Applications of Mechanism- and Auction-Design to Partnership Dissolution and Sequential Sale

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Contents

1 General Introduction

According to the First Theorem of Welfare Economics, Walrasian allocations are efficient. This result hinges on the assumption that the quality of traded goods is exogenously fixed and known to all market participants. As Akerlof shows in his celebrated "lemons" paper, allocations in a price taking world do not have to be efficient if the quality of goods is uncertain and private information. The selling decision of a seller owning a good with a quality unknown to a buyer contains information about the good's quality; the phenomenon of adverse selection may occur: the average quality of goods traded is less than the average quality of goods in the market. In extreme cases there may exist no price at which trade takes place with positive probability. The question is then if there are mechanisms which lead to efficient allocations despite the presence of uncertainty.

If agents' preferences only depend on own information the Clarke-Groves-Vickrey mechanism[1] is efficient. In this mechanism an agent internalizes the externalities she exerts on others. In general, however, this mechanism has to be subsidized, since transfer payments that are necessary to make agents reveal their preferences cannot be financed by the (expected) gains from trade. Hence it cannot be applied to decentralized markets. If one combines the requirements of efficiency and budget balancedness, Myerson and Satterthwaite [1983] have shown that there exists no such mechanism. One cannot expect efficient trade if uncertainty about other agents' preferences predominates. This impossibility result depends on the distribution of ownership rights of the traded good. If these are distributed more equally, as in a partnership, efficient trade might be possible, as shown by Cramton et al. [1987].

This dissertation analyzes efficient mechanism design in distinct settings. Sections 2 and 3 study environments in which agents' preferences depend on own private information (as in the work of Myerson and Satterthwaite [1983] and Cramton et al. [1987]) and on other agents' private information (as in the work of Akerlof [1970]). Section 2 derives and analyzes conditions for efficient trade of an object that can be possessed by more than

[1] See Clarke [1971], Groves [1973] and Vickrey [1961].

one agent (and therefore called partnership). For such an environment properties of a "natural" and simple trading mechanism, the double auction, are examined in the second part (section 3). Section 4 analyzes sequential auctions.

Section 2 originated from collaboration with Karsten Fieseler and section 4 from collaboration with Jörg Nikutta and Prof. Eyal Winter.

1.1 Partnerships, Lemons and Efficient Trade

In section two I address the question of whether efficient trade is possible in an economic environment where an agent's private information does not only effect own preferences but also preferences of trading partners (in this sense we have interdependent valuations). In many examples the private values framework, where an agent is certain about her own preferences and not about preferences of others, seems to be unrealistic. In one of the earliest examples of a market with asymmetric information, namely Akerlof's [1970] market for lemons, the seller has information about the quality of the traded good which naturally influences the buyer's valuation. In addition situations are imaginable where even the buyer might be better informed about characteristics of the traded good which influence the seller's preferences. For example this might be the case if a seller trades with an expert.

The economic model used in section two (which focuses on the trade of one good) allows for private taste as in Myerson and Satterthwaite's [1983] bilateral trade model with private valuations **and** for private information about the good's characteristics as in Akerlof's example. Hence I use a model of interdependent valuations where preferences may depend on one's own and other agents' private information. The main part of section two is attributed to an environment, where several agents own a single good together (a partnership) which shall be dissolved efficiently (given to the partner with the highest valuation). Especially in such an environment the interdependent valuations assumption is more realistic than a private values assumption (which is predominant in the mechanism-design literature on partnership dissolution). Think of a firm that is divided into several departments. It seems quite natural that different partners responsible for different departments gathered private information that also affects other partners' valuations for the entire partnership.

An important insight in section two is that efficient trade can be easier or harder to achieve with interdependent valuations than with private valuations depending on whether the influence of an agent's private information on other agents' preferences is positive or negative. If an agent's private information affects other agents' valuations and the own valuation in the same direction, i.e. if agents' valuations are positively correlated, then it is more difficult to obtain efficient trade[2]. The reason is that agents have to be compensated for facing a winner's curse: if e.g. the buyer receives the good she is aware of the fact that the seller's information is likely to be "bad" (otherwise it would have been efficient not to trade) which implies that her expected valuation conditional on receiving the good is lower than her unconditional expected valuation of the good. Hence compensatory payments to agents are higher (than in a comparable private values framework) and budget balanced and efficient mechanisms are "more difficult" to construct. In the bilateral trade environment with one buyer and a seller we find that efficient trade is only possible if there exists a unique price for which agents are willing to trade (whatever their information is). In contrast to the private values case this implies that trade might be impossible even though it is common knowledge that it is always efficient. In the partnership model it can be shown that there might exist partnerships that can never be dissolved efficiently (regardless of the distribution of private information), which is not possible in a private values framework. If valuations are negatively correlated we find that efficient trade is "easier to achieve" since the winner's curse (which in this case is a winner's blessing) reduces compensatory payments that need to be paid to agents. For example in the bilateral trade setting efficient trade might be possible even if agents' valuations are distributed on the same interval.

[2] A more precise formulation of this connection is given in section 2.

1.2 Partnerships and Double Auctions with Interdependent Valuations

Section three takes over the partnership environment of section two[3] and analyses a prominent dissolution mechanism, the double auction. The double auction is used in practice and is widely analyzed in the literature on partnership dissolution in private values settings. Given a partnership between two agents in equal shares, they are asked to submit a bid for the entire partnership. The partner with the higher bid receives the entire partnership and pays a convex-combination of the two bids to her partner as a price for selling her property rights. Such an auction can dissolve the partnership efficiently in a (symmetric) private values environment. An implication of the results of section two is that a double auction might fail to be an efficient dissolution mechanism, if the influence of the other agent's information on the own valuation is too strong (and goes in the same direction). This is due to a winners' and losers' curse. If a partner wins, it implies that her partner's information about the value of the partnership was more pessimistic than the own information, hence an agent has to bid cautiously. If she loses it is the other way round and she should bid aggressively. Since a partner has to account for both effects at the same time it might be that partners having "average" information fail to do so. Hence they might regret participation in a double auction if they are able to sustain the status quo by nonparticipation. For that case I derive symmetric pooling equilibria. They have the property that partners with information indicating an "average" valuation do not participate and that if both partners have "extreme" information trade takes place, i.e. the partnership is allocated to the agent with the higher valuation. In addition it is shown that even though some types of agents prefer not to participate in the auction, ex-ante there is always a positive probability that the partnership is dissolved efficiently. The efficiency loss due to nonparticipation is analyzed exemplarily and it is shown that it can be reduced if a more complex mechanism which, in contrast to a double auction, is

[3] More precisely section three concentrates on the case of two partners each owning half of the partnership.

not independent of specifications of the environment[4] is used.

Another way to overcome these inefficiencies is to enforce agents' participation. If e.g. agents not willing to participate can be punished by an outside institution, efficiency loss due to pooling will not occur. Similarly, if agents have to decide on a good they do not own yet, like a legacy, the testator might demand heritages to participate in the allocation mechanism in order to be considered. For these cases, where participation problems do not occur, it can be shown that the efficient equilibrium is the only equilibrium in pure strategies. This is surprising, since in the bilateral trade setting the double auction possesses a continuum of equilibria. In a partnership setting all types of agents (almost) always can benefit from trade (either as buyer or seller) and therefore it can be shown that bidding functions have to be strictly increasing and are determined uniquely by first order and initial conditions.

1.3 Discounting in Sequential Auctions

The market analyzed in the fourth section is different from those in section two and three. Firstly, there is more than one good offered for trade (there still is one seller though). Secondly, not all market participants have private information about the own valuation for the goods, i.e. it is assumed that the seller's valuation for the goods is common knowledge. Thirdly, there is more than one buyer and on the buyers' side we have private valuations (and not interdependent valuations) for the goods, i.e. each buyer has private information about her own (and only about her own) preferences. Instead of analyzing direct-revelation mechanisms (as in section two) I concentrate on auction mechanisms that are actually used in practice and investigate some of their properties (as in section three).

As in many "real life" auctions the goods are assumed to be auctioned sequentially, i.e. one good is sold after the other in subsequent first- or second price auctions. More precisely, in each auction all buyers who have not already received an object submit a

[4] I.e. not independent of the valuation function and the distribution of information.

bid (buyers have unit demand), the buyer who submitted the highest bid wins the object sold in that auction and pays her bid (in a first-price auction) or the highest bid of all other buyers to the seller (in a second price auction). Since the time difference between subsequent auctions may be significant (as e.g. in real-estate or liquidity auctions) or since more valuable objects are sold first (as can be observed in art auctions) I assume that a buyer's valuation declines for objects sold in later auctions. Efficient equilibria of these sequential auctions are derived if either prices of previous auctions are announced or no information about previous bidding is made public between auctions. For a large class of preferences that express declining valuations it is shown that prices decline as well. They decline even sharper than valuations[5] which is in contrast to results from models analyzed in the literature where all objects have the same value to a bidder and prices remain constant. Reduction of competition in later rounds (where only bidders participate that have not already won an object) cannot be arbitraged away as in a situation where valuations remain constant and (expected) prices are the same in all auctions.

The results can be transferred to situations where valuations for objects and payments are discounted with the same discount factor (between auctions). In this case I find that the sequence of actual prices is a supermartingale. Prices drift down on average even though nominal valuations stay constant. Another interpretation of such an environment is that forthcoming auctions only take place with a certain probability (which can be interpreted as a discount factor) which is common knowledge to all bidders and can therefore also be applied to situations with supply uncertainty. Furthermore it is shown that a large class of sequential bidding mechanisms that are efficient generate the same expected prices in each round.

The results of section four are supported by empirical findings observing declining prices in sequential auctions of identical objects and give a new explanation for the so called "declining price anomaly".

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[5] This statement is a bit vague. If valuations are discounted with some discount factor δ between auctions, prices decline with a lower discount factor. For more general preferences a similar statement applies.

2 Partnerships, Lemons and Efficient Trade

2.1 Introduction

We inquire whether efficient trade can take place in environments where the agents' valuations depend on their own private information and on the private information of other agents. Such interdependence is natural in many trading situations, e.g., when a seller has private information about the quality of the good which influences the valuations of both the seller and a potential buyer. Especially in situations where property rights are initially dispersed among several agents (e.g., a partnership) it is natural to assume that each agent has private information that also determines the other agents' valuations. For an illustration, consider the situation where each partner is responsible for a particular project (or client, or operative part of the business, etc...) and where the projects are not related to each other. It is clear that an estimate of the value of the entire business can be made only by having information on all projects. In addition to the "standard" case where private information influences all agents' valuations in the same direction (e.g., if this information is about quality) situations where "good" news for one agent turns out to be "bad" news for other agents are conceivable[1].

This section can be divided in two parts. We first show how results from the auction and mechanism design literature with private values can be adapted in order to analyze the possibility of efficient trade in models with interdependent values. The second part illustrates the advantages of this approach by analyzing in detail several trading situations with interdependent values, and in particular the dissolution of a partnership.

The analysis employs three main steps:

1. If signals are independent, we show that a Revenue-Equivalence-Theorem (in the tradition of Myerson's [1981] pioneering contribution) holds for incentive compatible

[1] This may be the case if agents have contrary interests or intentions with regard to further use of the traded good or partnership.

mechanisms in the interdependent valuation case[2].

2. We next construct a value-maximizing, incentive-compatible mechanism. The standard Clarke-Groves-Vickrey (CGV) approach calls for transfers to agent i that depend on the sum of the utilities of the other agents (in the implemented alternative). But here such transfers will depend on i's report, thus destroying incentives for truthful revelation. Hence, we have to use a refinement of the CGV approach. We adapt for our purposes the mechanism described in Maskin [1992] for a one-sided auction setting with one indivisible unit. Achieving incentive compatible value maximization is easy: the construction hinges on a single-crossing property which ensures that the value-maximizing allocation is monotone in the agents' signals.

3. Finally, using revenue equivalence, we note that it suffices to analyze the conditions under which generalized CGV mechanisms (which are incentive compatible and value-maximizing) satisfy individual rationality and budget-balancedness. For private values, a similar approach has been used by Williams [1999] and Krishna and Perry [1998].

It is important to note that in the recent literature on one-sided auction settings with interdependent values[3], the seller (whose private information does not play a role) is a "residual claimant" and receives all payments from the buyers. Budget-balancedness is therefore costless and it is automatically satisfied. Hence, the type of problem posed here is completely absent in that literature[4].

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[2] Already Myerson himself allowed for a simple form of interdependent valuations, the so-called "revision effects".

[3] See Maskin [1992], Jehiel, Moldovanu and Stacchetti [1996], Ausubel [1997], Dasgupta and Maskin [2000], and Perry and Reny [1999].

[4] Ignoring individual rationality or budget balancedness, Jehiel and Moldovanu [1998] show that value-maximization is, per-se, inconsistent with incentive compatibility if valuations are interdependent and if different coordinates of a multi-dimensional signal influence utilities in different alternatives (as in a general model of multi-object auctions).

Our first application concerns the dissolution of a partnership. Cramton, Gibbons and Klemperer [1987] look at situations where each one of several agents owns a fraction of a good, and where agents have independent private values. Assuming symmetric distributions of agents' valuations, they prove that efficient trade is always possible if the agents' initial shares are equal[5].

We analyze a model that uses the symmetry assumptions made by Cramton et. al., and where the private and common value components are separable. A comparison of the cases with private and interdependent values reveals that a crucial role is played by the sign of the derivatives of the common value components (note that the private values case is exactly characterized by setting these derivatives equal to zero.)

If valuations are increasing functions of other agents' signals, it is more difficult to achieve efficient trade with interdependent values than with private values, since the information revealed ex-post is always "bad news" and the agents must be cautious in order to avoid the respective (i.e. winner's or loser's) curses. More precisely the extend of the winner's and loser's curses, i.e. the value of the derivative of the agents' valuation functions with respect to other agents' information, determines the level of transfer payments to agents. In a mechanism that implements an (ex-post) efficient allocation these have to be high enough to ensure that agents prefer to participate rather than to stay out of the mechanism. Therefore these have to be higher if the curses are more severe also meaning that it becomes "more difficult" to refinance these payments by realized gains from trade. Thus efficient trade(without subsidies) is "more likely to take place" if the curses are less severe[6]. Even if initial shares are equal, it is not always possible to dissolve a partnership efficiently. Surprisingly, this result continues to hold for arbitrarily small common value components. Indeed, for any symmetric and separable valuations that are increasing in other agents' signals, we can construct a symmetric distribution function such that effi-

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[5]Schweizer [1998] has generalized this result by showing that, even if agents' types are not drawn from the same distribution, there always exists an initial distribution of property rights such that, ex-post, the partnership can be efficiently dissolved.

[6]See section 2.4 for a precise formulation of this intuition.

cient dissolution is impossible, no matter what the initial distribution of property rights is.

If valuations are decreasing functions of other agents' signals, the additional information revealed ex-post is always a "blessing", and it turns out that it is easier to achieve efficient trade with interdependent valuations. We show that, in this case, there exists an open set of partnerships (around the equal partnership) that can always be dissolved efficiently.

An important special case of the partnership model is the situation where, ex-ante, the property rights belong to one agent. In a bilateral private values framework, Myerson and Satterthwaite [1983] show that efficient trade is possible only in a setting where it is common knowledge that the buyer's lowest valuation exceeds the seller's highest valuation. The introduction of interdependent values allows us to connect the Myerson-Satterthwaite result to Akerlof's famous market for lemons [1970]. Akerlof examines a bilateral trading situation where only the seller has private information, but this information influences both traders' valuations. He gives an example where efficient trade is not possible even if it is common knowledge that the buyer's valuation always exceeds the seller's valuation[7].

With extreme ex-ante ownership, the "worst-off" types of traders are unambiguously defined, and we can relax some assumptions made for the analysis of partnerships. We display a general existence condition for efficient trade that generalizes Myerson-Satterthwaite's classical contribution. We also show how efficient trade can take place (even if its possibility is not common knowledge) if agents' valuations are decreasing in other agents' signals. This positive result complements the negative result obtained by Gresik [1991][8] for the case of valuations that increase in other agents' signals. Gresik

[7] In that environment it suffices to analyze simple fixed-price mechanisms. Because private information in this section is two-sided, we cannot restrict attention to price mechanisms, and the analysis is more complex. A detailed analysis of the Akerlof one-sided example using mechanism design techniques can be found in Myerson [1985] and Samuelson [1984], which constructs second-best mechanisms.

[8] Other work focused on impossibility results: Spier [1994] and Schweizer [1989] study models of pretrial negotiation where the outcome of a trial depends on both parties' signals. They observe that not going to trial (which is efficient) cannot occur with probability one. Bester and Wärneyard [1998] study a model

derived an existence condition for efficient trade as a by-product of his characterization of second-best bilateral mechanisms for several traders with interdependent values[9].

Section 2 is organized as follows: In section 2.2 we describe the model. In section 2.3 we construct value-maximizing, incentive compatible mechanisms for interdependent valuations, and we use a revenue equivalence result in order to derive conditions under which such mechanisms are budget balanced and individually rational. In section 2.4 we generalize the Cramton et al. [1987] environment to the case with interdependent valuations. In section 2.5 we briefly look at the case of bilateral trade. Concluding comments are gathered in Section 2.6. All proofs are relegated to Appendix A.1.

2.2 The Model

There are n risk-neutral agents and one good. Each agent i owns a fraction α_i of the good, where $0 \leq \alpha_i \leq 1$ and $\sum_{i=1}^{n} \alpha_i = 1$. We denote by θ_i the type of agent i , by θ the vector $\theta = (\theta_1, ..., \theta_n)$, and by θ_{-i} the vector $\theta_{-i} = (\theta_1, ..., \theta_{i-1}, \theta_{i+1}, ..., \theta_n)$. Types are independently distributed. Type θ_i is drawn according to a commonly known density function f_i with support $[\underline{\theta}_i, \overline{\theta}_i]$. The density f_i is continuous and positive (a.e.), with distribution F_i .

The valuation of agent i for the entire good is given by the function $v_i(\theta_i, \theta_{-i})$, where the arguments are always ordered by the agents' indices: $v_i(\theta_i, \theta_{-i}) = v_i(\theta_1, ..., \theta_n)$. The function $v_i(\theta_i, \theta_{-i})$ is strictly increasing in θ_i, and continuously differentiable. We further assume the following single crossing property (SCP):

$$v_{i,i} > v_{j,i} \qquad \forall\, i, j \neq i. \tag{SCP}$$

of conflict resolution where agents are uncertain about each other's fighting potential, and observe that conflict must arise with positive probability even if peaceful settlement is always efficient.

[9] Since the result is obtained via the solution of a variational problem, Gresik's approach depends on certain assumptions about virtual valuations. Such assumptions are not needed in our treatment.

where $v_{i,x}(\theta_1, ...\theta_n)$ denotes the $x'th$ partial derivative of $v_i(\theta_1, ..., \theta_n)$. This assumption[10] guarantees that the functions $v_i(\cdot, \theta_{-i})$ and $v_j(\cdot, \theta_{-i})$ are equal for at most one θ_i.

Agents have utility functions of the form $q_i v_i + m_i$ where q_i and m_i represent the share of the good and the money owned by i, respectively.

By the revelation principle, it suffices to analyze direct revelation mechanisms (DRM). In a DRM agents report their types, relinquish their shares α_i of the good, and then receive a payment $t_i(\theta)$ and a share $s_i(\theta)$ of the entire good. A DRM is therefore a game form $\Gamma = \left(\left[\underline{\theta}_1, \overline{\theta}_1\right], ..., \left[\underline{\theta}_n, \overline{\theta}_n\right], s, t\right)$, where $s(\theta) = (s_1(\theta), ..., s_n(\theta))$ is a vector with components $s_i : \times_{j=1}^n \left[\underline{\theta}_j, \overline{\theta}_j\right] \mapsto [0,1]$ such that $\sum_{i=1}^n s_i(\theta) = 1 \ \forall \ \theta$, and $t(\theta) = (t_1(\theta), ..., t_n(\theta))$ is a vector with components $t_i : \times_{j=1}^n \left[\underline{\theta}_j, \overline{\theta}_j\right] \mapsto \mathbb{R}$. We call the s and t the allocation rule and the payments, respectively. To simplify notation, we refer to the pair (s,t) as a DRM if it is clear which strategy sets $\left[\underline{\theta}_i, \overline{\theta}_i\right]$ are meant.

A mechanism (s,t) *implements* the allocation rule s if truth-telling is a Bayes-Nash-equilibrium in the game induced by Γ and by the agents' utility functions. Such a mechanism is called *incentive compatible* (IC). A mechanism is (ex post) *efficient* (EF) if it implements an allocation rule where the agent with the highest valuation of the good always gets the entire good[11]. A mechanism is called (ex-ante) *budget balanced* (BB) if a designer doesn't expect to pay subsidies to the agents, e.g. $E_\theta\left[\sum_{i=1}^n t_i(\theta)\right] \leq 0$. We call a mechanism (interim) *individual rational* (IR) if every agent i who knows his type θ_i wants to participate in the mechanism, given that all players report their types truthfully, e.g. if $U_i(\theta_i) \geq 0$ for all θ_i, $i = 1, .., n$, where $U_i(\theta_i)$ is the utility type θ_i expects to achieve by participating in the mechanism.

[10] Maskin [1992] shows that, without this assumption, the value-maximizing allocation may fail to be monotone in types, and hence it may be impossible to implement it. If that allocation just happens to be monotone (as in Akerlof's original example where SCP is not satisfied, but where the value-maximizing allocation is constant) our main results also go through.

[11] A more appropriate name for this property is *value-maximization,* since efficiency combines in fact **all** properties listed here. But we keep the common jargon.

We denote characteristic functions as follows:

$$\mathbf{1}\,(\text{statement}) := \begin{cases} 1, & \text{if statement is true} \\ 0, & \text{if statement is false} \end{cases} \quad \text{or } \mathbf{1}_A\,(x) := \begin{cases} 1, & \text{if } \quad x \in A \\ 0, & \text{if } \quad x \notin A. \end{cases}$$

2.3 Efficiency and Incentive Compatibility

Our analysis uses three main ideas

- A revenue equivalence result implies that any two EF and IC mechanisms yield, up to a constant, the same interim expected transfers.
- We display generalized Groves mechanisms that satisfy EF and IC for the case of interdependent values.
- By revenue equivalence, it is enough to check under which conditions a generalized Groves mechanism satisfies BB and IR to obtain general conditions for the existence of EF, IC BB and IR mechanisms[12].

Krishna and Perry [1998] and Williams [1999] have used the same combination for the analysis of efficient trade in buyer-seller settings with private values.

2.3.1 The Revenue-Equivalence-Theorem

The Revenue-Equivalence-Theorem constitutes the basis of most results in the mechanism-design literature with quasi-linear utility functions, risk-neutral agents and independent types. It states that expected payments are (up to a constant) the same in all IC mechanisms that implement the same allocation. Its proof can be easily extended to environments with interdependent valuations[13].

[12] Note that mechanisms that satisfy IC and EF without belonging to the CGV class do indeed exist. For instance, Cramton et al. [1987] show that a double auction (which is not a CGV mechanism) dissolves a partnership effciently when agents have private values.

[13] Various such extensions can be found in Myerson (1981), Jehiel et.al. (1996), Jehiel and Moldovanu (1999), and Krishna and Maenner (1999). None of these results covers the present setting.

We first need some notation: The interim utility of agent i with type θ_i which participates and announces the type $\widehat{\theta}_i$ (while all the other agents report truthfully) is given by

$$
\begin{aligned}
U_i\left(\theta_i,\widehat{\theta}_i\right) &= E_{\theta_{-i}}\left[v_i\left(\theta_i,\theta_{-i}\right)\ s_i\left(\widehat{\theta}_i,\theta_{-i}\right)-\alpha_i v_i\left(\theta_i,\theta_{-i}\right)+t_i\left(\widehat{\theta}_i,\theta_{-i}\right)\right] \\
&= : V_i\left(\theta_i,\widehat{\theta}_i\right)+E_{\theta_{-i}}\left[t_i\left(\widehat{\theta}_i,\theta_{-i}\right)\right] \\
&= : V_i\left(\theta_i,\widehat{\theta}_i\right)+T_i\left(\widehat{\theta}_i\right).
\end{aligned}
$$

To simplify notation we write:

$$U_i\left(\theta_i\right) := U_i\left(\theta_i,\theta_i\right).$$

Theorem 1 *Assume that $v_i\left(\theta_i,\theta_{-i}\right)$ is continuously differentiable in each component and that for all $\widehat{\theta}_i \in [\underline{\theta}_i,\overline{\theta}_i]$ $\lim_{\theta_i\to\widehat{\theta}_i} s_i\left(\theta_i,\theta_{-i}\right) = s_i\left(\widehat{\theta}_i,\theta_{-i}\right)$ for almost every θ_{-i}. Then, for every IC mechanism (s,t), the interim expected utility of agent i in a truth-telling equilibrium can be written as:*

$$
\begin{aligned}
U_i\left(\theta_i\right) &= U_i(\underline{\theta}_i)+\int_{\underline{\theta}_i}^{\theta_i} V_{i,1}\left(x,x\right)\ dx \\
&= U_i(\underline{\theta}_i)+\int_{\underline{\theta}_i}^{\theta_i} E_{\theta_{-i}}\left[v_{i,1}\left(x,\theta_{-i}\right)\left(s_i\left(x,\theta_{-i}\right)-\alpha_i\right)\right]dx
\end{aligned}
$$

Corollary 1 *Let (s,t) be an IC mechanism. If there is no BB and IR mechanism of the form $(s,t+q)$ where $q:=(q_1,...,q_n)$ is an arbitrary vector of constants, then there are no IR and BB mechanisms that implement s.*

We now examine under which conditions we can find an BB and IR mechanism in the class of mechanisms of the form $(s,t+q)$ where (s,t) is an IC mechanism. Let $\widetilde{\theta}_i$ be the "worst off" type of agent i in the mechanism (s,t). This is defined by[14]

$$U_i(\widetilde{\theta}_i) \leq U_i\left(\theta_i\right) \qquad \forall\theta_i.$$

[14]From the proof of Theorem 1 we have that U_i is continuous which implies that $\widetilde{\theta}_i$ is well defined.

Theorem 2 *Let (s,t) be an IC mechanism and $T_i, U_i, \widetilde{\theta}_i$ be the associated interim payments, interim utilities and "worst off" types, respectively. There exists an IC, BB and IR mechanism that implements s if and only if*

$$\sum_{i=1}^{n} E_{\theta_i}\left[T_i(\theta_i)\right] \leq \sum_{i=1}^{n} U_i(\widetilde{\theta}_i).$$

The worst type's utility $U_i(\widetilde{\theta}_i)$ can also be viewed as a maximal entry fee that can be collected from agent i in the mechanism (s,t) such that every type of agent i still participates. If these entry fees cover the expected payments needed to ensure IC then (and only then) there exists an IR and BB mechanism that implements s. Such a mechanism is then given by $(s, t+q)$ with $q = (q_1, ..., q_n) = (-U_1(\widetilde{\theta}_1), ..., -U_n(\widetilde{\theta}_n))$.

In the sequel we focus on EF mechanisms. For a given trading situation it suffices to analyze the allocation rule s^* given by:

$$s_i^*(\theta) := \begin{cases} 1, & \text{if} \quad i = m(\theta) \\ 0, & \text{if} \quad i \neq m(\theta) \end{cases},$$

where $m(\theta) := \max\{j |\ j \in \arg\max_i \ v_i(\theta)\}$.

Any two efficient allocation rules differ only in the tie breaking rule and coincide a.e. We can apply Theorem 1 for the efficient allocation rule s^* since, for all $\widehat{\theta}_i$, we have $\lim_{\theta_i \to \widehat{\theta}_i} s_i^*(\theta_i, \theta_{-i}) = s_i^*\left(\widehat{\theta}_i, \theta_{-i}\right)$ for almost all θ_{-i}.

2.3.2 The generalized Groves mechanism

We now display a mechanism which applies to the interdependent values case the idea behind Groves mechanisms. Variations on this idea have been used to construct value-maximizing auctions by Ausubel [1997], Dasgupta and Maskin [2000] , Jehiel and Moldovanu [1998] and Perry and Reny [1999].

Theorem 3 *Let s^* be an efficient allocation rule, and let the payments t^* be given by*

$$t_i^*(\theta) := \begin{cases} 0, & \textit{if} \quad s_i^*(\theta) = 1 \\ v_i(\theta_i^*(\theta_{-i}), \theta_{-i}), & \textit{if} \quad s_i^*(\theta) \neq 1 \end{cases},$$

where $\theta_i^*(\theta_{-i})$ *is defined by*

$$v_i(\theta_i^*(\theta_{-i}), \theta_{-i}) = \max_{j\neq i} v_j(\theta_i^*(\theta_{-i}), \theta_{-i})$$

if the equation has a solution, and by

$$\theta_i^*(\theta_{-i}) := \begin{cases} \overline{\theta}_i, & \text{if } \; v_i\left(\overline{\theta}_i, \theta_{-i}\right) < \max_{j\neq i} v_j\left(\overline{\theta}_i, \theta_{-i}\right) \\ \underline{\theta}_i, & \text{if } \; v_i\left(\underline{\theta}_i, \theta_{-i}\right) > \max_{j\neq i} v_j\left(\underline{\theta}_i, \theta_{-i}\right) \end{cases}$$

if it does not[15]*. Then* (s^*, t^*) *is incentive compatible*[16]*.*

2.3.3 The existence condition

We now have all the needed tools. Theorem 2 shows that an IC,EF,IR and BB mechanism exists if and only if

$$\sum_{i=1}^{n} E_{\theta_i}[T_i(\theta_i)] \leq \sum_{i=1}^{n} U_i\left(\widetilde{\theta}_i\right)$$

for an arbitrary IC and EF mechanism. For the mechanism constructed in Theorem 3 we have

$$T_i(\theta_i) = E_{\theta_{-i}}\left[v_i(\theta_i^*(\theta_{-i}), \theta_{-i}) \; \mathbf{1}\left(v_i(\theta_i, \theta_{-i}) < \max_{j\neq i} \; v_j(\theta_i, \theta_{-i})\right)\right].$$

Therefore we can find an IC, EF, IR and BB mechanism if and only if[17]

$$\sum_{i=1}^{n} E_{\theta}\left[v_i(\theta_i^*(\theta_{-i}), \theta_{-i}) \, \mathbf{1} \; \left(v_i(\theta_i, \theta_{-i}) < \max_{j\neq i} \; v_j(\theta_i, \theta_{-i})\right)\right] \leq \sum_{i=1}^{n} U_i\left(\widetilde{\theta}_i\right). \tag{1}$$

---

[15] If $\theta_i^*(\theta_{-i})$ does not exist, it can be arbitrarily chosen out of $\left[\underline{\theta}_i, \overline{\theta}_i\right]$. The definition given here simplifies calculations in the next section.

[16] Note that truthtelling is not an equilibrium in dominant strategies, but it is an *ex-post* equilibrium, i.e., it is an equilibrium no matter what the distributions of agents' types are.

[17] The existence condition for an IC, EF, IR and *ex-post* budget balanced mechanism is the same: by applying the ideas of Arrow [1979], d'Aspremont and Gérard-Varet [1979] we can find an expected externality mechanism that is ex-post budget-balanced and results in the same interim utilities and payments as the generalized Groves mechanism. In the expected externality mechanism, however, truthtelling is not an ex-post equilibrium.

2.4 Dissolving a Partnership

We now apply the above findings to the dissolution of a partnership. We make the following assumptions:

A1 Types are drawn independently from the same distribution function, e.g.

$$F_i = F, \quad \underline{\theta}_i = \underline{\theta}, \quad \overline{\theta}_i = \overline{\theta} \quad \forall i.$$

A2 The valuation functions $v_i(\theta_1, ..., \theta_n)$ have the following form:

$$v_i(\theta_1, ..., \theta_n) = g(\theta_i) + \sum_{j \neq i} h(\theta_j),$$

where g, h are continuously differentiable, g is strictly increasing, and $g' > h'$.

To simplify notation, we write $h(\theta_{-i}) := \sum_{j \neq i} h(\theta_j)$.

These conditions constitute a natural and simple generalization of the symmetry assumption in Cramton et al. [1987]. Condition A2 is also needed for computational reasons: it allows an explicit characterization of the "worst off" participating types, which otherwise become complex functions of the model's parameters[18] (including valuation functions).

By the single crossing property and A2, we obtain:

$$v_i(\theta_1, ..., \theta_n) > v_j(\theta_1, ..., \theta_n) \Leftrightarrow \theta_i > \theta_j, \tag{S1}$$

$$v_i(\theta_1, ..., \theta_n) = v_j(\theta_1, ..., \theta_n) \Leftrightarrow \theta_i = \theta_j. \tag{S2}$$

An EF and IC mechanism is given by (s^*, t^*) of Theorem 3. Because of S1, S2 and A2 we have $\theta_i^*(\theta_{-i}) = \max_{j \neq i} \theta_j$.

[18] This assumption does not restrict a main message of this section which states that general possibility results like in private values environments cannot be achieved if valuation functions are increasing in other agents' types.

Theorem 4 *The "worst off" type of agent i depends on her share α_i and is given by* $\widetilde{\theta}_i = F^{-1}(\alpha_i^{\frac{1}{n-1}})$.

1) An EF, IC, BB and IR mechanism exists if and only if:

$$\sum_{i=1}^{n}\left(\int_{\widetilde{\theta}_i}^{\overline{\theta}} g(\theta)dF^{n-1}(\theta) - \int_{\underline{\theta}}^{\overline{\theta}} g(\theta)F(\theta)\,dF^{n-1}(\theta)\right) + \int_{\underline{\theta}}^{\overline{\theta}} h'(\theta)\left(F^{n}(\theta) - F(\theta)\right)d\theta \geq 0. \tag{2}$$

2) The set of $(\alpha_1, ..., \alpha_n)$ for which EF, IC, BB and IR mechanisms exist is either empty or a symmetric, convex set around $\left(\frac{1}{n}, ..., \frac{1}{n}\right)$.

Condition 2 reduces to that given in Cramton et al. [1987] if $g(\theta_i) = \theta_i$ and $h(\theta_i) \equiv 0$. For that case, they also show that the condition is always fulfilled if $\alpha_1 = \cdots = \alpha_n = \frac{1}{n}$. Observe that the additional term containing the common value component is negative if $h' > 0$ and positive if $h' < 0$. Cramton's et. al. [1987] result implies that, in the latter case, a partnership can be efficiently dissolved if the initial property rights are distributed equally.

Example 1 *To see that both cases in Theorem 4-2 can occur, consider a setting with two agents such that:*

$$\begin{aligned} v_i(\theta_1, \theta_2) &= a\ \theta_i + b\ \theta_{-i}, \qquad a > b > 0, \\ f(\theta_i) &= 1_{[0,1]}(\theta_i)\,;\ \alpha_1 = \alpha_2 = \frac{1}{2}. \end{aligned}$$

Condition 2 reduces to:

$$2a\left(\int_{\frac{1}{2}}^{1} \theta_i\ d\theta_i - \int_0^1 \theta_i^2 d\theta_i\right) + b\int_0^1 \left(\theta_i^2 - \theta_i\right)\ d\theta_i = \frac{1}{12}a - \frac{1}{6}b \geq 0$$

The set of shares (α_1, α_2) for which an EF, IC, IR and BB mechanism exists is empty if and only if $0 < a < 2b$. ■

In the following we compare the existence condition for two settings with different valuation functions but the same interim valuations for agents (of the same type). For this purpose, consider two trading situation with two agents (each) given by $\left(F, \alpha_1, g, h_1, \underline{\theta}, \overline{\theta}\right)$ and $\left(F, \alpha_1, g, h_2, \underline{\theta}, \overline{\theta}\right)$. Assume that

$$\begin{aligned} h_1' &> h_2' \text{ and} &(3)\\ E[h_1] &= E[h_2]. &(4)\end{aligned}$$

Observe that, for all types of agent i, the interim valuation $E_{\theta_{-i}}\left[v_i\left(\theta_i, \theta_{-i}\right)\right]$ and the interim expected utility $U_i\left(\theta_i\right)$ are identical in both settings. Therefore the "worst off" types are also identical and we can collect the same entry fee in both settings while insuring participation of all types. But, according to Theorem 3, the needed (expected) transfers are given by

$$E_\theta\left[g\left(\max_i \ \theta_i\right)\right] + E_\theta\left[h_j\left(\max_i \ \theta_i\right)\right], \quad j = 1, 2,$$

and therefore different. Because of (3), (4) and since the distribution of the first order statistic F^2 stochastically dominates F, i.e. $F^2 \leq F$, we have that $E_\theta\left[h_1\left(\max_i \ \theta_i\right)\right] > E_\theta\left[h_2\left(\max_i \ \theta_i\right)\right]$. Hence if we can find EF, IC, IR and BB mechanisms in $\left(F, \alpha_1, g, h_1, \underline{\theta}, \overline{\theta}\right)$ we will also be able to find such mechanisms in $\left(F, \alpha_1, g, h_2, \underline{\theta}, \overline{\theta}\right)$. In this sense it is more expensive and less "likely" to get efficient trade if the derivative of h is higher, reflecting the intuition that the more severe winner's and loser's curses are the higher necessary compensatory payments have to be and the more difficult it is to refinance these payments by possible gains from trade. If $h' = 0$ we are in a private values environment. Hence, if $h' > 0$, the generalized Groves mechanism is more expensive than the standard Groves mechanism[19] (in a comparable private values setting where $v_i = g + E[h]$), and efficiency is harder to achieve with interdependent values than with private values. If $h' < 0$, exactly the opposite occurs: efficiency is easier to achieve with interdependent values.

Our next result shows that efficient trade is possible for any valuation functions where $h' \leq 0$ and for any distribution function F, if each individual share is not too small. For

[19] Bergemann and Välimäki [2000] focus on the differences between transfers in the CGV mechanisms in the private and interdependent values cases in order to compare the resulting incentives for information acquisition.

example, if there are two partners, efficient dissolution is always possible if the smaller share is at least 25%.

Theorem 5 *Let* $\alpha_1 \leq \cdots \leq \alpha_n$ *, and assume that, for all* $i = 1, ..., n-1$*, we have* $\sum_{j=1}^{i} \alpha_j \geq \left(\frac{i}{n}\right)^n$*. Then, for any valuation function* $v_i(\theta_i, \theta_{-i}) = g(\theta_i) + h(\theta_{-i})$ *with* $h'(\theta_{-i}) \leq 0$ *and for any distribution function* F*, the partnership can be dissolved efficiently.*

It is a-priori plausible that the above insight continues to hold if the derivative of the common value component is positive, but sufficiently small. We next show, however, that this is not the case: even if that derivative is arbitrarily small but positive, there exist distribution functions such that an equal partnership cannot be dissolved efficiently.

Theorem 6 *For any valuation function* $v_i(\theta_i, \theta_{-i}) = g(\theta_i) + h(\theta_{-i})$ *with* $h'(\theta_{-i}) > 0$ *there exists a distribution function* F *such that the equal partnership cannot be efficiently dissolved. By Theorem 4-2, for this* F *there is no ex-ante distribution of shares that leads to efficient trade.*

A distribution F with the above property puts mass on types close to the extremities of the types' interval. For these types, the payment difference between a standard Groves mechanism and the generalized Groves mechanism is relatively large. Since the later is much more costly, inefficiency occurs.

2.5 Bilateral Trade

We now briefly look at the case of two agents, one who a-priori owns the whole good (the seller) and another one who wants to buy the good (the buyer). We denote the agents by S and B for seller and buyer, respectively, so that $i \in \{S, B\}$, $\alpha_S = 1$ and $\alpha_B = 0$.

For this special case the "worst off" types do not depend on the functional form of the valuation functions[20], and we can allow for general valuations (as introduced in section 2.2).

[20]The "worst off" seller is always a seller of type $\overline{\theta}_S$ and the "worst off" buyer is always a buyer of type $\underline{\theta}_B$.

The following Theorem exhibits a condition under which efficient trade is possible.

Theorem 7 *An EF, IC, BB and IR mechanism exists if and only if*

$$\int_{\underline{\theta}_S}^{\overline{\theta}_S}\int_{\underline{\theta}_B}^{\overline{\theta}_B}\left(v_S\left(\theta_S^*\left(\theta_B\right),\theta_B\right)-v_B\left(\theta_S,\theta_B^*\left(\theta_S\right)\right)\right)\times$$

$$\mathbf{1}\left(s_B^*\left(\theta\right)=1\right)f_B\left(\theta_B\right)d\theta_B f_S\left(\theta_S\right)d\theta_S\leq 0.$$

Assume that agents' valuations are increasing in other agents' types, i.e. $v_{S,B}\geq 0, v_{B,S}\geq 0$. *An EF, IC, BB and IR mechanism exists if and only if:*

$$E_{\theta_S}\left[v_B\left(\theta_S,\underline{\theta}_B\right)\right]\geq E_{\theta_B}\left[v_S\left(\overline{\theta}_S,\theta_B\right)\right]\quad or\quad v_B\left(\underline{\theta}_S,\overline{\theta}_B\right)\leq v_S\left(\underline{\theta}_S,\overline{\theta}_B\right)$$

In other words, if valuations are increasing in the other agent's type, efficient trade is only possible if a price p exists such that we can always have trade at this price, i.e. if.

$$E_{\theta_S}\left[v_B\left(\theta_S,\underline{\theta}_B\right)\right]\geq p\geq E_{\theta_B}\left[v_S\left(\overline{\theta}_S,\theta_B\right)\right].$$

On the other hand, if the negative dependence of agents' valuations on the other agent's type is strong enough, then efficient trade is possible even if the distributions of types have overlapping support (i.e., even if the possibility of efficient trade is not common knowledge)[21]. This phenomenon is illustrated below.

Example 2 *Assume that valuations are:*

$$v_B\left(\theta_S,\theta_B\right)=a\theta_B+b\theta_S;\ v_S\left(\theta_S,\theta_B\right)=a\theta_S+b\theta_B$$

with $a>b$, $a>0$, *and assume that* $f_B\left(\theta_B\right)=\mathbf{1}_{[0,1]}\left(\theta_B\right),\ f_S\left(\theta_S\right)=\mathbf{1}_{[0,1]}\left(\theta_S\right).$ *Theorem 3 shows that the following mechanism is IC and EF:*

$$\begin{aligned} s_B^*\left(\theta\right) &= \begin{cases} 1 & if \quad \theta_B\geq\theta_S \\ 0 & if \quad \theta_B<\theta_S \end{cases} \\ t_S\left(\theta\right) &= \left(a+b\right)\theta_B\ \ s_B^*\left(\theta\right);\ t_B\left(\theta\right)=-\left(a+b\right)\theta_S\ \ s_B^*\left(\theta\right) \end{aligned}$$

[21]Gresik (1991) generally concludes that efficient trade is impossible. But some parts of his analysis hold in fact only for settings where valuations increase in the other agent's signal.

Because worst-off types never trade, the mechanism designer cannot collect entry fees. He has to pay (in expectation):

$$(a+b)\int_0^1\int_0^1 (\theta_B-\theta_S) \quad s_B^*(\theta_B,\theta_S)\,d\theta_S d\theta_B = \frac{1}{6}(a+b)\,.$$

For $b>0$ it is more costly to achieve efficiency than in the private values case (where the mechanism designer has to pay $\frac{1}{6}a$). For $b=-a$, everybody tells the truth without receiving any payments at all, so that BB and IR are also fulfilled. For $b<-a$ the designer can even extract money from the traders! ■

2.6 Conclusion

Analyzing the possibility of an efficient dissolution of a partnership, we have highlighted the similarities and differences between the private value case and the case with interdependent valuations. Our analysis generalizes and reformulates well-known results for private values environments: The Myerson-Satterthwaite impossibility result and the possibility results in Cramton et al. [1987]. We showed how the comparison of the private and interdependent cases crucially depends on whether valuations are increasing or decreasing in other agents' signals.

For the Myerson-Satterthwaite and Akerlof "extreme-ownership" settings second-best mechanisms have been exhibited in the literature (see Myerson and Satterthwaite [1983], Samuelson [1984], Gresik [1991]). The construction of a second-best mechanisms for the partnership model with interdependent values is still an open question. First steps have been undertaken by Jehiel and Pauzner [1999]. They study second-best mechanisms in a setting with a single informed partner and show that the second-best allocation method coincides with the ex-post efficient allocation only outside an interior interval of types where no trade takes place.

3 Partnerships and Double Auctions with Interdependent Valuations

3.1 Introduction

When two partners own a firm together and want to dissolve the partnership they face the problem of choosing a "good" way to do so. A common situation is one where different partners are responsible for different parts or departments of their firm. It is natural to assume that they gain different information that helps them valuing their partnership. A partner's valuation might depend only on own private information (private valuation) or also on the other partner's private information (interdependent valuation).

A widely analyzed mechanism used for dissolving partnerships is the $k-$double auction. In the $k-$double auction the partners each submit a sealed bid and the entire partnership is given to the partner with the higher bid. Her payment to the other partner is a convex combination of her own bid (b_H) and her partner's bid (b_L), i.e. the payment is given by $kb_L + (1-k)\, b_H$ where $k \in [0,1]$. These simple rules do not depend on a specific valuation structure.

In this section I will analyze the k-double auction with interdependent valuations. In contrast to the private valuation case participation turns out to be a problem. With private valuations partners always participate voluntarily since they can assure themselves a positive payoff by bidding their own valuation. Bidding one's own valuation is always profitable since a partner sells her share if payments exceed her valuation and she buys the other agent's share if payments are below her valuation.

With interdependent valuations partners do not participate automatically. Partners do not know their true valuation for the firm (since this depends on information private to the other partner). Bidding the own valuation thus assuring positive net-payoffs is no longer an available strategy. Even worse a bidder has to take into account a winner's and a loser's curse at the same time. If a partner wins this is "bad news" for her since this indicates that her partner's and therefore her own valuation are likely to be low. On the other hand, if she loses this again is "bad news" since this suggests that her partner's information indicates a high value of the firm. A bidder might not be able to correct for the winner's curse (by lowering her bid) and the loser's curse (by raising her bid) at the same time and therefore might not want to participate in the k-double auction.

If the mechanism designer can *force* agents to participate, i.e. if partners' participation constraints can be neglected[1] I find that the double auction dissolves the partnership efficiently[2], i.e. it is ex-post efficient. Moreover, with forced participation the double auction possesses a unique equilibrium (in pure strategies). If participation is *voluntary*, i.e. if partners' participation constraints cannot be neglected, we have to specify nonparticipation as a strategy that implements the status quo, i.e. the situation that occurs if a partner does not want to dissolve the partnership. This is done by allowing for vetoing against the dissolution by submitting a "No" instead of a (real numbered) bid[3]. I will call this modified auction $k-$double auction with veto. I will show that the $k-$double auction with veto possesses symmetric equilibria and nonvetoing takes place with positive probability. Moreover if there exists a dissolution mechanism that is individually rational, does not need to be subsidized and is ex-post efficient, then the k-double auction with veto has all these properties as well. If vetoing occurs, partners with information that indicates an average value for the partnership prevent the dissolution. Intuitively, in that case the simultaneous winner's and loser's curse is most severe. If a partner's information suggests either high or low values for the firm, a dissolution where the partner with the higher value receives the firm, takes place. Nonparticipation (i.e. vetoing) decreases efficiency since private information of nonparticipants cannot be exploited in the mechanism. Still, gains from trade are realized and the k-double auction with veto can be implemented without

[1] Or, equivalently, can be modified in a way that they are not binding, e.g. by threatening to take away the partnership in case of nonparticipation.

[2] As in most of the auction literature I have to assume certain symmetry conditions to obtain this and other results.

[3] This is equivalent to a two stage game, where partners first decide whether to participate or not and then (if both are participating) submit bids.

knowledge about the specifications of the valuation structure. Moreover, I will show by means of an example that more complex mechanisms that do rely on the specifications of the valuations can be more efficient than the k-double auction with veto.

These results are obtained for a symmetric framework with interdependent valuations. I assume an environment with symmetric partners to highlight the effects of nonparticipation on efficiency. In a model with asymmetric partners there might be efficiency losses due to asymmetries in the distribution of valuations[4].

Initially, the partnership is divided in parts of equal size owned by two partners. I assume that partner i has private information denoted by a type θ_i that effects her own and her partner's value of the partnership. The types are independently distributed according to a known distribution function F. Partner $i's$ valuation for the entire firm v_i does depend on both types: $v_i(\theta_i, \theta_{-i})$. I additionally assume symmetry in the valuation structure, i.e. agents are ex-ante indistinguishable. To compute equilibria in those cases where some types prefer not to participate I additionally assume separability of the valuation functions. Partners are risk-neutral and the utility of agent i who owns a share of β_i in the entire partnership and has money m_i is quasilinear and given by $u_i = \beta_i v_i(\theta_1, \theta_2) + m_i$.

The model of this section is based on Cramton et al. [1987] and generalizes their setting to interdependent valuations. In McAfee [1992] special $k-$double auctions are compared to other simple dissolution mechanisms for the equal partnership case. De Frutos [2000] compares efficiency and revenue of the $k-$double auction, $k = 0, 1$ for the equal partnership and asymmetrically distributed valuations. These papers restrict attention to an independent private values framework[5].

In addition to the literature on double auctions for the equal partnership there exists a literature on $k-$double auctions in bilateral trade environments with a buyer and a seller (which can be seen as an extreme case of a partnership where property rights belong to one agent, the seller). Leininger et al. [1989] and Satterthwaite and Williams [1989] show

[4] This problem has been analyzed by de Frutos [2000] for a private values model.

[5] Cramton et al. and de Frutos assume risk neutral agents whereas McAfee allows for CARA-utility functions.

that in the buyer/ seller setting $k-$double auctions possess a continuum of pure strategy equilibria[6] if $k \in (0,1)$. These can be ranked from equilibria that realize no gains from trade to equilibria that are incentive efficient[7]. The uniqueness result for equilibria in this section shows that multiplicity of equilibria is not necessarily present in $k-$double auctions if property rights are distributed more equally.

Bulow et al. [1999] analyze special cases of the $k-$double auction in a common values model with uniform distribution of types. They analyze the effects of an unequal distribution of ownership rights on bidders advantages in a first-price ($k = 0$) or second price ($k = 1$) double auction. The question of participation is not addressed in this paper. Engelbrecht-Wiggans [1994] computes equilibria of a first- and second-price double auction in a model with affiliated values. Since bidders do not possess ownership rights in the auctioned good, participation is always assured in his model.

Neglecting the problem of participation and allowing for subsidies to agents, Jehiel and Moldovanu [1998] show that as long as agents' private information is one-dimensional, an ex-post efficient mechanism to dissolve the partnership can always be found[8]. They also show that a refinement of the Clarke-Groves-Vickrey approach can be used to get an efficient and incentive compatible direct mechanism (this refinement has also been derived in Dasgupta and Maskin [2000]). In section 2 this mechanism is used to analyze whether in a partnership model with interdependent valuations there exist mechanisms that are ex-post efficient, incentive compatible, individually rational and budget balanced.

This section is organized as follows: In section 3.2, I will introduce the model of interdependent valuations. In section 3.3, I will derive the results for the k-double auction in situations with forced participation. I will also characterize those situations (with voluntary participation) in which nonparticipation in the k-double auction does not occur.

[6]If $k = 1$ or $k = 0$ there exists an unique equilibrium.

[7]For the existence of incentive efficient equilibria the assumption of uniformly distributed valuations is needed.

[8]They also show that, in general, efficiency is inconsistent with information revelation if private information is multidimensional.

In section 3.4, I will analyze those cases of separable valuation functions in which a double auction is not individually rational, i.e. in the k-double auction with veto vetoing takes place. For an example I will demonstrate how mechanisms can be obtained that are more efficient. Section 3.5 will be the conclusion. Proofs can be found in appendix A.2.

3.2 The Model

Two risk-neutral agents each own an equal share in a partnership. Each agent i has private information represented by a type θ_i which influences her own and her partner's valuation for the partnership. By θ_{-i} I denote the type of the agent other than i. Agent $i's$ valuation for the entire partnership is given by $v_i(\theta_1, \theta_2)$, which I assume to be continuously differentiable in every argument. I assume a symmetric environment: The types of the agents are drawn from the same distribution function F, and valuation functions are symmetric[9]:

$$v_1(\theta_1, \theta_2) = v_2(\theta_2, \theta_1). \tag{5}$$

Note that symmetry assumptions of this type are necessary to directly compute equilibria of the considered auctions and can for example also be found in Cramton et al. [1987] McAfee [1992] and Engelbrecht-Wiggans [1994]. The distribution function F is strictly increasing and differentiable with derivative f. The support of f is given by $[\underline{\theta}, \overline{\theta}] \subset \mathbb{R}$. Agents' types are independently distributed. The valuation function v_i is strictly increasing in θ_i and increasing in θ_{-i}. I denote the partial derivative of v_i with respect to its j'th component with $v_{i,j}$ and assume that

$$v_{1,1} > v_{2,1}. \tag{6}$$

Because of (5), this is equivalent to $v_{2,2} > v_{1,2}$. Condition (6) is a common assumption in interdependent valuations environments. It ensures the existence of efficient and incentive compatible mechanisms[10]. Given a realization of types the utility of agent i who owns β_i

[9] In the case of separable valuation functions (5) states: $v_1(\theta_1, \theta_2) = g(\theta_1) + h(\theta_2)$ and $v_2(\theta_1, \theta_2) = g(\theta_2) + h(\theta_1)$.

[10] For a discussion of this assumption see Dasgupta and Maskin [2000] or Jehiel and Moldovanu [1998].

in the entire partnership and has money m_i is quasilinear and given by

$$u_i = \beta_i v_i(\theta_1, \theta_2) + m_i.$$

Characteristic functions are defined as follows:

$$\mathbf{1}(\text{statement}) := \begin{cases} 1, & \text{if statement is true} \\ 0, & \text{if statement is false.} \end{cases}$$

3.3 The $k-$Double Auction with Forced Participation

The *k-double auction* is a Bayesian game where the strategy spaces of the agents are given by the set of functions $b : [\underline{\theta}, \overline{\theta}] \mapsto \mathbb{R}$. Given her type θ_i, agent i submits a bid $b_i(\theta_i) \in \mathbb{R}$. Denote the index of the agent who submits the higher bid by H and the index of the other agent by L. Given the bids b_L and b_H and the parameter $k \in [0,1]$, the agent with the higher bid gets the entire partnership and pays to the other agent the amount $\frac{1}{2}((1-k)b_H + kb_L)$. In case both agents submit the same bid the partnership is given to each agent with probability $\frac{1}{2}$ and the "winning bidder" pays 0 to the other agent. Note that such an auction is always ex-post budget balanced since agent L gets what agent H pays. Assume that agent $-i$ bids according to $b_{-i}(\theta_{-i})$. The interim utility of a type θ_i agent who bids b_i is given by

$$\begin{aligned} U_i(\theta_i, b_i) &= \frac{1}{2} E_{\theta_{-i}} \left[(v_i(\theta_i, \theta_{-i}) - (1-k) b_i - k b_{-i}(\theta_{-i})) \quad \mathbf{1}(b_i > b_{-i}(\theta_{-i})) \right] \\ &+ \frac{1}{2} E_{\theta_{-i}} \left[((1-k) b_{-i}(\theta_{-i}) + k b_i - v_i(\theta_i, \theta_{-i})) \quad \mathbf{1}(b_i < b_{-i}(\theta_{-i})) \right]. \end{aligned}$$

The equilibrium concept used is that of pure Bayes-Nash-equilibrium (*BNE*). A BNE $(b_1(\theta_1), b_2(\theta_2))$ is individually rational if for $i = 1, 2$

$$U_i(\theta_i, b_i(\theta_i)) \geq 0, \qquad \forall \theta_i. \tag{IR}$$

The $k-$double auction is called individually rational if there exists an individually rational BNE. Individual rationality assures that the (expected) payoff of any type who participates in the k-double auction exceeds her (expected) valuation for her part of the partnership.

A BNE $(b_1(\theta_1), b_2(\theta_2))$ is ex-post efficient if for all $(\theta_1, \theta_2) \in [\underline{\theta}, \overline{\theta}]^2$ we have :

$$v_1(\theta_1, \theta_2) > v_2(\theta_1, \theta_2) \Rightarrow b_1(\theta_1) > b_2(\theta_2) \tag{EF}$$

which, because of $\theta_1 > \theta_2 \Leftrightarrow v_1(\theta_1, \theta_2) > v_2(\theta_1, \theta_2)$, is equivalent to

$$\theta_1 > \theta_2 \Rightarrow b_1(\theta_1) > b_2(\theta_2).$$

The $k-$double auction is called ex-post efficient if there exists an ex-post efficient BNE.

The next Theorem shows that there exists a unique BNE of the k-double auction. This equilibrium is symmetric. For simplicity I use the following notation:

$$V_i(\theta_i) := v_i(\theta_i, \theta_i), \quad V_i'(\theta_i) := \frac{dV_i(\theta_i)}{d\theta_i}.$$

Theorem 8 *The $k-$double auction has a unique equilibrium bidding strategy in pure strategies given by*

$$b(\theta_i) = V_i(\theta_i) - \frac{\int_{F^{-1}(k)}^{\theta_i} V_i'(u)(F(u) - k)^2 du}{(F(\theta_i) - k)^2}. \tag{7}$$

The rules of the k-double auction constrain partners' strategies in a way that does not allow for nonparticipation, which would be optimal if (IR) is violated and agents could sustain the status quo by nonparticipation, i.e. if participation is voluntary. Then strategy (7) only describes a bidder's behavior if (IR) is not binding. It also describes bidding behavior if we can neglect (IR), i.e. if partners can be forced to participate[11].

The uniqueness result is in contrast to a setting where one agent (the seller) owns the entire good. Leininger et al. [1989] and Satterthwaite and Williams [1989] show that in such a setting there exist a continuum of equilibria of the $k-$double auction if $k \in (0,1)$. That multiplicity is generated by the fact that high type sellers and low type buyers never trade (i.e. the seller's bid is higher than the buyer's bid). Equilibrium bidding functions have to be "locally optimal", i.e. it should not be profitable to slightly increase

[11]This can be modelled by replacing (IR) with a weaker individual rationality constraint. E.g., if we can take away a partner's share if she does not participate, the individual rationality constraint is given by $U_i(\theta_i, b_i(\theta_i)) \geq -\frac{1}{2}E_{\theta_{-i}}[v_i(\theta_i, \theta_{-i})]$, which is always fulfilled by 7.

or decrease the own bid in order to win (or lose) against other agent's types submitting a bid equal or close to the own bid. If types never trade, their bidding strategies are not determined by this local optimality property and can be chosen from a continuum of possibilities. Furthermore the set of types that never trade is not uniquely determined which involves further possibilities for multiplicity. Nevertheless, in a partnership model where the object is not owned by only one bidder, each partner is buyer and seller at the same time, since trade occurs whenever the other partner's bid differs from the own bid. This is the case for almost all partners' types because if there was a positive measure of both partners' types submitting the same bid, bidders would prefer to increase or decrease their bid slightly since this would hardly change payments but significantly change their probability of winning or loosing. Therefore (almost) all types do always trade and local optimality conditions determine their bids.

Note that (7) is strictly increasing and therefore the $k-$double auction is ex-post efficient[12]. In the private values case, i.e. if $v_{i,-i} = 0$, $i = 1, 2$, any BNE of a k-double auction must be individually rational because by bidding exactly her valuation each agent can guarantee herself a positive outcome of the auction regardless of the bid of the other bidder. Independent of k, she never pays more than her valuation for the other agent's share if she wins and if she loses she never gets less than her valuation for the part of the partnership she sells[13]. In general, it is not possible for a partner to bid her true valuation, which depends on private information of the other partner. Therefore a partner might risk to lose her share for a payment that is smaller than her valuation. As shown below, this is exactly what happens if the influence of the other agent's information on the own valuation is high. The intuition behind this observation is that a bidder faces a winner's and a loser's curse. If she wins she risks to pay too much for the partnership since winning indicates a low partner's type and therefore a low valuation of the partnership.

[12] This Theorem generalizes results in Cramton et al [1987] and furthermore shows that there cannot exist equilibria that are not ex-post efficient.

[13] Note that this argument does neither depend on the assumption of equal distribution of ownership rights nor on independence of types.

If she loses, this is again "bad news" for her since this indicates a high type of the partner and therefore a high value of the partnership. Since a bidder has to take these winner's and loser's curses into account at the same time, she cannot correct for these in a way that prevents her from making losses. Since individual rationality is not assured and nonparticipation is not allowed for in the rules of the k-double auction (as defined above) the rules have to be extended for environments where participation is voluntary. Before this is done in the next section, the following Theorem characterizes environments where the ex-post efficient equilibrium (7) is individually rational.

Theorem 9 *A $k-$double auction is individually rational if and only if there exists an ex-post efficient, incentive compatible, individually rational and budget balanced direct revelation mechanism.*

A condition for the existence of ex-post efficient, incentive compatible, individually rational and budget balanced mechanisms is given in Theorem 4 for the class of separable valuation functions:

$$v_i(\theta_i, \theta_{-i}) = g(\theta_i) + h(\theta_{-i}). \tag{8}$$

From Theorem 4 we get:

Theorem 10 *Given valuation functions of the form* (8), *the $k-$double auction is individually rational if and only if*

$$2\int_{F^{-1}\left(\frac{1}{2}\right)}^{\overline{\theta}} g(\theta)\ f(\theta)\ d\theta - 2\int_{\underline{\theta}}^{\overline{\theta}} g(\theta)\ f(\theta)\ F(\theta)\ d\theta + \\ + \int_{\underline{\theta}}^{\overline{\theta}} h'(\theta)\ \left(F^2(\theta) - F(\theta)\right)\ d\theta \geq 0.$$

Note that this existence condition depends on v and F whereas the $k-$double auction is a simple mechanism in a sense that it does not depend on the specifications of the agents' valuations and can therefore be applied universally. Nevertheless, if participation in the $k-$double auction is voluntary a mechanism designer who is not familiar with these specifications does not know whether the partners will participate. In the next section I will extend the rules of the $k-$double auction to obtain a mechanism that is always individually rational, does not depend on specifications of the valuation structures and is ex-post efficient whenever there exists an ex-post efficient, individually rational, budget balanced and incentive compatible mechanism.

3.4 The $k-$Double Auctions with Veto

From Theorem 10 we know that the (unique) equilibrium (7) of the k-double auction might not be individually rational. Therefore we have to extend its rules for situations where participation is voluntary to give partners a strategy that maintains the status quo. I extend the strategy spaces in such a way that every agent has the right to say "No" (write "No" in the sealed bid). The agents' strategy spaces are given by the set of functions:

$$\{b \mid b : [\underline{\theta}, \overline{\theta}] \mapsto \mathbb{R} \cup \{\mathrm{N}\}\}.$$

The outcome of the game is defined as follows: If $b_1 =$N or $b_2 =$N then the partnership is not dissolved (or equivalently, each agent gets the partnership with probability $\frac{1}{2}$). In any other case, the partnership is given to the agent with the higher bid. She pays $\frac{1}{2}\left((1-k)\max\{b_1, b_2\} + k\min\{b_1, b_2\}\right)$, $k \in [0,1]$, to the other agent. I call this Bayesian game the $k-$*double auction with veto*. Vetoing is a way of modelling nonparticipation and blocking the dissolution. An equivalent way of extending the rules of the k-double auction is as follows: The k-double auction is modelled as a two stage game. In the first stage each agent decides whether to participate in the 2nd stage or not. If at least one agent decides not to participate in the 2nd stage, the partnership is not dissolved. Otherwise in the 2nd stage a $k-$double auction (without veto) is run.

Note that the $k-$double auction with veto is always individually rational, because every type can veto and therefore assure that she never makes losses. Furthermore, if the $k-$double auction (without veto) is individually rational its equilibrium is also an equilibrium of the $k-$double auction with veto. It is easy to see that the $k-$double auction with veto has at least one further equilibrium which does not realize any gains from trade: Always vetoing.

In the following I will restrict my attention to environments where the $k-$double auction is not individually rational. To get a precise characterization of symmetric Bayes-Nash-equilibria, I will also restrict the analysis to the case of additively separable valuation functions, i.e. agent i's valuation for the entire partnership is given by the function $v_i(\theta_1, \theta_2) = g(\theta_i) + h(\theta_{-i})$. I assume g, h to be twice differentiable with $g' > h' \geq 0$ and the existence condition in Theorem 10 not to hold, i.e.

$$2\int_{F^{-1}\left(\frac{1}{2}\right)}^{\overline{\theta}} g(\theta)\ f(\theta)\ d\theta - 2\int_{\underline{\theta}}^{\overline{\theta}} g(\theta)\ f(\theta)\ F(\theta)\ d\theta + \\ + \int_{\underline{\theta}}^{\overline{\theta}} h'(\theta)\ \left(F^2(\theta) - F(\theta)\right) d\theta < 0.$$

I will show that apart from the equilibrium where all types veto there exist further symmetric equilibria that realize gains from trade if we choose $k = \frac{1}{2}$. In these equilibria the types close to the boundaries of the support of agents' types want the partnership to be dissolved whereas types around $F^{-1}\left(\frac{1}{2}\right)$ prefer to veto. The intuition behind these equilibria is as follows. Agents with types close to $F^{-1}\left(\frac{1}{2}\right)$ are the "worst off" types in the $k-$double auction mechanism, i.e. these types have the lowest interim utility of participating in the $k-$double auction. This is due to the fact that these types are (almost) equally likely to be buyer or seller of a share. In each case the expected gains from trade (i.e. the expected differences in agents' valuations) are small compared to types close to the boundaries of the support of types. Therefore the types around $F^{-1}\left(\frac{1}{2}\right)$ are vetoing in the $\frac{1}{2}-$double auction with veto and types close to $\underline{\theta}$ or $\overline{\theta}$ do not veto. Indeed the following Theorem 11 shows that all types in an interval $[c, d]$ around $F^{-1}\left(\frac{1}{2}\right)$ prefer to veto. This interval is determined by the fact that agents with type c or d are indifferent between vetoing and non-vetoing. Theorem 11 summarizes these results and formulates necessary conditions for c and d. These can always be fulfilled, as shown in Theorem 12.

Theorem 11 *Let $c, d \in [\underline{\theta}, \overline{\theta}]$ be a solution of the following equations:*

$$\begin{aligned} 1 &= F(c) + F(d) \\ 0 &= \frac{1}{F(c)} \int_{\underline{\theta}}^{c} (g(t) + h(t))(F(t) - F(c)) f(t)\, dt \\ &+ \frac{1}{F(c)} \int_{d}^{\overline{\theta}} (g(t) + h(t))(F(t) - F(d)) f(t)\, dt \\ &+ \int_{\underline{\theta}}^{c} g(t) f(t)\, dt - \int_{d}^{\overline{\theta}} g(t) f(t)\, dt. \end{aligned} \tag{9}$$

Then the following bidding strategy constitutes a symmetric Bayes-Nash-equilibrium of the $\frac{1}{2}-$double auction with veto:

$$b(\theta_i) = \begin{cases} g(\theta_i) + h(\theta_i) - \frac{\int_c^{\theta_i} (g'(t) + h'(t))(F(t) - F(c))^2 dt}{(F(\theta_i) - F(c))^2} & \text{if } \theta_i \in [\underline{\theta}, c) \\ N & \text{if } \theta_i \in [c, d] \\ g(\theta_i) + h(\theta_i) - \frac{\int_d^{\theta_i} (g'(t) + h'(t))(F(t) - F(d))^2 dt}{(F(\theta_i) - F(d))^2} & \text{if } \theta_i \in (d, \overline{\theta}]. \end{cases} \tag{10}$$

Instead of directly verifying that a deviation of the given strategy cannot be profitable if the other agent sticks to it, I use the Revenue-Equivalence-Theorem (Theorem 17, see the Appendix) for an indirect proof. In connection with the Revelation Principle the Revenue-Equivalence-Theorem provides conditions on bidding strategies that assure that it is never profitable to imitate (and bid according to) a different type. I will show that these conditions are fulfilled by (10) and I will additionally show that deviating to a bid outside the range of (10) cannot be profitable[14].

Obviously the concept of a double auction with veto is only meaningful if there exist equilibria that realize gains from trade, i.e. that do not (always) sustain the status quo like the always vetoing equilibrium.

Theorem 12 *Every $\frac{1}{2}-$double auction with veto has a symmetric equilibrium where not vetoing occurs with positive probability.*

An important feature of the $k-$double auction with veto is the independence of its rules of v and F. A mechanism designer can run the auction and obtain the best possible

[14] The details of these arguments can be found in the appendix.

outcome (in terms of efficiency) if in general this outcome can be achieved by some budget balanced and individually rational mechanism. If this is not possible, the set of types that do not want to dissolve the partnership is determined by the agents themselves, depending on their knowledge about v and F. The allocation resulting from the $\frac{1}{2}$−double auction with veto can be seen in Figure 1 which shows the portion of the entire partnership agent 1 receives[15] (e.g. in the area indicated with $\frac{1}{2}$ agent 1 gets half of the partnership). Since

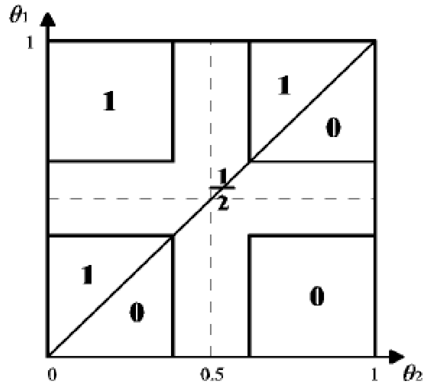

Figure 1: Allocation of $\frac{1}{2}$−double auction

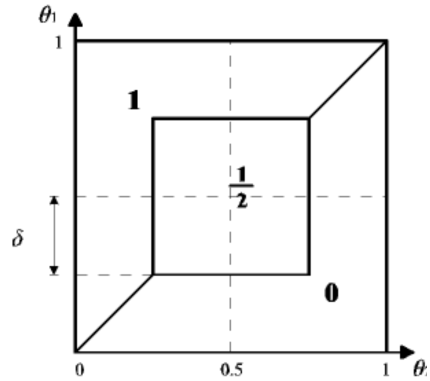

Figure 2: Performance improving allocation

bids only depend on one's own information it is possible that high or low types do not

[15]Figure 1 illustrates the case where $\underline{\theta}$=0, $\overline{\theta} = 1$, $c = d$ and $F^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}$.

a	b	G^{DA}	G^{TB}
1	.9	0.00169	0.00775
1	.7	0.02765	0.04459
1	.6	0.05248	0.06477
1	.55	0.06725	0.07437
1	.51	0.08002	0.08162

Table 1: Performance of $\frac{1}{2}$-double auction

trade if their partner has an "average" type. As a consequence inefficiencies might occur. For an example with linear valuation functions ($v_i = a\theta_i + b\theta_{-i}$) and uniform distribution of types, Figure 2 shows an allocation which[16] can be implemented by a mechanism that is budget balanced and individually rational and realizes more gains from trade[17] than the $\frac{1}{2}-$double auction with veto[18]. Table 1 compares the gains from trade of the $\frac{1}{2}-$double auction with veto (G^{DA}) with those generated by a mechanism resulting in the allocation given by Figure 2 (G^{TB}). Especially in cases where the influence of the other agent's type on the own valuation is high the loss in performance is considerable whereas if it is known that ex-post efficiency can almost be achieved the difference in performances becomes small. Note that these observations are in contrast to results showing the optimality, i.e. incentive efficiency of the double auction in a buyer-seller setting in Myerson and Satterthwaite [1983] or Chatterjee and Samuelson [1983]. If ex-post efficiency cannot be obtained, the simplicity of the rules of the $k-$double auction in general prevents an optimal allocation, even if environments are symmetric and linear.

[16]By choosing δ in Figure 2 in an appropriate way.

[17]Gains from trade are measured as the sum of the agents' ex-ante utilities of participating in the mechanism.

[18]It is not known how the incentive efficient mechanism looks like in this environment. For the linear environment (uniform distribution, linear valuation functions) the mechanism described by Figure 2 (and optimally chosen δ) is indeed the best performing mechanism the author could find.

3.5 Conclusion

The $k-$double auction is a favorable mechanism to dissolve a partnership since it has simple rules that do not depend on specifications of the agents' valuations. If the interdependent components of the valuation functions are small it can be applied without worrying about agents' participation decisions. Since this is not true any more if the influence of the other agent's information becomes larger, the rules of the $k-$double auction have to allow for blocking the dissolution. This is done by giving partners the possibility to veto against a dissolution. Symmetric equilibria of this k-double auction with veto are derived for $k = \frac{1}{2}$ and it is shown that even though the mechanism is not always optimal, it succeeds in realizing gains from trade. The rules remain simple and the mechanism designer does not need to know the distribution of types to determine those types not willing to dissolve the partnership. This is done by the participating agents themselves. An exemplarily comparison with another dissolution mechanism shows that (in contrast to a private valuations model) the mechanism designer can construct more efficient mechanisms if she is familiar with specifications of the valuations.

4 Discounting in Sequential Auctions

4.1 Introduction

If multiple objects are sold in auctions, this is often done sequentially – one object after the other. Auctions of wine cases (see Ashenfelter [1989]) and condominiums (Ashenfelter and Genesove [1992]) are two examples that have been investigated in the empirical literature, but a wide range of other commodities are traded in this manner, among others, through a variety of internet companies that conduct online auctions. In most of these sites items are put up in an auction and buyers are invited to submit bids within a specified deadline after which the items are replaced with new (identical) ones for a new round of auctions. An important empirical observation that can be made in some of the markets of sequential auctions reported in the empirical literature is that prices paid in later auctions are lower than those paid in earlier ones even when there is no significant decline in bidders' valuations across periods. In contrast, theoretical results going back to Weber [1983] indicate that if the objects are identical and agents have unit demand, prices should remain constant across periods since bidders would arbitrage away differences in prices. Prices could even increase if bidders can reduce the winner's curse in later auctions by learning about the true value of the object through the information acquired in earlier periods. Because of the discrepancy with the theory, the empirical observation of declining prices is often referred to as the declining price anomaly.

If the time lag between periods is significant, like e.g., real estate auctions, bidders might be willing to pay a premium for winning the object early, because its value is discounted across periods. In this section we study a model of sequential auctions with value depreciation. We find that if objects are sold sequentially by either first or second-price (sealed bid) auctions and if bidders' valuations decline across periods, then (expected) prices decline as well. More interestingly, prices are declining even if corrected for the decline in valuations. Indeed we demonstrate that a slight decline in valuations across periods can result in a substantial decline in prices. Hence our results also shed light on the empirical observation of the declining price anomaly when the valuation decline across periods is only marginal. Our result not only applies to expected prices, but also to conditional prices, i.e., conditional on the current price, the expected price at any future period must fall below the current price. In other words the stochastic process that governs the price development is a supermartingale.

Two different intuitions support the result: First if valuations are discounted over time and if agents who value the object more suffer more from delay (in absolute terms), then they will bid more aggressively in earlier auctions, win and leave the market where all remaining bidders now have lower (initial) valuations. So as we move from one period to another bidders tend to bid less not only because of the discounting but also because the non-discounted valuations of those who remain in the market are lower than in earlier periods.

Secondly, if equilibrium prices declined only by the discount factor (and not more), then a bidder's expected utility from participating in an auction conditional on winning declines exactly by the discount factor. Since the probability of winning at a given period is not affected by the discount factor, the (non-conditional) expected utility declines as well. This means that bidders can do better by increasing their bids at earlier periods, thereby deviating from the putative equilibrium.

An important role in our analysis plays the information policy of the seller. We develop our price dynamic for two information structures, one in which no information at all leaks between periods, and the second which assumes that the seller announces prices after each period. We also show that these information structures yield efficient equilibria. We then argue that the same price dynamic holds for other sequential bidding mechanisms and information structures. We use a revenue equivalence approach for this result. Specifically, we show that any two sequential mechanisms which are efficient and in which information is revealed about bidders who already received an object (and are therefore out of the market), yield the same price dynamic which is therefore also identical to the price dynamic of our benchmark models.

Declining valuations in sequential auctions can be interpreted in more than one way. They can reflect a real cost of delay when time lag between periods is sufficiently large, and when the auction involves an investment instrument. Auctions on real estate might be relevant in this respect. Declining valuations can also model situations in which the objects are not completely identical as they differ in quality. This will be the case if the seller himself decides to sell the quality ranked objects in a decreasing order so as to avoid inefficiencies and revenue losses due to bottom fishing (see Gale and Hausch [1994]). Finally, we note that a slight modification of our model can also be interpreted as a sequential auction of identical objects when bidders face some uncertainty about the continuation of the auction process in the next period, where the uncertainty is represented by a constant and common knowledge probability of continuation.

In our model k objects are sold sequentially in a first- or second-price auction. There are $n > k$ bidders participating in the first auction and in subsequent auctions all bidders that have not already won an object (bidders have unit demand) participate. Entry of bidders in subsequent auctions does not take place. Bidders are risk neutral and their valuations are functions of (one-dimensional) private information and the rank number of the auction where an object is sold. Private information is distributed identically and independently, i.e. our model is a symmetric independent private values model.

The findings of this section add to the already considerable literature on price trends in sequential auctions. Weber [1983] and Milgrom and Weber [2000] consider sequential auctions in a framework similar to ours but with constant valuations across periods. We consider a declining valuation model and concentrate on the independent values case. Milgrom and Weber [2000] specify the bidding equilibria in their models[1] but provide a proof only for the first-price auction with price announcements pointing out the problems they face with the proofs for the other models. In particular they point out that with price announcements second-price auctions reveal information about remaining bidders and thus break the symmetry. We show that this problem is tackled in our framework of independent values. Several empirical papers report on price decline in sequential auctions: Ashenfelter [1989] first observed this trend in wine auctions followed by Ashenfelter and Genesove [1992] for real estate auctions, Jones et al. [1996] for wool auctions and

[1]They allow for affiliation of types in their model.

recently van den Berg et al. [2001] for flower auctions. Some papers address the issue using a theoretical model including McAfee and Vincent [1993] who use risk aversion to explain the anomaly and von der Fehr [1994] who assumes participation costs. Another interesting empirical paper is Beggs and Graddy [1997] who observed that in art auctions if paintings are sold in a decreasing order relative to the seller's price estimates, then prices decline relative to these estimates. They propose an explanation to this observation with a two period second-price auction model but they neglect informational issues that can only be incorporated in a multiple period model.

This section is organized as follows: In section 4.1.1 we present the main ideas in a simple toy-model with complete information, two periods and three bidders. The general model is introduced in section 4.2. In section 4.3 we derive equilibria for sequential first- and second-price auctions with and without price announcements. In addition we give properties of the trend of prices, e.g. we show that the sequence of prices is a supermartingale, that expected prices decline even if corrected for the discounting and that for the important case of devaluation with discount factors the sequence of corrected prices is a supermartingale as well. A revenue-equivalence result transfers these findings to other sequential bidding mechanisms. An important modification modelling an environment where bidders face uncertainty about the continuation of auctions is also addressed. Section 4.4 gives comparative statics results on the sequences of expected prices and shows exemplarily that even a small decline in valuations might induce a relatively large decline in prices. Section 4.5 is the conclusion. Proofs can be found in appendix A.3.

4.1.1 An illustrative example: Constant discount factor and complete information

Consider the following simple model: There are two objects for sale in two subsequent second-price auctions. Three bidders are participating in these auctions. The valuation of bidder i for the object sold in the first auction is given by her type θ_i, her valuation for the object sold in the second auction is given by $\delta\theta_i$ where $\delta \in [0, 1]$ is a commonly known discount factor (that is the same for all bidders). The marginal valuation for getting a second object is 0, hence the bidder who wins the first auction does not participate in the second auction (or equivalently submits a bid of zero). Assume that bidders' types are common knowledge. We write $\theta_{(1)}$ for the highest, $\theta_{(2)}$ for the median and $\theta_{(3)}$ for the lowest of the types. The last auction is a normal second-price auction and it is a dominant strategy for the two remaining bidders to bid their valuations, i.e. $\delta\theta_i$. The optimal bid in the first auction is $b(\theta) = \theta - \delta\theta + \delta\theta_{(3)}$. This means that it is optimal to bid one's own valuation minus the utility of winning the second auction. If we have $\delta = 1$, i.e. the valuation for an object does not decrease over time, both objects are sold for the same price $p = \theta_{(3)}$. This is due to the fact that a bidder can arbitrage away differences in prices in the two auctions: if for example the price in the first auction were higher the winning bidder would do better by loosing the first auction and winning the second at a lower price. If valuations for the object are decreasing between the two auctions ($\delta \in (0,1)$) then the price paid in the first auction ($(1-\delta)\theta_{(2)} + \delta\theta_{(3)}$) is higher than the price in the second auction ($\delta\theta_{(3)}$). It is even higher than the "real" price of the second object since $(1-\delta)\theta_{(2)} + \delta\theta_{(3)} > \theta_{(3)}$. This means that the devaluation in prices does not follow the devaluation in valuations of the objects, in fact it is stronger. The intuition for this observation relies on the fact that bidders would prefer winning the first auction for a price of p instead of winning the second auction for a price of δp. The first would give an utility of $\theta - p$ whereas the latter would only give $\delta(\theta - p) < (\theta - p)$. Hence bidding in the first auction is more aggressive and results in higher prices. Bidders with higher types face a higher devaluation of the object, as a result their willingness to pay decreases more than that of low-type bidders.

Consider now the asymmetric information case. The intuition behind this is similar to the complete information case. Assume that bidders' types are drawn independently from the same distribution function. We will show in section 4.3 that for two periods the bidding function is $b_1(\theta) = \theta - \delta\theta + \delta E\left[\theta_{(3)}|\theta_{(2)} = \theta\right]$ in the first period and $b_2(\theta) = \delta\theta$ in the second period and therefore a direct analog to the complete information case: Bids in the first auction are a buyer's valuation minus her expected outside option conditional on winning, i.e. the outside option is the (discounted) expected gain of the second auction conditional on winning. Expected prices for the first and second object are given by $E[\theta_{(2)} - \delta\theta_{(2)} + \delta\theta_{(3)}]$ and $\delta E[\theta_{(3)}]$, respectively. As in the complete information case we have $E[\theta_{(2)} - \delta\theta_{(2)} + \delta\theta_{(3)}] > E[\theta_{(3)}]$ if $\delta \in (0,1)$, i.e. "real" prices decline. For more than two periods the equilibrium analysis for the asymmetric information case is more difficult since it turns out that the information revealed by the seller between the auctions is crucial to determine equilibrium strategies. Indeed such information does not effect the equilibrium in the two- period second-price auction case as it is a dominant strategy to bid one's true valuation in the second period.

4.2 A Model with Private Valuations

There are n risk neutral buyers $i = 1, ..., n$ and one seller who offers $k \leq n$ indivisible objects for sale. The seller uses a sequential first- or second-price auction, i.e. the objects are sold sequentially in periods: Each period consists of a first- or second-price auction for one of the objects. The entire selling process is called sequential first-price auction (sequential second-price auction) if in every period a first-price auction (second-price auction) is conducted. The seller's valuation for an object is assumed to be zero in all periods. The discount rate for money is normalized to 0. Buyer i's private valuation for the object auctioned in the first period is given by her type θ_i. The types θ_i are assumed to be independently distributed on $[\underline{\theta},\overline{\theta}]$ with $\underline{\theta} \geq 0$ and are drawn according to a common distribution function F with continuous and strictly positive density f. We write $\theta_{(i)}$ for the $i'th$ highest type among $\theta_1, ..., \theta_n$, i.e. $\theta_{(i)}$ denotes the i'th order statistic of $\theta_1, ..., \theta_n$. A buyer's valuation for an object sold in a later period is a function of the type and the rank number of the period. We assume that the devaluation can be described by the same functions D_l for each bidder, whereas $D_l(\theta_i)$ denotes bidder $i's$ valuation for the object sold in period l given that her type is θ_i. We assume the following properties of D_l :

A1 Normalization: $D_1(\theta) = \theta$, $D_l(\theta) \geq 0$ for all l, θ

A2 Time is valuable: For all l we have that $D_l(\theta) \geq D_{l+1}(\theta)$

A3 Objects are more valuable for higher types: For all l we have that $D_l(\theta)$ is strictly increasing in θ

A4 Continuity: $D_l(\theta)$ is continuous for all l

A5 Increasing loss to delay: $D_l(\theta) - D_{l+1}(\theta)$ is weakly increasing in θ for all l.

Example 3 *Standard discounting is given by* $D_l(\theta) = \delta^{l-1}\theta$ *for* $\delta \in (0, 1]$.

The last assumption states that higher types face higher devaluation (in absolute terms). Note the similarity of these conditions with those imposed on time preferences in Rubinstein's bargaining model [1982]. However our notion of the discounting function is more general in that it does not assume stationarity (an essential property in Rubinstein's time preferences), i.e. the degree of discounting may change in time.

4.3 Equilibria and Price Trends

We allow for two different information policies pursued by the seller: she can either reveal the winning price at the end of each period or she can reveal no information at all. We use the terms "auction with price announcement" for the former case and "auction without price announcement" for the latter one.

Each bidder has unit demand, therefore the number of active bidders[2] in period l is $n - l + 1$. We restrict our attention to symmetric (Bayes-Nash-) equilibria.

The following Theorem characterizes the symmetric equilibrium of the sequential second and first-price auctions with and without price announcements:

Theorem 13 *The symmetric equilibrium bidding strategy for a type* θ*-bidder in the* l*'th period of a sequential first-price auction with or without price announcements is given by* b_l *defined recursively:*

$$\begin{aligned} b_k(\theta) &= E\left[D_k\left(\theta_{(k+1)}\right) \mid \theta_{(k)} = \theta\right], \\ b_l(\theta) &= E\left[D_l\left(\theta_{(l+1)}\right) - D_{l+1}\left(\theta_{(l+1)}\right) + b_{l+1}\left(\theta_{(l+1)}\right) \mid \theta_{(l)} = \theta\right]. \end{aligned}$$

[2] Bidders who already received an object either stay away or bid zero.

The symmetric equilibrium bidding strategy for a type θ-bidder in the sequential second-price auction with or without price announcements is given by the following recursive definition:

$$
\begin{aligned}
b_k(\theta) &= D_k(\theta), \\
b_l(\theta) &= D_l(\theta) - D_{l+1}(\theta) + E\left[b_{l+1}\left(\theta_{(l+2)}\right) \mid \theta_{(l+1)} = \theta\right].
\end{aligned}
$$

These equilibria exhibit some interesting properties. First, bidding functions are strictly increasing, i.e. bidders of a higher type receive their object earlier. This implies that the sequential auctions are ex-post efficient. Furthermore, in the second-price auction we find that bidders shade their bids, i.e. $b_l < D_l$, except for the last period. Note that the bidding functions do not depend on the history of the game up to the current period. Since types are independent, the only relevant information (used for updating beliefs about remaining bidders' types) in period l of the first-price auction is the type of the bidder who won period $l-1$ since this is the $l-1$'th highest type. Every bidder can deduce this information by inverting the bidding function, since prices are announced. The situation in the second-price auction is more complex due to the fact that the bidder who sets the price in period $l-1$ participates in period l and therefore others might know her type. Theorem 13 shows that this does not lead to inefficiencies due to pooling.

A main insight from Weber [1983] is that bidders in earlier periods anticipate lower competition (due to a decreasing number of participants) in forthcoming periods and consequently bid less aggressively in earlier periods. As a result of this behavior price differences between periods are arbitraged away, i.e. the (expected) price is the same in each period (and equals the expected valuation of the $k+1'$th highest bidder). Moreover the sequence of prices is a martingale, i.e. prices are constant on average over time. These results hold for the sequential first- and second-price auctions. In our model with time preferences prices drift down over time. The following Theorem shows the link between (expected) prices in the subsequent period and observed prices in the actual period.

Denote by $D_{l,l+1} := D_{l+1} \circ D_l^{-1}$ the discount function from period l to period $l+1$, i.e. $D_{l,l+1}(v)$ denotes a bidders valuation in period $l+1$ if her valuation is v in period

l. Moreover denote by p_l the price of the l'th period in a sequential first- or second-price auction, i.e. $p_l = b_l\left(\theta_{(l)}\right)$ in a sequential first-price auction and $p_l = b_l\left(\theta_{(l+1)}\right)$ in a sequential second-price auction.

Theorem 14

1. *In a sequential first-price auction given a price p_l in period l the expected corrected price in period $l+1$ is always lower than p_l, i.e.*

$$E\left[D_{l,l+1}^{-1}\left(p_{l+1}\right) \mid p_l\right] \leq p_l. \tag{11}$$

2. *In a sequential second-price auction given a price p_l in period l the corrected expected price in period $l+1$ is always lower than $p_{l,}$ i.e.*

$$D_{l,l+1}^{-1}\left(E\left[p_{l+1} \mid p_l\right]\right) \leq p_l. \tag{12}$$

For the sequential first-price auction we obtain a comparison of the actual price in period l and the expected discounted price in period $l+1$. The price determining bid in round l contains the expected utility of the second highest bidder in that period given p_l. This can be compared to the expected corrected price of round $l+1$ given the price p_l. Since in a sequential second-price auction bidders are bidding their own valuation minus their own expected outside option conditional on being the price setting bidder in period l and since this depends on the expected price of the next period (conditional on being the price setting bidder) we are able to directly compare the actual price and the expected price of the subsequent period conditional on the actual price.

Obviously we get statement (11) for second-price auctions if $D_{l,l+1}$ is convex (using Jensen's inequality) and (12) for first-price auctions if $D_{l,l+1}$ is concave.
Theorem 14 has some important implications including the fact that the price trends exposed in Theorem 14 are carried over to the (non conditional) expected prices. This is summarized in the following Corollary.

Corollary 2 1. *The sequence of prices $(p_l)_{l\leq k}$ is a supermartingale.*

2. *The sequence of expected prices is the same for a sequential first- and second-price auction and we have*

$$E[D_{l,l+1}^{-1}(p_{l+1})] \leq E[p_l]$$

and

$$E[p_{l+1}] \leq E[D_{l,l+1}(p_l)].$$

If $D_{l,l+1}$ *is concave we have*

$$D_l^{-1}(E[p_l]) \geq D_{l+1}^{-1}(E[p_{l+1}]).$$

3. *If we have a discount factor of* δ_l *for discounting from period* l *to* $l+1$*, i.e.* $D_l(\theta) = \prod_{i=1}^{l-1} \delta_i \theta$, $D_{l,l+1}(x) = \delta_l x$*, then the sequence* $\left(D_l^{-1}(p_l)\right)_{l \leq k}$ *is a supermartingale. Moreover we find that (11) and (12) hold.*

We now wish to argue that the results on the trend of expected prices in Corollary 2 are valid for a larger class of sequential auctions. Due to the Revenue-Equivalence-Theorem the seller's expected revenue is the same for the sequential first- and second-price auction and for other mechanisms that implement the efficient allocation and results in the same utility level for a type-$\underline{\theta}$-bidder. To obtain equivalence of the trends of expected prices for sequential auctions, however, we need to modify the Revenue-Equivalence-Theorem to make it applicable to single periods rather than to the entire sequential mechanism. For the sequential first- and second-price auction we find that the expected price of a given period is the same for these auction formats. We show that this result also holds for other sequential auction mechanisms that have efficient equilibria. A sequential auction mechanism is a mechanism[3] in which bidders submit bids in each period and the object (sold in that period) is given to the bidder with the highest bid. Payments to the seller depend on the submitted bids. We consider four properties of sequential auction mechanisms:

[3]For a more formal definition, see the proof of Theorem 15.

P1 The mechanism results in an efficient allocation of the objects, i.e., the $l'th$ object is sold to the bidder with the l'th highest type.

P2 A bidder of type $\underline{\theta}$ has zero expected payments in each period.

P3 The information policy is such that after each period the type of the winning bidder in that period is announced.

P4 Each bidder's (expected) payment (in a period) only depends on her own bid (of that period), like in an all-pay auction.

The following Theorem shows revenue equivalence in each period for a large class of efficient sequential auction mechanisms.

Theorem 15 *Given two sequential auction mechanisms both satisfying either P1, P2, P3 or P1, P2, P4 then the expected sum of payments in the l'th period (i.e. for the l'th object) is the same in both mechanisms.*

We wish to argue that all mechanisms fulfilling the assumptions of Theorem 15 and the sequential first- and second-price auctions discussed in this section result in the same expected payments as a Clarke-Groves mechanism, i.e. expected payments in a given period l of a sequential auction mechanism also equal (expected) payments in a Clarke-Groves mechanism made by the l'th highest type. Recall that in a Clarke-Groves mechanism agents report their types, the efficient allocation and payments, reflecting agents' externalities imposed on others, are implemented. In our model k objects with values $D_l(\theta)$, $l = 1, \ldots, k$, to a type-θ-agent are allocated. Hence in a Clarke-Groves mechanism the l'th object is given to the agent with the l'th highest type. The payments of the agent with the l'th highest type only depend on other agents' types and are given by

$$\begin{cases} D_l\left(\theta_{(l+1)}\right) + \sum_{j=l+1}^{k}\left(D_j\left(\theta_{(j+1)}\right) - D_j\left(\theta_{(j)}\right)\right) & \text{if} \quad l \leq k \\ 0 & \text{if} \quad l > k. \end{cases}$$

Consider now the following sequential version of the Clarke-Groves mechanism: In period l bidders submit types $\widehat{\theta}_i$, the bidder with the highest type wins the l'th object and pays

$D_l\left(\widehat{\theta}^l_{(2)}\right)+\sum_{j=2}^{k-l+1}\left(D_j\left(\widehat{\theta}^l_{(j+1)}\right)-D_j\left(\widehat{\theta}^l_{(j)}\right)\right)$, where $\widehat{\theta}^l_{(j)}$ is the j'th highest announced type in period l. After each period the highest announced type is made public and the winning bidder of that period quits the mechanism (i.e. does not participate in subsequent periods). Since truthtelling is a dominant strategy in the Clarke-Groves mechanism, this is also true for its sequential version (which results in the same allocation and payments). Hence the Clarke-Groves mechanism yields the same expected payments (associated with any object l) as in any of the auction mechanisms for which the equivalent result of Theorem 15 applies.

Our previous analysis can be translated to a model with no discounting but there is uncertainty about whether further periods will take place in the future. Assume that bidders in period l expect a continuation of the auction process with probability δ_l, i.e. with probability $1-\delta_l$ period l was the last period. The probabilities δ_l are assumed to be common knowledge. We refer to this model as sequential auction with uncertain renewal. Since agents are assumed to be risk-neutral, this model is equivalent to a model in which discounting exists but applies to both payments and valuations with the same discount factor δ_l (between period l and $l+1$). Formally if $\delta_1,\ldots,\delta_{k-1}$ are the inter-period discount rates then winning an object in period $l+1$ for the (nominal) price of p_{l+1} yields an utility level of $\delta_l\left(\theta-p_{l+1}\right)$ for a type-$\theta-$agent finding herself in period l.

Corollary 3 *The equilibrium bidding function for the sequential first-price auction (with or without price announcement) with uncertain renewal is given by*

$$\begin{aligned} b_l(\theta) &= E\left[\theta_{(l+1)}-\delta_l\left(\theta_{(l+1)}-b_l\left(\theta_{(l+1)}\right)\right)\ \mid \theta_{(l)}=\theta\right], \\ b_k(\theta) &= E\left[\theta_{(k+1)}\right|\ \theta_{(k)}=\theta\right]. \end{aligned}$$

For the sequential second-price auction with uncertain renewal it is given by

$$\begin{aligned} b_l(\theta) &= \theta-\delta_l\left(\theta-E\left[b_{l+1}\left(\theta_{(l+2)}\right)\mid \theta_{(l+1)}=\theta\right]\right) \\ b_k(\theta) &= \theta. \end{aligned}$$

The sequence of actual prices $(p_l)_{l\leq k}$ with $p_l=b_l\left(\theta_{(l)}\right)$ for the first-price auction and

$p_l = b_l\left(\theta_{(l+1)}\right)$ *for the second-price auction is a supermartingale, i.e. given any realization* p_l *expected prices drift down on average.*

4.4 Comparative Statics

In this section we study how changes in various parameters of the model (e.g., the number of bidders, the discount factor and the distribution of valuations) affect the price dynamic. In particular we will show that the price decline that cannot be explained directly by the discount function can be substantial, i.e. a negligible decline in valuations can result in a major price decline. We start however with some observations on the effect of increasing the number of bidders.

If we denote by $\overline{p}_l$ the expected price in the l'th period, i.e. $\overline{p}_l = E\left[b_l\left(\theta_{(l)}\right)\right]$ for the sequential first price auction, we get as a direct consequence of Theorem 13

$$\begin{aligned}\overline{p}_l &= E\left[D_l\left(\theta_{(l+1)}\right) - D_{l+1}\left(\theta_{(l+1)}\right)\right] + \overline{p}_{l+1} \text{ for } l < k \text{ and} \\ \overline{p}_k &= E\left[D_k\left(\theta_{(k+1)}\right)\right].\end{aligned} \tag{13}$$

If we fix the number of objects k and denote by $\overline{p}_l(n)$ the expected price in the l'th auction if the number of bidders in the first auction was n then we have

$$\begin{aligned}\overline{p}_l(n) &< \overline{p}_l(n+1), \\ \lim_{n\to\infty} \overline{p}_l(n) &= D_l\left(\overline{\theta}\right).\end{aligned}$$

This is a standard result stating that prices increase in the number of bidders and converge to the highest type's valuation of the good.

For further analysis we confine our attention to the case where $D_l(\theta) := \delta^{l-1}\theta$. If the number of bidders becomes large (for a fixed number of goods), we find that expected prices decrease approximately with the discount factor, i.e.

$$\lim_{n\to\infty} \frac{\overline{p}_l(n)}{\overline{p}_{l-1}(n)} = \delta \quad \text{for all } l \leq k.$$

For a fixed number of periods and objects the difference in prices

$$\overline{p}_l - \overline{p}_{l+1} = \delta^{l-1}(1-\delta)\, E\left[\theta_{(l+1)}\right]$$

is decreasing in l. The development of the devaluation of prices, i.e. the sequence $\left(\frac{\overline{p}_l}{\overline{p}_{l-1}}\right)_{l\leq k}$, depends on the distribution of types, or more precisely on the expected values of the l'th order statistics.

Even though both declining and increasing trends of price devaluations might occur, it turns out that for high discount factors we will see a decrease in (relative) price devaluations $\frac{\overline{p}_l}{\overline{p}_{l-1}}$. The observations concerning the trends of devaluations of expected prices are summarized in the next Theorem.

Theorem 16 *Fix a number of bidders n and a number of objects $2 < k < n$.*

1. *For every distribution F there exists a threshold $\underline{\delta} < 1$ such that for all $\delta < 1$ with $\delta > \underline{\delta}$ the sequence $\left(\frac{\overline{p}_l}{\overline{p}_{l-1}}\right)_{l\leq k}$ is increasing.*
2. *If the sequence $\left(\frac{E\left[\theta_{(l)}\right]}{E\left[\theta_{(l-1)}\right]}\right)_{l\leq k}$ is increasing (decreasing) and if for $\delta \in (0,1)$ we find that*
$$\frac{E\left[\theta_{(k+1)}\right]}{(1-\delta)E\left[\theta_{(k)}\right]+\delta E\left[\theta_{(k+1)}\right]} \underset{(<)}{>} \frac{E\left[\theta_{(k)}\right]}{E\left[\theta_{(k-1)}\right]}$$
then the sequence $\left(\frac{\overline{p}_l}{\overline{p}_{l-1}}\right)_{l\leq k}$ is increasing (decreasing).

In the following we determine (for an example) the share of the total price decrease that can be directly explained by the decline in valuations. In a setting with two objects and discount factor δ (i.e. $D_2(\theta) = \delta\theta$) we are interested in the share of the change in expected prices if they declined with $\delta < 1$ in proportion to the total decrease in expected prices $\overline{p}_1 - \overline{p}_2$, i.e. we are interested in $\frac{(1-\delta)\overline{p}_1}{\overline{p}_1-\overline{p}_2}$. Because of (13) we have
$$\frac{(1-\delta)\overline{p}_1}{\overline{p}_1-\overline{p}_2} = \frac{\overline{p}_1}{E[\theta_{(2)}]} = \frac{(1-\delta)E[\theta_{(2)}]+\delta E[\theta_{(3)}]}{E[\theta_{(2)}]}.$$
Therefore the portion of the total price difference that cannot be directly explained by discounting is greater if δ is higher. The share that can be explained by the direct effect (of discounting) is always higher than $\frac{E[\theta_{(3)}]}{E[\theta_{(2)}]}$, but can be arbitrarily close to this value. If we have $\underline{\theta}= 0$, it is therefore possible that this share become arbitrarily small[4].

[4] This can be achieved by a distribution that has its mass concentrated on 0 and 1. If the mass concentrated in a small environment of 0 becomes large, $\frac{E[\theta_{(3)}]}{E[\theta_{(2)}]}$ is close to zero.

Example 4 *Consider the case of a two point distribution*[5] *where the type is 0 with probability* q *and 1 with probability* $1-q$. *If we have two objects (13) yields:*

$$\overline{p}_1 = (1-q)^3 + 3(1-\delta)(1-q)^2 q,$$
$$\overline{p}_2 = (1-q)^3.$$

In Figure 3 we display the case $q=\frac{3}{4}$ *for* $\delta \in [0,1]$. *The upper graph shows the expected price of the first period* $\overline{p}_1$; *the graph in the middle shows the discounted expected price of the first auction, i.e.* $\delta\overline{p}_1$; *and the lower graph the expected price of the second auction* $\overline{p}_2$ *in dependance of* δ. *For instance if we have* $\delta = 0.99$ *we get that* $\frac{\overline{p}_2}{\overline{p}_1} \approx 0.91$ *and* $\frac{1-\delta}{1-\frac{\overline{p}_2}{\overline{p}_1}} \approx \frac{1}{9}$, *indicating that* $\frac{8}{9}$ *of the total price drop cannot be explained directly by the discount factor. This shows that the indirect effect can be substantial and might offer an explanation for declining prices in some of the real life auctions where discounting at first sight seems to be too small to have a significant impact on the trend of prices.*

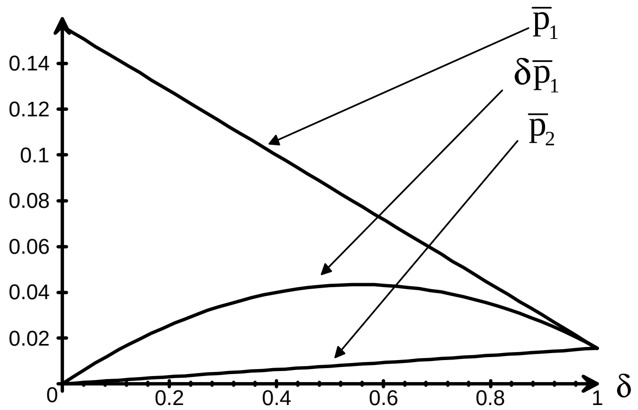

Figure 3: Graph for Example 4.

[5]Such a distribution does not fulfill the assumptions made in this section, nevertheless it can be approximated arbitrarily close by continuous distributions.

4.5 Conclusion

We show that in sequential first- and second price auctions with or without price announcements we have declining prices if valuations decrease for objects sold in later auctions. Even if we account for the general decrease in valuations, which is given by a common general "discount" function D_l, $l = 1, \ldots, k$, expected prices decline in later auctions. Important for this result is the assumption that valuations for higher types decrease stronger in absolute terms than that of lower types, thus increasing competition in the earlier auctions. If valuations remained constant, the increase in competition due to discounting could be arbitraged away. In contrast if valuations decline with the rank number of the auction this is no longer the case. Even if decline in valuations is relatively small it can have a substantial effect on the development of prices, hence our model might also explain the "price anomaly" for environments where discounting seems to be negligible at first sight. A revenue equivalent result shows that our findings translate to a large class of sequential selling mechanisms and therefore are applicable to other market mechanisms as well.

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A Appendix

A.1 Partnerships, Lemons and Efficient Trade

Proof of Theorem 1: The proof follows that for private values (see e.g. Myerson [1981]). There is one additional argument needed to justify the differentiation under the integral. Incentive compatibility implies:

$$U_i(\theta_i,\theta_i) \geq U_i\left(\theta_i,\widehat{\theta}_i\right) \text{ and } U_i\left(\widehat{\theta}_i,\widehat{\theta}_i\right) \geq U_i\left(\widehat{\theta}_i,\theta_i\right) \quad \forall\ \theta_i,\widehat{\theta}_i.$$

We therefore obtain the following inequalities:

$$V_i\left(\widehat{\theta}_i,\widehat{\theta}_i\right) - V_i\left(\theta_i,\widehat{\theta}_i\right) \geq U_i\left(\widehat{\theta}_i,\widehat{\theta}_i\right) - U_i(\theta_i,\theta_i) \geq V_i\left(\widehat{\theta}_i,\theta_i\right) - V_i(\theta_i,\theta_i).$$

Dividing by $\widehat{\theta}_i - \theta_i$ gives

$$E_{\theta_{-i}}\left[\frac{v_i\left(\widehat{\theta}_i,\theta_{-i}\right) - v_i(\theta_i,\theta_{-i})}{\widehat{\theta}_i-\theta_i}\left(s_i\left(\widehat{\theta}_i,\theta_{-i}\right)-\alpha_i\right)\right] \geq \frac{U_i\left(\widehat{\theta}_i,\widehat{\theta}_i\right) - U_i(\theta_i,\theta_i)}{\widehat{\theta}_i-\theta_i}$$
$$\geq E_{\theta_{-i}}\left[\frac{v_i\left(\widehat{\theta}_i,\theta_{-i}\right) - v_i(\theta_i,\theta_{-i})}{\widehat{\theta}_i-\theta_i}\left(s_i(\theta_i,\theta_{-i})-\alpha_i\right)\right].$$

Because v_i is continuously differentiable and because $\lim_{\widehat{\theta}_i\to\theta_i} s_i(\theta_i,\theta_{-i}) = s_i\left(\widehat{\theta}_i,\theta_{-i}\right)$ a.e. we can take the limit $\widehat{\theta}_i \to \theta_i$ and apply the Dominated Convergence Theorem to obtain

$$E_{\theta_{-i}}\left[v_{i,1}(\theta_i,\theta_{-i})\left(s_i(\theta_i,\theta_{-i})-\alpha_i\right)\right] \geq \frac{dU(\theta_i)}{d\theta_i} \geq E_{\theta_{-i}}\left[v_{i,1}(\theta_i,\theta_{-i})\left(s_i(\theta_i,\theta_{-i})-\alpha_i\right)\right]$$

and therefore that $U(\theta_i)$ is differentiable with

$$\frac{dU(\theta_i)}{d\theta_i} = E_{\theta_{-i}}\left[v_{i,1}(\theta_i,\theta_{-i})\left(s_i(\theta_i,\theta_{-i})-\alpha_i\right)\right].$$

Q.E.D.

Proof of Corollary1: Fix an IC mechanism that implements s and has the payment functions $t_i(\theta_1,...,\theta_n)$, $i = 1,...,n.$ Observe that a mechanism $(s,t+r)$ with $r = (r_1,...,r_n)$, where r_i is an arbitrary constant also implements s. Consider an arbitrary mechanism that implements s and has payment functions $w_i(\theta)$ and interim payment functions $W_i(\theta_i) := E_{\theta_{-i}}[w_i(\theta_i,\theta_{-i})]$. Denote by $U_i^w(\theta_i)$ and by $U_i^{t+r}(\theta_i)$ the interim equilibrium utilities of agents participating in (s,w) and $(s,t+r)$, respectively. Because the interim utilities of the participating agents are (up to a constant) the same for all IC mechanisms that implement s, we can find constants q_i such that $U_i^w(\theta_i) = U_i^{t+q}(\theta_i)$.This means that for every IC mechanism (s,w) we can find a mechanism $(s,t+q)$ that is equivalent to (s,w) in terms of

interim utilities. This leads to the following important observation: If the mechanism (s, w) is BB and IR, then the mechanism $(s, t+q)$ is also BB and IR. To check this note that $U_i^{t+q}(\theta_i) = U_i^w(\theta_i) \geq 0$ and that $\sum_{i=1}^n E_{\theta_i}[T_i(\theta_i) + q_i] = \sum_{i=1}^n E_{\theta_i}\left[U_i^{t+q}(\theta_i) - V_i(\theta_i, \theta_i)\right] = \sum_{i=1}^n E_{\theta_i}[W_i(\theta_i)] \leq 0$.
Q.E.D.

Proof of Theorem 2: Given a mechanism (s, t) that implements s, let $\widetilde{\theta}_i$ be the "worst off" type of agent i. Let $q = (q_1, ..., q_n)$ be a vector of constants. Because of $U_i^{t+q}(\theta_i) = U_i^t(\theta_i) + q_i$ the "worst off" type of player i in the mechanism $(s, t+q)$ is also given by $\widetilde{\theta}_i$. We are looking for constants q_i such that the mechanism $(s, t+q)$ is BB and IR, i.e., $\sum_{i=1}^n (E_{\theta_i}[T_i(\theta_i)] + q_i) \leq 0$ and $U_i^{t+q}\left(\widetilde{\theta}_i\right) = U_i^t\left(\widetilde{\theta}_i\right) + q_i \geq 0 \quad \forall i$. These conditions can hold if and only if $\sum_{i=1}^n E_{\theta_i}[T_i(\theta_i)] \leq \sum_{i=1}^n U_i^t\left(\widetilde{\theta}_i\right)$.
Q.E.D.

Proof of Theorem 3: Consider agent i and assume that all agents other than i report their types θ_{-i} truthfully. Assume first, that $\forall\ \theta_i$ we have $v_i(\theta_i, \theta_{-i}) > \max_{j \neq i} v_j(\theta_i, \theta_{-i})$ or $v_i(\theta_i, \theta_{-i}) < \max_{j \neq i} v_j(\theta_i, \theta_{-i})$. Then i's report does not change the allocation s^*. Because payments do not depend on θ_i, it is optimal for i to report truthfully.
Assume now that $v_i(\theta_i^*, \theta_{-i}) = \max_{j \neq i} v_j(\theta_i^*, \theta_{-i})$ for $\theta_i^* \in \left[\underline{\theta}_i, \overline{\theta}_i\right]$, and that the true type of agent i is θ_i. We distinguish several cases:

1. $\theta_i > \theta_i^*$: Any report $\widehat{\theta}_i > \theta_i^*$ does not change the allocation (agent i still gets the good) because we have

$$v_i(\theta_i^*, \theta_{-i}) = \max_{j \neq i}\ v_j(\theta_i^*, \theta_{-i}) \text{ and } v_{i,i}(\theta_i, \theta_{-i}) > v_{j,i}(\theta_i, \theta_{-i}) \qquad \forall\ j \neq i$$

$$\Rightarrow v_i\left(\widehat{\theta}_i, \theta_{-i}\right) > \max_{j \neq i}\ v_j\left(\widehat{\theta}_i, \theta_{-i}\right).$$

   Payments are not affected by reporting $\widehat{\theta}_i > \theta_i$ either. If i reports $\widehat{\theta}_i < \theta_i^*$ he won't get the good any more but receives the payment $v_i(\theta_i^*, \theta_{-i}) < v_i(\theta_i, \theta_{-i})$ and therefore it is optimal to report θ_i instead of $\widehat{\theta}_i$. If $\widehat{\theta}_i = \theta_i^*$ agent i gets either $v_i(\theta_i^*, \theta_{-i})$ or $v_i(\theta_i, \theta_{-i})$. So he cannot improve his payoff by lying.

2. $\theta_i < \theta_i^*$: As long as i announces $\widehat{\theta}_i < \theta_i^*$ he doesn't change the allocation because we have $v_i\left(\widehat{\theta}_i, \theta_{-i}\right) < \max_{j \neq i}\ v_j\left(\widehat{\theta}_i, \theta_{-i}\right)$. If $\widehat{\theta}_i > \theta_i^*$ he will get the good but values it $v_i(\theta_i, \theta_{-i})$ which is less than the payment he gets by reporting truthfully, $v_i(\theta_i^*, \theta_{-i})$. As above truth-telling yields at least the same as reporting $\widehat{\theta}_i = \theta_i^*$.

3. $\theta_i = \theta_i^*$: In this case agent i will always have the utility $v_i(\theta_i^*, \theta_{-i})$ (independent of his announcement) and therefore optimally reports the truth.

Q.E.D.

Proof of Theorem 4: We determine the "worst off" type $\widetilde{\theta_i} = \arg\min_{\theta_i} U_i(\theta_i)$ of agent i. The interim expected utility of agent i with type θ_i is given by:

$$\begin{aligned} U_i(\theta_i) &= E_{\theta_{-i}}\Big[v_i(\theta_i,\theta_{-i})\ \mathbf{1}(\theta_i > \max_{j\neq i}\ \theta_j)+ \\ &\quad +v_i\left(\max_{j\neq i}\ \theta_j,\theta_{-i}\right)\ \mathbf{1}(\theta_i < \max_{j\neq i}\ \theta_j) - \alpha_i v_i(\theta_i,\theta_{-i})\Big]. \end{aligned}$$

The first order condition of the minimization problem gives

$$\begin{aligned} 0 &= E_{\theta-i}\left[\frac{\partial}{\partial\ \theta_i} v_i(\theta_i,\theta_{-i})\ \left(\mathbf{1}(\theta_i > \max_{j\neq i}\ \theta_j) - \alpha_i\right)\right] \\ &= g'(\theta_i)\ E_{\theta_{-i}}[\mathbf{1}(\theta_i > \max_{j\neq i}\ \theta_j) - \alpha_i] = g'(\theta_i)\ (F^{n-1}(\theta_i) - \alpha_i) \end{aligned}$$

which yields $\widetilde{\theta_i} = F^{-1}(\alpha_i^{\frac{1}{n-1}})$. This is the only minimum because $F^{n-1}(\theta_i) - \alpha_i$ is negative for $\theta_i < \widetilde{\theta_i}$ and positive for $\theta_i > \widetilde{\theta_i}$.

1) We have to show that conditions (1) and (2) are equivalent. The interim utility $U_i\left(\widetilde{\theta_i}\right)$ is given by

$$U_i\left(\widetilde{\theta_i}\right) =$$

$$E_{\theta_{-i}}\left[v_i(\widetilde{\theta}_i,\theta_{-i})\ \mathbf{1}(\widetilde{\theta}_i > \max_{j\neq i}\ \theta_j) + v_i\left(\max_{j\neq i}\ \theta_j,\theta_{-i}\right)\ \mathbf{1}(\widetilde{\theta}_i < \max_{j\neq i}\ \theta_j) - \alpha_i v_i(\widetilde{\theta}_i,\theta_{-i})\right].$$

Using $\theta_i^*(\theta_{-i}) = \max_{j\neq i}\ \theta_j$ condition (1) writes:

$$\begin{aligned} &\sum_{i=1}^{n} E_{\theta_{-i}}\left[v_i(\widetilde{\theta}_i,\theta_{-i})\ \mathbf{1}(\widetilde{\theta}_i > \max_{j\neq i}\ \theta_j) + v_i\left(\max_{j\neq i}\ \theta_j,\theta_{-i}\right)\ \mathbf{1}(\widetilde{\theta}_i < \max_{j\neq i}\ \theta_j) - \alpha_i v_i(\widetilde{\theta}_i,\theta_{-i})\right] \\ &\qquad - \sum_{i=1}^{n} E_\theta\left[v_i\left(\max_{j\neq i}\ \theta_j,\theta_{-i}\right)\mathbf{1}\ \left(v_i(\theta_i,\theta_{-i}) < \max_{j\neq i}\ v_j(\theta_i,\theta_{-i})\right)\right] \geq 0. \end{aligned}$$

Using separability and symmetry of the valuation functions, we obtain (for some i)

$$\begin{aligned} &\sum_{j=1}^{n}\left[\int_{\widetilde{\theta}_j}^{\overline{\theta}} g(\theta)\ dF^{n-1}(\theta) - \int_{\underline{\theta}}^{\overline{\theta}} g(\theta)\ F(\theta)\ dF^{n-1}(\theta)\right] \\ &+ (n-1)\ E_{\theta_{-i}}\left[\sum_{j\neq i} h(\theta_j)\right] - n\ E_{\theta_{-i}}\left[\sum_{j\neq i} h(\theta_j)\ F\left(\max_{j\neq i}\ \theta_j\right)\right] \geq 0. \end{aligned}$$

Integration by parts gives

$$\begin{aligned} &(n-1)\ E_{\theta_{-i}}\left[\sum_{j\neq i} h(\theta_j)\right] - n\ E_{\theta_{-i}}\left[\sum_{j\neq i} h(\theta_j)\ F\left(\max_{j\neq i}\ \theta_j\right)\right] \\ =\ &(n-1)\ n\int_{\underline{\theta}}^{\overline{\theta}} h(\theta)\ \left[\int_{\theta}^{\overline{\theta}} F^{n-2}(M)\, f(M)\, dM - \frac{1}{n}\right]\ f(\theta)\, d\theta. \end{aligned}$$

Using

$$\int_{\theta}^{\overline{\theta}} F^{n-2}(M)\, f(M)\, dM = \frac{1 - F^{n-1}(\theta)}{n-1}$$

and

$$n \int_{\underline{\theta}}^{\overline{\theta}} h(\theta)\, F^{n-1}(\theta)\;\; f(\theta)\;\; d\theta = h\left(\overline{\theta}\right) - \int_{\underline{\theta}}^{\overline{\theta}} h'(\theta)\, F^{n}(\theta)\, d\theta$$

we get the wished result.

2) Consider $\psi : [0,1]^n \mapsto \mathbb{R}$ with

$$\begin{aligned} \psi(\alpha_1, ..., \alpha_n) \;\; &= \;\; \sum_{i=1}^{n} \left(\int_{\widetilde{\theta}_i}^{\overline{\theta}} g(\theta)\;\; dF^{n-1}(\theta)\;\; d\theta - \int_{\underline{\theta}}^{\overline{\theta}} g(\theta)\;\; F(\theta)\;\; dF^{n-1}(\theta)\;\; d\theta \right) + \\ &+ \int_{\underline{\theta}}^{\overline{\theta}} h'(\theta)\;\; \left(F^{n}(\theta) - F(\theta)\right)\; d\theta. \end{aligned}$$

The function ψ is symmetric in its arguments and concave (recall that $\alpha_i = F^{n-1}\left(\widetilde{\theta}_i\right)$):

$$\begin{aligned} \frac{\partial \psi}{\partial \alpha_i} \;\; &= \;\; -\; g(\widetilde{\theta}_i)\;\; (n-1)\, F^{n-2}\left(\widetilde{\theta}_i\right) f(\widetilde{\theta}_i) \frac{d\widetilde{\theta}_i}{d\alpha_i} = -g\left(\widetilde{\theta}_i\right) \frac{dF^{n-1}\left(\widetilde{\theta}_i\right)}{d\alpha_i} = -g\left(\widetilde{\theta}_i\right), \\ \frac{\partial^2 \psi}{\partial \alpha_i^2} \;\; &= \;\; -g'\left(\widetilde{\theta}_i\right)\; \frac{d\widetilde{\theta}_i}{d\alpha_i} < 0. \end{aligned}$$

Because of symmetry, ψ takes its maximum on the simplex $\sum_{i=1}^{n} \alpha_i = 1$ at $\frac{1}{n}, ..., \frac{1}{n}$, and the set of $(\alpha_1, ..., \alpha_n)$ that lead to positive values of ψ is either symmetric and convex, or empty.

Q.E.D.

Proof of Theorem 5: We have $\widetilde{\theta}_i = F^{-1}\left(\alpha_i^{\frac{1}{n-1}}\right)$ and therefore $\widetilde{\theta}_1 \leq \cdots \leq \widetilde{\theta}_n$. Integration by parts in condition 2 yields:

$$\sum_{i=1}^{n}\left(g\left(\overline{\theta}\right)-\alpha_i g\left(\widetilde{\theta}_i\right)-\int_{\widetilde{\theta}_i}^{\overline{\theta}} g'\left(\theta\right)F^{n-1}\left(\theta\right)d\theta\right)$$
$$-\left(n-1\right)g\left(\overline{\theta}\right)+\left(n-1\right)\int_{\underline{\theta}}^{\overline{\theta}} g'\left(\theta\right)F^{n}\left(\theta\right)d\theta$$
$$+\int_{\underline{\theta}}^{\overline{\theta}} h'\left(\theta\right)\ \left(F^{n}\left(\theta\right)-F(\theta)\right)\ d\theta$$
$$\geq \sum_{i=1}^{n}\int_{\widetilde{\theta}_i}^{\overline{\theta}} g'\left(\theta\right)\ \left(\alpha_i-F^{n-1}\left(\theta\right)\right)\ d\theta+\left(n-1\right)\int_{\underline{\theta}}^{\overline{\theta}} g'\left(\theta\right)\ F^{n}\left(\theta\right)\ d\theta$$
$$= \int_{\widetilde{\theta}_n}^{\overline{\theta}} g'\left(\theta\right)\left(1-nF^{n-1}\left(\theta\right)+\left(n-1\right)F^{n}\left(\theta\right)\right)d\theta+\sum_{i=1}^{n-1}\int_{\widetilde{\theta}_i}^{\widetilde{\theta}_n} g'\left(\theta\right)\ \left(\alpha_i-F^{n-1}\left(\theta\right)\right)\ d\theta$$
$$+\left(n-1\right)\int_{\underline{\theta}}^{\widetilde{\theta}_n} g'\left(\theta\right)\ F^{n}\left(\theta\right)\ d\theta$$
$$\geq \sum_{i=1}^{n-1}\int_{\widetilde{\theta}_i}^{\widetilde{\theta}_{i+1}} g'\left(\theta\right)\ \left(\sum_{j=1}^{i}\alpha_j-iF^{n-1}\left(\theta\right)+\left(n-1\right)F^{n}\left(\theta\right)\right)\ d\theta$$

Since for all $i,\ 1\leq i\leq n-1,\ iF^{n-1}\left(\theta\right)-\left(n-1\right)F^{n}\left(\theta\right)\leq\left(\frac{i}{n}\right)^{n}$ the last expression is non-negative. Q.E.D.

Proof of Theorem 6: It is sufficient to show that for any function $v_i=g\left(\theta_i\right)+h\left(\theta_{-i}\right)$ with $h'>0$ there exists a distribution function F such that efficient trade fails for $\alpha_1=\cdots=\alpha_n=\frac{1}{n}$. Integration by parts shows that condition 2 is equivalent to:

$$\int_{\widetilde{\theta}}^{\overline{\theta}} g'\left(\theta\right)\ \left(1-nF^{n-1}\left(\theta\right)+\left(n-1\right)F^{n}\left(\theta\right)\right)\ d\theta+\left(n-1\right)\int_{\underline{\theta}}^{\widetilde{\theta}} g'\left(\theta\right)\ F^{n}\left(\theta\right)\ d\theta$$
$$+\int_{\underline{\theta}}^{\overline{\theta}} h'\left(\theta\right)\ \left(F^{n}\left(\theta\right)-F(\theta)\right)\ d\theta\geq 0.$$

Let $a:=\max_{\theta\in\left[\underline{\theta},\overline{\theta}\right]}g'(\theta)>0$ and $b:=\min_{\theta\in\left[\underline{\theta},\overline{\theta}\right]}h'(\theta)>0$. Since $1-nF^{n-1}\left(\theta\right)+\left(n-1\right)F^{n}\left(\theta\right)\geq 0$ it suffices to show that there exists a distribution F such that

$$a\int_{\widetilde{\theta}}^{\overline{\theta}}\ \left(1-nF^{n-1}\left(\theta\right)+\left(n-1\right)F^{n}\left(\theta\right)\right)\ d\theta+a\left(n-1\right)\int_{\underline{\theta}}^{\widetilde{\theta}}\ F^{n}\left(\theta\right)\ d\theta$$
$$+b\int_{\underline{\theta}}^{\overline{\theta}}\ \left(F^{n}\left(\theta\right)-F(\theta)\right)\ d\theta<0. \tag{14}$$

We first show that this is the case for the discontinuous distribution F^* given by:

$$F^*\left(\theta\right)=\begin{cases}\left(\frac{1}{n}\frac{b}{(n-1)a+b}\right)^{\frac{1}{n-1}} & \text{if}\quad \theta\in\left[\underline{\theta},\overline{\theta}\right)\\ 1 & \text{if}\quad \theta=\overline{\theta}\end{cases}.$$

We set $\widetilde{\theta} = \overline{\theta}$ because $F^*(\theta) < \left(\frac{1}{n}\right)^{\frac{1}{n-1}}$ for all $\theta < 1$. Calculating (14) for F^* yields:

$$\begin{aligned}
& a(n-1)\int_{\underline{\theta}}^{\overline{\theta}} \left(\frac{1}{n}\frac{b}{(n-1)a+b}\right)^{\frac{n}{n-1}} d\theta \\
& +b\int_{\underline{\theta}}^{\overline{\theta}} \left(\left(\frac{1}{n}\frac{b}{(n-1)a+b}\right)^{\frac{n}{n-1}} - \left(\frac{1}{n}\frac{b}{(n-1)a+b}\right)^{\frac{1}{n-1}}\right) d\theta \\
= & \left(\overline{\theta}-\underline{\theta}\right)\left[(a(n-1)+b)\left(\frac{1}{n}\frac{b}{(n-1)a+b}\right) - b\right]\left(\frac{1}{n}\frac{b}{(n-1)a+b}\right)^{\frac{1}{n-1}} < 0.
\end{aligned}$$

We now construct a sequence off cumulative distribution functions that are feasible and "arbitrarily close" to F^*. Therefore, such distribution functions will also violate the existence condition. Let $\theta_M := \frac{(\overline{\theta}-\underline{\theta})}{2}$ and $K := \left(\frac{1}{n}\frac{b}{(n-1)a+b}\right)^{\frac{1}{n-1}}$, and consider the following sequence of cumulative distribution functions[1] F_m for odd $m > 1$ [2]:

$$F_m(\theta) = \begin{cases} \left(\frac{\theta-\theta_M}{\overline{\theta}-\theta_M}\right)^m (1-K)+K & \text{if} \quad \theta \geq \theta_M \\ K\left(\frac{\theta-\theta_M}{\theta_M-\underline{\theta}}\right)^m + K & \text{if} \quad \theta < \theta_M. \end{cases}$$

For m large enough F_m satisfies condition (14) , which completes the proof.

Q.E.D.

Proof of Theorem 7: The generalized Groves mechanism is given by (see Theorem 3):

$$\begin{aligned}
s_S^*(\theta_S,\theta_B) &= \begin{cases} 1, & \text{if} \quad v_S(\theta_S,\theta_B) \geq v_B(\theta_S,\theta_B) \\ 0, & \text{if} \quad v_S(\theta_S,\theta_B) < v_B(\theta_S,\theta_B) \end{cases}, \\
s_B^*(\theta_S,\theta_B) &= 1 - s_S^*(\theta_S,\theta_B).
\end{aligned}$$

$$t_i^*(\theta) := \begin{cases} 0, & \text{if} \quad s_i^*(\theta) = 1 \\ v_i(\theta_i^*(\theta_{-i}),\theta_{-i}), & \text{if} \quad s_i^*(\theta) \neq 1 \end{cases}, \quad i = S, B,$$

where $\theta_i^*(\theta_{-i})$ is defined by

$$\theta_B^*(\theta_S) = \begin{cases} \theta_B^* : v_B(\theta_S,\theta_B^*) = v_S(\theta_S,\theta_B^*), & \begin{array}{l} \text{if } v_B\left(\theta_S,\overline{\theta}_B\right) > v_S\left(\theta_S,\overline{\theta}_B\right) \\ \text{and } v_B(\theta_S,\underline{\theta}_B) < v_S(\theta_S,\underline{\theta}_B) \end{array} \\ \overline{\theta}_B, & \text{if } v_B\left(\theta_S,\overline{\theta}_B\right) < v_S\left(\theta_S,\overline{\theta}_B\right) \\ \underline{\theta}_B, & \text{if } v_B(\theta_S,\underline{\theta}_B) > v_S(\theta_S,\underline{\theta}_B) \end{cases}$$

[1]Observe that they are strictly increasing and differentiable.

[2]This will yield $\widetilde{\theta}_m = F^{-1}\left(\frac{1}{m^{\frac{1}{m-1}}}\right)$.

$$\theta_S^*(\theta_B) = \begin{cases} \theta_S^* : v_B(\theta_S^*, \theta_B) = v_S(\theta_S^*, \theta_B), & \text{if } v_B(\underline{\theta}_S, \theta_B) > v_S(\underline{\theta}_S, \theta_B) \\ & \text{and } v_B(\overline{\theta}_S, \theta_B) < v_S(\overline{\theta}_S, \theta_B) \\ \overline{\theta}_S, & \text{if } v_B(\overline{\theta}_S, \theta_B) > v_S(\overline{\theta}_S, \theta_B) \\ \underline{\theta}_S, & \text{if } v_B(\underline{\theta}_S, \theta_B) < v_S(\underline{\theta}_S, \theta_B). \end{cases}$$

We can distinguish three different cases.

1. Consider the case where $v_B(\overline{\theta}_S, \underline{\theta}_B) \geq v_S(\overline{\theta}_S, \underline{\theta}_B)$. This implies $v_B(\theta_S, \theta_B) \geq v_S(\theta_S, \theta_B)\ \forall \theta_S, \theta_B$ and therefore the efficient choice is $s_B^*(\theta) = 1$. Hence the payments are given by $t_B^* = 0$ and $t_S^*(\theta_S, \theta_B) = v_S(\overline{\theta}_S, \theta_B)$. Because payments are constant and v_S, v_B are monotone, the worst off types of the seller and the buyer are $\widetilde{\theta}_S = \overline{\theta}_S$ and $\widetilde{\theta}_B = \underline{\theta}_B$, respectively. For the interim utilities of the worst off types of the agents, we obtain $U_S(\overline{\theta}_S) = 0$ and $U_B(\underline{\theta}_B) = E_{\theta_S}[v_B(\theta_S, \underline{\theta}_B)]$. The condition of Theorem 2 therefore simplifies to

$$E_{\theta_S}[v_B(\theta_S, \underline{\theta}_B)] \geq E_{\theta_B}\left[v_S(\overline{\theta}_S, \theta_B)\right].$$

2. Consider now the case $v_B(\underline{\theta}_S, \overline{\theta}_B) \leq v_S(\underline{\theta}_S, \overline{\theta}_B)$. Here "no trade" is always efficient and therefore we have $t_S^* = 0$ and $E_\theta[t_B^*(\theta_S, \theta_B)] = E_{\theta_S}\left[v_B(\overline{\theta}_B, \theta_S)\right] = U_S\left(\widetilde{\theta}_S\right)$ which is just the condition of Theorem 2 in this case.

3. We finally analyze the case where $v_B(\underline{\theta}_S, \overline{\theta}_B) > v_S(\underline{\theta}_S, \overline{\theta}_B)$ and $v_B(\overline{\theta}_S, \underline{\theta}_B) < v_S(\overline{\theta}_S, \underline{\theta}_B)$. Before we can analyze it in detail we need the following connection:

$$s_B^*(\theta) = 1 \Rightarrow \theta_S^*(\theta_B) \geq \theta_S\ \wedge\ \theta_B^*(\theta_S) \leq \theta_B. \tag{15}$$

This is a consequence of the definition of $\theta_S^*(\theta_B)$ and $\theta_B^*(\theta_S)$. We now prove (15). By $s_B^*(\theta) = 1$, we have $v_S(\theta_S, \theta_B) < v_B(\theta_S, \theta_B)$. We can distinguish two cases. If $v_B(\overline{\theta}_S, \theta_B) > v_S(\overline{\theta}_S, \theta_B)$ then we have $\theta_S^*(\theta_B) = \overline{\theta}_S \geq \theta_S$. If $v_B(\overline{\theta}_S, \theta_B) \leq v_S(\overline{\theta}_S, \theta_B)$ we have $v_S(\theta_S^*(\theta_B), \theta_B) - v_B(\theta_S^*(\theta_B), \theta_B) = 0$ because $v_S(\theta_S, \theta_B) < v_B(\theta_S, \theta_B)$. Furthermore, $v_S(\theta_S, \theta_B) - v_B(\theta_S, \theta_B)$ is strictly increasing in θ_S (by SCP). Hence again we get $\theta_S^*(\theta_B) > \theta_S$. The other implication of (15) follows by a similar argument.

We now identify the worst-off types and calculate their interim expected utilities. The seller's interim utility is given by

$$\begin{aligned} U_S(\theta_S) &= E_{\theta_B}[-v_S(\theta_S, \theta_B)\ \mathbf{1}(s_B^*(\theta) = 1) + t_S(\theta_S, \theta_B)] \\ &= E_{\theta_B}[(-v_S(\theta_S, \theta_B)\ + v_S(\theta_S^*(\theta_B), \theta_B))\ \mathbf{1}(s_B^*(\theta) = 1)]. \end{aligned}$$

Because $v_S(\theta_S, \theta_B)$ is strictly monotone increasing in θ_S and because $\mathbf{1}(s_B^*(\theta) = 1)$ is monotone

decreasing in θ_S, we obtain that $U_S(\theta_S)$ is monotone decreasing in θ_S and therefore the worst off type is $\overline{\theta}_S$. We can distinguish two cases (remember that we assume $v_B\left(\underline{\theta}_S, \overline{\theta}_B\right) > v_S\left(\underline{\theta}_S, \overline{\theta}_B\right)$ and $v_B\left(\overline{\theta}_S, \underline{\theta}_B\right) < v_S\left(\overline{\theta}_S, \underline{\theta}_B\right)$).
If $v_B\left(\overline{\theta}_S, \overline{\theta}_B\right) \geq v_S\left(\overline{\theta}_S, \overline{\theta}_B\right)$, we get

$$U_S\left(\overline{\theta}_S\right) = \int_{\theta_B^*\left(\overline{\theta}_S\right)}^{\overline{\theta}_B} \left(v_S\left(\overline{\theta}_S, \theta_B\right) - v_S\left(\overline{\theta}_S, \theta_B\right)\right) f_B(\theta_B)\, d\theta_B = 0,$$

because $\theta_B \in \left[\theta_B^*\left(\overline{\theta}_S\right), \overline{\theta}_B\right] \Rightarrow \theta_S^*(\theta_B) = \overline{\theta}_S$.
If $v_B\left(\overline{\theta}_S, \overline{\theta}_B\right) < v_S\left(\overline{\theta}_S, \overline{\theta}_B\right)$, we get

$$U_S\left(\overline{\theta}_S\right) = \int_{\underline{\theta}_B}^{\overline{\theta}_B} \left(v_S\left(\theta_S^*(\theta_B), \theta_B\right) - v_S\left(\overline{\theta}_S, \theta_B\right)\right) f_B(\theta_B)\, d\theta_B = 0.$$

The buyer's interim utility is given by

$$\begin{aligned} U_B(\theta_B) &= E_{\theta_S}\left[v_B(\theta_S, \theta_B)\ \mathbf{1}\left(s_B^*(\theta) = 1\right) + v_B\left(\theta_S, \theta_B^*(\theta_S)\right)\ \mathbf{1}\left(s_B^*(\theta) = 0\right)\right] \\ &= E_{\theta_S}\left[\left(v_B(\theta_S, \theta_B)\ - v_B\left(\theta_S, \theta_B^*(\theta_S)\right)\right)\ \mathbf{1}\left(s_B^*(\theta) = 1\right)\right] + \\ &\quad + E_{\theta_S}\left[v_B\left(\theta_S, \theta_B^*(\theta_S)\right)\right]. \end{aligned}$$

This is increasing in θ_B and therefore the buyer's worst off type is $\underline{\theta}_B$ .We again distinguish two cases.
If $v_B\left(\underline{\theta}_S, \underline{\theta}_B\right) \geq v_S\left(\underline{\theta}_S, \underline{\theta}_B\right)$, this implies

$$\begin{aligned} U_B\left(\underline{\theta}_B\right) &= \int_{\underline{\theta}_S}^{\theta_S^*\left(\underline{\theta}_B\right)} \left(v_B\left(\theta_S, \underline{\theta}_B\right) - v_B\left(\theta_S, \underline{\theta}_B\right)\right) f_S(\theta_S)\, d\theta_S + \\ &\quad + \int_{\underline{\theta}_S}^{\overline{\theta}_S} v_B\left(\theta_S, \theta_B^*(\theta_S)\right) f_S(\theta_S)\, d\theta_S \\ &= \int_{\underline{\theta}_S}^{\overline{\theta}_S} v_B\left(\theta_S, \theta_B^*(\theta_S)\right) f_S(\theta_S)\, d\theta_S. \end{aligned}$$

If $v_B\left(\underline{\theta}_S, \underline{\theta}_B\right) < v_S\left(\underline{\theta}_S, \underline{\theta}_B\right)$, we get

$$\begin{aligned} U_B\left(\underline{\theta}_B\right) &= \int_{\underline{\theta}_S}^{\underline{\theta}_S} \left(v_B\left(\theta_S, \underline{\theta}_B\right) - v_B\left(\theta_S, \theta_B^*(\theta_S)\right)\right) f_S(\theta_S)\, d\theta_S + \\ &\quad + \int_{\underline{\theta}_S}^{\overline{\theta}_S} v_B\left(\theta_S, \theta_B^*(\theta_S)\right) f_S(\theta_S)\, d\theta_S \\ &= \int_{\underline{\theta}_S}^{\overline{\theta}_S} v_B\left(\theta_S, \theta_B^*(\theta_S)\right) f_S(\theta_S)\, d\theta_S. \end{aligned}$$

Hence the sum of worst-off types' interim utilities is

$$U_S\left(\widetilde{\theta}_S\right) + U_B\left(\widetilde{\theta}_B\right) = \int_{\underline{\theta}_S}^{\overline{\theta}_S} v_B\left(\theta_S, \theta_B^*(\theta_S)\right) f_S(\theta_S)\, d\theta_S.$$

Now we calculate

$$
\begin{aligned}
& E_{\theta_S}\left[T_S\left(\theta_S\right)\right]+E_{\theta_B}\left[T_B\left(\theta_B\right)\right] \\
& =\int_{\underline{\theta}_S}^{\overline{\theta}_S} \int_{\underline{\theta}_B}^{\overline{\theta}_B} v_S\left(\theta_S^*\left(\theta_B\right), \theta_B\right) \mathbf{1}\left(s_B^*(\theta)=1\right) f_B\left(\theta_B\right) d\theta_B f_S\left(\theta_S\right) d\theta_S \\
& +\int_{\underline{\theta}_S}^{\overline{\theta}_S} \int_{\underline{\theta}_B}^{\overline{\theta}_B} v_B\left(\theta_S, \theta_B^*\left(\theta_S\right)\right) \mathbf{1}\left(s_B^*(\theta)=0\right) f_B\left(\theta_B\right) d\theta_B f_S\left(\theta_S\right) d\theta_S \\
& =\int_{\underline{\theta}_S}^{\overline{\theta}_S} \int_{\underline{\theta}_B}^{\overline{\theta}_B}\left(v_S\left(\theta_S^*\left(\theta_B\right), \theta_B\right)-v_B\left(\theta_S, \theta_B^*\left(\theta_S\right)\right)\right) \mathbf{1}\left(s_B^*(\theta)=1\right) f_B\left(\theta_B\right) d\theta_B f_S\left(\theta_S\right) d\theta_S \\
& +\int_{\underline{\theta}_S}^{\overline{\theta}_S} v_B\left(\theta_S, \theta_B^*\left(\theta_S\right)\right) f_S\left(\theta_S\right) d\theta_S .
\end{aligned}
$$

Therefore the existence condition of Theorem 2:

$$E_{\theta_S}\left[T_S\left(\theta_S\right)\right]+E_{\theta_B}\left[T_B\left(\theta_B\right)\right] \leq U_B\left(\widetilde{\theta}_B\right)+U_S\left(\widetilde{\theta}_S\right)$$

is equivalent to

$$
\begin{aligned}
& \int_{\underline{\theta}_S}^{\overline{\theta}_S} \int_{\underline{\theta}_B}^{\overline{\theta}_B}\left(v_S\left(\theta_S^*\left(\theta_B\right), \theta_B\right)-v_B\left(\theta_S, \theta_B^*\left(\theta_S\right)\right)\right) \mathbf{1}\left(s_B^*(\theta)=1\right) f_B\left(\theta_B\right) d\theta_B f_S\left(\theta_S\right) d\theta_S \\
& \leq 0 .
\end{aligned}
$$

To show the second statement of the Theorem assume now that agent's valuations are increasing in other agent's types. Given the first statement of the Theorem we only have to show that:

1. $$\int_{\underline{\theta}_S}^{\overline{\theta}_S} \int_{\underline{\theta}_B}^{\overline{\theta}_B}\left(v_S\left(\theta_S^*\left(\theta_B\right), \theta_B\right)-v_B\left(\theta_S, \theta_B^*\left(\theta_S\right)\right)\right) \mathbf{1}\left(s_B^*(\theta)=1\right) f_B\left(\theta_B\right) d\theta_B f_S\left(\theta_S\right) d\theta_S>0$$

   if $v_B\left(\underline{\theta}_S, \overline{\theta}_B\right)>v_S\left(\underline{\theta}_S, \overline{\theta}_B\right)$ and $v_B\left(\overline{\theta}_S, \underline{\theta}_B\right)<v_S\left(\overline{\theta}_S, \underline{\theta}_B\right)$.

2. The condition $E_{\theta_S}\left[v_B\left(\theta_S, \underline{\theta}_B\right)\right] \geq E_{\theta_B}\left[v_S\left(\overline{\theta}_S, \theta_B\right)\right]$ can only be satisfied in the trivial case $v_B\left(\theta_S, \theta_B\right)>v_S\left(\theta_S, \theta_B\right) \; \forall \theta_S, \theta_B$.

The second statement is equivalent to

$$\exists \theta_S, \theta_B \text{ with } v_B\left(\theta_S, \theta_B\right) \leq v_S\left(\theta_S, \theta_B\right) \;\; \Rightarrow E_{\theta_S}\left[v_B\left(\theta_S, \underline{\theta}_B\right)\right]<E_{\theta_B}\left[v_S\left(\overline{\theta}_S, \theta_B\right)\right]$$

which holds because of

$$
\begin{aligned}
\exists \theta_S, \theta_B \text{ with } v_B\left(\theta_S, \theta_B\right) & \leq v_S\left(\theta_S, \theta_B\right) \\
& \Rightarrow v_B\left(\overline{\theta}_S, \underline{\theta}_B\right) \leq v_S\left(\overline{\theta}_S, \underline{\theta}_B\right) \\
& \Rightarrow E_{\theta_S}\left[v_B\left(\overline{\theta}_S, \underline{\theta}_B\right)\right] \leq E_{\theta_B}\left[v_S\left(\overline{\theta}_S, \underline{\theta}_B\right)\right] \\
& \Rightarrow E_{\theta_S}\left[v_B\left(\theta_S, \underline{\theta}_B\right)\right]<E_{\theta_B}\left[v_S\left(\overline{\theta}_S, \theta_B\right)\right] .
\end{aligned}
$$

To prove the first statement we show that $v_S\left(\theta_S^*\left(\theta_B\right),\theta_B\right) > v_B\left(\theta_S,\theta_B^*\left(\theta_S\right)\right)$ if $s_B^*\left(\theta\right) = 1$ (modulo sets of measure zero).

If $\theta_S^*\left(\underline{\theta}_B\right) < \theta_S$ and $\theta_B < \theta_B^*\left(\overline{\theta}_S\right)$ we have

$$v_B\left(\theta_S,\theta_B^*\left(\theta_S\right)\right) = v_S\left(\theta_S,\theta_B^*\left(\theta_S\right)\right) < v_S\left(\theta_S^*\left(\theta_B\right),\theta_B\right).$$

If $\theta_S^*\left(\underline{\theta}_B\right) < \theta_S$ and $\theta_B^*\left(\overline{\theta}_S\right) < \theta_B$ we have

$$v_B\left(\theta_S,\theta_B^*\left(\theta_S\right)\right) = v_S\left(\theta_S,\theta_B^*\left(\theta_S\right)\right) < v_S\left(\overline{\theta}_S,\theta_B^*\left(\overline{\theta}_S\right)\right) < v_S\left(\overline{\theta}_S,\theta_B\right) = v_S\left(\theta_S^*\left(\theta_B\right),\theta_B\right).$$

If $\theta_S < \theta_S^*\left(\underline{\theta}_B\right)$ and $\theta_B < \theta_B^*\left(\overline{\theta}_S\right)$ we have

$$\begin{aligned} v_B\left(\theta_S,\theta_B^*\left(\theta_S\right)\right) &= v_B\left(\theta_S,\underline{\theta}_B\right) < v_B\left(\theta_S^*\left(\underline{\theta}_B\right),\underline{\theta}_B\right) = v_S\left(\theta_S^*\left(\underline{\theta}_B\right),\underline{\theta}_B\right) < v_S\left(\theta_S^*\left(\underline{\theta}_B\right),\theta_B\right) \\ &= v_S\left(\theta_S^*\left(\theta_B\right),\theta_B\right). \end{aligned}$$

If $\theta_S < \theta_S^*\left(\underline{\theta}_B\right)$ and $\theta_B^*\left(\overline{\theta}_S\right) < \theta_B$ we have

$$v_B\left(\theta_S,\theta_B^*\left(\theta_S\right)\right) = v_B\left(\theta_S,\underline{\theta}_B\right) < v_B\left(\theta_S^*\left(\underline{\theta}_B\right),\underline{\theta}_B\right) = v_S\left(\theta_S^*\left(\underline{\theta}_B\right),\underline{\theta}_B\right) < v_S\left(\theta_S^*\left(\theta_B\right),\theta_B\right).$$

Q.E.D.

A.2 Partnerships and Double Auctions with Interdependent Valuations

I will use the notations and definitions introduced in section 2.2. In a direct revelation mechanisms (DRM) agents report their types, relinquish their share of the partnership, and then receive a payment $t_i(\theta)$ and a share $s_i(\theta)$ of the entire partnership. A DRM is therefore a game form $\Gamma = \left(\left[\underline{\theta}_1, \overline{\theta}_1\right], \left[\underline{\theta}_2, \overline{\theta}_2\right], s, t\right)$, where $s(\theta) = (s_1(\theta), s_2(\theta))$ is a vector with components $s_i : \left[\underline{\theta}, \overline{\theta}\right]^2 \mapsto [0,1]$ such that $s_1(\theta) + s_2(\theta) = 1$ for all θ, and $t(\theta) = (t_1(\theta), t_2(\theta))$ is a vector with components $t_i : \left[\underline{\theta}, \overline{\theta}\right]^2 \mapsto \mathbb{R}$. I call s the allocation rule and t the payments. I refer to the pair (s,t) as a DRM if it is clear which strategy sets $\left[\underline{\theta}, \overline{\theta}\right]$ are meant.
For some proofs I need a generalization of the Revenue-Equivalence-Theorem to environments with interdependent valuations of the form $v_i(\theta_1, \theta_2) = g(\theta_i) + h(\theta_{-i})$.

Theorem 17 *(Revenue-Equivalence-Theorem)*
A DRM (s,t) is incentive compatible if and only if the following holds for $i = 1,2$:
a) $\overline{s}_i(\theta_i) := \int_{\underline{\theta}}^{\overline{\theta}} \left(s_i(\theta_i, \theta_{-i}) - \frac{1}{2}\right) f(\theta_{-i})\, d\theta_{-i}$ *is increasing in* θ_i,
b) For all $\widetilde{\theta}_i, \theta_i \in \left[\underline{\theta}, \overline{\theta}\right]$ *we have:* $U_i(\theta_i) = U_i\left(\widetilde{\theta}_i\right) + \int_{\widetilde{\theta}_i}^{\theta_i} g'(t)\, \overline{s}_i(t)\, dt$

Proof. The proof is almost identical to the independent private values case and therefore omitted. ■

A.2.1 Proofs

Proof of Theorem 8:
In this proof I will denote an equilibrium of the $k-$double auction by $(b_1(\theta_1), b_2(\theta_2))$ where $b_i(\theta_i)$ denotes the equilibrium bidding strategy of agent i. The agent other than i is denoted by $-i$. Throughout the proof I will assume $k \in (0,1)$. The cases $k = 1$ and $k = 2$ are indeed simpler to prove and can be found for a similar model in Bulow et al. [1999].
I will summarize the different steps to illustrate the logic behind the whole proof: In the first step I will show that the equilibrium has to fulfill a (symmetric) system of differential equations if it is continuous and strictly increasing. In the 2nd step I will show that an equilibrium bidding strategy $b_i(\theta_i)$ can only be decreasing if there is a gap in $b_{-i}(\theta_{-i})$ at $\theta_{-i} = F^{-1}(k)$. In step 3, I will show that there cannot be atoms (i.e. a positive measure of types submitting the same bid) in the equilibrium bidding functions of both agents at the same bid. In the 4th step I will show that the bids of the highest types have to be the same for both bidders and that this is also the case for the bids of the lowest types, i.e. $b_1(\underline{\theta}) = b_2(\underline{\theta})$ and $b_1\left(\overline{\theta}\right) = b_2\left(\overline{\theta}\right)$. We also get that $b_i(\underline{\theta}) > v_i(\underline{\theta}, \underline{\theta})$ and $b_i\left(\overline{\theta}\right) < v_i\left(\overline{\theta}, \overline{\theta}\right)$. I will derive necessary conditions for equilibrium bidding functions to have atoms (step 5) or gaps (step 6). Step 7 and 8 will show that the differential equations determine a unique solution if starting from an initial condition $b_i(\underline{\theta}) = \underline{b}$ (or

$b_i\left(\overline{\theta}\right)=\overline{b}$) we increase (decrease) θ_i as long as either $\theta_i = F^{-1}(k)$ or $b_i(\theta_i) = v_i(\theta_i, \theta_i)$. In the 9th step I will show that for $\theta_i = F^{-1}(k)$ we get that $b_i(\theta_i) = v_i(\theta_i, \theta_i)$ and furthermore that even at $\theta_i = F^{-1}(k)$ the equilibrium bidding strategies are continuous. Hence the equilibrium bidding strategies are strictly increasing (have no atoms) and are continuous (have no gaps). This shows that the equilibrium has to fulfill the symmetric system of differential equations derived in step 1 and therefore is symmetric. In the last step (10) I will show that it is unique, i.e. only one possible initial condition $b_i(\underline{\theta}) = \underline{b}$ and $b_i\left(\overline{\theta}\right) = \overline{b}$ can be fulfilled.

The steps in detail:

1. Assume that for a given range $\left(b^L, b^H\right)$ b_1, b_2 are continuous and strictly increasing on $b_1^{-1}\left(\left(b^L, b^H\right)\right)$ and $b_2^{-1}\left(\left(b^L, b^H\right)\right)$ respectively and all types of player i that are lower than all types in $b_i^{-1}\left(\left(b^L, b^H\right)\right)$ bid below b^L and all types of player i that are higher than all types in $b_i^{-1}\left(\left(b^L, b^H\right)\right)$ bid above b^H.

   The utility of a type θ_i-bidder submitting a bid $b \in \left(b^L, b^H\right)$ is given by

$$\begin{aligned} U_i(\theta_i, b) &= \frac{1}{2}\int_{\underline{\theta}}^{b_{-i}^{-1}(b)} \left(v_i(\theta_1, \theta_2) - \left((1-k)b + k b_{-i}(\theta_{-i})\right)\right) f(\theta_{-i})\, d\theta_{-i} \\ &\quad + \frac{1}{2}\int_{b_{-i}^{-1}(b)}^{\overline{\theta}} \left(kb + (1-k) b_{-i}(\theta_{-i}) - v_i(\theta_1, \theta_2)\right) f(\theta_{-i})\, d\theta_{-i}. \end{aligned}$$

   Because b_1, b_2 are continuous and strictly increasing and therefore a.e. differentiable on $b_1^{-1}\left(\left(b^L, b^H\right)\right)$ and $b_2^{-1}\left(\left(b^L, b^H\right)\right)$ respectively, the same is true for the inverse functions $b_i^{-1}(b)$ on $\left(b^L, b^H\right)$. Differentiating with respect to b yields the following local first order conditions:

$$\begin{aligned} \left(v_1\left(b_1^{-1}(b), b_2^{-1}(b)\right) - b\right) f\left(b_2^{-1}(b)\right) \frac{\partial b_2^{-1}(b)}{\partial b} - \frac{1}{2}\left(F\left(b_2^{-1}(b)\right) - k\right) &= 0, \\ \left(v_2\left(b_1^{-1}(b), b_2^{-1}(b)\right) - b\right) f\left(b_1^{-1}(b)\right) \frac{\partial b_1^{-1}(b)}{\partial b} - \frac{1}{2}\left(F\left(b_1^{-1}(b)\right) - k\right) &= 0. \end{aligned} \tag{16}$$

   Note that this system of differential equations is given for the inverse functions of the equilibrium bidding functions. Since it is Lipschitz-continuous if $v_1\left(b_1^{-1}(b), b_2^{-1}(b)\right) \neq b$ and $v_2\left(b_1^{-1}(b), b_2^{-1}(b)\right) \neq b$ it uniquely determines b_i^{-1} given an "initial condition" on intervals where b_i^{-1} are strictly increasing, continuous and $v_1\left(b_1^{-1}(b), b_2^{-1}(b)\right) \neq b$, $v_2\left(b_1^{-1}(b), b_2^{-1}(b)\right) \neq b$. In particular the functions b_i are also uniquely determined on the range of such an interval. If we have $b_1^{-1} = b_2^{-1}$ we have Lipschitz-continuity if $v_1\left(b_1^{-1}(b), b_1^{-1}(b)\right) \neq b$ or equivalently $b_1(\theta) \neq v_1(\theta, \theta)$.

2. If b_i is (locally) decreasing, i.e. if we have $b_i(\theta_i^*) > b_i(\theta_i^{**})$ for some $\theta_i^* < \theta_i^{**}$ the following holds:

$$\Pr_{\theta_{-i}}\{b_{-i}(\theta_{-i}) \in [b_i(\theta_i^{**}), b_i(\theta_i^*)]\} = 0$$

and

$$\Pr_{\theta_{-i}}\{b_{-i}(\theta_{-i}) < b_i(\theta_i^{**})\} = k.$$

The proof of this statement follows standard revealed preferences arguments.

3. It is impossible that a positive measure of types of agents 1 and 2 submit the same bid, i.e. for all b we have:

$$\Pr_{\theta_i}\{b_i(\theta_i) = b\} > 0 \Rightarrow \Pr_{\theta_{-i}}\{b_{-i}(\theta_{-i}) = b\} = 0.$$

If the contrary statement was true, a bidder would prefer to increase or decrease her bid slightly since this would hardly change payments but significantly change her probability of winning or loosing[3].

4. We have

$$b_1(\underline{\theta}) = b_2(\underline{\theta}) \neq b_1(\overline{\theta}) = b_2(\overline{\theta}),$$

$$b_i(\theta) \in [b_1(\underline{\theta}), b_1(\overline{\theta})]$$

and

$$b_1(\underline{\theta}) > v_1(\underline{\theta}, \underline{\theta}), \quad b_1(\overline{\theta}) < v_1(\overline{\theta}, \overline{\theta}).$$

Assume without loss in generality that $\inf_\theta b_1(\theta) > \inf_\theta b_2(\theta)$ (I allow the last value to be $-\infty$). Then it is profitable for a type of player 2 who bids below $\inf_\theta b_1(\theta)$ to increase her bid such that she still loses against all types of player 1. Therefore we must have

$$\inf_\theta b_1(\theta) = \inf_\theta b_2(\theta).$$

[3]This is in contrast to the case analyzed in Leininger et al. [1989] where a buyer and a seller trade a good using an $\frac{1}{2}-$double auction. There in case of equal bids, the good is allocated to the buyer and it is possible to construct step function equilibria. For a buyer it is either optimal to bid $\underline{\theta}$ or to bid in the range of the seller's strategy, since otherwise a buyer could lower the price without changing winning probabilities by lowering her bid. A similar argument holds for the seller and it can be shown that such step functions may form an equilibrium. In the model of this section a given bid is (almost) always higher or lower than a bidder's valuation hence it is always optimal to trade which does not happen if partners submit the same bid.

The monotonicity condition 2. implies that we must have

$$b_1(\underline{\theta}) = b_2(\underline{\theta}) = \inf_{\theta} b_1(\theta) = \inf_{\theta} b_2(\theta).$$

An analogues argument shows that

$$b_1(\overline{\theta}) = b_2(\overline{\theta}) = \sup_{\theta} b_1(\theta) = \sup_{\theta} b_2(\theta).$$

We also obtain $b_1(\underline{\theta}) \neq b_1(\overline{\theta})$ from 2.
In addition we know that $b_1(\underline{\theta}) > v_1(\underline{\theta}, \underline{\theta})$. This is because we cannot have an atom at $b_1(\underline{\theta})$ in both agents' strategies and therefore if we had $b_1(\underline{\theta}) \leq v_1(\underline{\theta}, \underline{\theta})$ at least for one agent raising her bid by a small ε gains $\frac{1}{2}k\varepsilon$ when she sells (with probability close to one) and loses less then $\frac{1}{2}(1-k)\varepsilon$ when she buys (with arbitrarily small probability). A similar reasoning shows that $b_1(\overline{\theta}) < v_1(\overline{\theta}, \overline{\theta})$.

5. It is only possible to have an atom at $\widetilde{b}$ (i.e. a positive measure of types bidding $\widetilde{b}$) in the bidding function of agent i if there is either a gap in the equilibrium bidding function of the other agent below or above $\widetilde{b}$ or if for $\theta_{-i} := b_{-i}^{-1}(\widetilde{b})$ we have $E_{\theta_i}[v_{-i}(\theta_1, \theta_2) \mid b_i(\theta_i) = \widetilde{b}] = \widetilde{b}$. This is because a small change in the bid for types bidding close to $\widetilde{b}$ does hardly change payments but significantly changes the probability of winning and losing. Therefore it is profitable to increase the bid slightly above $\widetilde{b}$ instead of bidding just below $\widetilde{b}$ if the expected value for the partnership is higher than its price, i.e. if $E_{\theta_i}[v_{-i}(\theta_1, \theta_2) \mid b_i(\theta_i) = \widetilde{b}] > \widetilde{b}$, or to lower the bid from just above $\widetilde{b}$ to just below $\widetilde{b}$ if $E_{\theta_i}[v_{-i}(\theta_1, \theta_2) \mid b_i(\theta_i) = \widetilde{b}] < \widetilde{b}$.

6. I will show that if there is a gap between b^* and b** in the equilibrium bidding function of agent i and we have

$$(1-k) \Pr_{\theta_i}\{b_i(\theta_i) \leq b^*\} < k \Pr_{\theta_i}\{b_i(\theta_i) \geq b^{**}\}.$$

Then we must have

$$(1-k) \Pr_{\theta_{-i}}\{b_{-i}(\theta_{-i}) \leq b^*\} \geq k \Pr_{\theta_{-i}}\{b_{-i}(\theta_{-i}) \geq b^{**}\}.$$

I assume without loss in generality that $\Pr_{\theta_i}\{b_i(\theta_i) \in (b^*, b^{**})\} = 0$ and $\Pr_{\theta_i}\{b_i(\theta_i) \in (b^* - \varepsilon, b^*]\} > 0$, $\Pr_{\theta_i}\{b_i(\theta_i) \in [b^{**}, b^{**} + \varepsilon)\} > 0$ for all $\varepsilon > 0$. Note that $(1-k)\Pr_{\theta_i}\{b_i(\theta_i) \leq b^*\} < k \Pr_{\theta_i}\{b_i(\theta_i) \geq b^{**}\}$ implies that there is also a gap in the bidding function of agent $-i$ between b^* and b^{**} since a sufficiently small increase of a bid within the interval (b^*, b^{**}) of agent $-i$ leads to higher expected payments without changing the winning (and loosing) probability. Because of 3. we cannot have atoms at b^* in the equilibrium bidding strategies of both players. Therefore if we had $(1-k)\Pr_{\theta_{-i}}\{b_{-i}(\theta_{-i}) \leq b^*\} < k \Pr_{\theta_{-i}}\{b_{-i}(\theta_{-i}) \geq b^{**}\}$

at least one player could gain by increasing her bid from b^* to just below b^{**} since this leads to higher expected payments without changing the winning (and loosing) probability.
Similar arguments show that if we have

$$(1-k)\Pr_{\theta_i}\{b_i(\theta_i)\leq b^*\} > k\Pr_{\theta_i}\{b_i(\theta_i)\geq b^{**}\}$$

then we must have

$$(1-k)\Pr_{\theta_{-i}}\{b_{-i}(\theta_{-i})\leq b^*\} \leq k\Pr_{\theta_{-i}}\{b_{-i}(\theta_{-i})\geq b^{**}\}.$$

7. This part shows that starting from an initial bid $b_1(\underline{\theta}) = \underline{b} > v_i(\underline{\theta},\underline{\theta})$, it is possible to uniquely continue the solution of the differential equation (16) by increasing θ til either $\theta = F^{-1}(k)$ or $b_1(\theta) = v_i(\theta,\theta)$.
Define $\theta^* = \arg\sup\{\theta < F^{-1}(k) \mid b_1(x) = b_2(x) > v_i(x,x)$ for all $x \leq \theta^*\}$ (note that because of 4. θ^* is well defined). I will show that either $\theta^* = F^{-1}(k)$ or $b_1(\theta^*) = v_i(\theta^*,\theta^*)$. If this were not the case we could find θ_ε arbitrarily close to θ^* with $\theta_\varepsilon \in \left(\theta^*, F^{-1}(k)\right)$ and $v_i(\theta,\theta) < b_i(\theta) < b_{-i}(\theta)$ for all[4] $\theta \in (\theta^*,\theta_\varepsilon]$. Note that we cannot have a gap after[5] $b \leq b_i(\theta_\varepsilon)$. In addition we cannot have atoms in the equilibrium bidding functions at $b \in [b_i(\theta^*), b_i(\theta_\varepsilon)]$ if θ_ε is sufficiently close to θ^*. Assume there existed an interval $[\theta_j^D, \theta_j^U]$, $\theta_j^D \in [\theta^*,\theta_\varepsilon]$ of agent j bidding $\widetilde{b} \in [b_i(\theta^*), b_i(\theta_\varepsilon)]$. Since there are no gaps in the bidding function of $-j$ after or before $\widetilde{b}$ we have (because of 5.) for $\widetilde{\theta}_{-j} = b_{-j}^{-1}\left(\widetilde{b}\right) \leq \theta_\varepsilon$ that

$$E_{\theta_j}[v_{-j}\left(\theta_j,\widetilde{\theta}_{-j}\right)\ \mathbf{1}\left(b_j(\theta_j) = \widetilde{b}\right)] = \widetilde{b}.$$

Since we have $v_{-j}(\theta_\varepsilon,\theta_\varepsilon) < b_i(\theta_\varepsilon) < b_{-i}(\theta_\varepsilon)$ this implies $v_{-j}\left(\theta_j^U,\widetilde{\theta}_{-j}\right) > \widetilde{b}$. Therefore if θ_ε is chosen arbitrarily close to θ^* (which is possible) we have $\theta_j^U > \widetilde{\theta}_{-j}$ which implies $v_j\left(\theta_j^U,\widetilde{\theta}_{-j}\right) > v_{-j}\left(\theta_j^U,\widetilde{\theta}_{-j}\right) > \widetilde{b}$.
On the other hand we have $\widetilde{\theta}_{-j} < F^{-1}(k)$ hence type θ^U of agent j wins with a probability smaller than k and loses with a probability greater than k and can therefore improve by raising her bid from $\widetilde{b} < v_j\left(\theta_j^U,\widetilde{\theta}_{-j}\right)$ (and winning against types close to $\widetilde{\theta}_{-j}$ where winning is profitable because of $\widetilde{b} < v_j\left(\theta_j^U,\widetilde{\theta}_{-j}\right)$). Since there are neither gaps nor atoms in $[b_i(\theta^*), b_i(\theta_\varepsilon)]$ (16) prescribes a symmetric solution[6] for $\theta \in (\theta^*,\theta_\varepsilon)$ which is a contradiction to the definition of θ^* and this part

[4]Note that because of 2. bidding strategies cannot decrease in a neighborhood of θ^*.

[5]If we had a gap one of the bidders could improve by increasing her bid from b into the gap.

[6]In fact if b_1^{-1}, b_2^{-1} are differentiable (a.e.) they have to fulfill (16). This is the case since b_1 and b_2 are strictly increasing and continuous (in the considered range) and therefore the same is true for b_1^{-1}, b_2^{-1}.

of the proof is complete.

Since b_1, b_2 can have neither gaps nor atoms in this range (because of 3. and 6.) and are strictly increasing (because of 2.) the same holds for b_1^{-1}, b_2^{-1}. We can therefore, starting from an initial bid $b_1(\underline{\theta}) = \underline{b} > v_i(\underline{\theta}, \underline{\theta})$, uniquely continue the solution of the differential equation (16) by increasing θ until either $\theta = F^{-1}(k)$ or $b_1(\theta) = v_i(\theta, \theta)$. The same reasoning shows that starting with $b_1(\overline{\theta}) = b_2(\overline{\theta}) < v(\overline{\theta}, \overline{\theta})$ if we decrease the type θ_i, b_1 and b_2 are uniquely determined by (16) (and therefore symmetric) as long as $\theta_i > F^{-1}(k)$and $b_i(\theta_i) < v_i(\theta_i, \theta_i)$.

8. If we can show for equilibrium bidding strategies $b_1(\theta_1)$, $b_2(\theta_2)$ that $\theta_i \neq F^{-1}(k)$ implies that $b_i(\theta_i) \neq v_i(\theta_i, \theta_i)$ we have shown that any equilibrium is given by $b_1(\underline{\theta})$ and $b_1(\overline{\theta})$ and the differentiable solution of (16) for $\theta_1 \neq F^{-1}(k)$. Assume without loss in generality that $b_i(\theta_i) > v_i(\theta_i, \theta_i)$ for all $\theta_i < \theta_i^*$ and we have $b_i(\theta_i^*) = v_i(\theta_i^*, \theta_i^*)$ and $\theta_i^* < F^{-1}(k)$. Arguments similar to those used in 7. show that there are neither gaps nor atoms in a small environment around $b_i(\theta_i^*)$ which implies that (16) is valid. Even though its solution is not necessarily unique any more (because $b_i(\theta_i^*) = v_i(\theta_i^*, \theta_i^*)$) we can deduce from (16) that at least for $\theta_i > \theta_i^*$ and close to θ_i^* the derivatives of $b_i(\theta_i)$ are decreasing which is in contrast to 2. and therefore not possible. Again a similar argument shows that we cannot have $b_i(\theta_i^*) = v_i(\theta_i^*, \theta_i^*)$ and $\theta_i^* > F^{-1}(k)$.
   Because of continuity of b_i and v we have $b_i(\theta_i) > v_i(\theta_i, \theta_i)$ for $\theta_i < F^{-1}(k)$ and $b_i(\theta_i) < v_i(\theta_i, \theta_i)$ for $\theta_i > F^{-1}(k)$ which implies (because of 2.) that $b_i(\theta_i)$ is continuous at $\theta_i = F^{-1}(k)$ and we have $b_i(F^{-1}(k)) = v_i(F^{-1}(k), F^{-1}(k))$.

9. From the previous steps we know that any equilibrium (b_1, b_2) has to be symmetric, strictly increasing and must be a solution of the symmetric system of differential equations given by (16). Furthermore an equilibrium is uniquely determined by (16) and the initial conditions $\underline{\theta} = b_i^{-1}(\underline{b})$ and $\overline{\theta} = b_i^{-1}(\overline{b})$ where $\underline{b}$ and $\overline{b}$ denote the lowest and highest bid since $v_i(\theta, \theta) = b(\theta) \Leftrightarrow \theta = F^{-1}(k)$. Therefore any equilibrium must also be a solution of the following differential equation, which is directly derived from (16) by using the symmetry property of the equilibrium:

$$(v_i(\theta, \theta) - b(\theta)) - \frac{1}{2} \frac{F(\theta) - k}{f(\theta)} \frac{db(\theta)}{d\theta} = 0.$$

   This is a linear differential equation and it is easy to verify that its solution for $\theta \neq F^{-1}(k)$ must have the following form[7]:

$$b(\theta) = v_i(\theta, \theta) - \frac{\int_c^\theta V'(t)(F(t) - k)^2 dt}{(F(\theta) - k)^2}, \quad c \in \mathbb{R}.$$

   Since for any equilibrium we have $b(F^{-1}(k)) = v(F^{-1}(k), F^{-1}(k))$ we must have $c = F^{-1}(k)$ and therefore the only possible candidate for an equilibrium is given by 7. Checking the second

[7]The differential equation can be transformed to $\frac{d}{d\theta}(Q(\theta) b(\theta)) = V(\theta) \frac{dQ(\theta)}{d\theta}$ with $Q(\theta) := (F(\theta) - k)^2$. This can be solved by integration.

order condition (which can be done by straight forward calculations) reveals that $(b_1(\theta), b_2(\theta))$ with $b_1(\theta) = b_2(\theta) = b(\theta)$ according to (7) indeed constitutes an equilibrium.

Q.E.D.

Proof of Theorem 9:

The sufficient part is obvious.

Because of Theorem 1 we know that the agents' interim utilities by participation in a mechanism that implements the efficient allocation s^* of the partnership must have the following representations, where

$$\begin{aligned} Q(\theta_i) &= \int_{\widetilde{\theta}_i}^{\theta_i} E_{\theta_{-i}}\left[v_{i,1}(t,\theta_{-i})\left(s_i^*(t,\theta_{-i}) - \frac{1}{2}\right)\right] dt \\ \text{and} \quad R(\theta_i) &= E_{\theta_{-i}}\left[v_i(\theta_i,\theta_{-i})\left(s_i^*(\theta_i,\theta_{-i}) - \frac{1}{2}\right)\right] \end{aligned}$$

do not depend on payments[8]:

1. $U_i(\theta_i) = U_i\left(\widetilde{\theta}_i\right) + Q(\theta_i)$, where $\widetilde{\theta}_i$ denotes the type for which participation is most costly/ least profitable[9].

2. $U_i(\theta_i) = R(\theta_i) + T_i(\theta_i)$, where $T_i(\theta_i)$ are the expected payments to a type θ_i agent.

If there exists an incentive compatible, efficient, budget balanced and individually rational mechanism this mechanism satisfies

$$U_i^M\left(\widetilde{\theta}_i\right) = E_{\theta_i}[R(\theta_i) - Q(\theta_i) + T_i^M(\theta_i)] \geq 0$$

and because of budget balancedness we have $E_\theta[T_1^M(\theta_1) + T_2^M(\theta_2)] \leq 0$ and therefore

$$U_1^M\left(\widetilde{\theta}_1\right) + U_2^M\left(\widetilde{\theta}_2\right) \leq 2E_{\theta_i}[R(\theta_i) - Q(\theta_i)].$$

Since the $k-$double auction is budget balanced and efficient (as a result of Theorem 8) the interim utilities of the "worst-off" types $U_1^{DA}\left(\widetilde{\theta}_1\right) = U_2^{DA}\left(\widetilde{\theta}_2\right)$ in the double auction must satisfy:

$$\begin{aligned} U_1^{DA}\left(\widetilde{\theta}_1\right) + U_2^{DA}\left(\widetilde{\theta}_2\right) &= E_\theta[2R(\theta_1) - 2Q(\theta_1) + T_1^M(\theta_1) + T_2^M(\theta_2)] \\ &= 2E_{\theta_i}[R(\theta_1) - Q(\theta_1)] \\ &\geq U_1^M\left(\widetilde{\theta}_1\right) + U_2^M\left(\widetilde{\theta}_2\right) \geq 0. \end{aligned}$$

Q.E.D.

[8] Because of symmetry Q and R are independent of i.

[9] Note that $\widetilde{\theta}_i$ is the same for all efficient mechanisms. It can easily be shown that $\widetilde{\theta}_i = F^{-1}\left(\frac{1}{2}\right)$.

Proof of Theorem 11:

Instead of directly verifying that a deviation of the given strategy cannot be profitable if the other agent sticks to it, I use the Revenue-Equivalence-Theorem (Theorem 17, page 73) for an indirect proof. Given an allocation rule s, the Revenue-Equivalence-Theorem determines (up to a type-independent constant) the payments (depending on the agents' reported types) necessary and sufficient to implement s in a truthtelling equilibrium. By the revelation principle in any (indirect) mechanism that implements s the expected payments to agents in an equilibrium have to equal those given by the Revenue-Equivalence-Theorem (up to a type independent constant). Furthermore, if the expected payments to agents induced by a candidate of an equilibrium (i.e. (10)) of an indirect mechanism that implements s equal those given by the Revenue-Equivalence-Theorem we know that imitating the strategy of a different type cannot be profitable. Therefore I have to show that the payments induced by the given strategies of the double-auction with veto equal those of a direct mechanism that implements the same allocation as the suggested equilibrium strategies. If in addition I can show that deviating to a bid outside the range of (10) cannot be profitable (10) has to constitute an equilibrium.

I will split the proof in four steps:

1. I will show that the condition

$$F(c) + F(d) = 1$$

is necessary for the induced allocation to result from equilibrium bidding behavior.

2. For general $c, d \in [\underline{\theta}, \overline{\theta}]$ with $F(c) + F(d) = 1$, I will calculate the expected payments of a direct mechanism that implements the allocation that would result from bidding according to (10).

3. For general $c, d \in [\underline{\theta}, \overline{\theta}]$ with $F(c) + F(d) = 1$, I will calculate the expected payments induced by (10) and will show that these equal the payments derived in step 2. if (and only if) (9) holds.

4. I will show that no type has an incentive to bid outside the range of $b(\theta_i)$ as defined by (10).

Step 1: If the agents bid according to (10) this would result in the following allocation:

$$s_{i,c,d}(\theta) := \begin{cases} 1 & if \quad \theta_i > \theta_{-i} \text{ and } \theta_i, \theta_{-i} \notin [c,d] \\ \frac{1}{2} & if \quad \theta_i \in [c,d], \ \theta_{-i} \in [c,d] \\ 0 & if \quad \theta_i \leq \theta_{-i} \text{ and } \theta_i, \theta_{-i} \notin [c,d], \end{cases} \qquad i = 1, 2.$$

Because of the Revenue-Equivalence-Theorem this allocation can only be implemented if

$$\overline{s}_{i,c,d}(\theta_i) = \int_{\underline{\theta}}^{\overline{\theta}} \left(s_{i,c,d}(\theta_i, \theta_{-i}) - \frac{1}{2} \right) f(\theta_{-i}) \, d\theta_{-i}$$

is increasing in θ_i. This is the case iff $F(c)+F(d)=1$. To see this note that for $\theta_i \in [\underline{\theta}, c]$ we have

$$\begin{aligned}
&\int_{\underline{\theta}}^{\overline{\theta}} \left(s_{i,c,d}(\theta_i, \theta_{-i}) - \frac{1}{2} \right) f(\theta_{-i})\, d\theta_{-i} \\
=\;& F(\theta_i) + \frac{1}{2}(F(d) - F(c)) - \frac{1}{2}.
\end{aligned}$$

Similarly for $\theta_i \in [d, 1]$ we have

$$\begin{aligned}
&\int_{\underline{\theta}}^{\overline{\theta}} \left(s_{i,c,d}(\theta_i, \theta_{-i}) - \frac{1}{2} \right) f(\theta_{-i})\, d\theta_{-i} \\
=\;& F(\theta_i) - \frac{1}{2}(F(d) - F(c)) - \frac{1}{2}.
\end{aligned}$$

Therefore $\overline{k}_{i,c,d}(\theta_i)$ is increasing iff

$$\begin{aligned}
F(\theta_i) + \frac{1}{2}(F(d) - F(c)) - \frac{1}{2} &\leq 0 \quad \forall \theta_i \leq c \Leftrightarrow F(c) + F(d) \leq 1 \\
F(\theta_i) - \frac{1}{2}(F(d) - F(c)) - \frac{1}{2} &\geq 0 \quad \forall \theta_i \geq d \Leftrightarrow F(c) + F(d) \geq 1
\end{aligned}$$

Which means that $F(c) + F(d) = 1$.

Step 2: Because of step 1 and Theorem 17 (s,t) is IC iff we have for an arbitrary type $\widetilde{\theta}_i$:

$$U_i(\theta_i) = U_i\left(\widetilde{\theta}_i\right) + \int_{\widetilde{\theta}_i}^{\theta_i} g'(t)\, \overline{s}_{i,c,d}(t)\, dt.$$

Note that this implies that all agents of type $\theta_i \in [c,d]$ must get the same interim utility, which we denote by K. Because of

$$T_i(\theta_i) = U(\theta_i) - E_{\theta_{-i}}\left[(g(\theta_i) + h(\theta_{-i})) \left(s_{i,c,d}(\theta_i, \theta_{-i}) - \frac{1}{2} \right) \right]$$

it follows immediately that $T_i(\theta_i) = K$ if $\theta_i \in [c,d]$. If $\theta_i \in [\underline{\theta}, c] \cup [d, \overline{\theta}]$ straight forward calculations result in

$$\begin{aligned}
T_i(\theta_i) \;=\;& K - (g(\theta_i) + h(\theta_i)) F(\theta_i) + \int_c^{\theta_1} g'(t) F(t)\, dt + \frac{1}{2} \int_c^{\theta_i} h'(t) F(t)\, dt \\
&+ \frac{1}{2} \int_{\underline{\theta}}^{\theta_1} h'(t) F(t)\, dt - \frac{1}{2} \int_d^{\overline{\theta}} h'(t) F(t)\, dt \\
&+ \frac{1}{2} h\left(\overline{\theta}\right) - \frac{1}{2} h(d) F(d) + g(c) F(c) + \frac{1}{2} h(c) F(c),
\end{aligned} \tag{17}$$

if $\theta_i \in [c,d]$ and if $\theta_i \in [d, \overline{\theta}]$ then we get

$$\begin{aligned}
T_i(\theta_i) \;=\;& K - (g(\theta_i) + h(\theta_i)) F(\theta_i) + \int_d^{\theta_1} g'(t) F(t)\, dt + \frac{1}{2} \int_{\underline{\theta}}^{c} h'(t) F(t)\, dt \\
&+ \frac{1}{2} \int_d^{\theta_1} h'(t) F(t)\, dt - \frac{1}{2} \int_{\theta_1}^{\overline{\theta}} h'(t) F(t)\, dt \\
&+ \frac{1}{2} h\left(\overline{\theta}\right) + \frac{1}{2} h(d) F(d) + g(d) F(d) - \frac{1}{2} h(c) F(c).
\end{aligned} \tag{18}$$

Step 3: We have to check whether the expected payments induced by bidding according to $b_i(\theta_i)$ defined by (10) equal those derived in the previous step. If this is the case we know that no agent can profit by deviating to another bid in the range of the given bidding function or by vetoing in case he has not vetoed before.

First note that $b(\theta_i)$ is strictly increasing in $\theta_i \in [\underline{\theta}, c) \cup (d, \overline{\theta}]$. Using l'Hôpital's rule we get

$$\lim_{\theta_i \to c} b_i(\theta_i) = g(c) + h(c)$$

and

$$\lim_{\theta_i \to d} b_i(\theta_i) = g(d) + h(d).$$

In a next step I will show that the expected payments to the agents equal those derived in step 2. For $\theta_i \in [c, d]$ this is obviously the case if $K = 0$. Consider first the case $\theta_i \in [\underline{\theta}, c)$. Given the rules of the auction we have:

$$\begin{aligned}
T_i(\theta_i) &= -\int_{\underline{\theta}}^{\theta_i} \frac{b(\theta_i) + b(\theta_{-i})}{4} f(\theta_{-i})\, d\theta_{-i} + \int_{\theta_i}^{c} \frac{b(\theta_i) + b(\theta_{-i})}{4} f(\theta_{-i})\, d\theta_{-i} \\
&\quad + \int_{d}^{\overline{\theta}} \frac{b(\theta_i) + b(\theta_{-i})}{4} f(\theta_{-i})\, d\theta_{-i} \\
&= -\frac{1}{2}(g(\theta_i) + h(\theta_i)) F(\theta_i) + \frac{1}{2}(g(\theta_i) + h(\theta_i)) F(c) \\
&\quad + \frac{1}{2} \frac{\int_c^{\theta_i} (g'(t) + h'(t)) (F(t) - F(c))^2\, dt}{F(\theta_i) - F(c)} \\
&\quad - \frac{1}{4} \int_{\underline{\theta}}^{\theta_i} (g(t) + h(t)) f(t)\, dt + \frac{1}{4} \int_{\theta_i}^{c} (g(t) + h(t)) f(t)\, dt \\
&\quad + \frac{1}{4} \int_{d}^{\overline{\theta}} (g(t) + h(t)) f(t)\, dt \\
&\quad + \frac{1}{4} \int_{\underline{\theta}}^{\theta_i} \frac{\int_c^{\theta_{-i}} (g'(t) + h'(t)) (F(t) - F(c))^2\, dt}{(F(\theta_{-i}) - F(c))^2} f(\theta_{-i})\, d\theta_{-i} \\
&\quad - \frac{1}{4} \int_{\theta_i}^{c} \frac{\int_c^{\theta_{-i}} (g'(t) + h'(t)) (F(t) - F(c))^2\, dt}{(F(\theta_{-i}) - F(c))^2} f(\theta_{-i})\, d\theta_{-i} \\
&\quad - \frac{1}{4} \int_{d}^{\overline{\theta}} \frac{\int_d^{\theta_{-i}} (g'(t) + h'(t)) (F(t) - F(c))^2\, dt}{(F(\theta_{-i}) - F(d))^2} f(\theta_{-i})\, d\theta_{-i} \Bigg).
\end{aligned}$$

Integration by parts and using the fact that

$$\lim_{\theta_i \to c} \frac{\int_c^{\theta_i} (g'(t) + h'(t)) (F(t) - F(c))^2\, dt}{F(\theta_i) - F(c)} = \lim_{\theta_i \to c} \frac{(g'(\theta_i) + h'(\theta_i)) (F(\theta_i) - F(c))^2\, dt}{f(\theta_i)} = 0$$

gives:

$$
\begin{aligned}
T_i(\theta_i) \;=\;& -\left(g(\theta_i)+h(\theta_i)\right)F(\theta_i)+\int_c^{\theta_i} g'(t)\,F(t)\,dt+\frac{1}{2}\int_c^{\theta_i} h'(t)\,F(t)\,dt\\
&+\frac{1}{2}\int_{\underline{\theta}}^{\theta_i} h'(t)\,F(t)\,dt+\frac{1}{2}\int_{\underline{\theta}}^{c} g'(t)\,F(t)\,dt\\
&+\frac{1}{4}\left(g(\underline{\theta})+h(\underline{\theta})\right)F(c)+\frac{1}{2}\left(g(c)+h(c)\right)F(c)\\
&-\frac{1}{4}\frac{\int_c^{\underline{\theta}}\left(g'(t)+h'(t)\right)\left(F(t)-F(c)\right)^2 dt}{F(c)}+\frac{1}{4}\int_d^{\overline{\theta}}\left(g(t)+h(t)\right)f(t)\,dt\\
&+\frac{1}{4}\frac{\int_d^{\overline{\theta}}\left(g'(t)+h'(t)\right)\left(F(t)-F(d)\right)^2 dt}{F(c)}\\
&-\frac{1}{4}\int_d^{\overline{\theta}}\left(g'(t)+h'(t)\right)\left(F(t)-F(d)\right)dt.
\end{aligned}
$$

This equals the expected payments given in (17) iff

$$
\begin{aligned}
&\frac{1}{2}\int_{\underline{\theta}}^{c} g'(t)\,F(t)\,dt+\frac{1}{4}\left(g(\underline{\theta})+h(\underline{\theta})\right)F(c)+\frac{1}{2}\left(g(c)+h(c)\right)F(c)\\
&-\frac{1}{4}\frac{\int_c^{\underline{\theta}}\left(g'(t)+h'(t)\right)\left(F(t)-F(c)\right)^2 dt}{F(c)}+\frac{1}{4}\int_d^{\overline{\theta}}\left(g(t)+h(t)\right)f(t)\,dt\\
&+\frac{1}{4}\frac{\int_d^{\overline{\theta}}\left(g'(t)+h'(t)\right)\left(F(t)-F(d)\right)^2 dt}{F(c)}-\frac{1}{4}\int_d^{\overline{\theta}}\left(g'(t)-h'(t)\right)\left(F(t)-F(d)\right)dt\\
=\;&\frac{1}{2}\int_d^{\overline{\theta}} h(t)\,f(t)\,dt+g(c)\,F(c)+\frac{1}{2}h(c)\,F(c)\\
\Leftrightarrow\;&\frac{1}{4}\frac{\int_c^{\underline{\theta}}\left(g'(t)+h'(t)\right)\left(F(t)-F(c)\right)^2 dt}{F(c)}-\frac{1}{4}\frac{\int_d^{\overline{\theta}}\left(g'(t)+h'(t)\right)\left(F(t)-F(d)\right)^2 dt}{F(c)}\\
&-\frac{1}{2}\int_{\underline{\theta}}^{c} g'(t)\,F(t)\,dt+\frac{1}{2}\int_d^{\overline{\theta}} g'(t)\,F(t)\,dt\\
&-\frac{1}{4}\left(\left(g(\underline{\theta})+h(\underline{\theta})\right)F(c)+\left(g(\overline{\theta})+h(\overline{\theta})\right)F(d)\right)\\
&-\frac{1}{4}\left(g(\overline{\theta})-h(\overline{\theta})\right)+\frac{1}{2}g(d)\,F(d)+\frac{1}{2}g(c)\,F(c)\\
=\;&0. \qquad (19)
\end{aligned}
$$

Similar calculations reveal that the expected payments a player of type $\theta_i \in (d,\overline{\theta}]$ can expect by participating in the auction equals the expected payments given by (18) under the same condition. Therefore the expected payments in the double auction with veto equal those derived in the previous step iff (19) holds which is equivalent to (9).

Step 4: It remains to show that no type has an incentive to change his bid to a number out of the set: $(-\infty, b(\underline{\theta})) \cup [b(c), b(d)] \cup (b(\overline{\theta}), \infty)$ (I define $b(c) := \lim_{\theta\to c} b(\theta) = g(c)+h(c)$ and $b(d) := \lim_{\theta\to d} b(\theta) = g(d)+h(d)$). A bidder would always prefer $b(\underline{\theta})$ to any bid in $(-\infty, b(\underline{\theta}))$ because in either case he never gets the partnership but he receives more money if he bids $b(\underline{\theta})$ instead of bidding a number in $(-\infty, b(\underline{\theta}))$. For a similar reason he would never bid a number in $(b(\overline{\theta}), \infty)$. To see why it is never profitable to bid $\widetilde{b} \in [b(c), b(d)]$ note first that the utility of a bidder having type θ_i and bidding $\widetilde{b} \in [b(c), b(d)]$ gives him utility

$$U_i\left(\theta_i, \widetilde{b}\right) = \frac{1}{2}\int_{\underline{\theta}}^{c}\left(h(\theta_{-i}) - \frac{b(\theta_{-i})}{2}\right) f(\theta_{-i})\, d\theta_{-i} - \frac{1}{2}\int_{d}^{\overline{\theta}}\left(h(\theta_{-i}) - \frac{b(\theta_{-i})}{2}\right)$$

which does not depend on $\widetilde{b}$ as long as $\widetilde{b} \in [b(c), b(d)]$. On the other hand we know from the calculations above, that the bidder has no incentive to deviate to bidding $b(c)$ or $b(d)$ (because the above calculations do not use the fact that types c and d veto instead of bidding $b(c)$ and $b(d)$). Therefore he has no incentive to bid $\widetilde{b} \in [b(c), b(d)]$.

Q.E.D.

Proof of Theorem 12:

It has to be shown that there always exists a solution $c, d \in (\underline{\theta}, \overline{\theta})$ to the equations:

$$\begin{aligned}
1 &= F(c) + F(d) \\
0 &= \frac{1}{2F(c)}\int_{\underline{\theta}}^{c}(g(t) + h(t))(F(t) - F(c))\, f(t)\, dt \\
&\quad + \frac{1}{2F(c)}\int_{d}^{\overline{\theta}}(g(t) + h(t))(F(t) - F(d))\, f(t)\, dt \\
&\quad + \frac{1}{2}\int_{\underline{\theta}}^{c} g(t)\, f(t)\, dt - \frac{1}{2}\int_{d}^{\overline{\theta}} g(t)\, f(t)\, dt.
\end{aligned}$$

Because of the strict monotonicity of F we can combine these equations to:

$$\begin{aligned}
Q(c) &= \frac{1}{2F(c)}\int_{\underline{\theta}}^{c}(g(t) + h(t))(F(t) - F(c))\, f(t)\, dt \\
&\quad + \frac{1}{2F(c)}\int_{F^{-1}(1-F(c))}^{\overline{\theta}}(g(t) + h(t))(F(t) - 1 + F(c))\, f(t)\, dt \\
&\quad + \frac{1}{2}\int_{\underline{\theta}}^{c} g(t)\, f(t)\, dt - \frac{1}{2}\int_{F^{-1}(1-F(c))}^{\overline{\theta}} g(t)\, f(t)\, dt \\
&= 0.
\end{aligned}$$

As a next step I calculate the value of $Q\left(F^{-1}\left(\frac{1}{2}\right)\right)$.

$$\begin{aligned}
Q\left(F^{-1}\left(\frac{1}{2}\right)\right) &= \int_{\underline{\theta}}^{\overline{\theta}} (g(t)+h(t))\left(F(t)-\frac{1}{2}\right) f(t)\,dt + \frac{1}{2}\int_{\underline{\theta}}^{F^{-1}\left(\frac{1}{2}\right)} g(t) f(t)\,dt \\
&\quad -\frac{1}{2}\int_{F^{-1}\left(\frac{1}{2}\right)}^{\overline{\theta}} g(t) f(t)\,dt \\
&= \int_{\underline{\theta}}^{\overline{\theta}} g(t)F(t)f(t)\,dt - \int_{F^{-1}\left(\frac{1}{2}\right)}^{\overline{\theta}} g(t)f(t)\,dt - \frac{1}{2}\int_{\underline{\theta}}^{\overline{\theta}} h(t)f(t)\,dt \\
&\quad +\int_{\underline{\theta}}^{\overline{\theta}} h(t)F(t)f(t)\,dt \\
&= \int_{\underline{\theta}}^{\overline{\theta}} g(t)F(t)f(t)\,dt - \int_{F^{-1}\left(\frac{1}{2}\right)}^{\overline{\theta}} g(t)f(t)\,dt + \frac{1}{2}\int_{\underline{\theta}}^{\overline{\theta}} h'(t)F(t)\,dt \\
&\quad -\frac{1}{2}\int_{\underline{\theta}}^{\overline{\theta}} h'(t)F^2(t)\,dt.
\end{aligned}$$

Because I assumed the existence condition in Theorem 10 not to hold we have

$$Q\left(F^{-1}\left(\frac{1}{2}\right)\right) > 0.$$

On the other hand we have $Q(\underline{\theta}) = 0$. Because of the continuity of $Q(c)$ we have proven the statement if we can show that for an arbitrary $\varepsilon > 0$ we have $Q'(c) < 0$ for $c \in (\underline{\theta}, \varepsilon)$. Using

$$\frac{dF^{-1}(1-F(c))}{dc} = -\frac{f(c)}{f\left(F^{-1}(1-F(c))\right)}$$

we get

$$\begin{aligned}
Q'(c) &= -\frac{f(c)\int_{\underline{\theta}}^{c}(g(t)+h(t))F(t)f(t)\,dt}{2F^2(c)} \\
&\quad -\frac{f(c)\int_{F^{-1}(1-F(c))}^{\overline{\theta}}(g(t)+h(t))(F(t)-1)f(t)\,dt}{2F^2(c)} \\
&\quad +\frac{1}{2}g(c)f(c) - \frac{1}{2}g\left(F^{-1}(1-F(c))\right)f(c)
\end{aligned}$$

and hence using l'Hôpital's rule (for which we need that f'' exists):

$$\begin{aligned}
\lim_{c\to\underline{\theta}} Q'(c) &= -\frac{1}{4}(g(\underline{\theta})+h(\underline{\theta}))f(\underline{\theta}) + \frac{1}{4}\left(g(\overline{\theta})+h(\overline{\theta})\right)f(\underline{\theta}) \\
&\quad +\frac{1}{2}g(\underline{\theta})f(\underline{\theta}) - \frac{1}{2}g(\overline{\theta})f(\underline{\theta}) \\
&= \frac{1}{4}f(\underline{\theta})\left[\left(g(\underline{\theta})-g(\overline{\theta})\right) - \left(h(\underline{\theta})-h(\overline{\theta})\right)\right] \\
&< 0
\end{aligned}$$

where the last inequality results because of the assumption that $g' > h'$.

Q.E.D.

A.3 Discounting in Sequential Auctions

Proof of Theorem 13:

First we prove the statement for the **sequential first price auction without price announcements**. The proof for the sequential second price auction without price announcements follows similar arguments and is omitted here.

We have to prove that it is never beneficial for a bidder to imitate a type different to her own type θ in some of the periods of the auction. The expected utility of a type θ bidder who always bids according to $b_l(\theta)$ in period l (if she is still in the auction) and who faces bidders following the same strategies b_l is given by:

$$U(\theta) = (\theta - b_1(\theta)) F^{n-1}(\theta) + (n-1)(D_2(\theta) - b_2(\theta)) \int_\theta^{\overline{\theta}} F^{n-2}(\theta) f(x_1) dx_1$$
$$+ \sum_{i=3}^{k} (D_i(\theta) - b_i(\theta)) \int_\theta^{\overline{\theta}} \int_\theta^{x_1} \cdots \int_\theta^{x_{i-2}} \frac{(n-1)!}{(n-i)!} F^{n-i}(\theta) f(x_{i-1}) dx_{i-1} \dots f(x_1) dx_1.$$

A bidder who deviates from the strategies b_l by bidding as if she were of type θ^l in period l (if she did not win in period $m < l$), expects a utility of

$$U\left(\theta, \theta^1, \dots, \theta^k\right)$$
$$= \left(\theta - b_1\left(\theta^1\right)\right) F^{n-1}\left(\theta^1\right) + (n-1)\left(D_2(\theta) - b_2\left(\theta^2\right)\right) \int_{\theta^1}^{\overline{\theta}} F^{n-2}\left(\min\left(x_1, \theta^2\right)\right) f(x_1) dx_1$$
$$+ \sum_{i=3}^{k} \left(D_i(\theta) - b_i\left(\theta^i\right)\right)$$
$$+ \int_{\theta^1}^{\overline{\theta}} \int_{\theta^2}^{\max\left(x_1, \theta^2\right)} \cdots \int_{\theta^{i-1}}^{\max\left(x_{i-2}, \theta^{i-1}\right)} \frac{(n-1)!}{(n-i)!} F^{n-i}\left(\min\left(x_{i-1}, \theta^i\right)\right) f(x_{i-1}) dx_{i-1} \dots f(x_1) dx_1.$$

Obviously it cannot be beneficial for a bidder to bid in period l if she already won in a previous period. Note that a bidder does not learn relevant information between the periods apart from whether or not she is still in the auction.

We show that $U(\theta) \geq U\left(\theta, \theta^1, \dots, \theta^k\right)$ for all $\left(\theta, \theta^1, \dots, \theta^k\right) \in [\underline{\theta}, \overline{\theta}]^{k+1}$, we show that for $l = 1$ and

$x_{l-1} = x_0 = \overline{\theta}$ we have

$$\begin{aligned}
&\left(D_l(\theta) - b_l\left(\theta^l\right)\right) F^{n-l}\left(\theta^l\right) \qquad (20)\\
&+ (n-l)\left(D_{l+1}(\theta) - b_{l+1}\left(\theta^{l+1}\right)\right) \int_{\theta^l}^{x_{l-1}} F^{n-l-1}\left(\min\left(x_l, \theta^{l+1}\right)\right) f(x_l)\, dx_l \\
&+ \sum_{i=l+2}^{k} \left(D_i(\theta) - b_i\left(\theta^i\right)\right) \frac{(n-l)!}{(n-i)!} \\
&\times \int_{\theta^l}^{x_{l-1}} \int_{\theta^{l+1}}^{\max\left(x_l, \theta^{l+1}\right)} \cdots \int_{\theta^{i-1}}^{\max\left(x_{i-2}, \theta^{i-1}\right)} F^{n-i}\left(\min\left(x_{i-1}, \theta^i\right)\right) f(x_{i-1})\, dx_{i-1} \ldots f(x_l)\, dx_l \\
\leq\ & \left(D_l(\theta) - b_l(\theta)\right) F^{n-l}(\theta) + (n-l)\left(D_{l+1}(\theta) - b_{l+1}(\theta)\right) \int_{\theta}^{x_{l-1}} F^{n-l-1}(\theta) f(x_l)\, dx_l \\
&+ \sum_{i=l+2}^{k} \left(D_i(\theta) - b_i(\theta)\right) \frac{(n-l)!}{(n-i)!} \int_{\theta}^{x_{l-1}} \cdots \int_{\theta}^{x_{i-2}} F^{n-i}(\theta) f(x_{i-1})\, dx_{i-1} \ldots f(x_l)\, dx_l.
\end{aligned}$$

Since we have [10]

$$\left(D_l(\theta) - b_l\left(\theta^l\right)\right) F^{n-l}\left(\theta^l\right) = \int_{\underline{\theta}}^{\theta^l} \left(D_l(\theta) - D_l(x) + D_{l+1}(x) - b_{l+1}(x)\right) dF^{n-l}(x)$$

(20) is equivalent to:

$$\begin{aligned}
&\int_{\underline{\theta}}^{\theta^l} \left(D_l(\theta) - D_l(x_l) + D_{l+1}(x_l) - b_{l+1}(x_l)\right) F^{n-l-1}(x_l) f(x_l)\, dx_l \\
&+ \int_{\theta^l}^{x_{l-1}} \left(\left(D_{l+1}(\theta) - b_{l+1}\left(\theta^{l+1}\right)\right) F^{n-l-1}\left(\min\left(x_l, \theta^{l+1}\right)\right)\right. \\
&+ \sum_{i=l+2}^{k} \left(D_i(\theta) - b_i\left(\theta^i\right)\right) \frac{(n-l-1)!}{(n-i)!} \\
&\left.\times \int_{\theta^{l+1}}^{\max\left(x_l, \theta^{l+1}\right)} \cdots \int_{\theta^{i-1}}^{\max\left(x_{i-2}, \theta^{i-1}\right)} F^{n-i}\left(\min\left(x_{i-1}, \theta^i\right)\right) f(x_{i-1})\, dx_{i-1} \ldots f(x_{l+1})\, dx_{l+1}\right) f(x_l)\, dx_l \\
\leq\ & \int_{\underline{\theta}}^{\theta} \left(D_l(\theta) - D_l(x_l) + D_{l+1}(x_l) - b_{l+1}(x_l)\right) F^{n-l-1}(x_l) f(x_l)\, dx_l \\
&+ \int_{\theta}^{x_{l-1}} \left(\left(D_{l+1}(\theta) - b_{l+1}(\theta)\right) F^{n-l-1}(\theta) + \sum_{i=l+2}^{k} \left(D_i(\theta) - b_i(\theta)\right) \frac{(n-l-1)!}{(n-i)!}\right. \\
&\left.\times \int_{\theta}^{x_l} \cdots \int_{\theta}^{x_{i-2}} F^{n-i}(\theta) f(x_{i-1})\, dx_{i-1} \ldots f(x_{l+1})\, dx_{l+1}\right) f(x_l)\, dx_l.
\end{aligned}$$

Therefore it suffices to show that for all $\left(\theta, \theta^{l+1}, \ldots, \theta^k\right) \in [\underline{\theta}, \overline{\theta}]^{k-l+1}$ the following three statements hold:

---

[10] This can easily be seen by using the following representation of b_i:

$$b_l(\theta) = \frac{1}{F^{n-l}(\theta)} \int_{\underline{\theta}}^{\theta} \left(D_l(x) - D_{l+1}(x) + b_{l+1}(x)\right) dF^{n-l}(x).$$

1. for $x_l \geq \theta$

$$\begin{aligned}
& \left(D_l(\theta) - D_l(x_l) + D_{l+1}(x_l) - b_{l+1}(x_l)\right) F^{n-l-1}(x_l) \\
\leq\ & \left(D_{l+1}(\theta) - b_{l+1}(\theta)\right) F^{n-l-1}(\theta) \\
& + \sum_{i=l+2}^{k} \left(D_i(\theta) - b_i(\theta)\right) \int_\theta^{x_l} \cdots \int_\theta^{x_{i-2}} \frac{(n-l-1)!}{(n-i)!} F^{n-i}(\theta) f(x_{i-1})\, dx_{i-1} \ldots f(x_{l+1})\, dx_{l+1}
\end{aligned} \tag{21}$$

2. for $x_l \leq \theta$

$$\begin{aligned}
& D_{l+1}(\theta) - b_{l+1}\left(\theta^{l+1}\right) F^{n-l-1}\left(\min\left(x_l, \theta^{l+1}\right)\right) \\
& + \sum_{i=l+2}^{k} \left(D_i(\theta) - b_i\left(\theta^i\right)\right) \frac{(n-l-1)!}{(n-i)!} \\
& \int_{\theta^{l+1}}^{\max\left(x_l, \theta^{l+1}\right)} \cdots \int_{\theta^{i-1}}^{\max\left(x_{i-2}, \theta^{i-1}\right)} F^{n-i}\left(\min\left(x_{i-1}, \theta^i\right)\right) f(x_{i-1})\, dx_{i-1} \ldots f(x_{l+1})\, dx_{l+1} \\
\leq\ & \left(D_l(\theta) - D_l(x_l) + D_{l+1}(x_l) - b_{l+1}(x_l)\right) F^{n-l-1}(x_l)
\end{aligned} \tag{22}$$

3. for $x_l \geq \theta$

$$\begin{aligned}
& \left(D_{l+1}(\theta) - b_{l+1}\left(\theta^{l+1}\right)\right) F^{n-l-1}\left(\min\left(x_l, \theta^{l+1}\right)\right) \\
& + \sum_{i=l+2}^{k} \left(D_i(\theta) - b_i\left(\theta^i\right)\right) \frac{(n-l-1)!}{(n-i)!} \\
& \times \int_{\theta^{l+1}}^{\max\left(x_l, \theta^{l+1}\right)} \cdots \int_{\theta^{i-1}}^{\max\left(x_{i-2}, \theta^{i-1}\right)} F^{n-i}\left(\min\left(x_{i-1}, \theta^i\right)\right) f(x_{i-1})\, dx_{i-1} \ldots f(x_{l+1})\, dx_{l+1} \\
\leq\ & \left(D_{l+1}(\theta) - b_{l+1}(\theta)\right) F^{n-l-1}(\theta) \\
& + \sum_{i=l+2}^{k} \left(D_i(\theta) - b_i(\theta)\right) \frac{(n-l-1)!}{(n-i)!} \int_\theta^{x_l} \cdots \int_\theta^{x_{i-2}} F^{n-i}(\theta) f(x_{i-1})\, dx_{i-1} \ldots f(x_{l+1})\, dx_{l+1}
\end{aligned} \tag{23}$$

This is done by three induction arguments[11]. We proof the following statements by induction. To simplify notation, set $D_l = b_l = 0$ for $l > k$.

[11] The induction is over l starting from $l = k$ going backwards to $l = 1$.

1. Subtracting $\left(D_l(\theta) - b_{l+1}(\theta)\right) F^{n-l-1}(\theta)$ on both sides of (21) gives

$$\begin{aligned}
& \left(D_l(\theta) - D_l(x_l) + D_{l+1}(x_l) - b_{l+1}(x_l)\right) F^{n-l-1}(x_l) - \left(D_{l+1}(\theta) - b_{l+1}(\theta)\right) F^{n-l-1}(\theta) \\
= & \int_{\underline{\theta}}^{x_l} \left(D_l(\theta) - D_l(x_l) + D_{l+1}(x_l) - D_{l+1}(y) + D_{l+2}(y) - b_{l+2}(y)\right) dF^{n-l-1}(y) \\
& - \int_{\underline{\theta}}^{\theta} \left(D_{l+1}(\theta) - D_{l+1}(y) + D_{l+2}(y) - b_{l+2}(y)\right) dF^{n-l-1}(y) \\
= & \left(D_l(\theta) - D_l(x_l) + D_{l+1}(x_l) - D_{l+1}(\theta)\right) F^{n-l-1}(x_l) \\
& + \int_{\theta}^{x_l} \left(D_{l+1}(\theta) - D_{l+1}(x_{l+1}) + D_{l+2}(x_{l+1}) - b_{l+2}(x_{l+1})\right) dF^{n-l-1}(x_{l+1}) \\
\leq & \int_{\theta}^{x_l} \Big((n-l-1)\left(D_{l+2}(\theta) - b_{l+2}(\theta)\right) F(\theta)^{n-l-2} + \\
& + \sum_{i=l+3}^{k} \frac{(n-(l+1))!}{(n-i)!} \left(D_i(\theta) - b_i(\theta)\right) F(\theta)^{n-i} \\
& \times \int_{\theta}^{x_{l+1}} \cdots \int_{\theta}^{x_{i-2}} f(x_{i-1})\, dx_{i-1} \ldots f(x_{l+2})\, dx_{l+2} \Big) f(x_{l+1})\, dx_{l+1}.
\end{aligned}$$

Since $D_l(\theta) - D_l(x_l) + D_{l+1}(x_l) - D_{l+1}(\theta) \leq 0$ this is true if for all $x_{l+1} \geq \theta$ we have

$$\begin{aligned}
& \left(D_{l+1}(\theta) - D_{l+1}(x_{l+1}) + D_{l+2}(x_{l+1}) - b_{l+2}(x_{l+1})\right) F^{n-l-2}(x_{l+1}) \\
& \leq \left(D_{l+2}(\theta) - b_{l+2}(\theta)\right) F^{n-l-2}(\theta) + \\
& + \sum_{i=l+3}^{k} \left(D_i(\theta) - b_i(\theta)\right) \frac{(n-l-2)!}{(n-i)!} \int_{\theta}^{x_{l+1}} \cdots \int_{\theta}^{x_{i-2}} F^{n-i}(\theta) f(x_{i-1})\, dx_{i-1} \ldots f(x_{l+2})\, dx_{l+2}.
\end{aligned}$$

For $l = k$ and $x_k \geq \theta$ this is true since $D_k(\theta) - D_k(x_k) \leq 0$.

2. For $x_l \leq \theta^{l+1}$ this is true since b_{l+1} is increasing and we have

$$D_l(\theta) - D_{l+1}(\theta) - \left(D_l(x_l) - D_{l+1}(x_l)\right) \geq 0.$$

Assume now that $x_l > \theta^{l+1}$. Then (22) is equivalent to

$$\begin{aligned}
& \int_{\theta^{l+1}}^{x_l} \Bigg((n-l-1)\left(D_{l+2}(\theta) - b_{l+2}\left(\theta^{l+2}\right)\right) F^{n-l-2}\left(\min\left(x_{l+1}, \theta^{l+2}\right)\right) \\
& + \sum_{i=l+3}^{k} \left(D_i(\theta) - b_i\left(\theta^i\right)\right) \frac{(n-l-1)!}{(n-i)!} \\
& \times \int_{\theta^{l+2}}^{\max\left(x_{l+1}, \theta^{l+2}\right)} \cdots \int_{\theta^{i-1}}^{\max\left(x_{i-2}, \theta^{i-1}\right)} F^{n-i}\left(\min\left(x_{i-1}, \theta^i\right)\right) f(x_{i-1})\, dx_{i-1} \ldots f(x_{l+2})\, dx_{l+2} \Bigg) \\
& f(x_{l+1})\, dx_{l+1} \\
\leq \; & \underbrace{\left(D_l(\theta) - D_l(x_l) + D_{l+1}(x_l) - D_{l+1}(\theta)\right)}_{\geq 0} F^{n-l-1}(x_l) \\
& + \int_{\theta^{l+1}}^{x_l} \left(D_{l+1}(\theta) - D_{l+1}(y) + D_{l+2}(y) - b_{l+2}(y)\right) dF^{n-l-1}(y).
\end{aligned}$$

This is true if for $x_{l+1} \leq \theta$ we have that

$$\begin{aligned}
& (n-l-1)\left(D_{l+2}(\theta) - b_{l+2}\left(\theta^{l+2}\right)\right) F^{n-l-2}\left(\min\left(x_{l+1}, \theta^{l+2}\right)\right) \\
& + \sum_{i=l+3}^{k} \left(D_i(\theta) - b_i\left(\theta^i\right)\right) \frac{(n-l-1)!}{(n-i)!} \\
& \times \int_{\theta^{l+2}}^{\max\left(x_{l+1}, \theta^{l+2}\right)} \cdots \int_{\theta^{i-1}}^{\max\left(x_{i-2}, \theta^{i-1}\right)} F^{n-i}\left(\min\left(x_{i-1}, \theta^i\right)\right) f(x_{i-1})\, dx_{i-1} \ldots f(x_{l+2})\, dx_{l+2} \\
\leq\ & \left(D_{l+1}(\theta) - D_{l+1}(x_{l+1}) + D_{l+2}(x_{l+1}) - b_{l+2}(x_{l+1})\right) F^{n-l-2}(x_{l+1}).
\end{aligned}$$

For $l = k$ the statement is true since $0 \leq (D_k(\theta) - D_k(x_k)) F^{n-(k+1)}(x_k)$ for $x_k \leq \theta$.

3. To show that (23) holds we again consider two cases: If $x_l \leq \theta^{l+1}$ the statement reduces to

$$\begin{aligned}
& \left(D_{l+1}(\theta) - b_{l+1}\left(\theta^{l+1}\right)\right) F^{n-l-1}(x_l) \\
\leq\ & \left(D_{l+1}(\theta) - b_{l+1}(\theta)\right) F^{n-l-1}(\theta) \\
& + \sum_{i=l+2}^{k} \left(D_i(\theta) - b_i(\theta)\right) \frac{(n-l-1)!}{(n-i)!} F^{n-i}(\theta) \\
& \times \int_{\theta}^{x_l} \cdots \int_{\theta}^{x_{i-2}} f(x_{i-1})\, dx_{i-1} \ldots f(x_{l+1})\, dx_{l+1}.
\end{aligned}$$

This holds if

$$\begin{aligned}
& \left(D_{l+1}(\theta) - b_{l+1}(x_l)\right) F^{n-l-1}(x_l) - \left(D_{l+1}(\theta) - b_{l+1}(\theta)\right) F^{n-l-1}(\theta) \qquad (24) \\
=\ & \int_{\theta}^{x_l} \left(D_{l+1}(\theta) - D_{l+1}(x_{l+1}) + D_{l+2}(x_{l+1}) - b_{l+2}(x_{l+1})\right) dF^{n-l-1}(x_{l+1}) \\
\leq\ & \int_{\theta}^{x_l} \left(\sum_{i=l+2}^{k} \left(D_i(\theta) - b_i(\theta)\right) \frac{(n-l-1)!}{(n-i)!} F(\theta)^{n-i} \right. \\
& \left. \times \int_{\theta}^{x_{l+1}} \cdots \int_{\theta}^{x_{i-2}} f(x_{i-1})\, dx_{i-1} \ldots f(x_{l+2})\, dx_{l+2}\right) f(x_{l+1})\, dx_{l+1}
\end{aligned}$$

which again is true if for $x_{l+1} \geq \theta$ we get

$$\begin{aligned}
& \left(D_{l+1}(\theta) - D_{l+1}(x_{l+1}) + D_{l+2}(x_{l+1}) - b_{l+2}(x_{l+1})\right) F^{n-l-2}(x_{l+1}) \\
\leq\ & \left(D_{l+2}(\theta) - b_{l+2}(x_{l+1})\right) F^{n-l-2}(x_{l+1}) \\
\leq\ & \left(D_{l+2}(\theta) - b_{l+2}(\theta)\right) F^{n-l-2}(\theta) + \\
& \sum_{i=l+3}^{k} \left(D_i(\theta) - b_i(\theta)\right) \frac{(n-(l+2))!}{(n-i)!} F(\theta)^{n-i} \int_{\theta}^{x_{l+1}} \cdots \int_{\theta}^{x_{i-2}} f(x_{i-1})\, dx_{i-1} \ldots f(x_{l+2})\, dx_{l+2}.
\end{aligned}$$

Therefore by induction (24) must hold for $x_l > \theta$ if it is true for $l = k$ and $x_l \geq \theta$ which equals

$$\left(D_k(\theta) - b_k(x_k)\right) F^{n-k}(x_k) \leq \left(D_k(\theta) - b_k(\theta)\right) F^{n-k}(\theta).$$

If we have $x_l \geq \theta^{l+1}$, we have to show that

$$\begin{aligned}
& \left(D_{l+1}(\theta) - b_{l+1}\left(\theta^{l+1}\right)\right) F^{n-(l+1)}\left(\theta^{l+1}\right) + \\
& + (n-l-1)\left(D_{l+2}(\theta) - b_{l+2}\left(\theta^{l+2}\right)\right) \int_{\theta^{l+1}}^{x_l} F^{n-l-2}\left(\min\left(x_{l+1}, \theta^{l+2}\right)\right) f(x_{l+1})\, dx_{l+1} + \\
& + \sum_{i=l+3}^{k} \left(D_i(\theta) - b_i\left(\theta^i\right)\right) \int_{\theta^{l+1}}^{x_l} \int_{\theta^{l+2}}^{\max\left(x_{l+1}, \theta^{l+2}\right)} \cdots \int_{\theta^{i-1}}^{\max\left(x_{i-2}, \theta^{i-1}\right)} \frac{(n-l-1)!}{(n-i)!} \\
& F^{n-i}\left(\min\left(x_{i-1}, \theta^i\right)\right) f(x_{i-1})\, dx_{i-1} \ldots f(x_{l+2})\, dx_{l+2} f(x_{l+1})\, dx_{l+1} \\
\leq \;& \left(D_{l+1}(\theta) - b_{l+1}(\theta)\right) F^{n-l-1}(\theta) + \\
& + (n-l-1)\left(D_{l+2}(\theta) - b_{l+2}(\theta)\right) \int_{\theta}^{x_l} F^{n-l-2}(\theta) f(x_{l+1})\, dx_{l+1} + \\
& + \sum_{i=l+3}^{k} \left(D_i(\theta) - b_i(\theta)\right) \frac{(n-l-1)!}{(n-i)!} F(\theta)^{n-i} \int_{\theta}^{x_l} \cdots \int_{\theta}^{x_{i-2}} f(x_{i-1})\, dx_{i-1} \ldots f(x_{l+1})\, dx_{l+1}.
\end{aligned}$$

This is statement (20) formulated for $l+1$. Therefore the Theorem holds for general l if (20) holds for $l = k$ which is the case since

$$\left(D_k(\theta) - b_k\left(\theta^k\right)\right) F^{n-k}\left(\theta^k\right) \leq \left(D_k(\theta) - b_k(\theta)\right) F^{n-k}(\theta).$$

Proof for the sequential second-price auction with price announcements:
We write $v_l(\theta; x_1 \ldots x_{n-l})$ for the utility of a buyer with type θ, who finds herself in period l given her remaining opponents have types $x_{1,\ldots,}x_{n-l}$ and everyone announces her type truthfully. If $x_i < \theta$ for all $i = 1..n-l$ we have

$$v_l(\theta; x_1, \ldots, x_{n-l})) = D_l(\theta) - b_l(\max\{x_1, \ldots, x_{n-l}\})$$

since the θ-type bidder wins the l'th auction.
To avoid tedious case distinctions in the recursive formulas used in this proof we define $D_l = b_l = 0$ for $l > k$. We show by induction that it is optimal to bid according to $b_l(\theta_i)$ in period l if it is optimal to bid according to b_m in period m for $m > l$ and if all other bidders (always) bid according to b_l.
In period $l = k$ bidding $D_k(\theta_i)$ is a dominant strategy. To show that it is optimal to bid according to b_l in period l we distinguish two cases:

Case 1 It is optimal to bid according to b_l for the bidder who submitted the highest bid in period $l-1$.

Case 2 It is optimal to bid according to b_l for bidders who did not submit the highest bid in period $l-1$.

Case 1:
The expected utility of a bidder in period l who sets the price in period $l-1$ does only depend on her

type θ, her bid in period l given by[12] $b_l\left(\widehat{\theta}\right)$ and in period $l-1$ given by $b_{l-1}\left(\widetilde{\theta}\right)$. Her bid in period $l-1$ influences her expected utility since she updates her beliefs about other agents' types distributions by inferring that these are given by $F[\theta|\ \theta \leq \widetilde{\theta}] = \frac{F(\theta)}{F(\widetilde{\theta})}$ for $\theta \leq \widetilde{\theta}$. Bids in periods 1 to $l-2$ have no influence since all relevant information about other agents' types is given by the fact that these are smaller than[13] $\widetilde{\theta}$. A bidder's expected utility in period l if she is type θ, bids as if she were of type $\widehat{\theta}$ (in period l), submitted $b_{l-1}\left(\widetilde{\theta}\right)$ in period $l-1$ and $b_m(\theta)$ in periods $m > l$ is given by:

$$
\begin{aligned}
& U_l\left(\theta,\widehat{\theta},\widetilde{\theta}\right) \\
= \; & \frac{n-l}{F^{n-l}\left(\widetilde{\theta}\right)} \int_{\underline{\theta}}^{\widehat{\theta}} \left[D_l(\theta) - b_l(x_1)\right] F^{n-l-1}(x_1) f(x_1)\, dx_1 \\
& + \frac{n-l}{F^{n-l}\left(\widetilde{\theta}\right)} \int_{\widehat{\theta}}^{\widetilde{\theta}} \int_{\underline{\theta}}^{\min\{x_1,\theta\}} \ldots \int_{\underline{\theta}}^{\min\{x_1,\theta\}} v_{l+1}(\theta; x_2 \ldots x_{n-l}) f(x_{n-l})\, dx_{n-l} \cdots f(x_1)\, dx_1 \\
& + \frac{n-l}{F^{n-l}\left(\widetilde{\theta}\right)} (n-l-1) \int_{\widehat{\theta}}^{\widetilde{\theta}} \int_{\min\{x_1,\theta\}}^{x_1} \int_{\underline{\theta}}^{x_2} \ldots \int_{\underline{\theta}}^{x_2} v_{l+1}(\theta; x_2 \ldots x_{n-l}) f(x_{n-l})\, dx_{n-l} \cdots f(x_1)\, dx_1.
\end{aligned}
$$

The first addend describes the case where the bidder wins in period l. The second addend describes the case where she does not win period l but wins period $l+1$. The last part is the case where she neither wins period l nor period $l+1$.

We show that $\frac{\partial}{\partial\widehat{\theta}} U_l\left(\theta,\widehat{\theta},\widetilde{\theta}\right) \geq 0$ for $\widehat{\theta} < \theta$ and $\frac{\partial}{\partial\widehat{\theta}} U_l\left(\theta,\widehat{\theta},\widetilde{\theta}\right) \leq 0$ for $\widehat{\theta} > \theta$ if the same is true for period $l+1$. Since

$$
b_l\left(\widehat{\theta}\right) = D_l\left(\widehat{\theta}\right) - \frac{1}{F^{n-l-1}\left(\widehat{\theta}\right)} \int_{\underline{\theta}}^{\widehat{\theta}} \ldots \int_{\underline{\theta}}^{\widehat{\theta}} v_{l+1}\left(\widehat{\theta}; x_2 \ldots x_{n-l}\right) f(x_{n-l})\, dx_{n-l} \cdots f(x_2)\, dx_2
$$

we have to determine the sign of

$$
\begin{aligned}
& \frac{F^{n-l}\left(\widetilde{\theta}\right)}{(n-l) f\left(\widehat{\theta}\right)} \frac{d}{d\widehat{\theta}} U_l\left(\theta,\widehat{\theta},\widetilde{\theta}\right) \qquad (25) \\
= \; & \int_{\underline{\theta}}^{\widehat{\theta}} \ldots \int_{\underline{\theta}}^{\widehat{\theta}} \left[D_l(\theta) - D_l\left(\widehat{\theta}\right) + v_{l+1}\left(\widehat{\theta}; x_2 \ldots x_{n-l}\right)\right] f(x_{n-l})\, dx_{n-l} \cdots f(x_2)\, dx_2 \\
& - \int_{\underline{\theta}}^{\min\left(\widehat{\theta},\theta\right)} \ldots \int_{\underline{\theta}}^{\min\left(\widehat{\theta},\theta\right)} v_{l+1}(\theta; x_2 \ldots x_{n-l}) f(x_{n-l})\, dx_{n-l} \cdots f(x_2)\, dx_2 \\
& - (n-l-1) \int_{\min\left(\widehat{\theta},\theta\right)}^{\widehat{\theta}} \int_{\underline{\theta}}^{x_2} \ldots \int_{\underline{\theta}}^{x_2} v_{l+1}(\theta; x_2 \ldots x_{n-l}) f(x_{n-l})\, dx_{n-l} \cdots f(x_2)\, dx_2
\end{aligned}
$$

[12] Note that bidding outside the range of b_l has the same effect as bidding $b_l(\theta) = \underline{\theta}$ or $b_l(\theta) = \overline{\theta}$.

[13] This is due to independence of types.

First assume[14] $\widehat{\theta} > \theta$: We have to show that (25)<0 for any $\widehat{\theta} > \theta$. This is done by induction over the stages. For $l = k$ (25) becomes

$$\int_{\underline{\theta}}^{\widehat{\theta}} ... \int_{\underline{\theta}}^{\widehat{\theta}} \left[D_k(\theta) - D_k\left(\widehat{\theta}\right)\right] f(x_{n-k})\, dx_{n-k} \cdots f(x_2)\, dx_2,$$

which is smaller than zero, since $\widehat{\theta} > \theta$ and since D_k is strictly increasing.

We show that for any $\widehat{\theta} > \theta$ the following reformulation of (25) is negative:

$$\begin{aligned}
&\int_{\underline{\theta}}^{\theta} ... \int_{\underline{\theta}}^{\theta} \Big[D_l(\theta) - D_l\left(\widehat{\theta}\right) + v_{l+1}\left(\widehat{\theta}; x_2...x_{n-l}\right) \\
&-v_{l+1}(\theta; x_2...x_{n-l})\Big] f(x_{n-l})\, dx_{n-l} \cdots f(x_2)\, dx_2 \\
&+(n-l-1) \int_{\theta}^{\widehat{\theta}} \int_{\underline{\theta}}^{x_2} ... \int_{\underline{\theta}}^{x_2} \Big[D_l(\theta) - D_l\left(\widehat{\theta}\right) + v_{l+1}\left(\widehat{\theta}; x_2...x_{n-l}\right) \\
&-v_{l+1}(\theta; x_2...x_{n-l})\Big] f(x_{n-l})\, dx_{n-l} \cdots f(x_2)\, dx_2
\end{aligned} \tag{26}$$

In the first integral of (26) both θ and $\widehat{\theta}$ are always greater than x_i, $i = 2, \ldots, n-l$, therefore we have that the first integral of (26) equals[15]

$$\int_{\underline{\theta}}^{\theta} ... \int_{\underline{\theta}}^{\theta} (D_l(\theta) - D_{l+1}(\theta)) - \left(D_l\left(\widehat{\theta}\right) - D_{l+1}\left(\widehat{\theta}\right)\right) f(x_{n-l})\, dx_{n-l} \cdots f(x_2)\, dx_2$$

Since $\widehat{\theta} > \theta$ and $D_l - D_{l+1}$ is increasing this is non-positive. It therefore remains to show, that

$$\begin{aligned}
0 \quad \geq \quad & (n-l-1) \int_{\theta}^{\widehat{\theta}} \int_{\underline{\theta}}^{x_2} ... \int_{\underline{\theta}}^{x_2} \Big[D_l(\theta) - D_l\left(\widehat{\theta}\right) \\
& + v_{l+1}\left(\widehat{\theta}; x_2...x_{n-l}\right) - v_{l+1}(\theta; x_2...x_{n-l})\Big] f(x_{n-l})\, dx_{n-l} \cdots f(x_2)\, dx_2.
\end{aligned}$$

Since $\widehat{\theta} > x_i$ (a.e.) for $i = 2, \ldots, n-l$ and since x_2 denotes the highest of the other bidders' types

[14] This is the more complicated case since "overbidding", i.e. overstating her own type might lead to winning round l instead of winning at some later round.

[15] Note that we have

$$v_{l+1}\left(\widehat{\theta}; x_2...x_{n-l}\right) = D_{l+1}\left(\widehat{\theta}\right) - b_{l+1}(\max\{x_2...x_{n-l}\})$$

and

$$v_{l+1}(\theta; x_2...x_{n-l}) = D_{l+1}(\theta) - b_{l+1}(\max\{x_2...x_{n-l}\}).$$

(remaining in the auction) we have

$$\begin{aligned} v_{l+1}\left(\widehat{\theta};x_2...x_{n-l}\right) &= D_{l+1}\left(\widehat{\theta}\right)-b_{l+1}\left(x_2\right) \\ &= D_{l+1}\left(\widehat{\theta}\right)-D_{l+1}\left(x_2\right) \\ &\quad +\frac{1}{F^{n-l-2}\left(x_2\right)}\int_{\underline{\theta}}^{x_2}...\int_{\underline{\theta}}^{x_2} v_{l+2}\left(x_2;\widetilde{x}_3,...,\widetilde{x}_{n-l}\right)f\left(\widetilde{x}_{n-l}\right)d\widetilde{x}_{n-l}\cdots f\left(\widetilde{x}_3\right)d\widetilde{x}_3 \end{aligned}$$

In addition we have $\theta < x_2$ consequently $v_{l+1}\left(\theta;x_2...x_{n-l}\right)=v_{l+2}\left(\theta;x_3...x_{n-l}\right)$. Hence we have to show that the following equation is negative:

$$\begin{aligned} & (n-l-1)\int_{\theta}^{\widehat{\theta}}\int_{\underline{\theta}}^{x_2}...\int_{\underline{\theta}}^{x_2}\Big[D_l(\theta)-D_l\left(\widehat{\theta}\right)+D_{l+1}\left(\widehat{\theta}\right)-D_{l+1}\left(x_2\right) \\ & +\frac{1}{F^{n-l-2}\left(x_2\right)}\int_{\underline{\theta}}^{x_2}..\int_{\underline{\theta}}^{x_2} v_{l+2}\left(x_2;\widetilde{x}_3...\widetilde{x}_{n-l}\right)f\left(\widetilde{x}_{n-l}\right)d\widetilde{x}_{n-l}\cdots f\left(\widetilde{x}_3\right)d\widetilde{x}_3 \\ & -v_{l+2}\left(\theta;x_3...x_{n-l}\right)\Big]f\left(x_{n-l}\right)dx_{n-l}\cdots f\left(x_2\right)dx_2 \\ = \; & (n-l-1)\int_{\theta}^{\widehat{\theta}}\int_{\underline{\theta}}^{x_2}...\int_{\underline{\theta}}^{x_2}\Big[D_l(\theta)-D_l\left(\widehat{\theta}\right)+D_{l+1}\left(\widehat{\theta}\right)-D_{l+1}\left(x_2\right) \\ & +v_{l+2}\left(x_2;x_3...x_{n-l}\right)-v_{l+2}\left(\theta;x_3...x_{n-l}\right)\Big]f\left(x_{n-l}\right)dx_{n-l}\cdots f\left(x_2\right)dx_2. \end{aligned} \tag{27}$$

Note that since D_l-D_{l+1} is increasing we have $D_l(\theta)-D_l\left(\widehat{\theta}\right)+D_{l+1}\left(\widehat{\theta}\right)\leq D_{l+1}(\theta)$ and therefore

$$\begin{aligned} & (27) \\ \leq \; & (n-l-1)\int_{\theta}^{\widehat{\theta}}\int_{\underline{\theta}}^{x_2}...\int_{\underline{\theta}}^{x_2}\left[D_{l+1}(\theta)-D_{l+1}\left(x_2\right)\right. \\ & \left.+v_{l+2}\left(x_2;x_3...x_{n-l}\right)-v_{l+2}\left(\theta;x_3...x_{n-l}\right)\right]f\left(x_{n-l}\right)dx_{n-l}\cdots f\left(x_2\right)dx_2 \\ = \; & (n-l-1)\int_{\theta}^{\widehat{\theta}}\Bigg[\int_{\underline{\theta}}^{\theta}...\int_{\underline{\theta}}^{\theta}\left[D_{l+1}(\theta)-D_{l+1}\left(x_2\right)\right. \\ & \left.+v_{l+2}\left(x_2;x_3...x_{n-l}\right)-v_{l+2}\left(\theta;x_3...x_{n-l}\right)\right]f\left(x_{n-l}\right)dx_{n-l}\cdots f\left(x_3\right)dx_3 \\ & +(n-l-2)\int_{\theta}^{x_2}\int_{\theta}^{x_3}...\int_{\theta}^{x_3}\left[D_{l+1}(\theta)-D_{l+1}\left(x_2\right)\right. \\ & \left.+v_{l+2}\left(x_2;x_3...x_{n-l}\right)-v_{l+2}\left(\theta;x_3...x_{n-l}\right)\right]f\left(x_{n-l}\right)dx_{n-l}\cdots f\left(x_3\right)dx_3\Bigg]f\left(x_2\right)dx_2. \end{aligned}$$

The integrand of the outer integral is smaller than zero by induction[16] since $x_2 > \theta$. This completes the case $\theta < \widehat{\theta}$.

Assume now that $\widehat{\theta} < \theta$. We have to show that $\frac{d}{d\theta}U_l\left(\theta,\widehat{\theta},\widetilde{\theta}\right)\geq 0$. Because of (25) we have to show that the following is not negative:

$$\begin{aligned} & \int_{\underline{\theta}}^{\widehat{\theta}}...\int_{\underline{\theta}}^{\widehat{\theta}}\left[D_l(\theta)-D_l\left(\widehat{\theta}\right)+v_{l+1}\left(\widehat{\theta};x_2...x_{n-l}\right)-v_{l+1}\left(\theta;x_2...x_{n-l}\right)\right]f\left(x_{n-l}\right)dx_{n-l}\cdots f\left(x_2\right)dx_2 \\ = \; & \int_{\underline{\theta}}^{\widehat{\theta}}...\int_{\underline{\theta}}^{\widehat{\theta}}\left[D_l(\theta)-D_{l+1}(\theta)-\left(D_l\left(\widehat{\theta}\right)-D_{l+1}\left(\widehat{\theta}\right)\right)\right]f\left(x_{n-l}\right)dx_{n-l}\cdots f\left(x_2\right)dx_2. \end{aligned}$$

---

[16] Obviously it is smaller than zero if $l=k-1$.

This is nonnegative since $D_l - D_{l+1}$ is increasing.

Case 2:

Assume y to be the highest of the other bidders' types which is known since it can be inferred from the announced price of the previous period. We show that for $\theta > y$ it is optimal for bidder i to win period l, which implies that bidding according to b_l is optimal (given it is optimal to bid according to b_m in forthcoming periods $m > l$). If $\theta < y$, bidder i finds it optimal not to win period l which is achieved by bidding according to b_l as well.

If bidder i wins this period her profit is given by

$$D_l(\theta) - b_l(y).$$

If she does not win this period her profit is

$$\begin{aligned}&\frac{1}{F^{n-l-1}(y)}\int_{\underline{\theta}}^{\min\{y,\theta\}}\ldots\int_{\underline{\theta}}^{\min\{y,\theta\}} v_{l+1}(\theta; x_2 \ldots x_{n-l}) f(x_{n-l}) dx_{n-l}\cdots f(x_2) dx_2 \\ &+\frac{1}{F^{n-l-1}(y)}(n-l-1)\int_{\min\{y,\theta\}}^{y}\int_{\underline{\theta}}^{x_2}\ldots\int_{\underline{\theta}}^{x_2} v_{l+1}(\theta; x_2 \ldots x_{n-l}) f(x_{n-l}) dx_{n-l}\cdots f(x_2) dx_2.\end{aligned}$$

Hence the difference in utility is

$$\begin{aligned}&D_l(\theta) - b_l(y) \\ &-\frac{1}{F^{n-l-1}(y)}\int_{\underline{\theta}}^{\min\{y,\theta\}}\ldots\int_{\underline{\theta}}^{\min\{y,\theta\}} v_{l+1}(\theta; x_2 \ldots x_{n-l}) f(x_{n-l}) dx_{n-l}\cdots f(x_2) dx_2 \\ &-\frac{1}{F^{n-l-1}(y)}(n-l-1)\int_{\min\{y,\theta\}}^{y}\int_{\underline{\theta}}^{x_2}\ldots\int_{\underline{\theta}}^{x_2} v_{l+1}(\theta; x_2 \ldots x_{n-l}) f(x_{n-l}) dx_{n-l}\cdots f(x_2) dx_2.\end{aligned} \tag{28}$$

This equation has the same sign as (25) if $\widehat{\theta} = y$. Therefore we already proved in case 1 that (28) is negative if $\theta < y$ which shows that bidder i prefers to lose in the $l'th$ period which she achieves by bidding $b_l(\theta)$. If $\theta > y$ we know that (28) is positive, i.e. winning period l is optimal. This again is achieved by bidding according to b_l.

Proof for the first price auction with price announcements:

Note first that the bidding strategies b_l are strictly increasing. At period l a remaining bidder knows the types of the $l-1$ bidders who already won in previous periods (i.e. $\theta_{(1)}, \ldots, \theta_{(l-1)}$).Since types are independent the only relevant information (used for updating beliefs) is given by $\theta_{(l-1)}$. The expected utility of a bidder at the last period $l = k$ who knows $\theta_{(k-1)}$, faces bidders bidding according to b_k and who bids as if she were of type $\widehat{\theta}$ instead of θ is given by:

$$U_k\left(\theta, \widehat{\theta}, \theta_{(k-1)}\right) = \frac{F^{n-k}\left(\min\left(\widehat{\theta}, \theta_{(k-1)}\right)\right)}{F^{n-k}\left(\theta_{(k-1)}\right)}\left[D_k(\theta) - b_k\left(\widehat{\theta}\right)\right].$$

Therefore period k is like a first price auction with $n-k+1$ bidders whose types are distributed according to the (conditional) distribution $\frac{F(\theta)}{F\left(\theta_{(k-1)}\right)}$ and whose valuations are given by $D_k(\theta)$. Standard arguments show the optimality of bidding $b_k\left(\min\left(\theta,\theta_{(k-1)}\right)\right)$.

Let us assume that bidding according to $b_l\left(\min\left(\theta,\theta_{(l-1)}\right)\right)$is optimal at stages $l+1,\ldots,k$ and that all other bidders bid according to b_l. Obviously it is never profitable for a bidder at period l to bid as if she were of a type $\widehat{\theta}>\theta_{(l-1)}$. The expected utility of a bidder at period l knowing $\theta_{(l-1)}$ and bidding as if she were of type $\widehat{\theta}\leq\theta_{(l-1)}$ instead of θ in period l and bidding truthfully in all forthcoming periods is denoted by $U_l\left(\theta,\widehat{\theta},\theta_{(l-1)}\right)$ and given by:

$$\begin{aligned} U_l\left(\theta,\widehat{\theta},\theta_{(l-1)}\right) &= \frac{1}{F^{n-l}\left(\theta_{(l-1)}\right)}\left[F^{n-l}\left(\widehat{\theta}\right)\left(D_l(\theta)-b_l\left(\widehat{\theta}\right)\right)\right. \\ &\quad \left.+(n-l)\int_{\widehat{\theta}}^{\theta_{(l-1)}}\int_{\underline{\theta}}^{x}\cdots\int_{\underline{\theta}}^{x}U_{l+1}(\theta,\theta,x)\,f\left(\theta_{(n)}\right)d\theta_{(n)}\ldots f\left(\theta_{(l+1)}\right)d\theta_{(l+1)}f(x)\,dx\right]. \end{aligned}$$

To show that bidding according to b_l is optimal we have to analyze $\frac{\partial U_l\left(\theta,\widehat{\theta},\theta_{(l-1)}\right)}{\partial\widehat{\theta}}$. For $\widehat{\theta}\leq\theta_{(l+1)}$ we have:

$$\begin{aligned} \frac{\partial U_l\left(\theta,\widehat{\theta},\theta_{(l-1)}\right)}{\partial\widehat{\theta}} &= \frac{1}{F^{n-l}\left(\theta_{(l-1)}\right)}\left[(n-l)F^{n-l-1}\left(\widehat{\theta}\right)f\left(\widehat{\theta}\right)\left(D_l(\theta)-b_l\left(\widehat{\theta}\right)\right)-F^{n-l}\left(\widehat{\theta}\right)\frac{db_l\left(\widehat{\theta}\right)}{d\widehat{\theta}}\right. \\ &\quad \left.-(n-l)f\left(\widehat{\theta}\right)\int_{\underline{\theta}}^{\widehat{\theta}}\cdots\int_{\underline{\theta}}^{\widehat{\theta}}U_{l+1}(\theta,\theta,x)\,f\left(\theta_{(n)}\right)d\theta_{(n)}\ldots f\left(\theta_{(l+1)}\right)d\theta_{(l+1)}\right] \\ &= \frac{1}{F^{n-l}\left(\theta_{(l-1)}\right)}\left[(n-l)F^{n-l-1}\left(\widehat{\theta}\right)f\left(\widehat{\theta}\right)D_l(\theta)-\frac{d}{d\widehat{\theta}}\left(b_l\left(\widehat{\theta}\right)F^{n-l}\left(\widehat{\theta}\right)\right)\right. \\ &\quad \left.-(n-l)f\left(\widehat{\theta}\right)\int_{\underline{\theta}}^{\widehat{\theta}}\cdots\int_{\underline{\theta}}^{\widehat{\theta}}U_{l+1}(\theta,\theta,x)\,f\left(\theta_{(n)}\right)d\theta_{(n)}\ldots f\left(\theta_{(l+1)}\right)d\theta_{(l+1)}\right] \\ &= \frac{(n-l)}{F^{n-l}\left(\theta_{(l-1)}\right)}f\left(\widehat{\theta}\right)\left[\left(D_l(\theta)-D_l\left(\widehat{\theta}\right)+D_{l+1}\left(\hat{\theta}\right)-b_{l+1}\left(\widehat{\theta}\right)\right)F^{n-l-1}\left(\widehat{\theta}\right)\right. \\ &\quad \left.-\int_{\underline{\theta}}^{\widehat{\theta}}\cdots\int_{\underline{\theta}}^{\widehat{\theta}}U_{l+1}(\theta,\theta,x)\,f\left(\theta_{(n)}\right)d\theta_{(n)}\ldots f\left(\theta_{(l+1)}\right)d\theta_{(l+1)}\right]. \end{aligned}$$

The last equation holds since

$$\frac{d}{d\theta}\left(b_l(\theta)F^{n-l}(\theta)\right)=(n-l)F^{n-l-1}(\theta)f(\theta)\left(D_l(\theta)-D_{l+1}(\theta)+b_{l+1}(\theta)\right).$$

If $\widehat{\theta}<\theta$ we get $\frac{\partial U_m\left(\theta,\widehat{\theta},H_m\right)}{\partial\widehat{\theta}}>0$ since

$$\frac{\partial U_l\left(\theta,\widehat{\theta},\theta_{(l-1)}\right)}{\partial\widehat{\theta}}=\frac{(n-l)F^{n-l-1}\left(\widehat{\theta}\right)f\left(\widehat{\theta}\right)}{F^{n-l}\left(\theta_{(l-1)}\right)}\left(D_l(\theta)-D_{l+1}(\theta)-\left(D_l\left(\widehat{\theta}\right)-D_{l+1}\left(\widehat{\theta}\right)\right)\right)$$

and since D_l-D_{l+1} is increasing.

If $\widehat{\theta}>\theta$ we can conclude $\frac{\partial U_l\left(\theta,\widehat{\theta},\theta_{(l-1)}\right)}{\partial\widehat{\theta}}<0$ from $\frac{\partial U_{l+1}\left(\theta,\widehat{\theta},\theta_{(l)}\right)}{\partial\widehat{\theta}}<0$ by separating those situation where a bidder wins the next period and those where she does not win neither this period nor the next period:

$$\begin{aligned}
\frac{\partial U_l\left(\theta,\widehat{\theta},\theta_{(l-1)}\right)}{\partial\widehat{\theta}} &= \frac{(n-l)\,f\left(\widehat{\theta}\right)}{F^{n-l}\left(\theta_{(l-1)}\right)}\Big[F^{n-l-1}\left(\widehat{\theta}\right)\left(D_l(\theta)-D_l\left(\widehat{\theta}\right)+D_{l+1}\left(\widehat{\theta}\right)-b_{l+1}\left(\widehat{\theta}\right)\right) \\
&\quad -\int_{\underline{\theta}}^{\widehat{\theta}}\cdots\int_{\underline{\theta}}^{\widehat{\theta}} U_{l+1}\left(\theta,\theta,\widehat{\theta}\right) f\left(\theta_{(n)}\right)d\theta_{(n)}\ldots f\left(\theta_{(l+1)}\right)d\theta_{(l+1)}\Big] \\
&= \frac{(n-l)\,f\left(\widehat{\theta}\right)}{F^{n-l}\left(\theta_{(l-1)}\right)}\Big[F^{n-l-1}\left(\widehat{\theta}\right)\left(D_l(\theta)-D_l\left(\widehat{\theta}\right)+D_{l+1}\left(\widehat{\theta}\right)-b_{l+1}\left(\widehat{\theta}\right)\right) \\
&\quad -F^{n-l-1}(\theta)\left(D_{l+1}(\theta)-b_{l+1}(\theta)\right) \\
&\quad -(n-l-1)\int_{\theta}^{\widehat{\theta}}\int_{\underline{\theta}}^{x}\cdots\int_{\underline{\theta}}^{x} U_{l+2}(\theta,\theta,x)\, f\left(\theta_{(n)}\right)d\theta_{(n)}\ldots f\left(\theta_{(l+2)}\right)d\theta_{(l+2)}f(x)\,dx\Big] \\
&= \cdot\frac{(n-l)\,f\left(\widehat{\theta}\right)}{F^{n-l}\left(\theta_{(l-1)}\right)}\Big[F^{n-l-1}\left(\widehat{\theta}\right)\left(D_l(\theta)-D_l\left(\widehat{\theta}\right)+D_{l+1}\left(\widehat{\theta}\right)\right) \\
&\quad -\int_{\theta}^{\widehat{\theta}}\left(D_{l+1}(x)-D_{l+2}(x)+b_{l+2}(x)\right)dF^{n-l-1}(x)\,dx - F^{n-l-1}(\theta)\,D_{l+1}(\theta) \\
&\quad -(n-l-1)\int_{\theta}^{\widehat{\theta}}\int_{\underline{\theta}}^{x}\cdots\int_{\underline{\theta}}^{x} U_{l+2}(\theta,\theta,x)\, f\left(\theta_{(n)}\right)d\theta_{(n)}\ldots f\left(\theta_{(l+2)}\right)d\theta_{(l+2)}f(x)\,dx\Big] \\
&= \cdot\frac{(n-l)\,f\left(\widehat{\theta}\right)}{F^{n-l}\left(\theta_{(l-1)}\right)}\Big[F^{n-l-1}\left(\widehat{\theta}\right)\left(D_l(\theta)-D_l\left(\widehat{\theta}\right)+D_{l+1}\left(\widehat{\theta}\right)-D_{l+1}(\theta)\right) \\
&\quad +\int_{\theta}^{\widehat{\theta}}(n-l-1)\,f(x)\Big[\left(D_{l+1}(\theta)-D_{l+1}(x)+D_{l+2}(x)-b_{l+2}(x)\right)F^{n-l-2}(x) \\
&\quad -\int_{\underline{\theta}}^{x}\cdots\int_{\underline{\theta}}^{x} U_{l+2}(\theta,\theta,x)\, f\left(\theta_{(n)}\right)d\theta_{(n)}\ldots f\left(\theta_{(l+2)}\right)d\theta_{(l+2)}\Big]\,dx\Big]
\end{aligned}$$

Since $\theta < \widehat{\theta}$ this is negative if

$$\begin{aligned}
&\left(D_{l+1}(\theta)-D_{l+1}(x)+D_{l+2}(x)-b_{l+2}(x)\right)F^{n-l-2}(x) \\
&-\int_{\underline{\theta}}^{x}\cdots\int_{\underline{\theta}}^{x} U_{l+2}(\theta,\theta,x)\, f\left(\theta_{(n)}\right)d\theta_{(n)}\ldots f\left(\theta_{(l+2)}\right)d\theta_{(l+2)} < 0
\end{aligned}$$

for $\theta < x$ which is fulfilled since $\frac{\partial U_{l+1}\left(\theta,x,\theta_{(l)}\right)}{\partial x} < 0$ for $\theta < x$. The validity of this argument for period k follows from the concavity of $U_k\left(\theta,\widehat{\theta},\theta_{(k-1)}\right)$ since as argued above the $k'th$ period is like a "normal" first price auction[17]. Since it is obviously never optimal to bid higher than $b_m\left(\theta_{(m-1)}\right)$ we have shown that bidding according to $b_m\left(\min\left(\theta,\theta_{(m-1)}\right)\right)$ is optimal on each period $m \geq k$.
Q.E.D.

Proof of Theorem 14:

[17]For the induction argument to be carried out to the last stage we have to define $D_l = b_l = 0$ for $l < k$.

For the sequential second-price auction we have

$$b_l(\theta) = D_l(\theta) - D_{l+1}(\theta) + E\left[b_{l+1}\left(\theta_{(l+2)}\right) \mid \theta_{(l+1)} = \theta\right].$$

We define p_0 by $p_l = D_l(p_0)$ and get

$$\begin{aligned} & E\left[b_{l+1}\left(\theta_{(l+2)}\right) \mid b_l\left(\theta_{(l+1)}\right) = p_l\right] \\ = \; & D_l(p_0) - D_l\left(b_l^{-1}\left(D_l(p_0)\right)\right) + D_{l+1}\left(b_l^{-1}\left(D_l(p_0)\right)\right) \\ \leq \; & D_{l+1}(p_0). \end{aligned}$$

Where the inequality holds because $D_l - D_{l+1}$ is increasing and because $D_l > b_l$. The inequality is strict if $D_l - D_{l+1}$ is strictly increasing.

For the sequential first-price auction define $\widetilde{\theta}$ by the type who sets the price p_l, i.e.

$$b_l\left(\widetilde{\theta}\right) = E\left[D_l\left(\theta_{(l+1)}\right) - D_{l+1}\left(\theta_{(l+1)}\right) + b_{l+1}\left(\theta_{(l+1)}\right) \mid \theta_{(l)} = \widetilde{\theta}\right] = p_l.$$

We have that $b_{l+1} \leq D_{l+1}$ and therefore $D_{l+1}^{-1}\left(b_{l+1}\left(\theta_{(l+1)}\right)\right) \leq \theta_{(l+1)}$. Since $D_l - D_{l+1}$ is increasing, this implies

$$D_l\left(\theta_{(l+1)}\right) - D_{l+1}\left(\theta_{(l+1)}\right) + D_{l+1}\left(D_{l+1}^{-1}\left(b_{l+1}\left(\theta_{(l+1)}\right)\right)\right) \geq D_l\left(D_{l+1}^{-1}\left(b_{l+1}\left(\theta_{(l+1)}\right)\right)\right)$$

which yields

$$E\left[D_l\left(\theta_{(l+1)}\right) - D_{l+1}\left(\theta_{(l+1)}\right) + b_{l+1}\left(\theta_{(l+1)}\right) \mid \theta_{(l)} = \widetilde{\theta}\right] \geq E\left[D_l\left(D_{l+1}^{-1}\left(b_{l+1}\left(\theta_{(l+1)}\right)\right)\right) \mid \theta_{(l)} = \widetilde{\theta}\right].$$

Q.E.D.

Proof of Corollary 2:

1. This follows directly from Theorem 14 and the fact that $D_{l,l+1}^{-1}(x) \geq x$.

2. The fact that expected prices in period l are the same for both auction formats can be directly deduced from the bidding functions by a simple induction argument. The other statements are only shown for the first-price auction since the same arguments apply for the second-price auction. We have that

   $$E[b_l\left(\theta_{(l)}\right)] = E[D_l\left(\theta_{(l+1)}\right) - D_{l+1}\left(\theta_{(l+1)}\right) + b_{l+1}\left(\theta_{(l+1)}\right)].$$

   Since $D_l - D_{l+1}$ is increasing and $b_{l+1} \leq D_{l+1}$ we obtain

   $$E[D_l\left(\theta_{(l+1)}\right) - D_{l+1}\left(\theta_{(l+1)}\right) + D_{l+1}\left(D_{l+1}^{-1}\left(b_{l+1}\left(\theta_{(l+1)}\right)\right)\right)] \geq E[D_l\left(D_{l+1}^{-1}\left(b_{l+1}\left(\theta_{(l+1)}\right)\right)\right)].$$

   Similarly we have that

   $$E[b_l\left(\theta_{(l)}\right) - D_l\left(\theta_{(l+1)}\right) + D_{l+1}\left(\theta_{(l+1)}\right)] \leq E[D_{l+1}\left(D_l^{-1}\left(b_l\left(\theta_{(l)}\right)\right)\right)].$$

   The last statement follows from Jensen's inequality.

3. The results from the linearity of $D_{l,l+1}$and Theorem 14.

Q.E.D.

Proof of Theorem 15:

A sequential $k-$period auction is given by the strategy set $\mathbf{R}^{+}$, the sets of participating bidders of period l, $H_l \subseteq \{1,\ldots,n\}$, the sellers information policy, allocation functions $s = (s_1,\ldots,s_k)$ and payment functions $t = (t_1,\ldots,t_k)$ specified as follows: In period l all bidders submit bids $b_{l,i} \in \mathbf{R}^{+}$. The allocation function $s_l : \mathbf{R}^n \mapsto \{1,\ldots,n\}$ allocates the l'th object to the highest participating bidder of that period[18], i.e. $s_l(b_{l,1},\ldots,b_{l,n}) = \arg\max_{i\in H_l} b_{l,i}$. Bidder i has to make a payment to the seller which is given by $-t_{l,i}(b_{l,1},\ldots,b_{l,n})$ whereas this is zero for non-participating bidders (i.e. $t_{l,i} = 0$ if $i \notin H_l$) and does not depend on bids of non-participating bidders (i.e. for all $j \notin H_l$ we have $t_{l,i}(b_{l,j}, b_{l,-j}) = t_{l,i}\left(\widetilde{b}_{l,j}, b_{l,-j}\right)$ for all $i \in H_l$ and $b_{l,j}, \widetilde{b}_{l,j} \in \mathbf{R}^{+}$). In period $l+1$ the set of participating bidders is given by $H_{l+1} = H_l \backslash \{s_l\}$. Before period l information concerning the winning type of the previous period might be revealed to all agents (e.g. if we have efficient equilibria the seller can do so by announcing the highest bid of the previous period). The information policy is common knowledge.

Consider period l of a $k-$period sequential auction where everyone bids according to an efficient equilibrium in previous periods. The belief about other types' distribution of an agent who participates in period l depends on the previously observed history and on her own type. We will denote this distribution (for an agent i) by $F_{l,i}(\theta_i, \theta_{-i})$. If the winner's type of period $l-1$ is known $F_{l,i}$ does only depend on this type since types are distributed independently. If no announcements are made, $F_{l,i}$ only depends on θ_i (since we assumed truthful bidding in previous periods). In period l no agent should have an incentive to bid as if she were of a different type given all other agents stick to the equilibrium. We denote by $U_{l,i}\left(\theta_i, \widehat{\theta}_i\right)$ the expected utility of an agent $i \in H_l$ of type θ_i in period l who behaves subsequently as if she were of type $\widehat{\theta}_i$ and who faces agents that are bidding according to the equilibrium (and do not imitate other types). In addition we denote by -$t_{l,i}\left(\widehat{\theta}_i, \theta_{-i}\right)$ the payment of a bidder who bids as if she were of type $\widehat{\theta}_i$ in period l (given the other agents bid according to their equilibrium strategies). We have that

$$
\begin{aligned}
& U_{l,i}\left(\theta_i, \widehat{\theta}_i\right) \qquad (29)\\
= \; & D_l(\theta_i) E_{\theta_{-i}}\left[\mathbf{1}\left(\widehat{\theta}_i > \theta_{(l)}\right) \mid F_{l,i}(\theta_i, \theta_{-i})\right] + E_{\theta_{-i}}\left[t_{l,i}\left(\widehat{\theta}_i, \theta_{-i}\right) \mid F_{l,i}(\theta_i, \theta_{-i})\right] \\
& + \sum_{j=l+1}^{k}\left[D_j(\theta_i) E_{\theta_{-i}}\left[\mathbf{1}\left(\theta_{(j-1)} > \widehat{\theta}_i > \theta_{(j)}\right) \mid F_{l,i}(\theta_i, \theta_{-i})\right] + E_{\theta_{-i}}\left[t_{j,i}\left(\widehat{\theta}_i, \theta_{-i}\right) \mid F_{l,i}(\theta_i, \theta_{-i})\right]\right].
\end{aligned}
$$

Consider the case where the winning type of the previous period is known, i.e. $F_{l,i}(\theta_i, \theta_{-i}) = F_{l,i}(\theta_{-i})$. Since imitating another type cannot be profitable we have that

[18] If there is more than a single highest bidder any tie-breaking rule can be applied.

$U_{l,i}(\theta_i) := U_{l,i}(\theta_i,\theta_i) = \max_{\widehat{\theta}_i} U_{l,i}\left(\theta_i,\widehat{\theta}_i\right)$ and therefore the Envelope-Theorem yields:

$$\frac{dU_{l,i}(\theta_i)}{d\theta_i} = \sum_{j=l}^{k} \frac{dD_j(\theta_i)}{d\theta_i} E_{\theta_{-i}}\left[\mathbf{1}\left(\theta_{(j-1)} > \theta_i > \theta_{(j)}\right) \mid F_{l,i}(\theta_{-i})\right]. \tag{30}$$

Combining (29) and (30) yields

$$\begin{aligned} E_{\theta_{-i}}\left[t_{l,i}(\theta_i,\theta_{-i}) \mid F_{l,i}(\theta_{-i})\right] &= \sum_{j=l}^{k} D_j(\theta_i) E_{\theta_{-i}}\left[\mathbf{1}\left(\theta_{(j-1)} > \theta_i > \theta_{(j)}\right) \mid F_{l,i}(\theta_{-i})\right] \\ &+ \sum_{j=l+1}^{k} E_{\theta_{-i}}\left[t_{j,i}(\theta_i,\theta_{-i}) \mid F_{l,i}(\theta_i,\theta_{-i})\right] - U_{l,i}(\underline{\theta}) \\ &- \int_{\underline{\theta}}^{\theta_i}\left(\sum_{j=l}^{k} \left.\frac{dD_j}{d\theta_i}\right|_{\theta_i=t} E_{\theta_{-i}}\left[\mathbf{1}\left(\theta_{(j-1)} > t > \theta_{(j)}\right) \mid F_{l,i}(\theta_{-i})\right]\right) dt. \end{aligned}$$

The ex-ante expected payment an agent has to make in period l is therefore given by:

$$\begin{aligned} -E_{\theta}\left[t_{l,i}(\theta)\right] &= -E_{\theta}\left[E_{\theta_{-i}}\left[t_{l,i}(\theta_i,\theta_{-i}) \mid F_{l,i}(\theta_{-i})\right]\right] = \\ & -\sum_{j=l}^{k} D_j(\theta_i) E_{\theta}\left[\mathbf{1}\left(\theta_{(j-1)} > \theta_i > \theta_{(j)}\right)\right] + \sum_{j=l+1}^{k} E_{\theta}\left[t_{j,i}(\theta)\right] + U_{l,i}(\underline{\theta}) \\ & + E_{\theta_i}\left[\int_{\underline{\theta}}^{\theta_i}\left(\sum_{j=l}^{k} \frac{dD_j}{d\theta_i}(t) E_{\theta_{-i}}\left[\mathbf{1}\left(\theta_{(j-1)} > t > \theta_{(j)}\right)\right]\right) dt\right]. \end{aligned}$$

Therefore we also have that $E_{\theta}\left[t_{l,i}(\theta)\right]$ is the same for all sequential auctions (with announcement of winning types) if this is true for $E_{\theta}\left[t_{j,i}(\theta)\right],\ j = l+1,\ldots,n$. By induction we can conclude that this indeed must be the case.

If no announcements are made, beliefs are updated by using the information that all winning types of previous periods are higher than the own type (i.e. $F_{l,i}$ only depends on her own type θ_i). In this case we have

$$E_{\theta_{-i}}\left[t_{l,i}\left(\widehat{\theta}_i,\theta_{-i}\right) \mid F_{l,i}(\theta_i,\theta_{-i})\right] = t_{l,i}\left(\widehat{\theta}_i\right)$$

and the Envelope-Theorem and the argumentation above apply to this case as well.

Q.E.D.

Proof of Corollary 3: The Corollary is an immediate consequence of the proof of Theorem 13 and Corollary 2. We have (expected) payoffs which are identical to a sequential auction (as analyzed in the proof of Theorem 13) in a period l up to a factor of $\prod_{i=1}^{l-1}\delta_i$ if we set $D_1(\theta) = \theta$ and $D_{l+1}(\theta) = \delta_l D_l(\theta)$. Therefore the analysis is the same as in Theorem 13.

Q.E.D.

Proof of Theorem 16:

From formula (13) we observe that $\overline{p}_l = \left(\delta^{l-1} - \delta^l\right) E[\theta_{(l+1)}] + \overline{p}_{l+1}$. Hence

$$\begin{aligned} \frac{p_{l+1}}{p_l} &> \frac{p_l}{p_{l-1}} \\ \Leftrightarrow \quad \frac{p_{l+1}}{p_l} &> \frac{\left(\delta^{l-1} - \delta^l\right) E[\theta_{(l+1)}] + p_{l+1}}{\left(\delta^{l-2} - \delta^{l-1}\right) E[\theta_{(l)}] + p_l} \\ \Leftrightarrow \quad .\frac{p_{l+1}}{p_l} &> \frac{\left(\delta^{l-1} - \delta^l\right) E[\theta_{(l+1)}]}{\left(\delta^{l-2} - \delta^{l-1}\right) E[\theta_{(l)}]}. \end{aligned} \tag{31}$$

1) Because of (31) we have to show that for every distribution there exists $\delta \in (0,1)$ with $\frac{\overline{p}_{l+1}}{\overline{p}_l} > \delta \frac{E[\theta_{(l+1)}]}{E[\theta_{(l)}]}$ for all $l \leq k$. Let m be defined by $m = \arg\max_{l \leq k} \frac{E[\theta_{(l+1)}]}{E[\theta_{(l)}]}$. We know that $\frac{E[\theta_{(m+1)}]}{E[\theta_{(m)}]} < 1$. Since for $\delta \to 1$ we have $\frac{\overline{p}_{l+1}}{\overline{p}_l} \to 1$ for all l there exists a δ sufficiently close to 1 such that $\frac{\overline{p}_{l+1}}{\overline{p}_l} > \frac{E[\theta_{(m+1)}]}{E[\theta_{(m)}]}$ for all $l \leq k$.

2) We have $\frac{\overline{p}_k}{\overline{p}_{k-1}} \underset{(<)}{>} \frac{\overline{p}_{k-1}}{\overline{p}_{k-2}}$ because of $\frac{\overline{p}_k}{\overline{p}_{k-1}} = \delta \frac{E\left[\theta_{(k+1)}\right]}{(1-\delta)E\left[\theta_{(k)}\right] + \delta E\left[\theta_{(k+1)}\right]}$ the assumption and (31). Similarly to (31) we have that

$$\frac{\overline{p}_{l+1}}{\overline{p}_l} \underset{(<)}{>} \frac{\overline{p}_l}{\overline{p}_{l-1}} \Leftrightarrow \frac{\overline{p}_l}{\overline{p}_{l-1}} \underset{(<)}{>} \delta \frac{E[\theta_{(l+1)}]}{E[\theta_{(l)}]}.$$

Therefore if $\frac{\overline{p}_{l+1}}{\overline{p}_l} \underset{(<)}{>} \frac{\overline{p}_l}{\overline{p}_{l-1}}$ we have (from the assumptions) that $\frac{\overline{p}_l}{\overline{p}_{l-1}} \underset{(<)}{>} \delta \frac{E[\theta_{(l+1)}]}{E[\theta_{(l)}]} \underset{(<)}{>} \delta \frac{E[\theta_{(l)}]}{E[\theta_{(l-1)}]}$ which implies that $\frac{\overline{p}_l}{\overline{p}_{l-1}} \underset{(<)}{>} \frac{\overline{p}_{l-1}}{\overline{p}_{l-2}}$. The statement therefore follows by induction.

Q.E.D.

Hiermit erkläre ich, dass ich die Dissertation selbständig angefertigt und mich anderer als der in ihr angegebenen Hilfsmittel nicht bedient habe, insbesondere, dass aus anderen Schriften Entlehnungen, soweit sie in der Dissertation nicht ausdrücklich als solche gekennzeichnet und mit Quellenangaben versehen sind, nicht stattgefunden haben.

Mannheim, den 13.1.2002 Thomas Kittsteiner

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