_h^k | k \in \mathbb{Z}\}, \tag{430}$$
$$\mathrm{Aut}\,(B_3, L) = \{\pm (M_h|_{B_3})^k | k \in \mathbb{Z}\} \times \mathrm{Sym}(3), \tag{431}$$

where $B_3 := \ker\,(\Phi_7 : Ml(f)_{\mathbb{C}} \to Ml(f)_{\mathbb{C}}) \cap Ml(f)$.

Proof. Note that B_3 is a primitive sublattice of $Ml\,(f)$ of rank 12. The lattices $B_1 \cap B_3$ and B_2 are M_h-invariant primitive sublattices of B_3 of rank 6. Together they generate B_3. It holds

$$(B_1)_{\mathbb{C}} = \ker(\Phi_{21}(M_h)) \oplus (B_1 \cap B_3)_{\mathbb{C}}. \tag{432}$$

Now every automorphism $g \in G_{\mathbb{Z}}$ with $g|_{B_3} = \pm(M_h|_{B_3})^k$ for some $k \in \mathbb{Z}$ restricts to an automorphism of B_1, because of $B_2 \subseteq B_3$ and equation (432). So we are able to apply Lemma A.2.3 to the cyclic block B_1. This then yields $g|_{B_1} = \pm(M_h|_{B_1})^l$ for some $l \in \mathbb{Z}$. Since $g|_{B_3} = \pm(M_h|_{B_3})^k$, it has to be $k \equiv l \mod \operatorname{lcm}(m : \Phi_m|\Phi_7)$. So the automorphism g which acts on B_3 like some power of the monodromy is just $g = \pm M_h^k$. In other words it holds

$$\{g \in G_{\mathbb{Z}} \,|\, g|_{B_3} = \pm(M_h|_{B_3})^k\} = \{\pm M_h^k \,|\, k \in \mathbb{Z}\}. \tag{433}$$

So it remains to the determine the group $\operatorname{Aut}(B_3, L)$. Therefore, we use a trick and do the computations with the quasihomogeneous polynomial $x^3 + xy^2 + z^7$, which is right equivalent to $x^3 + y^3 + z^7$. This new polynomial is the sum of singularities in different variables of types D_4 and A_6. So we are in the situation of Lemma A.2.7 (c) and we can apply it with $l = 6$ and $m = 2$. This yields

$$\operatorname{Aut}(B_3, L) = \{\pm(M_h|_{B_3})^k \,|\, k \in \mathbb{Z}\} \times \operatorname{Sym}(3). \tag{434}$$

□

Now we have to prove the Torelli Conjecture for the singularity at question. The singularity has a semiuniversal unfolding. This semiuniversal unfolding can be restricted to the μ-constant stratum $T_- := \mathbb{C}^5$ and we obtain

$$F_t^{(\mu)}(x,y,z) = f(x,y,z) + t_1 xyz^5 + t_2 yz^5 + t_3 xz^5 + t_4 xyz^4 + t_5 xyz^3 \tag{435}$$

with $t := (t_1, t_2, t_3, t_4, t_5) \in T_-$ (cf. [GHP97, Example 2.8]). Note that this family contains all right equivalence classes of semiquasihomogeneous singularities with principal part $f(x,y,z)$.

Theorem 4.6.2. *For $f_0 = x^3 + y^3 + z^7$ it holds*

$$G^{smar}(f_0) \times \{\pm \operatorname{id}\} = G^{mar}(f_0) = G_{\mathbb{Z}} \cong \{\pm M_h^k \,|\, k \in \mathbb{Z}\} \times \operatorname{Sym}(3), \tag{436}$$

i.e. the respective moduli space $M_\mu^{mar} \cong T_- = \mathbb{C}^5$ of marked singularities is connected. So the statement and the conjecture are both true here. Moreover the period map $BL : M_\mu^{mar} \to D_{BL}(f_0)$ is injective, so the strong global Torelli Conjecture 3.2.2 (a) is true.

Proof. We choose the quasihomogeneous singularity $f_0 = x^3 + y^3 + z^7$ with trivial marking $\pm\,\mathrm{id}$ in T_- as reference singularity. Then all elements of T_- become marked singularities, because T_- is simply connected. So the period map $T_- \to D_{BL}, t \mapsto H_0''\left(F_t^{(\mu)}(x,y,z)\right)$ is well-defined. The singularities $F_t^{(\mu)}(x,y,z)$ with parameters t in $T_- \setminus \{0\}$ are semiquasihomogeneous and only f_0 is quasihomogeneous. The spectral numbers and the structure of the spaces C^α are described in Example 2.2.3. Thus the period map is

$$T_- \to H_0''\left(F_t^{(\mu)}(x,y,z)\right) \subseteq D_{BL}, t \mapsto H_0''\left(F_t^{(\mu)}(x,y,z)\right) \subseteq \bigoplus_{\frac{-4}{21}\leq\alpha\leq\frac{4}{21}} C^\alpha \oplus V^{>(\alpha_{24}-1)} \tag{437}$$

The classifying space D_{BL} is of dimension 6. An element in it is uniquely determined by the sections $s\left(\omega_0, \frac{-3}{21}\right)(t)$, $s\left(\omega_0, \frac{-2}{21}\right)(t)$, $s\left(\omega_0, \frac{1}{21}\right)(t)$, $s\left(\omega_0, \frac{4}{21}\right)(t)$ and $s\left(z\omega_0, \frac{1}{21}\right)(t)$, where $\omega_0 := dxdydz$. Using Proposition A.1.4 we obtain the following representations of these sections

$$s\left(\omega_0, \frac{-3}{21}\right)(t) = t_2 b_1 + t_3 b_2, \tag{438}$$
$$s\left(\omega_0, \frac{-2}{21}\right)(t) = (t_5 + \text{polynomial w/ complex coeff. in } t_2 \text{ and } t_3)\, b_3, \tag{439}$$
$$s\left(\omega_0, \frac{1}{21}\right)(t) = (t_4 + \text{polynomial w/ complex coeff. in } t_5,\ t_2 \text{ and } t_3)\, b_4, \tag{440}$$
$$s\left(\omega_0, \frac{4}{21}\right)(t) = (t_1 + \text{polynomial w/ complex coeff. in } t_4,\ t_5,\ t_2 \text{ and } t_3)\, b_5, \tag{441}$$
$$s\left(z\omega_0, \frac{1}{21}\right)(t) = (\text{polynomial w/ complex coeff. in } t_5,\ t_2 \text{ and } t_3)\, b_4, \tag{442}$$

where

$$b_1 = -\partial_\tau s\left[yz^5\omega_0\right](0) \in C^{\frac{-3}{21}}, \tag{443}$$
$$b_2 = -\partial_\tau s\left[xz^5\omega_0\right](0) \in C^{\frac{-3}{21}}, \tag{444}$$
$$b_3 = -\partial_\tau s\left[xyz^3\omega_0\right](0) \in C^{\frac{-2}{21}}, \tag{445}$$
$$b_4 = -\partial_\tau s\left[xyz^4\omega_0\right](0) \in C^{\frac{1}{21}}, \tag{446}$$
$$b_5 = -\partial_\tau s\left[xyz^5\omega_0\right](0) \in C^{\frac{4}{21}}. \tag{447}$$

Now it is easy to see that two elements $(t_1,t_2,t_3,t_4,t_5), \left(\widetilde{t}_1,\widetilde{t}_2,\widetilde{t}_3,\widetilde{t}_4,\widetilde{t}_5\right) \in T_-$ with the same image under the period map must have $t_2 = \widetilde{t}_2$ and $t_3 = \widetilde{t}_3$ by (438). Plugging this into the equations (439), (440), (441) succeedingly we obtain $(t_1,t_2,t_3,t_4,t_5) = \left(\widetilde{t}_1,\widetilde{t}_2,\widetilde{t}_3,\widetilde{t}_4,\widetilde{t}_5\right)$. So the map $T_- \to D_{BL}$ is injective. Thus, the marked Brieskorn lattices of the marked singularities in T_- are all different and the marked singularities are all not right equivalent. So this yields an embedding $T_- \hookrightarrow \left(M_\mu^{mar}\right)^0$. On the other hand, we have the immersion $BL : \left(M_\mu^{mar}\right)^0 \to D_{BL}$ (see Theorem 3.2.1). All in all, the situation looks as follows

$$\begin{tikzcd} T_- \arrow[r, hook] \arrow[d, "\cong"] & (M_\mu^{mar})^0 \arrow[d, "BL"] \\ D_{BL}^{sub} \arrow[r, hook] & D_{BL} \end{tikzcd} \tag{448}$$

where $D_{BL}^{sub} \subseteq D_{BL}$ is the image of T_- under the period map $T_- \to D_{BL}$. It rests to show that $\left(M_\mu^{mar}\right)^0 = M_\mu^{mar}$, i.e. $G^{mar}(f_0) = G_{\mathbb{Z}}$.
First of all we notice that there is a good $\mathbb{C}^*$-action on T_- given by

$$c * (t_1, t_2, t_3, t_4, t_5) = \left(c^{-24} t_1, c^{-3} t_2, c^{-3} t_3, c^{-15} t_4, c^{-6} t_5\right) \tag{449}$$

with $c \in \mathbb{C}^*$. It commutes with the action of $G_{\mathbb{Z}}$. This yields

$$\begin{aligned} G^{mar} = \mathrm{Stab}_{G_{\mathbb{Z}}}([(f_0, \pm\mathrm{id})]) &=_{\text{Prop. 3.1.6 (g)}} G_{\mathcal{R}}^{mar}(f_0) \\ &=_{\text{Prop. 3.1.6 (d)}} G_{\mathcal{R}}^{smar}(f_0) \times \{\pm\mathrm{id}\} \\ &\cong_{\text{Prop. 3.1.6 (c)+Thm. A.1.5}} \mathrm{Stab}_{G_w}(f_0) \times \{\pm\mathrm{id}\}. \end{aligned} \tag{450}$$

The stabilizer group $\mathrm{Stab}_{G_w}(f_0)$ contains exactly 126 elements by [Ba03, Example 7][4] and, thus, G^{mar} contains $2 \cdot 126$ elements. So it suffices to show that $G_{\mathbb{Z}}$ contains at most those $2 \cdot 126$ elements.
Therefore, we consider the canonical map $G_{\mathbb{Z}} \to \mathrm{Aut}(B_3, L)$. The kernel of this map is

$$\ker(G_{\mathbb{Z}} \to \mathrm{Aut}(B_3, L)) =_{\text{Prop. 4.6.1}} \{\pm M_h^k \mid \pm (M_h|_{B_3})^k = \mathrm{id}\} = \{\mathrm{id}, M_h^7, M_h^{14}\}. \tag{451}$$

Thus, the cardinality of the kernel is $|\ker(G_{\mathbb{Z}} \to \mathrm{Aut}(B_3, L))| = 3$. In total we obtain

$$2 \cdot 126 = |G^{mar}| \le |G_{\mathbb{Z}}| = 3 \cdot |\mathrm{im}(G_{\mathbb{Z}} \to \mathrm{Aut}(B_3, L))| \tag{452}$$

$$\le 3 \cdot |\mathrm{Aut}(B_3, L)| =_{\text{Prop. 4.6.1}} 3 \cdot 14 \cdot 6 = 2 \cdot 126. \tag{453}$$

So it is $\mathrm{im}(G_{\mathbb{Z}} \to \mathrm{Aut}(B_3, L)) = \mathrm{Aut}(B_3, L)$, $G^{mar} = G_{\mathbb{Z}} = \{\pm M_h^k \mid k \in \mathbb{Z}\} \times \mathrm{Sym}(3)$ and $\left(M_\mu^{mar}\right)^0 = M_\mu^{mar}$. □

I believe that the methodology used above can be easily adapted to treat all singularities of the form $x^3 + y^3 + z^l$ where l is not divisible by 3. Those are singularities in the U_p-series of Arnold. So the Torelli Conjectures 3.2.2 (a) and 3.2.2 (b) should be true for the respective infinite subseries of the U_p-series as well.

4.7 OVERVIEW AND OUTLOOK

In this chapter we have seen (among other things) proofs of the Torelli results for all types of singularities on Arnold's list [Ar76] of singularities of modality ≤ 2. Above that we have considered the case U_{24}. In particular, we know now that the Torelli Conjectures 3.2.2 (a) and 3.2.2 (b) and the Conjectures 3.2.2 (c) and 3.1.2 are true in all aforementioned cases. This closes some long-standing gaps as described in Remark 4.4.6. Note that, it was shown en passant that Orlik's Conjecture A.2.2 is true in the cases where it applies. A general understanding of the analytic behavior of singularities inside some μ-constant family is still lacking and no general proof that the period maps LBL and BL are injective is in

4 Note that there is a misprint in [GHP97, Example 2.8]. Namely, the true size of the stabilizer group in [Ba03] differs by a factor of 7 from the size given in [GHP97].

sight. However I am sure that the framework with the moduli space of marked singularities M_μ^{mar} is the most promising in order to answer these questions. What is especially surprising is that M_μ^{mar} is not connected in general (in contrast to Conjecture [He11, Conjecture 3.2 (a)]) and that $G_{\mathbb{Z}}$ is rather small. The latter means that $(Ml\,(f)\,, L)$ is rigid, i.e. the condition on automorphisms to respect L is a rather strong condition.

We list all data which is currently available in the literature. This includes in particular the results on Brieskorn-Pham singularities with pairwise coprime exponents and the results on semiquasihomogeneous singularities with weights $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ from [He98]. Note that for Brieskorn-Pham singularities with pairwise coprime exponents the strong global Torelli Conjecture 3.2.2 (a) is also true (see [He11, Remarks 8.5]). We start with the classifying space D_{BL}. The number $N_{BL} \in \mathbb{Z}_{\geq 0}$ in Table 12 depends on the type of the singularity, but it is always strictly smaller than $\frac{1}{4}\mu^2$ (cf. Section 2.2). In many cases the exact numerical value of N_{BL} can be found in the respective sections of Chapter 4.

Singularity family	D_{BL}
Simple sing.	$\{pt\}$
Simple elliptic sing. $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$	$\mathbb{H}$
Hyperbolic sing. $T_{p,q,r}$	$\mathbb{C}$
Exceptional unimodal sing.	$\mathbb{C}$
Exceptional bimodal sing.	$\mathbb{C}^2$
Quadrangle sing.	$\mathbb{H} \times \mathbb{C}$
Generic bimodal series, e.g. $E_{3,p}$ with $18 \not| p$	$\mathbb{C}^{N_{BL}}$
Bimodal subseries, e.g. $E_{3,p}$ with $18 \mid p$	$\mathbb{H} \times \mathbb{C}^{N_{BL}}$
Brieskorn-Pham sing. with coprime exponents	$\mathbb{C}^{N_{BL}}$
Semiquasihom. sing. with weights $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$	$\{z \in \mathbb{C}^4 \mid \lvert z \rvert < 1\} \times \mathbb{C}$
U_{24}, i.e. $x^3 + y^3 + z^7$	$\mathbb{C}^6$

Table 12: Overview of the results for D_{BL}

Now we list the groups $G_{\mathbb{Z}}/\{\pm M_h^k \mid k \in \mathbb{Z}\}$ (cf. Table 13). This group (resp. the group $G_{\mathbb{Z}}$) was not computed in [He98], because there the proof of the Torelli result used some special classical result on cubics in three-dimensional projective space $P^3\mathbb{C}$ instead. Now we make a quick ad-hoc computation to gain at least some information on the group $G_{\mathbb{Z}}$ for semiquasihomogeneous singularities f with weights $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and to show that $G_{\mathbb{Z}}$ is again "small" and $(Ml\,(f)\,, L)$ is rigid in this case.

Proposition 4.7.1. *For semiquasihomogeneous singularities with weights $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ Orlik's Conjecture A.2.2 is true and the lattice $(Ml\,(f)\,, L)$ is "somewhat rigid".*

Proof. For such singularities the Stokes matrix S is given as

$$S := S\left(x_0^3 + x_1^3 + x_2^3 + x_3^3\right) = S\left(x_0^3\right) \otimes S\left(x_1^3\right) \otimes S\left(x_2^3\right) \otimes S\left(x_3^3\right). \tag{454}$$

From this Stokes matrix we can compute the monodromy operator M_h on the Milnor lattice via the formulas in Chapter 2. The characteristic polynomial of the monodromy is $p_{ch} = \Phi_1^6\Phi_3^5$. We can compute that

$$\begin{aligned} Ml\left(f\right) = &\bigoplus_{i=0}^{2}\mathbb{Z}\cdot M_h^i\left(\delta_4\right) \oplus \bigoplus_{i=0}^{2}\mathbb{Z}\cdot M_h^i\left(\delta_1\right) \oplus \bigoplus_{i=0}^{2}\mathbb{Z}\cdot M_h^i\left(\delta_8\right) \\ &\oplus \bigoplus_{i=0}^{2}\mathbb{Z}\cdot M_h^i\left(\delta_{11}\right) \oplus \bigoplus_{i=0}^{2}\mathbb{Z}\cdot M_h^i\left(\delta_{14}\right) \oplus \mathbb{Z}\cdot\left(\delta_6-\delta_7-\delta_{10}+\delta_{11}\right) \end{aligned} \tag{455}$$

is a decomposition of the Milnor lattice and that the monodromy with respect to this new basis has the form

$$M_M = \begin{pmatrix} S_{\Phi_3\Phi_1} & & & & & \\ & S_{\Phi_3\Phi_1} & & & & \\ & & S_{\Phi_3\Phi_1} & & & \\ & & & S_{\Phi_3\Phi_1} & & \\ & & & & S_{\Phi_3\Phi_1} & \\ & & & & & S_{\Phi_1} \end{pmatrix}. \tag{456}$$

This exactly says that Orlik's Conjecture A.2.2 is true. To prove the remaining bit of claim, we consider the space $Ml\left(f\right)_1$. It is 6-dimensional (cf. [He98, Section 5]). The respective lattice $Ml\left(f\right)_{1,\mathbb{Z}}$ is generated by the vectors $\gamma_1 := \delta_4 + M_h\left(\delta_4\right) + M_h^2\left(\delta_4\right)$, $\gamma_2 := \delta_1 + M_h\left(\delta_1\right) + M_h^2\left(\delta_1\right)$, $\gamma_3 := \delta_8 + M_h\left(\delta_8\right) + M_h^2\left(\delta_8\right)$, $\widetilde{\gamma_4} := \delta_{11} + M_h\left(\delta_{11}\right) + M_h^2\left(\delta_{11}\right)$, $\gamma_5 := \delta_{14} + M_h\left(\delta_{14}\right) + M_h^2\left(\delta_{14}\right)$, $\gamma_6 := \delta_6 - \delta_7 - \delta_{10} + \delta_{11}$. We then do a base change $\gamma_4 := 4\widetilde{\gamma_4} - 3\gamma_6$. Then the Seifert form with respect to those elements looks as follows

$$L\left(\begin{pmatrix}\gamma_1\\ \gamma_2\\ \vdots\\ \gamma_6\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\\ \vdots\\ \gamma_6\end{pmatrix}^T\right) = \begin{pmatrix} -6 & -3 & 3 & 0 & 0 & 0 \\ -3 & -6 & 3 & -12 & 3 & 0 \\ 3 & 3 & -6 & 0 & 0 & 0 \\ 0 & -12 & 0 & -60 & 0 & 0 \\ 0 & 3 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{pmatrix}. \tag{457}$$

Note that this matrix is negative definite. Now we can easily see that each automorphism on the Milnor lattice which respects L must map the following set of "big vectors" to itself:

$$\{a \in Ml(f)_1 \,|\, L(a,a) = -4\} = \{\pm\gamma_6\} = \{\pm\left(\delta_6-\delta_7-\delta_{10}+\delta_{11}\right)\}. \tag{458}$$

Since γ is a generator of the Orlik block $\mathbb{Z}\cdot\left(\delta_6-\delta_7-\delta_{10}+\delta_{11}\right)$, this means that every $g \in G_{\mathbb{Z}}$ acts like the identity (up to sign) on this block. This gives some rigidity for $\left(Ml\left(f\right), L\right)$. $\square$

Singularity family	$G_{\mathbb{Z}}/\{\pm M_h^k \mid k \in \mathbb{Z}\}$
Simple sing.	$\{\mathrm{id}\}$ or $\mathrm{Sym}\,(2)$ or $\mathrm{Sym}\,(3)$
Simple elliptic sing. $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$	a finite extension of $\mathrm{SL}(2;\mathbb{Z})$
Hyperbolic sing. $T_{p,q,r}$	a finite group
Exceptional unimodal sing.	$\{\mathrm{id}\}$ or $\mathrm{Sym}\,(2)$ or $\mathrm{Sym}\,(3)$
Exceptional bimodal sing.	$\{\mathrm{id}\}$ or $\mathrm{Sym}\,(2)$ or $\mathrm{Sym}\,(3)$
Quadrangle sing.	a triangle group
Generic bimodal series, e.g. $E_{3,p}$ with $18 \not| p$	a cyclic finite group
Bimodal subseries, e.g. $E_{3,p}$ with $18 \mid p$	an infinite Fuchsian group
Brieskorn-Pham sing. with coprime exponents	$\{\mathrm{id}\}$
Semiquasihom. sing. with weights $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$	?
U_{24}, i.e. $x^3 + y^3 + z^7$	$\mathrm{Sym}\,(3)$

Table 13: Overview of the results for $G_{\mathbb{Z}}$

Now we give descriptions of the moduli space of marked singularities $M_\mu^{mar}(f_0)$ for all different types of reference singularities f_0 that were computed so far (in this monograph and beyond).

Singularity family	Isom. class of $M_\mu^{mar}(f_0)$
Simple sing.	$\{pt\}$
Simple elliptic sing. $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$	$\mathbb{H}$
Hyperbolic sing. $T_{p,q,r}$	$\mathbb{C}$
Exceptional unimodal sing.	$\mathbb{C}$
Exceptional bimodal sing.	$\mathbb{C}^2$
Quadrangle sing.	$(\mathbb{H} \setminus \{\text{discrete set}\}) \times \mathbb{C}$
Generic bimodal series, e.g. $E_{3,p}$ with $18 \not| p$	$\mathbb{C}^* \times \mathbb{C}$
Bimodal subseries, e.g. $E_{3,p}$ with $18 \mid p$	∞-many copies of $\mathbb{C}^* \times \mathbb{C}$
Brieskorn-Pham sing. with coprime exponents	$\mathbb{C}^{\bmod (f)}$
Semiquasihom. sing. with weights $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$	?
U_{24}, i.e. $x^3 + y^3 + z^7$	$\mathbb{C}^5$

Table 14: Overview of the results for $M_\mu^{mar}(f_0)$

Finally, it is worth mentioning that there are ideas to study families of singularities that are not μ-constant and to formulate Torelli Conjectures in those cases as well. Such a family is for example given by functions that have a certain number μ of critical values that are all singularities of type A_1. This is the *semisimple case*. More precisely, in the unfinished and unpublished manuscript [GH18-2] the moduli space M_μ^{mar} is thickened to a μ-dimensional F-manifold M^{mar} which is locally at each point of M_μ^{mar} the base space of a semiuniversal unfolding. Torelli Conjectures for all the μ-homotopy strata of multigerms of singularities in M_μ^{mar} are formulated. Of course, this new Torelli Conjectures must be again verified. In the semisimple case this follows from a classical result of Looijenga and Deligne for the simple singularities and for the simple elliptic singularities it was done by Hertling and Roucairol (see [HR18] for further information). The general cases for other strata and other types of singularities are widely open. So this gives a fruitful area for future research on Torelli problems for isolated hypersurface singularities.

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APPENDIX

A.1 NEWTON NONDEGENERATE AND SEMIQUASIHOMOGENEOUS SINGULARITIES

Many of the singularities studied in this thesis have a nondegenerate Newton polytope and/or are (semi)quasihomogeneous. Hence, it makes sense to summarize some of the properties of those singularities in this section. We start with Newton nondegenerate singularities and set $\omega_0 := dx_0 \dots dx_n$.

A.1.0.1 *Newton nondegenerate singularities*

Consider a singularity given as $f = \sum_{i \in \mathbb{Z}^{n+1}_{\geq 0}} a_i x^i$. Then we have the following sets

$$\operatorname{supp}(f) := \{i \in \mathbb{Z}^{n+1}_{\geq 0} \mid a_i \neq 0\}, \tag{459}$$

$$\Gamma_+(f) := \text{convex hull of } \left(\bigcup_{i \in \operatorname{supp}(f)} \left(i + \mathbb{R}^{n+1}_{\geq 0} \right) \right) \subseteq \mathbb{R}^{n+1}, \tag{460}$$

$$\Gamma_{com}(f) := \{\sigma \mid \sigma \text{ is a compact face of } \Gamma_+(f)\}, \tag{461}$$

$$\Gamma_{com,n}(f) := \{\sigma \in \Gamma_{com}(f) \mid \dim \sigma = n\} \tag{462}$$

and the linear function $l_\sigma : \mathbb{R}^{n+1} \to \mathbb{R}$ with $\sigma \subseteq l_\sigma^{-1}(1)$ for $\sigma \in \Gamma_{com,n}(f)$. Here we refer to $\Gamma_+(f)$ as the *Newton polyhedron* of f. Finally we have the *Newton order* $\nu : \mathbb{C}\{x_0, x_1, \dots, x_n\} \to \mathbb{Q}_{\geq 0} \cup \{\infty\}$ given by

$$\nu\left(\sum_i b_i x^i\right) := \min\{l_\sigma(i) \mid \text{ all } i \text{ with } b_i \neq 0, \text{ all } \sigma \in \Gamma_{com,n}(f)\}. \tag{463}$$

This can be extended to a Newton order $\nu : \Omega^{n+1}_{\mathbb{C}^{n+1},0} \to \mathbb{Q}_{\geq 0} \cup \{\infty\}$ on the sheaf of $n+1$-forms via

$$\nu\left(\left(\sum_i b_i x^i\right) \cdot \omega_0\right) := \nu\left(\left(\sum_i b_i x^i\right) \cdot x_0 \dots x_n\right) \tag{464}$$

and to a Newton order $\overline{\nu} : H_0''(f) \to \mathbb{Q}_{\geq 0} \cup \{\infty\}$ on the Brieskorn lattice via

$$\overline{\nu} := \max\{\nu(\eta) \mid \eta \equiv \omega \mod df \wedge d\Omega^{n-1}_{\mathbb{C}^{n+1},0}\}. \tag{465}$$

Now we can define the main property of this subsection. Here we follow the notations and definitions of Kouchnirenko (see [Ko76]). Note that we will use multiindex notation for $i = (i_0, i_1, \dots, i_n)$ throughout this section.

Definition A.1.1. *For f as above we set $f_\sigma := \sum_{i\in\sigma} a_i x^i$. Then f is called* Newton nondegenerate *if for each $\sigma \in \Gamma_{com}(f)$ the Jacobi ideal of f_σ has no zero in $(\mathbb{C}^*)^{n+1}$. It is called* convenient *if it contains for each index $j \in \{0,\dots,n\}$ a monomial $x_j^{m_j}$ with $m_j \geq 2$.*

For convenient and Newton nondegenerate singularities we have the following characterization of the order of a form via its Newton order.

Theorem A.1.2 ([SaM88]). *Let f be a convenient and Newton nondegenerate singularity. Then for any $\omega \in \Omega^{n+1}_{\mathbb{C}^{n+1},0}$ the order is $\alpha(\omega) = \overline{\nu}(\omega) - 1$.*

From this theorem it is possible to deduce the following corollary.

Corollary A.1.3 ([He93, Satz 1.10]). *Let f be a convenient and Newton nondegenerate singularity. We define the index set $I(f)$ as*

$$I(f) := \{i \in \mathbb{Z}^{n+1}_{\geq 0} \,|\, \nu\left(x^i \cdot \omega_0\right) - 1 < s(f)\} \tag{466}$$

where $s(f) := \min\{\nu\left(\frac{\partial f}{\partial x_j} \cdot \omega_0\right) - 1 \,|\, j \in \{0,\dots,n\}\} > 0$.
Then it holds

(a) $\alpha(x^i \cdot \omega_0) = \nu(x^i \cdot \omega_0) - 1$ *for each* $i \in I(f)$,

(b) *the numbers* $\alpha(x^i \omega_0)$ *for* $i \in I(f)$ *are the spectral numbers in the open interval* $(-1, s(f))$,

(c) *for some* $g := \sum_i b_i x^i$ *it is*

$$\alpha(g \cdot \omega_0) = \begin{cases} \min\{\alpha(x^i \cdot \omega_0) \,|\, i \in I(f), b_i \neq 0\} & \text{, if an } i \in I(f) \text{ with } b_i \neq 0 \text{ exists,} \\ \geq s(f) & \text{, otherwise.} \end{cases}$$

We have similar characterizations of the order for so-called semiquasihomogeneous singularities. This will be subject of the next subsection.

A.1.0.2 *Semiquasihomogeneous singularities*

A singularity f is called *semiquasihomogeneous* with weights $w_1, \dots, w_n \in \mathbb{Q}_{>0}$ if

$$f = \sum_{i \in \mathbb{Z}^{n+1}_{\geq 0}} a_i x^i \text{ with } \deg_w x^i \geq 1 \text{ for all } i \text{ with } a_i \neq 0, \tag{467}$$

and its *principal part* $f_{qh} := \sum_{i:\deg_w x^i = 1} a_i x^i$ has an isolated singularity at 0. Moreover f is called *quasihomogeneous (of weighted degree one)* if it is semiquasihomogeneous and $f_{qh} = f$. For any singularity f that is semiquasihomogeneous but not quasihomogeneous, the 1-parameter family $f_{qh} + t \cdot (f - f_{qh})$ with $t := (t_1, \dots, t_l)$ is a μ-constant family (see [AGV85, Chapter 12]). We use the multiindex notations $t^k := t_1^{k_1} + \dots t_l^{k_l}$, $|k| := k_1 + \dots + k_l$ and $k! := k_1! \dots k_l!$ for $k := (k_1, \dots, k_l) \in \mathbb{Z}^l_{\geq 0}$. Then the following characterization of the order holds.

Proposition A.1.4. *Let f be a singularity.*

(a) If f is quasihomogeneous with weights $(w_0,\ldots,w_n)$ and $\omega = x^i\omega_0$ is a monomial differential form, then either $s\,[\omega]_0 = 0$ or $\alpha(\omega) = \deg_w(x^i x_0 \ldots x_n) - 1$ and $s[\omega]_0 = s(\omega,\alpha(\omega))$.

(b) If f is semiquasihomogeneous with weights $(w_0,\ldots,w_n)$, but not quasihomogeneous, and $\omega = x^i\omega_0$ is a monomial differential form, then

$$\alpha(\omega) \geq \deg_w\left(x^i x_0 \ldots x_n\right) - 1, \tag{468}$$

$$s\,[\omega]_0\,(0) = s\left(\omega, \deg_w\left(x^i x_0 \ldots x_n\right) - 1\right)(t), \tag{469}$$

$$s(\omega,\alpha)(t) = \sum_{|k|\geq 0} \frac{1}{k!} \cdot t^k \cdot (-\partial_\tau)^{|k|}\, s\left((f - f_{qh})^k \cdot \omega, \alpha + |k|\right)(t). \tag{470}$$

Note that the last expression is polynomial in t, because $\alpha\left((f-f_{qh})^k \cdot \omega\right) > \alpha + |k|$ for $|k| >> 0$.

We omit a proper proof of Proposition A.1.4 here. However some arguments can be found in [GH18, Section 7]. Finally we present an older result on the calculation of R_f in the case of quasihomogeneous singularities.

Theorem A.1.5 ([He02-2, Theorem 13.11])**.** *Let $f \in \mathbb{C}[x_0,\ldots,x_n]$ be a quasihomogeneous singularity with weights $w_0,\ldots,w_n \in \left(0,\frac{1}{2}\right]$. Assume that $w_0 \leq \ldots \leq w_{n-1} < \frac{1}{2}$. Then it holds*

$$R_f \cong \mathrm{Stab}_{G_w}(f) \tag{471}$$

where G_w is the algebraic group of quasihomogeneous coordinate changes, i.e. the automorphisms on $\mathbb{C}[x_0,\ldots,x_n]$ that respect the grading by the weights $w_0,\ldots,w_n$.

A.2 LATTICES

The main results of this monograph rely on extensive computations in integer lattices. In this subsection we summarize many important properties of these lattices.

Here a *($\mathbb{Z}$-)lattice* is a free $\mathbb{Z}$-module of rank $\mu \in \mathbb{Z}_{>0}$. It will usually come equipped with one (or more) bilinear forms. We call such a bilinear form *unimodular* if the determinant of the bilinear form has absolute value one with respect to a basis of the lattice. Furthermore, we call it *symmetric* if the matrix representation of the bilinear form is symmetric with respect to a basis of the lattice.
Now let Λ be a $\mathbb{Z}$-lattice of rank μ and $L : \Lambda \times \Lambda \to \mathbb{Z}$ be a unimodular bilinear form. We have a unique automorphism $M_h : \Lambda \to \Lambda$ given by

$$L(M_h(a), b) = -L(b,a) \tag{472}$$

for $a, b \in \Lambda$. This then immediately implies that the bilinear form L is M_h-invariant, i.e. $L(M_h(a), M_h(b)) = L(a,b)$ for all $a,b \in \Lambda$. As the notation suggests L will be the Seifert form and M_h will be the monodromy of an isolated hypersurface singularity in the main

part of the thesis. So we assume that M_h is semisimple and its eigenvalues are roots of unity. According to Theorem 2.1.1 the eigenvalues of the monodromy of isolated hypersurface singularities are always roots of unity.
For quasihomogeneous singularities and their Milnor lattices we have the following conjecture, which is due to the American mathematician Peter Orlik. Loosely speaking it says that the Milnor lattice is a sum of certain cyclic monodromy modules with certain characteristic polynomials $p_{\Lambda^{(j)}}$. Here we define for any subring $R \subseteq \mathbb{C}$ the tensor product $\Lambda_R := \Lambda \otimes_{\mathbb{Z}} R$. Moreover, for any monodromy invariant subspace $V \subseteq \Lambda_{\mathbb{C}}$ we denote by $E(V) \subseteq S^1$ the set of eigenvalues of M_h on V and by p_V its characteristic polynomial. For $\lambda \in E(V)$ we write $V_\lambda := \ker(M_h - \lambda\,\mathrm{id} : V \to V) \subseteq V$. For any monodromy invariant sublattice $\Lambda^{(1)} \subseteq \Lambda$ we write $E(\Lambda^{(1)}) := E(\Lambda^{(1)}_{\mathbb{C}})$ and $p_{\Lambda^{(1)}} := p_{\Lambda^{(1)}_{\mathbb{C}}}$ and $\Lambda^{(1)}_\lambda := (\Lambda^{(1)}_{\mathbb{C}})_\lambda$. For any product $p \in \mathbb{Z}[t]$ of cyclotomic polynomials with $p|p_{\Lambda^{(1)}}$ we write

$$\Lambda^{(1)}_{\mathbb{C},p} := \bigoplus_{\lambda:\, p(\lambda)=0} \Lambda^{(1)}_\lambda \quad \text{and} \quad \Lambda^{(1)}_p := \Lambda^{(1)}_{\mathbb{C},p} \cap \Lambda^{(1)}. \tag{473}$$

Then $\Lambda^{(1)}_p$ is a primitive[1] and monodromy invariant sublattice of $\Lambda^{(1)}$. Now the specific cyclic monodromy modules, which are important for the conjecture, are defined next.

Definition A.2.1. *Let (Λ, L, M_h) be as above. An* Orlik block *is a primitive and monodromy invariant sublattice $\Lambda^{(1)} \subseteq \Lambda$ with $\Lambda^{(1)} \supsetneqq \{0\}$ and with a cyclic generator, i.e. a lattice vector $e^{(1)} \in \Lambda^{(1)}$ with*

$$\Lambda^{(1)} = \bigoplus_{j=0}^{\deg p_{\Lambda^{(1)}}-1} \mathbb{Z} \cdot M_h^j(e^{(1)}). \tag{474}$$

Now we can formulate Orlik's Conjecture from 1972.

Conjecture A.2.2 ([Or72, Conjecture 3.1])**.** *Let (Λ, M_h) be the Milnor lattice with monodromy of a quasihomogeneous singularity. Let $k := \max(\dim \Lambda_\lambda \,|\, \lambda \in E(\Lambda))$. Then a decomposition $\Lambda = \bigoplus_{j=1}^k \Lambda^{(k)}$ into Orlik blocks $\Lambda^{(1)}, \ldots, \Lambda^{(k)}$ with $p_{\Lambda^{(j+1)}}|p_{\Lambda^{(j)}}$ for $0 \leq j < k$ exists. The $p_{\Lambda^{(1)}}, \ldots, p_{\Lambda^{(k)}}$ are unique. They are*

$$p_{\Lambda^{(j)}} = \prod_{\lambda \in E(\Lambda):\, \dim \Lambda_\lambda \geq j} (t - \lambda) \quad \textit{for } j = 1, \ldots, k. \tag{475}$$

Although Orlik's Conjecture is rather old, it is not very well understood. There are proofs of Orlik's Conjecture for quasihomogeneous cycle type singularities, see [Co82], and quasihomogeneous curve singularities, see [MW86]. But both proofs are not very well documented and it is unclear if they are completely correct. At least in the cases that we are considering in the main part of this monograph, i.e. quasihomogeneous singularities with modality ≤ 2, there Orlik's Conjecture is known to be true by [He93].

One of the working horses in the computation of lattice automorphism groups is the lemma below. It can be considered as an analogue of a number theoretic result of Kronecker

1 A sublattice $\Lambda^{(1)}$ of Λ is called *primitive* if $\Lambda/\Lambda^{(1)}$ has no torsion. For any sublattice $\Lambda^{(2)} \subseteq \Lambda$ there is a unique primitive sublattice $\Lambda^{(3)}$ with $\Lambda^{(3)}_{\mathbb{Q}} = \Lambda^{(2)}_{\mathbb{Q}}$. It is $\Lambda^{(3)} = \Lambda^{(2)}_{\mathbb{Q}} \cap \Lambda$ and it satisfies $[\Lambda^{(3)} : \Lambda^{(2)}] < \infty$.

(see [Co07, Proposition 3.3.9] for an up-to-date reference). It originates from [He11], but similar arguments can be already found in [He98, Chapter 6].

Lemma A.2.3 ([He11, Lemma 8.2])**.** *Suppose that Λ is a single Orlik block. For an eigenvalue of the monodromy we denote* $\mathrm{ord}\,(\lambda) := \min\{k \in \mathbb{N} \mid \lambda^k = 1\}$ *and set* $\mathrm{Ord} := \{\mathrm{ord}\,(\lambda)\,, \mid \lambda$ *eigenvalue of* $M_h\} \subseteq \mathbb{Z}_{\geq 1}$. *We assume for the set* $\mathrm{Ord} \subseteq \mathbb{Z}_{\geq 1}$ *that there are four sequences* $(m_i)_{i=1,\dots,|\,\mathrm{Ord}\,|}$, $(j(i))_{i=2,\dots,|\,\mathrm{Ord}\,|}$, $(p_i)_{i=2,\dots,|\,\mathrm{Ord}\,|}$, $(k_i)_{i=2,\dots,|\,\mathrm{Ord}\,|}$ *of numbers in* $\mathbb{Z}_{\geq 1}$ *and two numbers* $i_1, i_2 \in \mathbb{Z}_{\geq 1}$ *with* $i_1 \leq i_2 \leq |\,\mathrm{Ord}\,|$ *and with the properties:*

- $\mathrm{Ord} = \{m_1, \dots, m_{|\,\mathrm{Ord}\,|}\}$,
- p_i *is a prime number,* $p_i = 2$ *for* $i_1 + 1 \leq i \leq i_2$, $p_i \geq 3$ *else,*
- $j(i) = i - 1$ *for* $i_1 + 1 \leq i \leq i_2$, $j(i) < i$ *else,*
- $m_i = m_{j(i)}/p_i^{k_i}$.

Then it holds

$$\mathrm{Aut}(\Lambda, M_h, L) = \{\pm M_h^k \mid k \in \mathbb{Z}\}.$$

Note that, for an Orlik block $\Lambda^{(1)} \subseteq \Lambda$ with cyclic generator $e^{(1)}$ and $p_{\Lambda^{(1)}} = p_1 \cdot p_2$ where $\deg p_1 \geq 1$ and $\deg p_2 \geq 1$ the sublattice $\Lambda^{(1)}_{p_1}$ is also an Orlik block. It is generated by the cyclic generator

$$e^{(2)} := p_2(M_h)(e^{(1)}). \tag{476}$$

Moreover, if $\lambda \in E(\Lambda^{(1)})$ is an eigenvalue of the monodromy on $\Lambda^{(1)}$, then there is an eigenvector eigenvector as follows

$$v(e^{(1)}, \lambda) := \frac{p_{\Lambda^{(1)}}}{t - \lambda}(M_h)(e^{(1)}). \tag{477}$$

It holds

$$\begin{aligned} L(v(e^{(1)}, \lambda), v(e^{(1)}, \overline{\lambda})) &= L(v(e^{(1)}, \lambda), \frac{p_{\Lambda^{(1)}}}{t - \overline{\lambda}}(M_h)(e^{(1)})) \\ &= L(\frac{p_{\Lambda^{(1)}}}{t - \overline{\lambda}}(M_h^{-1})v(e^{(1)}, \lambda), e^{(1)}) \\ &= \frac{p_{\Lambda^{(1)}}}{t - \overline{\lambda}}(\overline{\lambda}) \cdot L(v(e^{(1)}, \lambda), e^{(1)}) \\ &= \frac{p_{\Lambda^{(1)}}}{t - \overline{\lambda}}(\overline{\lambda}) \cdot L(\frac{p_{\Lambda^{(1)}}}{t - \lambda}(M_h)(e^{(1)}), e^{(1)}). \end{aligned} \tag{478}$$

This allows us to formulate the following lemma. In contrast to Lemma A.2.3 it will not need L — but only M_h.

Lemma A.2.4. *Let* (Λ, M_h) *be as above. Let* $\Lambda^{(1)}, \dots, \Lambda^{(k)} \subseteq \Lambda$ *be Orlik blocks with cyclic generators* $e^{(1)}, \dots, e^{(k)}$ *and with*

$$\Lambda^{(1)} + \ldots + \Lambda^{(k)} = \Lambda^{(1)} \oplus \ldots \oplus \Lambda^{(k)}.$$

Consider an element $g \in \mathrm{Aut}(\Lambda^{(1)} \oplus \ldots \oplus \Lambda^{(k)}, M_h)$. *Then there are unique polynomials* $p_{ij} \in \mathbb{Z}[t]_{<\operatorname{rank}\Lambda^{(j)}}$ *for* $i,j = 1, \ldots, k$ *with*

$$g(e^{(j)}) = \sum_{i=1}^{k} p_{ij}(M_h)(e^{(i)}). \tag{479}$$

Suppose now that $p_0 \in \mathbb{Z}[t]$ *divides* $\gcd(p_{\Lambda^{(1)}}, \ldots, p_{\Lambda^{(k)}})$ *and that*

$$g = \mathrm{id} \quad \textit{on} \quad \Lambda^{(j)}_{p_{\Lambda^{(j)}}/p_0} \quad \textit{for any } j, \tag{480}$$

so that g *acts nontrivial only on* $(\Lambda^{(1)} \oplus \ldots \oplus \Lambda^{(k)})_{p_0}$. *Then*

$$p_{ij} = \delta_{ij} + \frac{p_{\Lambda^{(i)}}}{p_0} \cdot q_{ij} \tag{481}$$

for suitable polynomials $q_{ij} \in \mathbb{Z}[t]_{<\deg p_0}$. *Suppose furthermore that a unit root* ξ *satisfies* $p_0(\xi) = 0$. *Then* g *with respect to the eigenvectors* $v(e^{(1)}, \xi) \in \Lambda^{(1)}_{\xi}, \ldots, v(e^{(k)}, \xi) \in \Lambda^{(k)}_{\xi}$ *(defined in (477)) is given by*

$$g(v(e^{(j)}, \xi)) = \sum_{i=1}^{k} (\delta_{ij} + \frac{p_{\Lambda^{(j)}}}{p_0} \cdot q_{ij})(\xi) \cdot v(e^{(i)}, \xi) \tag{482}$$

Proof. It is sufficient to prove the part after (479). Everything else is trivial. Suppose that p_0 and g are as stated above. By assumption it holds

$$g(e^{(j)}) - e^{(j)} \in (\Lambda^{(1)} \oplus \ldots \oplus \Lambda^{(k)})_{p_0} \subseteq \bigoplus_{i=1}^{k} \Lambda^{(i)}_{\mathbb{C},p_0} = \bigoplus_{i=1}^{k} \frac{p_{\Lambda^{(i)}}}{p_0}(M_h)(\Lambda^{(i)}_{\mathbb{C}}).$$

Thus it is $p_{ij} - \delta_{ij} \in \frac{p_{\Lambda^{(i)}}}{p_0} \cdot \mathbb{C}[t]$ and even more $p_{ij} - \delta_{ij} \in \frac{p_{\Lambda^{(i)}}}{p_0} \cdot \mathbb{Z}[t]_{<\deg p_0}$.

Now the following calculation proves (482).

$$\begin{aligned}
g(v(e^{(j)}, \xi)) &= g\left(\frac{p_{\Lambda^{(j)}}}{t-\xi}(M_h)(e^{(j)})\right) = \frac{p_{\Lambda^{(j)}}}{t-\xi}(M_h)\left(g(e^{(j)})\right) \\
&= \frac{p_{\Lambda^{(j)}}}{t-\xi}(M_h)\left(\sum_{i=1}^{k}\left(\delta_{ij} + \frac{p_{\Lambda^{(i)}}}{p_0} \cdot q_{ij}\right)(M_h)(e^{(i)})\right) \\
&= \sum_{i=1}^{k}\left(\left(\delta_{ij} + \frac{p_{\Lambda^{(i)}}}{p_0} \cdot q_{ij}\right) \cdot \frac{p_{\Lambda^{(j)}}}{t-\xi}\right)(M_h)(e^{(i)}) \\
&= \sum_{i=1}^{k}\left(\delta_{ij} + \frac{p_{\Lambda^{(j)}}}{p_0} \cdot q_{ij}\right)(M_h)(v(e^{(i)}, \xi)) \\
&= \sum_{i=1}^{k}\left(\delta_{ij} + \frac{p_{\Lambda^{(j)}}}{p_0} \cdot q_{ij}\right)(\xi) \cdot v(e^{(i)}, \xi).
\end{aligned}$$

□

The next two lemmas will be useful for many of the quadrangle and bimodal series singularities. There it is $\Lambda^{(1)} = Ml\,(f)$ and $\Lambda = \widetilde{B_1} \oplus B_2$ (cf. with Section 4.4 and 4.5 for the notations).

Lemma A.2.5. *(a) Let $p = \prod_{i\in I}\Phi_{m_i}$ be a product of cyclotomic polynomials. Then $p(1)\equiv 1(2)$ if and only if all $m_i \in \mathbb{Z}_{\geq 1}\setminus\{2^k \mid k\in\mathbb{Z}_{\geq 0}\}$.*

(b) Let (Λ, L, M_h) be as above. Let $\Lambda^{(1)} \subseteq \Lambda$ be an M_h-invariant sublattice with $[\Lambda : \Lambda^{(1)}] = 2$. Write

$$\begin{aligned} p_\Lambda &= p_1\cdot p_2 \quad \text{with } p_j = \prod_{m\in J_j}\Phi_m \\ \text{and} \quad & J_1\subseteq \mathbb{Z}_{\geq 1}\setminus\{2^k \mid k\in\mathbb{Z}_{\geq 0}\},\quad J_2\subseteq\{2^k\mid k\in\mathbb{Z}_{\geq 0}\}. \end{aligned}$$

Then $J_2\neq\emptyset$, $p_2\neq 1$, and

$$\Lambda_p = \Lambda_p^{(1)} \qquad \text{for any } p \text{ with } p|p_1, \tag{483}$$

$$[\Lambda_p : \Lambda_p^{(1)}] = 2 \qquad \text{for any } p \text{ with } p_2|p. \tag{484}$$

Proof. (a) Everbody knows that $\Phi_{2^k}(t) = t^{2^{k-1}}+1$ for $k\geq 1$ and

$$t^{2^k\cdot q}-1 = (t^{2^k}-1)(t^{2^k(q-1)}+t^{2^k(q-2)}+\ldots+t^{2^k}+1). \tag{485}$$

Now for odd $q>1$, the second factor has at $t=1$ the odd value q. Therefore $\Phi_m(1)\equiv 1(2)$ for any m with $2^k|m|2^k\cdot q$ and $2^k\neq m$ with q odd.

(b) For an arbitrary element $\gamma\in\Lambda\setminus\Lambda^{(1)}$ it holds

$$\Lambda\setminus\Lambda^{(1)} = \gamma+\Lambda^{(1)}.$$

This set is M_h-invariant, because $\Lambda^{(1)}$ is M_h-invariant. Thus for any integer $k\in\mathbb{Z}_{\geq 1}$ it is $M_h^k(\gamma)\in\Lambda\setminus\Lambda^{(1)}$. By part (a) it is $p_1(1)\equiv 1(2)$. So we have $p_1(M_h)(\gamma)\in\Lambda\setminus\Lambda^{(1)}$ and, thus,

$$p_1(M_h)(\Lambda\setminus\Lambda^{(1)})\subseteq\Lambda\setminus\Lambda^{(1)}.$$

On the other hand it is

$$\begin{aligned} p_1(M_h)(\Lambda_{p_1}) &= \{0\}\subseteq\Lambda^{(1)}, \quad \text{thus } \Lambda_{p_1}\subseteq\Lambda^{(1)} \quad \text{and, thus, it holds (483).} \\ p_1(M_h)(\Lambda) &\subseteq \Lambda_{p_2}, \quad \text{thus } \Lambda_{p_2}\cap(\Lambda\setminus\Lambda^{(1)})\neq\emptyset \quad \text{and, thus, it holds (484).} \end{aligned}$$

□

A square root on the set $S^1\setminus\{-1\}$ is given via $\sqrt{e^{2\pi i\alpha}} := e^{\pi i\alpha}$ for $\alpha\in(-\frac12,\frac12)$.

Lemma A.2.6. *(a) Let $\lambda\in E(\Lambda)\setminus\{1\}$. Then the sesquilinear[2] form $h_\lambda:\Lambda_\lambda\times\Lambda_\lambda\to\mathbb{C}$ with*

$$h_\lambda(a,b) := \sqrt{-\lambda}\cdot L(a,\overline{b}) \tag{486}$$

2 This means linear in the first coordinate and semilinear in the other one.

is Hermitian, i.e. $h_\lambda(b,a) = \overline{h_\lambda(a,b)}$. *Especially,* $\sqrt{-\lambda}\cdot L(a,\overline{a}) \in \mathbb{R}$. *Together, these forms define a Hermitian form* $h := \bigoplus_{\lambda\in E(\Lambda)\setminus\{1\}} h_\lambda$.

(b) *Let* $V \subseteq \Lambda_{\mathbb{C}}$ *be a monodromy invariant subspace with* $1 \notin E(V)$. *The following two properties are equivalent.*

(α) $h|_V$ *is positive definite.*

(β) *The Hermitian form on* V *defined by* $(a,b) \mapsto L(a,\overline{b}) + L(\overline{b},a)$ *is positive definite.*

Proof. (a) Let $a,b \in V_\lambda$. Then we compute

$$\begin{aligned} h_\lambda(b,a) &= \sqrt{-\lambda}\cdot L(b,\overline{a}) = -\sqrt{-\lambda}\cdot L(M_h(\overline{a}),b) = -\sqrt{-\lambda}\cdot\overline{\lambda}\cdot L(\overline{a},b) \\ &= \sqrt{-\overline{\lambda}}\cdot\overline{L(a,\overline{b})} = \overline{\sqrt{-\lambda}\cdot L(a,\overline{b})} = \overline{h_\lambda(a,b)}. \end{aligned}$$

So $h_\lambda(a,b)$ is Hermitian and the rest of part (a) follows straight away.

(b) Let $\lambda \in E(V)$ and $a,b\in V$. We observe that $\sqrt{-\overline{\lambda}} + \sqrt{-\lambda} > 0$. Now we obtain

$$\begin{aligned} L(a,\overline{b}) + L(\overline{b},a) &= L(a,\overline{b}) + \overline{L(b,\overline{a})} \\ &= \sqrt{-\overline{\lambda}}\cdot h_\lambda(a,b) + \overline{\sqrt{-\overline{\lambda}}\cdot h_\lambda(b,a)} \\ &= (\sqrt{-\overline{\lambda}} + \sqrt{-\lambda})\cdot h_\lambda(a,b). \end{aligned}$$

Thus the form given by $L(a,\overline{b}) + L(\overline{b},a)$ is positive definite if and only if $h_\lambda(a,b)$ is. □

Finally we present a lemma which will be useful for some of the exceptional unimodal and bimodal singularities, as well as singularities of type U_{24}. It can be seen as a natural continuation of Lemma A.2.3 (cf. [GH17, Remark 4.3]). Note that it is stated slightly different than in [GH17, Lemma 4.2] here.

Lemma A.2.7. (a) *Let* $V_{\mathbb{Z}}$ *be a* $\mathbb{Z}$*-lattice of rank 2 with a* $\mathbb{Z}$*-lattice* $b = (b_1,b_2)$ *and a symmetric pairing* $L_{\mathbb{Z}}$ *of the form*

$$\begin{pmatrix} 2 & -1 \\ -1 & m \end{pmatrix}$$

for some $m \in \mathbb{Z}_{>1}$. *Moreover let* $\xi := e^{\frac{2\pi i}{l}}$ *with* $l \in \{3,4,5,6\}$ *if* $m=2$ *and* $\xi := e^{\frac{2\pi i}{l}}$ *with* $l\in\{3,4\}$ *if* $m \geq 3$. *We define* $V_{\mathbb{C}} := V_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{C}$, $V_{\mathbb{Z}[\xi]} := V_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{Z}[\xi] \subseteq V_{\mathbb{C}}$, *and extend* $L_{\mathbb{Z}}$ *sesquilinearly to* $V_{\mathbb{C}}$. *Then for* $m=2$

$$\{r\in V_{\mathbb{Z}[\xi]} \mid L_{\mathbb{C}}(r,r) = 2\} = \{\pm\xi^k \mid k\in\mathbb{Z}\}\times\{r\in V_{\mathbb{Z}} \mid L_{\mathbb{Z}}(r,r) = 2\} \tag{487}$$

and for $m \geq 3$

$$\begin{aligned} &\{r\in V_{\mathbb{Z}[\xi]} \mid L_{\mathbb{C}}(r,r) = m, r\notin \mathbb{Z}[\xi]b_1\} \\ =&\{\pm\xi^k \mid k\in\mathbb{Z}\}\times\{r\in V_{\mathbb{Z}} \mid L_{\mathbb{Z}}(r,r) = m, r\notin\mathbb{Z}b_1\}. \end{aligned} \tag{488}$$

(b) In the situation of (a) we have

$$\mathrm{Aut}\left(V_{\mathbb{Z}[\xi]}, L_{\mathbb{C}}\right) = \{\pm\xi^k \mid k \in \mathbb{Z}\} \cdot \mathrm{Aut}\left(V_{\mathbb{Z}}, L_{\mathbb{Z}}\right) \tag{489}$$

and

$$\begin{aligned} \mathrm{Aut}\left(V_{\mathbb{Z}}, L_{\mathbb{Z}}\right) \cong & \{\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\} \\ \cong & \{\pm \mathrm{id}\} \times \mathrm{Sym}\left(2\right) \ \textit{for } m \geq 3, & (490) \\ \mathrm{Aut}\left(V_{\mathbb{Z}}, L_{\mathbb{Z}}\right) \cong & \mathrm{Aut}\left(\textit{root lattice of type } A_2\right) \\ \cong & \{\pm \mathrm{id}\} \times \mathrm{Sym}\left(3\right) \ \textit{for } m = 2. & (491) \end{aligned}$$

(c) In the situation of the proof of Theorem 4.3.1 and Proposition 4.6.1 it is

$$\mathrm{Aut}\left(B_3, L\right) = \{\pm \left(M_h|_{B_3}\right)^k \mid k \in \mathbb{Z}\} \times U, \tag{492}$$

where U is Sym (3) *for U_{24} or as in* (105).

(d) In the situation of the proof of Theorem 4.3.1 the map $G_{\mathbb{Z}} \to \mathrm{Aut}\left(B_3, L\right)$ is surjective.

Proof. (a) Let $r = r_1 b_1 + r_2 b_2$ with $r_1, r_2 \in \mathbb{Z}[\xi]$. We obtain

$$\begin{aligned} L_{\mathbb{C}}(r,r) &= 2|r_1|^2 - (r_1\overline{r_2} + \overline{r_1}r_2) + m|r_2|^2 \\ &= |r_1|^2 + |r_1 - r_2|^2 + (m-1)|r_2|^2. \end{aligned} \tag{493}$$

At first we consider the case $l \in \{3,4,6\}$. In this case[3] it is $\mathbb{Z}[\xi] \cap \mathbb{R} = \mathbb{Z}$. So the three absolute values in (493) are non-negative integer. The sum of these absolute values is 2 if and only if

$$|r_1| = 1, r_2 = 0 \qquad \text{in the cases } m \geq 3, \tag{494}$$

$$\left.\begin{array}{l} (|r_1|, |r_2|) \in \{(1,0),(0,1),(1,1)\} \\ \text{and in the last case } r_1 = r_2 \end{array}\right\} \qquad \text{in the case } m = 2. \tag{495}$$

In the case $m \geq 3$ and in the case of an $r \notin \mathbb{Z}[\xi]\, b_1$, the sum of the three absolute values in (493) is m if and only if

$$(r_1 = 0, |r_2| = 1) \text{ or } (r_1 = r_2, |r_1| = 1). \tag{496}$$

Together with the number theoretic result $\mathbb{Z}[\xi] \cap S^1 = \{\pm\xi^k \mid k \in \mathbb{Z}\}$ of Kronecker, which was mentioned earlier, this shows part (a) for $l \in \{3,4\}$.
It remains to consider the case $(m,l) = (2,5)$. In that case we write

$$r_1 = r_{10} + r_{11}\xi + r_{12}\xi^2 + r_{13}\xi^3, \quad r_2 = r_{20} + r_{21}\xi + r_{22}\xi^2 + r_{23}\xi^3$$

3 Note that it holds $\mathbb{Z}\left[e^{\frac{2\pi i}{3}}\right] = \mathbb{Z}\left[e^{\frac{2\pi i}{6}}\right]$, e.g. see [He93, Kapitel 3 a)].

with $r_{ij} \in \mathbb{Z}$. Now we can compute

$$\begin{aligned}
L_{\mathbb{C}}(r,r) &= 2|r_1|^2 + 2|r_2|^2 - (r_1\overline{r_2} + \overline{r_1}r_2) \\
&= 2\left[\sum_{j=0}^{3} r_{1j}^2 + (\xi+\xi^4)\sum_{j=1}^{3} r_{1j}r_{1,j-1} + (\xi^2+\xi^3)\sum_{j-k\geq 2} r_{1j}r_{1k}\right] \\
&+ 2\left[\sum_{j=0}^{3} r_{2j}^2 + (\xi+\xi^4)\sum_{j=1}^{3} r_{2j}r_{2,j-1} + (\xi^2+\xi^3)\sum_{j-k\geq 2} r_{2j}r_{2k}\right] \\
&- \Bigg[2\sum_{j=0}^{3} r_{1j}r_{2j} + (\xi+\xi^4)\sum_{j=1}^{3}(r_{1j}r_{2,j-1} + r_{1,j-1}r_{2j}) \\
&\quad + (\xi^2+\xi^3)\sum_{j-k\geq 2}(r_{1j}r_{2k} + r_{1k}r_{2j})\Bigg] \\
&= A_1 + A_2 \cdot \frac{\sqrt{5}}{2} \quad \text{with integers } A_1, A_2 \in \mathbb{Z}.
\end{aligned}$$

In particular it holds

$$\begin{aligned}
A_1 &= \frac{1}{4}\sum_{j=0}^{3}\left[r_{1j}^2 + r_{2j}^2 + (r_{1j}-r_{2j})^2\right] \\
&+ \frac{1}{4}\sum_{j<k}\left[(r_{1j}-r_{1k}-r_{2j}+r_{2k})^2 + (r_{1j}-r_{1k})^2 + (r_{2j}-r_{2k})^2\right]. \qquad (497)
\end{aligned}$$

There are the following 8-tuples $(r_{10},\dots,r_{23}) \in \mathbb{Z}^8$ for which (497) takes the value 2:

$$\begin{aligned}
&\pm e_1,\dots,\pm e_8, \pm(e_1+e_5), \pm(e_2+e_6), \pm(e_3+e_7), \pm(e_4+e_8), \qquad (498)\\
&\pm(1,1,1,1,0,0,0,0), \pm(0,0,0,0,1,1,1,1), \pm(1,1,1,1,1,1,1,1).
\end{aligned}$$

Here $e_j = (\delta_{ij})_{i=1,\dots,8}$ for $j = 1,\dots,8$ denotes the standard basis of $\mathbb{Z}^8$. Moreover observe that $1+\xi+\xi^2+\xi^3 = -\xi^4$ and

$$\{r \in V_{\mathbb{Z}} \,|\, L_{\mathbb{Z}}(r,r) = 2\} = \{\pm b_1, \pm b_2, \pm(b_1+b_2)\}.$$

The tuples $(r_{10},\dots,r_{23})$ in (498) give precisely the elements $r = r_1b_1 + r_2b_2$ on the right hand side of (487). This finishes the proof of part (a) also for $(m,l) = (2,5)$.

(b) Let g be an element in $\mathrm{Aut}\left(V_{\mathbb{Z}[\xi]}, L_{\mathbb{C}}\right)$. Then g maps the sets (487) and (488) to themselves. The basis elements b_1 and b_2 are mapped to two elements in these sets with $L_{\mathbb{C}}\left(g\left(b_1\right), g\left(b_2\right)\right) = L_{\mathbb{Z}}\left(b_1, b_2\right) = -1$. Therefore g is up to a factor in $\{\pm\xi \,|\, k \in \mathbb{Z}\}$ an element of $\mathrm{Aut}\left(V_{\mathbb{Z}}, L_{\mathbb{Z}}\right)$ and, thus, (489) holds. Now in the case of $m \geq 3$ it is

$$\{r \in V_{\mathbb{Z}} \,|\, L_{\mathbb{Z}}\left(r,r\right) = 2\} = \{\pm b_1\}, \qquad (499)$$

$$\{r \in V_{\mathbb{Z}} \,|\, L_{\mathbb{Z}}\left(r,r\right) = m\} = \{\pm b_2, \pm\left(b_1+b_2\right)\}. \qquad (500)$$

So choosing the matrix representation with respect to b_1 and b_2 yields (490). The case $m = 2$ is very well known, e.g. see [BH16, Theorem 2.3]. It holds (491).

(c) First of all, we consider the singularities of types Z_{12} and Z_{18}. Those types appear already as curve singularities ($n = 1$), see Section 4.3. Now by [Ka96] any automorphism on $(\mathrm{Rad}\,(I)\,, L)$ maps the set $\{l_1, \ldots, l_r\}$ from Section 2.1 to itself. Here $\{l_1, l_2\}$ is a set of generators of $\mathrm{Rad}\,(I) = \ker\,(M_h^{curve} - \mathrm{id})$ and the respective Seifert form is

$$L|_{\mathrm{Rad}(I)} = \begin{cases} \begin{pmatrix} -4 & 1 \\ 1 & -3 \end{pmatrix} & \text{for } Z_{12}, \\ \begin{pmatrix} -6 & 1 \\ 1 & -3 \end{pmatrix} & \text{for } Z_{18}. \end{cases} \tag{501}$$

Hence the automorphisms on $(\mathrm{Rad}\,(I)\,, L)$ map the generators l_1 and l_2 to themselves (modulo sign) and it is $\mathrm{Aut}\,(\mathrm{Rad}\,(I)\,, L) = \{\pm\,\mathrm{id}\}$. Through stabilization $\mathrm{Rad}(I)$ becomes B_3 and the Seifert form L changes just the sign (cf. Section 2.1.3). Hence, it holds $\mathrm{Aut}\,(B_3, L) = \{\pm\,\mathrm{id}\}$. With $M_h|_{B_3} = -\,\mathrm{id}$ and $U = \{\mathrm{id}\}$ for Z_{12} and Z_{18} this yields (c) in these cases.
Now we consider the remaining cases $Q_{12}, Q_{16}, U_{12}, U_{16}$ and U_{24}. The normal forms of these quasihomogeneous (surface) singularities are sums of singularities in different variables of types A_l and D_{2m} with $(l, 2m)$ as in (502) (cf. Section 4.3 and Section 4.6).

	Q_{12}	Q_{16}	U_{12}	U_{16}	U_{24}
$(l, 2m)$	$(2, 6)$	$(2, 8)$	$(3, 4)$	$(4, 4)$	$(6, 4)$
	$A_2 \otimes D_6$	$A_2 \otimes D_8$	$A_3 \otimes D_4$	$A_4 \otimes D_4$	$A_6 \otimes D_4$

(502)

The singularity A_l (in one variable) has the characteristic polynomial $p_{ch}^{A_l} = (t^{l+1} - 1)/(t-1)$, the singularity D_{2m} (as a curve singularity) has the characteristic polynomial $p_{ch}^{D_{2m}} = (t^{2m-1} - 1)\Phi_1$. This can be computed for example from the spectral numbers given in Table 2. The Thom-Sebastiani result (cf. Section 2.1.3) applies with

$$\begin{aligned} (Ml(f), L) &\cong (Ml(A_l), L_{A_l}) \otimes (Ml(D_{2m}), L_{D_{2m}}), \\ M_h &\cong M_h^{A_l} \otimes M_h^{D_{2m}}, \end{aligned} \tag{503}$$

and it follows

$$\begin{aligned} p_2 &= p_{ch}^{A_l}, \\ (B_3, L) &\cong (Ml(A_l), L_{A_l}) \otimes (Ml(D_{2m})_{1,\mathbb{Z}}, L_{D_{2m}}), \\ M_h|_{B_3} &\cong M_h^{A_l} \otimes \mathrm{id}\,. \end{aligned} \tag{504}$$

Now we will consider automorphisms on $(Ml(D_{2m})_{1,\mathbb{Z}}, L_{D_{2m}})$. There exists a $\mathbb{Z}$-basis $\underline{b} = (b_1, b_2)$ (coming from the set $\{l_1, l_2, l_3\}$ as above) of $Ml(D_{2m})_{1,\mathbb{Z}}$ with

$$L_{D_{2m}}(\underline{b}^T, \underline{b}) = \begin{pmatrix} -2 & 1 \\ 1 & -m \end{pmatrix}. \tag{505}$$

So it is easy to see that the cardinality is $|\mathrm{Aut}\,(Ml(D_{2m})_{1,\mathbb{Z}}, L_{D_{2m}})| = 12$ for $m = 2$ and $|\mathrm{Aut}\,(Ml(D_{2m})_{1,\mathbb{Z}}, L_{D_{2m}})| = 4$ for $m \geq 3$. The pairings L and $L_{A_l} \otimes L_{D_{2m}}$ will be extended sesquilinearly from the $\mathbb{Z}$-lattices to the $\mathbb{C}$-vector spaces.

The $\mathbb{Z}$-lattice $Ml(A_l)$ is a cyclic monodromy module. Choose a generator e of it. Then $Ml(A_l) \otimes Ml(D_{2m})_{1,\mathbb{Z}}$ is a sum of two cyclic monodromy modules, and generators are $e \otimes b_1$ and $e \otimes b_2$. For any automorphism g of $(Ml(A_l) \otimes Ml(D_{2m})_{1,\mathbb{Z}}, M_h^{A_l} \otimes \mathrm{id})$ there are unique polynomials $g_1, g_2, g_3, g_4 \in \mathbb{Z}[t]$ of degree $\leq \deg p_2 - 1$ such that

$$\begin{pmatrix} g(v \otimes b_1) \\ g(v \otimes b_2) \end{pmatrix} = \begin{pmatrix} g_1(M_h^{A_l})(v) \otimes b_1 + g_3(M_h^{A_l})(v) \otimes b_2 \\ g_2(M_h^{A_l})(v) \otimes b_1 + g_4(M_h^{A_l})(v) \otimes b_2 \end{pmatrix} \tag{506}$$

for any $v \in Ml(A_l)$.

Now choose any eigenvalue ξ of $M_h^{A_l}$. Then $\mathbb{Z}[\xi]$ is a principal ideal domain. The space $\ker(M_h^{A_l} - \xi\,\mathrm{id}) \cap Ml(A_l)_{\mathbb{Z}[\xi]}$ is a free $\mathbb{Z}[\xi]$-module of rank 1. Choose a generating vector v. This choice gives an isomorphism from this space to $\mathbb{Z}[\xi]$. The spaces

$$\ker(M_h - \xi\,\mathrm{id}) \cap Ml(f)_{\mathbb{Z}[\xi]} \cong (\ker(M_h^{A_l} - \xi\,\mathrm{id}) \cap Ml(A_l)_{\mathbb{Z}[\xi]}) \otimes Ml(D_{2m})_{1,\mathbb{Z}[\xi]}$$

are free $\mathbb{Z}[\xi]$-modules of rank 2. The space on the right hand side has the $\mathbb{Z}[\xi]$-basis $(v \otimes b_1, v \otimes b_2) =: v \otimes \underline{b}$. Now (506) becomes

$$g(v \otimes \underline{b}) = v \otimes \underline{b} \cdot \begin{pmatrix} g_1(\xi) & g_2(\xi) \\ g_3(\xi) & g_4(\xi) \end{pmatrix}. \tag{507}$$

The pairing satisfies

$$(L_{A_l} \otimes L_{D_{2m}})(v \otimes \underline{b}) = L_{A_l}(v, v) \cdot \begin{pmatrix} -2 & 1 \\ 1 & -m \end{pmatrix}, \tag{508}$$

where $L_{A_l}(v, v) \in \mathbb{Z}[\xi] \cap \mathbb{R}_{>0}$. This space with this pairing is up to a scalar isomorphic to a pair $(V_{\mathbb{Z}[\xi]}, L_{\mathbb{C}})$ considered in the parts (a) and (b). Therefore by part (b), its group of automorphisms is isomorphic to $\{\pm\xi^k \mid k \in \mathbb{Z}\} \cdot \mathrm{Aut}(V_{\mathbb{Z}}, L_{\mathbb{Z}})$. Thus any element of $\mathrm{Aut}(B_3, L)$ restricts on $\ker(M_h - \xi\,\mathrm{id}) \cap Ml(f)_{\mathbb{Z}[\xi]}$ to such an automorphism. In all cases except U_{12} the characteristic polynomial p_2 is irreducible. Namely, it is Φ_3 for Q_{12} and Q_{16}, Φ_5 for U_{16} and Φ_7 for U_{24}. So all its zeros ξ are conjugate and we obtain

$$\begin{aligned} \mathrm{Aut}\,(B_3, L) &\cong \{\pm\,(M_h|_{B_3})^k \mid k \in \mathbb{Z}\} \cdot \mathrm{Aut}\,(V_{\mathbb{Z}}, L_{\mathbb{Z}}) \\ &\cong \{\pm\,(M_h|_{B_3})^k \mid k \in \mathbb{Z}\} \times U. \end{aligned} \tag{509}$$

This proves part (c) in those cases.
It remains to consider the case U_{12}. Here the polynomial is $p_{ch}^{A_3} = p_2 = \Phi_4\Phi_2$ and, thus, it is reducible. Now consider an automorphism g of

$$(Ml(A_l) \otimes Ml(D_{2m})_{1,\mathbb{Z}}, L_{A_l} \otimes L_{D_{2m}}).$$

It is determined by the polynomials $g_1, g_2, g_3, g_4 \in \mathbb{Z}[t]$ in (506). For $\xi = i$ and for $\xi = -1$ it gives an automorphism of $\mathbb{Z}[\xi]v \otimes b_1 \oplus \mathbb{Z}[\xi]v \otimes b_2$ which is given by a matrix $\begin{pmatrix} g_1(\xi) & g_2(\xi) \\ g_3(\xi) & g_4(\xi) \end{pmatrix}$ which is by part (b) in

$$\{\pm\xi^k \mid k \in \mathbb{Z}\} \quad \cdot \quad \Big\{\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \pm\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \\ \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pm\begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \pm\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}\Big\}. \tag{510}$$

By multiplying g with a suitable automorphism we can suppose that the matrix for $\xi = i$ is the identity matrix. Then

$$\begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} = \begin{pmatrix} 1 + (t^2+1)\widetilde{g}_1 & (t^2+1)\widetilde{g}_2 \\ (t^2+1)\widetilde{g}_3 & 1 + (t^2+1)\widetilde{g}_4 \end{pmatrix},$$

for some $\widetilde{g}_1, \widetilde{g}_2, \widetilde{g}_3, \widetilde{g}_4 \in \mathbb{Z}[t]$, so

$$\begin{pmatrix} g_1(-1) & g_2(-1) \\ g_3(-1) & g_4(-1) \end{pmatrix} = \begin{pmatrix} 1 + 2\widetilde{g}_1(-1) & 2\widetilde{g}_2(-1) \\ 2\widetilde{g}_3(-1) & 1 + 2\widetilde{g}_4(-1) \end{pmatrix}.$$

The only two possibilities are $\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. In the case of a minus sign $g \circ ((M_h^{A_l})^2 \otimes \mathrm{id}_{D_{2m}}) = -\mathrm{id}$, in the case of a plus sign $g = \mathrm{id}$. This finishes the proof of (c) also for singularities of type U_{12}.

(d) In Subsection 4.3.2 we compute the subgroup G^{mar} of $G_{\mathbb{Z}}$. It holds

$$G^{mar} = \{\pm (M_h)^k \mid k \in \mathbb{Z}\} \times U. \tag{511}$$

Thus we can conclude that $G_{\mathbb{Z}} \supseteq \{\pm (M_h)^k \mid k \in \mathbb{Z}\} \times U$ and that the map $G_{\mathbb{Z}} \to \mathrm{Aut}\,(B_3, L) \cong \{\pm (M_h|_{B_3})^k \mid k \in \mathbb{Z}\} \times U$ is surjectiv. □

A.3 FUCHSIAN GROUPS

Fuchsian groups, which are named after the German mathematician Lazarus Immanuel Fuchs, are certain groups of fractional linear transformations. For us Fuchsian groups will be important because the group $G_{\mathbb{Z}}(f)$ for quadrangle or bimodal series singularities f is a finite extension of a group belonging to this class (see Section 4.4 and 4.5). In order to make this monograph self-consistent we will recall the necessary parts of the theory of Fuchsian groups in the following. A nicely written outline on the rich theory of these groups is given in [Mi89, Chapter 1].

A *fractional linear transformation* is a rational function $z \mapsto \frac{az+b}{cz+d}$ in one complex variable z for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2;\mathbb{C})$. This can be extended to an automorphism of the Riemann sphere $P^1\mathbb{C} := \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. The group of such automorphisms is the projective linear group $\mathrm{PGL}(2;\mathbb{C})$. The non-identity elements of this group are of one of three different types. Namely, they are *parabolic*, if they have exactly one fixed point in $\hat{\mathbb{C}}$, *elliptic* or *hyperbolic*, otherwise. Besides fixed points also limit points will be studied.

Definition A.3.1. *Let Γ be a subgroup of $\mathrm{GL}(2;\mathbb{C})$. A point $z \in \hat{\mathbb{C}}$ is a* limit point *for the action of Γ if there is a point $w \in \hat{\mathbb{C}}$ and a sequence of distinct elements $g_1, g_2, \ldots \in \Gamma$ with $g_i(w) \to z$. The set of all limit points of Γ is called its* limit set*. We denote the limit set as $L(\Gamma)$.*

Note that the fixed points of the parabolic elements of the group Γ are contained in the limit set $L(\Gamma)$.

Definition A.3.2. *Let $C \subseteq \hat{\mathbb{C}}$ be the so-called* principal circle[4]*. Then a discrete subgroup Γ of $\mathrm{GL}(2;\mathbb{C})$ (resp. $\mathrm{PGL}(2;\mathbb{C})$) is called* Fuchsian*, if it maps C to itself. The group is called a Fuchsian group* of the first kind *if $L(\Gamma) = C$, and* of the second kind *otherwise.*

A very important class of Fuchsian groups are the so-called arithmetic Fuchsian groups. A Fuchsian group is *arithmetic* if it is commensurable to a group derived from a quaternion algebra (see [Sh71] for a precise definition). The Fuchsian groups that appear in the main part of this monograph are all arithmetic Fuchsian groups and of a certain form, which will be specified below. The upcoming results will build upon Dirichlet's Unit Theorem, which we recall in A.3.3.

Theorem A.3.3 (Dirichlet's Unit Theorem, see [BS66, Ch. 2, 3.1 Theorem 1] for part (a) and [BS66, Ch. 2, 4.3 Theorem 5] for part (b))**.** *Let K be an algebraic number field of degree $n = s + 2t$ over $\mathbb{Q}$ with s real embeddings $\sigma_j : K \to \mathbb{R}$, $j = 1, \ldots, s$, and $2t$ complex embeddings $\sigma_j : K \to \mathbb{C}$, $j = s+1, \ldots, n$, with $\sigma_{s+t+j} = \overline{\sigma_{s+j}}$ for $j = 1, \ldots, t$.*

(a) Define $\underline{\sigma} := (\sigma_1, \ldots, \sigma_{s+t}) : K \to \mathbb{R}^s \times \mathbb{C}^t \cong \mathbb{R}^n$. Any $\mathbb{Q}$-basis of the number field K maps to an $\mathbb{R}$-basis of $\mathbb{R}^n$. Thus the image under $\underline{\sigma}$ of any order $\mathcal{O} \subseteq K$ is a lattice of rank n in $\mathbb{R}^n$.

(b) Let $\mathcal{O} \subseteq K$ be an order. One can choose $r = s + t - 1$ units $a_1, \ldots, a_r \in \mathcal{O}^*$ such that any unit has a unique representation $\xi \cdot a_1^{k_1} \cdot \ldots \cdot a_r^{k_r}$ with $k_1, \ldots, k_r \in \mathbb{Z}$ and ξ a root of 1 in $\mathcal{O}$.

4 See [Le64, Section IV 7].

Theorem A.3.4. *Let $m \in \mathbb{Z}_{\geq 3}$, $\zeta := e^{2\pi i/m}$, $p_1 := \zeta + \overline{\zeta}$, and $w = w(\zeta) \in \mathbb{Q}(\zeta)$ with*

$$w(\zeta) > 0 \qquad (\textit{thus } w(\zeta) = w(\overline{\zeta}) \in \mathbb{Q}(p_1)), \tag{512}$$

$$w(\xi) < 0 \qquad \textit{for any primitive } m\textit{-th unit root } \xi \notin \{\zeta, \overline{\zeta}\}. \tag{513}$$

Then the matrix group

$$\Gamma := \{A \in \mathrm{GL}(2; \mathbb{Z}[\zeta]) \mid \begin{pmatrix} -1 & 0 \\ 0 & w \end{pmatrix} = A^T \begin{pmatrix} -1 & 0 \\ 0 & w \end{pmatrix} \overline{A}\} \tag{514}$$

is an arithmetic Fuchsian group. It preserves the circle

$$C = \{z \in \mathbb{C} \mid |z|^2 = w\}. \tag{515}$$

The map

$$\begin{aligned} \{(a, c, \delta) \in \mathbb{Z}[\zeta]^2 \times Eiw(\zeta) \mid |a|^2 - 1 = w \cdot |c|^2\} &\to \Gamma \\ (a, c, \delta) &\mapsto \begin{pmatrix} a & w\overline{c}\delta \\ c & \overline{a}\delta \end{pmatrix} \end{aligned} \tag{516}$$

is a bijection. Here it is $Eiw(\zeta) = \{\pm\zeta^k \mid k \in \mathbb{Z}\}$ (cf. [He93, Lemma 3.4]).

Proof. We consider a matrix $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2; \mathbb{Z}[\zeta])$. By definition the matrix A is in Γ if and only

$$\begin{pmatrix} -1 & 0 \\ 0 & w \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & w \end{pmatrix} \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix} = \begin{pmatrix} -a\overline{a} + wc\overline{c} & -b\overline{a} + wd\overline{c} \\ -\overline{a}b + w\overline{c}d & -b\overline{b} + wd\overline{d} \end{pmatrix}. \tag{517}$$

The determinant $\delta := \det(A) = ad - bc$ is a unit in $\mathbb{Z}[\zeta]$ and it has absolute value one. In particular, this means that $\delta \in Eiw(\zeta)$ by [He93, Lemma 3.4 c)]. Using the definition of δ and the equations from (517) we obtain the relations

$$\overline{a}\delta = \overline{a}(ad - bc) = (wc\overline{c} + 1)\, d - (wd\overline{c})\, d = d, \tag{518}$$

$$w\overline{c}\delta = w\overline{c}(ad - bc) = (\overline{a}b)\, a - (a\overline{a} - 1)\, b = b. \tag{519}$$

This settles the bijection in (516).

Now as the map in (516) is bijective we can work with its starting domain. We see that the defining equation

$$|a(\zeta)|^2 - 1 = w(\zeta) \cdot |c(\zeta)|^2 \tag{520}$$

of this starting domain is a Pell (like) equation for $a, c, w(\zeta) \in \mathbb{Z}[p_1]$. From this equation we get the following inequalities

$$\begin{aligned} 0 \leq & |c(\zeta)|^2 = (w(\zeta))^{-1} \cdot (|a(\zeta)|^2 - 1), \\ 1 \leq & |a(\zeta)| \end{aligned} \tag{521}$$

and for any primitive m-th unit root $\xi \notin \{\zeta, \overline{\zeta}\}$ it holds

$$\begin{aligned} 0 \leq & |c(\xi)|^2 = (-w(\xi))^{-1} \cdot (1 - |a(\xi)|^2) < (-w(\xi))^{-1}, \\ |a(\xi)| \leq & 1. \end{aligned} \tag{522}$$

The matrix $\begin{pmatrix} -1 & 0 \\ 0 & w \end{pmatrix}$ defines an indefinite Hermitian form with isotropic vectors $\begin{pmatrix} z \\ 1 \end{pmatrix}$ for $z \in C$. As $A \in \Gamma$ respects this Hermitian form, it maps the isotropic vectors $\begin{pmatrix} z \\ 1 \end{pmatrix}$ to other isotropic vectors. So the matrix $A \in \Gamma$, which was chosen arbitrarily, maps the circle $C \cong \{\begin{pmatrix} z \\ 1 \end{pmatrix} \mid z \in C\} \subseteq \hat{\mathbb{C}}$ to itself. It remains to prove that Γ is arithmetic and a discrete group. We start with the latter. This means we have to show that the set

$$P_1 := \{a \in \mathbb{Z}[\zeta] \mid \exists c \in \mathbb{Z}[\zeta] \text{ with } |a|^2 - 1 = w \cdot |c|^2\} \tag{523}$$

intersects each compact set $K \subseteq \mathbb{C}$ in a finite set. By Theorem A.3.3 there is an embedding $\underline{\sigma} = \left(\sigma_1, \ldots, \sigma_{\varphi(n)}\right) : \mathbb{Q}(\zeta) \to \mathbb{R}^{\varphi(n)}$ where $\varphi(n)$ denotes the number of positive integers up to n that are relatively prime to n. It maps $\mathbb{Z}[\zeta]$ to a lattice. Moreover it maps $P_1 \cap K$ to a subset of

$$\underline{\sigma}(\mathbb{Z}[\zeta]) \cap \left(K \times \{z \in \mathbb{C} \mid |z| \leq 1\}^{\frac{\varphi(n)}{2} - 1}\right) \tag{524}$$

as (522) holds. This is a finite set. Hence $P_1 \cap K$ is finite and Γ is a Fuchsian group. Finally Γ is an arithmetic Fuchsian group, because it is a Fuchsian group of the first kind and we have [Ta75, Theorem 2]. That it is a Fuchsian group of the first kind follows with some work by comparing the data above with the data in [Sh71, Section 9.2]. □

As mentioned earlier, the groups from Theorem A.3.4 turn up in Theorem 4.5.2 as quotients of the groups $G_{\mathbb{Z}}$ for the quadrangle singularities. There the first six of the eight elements $w(\zeta)$ in (217) in the case $r = 0$ will be used. Note that, via some straightforward computation in the principal ideal domains $\mathbb{Z}[\zeta]$ and $\mathbb{Z}[p_1]$ the first six of the eight elements $w(\zeta)$ in (217) with $r = 0$ can be written as follows.

$$\begin{aligned}
W_{1,0} &: \quad w(\zeta) = \frac{6}{(2-p_1)p_1} = \frac{1}{(2-p_1)(2+p_1)} \cdot 2p_1(p_1+2). \\
S_{1,0} &: \quad w(\zeta) = \frac{-2}{(-p_3)(-p_3-1)} = 1 \cdot 2p_1^3. \\
U_{1,0} &: \quad w(\zeta) = \frac{-3}{(2+p_7)(1-p_1)} = 1 \cdot p_1(p_1+2). \\
E_{3,0} &: \quad w(\zeta) = \frac{3(2-p_1)}{(p_1+2)(p_1-1)} = (2-p_1)^2 \cdot p_1(p_1+2). \\
Z_{1,0} &: \quad w(\zeta) = \frac{1}{-p_5} = 1 \cdot (-p_5)^{-1} = 1 \cdot (p_1-1). \\
Q_{2,0} &: \quad w(\zeta) = \frac{2-p_1}{p_1+1} = (2-p_1) \cdot \frac{1}{p_1+1}.
\end{aligned} \tag{525}$$

Here, in the case $U_{1,0}$ we use $m = 18$ instead of $m = 9$. So above it is $\zeta = e^{2\pi i/18}$ for $E_{3,0}$ and $U_{1,0}$. It is $p_3 := \zeta^3 + \overline{\zeta}^3$ for $m = 10, 14$, $p_5 := \zeta^5 + \overline{\zeta}^5$ for $m = 12, 14, 18$ and $p_7 := \zeta^7 + \overline{\zeta}^7$ for $m = 18$.

The groups from Theorem A.3.4 contain infinitely many elements — as the next lemma shows. Note that this lemma can be seen as a special case of [Sch06, Theorem 3].

Lemma A.3.5. *Let m, ζ, p_1 and w be as above. Then the set*

$$P_2 := \{(a,c) \in \mathbb{Z}[p_1] \mid a^2 - 1 = w \cdot c^2\} \tag{526}$$

contains infinitely many elements. In particular, if $w \in \mathbb{Z}[p_1]$ the set P_2 contains pairs (a,c) with $w \mid (a-1)$.

Proof. Without loss of generality we assume that $w \in \mathbb{Z}[p_1]$.[5] We will construct infinitely many units in $\mathbb{Z}[\sqrt{w}, p_1]^* \setminus \mathbb{Z}[p_1]^*$ and then, building on that, infinitely many elements in P_2.

According to (512) and (513) the algebraic number field $\mathbb{Q}(\sqrt{w}, p_1)$ has degree $\varphi(m)$ over $\mathbb{Q}$, two real embeddings and $\varphi(m) - 2$ complex embeddings. Now by Dirichlet's Unit Theorem A.3.3 the unit group $\mathbb{Z}[\sqrt{w}, p_1]^*$ of $\mathbb{Z}[\sqrt{w}, p_1]$ in $\mathbb{Q}(\sqrt{w}, p_1)$ contains a free Abelian group of rank $2 + \frac{\varphi(m)-2}{2} - 1 = \frac{\varphi(m)}{2}$. However the unit group $\mathbb{Z}[p_1]^*$ contains only a free Abelian group of rank $\frac{\varphi(m)}{2} - 1$. Thus there are infinitely many units $a_1 + \sqrt{w}c_1 \in \mathbb{Z}[\sqrt{w}, p_1]^*$ with $a_1 \neq 0$ and $c_1 \neq 0$. Note that if $a_1 + \sqrt{w}c_1 \in \mathbb{Z}[\sqrt{w}, p_1]^*$ is a unit also $a_1 - \sqrt{w}c_1$, $(a_1 + \sqrt{w}c_1)^2 = (a_1^2 + wc_1^2) + \sqrt{w}(2a_1c_1) =: (a_2 + \sqrt{w}c_2)$ and $h := (a_1 + \sqrt{w}c_1) \cdot (a_1 - \sqrt{w}c_1) \in \mathbb{Z}[p_1]^*$ are units. We obtain $(a_3, c_3) := (\frac{a_2}{h}, \frac{c_2}{h}) \in P_2$, because

$$\begin{aligned}
a_3^2 - wc_3^2 = h^{-2}\left(a_2^2 - wc_2^2\right) &= h^{-2}(a_2 + \sqrt{w}c_2)(a_2 - \sqrt{w}c_2) \\
&= h^{-2}(a_1 + \sqrt{w}c_1)^2 \cdot (a_1 - \sqrt{w}c_1)^2 = 1.
\end{aligned} \tag{527}$$

5 Suppose $w \in \mathbb{Q}(p_1) \setminus \mathbb{Z}[p_1]$. Then there is a $u \in \mathbb{Z}[p_1] \setminus \{0\}$ with $\widetilde{w} = w \cdot u^2 \in \mathbb{Z}[p_1]$. Now every pair $(a, \widetilde{c}) \in \mathbb{Z}[p_1]^2$ with $a^2 - 1 = \widetilde{w}\widetilde{c}^2$ gives a pair $(a, c) := (a, \widetilde{c}u) \in P_2$. So it is sufficient to consider $w \in \mathbb{Z}[p_1]$.

As mentioned earlier we have infinitely many units $a_1 + \sqrt{w}c_1$. But only finitely many units $a_1 + \sqrt{w}c_1$ can give rise to the same element (a_3, c_3). Hence, there are also infinitely many elements in P_2.
If $w \in \mathbb{Z}[p_1]$ then among those infinitely many elements there are pairs such that w divides the first element of the pair subtracted by one. Namely, from an element $(a_3, c_3) \in P_2$ with $c_3 \neq 0$ as above we can construct an element $(a_4, c_4) := (a_3^2 + wc_3^2, 2a_3c_3)$. Since

$$\begin{aligned} a_4^2 - wc_4^2 &= (a_4 + \sqrt{w}c_4) \cdot (a_4 - \sqrt{w}c_4) \\ &= (a_3 + \sqrt{w}c_3)^2 \cdot (a_3 - \sqrt{w}c_3)^2 = \left(a_3^2 - wc_3^2\right)^2 = 1 \end{aligned} \tag{528}$$

this is again in P_2 and $w|(a_4 - 1)$ with $a_4 - 1 = a_3^2 + wc_3^2 - 1 = 2wc_3^2$. □

If we consider the images in $\mathrm{PGL}(2;\mathbb{C})$ of the arithmetic Fuchsian groups from Theorem A.3.4 we end up in an even more specific class of groups. Note that we call a Fuchsian group a *(Schwarzian) triangle group* $\Delta(k, l, m)$ if it a has a defining presentation of the form

$$\Delta(k, l, m) = \left\langle A_1, A_2 \,|\, A_1^m = A_2^k = (A_1A_2)^l = \mathrm{id} \right\rangle. \tag{529}$$

According to a result of Kisao Takeuchi there are exactly 85 arithmetic Fuchsian groups that are triangle groups. In [Ta77] he lists all possible triples of integers (k, l, m) of those groups $\Delta(k, l, m)$. Five of these triples (resp. groups) appear in the context of the quadrangle singularities.

Theorem A.3.6. *The image in* $\mathrm{PGL}(2;\mathbb{C})$ *of the group* Γ *in Theorem A.3.4 for the following values of* m *and* w

	$W_{1,0}$	$S_{1,0}$	$E_{3,0}$ & $U_{1,0}$	$Z_{1,0}$	$Q_{2,0}$
m	12	10	18	14	12
w	$2p_1(p_1+2)$	$2p_1^3$	$p_1(p_1+2)$	$(-p_5)^{-1}$	$(p_1+1)^{-1}$

(530)

is a triangle group of the following type:

$W_{1,0}$	$S_{1,0}$	$E_{3,0}$ & $U_{1,0}$	$Z_{1,0}$	$Q_{2,0}$
$(2, 12, 12)$	$(2, 10, 10)$	$(2, 3, 18)$	$(2, 3, 14)$	$(2, 3, 12)$

(531)

Proof. The proof consists of three steps. All three steps together yield the claim. In Step 1 we will present two matrices A_1 and A_2 in Γ whose images in $\mathrm{PGL}(2;\mathbb{C})$ are elliptic and generate a Schwarzian triangle group of the claimed type in each of the cases. In Step 2 we will show that no matrix in Γ is closer to A_1 than A_2. This will be used in Step 3 where we prove that the images in $\mathrm{PGL}(2;\mathbb{C})$ of A_1 and A_2 generate the image of Γ in $\mathrm{PGL}(2;\mathbb{C})$.

Step 1: First of all, we check that the matrices A_1 and A_2 as described in (532) are in Γ (see (516) for a good characterization of the group Γ).

$$A_1 = \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{for all 5 cases.} \tag{532}$$

$$\begin{aligned}
W_{1,0}: \quad & A_2 = \begin{pmatrix} p_1+2 & -2p_1(p_1+2) \\ 1 & -(p_1+2) \end{pmatrix}, \quad \det A_2 = -1, \\
S_{1,0}: \quad & A_2 = \begin{pmatrix} (\zeta+1)p_1 & -2p_1^3\zeta \\ 1 & -(\zeta+1)p_1 \end{pmatrix}, \quad \det A_2 = -\zeta, \\
E_{3,0} \,\&\, U_{1,0}: \quad & A_2 = \begin{pmatrix} p_1+1 & -p_1(p_1+2) \\ 1 & -(p_1+1) \end{pmatrix}, \quad \det A_2 = -1, \\
Z_{1,0}: \quad & A_2 = p_1(1-\zeta^3)\cdot \begin{pmatrix} 1 & -(-p_5)^{-1} \\ 1 & -1 \end{pmatrix}, \quad \det A_2 = \zeta^3, \\
Q_{2,0}: \quad & A_2 = \begin{pmatrix} \zeta+1 & -\zeta \\ p_1+1 & -(\zeta+1) \end{pmatrix}, \quad \det A_2 = -\zeta.
\end{aligned} \tag{533}$$

Note that a matrix $A \in \mathrm{GL}(2;\mathbb{C})$ is elliptic if its eigenvalues λ_1 and λ_2 satisfy $\frac{\lambda_2}{\lambda_1} \in S^1$. Let $\begin{pmatrix} z_j \\ 1 \end{pmatrix}$ be an eigenvector with eigenvalue λ_j for $j = 1, 2$ (possibly $z_1 = 0$ and $z_2 = \infty$). Then the linear transformation of A is a rotation around the fixed point z_1 with angle $\alpha(A) = \arg \frac{\lambda_2}{\lambda_1}$. Now let $A \in \Gamma$ be elliptic. We assume that its eigenvalues λ_1, λ_2 are labelled in such a way that $|z_1| < |z_2|$. Then it is $|z_1|^2 < w$ and z_1 is in the interior of the circle C. In all five cases it holds

$$\lambda_1(A_1) = 1, \quad \lambda_2(A_1) = \zeta, \quad \alpha(A_1) = \frac{2\pi}{m}, \tag{534}$$
$$\mathrm{tr}(A_2) = 0, \quad \alpha(A_2) = \pi. \tag{535}$$

In (536) the eigenvalues λ_1, λ_2 and the angle $\alpha(A_1A_2)$ of the product A_1A_2 are given.

	λ_1	λ_2	α
$W_{1,0}$	ζ^4	ζ^3	$\frac{-2\pi}{12}$
$S_{1,0}$	ζ^4	ζ^3	$\frac{-2\pi}{10}$
$E_{3,0}$ & $U_{1,0}$	ζ^8	ζ^2	$\frac{-2\pi}{3}$
$Z_{1,0}$	$e^{2\pi i/6}\zeta^2$	$e^{-2\pi i/6}\zeta^2$	$\frac{-2\pi}{3}$
$Q_{2,0}$	ζ^6	ζ^2	$\frac{-2\pi}{3}$

(536)

We see that the images of A_1 and A_2 in $\mathrm{PGL}(2;\mathbb{C})$ generate a Schwarzian triangle group of type as in (531) (cf. [Le64, Section VII 1G]).

Step 2: Let $A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for any $A \in \Gamma$.

Claim 1: Any $A \in \Gamma$ with $c \neq 0$ satisfies $|a| \geq |a_2|$.

The proof of the Claim 1 makes the proof of Theorem A.3.4 more constructive. So at first we look for candidates $f \in \mathbb{Z}[p_1]$ of $|a|^2$ which are compatible with the inequalities (521) and (522) and which satisfy $f < |a_2|^2$. Then we will show that these candidates are not compatible with the equality $|a|^2 = 1 + w \cdot |c|^2$.

Denote by $\underline{\sigma}^{\mathbb{R}} = (\sigma_1^{\mathbb{R}}, \dots, \sigma_{\varphi(m)/2}^{\mathbb{R}}) : \mathbb{Q}(p_1) \to \mathbb{R}^{\varphi(m)/2}$ the tuple of the embeddings $\sigma_j^{\mathbb{R}} : \mathbb{Q}(p_1) \to \mathbb{R}$. Then $\underline{\sigma}^{\mathbb{R}}(\mathbb{Z}[p_1])$ is a $\mathbb{Z}$-lattice in $\mathbb{R}^{\varphi(m)/2}$. By the inequalities in (521) and (522) naive candidates f are the numbers $f = f(p_1)$ in $\mathbb{Z}[p_1]$ with

$$\underline{\sigma}^{\mathbb{R}}(f) \quad \in \quad (1, |a_2|^2) \ \times \ (0,1)^{\varphi(m)/2-1}. \tag{537}$$

The real numbers $p_1 = \zeta + \overline{\zeta}$ are $p_1 = \frac{\sqrt{5}+1}{2} \approx 1.6180339887$ for $m = 10$, $p_1 = \sqrt{3} \approx 1.7320508076$ for $m = 12$, $p_1 = 2\cos\frac{\pi}{7} \approx 1.8019377358$ for $m = 14$ and $p_1 = 2\cos\frac{\pi}{9} \approx 1.8793852416$ for $m = 18$. Thus the naive candidates are of the following forms.

$$\begin{array}{rcl}
W_{1,0} & : & f(p_1) = \alpha \cdot 1 + \beta \cdot p_1, \quad (\alpha, \beta) \in \{(2,1),(4,2),(6,3)\}. \\
S_{1,0} & : & f(p_1) = \alpha \cdot 1 + \beta \cdot p_1, \quad (\alpha, \beta) \in \{(2,2),(2,3)\}. \\
E_{3,0} & \& & U_{1,0} \ : \ \emptyset. \\
Z_{1,0} & : & \emptyset. \\
Q_{2,0} & : & \emptyset.
\end{array}$$

Now all these naive candidates can be excluded with the help of the condition

$$\mathrm{Norm}(|a|^2 - 1) = \mathrm{Norm}(w \cdot |c|^2) = \mathrm{Norm}(w) \cdot \mathrm{Norm}(|b|^2).$$

Here the norm is the norm in $\mathbb{Q}(p_1)$ and $\mathbb{Z}[p_1]$ with values in $\mathbb{Q}$ and $\mathbb{Z}$, respectively. In particular, in the two cases where naive candidates exist it is $\mathrm{Norm}(w) = -12$, $\mathrm{Norm}(1 + p_1) = -2$, $\mathrm{Norm}(3 + 2p_1) = -3$, $\mathrm{Norm}(5 + 3p_1) = -2$ for the case $W_{1,0}$ and $\mathrm{Norm}(w) = -4$, $\mathrm{Norm}(1 + 2p_1) = -1$, $\mathrm{Norm}(1 + 3p_1) = -5$ for the case $S_{1,0}$.

Step 3: It is sufficient to show the following Claim 2.

Claim 2: For any matrix $A_3 \in \Gamma$ with $c_3 \neq 0$, a number $k \in \mathbb{Z}$ exists such that the product

$$A_4 := A_3 \cdot A_1^{-k} A_2 A_1^k = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} \begin{pmatrix} a_2 & \zeta^{-k} b_2 \\ \zeta^k c_2 & d_2 \end{pmatrix} \tag{538}$$

satisfies

$$|c_4| < |c_3|, \qquad \text{here } c_4 = c_3 a_2 + \zeta^k d_3 c_2. \tag{539}$$

We can choose $k \in \mathbb{Z}$ such that

$$\beta := |\arg(c_3 a_2) - \arg(-\zeta^k d_3 c_2)| \leq \frac{\pi}{m}. \tag{540}$$

We observe that it is

$$\frac{|\zeta^k d_3 c_2|^2}{|c_3 a_2|^2} = \frac{|a_3|^2 \frac{|a_2|^2-1}{w(\zeta)}}{\frac{|a_3|^2-1}{w(\zeta)}|a_2|^2} = \frac{1-|a_2|^{-2}}{1-|a_3|^{-2}}. \tag{541}$$

The trivial inequality $1-|a_3|^{-2} < 1$ and the inequality $|a_3| \geq |a_2|$ from Step 2 give the inequalities

$$\left(1-|a_2|^{-2}\right)|c_3 a_2|^2 < |\zeta^k d_3 c_2|^2 \leq |c_3 a_2|^2. \tag{542}$$

Moreover it is

$$\sqrt{1-|a_2|^{-2}} < \cos\frac{\pi}{m}. \tag{543}$$

Therefore it is

$$\begin{aligned}
|c_4| &= |c_3 a_2|^2 (\sin\beta)^2 + (|c_3 a_2|\cos\beta - |d_3 c_2|)^2 \\
&< |c_3 a_2|^2 (\sin\frac{\pi}{m})^2 + \left(1-\sqrt{1-|a_2|^{-2}}\right)^2 \cdot |c_3 a_2|^2 \\
&= |c_3|^2 \cdot |a_2|^2 \left((\sin\frac{\pi}{m})^2 + \left(1-\sqrt{1-|a_2|^{-2}}\right)^2\right) \\
&\overset{(*)}{<} |c_3|^2.
\end{aligned} \tag{544}$$

The inequality $\overset{(*)}{<}$ follows in all of the five cases by dividing by $|c_3|^2$ and computing the numerical values on the left hand side in the forelast line of (544). The values of $|a_2|^2$ with a_2 as in (533) are

$$|a_2|^2 = 7+4\sqrt{3} \qquad \approx 13.9282032 \qquad \text{for } W_{1,0}, \tag{545}$$

$$|a_2|^2 = 5+2\sqrt{5} \qquad \approx 9.47213596 \qquad \text{for } S_{1,0}, \tag{546}$$

$$|a_2|^2 = 3+4\cos\frac{\pi}{9}+2\cos\frac{2\pi}{9} \qquad \approx 8.29085937 \qquad \text{for } E_{3,0} \text{ and } U_{1,0}, \tag{547}$$

$$|a_2|^2 = -8\left(\sin\frac{\pi}{14}-1\right)\left(\cos\frac{\pi}{7}\right)^2 \qquad \approx 5.04891734 \qquad \text{for } Z_{1,0}, \tag{548}$$

$$|a_2|^2 = 2+\sqrt{3} \qquad \approx 3.73205081 \qquad \text{for } Q_{2,0}. \tag{549}$$

□