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A Review and Some Complements on Quantile Risk Measures and Their Domain

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Abstract: In the present paper, we study quantile risk measures and their domain. Our starting point is that, for a probability measure Q on the open unit interval and a wide class $\mathcal{L}_Q$ of random variables, we define the quantile risk measure ϱ_Q as the map that integrates the quantile function of a random variable in $\mathcal{L}_Q$ with respect to Q. The definition of $\mathcal{L}_Q$ ensures that ϱ_Q cannot attain the value $+\infty$ and cannot be extended beyond $\mathcal{L}_Q$ without losing this property. The notion of a quantile risk measure is a natural generalization of that of a spectral risk measure and provides another view of the distortion risk measures generated by a distribution function on the unit interval. In this general setting, we prove several results on quantile or spectral risk measures and their domain with special consideration of the expected shortfall. We also present a particularly short proof of the subadditivity of expected shortfall.

Keywords: integrated quantile functions; quantile risk measures; spectral risk measures; subadditivity; value at risk; expected shortfall

1. Introduction

In the present paper, we study quantile risk measures and their domain. Our starting point is that, for a probability measure Q on the open unit interval and a wide class $\mathcal{L}_Q$ of random variables, we define the quantile risk measure ϱ_Q as the map that integrates the quantile function of a random variable in $\mathcal{L}_Q$ with respect to Q. The definition of $\mathcal{L}_Q$ ensures that ϱ_Q cannot attain the value $+\infty$ and cannot be extended beyond $\mathcal{L}_Q$ without losing this property. The notion of a quantile risk measure is a natural generalization of that of a spectral risk measure and provides another view of the distortion risk measures generated by a distribution function on the unit interval.

Quantile risk measures are thus mixtures of the values at risk at different levels and hence mixtures of a parametric family of risk measures. Such mixtures have already been considered by Acerbi (2002), who, however, gave little attention to the domain on which a given risk measure can be defined; he argued that in a real-world risk management application, the integral (defining a risk measure) will always be well-defined and finite. Nevertheless, Acerbi (2002) proposed a maximal class of random variables on which a given spectral risk measure is well-defined and finite. In the case of a spectral risk measure, the domain of a quantile risk measure proposed in the present paper contains the class proposed by Acerbi (2002) and turns out to be a convex cone, which is of interest with regard to the subadditivity of the risk measure.

In this paper, we review and partly extend known results on quantile risk measures, with particular attention to spectral risk measures and, in particular, expected shortfall, with emphasis on their maximal domain mentioned before. We deliberately adopt arguments from the literature, with appropriate modifications if necessary, but some of our proofs and results are new.

The literature on risk measures is vast and rapidly growing. A substantial part of the theory can be found in the monographs by Föllmer and Schied (2016), McNeil et al. (2015), Pflug and Römisch (2007) and Rüschendorf (2013) and in the references given in these books. Since the theory of risk measures is inspired by two sources, finance and insurance, the definitions of financial and insurance risk measures are slightly different, and the terminology is not fully consistent; for example, the use of the term expected shortfall is not generally agreed upon. In the present paper, we consider insurance risk measures, which are closely related to premium principles, and to avoid more ponderous expressions, we employ the short-term *quantile risk measure* for a well-defined class of risk measures.

This paper is organized as follows: We first fix some notation, recall some basic properties of the quantile function and present a couple of examples of distortion functions (Section 2). We then introduce quantile risk measures and provide several alternative representations of quantile risk measures and their domain, as well as conditions under which certain quantile risk measures can be compared (Section 3). In the next step, we consider spectral risk measures and characterize spectral risk measures within the class of all quantile risk measures (Section 4). We then present a particularly short proof of the subadditivity of expected shortfall and use this result to show that a quantile risk measure is subadditive if and only if it is spectral (Section 5). As a major issue of this paper, we proceed with a detailed comparison of the domain of a quantile risk measure with the classes of random variables proposed by Acerbi (2002) and Pichler (2013) in the spectral case (Section 6). Finally, and as a complement, we briefly discuss related integrated quantile functions occurring in the measurement of economic inequality (Section 7).

2. Preliminaries

We use the terms positive and increasing in the weak sense which admits equality in the inequalities defining these terms. For $B \subseteq \mathbb{R}$, we denote by χ_B the indicator function of B (such that $\chi_B(x) = 1$ if $x \in B$ and $\chi_B(x) = 0$ if $x \notin B$). Furthermore, we denote:

– by $\mathcal{B}(\mathbb{R})$ the σ-field of all Borel sets of $\mathbb{R}$,
– by $\mathcal{B}((0,1))$ the σ-field of all Borel sets of $(0,1)$ and
– by λ the Lebesgue measure on $\mathcal{B}(\mathbb{R})$ or its restriction to $\mathcal{B}((0,1))$.

By the correspondence theorem, there exists a bijection between the distribution functions on $\mathbb{R}$ and the probability measures on $\mathcal{B}(\mathbb{R})$ such that the probability measure Q^G corresponding to the distribution function G satisfies $Q^G[(x,y]] = G(y) - G(x)$ for all $x, y \in \mathbb{R}$ such that $x \leq y$. Correspondingly, there exists a bijection between the distribution functions on $(0,1)$ and the probability measures on $\mathcal{B}((0,1))$.

Throughout this paper, we consider a fixed probability space $(\Omega, \mathcal{F}, P)$ and random variables $(\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, and we denote:

– by $\mathcal{L}^0$ the vector lattice of all random variables,
– by $\mathcal{L}^1$ the vector lattice of all integrable random variables,
– by $\mathcal{L}^2$ the vector lattice of all square integrable random variables and
– by $\mathcal{L}^\infty$ the vector lattice of all almost surely bounded random variables.

Then, we have $\mathcal{L}^\infty \subseteq \mathcal{L}^2 \subseteq \mathcal{L}^1 \subseteq \mathcal{L}^0$. For a random variable $X \in \mathcal{L}^0$, we denote by F_X its distribution function $\mathbb{R} \to [0,1]$ given by:

$$F_X(x) := P[\{X \leq x\}]$$

and by $F_X^{\leftarrow}$ its (lower) quantile function $(0,1) \to \mathbb{R}$ given by:

$$F_X^{\leftarrow}(u) := \inf\Big\{x \in \mathbb{R} \;\Big|\; F_X(x) \geq u\Big\}$$

For $u \in (0,1)$ and $x \in \mathbb{R}$, the quantile function satisfies $F_X^{\leftarrow}(u) \leq x$ if and only if $u \leq F_X(x)$. Moreover, the quantile function is increasing and has the following properties:

Lemma 1. *Consider $X, Y \in \mathcal{L}^0$. Then:*

(1) *If $X \leq Y$, then $F_X^{\leftarrow} \leq F_Y^{\leftarrow}$.*
(2) *If $a \in \mathbb{R}_+$, then $F_{aX}^{\leftarrow} = a\, F_X^{\leftarrow}$.*
(3) *If $c \in \mathbb{R}$, then $F_{X+c}^{\leftarrow} = F_X^{\leftarrow} + c$.*
(4) *If X and Y are comonotone, then $F_{X+Y}^{\leftarrow} = F_X^{\leftarrow} + F_Y^{\leftarrow}$.*
(5) *$F_{X^+}^{\leftarrow} = (F_X^{\leftarrow})^+$.*

A function $D : [0,1] \to [0,1]$ is said to be a distortion function if it is increasing and continuous from the right and satisfies $D(0) = 0$ and $\sup_{u \in (0,1)} D(u) = 1$ (and hence, $D(1) = 1$). The restriction of a distortion function D to $(0,1)$ is a distribution function on $(0,1)$, and for simplicity, the probability measure corresponding to the restriction of D to $(0,1)$ will be referred to as the probability measure corresponding to D.

Example 1. *The terms attached to the following examples are the names of the risk measures resulting from the respective distortion functions.*

(1) *Expectation: The function $D^E : [0,1] \to [0,1]$ given by:*

$$D^E(u) := u$$

is a distortion function.

(2) *Value at risk: For $\alpha \in (0,1)$, the function $D^{\mathrm{VaR}_\alpha} : [0,1] \to [0,1]$ given by:*

$$D^{\mathrm{VaR}_\alpha}(u) := \chi_{[\alpha,1]}(u)$$

is a distortion function.

(3) *Expected shortfall: For $\alpha \in [0,1)$, the function $D^{\mathrm{ES}_\alpha} : [0,1] \to [0,1]$ given by:*

$$D^{\mathrm{ES}_\alpha}(u) := \frac{u-\alpha}{1-\alpha}\, \chi_{[\alpha,1]}(u)$$

is a distortion function; in particular, $D^{\mathrm{ES}_0} = D^E$.

(4) *Expected shortfall of higher degree: For $n \in \mathbb{N}$ and $\alpha \in [0,1)$, the function $D^{\mathrm{ES}_{n;\alpha}}(u) : [0,1] \to [0,1]$ given by:*

$$D^{\mathrm{ES}_{n;\alpha}}(u) := \left(\frac{u-\alpha}{1-\alpha}\right)^n \chi_{[\alpha,1]}(u)$$

is a distortion function; in particular, $D^{\mathrm{ES}_{1;\alpha}} = D^{\mathrm{ES}_\alpha}$.

(5) *Range value at risk: For $\alpha \in [0,1)$ and $\beta \in (0,\alpha)$, the function $D^{\mathrm{ES}_{\alpha,\beta}} : [0,1] \to [0,1]$ given by:*

$$D^{\mathrm{ES}_{\alpha,\beta}}(u) := \frac{u-\alpha+\beta}{1-\alpha}\, \chi_{[\alpha-\beta,1-\beta)}(u) + \chi_{[1-\beta,1]}(u)$$

is a distortion function; in particular, $\lim_{\beta \to 0} D^{\mathrm{ES}_{\alpha,\beta}}(u) = D^{\mathrm{ES}_\alpha}(u)$.

The distortion functions $D^{\mathrm{ES}_{n;\alpha}}$, and in particular D^{ES_α} and D^E, are convex, whereas D^{VaR_α} and $D^{\mathrm{ES}_{\alpha,\beta}}$ are not convex. Further distortion functions may be found e.g., in Hardy (2006).

Throughout this paper, we consider pairs (D, Q) consisting of a distortion function $D : [0, 1] \to [0, 1]$ and the probability measure $Q : \mathcal{B}((0, 1)) \to [0, 1]$ corresponding to D, and we use identical sub- or super-scripts for both, D and Q, in the case of a particular choice of D or Q.

3. Quantile Risk Measures

Define:

$$\mathcal{L}_Q := \left\{ X \in \mathcal{L}^0 \;\middle|\; \int_{(0,1)} (F_X^{\leftarrow}(u))^+ \, dQ(u) < \infty \right\}$$

Then, we have $\mathcal{L}^\infty \subseteq \mathcal{L}_Q$, and the map $\varrho_Q : \mathcal{L}_Q \to [-\infty, \infty)$ given by:

$$\varrho_Q[X] := \int_{(0,1)} F_X^{\leftarrow}(u) \, dQ(u)$$

is said to be a quantile risk measure.

For every $X \in \mathcal{L}^0$, we have $X \in \mathcal{L}_Q$ if and only if $X^+ \in \mathcal{L}_Q$, by Lemma 1. This implies that, for every $Z \in \mathcal{L}^0$ satisfying $Z \leq X$ for some $X \in \mathcal{L}_Q$, we have $Z \in \mathcal{L}_Q$. Lemma 1 also yields the following properties of a quantile risk measure:

Lemma 2. *Consider $X, Y \in \mathcal{L}_Q$. Then:*

(1) *If $X \leq Y$, then $\varrho_Q[X] \leq \varrho_Q[Y]$.*
(2) *If $a \in \mathbb{R}_+$, then $aX \in \mathcal{L}_Q$ and $\varrho_Q[aX] = a\,\varrho_Q[X]$.*
(3) *If $c \in \mathbb{R}$, then $X+c \in \mathcal{L}_Q$ and $\varrho_Q[X+c] = \varrho_Q[X] + c$.*
(4) *If X and Y are comonotone, then $X + Y \in \mathcal{L}_Q$ and $\varrho_Q[X + Y] = \varrho_Q[X] + \varrho_Q[Y]$.*

The quantile risk measure ϱ_Q is said to be subadditive if $\varrho_Q[X + Y] \leq \varrho_Q[X] + \varrho_Q[Y]$ holds for all $X, Y \in \mathcal{L}_Q$ such that $X + Y \in \mathcal{L}_Q$. We shall show that ϱ_Q is subadditive if and only if D is convex and that, in this case, $\mathcal{L}_Q$ is a convex cone; see Theorem 4 below.

To obtain alternative representations of a quantile risk measure and its domain, we need the following Lemma:

Lemma 3. *The identities:*

$$\int_{(0,1)} (F_X^{\leftarrow}(u))^+ \, dQ(u) = \int_{\mathbb{R}} x^+ \, dQ^{D \circ F_X}(x) = \int_{(0,\infty)} \Big(1 - (D \circ F_X)(x)\Big) \, d\lambda(x)$$

and:

$$\int_{(0,1)} (F_X^{\leftarrow}(u))^- \, dQ(u) = \int_{\mathbb{R}} x^- \, dQ^{D \circ F_X}(x) = \int_{(-\infty,0)} (D \circ F_X)(x) \, d\lambda(x)$$

hold for every $X \in \mathcal{L}^0$.

The following result is immediate from Lemma 3:

Theorem 1. *The domain of ϱ_Q satisfies:*

$$\mathcal{L}_Q = \left\{ X \in \mathcal{L}^0 \;\middle|\; \int_{\mathbb{R}} x^+ \, dQ^{D \circ F_X}(x) < \infty \right\} = \left\{ X \in \mathcal{L}^0 \;\middle|\; \int_{(0,\infty)} \Big(1 - (D \circ F_X)(x)\Big) \, d\lambda(x) < \infty \right\}$$

and the identities:

$$\varrho_Q[X] = \int_{\mathbb{R}} x \, dQ^{D \circ F_X}(x) = \int_{(0,\infty)} \Big(1 - (D \circ F_X)(x)\Big) \, d\lambda(x) - \int_{(-\infty,0)} (D \circ F_X)(x) \, d\lambda(x)$$

hold for every $X \in \mathcal{L}_Q$.

Because of the previous result, the quantile risk measure generated by the probability measure Q corresponds to the distortion risk measure generated by the distortion function D; the latter is also known as Wang's premium principle.

Example 2.

(1) *Expectation: The distortion function D^E satisfies $D^E \circ F_X = F_X$. Because of Theorem 1, this yields:*

$$\mathcal{L}_{Q^E} = \left\{ X \in \mathcal{L}^0 \;\middle|\; E[X^+] < \infty \right\}$$

and:

$$\varrho_{Q^E}[X] = E[X]$$

for every $X \in \mathcal{L}_{Q^E}$.

(2) *Value at risk: For $\alpha \in (0,1)$, the probability measure Q^{VaR_α} corresponding to D^{VaR_α} is the Dirac measure at α. This yields:*

$$\mathcal{L}_{Q^{\mathrm{VaR}_\alpha}} = \mathcal{L}^0$$

and:

$$\varrho_{Q^{\mathrm{VaR}_\alpha}}[X] = F_X^{\leftarrow}(\alpha)$$

for every $X \in \mathcal{L}_{Q^{\mathrm{VaR}_\alpha}}$; in particular, $\varrho_{Q^{\mathrm{VaR}_\alpha}}$ is finite. The quantile risk measure $\varrho_{Q^{\mathrm{VaR}_\alpha}}$ is called value at risk at level α and is usually denoted by VaR_α.

(3) *Expected shortfall: For $\alpha \in [0,1)$, the probability measure Q^{ES_α} corresponding to D^{ES_α} satisfies:*

$$Q^{\mathrm{ES}_\alpha} = \int \frac{1}{1-\alpha}\, \chi_{(\alpha,1)}(u)\, d\lambda(u)$$

Since $F_X^{\leftarrow}$ is increasing and $F_X^{\leftarrow}(\alpha)$ is finite for $\alpha \in (0,1)$, this yields, because of (1),

$$\begin{aligned} \mathcal{L}_{Q^{\mathrm{ES}_\alpha}} &= \left\{ X \in \mathcal{L}^0 \;\middle|\; \int_{(\alpha,1)} (F_X^{\leftarrow}(u))^+ \, d\lambda(u) < \infty \right\} \\ &= \left\{ X \in \mathcal{L}^0 \;\middle|\; \int_{(0,1)} (F_X^{\leftarrow}(u))^+ \, d\lambda(u) < \infty \right\} \\ &= \left\{ X \in \mathcal{L}^0 \;\middle|\; E[X^+] < \infty \right\} \\ &= \mathcal{L}_{Q^E} \end{aligned}$$

and:

$$\varrho_{Q^{\mathrm{ES}_\alpha}}[X] = \int_{(0,1)} F_X^{\leftarrow}(u)\, \frac{1}{1-\alpha}\, \chi_{(\alpha,1)}(u)\, d\lambda(u)$$

for every $X \in \mathcal{L}_{Q^{\mathrm{ES}_\alpha}}$. In particular, $\varrho_{Q^{\mathrm{ES}_0}} = \varrho_{Q^E}$, and $\varrho_{Q^{\mathrm{ES}_\alpha}}$ is finite for every $\alpha \in (0,1)$. The quantile risk measure $\varrho_{Q^{\mathrm{ES}_\alpha}}$ is called expected shortfall at level α and is usually denoted by ES_α.

(4) *Expected shortfall of higher degree: For $n \in \mathbb{N}$ and $\alpha \in [0,1)$, the probability measure $Q^{\mathrm{ES}_{n;\alpha}}$ corresponding to $D^{\mathrm{ES}_{n;\alpha}}$ satisfies:*

$$Q^{\mathrm{ES}_{n;\alpha}} = \int \frac{n}{1-\alpha} \left(\frac{u-\alpha}{1-\alpha} \right)^{n-1} \chi_{(\alpha,1)}(u)\, d\lambda(u)$$

This yields:

$$\mathcal{L}_{Q^{\mathrm{ES}_{n;\alpha}}} = \mathcal{L}_{Q^E}$$

and:

$$\varrho_{Q^{\mathrm{ES}_{n;\alpha}}}[X] = \int_{(0,1)} F_X^{\leftarrow}(u)\, \frac{n}{1-\alpha} \left(\frac{u-\alpha}{1-\alpha} \right)^{n-1} \chi_{(\alpha,1)}(u)\, d\lambda(u)$$

for every $X \in \mathcal{L}_{Q^{\mathrm{ES}_{n;\alpha}}}$. *In particular,* $\varrho_{Q^{\mathrm{ES}_{1;\alpha}}} = \varrho_{Q^{\mathrm{ES}_{\alpha}}}$, *and* $\varrho_{Q^{\mathrm{ES}_{n;\alpha}}}$ *is finite for every* $n \in \mathbb{N}$ *and* $\alpha \in (0,1)$. *The quantile risk measure* $\varrho_{Q^{\mathrm{ES}_{n;\alpha}}}$ *is called expected shortfall of degree* n *at level* α.

(5) *Range value at risk: For* $\alpha \in [0,1)$ *and* $\beta \in (0,\alpha)$, *the probability measure* $Q^{\mathrm{ES}_{\alpha,\beta}}$ *corresponding to* $D^{\mathrm{ES}_{\alpha,\beta}}$ *satisfies:*

$$Q^{\mathrm{ES}_{\alpha,\beta}} = \int \frac{1}{1-\alpha}\, \chi_{(\alpha-\beta,1-\beta)}(u)\, d\lambda(u)$$

This yields:

$$\mathcal{L}_{Q^{\mathrm{ES}_{\alpha,\beta}}} = \mathcal{L}^0$$

and:

$$\varrho_{Q^{\mathrm{ES}_{\alpha,\beta}}}[X] = \int_{(0,1)} F_X^{\leftarrow}(u)\, \frac{1}{1-\alpha}\, \chi_{(\alpha-\beta,1-\beta)}(u)\, d\lambda(u)$$

for every $X \in \mathcal{L}_{Q^{\mathrm{ES}_{\alpha,\beta}}}$. *In particular,* $\varrho_{Q^{\mathrm{ES}_{\alpha,\beta}}}$ *is finite for every* $\alpha \in (0,1)$ *and* $\beta \in (0,\alpha)$. *The quantile risk measure* $\varrho_{Q^{\mathrm{ES}_{\alpha,\beta}}}$ *is called range value at risk at levels* α *and* β; *see Cont et al. (2010) and Embrechts et al. (2017).*

The examples show that the domains of different quantile risk measures may be distinct.

Lemma 3 and Theorem 1 have several applications. For example, they provide a condition on D under which ϱ_Q is finite:

Corollary 1. *Assume that there exists some* $\delta \in (0,1)$ *such that* $D(u) = 0$ *holds for every* $u \in (0,\delta)$. *Then:*

$$\begin{aligned}
\mathcal{L}_Q &= \left\{ X \in \mathcal{L}^0 \,\middle|\, \int_{(0,1)} |F_X^{\leftarrow}(u)|\, dQ(u) < \infty \right\} \\
&= \left\{ X \in \mathcal{L}^0 \,\middle|\, \int_{\mathbb{R}} |x|\, dQ^{D\circ F_X}(x) < \infty \right\} \\
&= \left\{ X \in \mathcal{L}^0 \,\middle|\, \int_{(0,\infty)} \Big(1-(D\circ F_X)(x)\Big)\, d\lambda(x) + \int_{(-\infty,0)} (D\circ F_X)(x)\, d\lambda(x) < \infty \right\}
\end{aligned}$$

and ϱ_Q *is finite.*

Proof. For every $X \in \mathcal{L}^0$, the assumption yields:

$$\begin{aligned}
\int_{(0,1)} (F_X^{\leftarrow}(u))^-\, dQ(u) &= \int_{(-\infty,0)} (D\circ F_X)(x)\, d\lambda(x) \\
&= \int_{(-\infty,0)} (D\circ F_X)(x)\, \chi_{[\delta,1)}(F_X(x))\, d\lambda(x) \\
&= \int_{(-\infty,0)} (D\circ F_X)(x)\, \chi_{[F_X^{\leftarrow}(\delta),0)}(x)\, d\lambda(x) \\
&\leq (D\circ F_X)(0) \int_{(-\infty,0)} \chi_{[F_X^{\leftarrow}(\delta),0)}(x)\, d\lambda(x)
\end{aligned}$$

Since $F_X^{\leftarrow}(\delta)$ is finite, this proves the assertion. □

Theorem 1 also provides a condition for the comparison of the domains of quantile risk measures:

Corollary 2. *Assume that there exists some* $\delta \in (0,1)$ *such that* $D_1(u) \leq D_2(u)$ *holds for every* $u \in [\delta,1)$. *Then,* $\mathcal{L}_{Q_1} \subseteq \mathcal{L}_{Q_2}$.

Proof. For every $X \in \mathcal{L}^0$, we have:

$$\begin{aligned}
&\int_{(0,\infty)} \Big(1-(D_2 \circ F_X)(x)\Big)\, d\lambda(x) \\
&= \int_{(0,\infty)} \Big(1-(D_2 \circ F_X)(x)\Big)\, \chi_{(0,F_X^{\leftarrow}(\delta))}(x)\, d\lambda(x) + \int_{(0,\infty)} \Big(1-(D_2 \circ F_X)(x)\Big)\, \chi_{[F_X^{\leftarrow}(\delta),\infty)}(x)\, d\lambda(x) \\
&\leq \int_{(0,\infty)} \chi_{(0,F_X^{\leftarrow}(\delta))}(x)\, d\lambda(x) + \int_{(0,\infty)} \Big(1-(D_1 \circ F_X)(x)\Big)\, d\lambda(x)
\end{aligned}$$

Since $F_X^{\leftarrow}(\delta)$ is finite, Theorem 1 yields $\mathcal{L}_{Q_1} \subseteq \mathcal{L}_{Q_2}$. □

Corollary 3. *Assume that there exist some $n \in \mathbb{N}$ and $\alpha, \delta \in (0,1)$ such that:*

$$D^{\mathrm{ES}_{n,\alpha}}(u) \leq D(u) \leq D^E(u)$$

holds for every $u \in [\delta, 1)$. Then, $\mathcal{L}_Q = \mathcal{L}_{Q^E}$.

Proof. Because of Corollary 2, we have $\mathcal{L}_{Q^{\mathrm{ES}_{n,\alpha}}} \subseteq \mathcal{L}_Q \subseteq \mathcal{L}_{Q^E}$. Now, the assertion follows from $\mathcal{L}_{Q^{\mathrm{ES}_{n,\alpha}}} = \mathcal{L}_{Q^E}$. □

Combining Corollaries 1 and 3 yields a condition under which $\mathcal{L}_Q = \mathcal{L}_{Q^E}$ and ϱ_Q is finite. Corollary 2 also yields some further results on the comparison of quantile risk measures and their domains:

Corollary 4.

(1) *$D_1 \leq D_2$ if and only if $\varrho_{Q_2}[X] \leq \varrho_{Q_1}[X]$ holds for every $X \in \mathcal{L}_{Q_1} \cap \mathcal{L}_{Q_2}$, and in this case, $\mathcal{L}_{Q_1} \subseteq \mathcal{L}_{Q_2}$.*
(2) *$D \leq D^E$ if and only if $E[X] \leq \varrho_Q[X]$ holds for every $X \in \mathcal{L}_Q \cap \mathcal{L}_{Q^E}$, and in this case, $\mathcal{L}_Q \subseteq \mathcal{L}_{Q^E}$.*
(3) *If D is convex, then $\mathcal{L}_Q \subseteq \mathcal{L}_{Q^E}$ and $E[X] \leq \varrho_Q[X]$ holds for every $X \in \mathcal{L}_Q$.*
(4) *Consider $\alpha, \beta \in [0,1)$. Then, $\alpha \leq \beta$ if and only if $\varrho_{Q^{\mathrm{ES}_\alpha}}[X] \leq \varrho_{Q^{\mathrm{ES}_\beta}}[X]$ holds for every $X \in \mathcal{L}_{Q^E}$.*
(5) *The identity $E[X] = \inf_{\alpha \in (0,1)} \varrho_{Q^{\mathrm{ES}_\alpha}}[X]$ holds for every $X \in \mathcal{L}_{Q^E}$.*

Proof. Assume first that $D_1 \leq D_2$. Then Corollary 2 yields $\mathcal{L}_{Q_1} \subseteq \mathcal{L}_{Q_2}$ and Theorem 1 yields $\varrho_{Q_2}[X] \leq \varrho_{Q_1}[X]$ for every $X \in \mathcal{L}_{Q_1} \cap \mathcal{L}_{Q_2} = \mathcal{L}_{Q_1}$. Assume now that $\varrho_{Q_2}[X] \leq \varrho_{Q_1}[X]$ holds for every $X \in \mathcal{L}_{Q_1} \cap \mathcal{L}_{Q_2}$ and consider $u \in (0,1)$. Then, for any choice of $a, b \in \mathbb{R}$ such that $a < b$ and for every random variable X satisfying $P[\{X = a\}] = u = 1 - P[\{X = b\}]$, we have $X \in \mathcal{L}^\infty \subseteq \mathcal{L}_{Q_1} \cap \mathcal{L}_{Q_2}$. Straightforward computation yields $\varrho_{D_i}[X] = b - (b-a)D_i(u)$ for all $i \in \{1,2\}$, and hence, $D_1(u) \leq D_2(u)$. Since $u \in (0,1)$ was arbitrary, it follows that $D_1 \leq D_2$. This proves (1). Assertions (2)–(4) are immediate from (1), and Assertion (5) follows from the dominated convergence theorem. □

Assertion (1) of Corollary 4 extends a result of Wang et al. (2015), who considered risk measures that are defined on a common convex cone containing $\mathcal{L}^\infty$.

4. Spectral Risk Measures

A map $s : (0,1) \to \mathbb{R}_+$ is said to be a spectral function if it is increasing and satisfies $\int_{(0,1)} s(u)\, d\lambda(u) = 1$.

The quantile risk measure ϱ_Q is said to be a spectral risk measure if there exists a spectral function s such that:

$$Q = \int s(u)\, d\lambda(u)$$

Thus, if ϱ_Q is a spectral risk measure with spectral function s, then the domain of ϱ_Q satisfies:

$$\mathcal{L}_Q = \left\{ X \in \mathcal{L}^0 \;\middle|\; \int_{(0,1)} (F_X^{\leftarrow}(u))^+ \, s(u) \, d\lambda(u) < \infty \right\}$$

and the identity:

$$\varrho_Q[X] = \int_{(0,1)} F_X^{\leftarrow}(u) \, s(u) \, d\lambda(u)$$

holds for every $X \in \mathcal{L}_Q$. Note that the spectral function of a spectral risk measure is unique almost everywhere, by the Radon–Nikodym theorem.

Example 3.

(1) *Expectation: Since* $D^E(u) = u$*, we have:*

$$Q^E = \lambda$$

and the function $s^E : (0,1) \to \mathbb{R}_+$ *given by:*

$$s^E(u) := 1$$

is a spectral function. Therefore, ϱ_{Q^E} *is a spectral risk measure.*

(2) *Value at risk: For every* $\alpha \in (0,1)$, Q^{VaR_α} *is the Dirac measure at* α *and hence does not have a density with respect to* λ*. Therefore,* $\varrho_{Q^{\mathrm{VaR}_\alpha}}$ *is not a spectral risk measure.*

(3) *Expected shortfall: For every* $\alpha \in [0,1)$*, we have:*

$$Q^{\mathrm{ES}_\alpha} = \int \frac{1}{1-\alpha} \chi_{(\alpha,1)}(u) \, d\lambda(u)$$

and the function $s^{\mathrm{ES}_\alpha} : (0,1) \to \mathbb{R}_+$ *given by:*

$$s^{\mathrm{ES}_\alpha}(u) := \frac{1}{1-\alpha} \chi_{(\alpha,1)}(u)$$

is a spectral function. Therefore, $\varrho_{Q^{\mathrm{ES}_\alpha}}$ *is a spectral risk measure.*

(4) *Expected shortfall of higher degree: For every* $n \in \mathbb{N}$ *and* $\alpha \in [0,1)$*, we have:*

$$Q^{\mathrm{ES}_{n;\alpha}} = \int \frac{n}{1-\alpha} \left(\frac{u-\alpha}{1-\alpha} \right)^{n-1} \chi_{(\alpha,1)}(u) \, d\lambda(u)$$

and the function $s^{\mathrm{ES}_{n;\alpha}} : (0,1) \to \mathbb{R}_+$ *given by:*

$$s^{\mathrm{ES}_{n;\alpha}}(u) := \frac{n}{1-\alpha} \left(\frac{u-\alpha}{1-\alpha} \right)^{n-1} \chi_{(\alpha,1)}(u)$$

is a spectral function. Therefore, $\varrho_{Q^{\mathrm{ES}_{n;\alpha}}}$ *is a spectral risk measure.*

(5) *Range value at risk: For every* $\alpha \in [0,1)$ *and* $\beta \in (0,\alpha)$*, we have:*

$$Q^{\mathrm{ES}_{\alpha,\beta}} = \int \frac{1}{1-\alpha} \chi_{(\alpha-\beta,1-\beta)}(u) \, d\lambda(u)$$

and the function $s^{\mathrm{ES}_{\alpha,\beta}} : (0,1) \to \mathbb{R}_+$ *given by:*

$$s^{\mathrm{ES}_{\alpha,\beta}} = \frac{1}{1-\alpha} \chi_{(\alpha-\beta,1-\beta)}(u)$$

fails to be increasing and hence fails to be a spectral function. Therefore, $\varrho_{Q^{\mathrm{ES}_{\alpha,\beta}}}$ is not a spectral risk measure.

Our aim is to characterize the spectral risk measures within the class of all quantile risk measures. The following result is inspired by Gzyl and Mayoral (2008), who considered distortion risk measures on the positive cone of $\mathcal{L}^2$:

Theorem 2. *The following are equivalent:*

(a) *D is convex.*
(b) *There exists a spectral function s such that $Q = \int s(u)\, d\lambda(u)$.*
(c) *ϱ_Q is a spectral risk measure.*

In this case, every spectral function s representing Q satisfies $s = D'$ almost everywhere (with respect to λ).

Proof. Since $\lim_{u\to 0} D(u) = 0 = D(0)$ and $\lim_{u\to 1} D(u) = 1 = D(1)$, D is convex if and only if D is convex on $(0,1)$.

Assume first that (a) holds. The following arguments are taken from Aliprantis and Burkinshaw (1990, chp. 29). Since D is increasing, D is differentiable almost everywhere, and since D is convex, its derivative D' is increasing. Consider now an arbitrary interval $[u,v] \subseteq (0,1)$. Since D is convex, the restriction of D to $[u,v]$ is Lipschitz continuous, hence absolutely continuous and, thus, continuous and of bounded variation. Therefore, the restriction of Q to the σ-field of all Borel sets in $[u,v]$ is absolutely continuous with respect to the restriction of λ, and its Radon–Nikodym derivative agrees with D'. Since $[u,v] \subseteq (0,1)$ was arbitrary, it follows that Q is absolutely continuous with respect to λ, and since the Radon–Nikodym derivative $s : (0,1) \to \mathbb{R}_+$ of Q with respect to λ is unique almost everywhere, it follows that $s = D'$ almost everywhere. This yields the existence of an increasing function $s : (0,1) \to \mathbb{R}_+$ satisfying $Q = \int s(u)\, d\lambda(u)$. Therefore, (a) implies (b).

Assume now that (b) holds. Since s is increasing, we have, for any $u, v, w \in (0,1)$ such that $u < v < w$,

$$\frac{D(v) - D(u)}{v - u} = \frac{1}{v - u} \int_{(u,v]} s(t)\, d\lambda(t) \leq s(v) \leq \frac{1}{w - v} \int_{(v,w]} s(t)\, d\lambda(t) = \frac{D(w) - D(v)}{w - v}$$

which implies that D is convex. Therefore, (b) implies (a). □

The following result is inspired by Kusuoka (2001), who studied risk measures on $\mathcal{L}^\infty$:

Theorem 3. *If D is convex, then there exists a measure $\nu : \mathcal{B}([0,1)) \to [0,\infty]$ such that:*

$$\varrho_Q[X] = \int_{[0,1)} (1-\alpha)\, \varrho_{Q^{\mathrm{ES}_\alpha}}[X]\, d\nu(\alpha)$$

holds for every $X \in \mathcal{L}_Q$.

Proof. Without loss of generality, we may and do assume that s is continuous from the right. Define $s(0) := \inf_{u\in(0,1)} s(u)$. Then, there exists a unique σ-finite measure $\nu : \mathcal{B}([0,1)) \to [0,\infty]$ satisfying $\nu[[0,u]] = s(u)$ for all $u \in (0,1)$. Since the map $(0,1) \times [0,1) \to \mathbb{R} : (u,\alpha) \to F_X^{\leftarrow}(u)\, \chi_{[0,u]}(\alpha)$ is measurable and its positive part is integrable with respect to the product measure $\nu \otimes \lambda$, Fubini's theorem yields:

$$\begin{aligned} \varrho_Q[X] &= \int_{(0,1)} F_X^{\leftarrow}(u)\, s(u)\, d\lambda(u) \\ &= \int_{(0,1)} F_X^{\leftarrow}(u) \int_{[0,1)} \chi_{[0,u]}(\alpha)\, d\nu(\alpha)\, d\lambda(u) \end{aligned}$$

$$
\begin{aligned}
&= \int_{[0,1)} \int_{(0,1)} F_X^{\leftarrow}(u)\, \chi_{(\alpha,1)}(u)\, d\lambda(u)\, d\nu(\alpha) \\
&= \int_{[0,1)} (1-\alpha)\, \varrho_{Q^{\mathrm{ES}_\alpha}}[X]\, d\nu(\alpha)
\end{aligned}
$$

This proves the assertion. □

5. Subadditivity of Spectral Risk Measures

In the present section, we show that a quantile risk measure is subadditive if and only if its distortion function is convex. To prove that the convexity of the distortion function is sufficient for subadditivity of the quantile risk measure, we use Theorem 3. Since the expectation is additive and hence subadditive, it remains to show that the expected shortfall at any level is subadditive.

To establish subadditivity of the expected shortfall, we need the following lemma, which provides another representation of the values of the expected shortfall:

Lemma 4. *For every $\alpha \in (0,1)$, the identity:*

$$
\varrho_{Q^{\mathrm{ES}_\alpha}}[X] = F_X^{\leftarrow}(\alpha) + \frac{1}{1-\alpha} E\Big[\Big(X - F_X^{\leftarrow}(\alpha)\Big)^+\Big] = \inf_{c\in\mathbb{R}} \Big(c + \frac{1}{1-\alpha} E[(X-c)^+]\Big)
$$

holds for every $X \in \mathcal{L}_{Q^{\mathrm{ES}_\alpha}}$.

Lemma 4 is well-known and is frequently used to establish the subadditivity of expected shortfall on $\mathcal{L}^\infty$; see, e.g., Embrechts and Wang (2015), who used a general extension procedure to extend this result beyond $\mathcal{L}^\infty$. Here, we use Lemma 4 to establish the subadditivity of expected shortfall on its (maximal) domain $\mathcal{L}_{Q^{\mathrm{ES}_\alpha}}$ in a single step:

Lemma 5. *For every $\alpha \in [0,1)$, $\mathcal{L}_{Q^{\mathrm{ES}_\alpha}}$ is a convex cone and $\varrho_{Q^{\mathrm{ES}_\alpha}}$ is subadditive.*

Proof. Since $\mathcal{L}_{Q^{\mathrm{ES}_\alpha}} = \mathcal{L}_{Q^E}$, we see that $\mathcal{L}_{Q^{\mathrm{ES}_\alpha}}$ is a convex cone. Furthermore, since $Q^{\mathrm{ES}_0} = Q^E$, we see that $\varrho_{Q^{\mathrm{ES}_0}}$ is subadditive. Consider now $\alpha \in (0,1)$ and $X, Y \in \mathcal{L}_{Q^{\mathrm{ES}_\alpha}}$. Then, we have $X + Y \in \mathcal{L}_{Q^{\mathrm{ES}_\alpha}}$ and, for any $x, y \in \mathbb{R}$, Lemma 4 yields:

$$
\begin{aligned}
\varrho_{Q^{\mathrm{ES}_\alpha}}[X+Y] &\leq (x+y) + \frac{1}{1-\alpha} E\Big[\Big((X+Y) - (x+y)\Big)^+\Big] \\
&= x + y + \frac{1}{1-\alpha} E\Big[\Big((X-x) + (Y-y)\Big)^+\Big] \\
&\leq \Big(x + \frac{1}{1-\alpha} E[(X-x)^+]\Big) + \Big(y + \frac{1}{1-\alpha} E[(Y-y)^+]\Big)
\end{aligned}
$$

Now, minimization over $x, y \in \mathbb{R}$ and using Lemma 4 again yields: $\varrho_{Q^{\mathrm{ES}_\alpha}}[X+Y] \leq \varrho_{Q^{\mathrm{ES}_\alpha}}[X] + \varrho_{Q^{\mathrm{ES}_\alpha}}[Y]$. Therefore, $\varrho_{Q^{\mathrm{ES}_\alpha}}$ is subadditive for every $\alpha \in (0,1)$. □

The previous result provides the key for proving the main implication of the following theorem; see also Wang and Dhaene (1998), who considered distortion risk measures on the positive cone of $\mathcal{L}^1$ and used a proof based on comonotonicity.

Theorem 4. *The following are equivalent:*

(a) *D is convex.*
(b) *ϱ_Q is subadditive.*
(c) *$\mathcal{L}_Q$ is a convex cone, and ϱ_Q is subadditive.*

Proof. Assume first that (a) holds, and consider a spectral function s representing Q and the measure ν constructed in the proof of Theorem 3. Consider $X, Y \in \mathcal{L}_Q$ and $a \in \mathbb{R}_+$. Then, we have $aX \in \mathcal{L}_Q$. Moreover, since D is convex, Corollary 4 yields $X, Y \in \mathcal{L}_{Q^E}$. For every $\alpha \in [0,1)$, this yields $X, Y \in \mathcal{L}_{Q^{\mathrm{ES}_\alpha}}$; hence, $X+Y \in \mathcal{L}_{Q^{\mathrm{ES}_\alpha}}$, by Lemma 5; and thus, $X^+, Y^+, (X+Y)^+ \in \mathcal{L}_{Q^{\mathrm{ES}_\alpha}}$. Proceeding as in the proof of Theorem 3 and using Lemma 5 again, we obtain:

$$
\begin{aligned}
\int_{(0,1)} F^{\leftarrow}_{(X+Y)^+}(u)\, s(u)\, d\lambda(u) &= \int_{[0,1)} (1-\alpha)\, \varrho_{Q^{\mathrm{ES}_\alpha}}[(X+Y)^+]\, d\nu(\alpha) \\
&\leq \int_{[0,1)} (1-\alpha) \Big(\varrho_{Q^{\mathrm{ES}_\alpha}}[X^+] + \varrho_{Q^{\mathrm{ES}_\alpha}}[Y^+] \Big)\, d\nu(\alpha) \\
&= \int_{[0,1)} (1-\alpha)\, \varrho_{Q^{\mathrm{ES}_\alpha}}[X^+]\, d\nu(\alpha) + \int_{[0,1)} (1-\alpha)\, \varrho_{Q^{\mathrm{ES}_\alpha}}[Y^+]\, d\nu(\alpha) \\
&= \varrho_Q[X^+] + \varrho_Q[Y^+] \\
&< \infty
\end{aligned}
$$

This yields $(X+Y)^+ \in \mathcal{L}_Q$, and hence, $X+Y \in \mathcal{L}_Q$. Thus, $\mathcal{L}_Q$ is a convex cone, and Theorem 3 together with Lemma 5 implies that ϱ_Q is subadditive. Therefore, (a) implies (c). Obviously, (c) implies (b), and it follows from Example 4 below that (b) implies (a). □

For the discussion of the subsequent Example 4, we need the following lemma:

Lemma 6. *The following are equivalent:*

(a) *D is convex.*
(b) *The inequality:*

$$D(u) \leq \frac{1}{2}\Big(D(u-\varepsilon) + D(u+\varepsilon) \Big)$$

holds for all $u \in (0,1)$ and $\varepsilon \in (0, \min\{u, 1-u\})$.

Proof. Assume that (b) holds. Then, the inequality:

$$D\left(\frac{u+v}{2}\right) \leq \frac{1}{2}\Big(D(u) + D(v) \Big)$$

holds for all $u, v \in (0,1)$, and this implies that D is continuous on $(0,1)$. Since D is a distortion function, it follows that D is continuous on $[0,1]$, and now, the previous inequality implies that D is convex. Therefore, (b) implies (a). The converse implication is obvious. □

The bivariate distribution discussed in the following example was proposed by Wirch and Hardy (2002).

Example 4. *Assume that D is not convex. Then, Lemma 6 yields the existence of some $u \in (0,1)$ and $\varepsilon \in (0, \min\{u, 1-u\})$ such that:*

$$2\, D(u) > D(u-\varepsilon) + D(u+\varepsilon)$$

Consider random variables $X, Y \in \mathcal{L}^\infty$ whose joint distribution is given by the following table with $a \in (0, \infty)$:

x	y: $-(a+\varepsilon)$	$-(a+\varepsilon/2)$	0	$P[\{X=x\}]$	$P[\{X \leq x\}]$
$-(a+\varepsilon)$	$u-\varepsilon$	0	ε	u	u
0	0	ε	$1-u-\varepsilon$	$1-u$	1
$P[\{Y=y\}]$	$u-\varepsilon$	ε	$1-u$		
$P[\{Y \leq y\}]$	$u-\varepsilon$	u	1		

Then, the distribution of the sum $X+Y$ is given by the table:

z	$-2(a+\varepsilon)$	$-(a+\varepsilon)$	$-(a+\varepsilon/2)$	0
$P[\{X+Y=z\}]$	$u-\varepsilon$	ε	ε	$1-u-\varepsilon$
$P[\{X+Y\leq z\}]$	$u-\varepsilon$	u	$u+\varepsilon$	1

Because of Theorem 1, this yields:

$$\begin{aligned}\varrho_Q[X] &= -(a+\varepsilon)\,D(u)\\ \varrho_Q[Y] &= -(\varepsilon/2)\,D(u-\varepsilon)-(a+\varepsilon/2)\,D(u)\\ \varrho_Q[X+Y] &= -(a+\varepsilon)\,D(u-\varepsilon)-(\varepsilon/2)\,D(u)-(a+\varepsilon/2)\,D(u+\varepsilon)\end{aligned}$$

and hence:

$$\begin{aligned}\varrho_Q[X+Y] &= \varrho_Q[X]+\varrho_Q[Y]+(a+\varepsilon/2)\Big(2\,D(u)-D(u-\varepsilon)-D(u+\varepsilon)\Big)\\ &> \varrho_Q[X]+\varrho_Q[Y]\end{aligned}$$

Therefore, ϱ_Q fails to be subadditive.

6. On the Domain of a Quantile Risk Measure

In this section, we compare the domain:

$$\mathcal{L}_Q=\left\{X\in\mathcal{L}^0\;\middle|\;\int_{(0,1)}(F_X^{\leftarrow}(u))^+\,dQ(u)<\infty\right\}$$

of the quantile risk measure ϱ_Q with two other classes of random variables. Define:

$$\mathcal{L}_Q^{\text{Acerbi}}:=\left\{X\in\mathcal{L}^0\;\middle|\;\int_{(0,1)}|F_X^{\leftarrow}(u)|\,dQ(u)<\infty\right\}$$

and:

$$\mathcal{L}_Q^{\text{Pichler}}:=\left\{X\in\mathcal{L}^0\;\middle|\;\int_{(0,1)}F_{|X|}^{\leftarrow}(u)\,dQ(u)<\infty\right\}$$

In the case where Q is represented by a spectral function, these classes were introduced by Acerbi (2002) and Pichler (2013), respectively. We have $\mathcal{L}_Q^{\text{Acerbi}}\subseteq\mathcal{L}_Q$, and Corollary 1 provides a sufficient condition for $\mathcal{L}_Q^{\text{Acerbi}}=\mathcal{L}_Q$. Moreover, since $X^+\leq|X|$, we also have $\mathcal{L}_Q^{\text{Pichler}}\subseteq\mathcal{L}_Q$. Below, we shall show that $\mathcal{L}_Q^{\text{Pichler}}\subseteq\mathcal{L}_Q^{\text{Acerbi}}$ whenever D is convex. To this end, we need the following lemma:

Lemma 7. *Assume that D is convex and consider $X\in\mathcal{L}^0$. If $X^+\in\mathcal{L}_Q^{\text{Acerbi}}$ and $X^-\in\mathcal{L}_Q^{\text{Acerbi}}$, then $X\in\mathcal{L}_Q^{\text{Acerbi}}$.*

Proof. From $(F_X^{\leftarrow})^+=F_{X^+}^{\leftarrow}$ and $X^+\in\mathcal{L}_Q^{\text{Acerbi}}$, we obtain:

$$\int_{(0,1)}(F_X^{\leftarrow}(u))^+\,dQ(u)<\infty$$

To prove that the integral $\int_{(0,1)}(F_X^{\leftarrow}(u))^-\,dQ(u)$ is finite, as well, we need the upper quantile function $F_X^{\rightarrow}:(0,1)\to\mathbb{R}$ given by:

$$F_X^{\rightarrow}(u):=\sup\left\{x\in\mathbb{R}\;\middle|\;F_X(x)\leq u\right\}$$

The lower and upper quantile functions satisfy $F_X^{\leftarrow} \leq F_X^{\rightarrow}$, and we have:

$$(F_X^{\leftarrow}(u))^- = -F_X^{\leftarrow}(u)\,\chi_{(0,F_X(0)]}(u)$$

and:

$$F_{X^-}^{\leftarrow}(1-u) = -F_X^{\rightarrow}(u)\,\chi_{(0,F_X(0))}(u)$$

almost everywhere with respect to λ. Since D is convex and hence continuous, Q is absolutely continuous with respect to λ. This yields:

$$\begin{aligned} 0 &\leq \int_{(0,1)} \Big(F_X^{\rightarrow}(u) - F_X^{\leftarrow}(u)\Big)\, dQ(u) \\ &= \int_{(0,1)} \int_{\mathbb{R}} \chi_{[F_X^{\leftarrow}(u),F_X^{\rightarrow}(u))}(x)\, d\lambda(x)\, dQ(u) \\ &\leq \int_{\mathbb{R}} \int_{(0,1)} \chi_{\{F_X(x)\}}(u)\, dQ(u)\, d\lambda(x) \\ &= 0 \end{aligned}$$

and hence. $F_X^{\rightarrow} = F_X^{\leftarrow}$ almost everywhere with respect to Q. Consider now a spectral function s representing Q. Since s is positive and increasing, we obtain:

$$\begin{aligned} \int_{(0,1)} (F_X^{\leftarrow}(u))^-\, dQ(u) &= \int_{(0,1)} (-F_X^{\leftarrow}(u))\,\chi_{(0,F_X(0)]}(u)\, dQ(u) \\ &= \int_{(0,1)} (-F_X^{\rightarrow}(u))\,\chi_{(0,F_X(0))}(u)\, dQ(u) \\ &= \int_{(0,1)} (-F_X^{\rightarrow}(u))\,\chi_{(0,F_X(0))}(u)\, s(u)\, d\lambda(u) \\ &= \int_{(0,1)} F_{X^-}^{\leftarrow}(1-u)\, s(u)\, d\lambda(u) \\ &= \int_{(0,1)} F_{X^-}^{\leftarrow}(u)\, s(1-u)\, d\lambda(u) \\ &\leq \int_{(0,1/2)} F_{X^-}^{\leftarrow}(1/2)\, s(1-u)\, d\lambda(u) + \int_{(1/2,1)} F_{X^-}^{\leftarrow}(u)\, s(u)\, d\lambda(u) \\ &\leq F_{X^-}^{\leftarrow}(1/2) + \int_{(0,1)} F_{X^-}^{\leftarrow}(u)\, dQ(u) \end{aligned}$$

Since $X^- \in \mathcal{L}_Q^{\text{Acerbi}}$, the last expression is finite, and this yields:

$$\int_{(0,1)} (F_X^{\leftarrow}(u))^-\, dQ(u) < \infty$$

Therefore, we have $X \in \mathcal{L}_Q^{\text{Acerbi}}$. □

Theorem 5. *If D is convex, then $\mathcal{L}_Q^{\text{Pichler}} \subseteq \mathcal{L}_Q^{\text{Acerbi}}$.*

Proof. Consider $X \in \mathcal{L}_Q^{\text{Pichler}}$. Then, we have $|X| \in \mathcal{L}_Q^{\text{Pichler}}$, hence $X^+, X^- \in \mathcal{L}_Q^{\text{Pichler}}$, and thus, $X^+, X^- \in \mathcal{L}_Q^{\text{Acerbi}}$. Now, Lemma 7 yields $X \in \mathcal{L}_Q^{\text{Acerbi}}$. □

The following examples provide some further insight into the relationships between these three classes of random variables:

Example 5.

(1) *If $D = D^{\text{VaR}_\alpha}$, then $\mathcal{L}_Q^{\text{Pichler}} = \mathcal{L}_Q^{\text{Acerbi}} = \mathcal{L}_Q = \mathcal{L}^0$.*

(2) *If* $D = D^E$*, then* $\mathcal{L}_Q^{\text{Pichler}} = \mathcal{L}_Q^{\text{Acerbi}} = \mathcal{L}^1 \neq \mathcal{L}_Q$*.*
(3) *If* $D = D^{\text{ES}_\alpha}$ *for some* $\alpha \in (0,1)$*, then* $\mathcal{L}_Q^{\text{Pichler}} \neq \mathcal{L}^Q = \mathcal{L}_Q^{\text{Acerbi}}$*.*
(4) *Assume that there exists some* $\delta \in (0,1)$ *such that D satisfies:*

$$D(u) = u\,\chi_{[0,\delta)}(u) + \chi_{[\delta,1]}(u)$$

(and hence, fails to be convex). Then, every $X \in \mathcal{L}^0$ *satisfies:*

$$\int_{(0,1)} F_{|X|}^{\leftarrow}(u)\,dQ(u) < \infty \qquad \textit{and} \qquad \int_{(0,1)} (F_X^{\leftarrow}(u))^+\,dQ(u) < \infty$$

This yields $\mathcal{L}_Q^{\text{Pichler}} = \mathcal{L}^0 = \mathcal{L}_Q$*, as well as:*

$$\begin{aligned} \mathcal{L}_Q^{\text{Acerbi}} &= \left\{ X \in \mathcal{L}^0 \,\middle|\, \int_{(0,1)} (F_X^{\leftarrow}(u))^-\,dQ(u) < \infty \right\} \\ &= \left\{ X \in \mathcal{L}^0 \,\middle|\, \int_{(0,\delta)} (F_X^{\leftarrow}(u))^-\,d\lambda(u) < \infty \right\} \\ &= \left\{ X \in \mathcal{L}^0 \,\middle|\, \int_{(0,1)} (F_X^{\leftarrow}(u))^-\,d\lambda(u) < \infty \right\} \\ &= \left\{ X \in \mathcal{L}^0 \,\middle|\, E[X^-] < \infty \right\} \end{aligned}$$

such that $\mathcal{L}_Q^{\text{Pichler}} \neq \mathcal{L}_Q^{\text{Acerbi}}$ *and* $\mathcal{L}_Q^{\text{Acerbi}} \neq \mathcal{L}_Q$*.*

(5) *Assume that D satisfies:*

$$D(u) = \frac{1}{2}\sqrt{u}\,\chi_{[0,1/4)}(u) + u\,\chi_{[1/4,1]}(u)$$

Then, Corollary 3 yields $\mathcal{L}_Q = \mathcal{L}_{Q^E}$*. Moreover, straightforward calculation yields:*

$$\int_{(0,1)} F_{|X|}^{\leftarrow}(u)\,dQ(u) \le \lambda[(0, F_{|X|}^{\leftarrow}(1/4))] + \int_{[F_{|X|}^{\leftarrow}(1/4),\infty)} \Big(1 - (D \circ F_{|X|})(x)\Big)\,d\lambda(x)$$

and:

$$\int_{(0,1)} F_{|X|}^{\leftarrow}(u)\,d\lambda(u) \le \lambda[(0, F_{|X|}^{\leftarrow}(1/4))] + \int_{[F_{|X|}^{\leftarrow}(1/4),\infty)} \Big(1 - F_{|X|}(x)\Big)\,d\lambda(x)$$

Since:

$$\int_{[F_{|X|}^{\leftarrow}(1/4),\infty)} \Big(1 - (D \circ F_{|X|})(x)\Big)\,d\lambda(x) = \int_{[F_{|X|}^{\leftarrow}(1/4),\infty)} \Big(1 - F_{|X|}(x)\Big)\,d\lambda(x)$$

we see that $\mathcal{L}_Q^{\text{Pichler}} = \mathcal{L}^1 \neq \mathcal{L}_{Q^E} = \mathcal{L}_Q$*. Consider, finally, a random variable X satisfying:*

$$F_X(x) = \left(\frac{\beta}{-x}\right)^2 \chi_{(-\infty,-\beta)}(x) + \chi_{[-\beta,\infty)}(x)$$

for some $\beta \in (0,\infty)$*. Then,* $-X$ *has a Pareto distribution with finite expectation. This yields* $X \in \mathcal{L}^1 = \mathcal{L}_Q^{\text{Pichler}} \subseteq \mathcal{L}_Q$*. Since* $D(u) \ge (1/2)\sqrt{u}\,\chi_{[0,1/4)}(u)$*, we obtain:*

$$\begin{aligned} \int_{(0,1)} |F_X^{\leftarrow}(u)|\,dQ(u) &\ge \int_{(0,1)} (F_X^{\leftarrow}(u))^-\,dQ(u) \\ &= \int_{(-\infty,0)} (D \circ F_X)(x)\,d\lambda(x) \\ &\ge \int_{(-\infty,0)} \frac{1}{2}\sqrt{F_X(x)}\,\chi_{[0,1/4)}(F_X(x))\,d\lambda(x) \end{aligned}$$

$$
\begin{aligned}
&= \int_{(-\infty,0)} \frac{1}{2} \left(\frac{\beta}{-x}\, \chi_{(-\infty,-\beta)}(x) + \chi_{[-\beta,\infty)}(x) \right) \chi_{(-\infty,-2\beta)}(x)\, d\lambda(x) \\
&= \int_{(-\infty,-2\beta)} \frac{\beta}{-2\,x}\, d\lambda(x) \\
&= \frac{\beta}{2} \int_{(2\beta,\infty)} \frac{1}{z}\, d\lambda(z)
\end{aligned}
$$

and hence, $X \notin \mathcal{L}_Q^{\text{Acerbi}}$. *Therefore, any two of the three classes* $\mathcal{L}_Q$, $\mathcal{L}_Q^{\text{Acerbi}}$ *and* $\mathcal{L}_Q^{\text{Pichler}}$ *are distinct.*

7. Related Integrated Quantile Functions

Integrated quantile functions also occur in the measurement of economic inequality. To briefly give an idea of this topic, consider the class:

$$\mathcal{L}^{\text{Lorenz}} := \left\{ X \in \mathcal{L}^0 \;\middle|\; X \geq 0 \text{ and } E[X] = 1 \right\}$$

and the map $L : \mathcal{L}^{\text{Lorenz}} \times [0,1) \to \mathbb{R}$ given by:

$$L(X,t) := \int_{(0,1)} F_X^{\leftarrow}(u)\, \chi_{(0,t]}(u)\, d\lambda(u)$$

Then, for any $X \in \mathcal{L}^{\text{Lorenz}}$, the function $L_X : (0,1) \to [0,1]$ given by:

$$L_X(t) := \int_{(0,1)} F_X^{\leftarrow}(u)\, \chi_{(0,t]}(u)\, d\lambda(u)$$

is called the Lorenz curve of X. If the distribution of X is interpreted as the normalized income distribution of a population, then the value $L_X(t)$ represents the proportion of the poorest $100\,t$ percent of the population; see Rüschendorf (2013). On the other hand, for any $t \in (0,1)$ and with:

$$Q^{\text{Lorenz},t} := \int \chi_{(0,t]}(u)\, d\lambda(u)$$

the map $\varrho_{Q^{\text{Lorenz},t}} : \mathcal{L}^{\text{Lorenz}} \to [0,1]$ given by:

$$\varrho_{Q^{\text{Lorenz},t}}[X] := \int_{(0,1)} F_X^{\leftarrow}(u)\, dQ^{\text{Lorenz},t}(u)$$

can be used to compare the proportions of the poorest $100\,t$ percent of different populations. Moreover, the map $\varrho_{Q^{\text{Gini}}} : \mathcal{L}^{\text{Lorenz}} \to [0,1]$ given by:

$$\varrho_{Q^{\text{Gini}}}[X] := 2 \int_{(0,1)} (t - L_X(t))\, d\lambda(t)$$

is called the Gini index of X and can be used to measure the inequality of the incomes within a given population; see Bennett and Zitikis (2015) and Greselin and Zitikis (2015). Letting:

$$Q^{\text{Gini}} := \int (2u-1)\, d\lambda(u)$$

we obtain:

$$\varrho_{Q^{\text{Gini}}}[X] = \int_{(0,1)} F_X^{\leftarrow}(u)\, dQ^{\text{Gini}}(u)$$

Formally, each of the maps $\varrho_{Q^{\text{Lorenz},t}}$ and $\varrho_{Q^{\text{Gini}}}$ looks like a quantile risk measure, but it should be noted that the integrating measures $Q^{\text{Lorenz},t}$ fail to be probability measures and that Q^{Gini} is only a signed measure.

Because of these examples, it appears to be reasonable to extend the notion of a quantile risk measure ϱ_Q to the case of an arbitrary integrating measure or even an integrating signed measure $Q : \mathcal{B}((0,1)) \to \mathbb{R}$, although in the latter case, Property (1) of Lemma 2, would be lost.

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