Mean Field Limit of Many Particle System with Non-Lipschitz Force

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Abstract

In this thesis we consider the mean field limit of N-particle system induced both from social science application and from physical background. Based on establishing the ordinary differential equation of characteristics for transport equation, we handle the non-Lipschitz force in the non-linear partial differential equation, which is also a Vlasov-type equation.

The first part of the thesis is to review different kinds of kinetic particle systems to their corresponding Vlasov-related equations. We will summerize the literatures to date and give a whole picture of what the mean field limit is all about through very concrete examples and models done so far. Followed by the review, we will in the chapters to come present some novel ideas and methods that we used to tackle those problems during the years of research.

Further for the particle model, we investigate a two-dimensional pedestrian flow system and illustrate the probabilistic method in detailed steps to show that the N-particle pedestrian flow system can be represented by the one particle density function when N approaches infinity, which is so-called mean field equation. As regard to the wellposedness of the mean field equation, the weak solution is also presented with a more general setting for the equation, or in other words, for the non-Lipschitz force. Last but not least, we focus on the well-known relativistic Vlasov-Maxwell and present both the non-relativistic limit and mean field limit for the corresponding particle relativistic Vlasov-Maxwell model, which converges to what we know as the Vlasov-Poisson equation, with all the analysis in the last chapter.

Zusammenfassung

In dieser Arbeit betrachten wir die *mean field limit* des N-Partikelsystems, die sowohl aus sozialwissenschaftlichen Anwendungen als auch aus physikalischen Gründen induziert wird. Basierend auf der Festlegung der gewöhnlichen Differentialgleichung der *Characteristics* für die Transportgleichung, behandeln wir die Nicht-Lipschitz-Kraft in der nichtlinearen partiellen Differentialgleichung, die ebenfalls eine Vlasov-Gleichung ist.

Der erste Teil der Arbeit ist die Überprüfung verschiedener Arten von kinetischen Partikelsystemen zu ihren entsprechenden Vlasov-bezogenen Gleichungen. Wir werden die bisherigen Literaturen zusammenfassen und anhand sehr konkreter Beispiele und bisheriger Modelle ein Gesamtbild davon vermitteln, worum es bei der *mean field limit* geht. Nach der Überprüfung werden wir in den folgenden Kapiteln einige neue Ideen und Methoden vorstellen, mit denen wir diese Probleme in den Jahren der Forschung angegangen sind.

Weiterhin untersuchen wir für das Partikelmodell ein zweidimensionales Fußgängerströmungssystem und veranschaulichen die probabilistische Methode in detaillierten Schritten, um zu zeigen, dass das N-Partikel-Fußgängerströmungssystem durch die One-Particle-Density-Funktion dargestellt werden kann, wenn sich N der Unendlichkeit nähert, was eine sogenannte *mean field Gleichung* ist. Hinsichtlich der Wellposedness der *mean field Gleichung* wird der schwachen Lösung auch eine allgemeinere Einstellung für die Gleichung, also für die Nicht-Lipschitz-Kraft, gegeben. Letztens konzentrieren wir uns auf den bekannten relativistischen Vlasov-Maxwell und präsentieren sowohl die nicht-relativistische Grenze als auch die *mean field limit* für das entsprechende partikelrelativistische Vlasov-Maxwell-Modell, das mit der gesamten Analyse im letzten Kapitel zu dem konvergiert, was wir als Vlasov-Poisson-Gleichung kennen.

Contents

Chapter 1

Introduction

Collective behaviour of a large number of interacting individuals is a very ubiquitous and yet an important phenomenon, such as animal behaviours (flocking, swarming), pedestrian flow and many other social and natural sciences. Modelling these collective behaviours is no doubt an essential task and mathematically challenging research topic. Most of the literatures up to date follow the strategy of so-called mean field limit, which is from the Newtonian motion equations of particle description to its corresponding one-particle density equation. This limiting process is actually from microscopic level to mesoscopic level. One can also go further to the hydrodynamic limit, which is marcoscopic level.

Nature and human societies offer many examples of self-organized behavior. Ants form colonies, birds fly in flocks, mobile networks coordinate a rendezvous, and human opinions evolve into parties. These are simple examples of collective dynamics that tend to self-organize into large-scale clusters of colonies, flocks, parties, etc. [113].

In the following sections, we will review and investigate different kinds of microscopic particle models and their corresponding mesoscopic descriptions.

1.1 Background

Modeling the collective behaviour of a large number of interacting individuals is a very challenging problem in animal behaviour, pedestrian flow, cell adhesion and chemotaxis problems, and many other biological applications, see for instance [6, 12, 13, 94] and the literature therein [17].

Consider a system of identical point particles. If the total number of particles is large enough, the state of the system at the time t can be then described in the statistic sense by the distribution function $f \equiv f(t, x, v)$ in the one particle phase space, which represents the density of particles located at the position $x \in \mathbb{R}^n$ with velocity $v \in \mathbb{R}^n$ at the time t. Due to the practical application, we consider the dimension $n = 2$ or $n = 3$. Mean field equation is best understood by getting acquainted with the most famous examples of such equations listed below.

1.2 Vlasov-type Equations

Vlasov-Poisson

The Vlasov-Poisson system reads

$$\begin{cases} \partial_t f_p + v \cdot \nabla_x f_p + E_p \cdot \nabla_v f_p = 0, \\ E_p(t,x) = \dfrac{1}{4\pi} \displaystyle\int_{\mathbb{R}^3} \rho_p(t,y) \frac{x-y}{|x-y|^3}\, dy, \\ \rho_p(t,y) = \displaystyle\int_{\mathbb{R}^3} f_p(t,y,v)\, dv, \end{cases} \tag{1.2.1}$$

with the initial data $f_p(0,x,v) = f_0(x,v)$.

The Cauchy problem for the Vlasov-Poisson system has been the focus of countless works in the past few decades. For the Vlasov-Poisson system, global existence and uniqueness of classical solutions were obtained by Ukai and Okabe [119] in two dimensions. The three dimensional case is more delicate; global weak solutions with finite energy were first built by Arsenev [5]. Global existence and, in some cases, uniqueness, of more regular solutions were then separately established by Lion and Perthame [84] and by Pfaffelmoser [98] with different techniques. In both works the main issue consists in controlling the large plasma velocities for all time in order to propagate regularity properties of the solutions. In Lion and Perthame's work [84], this is achieved by constructing weak solutions with finite velocity moments of order higher than three

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^m f(t,x,v)\, dxdv < \infty, \quad m > 3,$$

which. by Sobolev embeddings, implies further bounds on the spatial density and on the electric field. In particular, if the solution admits finite moments of order m $>$ 6 then the electric field is uniformly bounded and uniqueness holds under some additional regularity assumptions on the initial density. On the other hand, the theory of DiPerna and Lions [40] ensures that such solutions are constant along the trajectories of a "generalized flow" defined in a weak sense. In contrast with the Eulerian approach of [84], the strategy of [98] relies on a careful analysis of the characteristics to control the growth of the velocity support and thereby obtain global existence and uniqueness of classical compactly supported solutions, which moreover propagate the regularity of the initial density [37].

We refer to the further improvements and developments by Schaeffer [107], Wollman [125], Castella [21], Loeper [85], Chen and Zhang [24]. Moreover, Gasser, Jabin and Perthame [45] established propagation of the velocity moments for $m > 2$ with an additional assumption on the space moments, and in [105], Salort proved existence and uniqueness of weak solutions even if $m < 6$. Finally, Pallard [92] recently combined Eulerian and Lagrangian points of view to establish existence of solutions propagating velocity moments for $m > 2$ and obtained an explicit polynomial in time bound on the moments [37].

Vlasov-Maxwell

The time evolution of the magnetized plasma when each charged particle is accelerated by Lorentz force which comes from the electromagnetic field created by all the other particles [50] is described by the relativistic Vlasov-Maxwell system:

$$\begin{cases} \partial_t f_m + \hat{v} \cdot \nabla_x f_m + (E_m + c^{-1}\hat{v} \times B_m) \cdot \nabla_v f_m = 0, \\ \partial_t E_m = c\nabla \times B_m - j_m, \quad \nabla \cdot E_m = \rho_m, \\ \partial_t B_m = -c\nabla \times E_m, \quad \nabla \cdot B_m = 0, \end{cases} \tag{1.2.2}$$

where $\hat{v} = \dfrac{v}{\sqrt{1 + c^{-2}v^2}}$, $\rho_m(t,x) = \displaystyle\int_{\mathbb{R}^3} f_m(t,x,v)\,dv$ and $j_m(t,x) = \displaystyle\int_{\mathbb{R}^3} \hat{v} f_m(t,x,v)\,dv$. The parameter c is the speed of light, (E_m, B_m) is the electro-magnetic field, and the distribution function $f_m(t,x,v) \geq 0$ describes the density of particles with position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$. The initial data

$$\begin{cases} f_m(0,x,v) = f_0(x,v), \\ E_m(0,x) = E_0(x), \\ B_m(0,x) = B_0(x), \end{cases} \tag{1.2.3}$$

satisfy the compatibility conditions $\nabla \cdot E_0(x) = \rho_0(x) = \displaystyle\int_{\mathbb{R}^3} f_0(x,v)\,dv$, $\nabla \cdot B_0(x) = 0$.

Local existence and uniqueness of classical solutions to this initial value problem for smooth and compactly supported data was established in [47]. These solutions can be extended globally in time provided the momentum support can be controlled, which has been done for data which are small [48] or close to neutral [46] or close to spherically symmetric [102]. Worthy of mentioning is that different approaches to the results in [47] were recently given in [9, 73]. In order to obtain the global solution, DiPerna and Lions weakened the solution concept to weak solutions. We refer to [39].

As was shown in [106] using an integral representation for the electric and magnetic field due to Glassey and Strauss [47], the solutions of relativistic Vlasov-Maxwell system converge in pointwise sense to solutions of the non-relativistic Vlasov-Poisson system (below) at the rate of $1/c$ as c tends to infinity. The Vlasov-Poisson system reads

$$\begin{cases} \partial_t f_p + v \cdot \nabla_x f_p + E_p \cdot \nabla_v f_p = 0, \\ E_p(t,x) = \dfrac{1}{4\pi} \displaystyle\int_{\mathbb{R}^3} \rho_p(t,y) \frac{x-y}{|x-y|^3} \, dy, \\ \rho_p(t,y) = \displaystyle\int_{\mathbb{R}^3} f_p(t,y,v) \, dv, \end{cases} \tag{1.2.4}$$

with the initial data $f_p(0,x,v) = f_0(x,v)$. We note that there are global existence results for classical solutions of the Vlasov-Poisson system [84, 98, 107].

However a more interesting and challenging question to consider is what the corresponding particle model of relativistic Vlasov-Maxwell equation is and whether we can rigorously in mathematics prove the mean filed limit (or in the large N limit). Up to our knowledge and at the time of this writing, taking both the mean filed limit and the non-relativistic limit (or classical limit) of Vlasov-Maxwell system together into account is rare in all literatures. Separately speaking of the mean field limit, Braun and Hepp [11] and Dobrushin [41] have proposed rigorous derivations of a system analogous to the Vlasov-Poisson system with a twice differentiable mollification of the Coulomb potential. Hauray and Jabin [63] have succeeded in treating the case of singular potentials, but not including the Coulomb singularity yet. Until recently, Lazarovici and Pickl [81] gave a probabilisitic proof of the mean field limit and propagation of chaos N-particle systems in three dimensions with Coulomb potential, which provide us with a very constructive idea of method.

On the other hand, writing down the corresponding N-particle model of the non-relativistic Vlasov-Maxwell system is a perplexing task because one needs to find the satisfying description of the electromagnetic self-interaction within the theory of classical electrodynamics [43, 70, 110]. The problem of deriving a regularized variant of the Vlasov-Maxwell system from a particle model was explicitly mentioned by Kiessling in [72]. Only after several years did Golse [50] establish the mean field limit of a N-particle system towards a regularized variant of the relativistic Vlasov-Maxwell system with the help of [43] by Elsken, Kiessling and Ricci.

In the present work, we want to combine the mean field limit and non-relativistic limit of the regularized relativistic Vlasov-Maxwell particle model to Vlasov-Poisson equation. The method we apply here is more or less along the line of [47, 50, 81] with the mollifications, the regularization procedure of which somehow removes the difficulties caused by the electromagnetic self-interaction forces. Unlike regularizing the Coulomb potential in the mean field limit established in [11, 41], the regularization of the self-interaction force in Vlasov-Maxwell system is more difficult since the electromagnetic field involves both a scalar and vector potentials [50]. The solutions of the relativistic Vlasov-Maxwell system, as was discussed by Glassey and Strauss in [47], are closely related with the wave equation, namely with Kirchhoff formula, which we also used in this paper. We would like to mention that there are other representations of the solutions of the relativistic Vlasov-Maxwell system, for example [9, 10], but they are all in fact equivalent.

Vlasov-Klein-Gordon

The relativistic Vlasov-Klein-Gordon describes a collisionless ensemble of particles moving at relativistic speeds, coupled to a Klein-Gordon field. Let $f = f(t, x, v) \geq 0$ denote the density of the particles in phase space, $\rho = \rho(t, x)$ their density in space, and $u = u(t, x)$ a scalar Klein-Gordon field, where $t \in \mathbb{R}_+$, $x \in \mathbb{R}^3$ and $v \in \mathbb{R}^3$ are the time, position and velocity respectively. The system reads

$$\begin{cases} \partial_t f_{KG} + \hat{v} \cdot \nabla_x f_{KG} - \nabla_x u \cdot \nabla_v f_{KG} = 0, \\ \partial_t^2 u - \Delta u + u = -\rho_{KG}, \\ \rho_{KG}(t, y) = \int_{\mathbb{R}^3} f_{KG}(t, y, v)\, dv, \end{cases} \tag{1.2.5}$$

with the initial data $f_{KG}(0, x, v) = f_0(x, v)$, $u(0, x) = u_1(x)$ and $\partial_t u(0, x) = u_2(x)$.
The study of this system was first initiated in [77], where the existence of global weak solutions for initial data with a size restriction was proven. This size restriction was necessary because the energy of the system is indefinite so that conservation of energy does not lead to a-priory bounds for general data [78]. Rein in [78] proved local-in-time existence of classical solutions and a continuation criterion which says that a solution can blow up only if the particle momenta become large. They also show that the classical solutions are global in time in the one-dimensional case.

Cucker-Smale Model

Flocking is a general phenomenon where autonomous agents reach a consensus based on the limited environmental information and simple rules (mainly three basic rules):

1. Separation - avoid crowding neighbours (short range repulsion);

2. Alignment (re-orientation)- steer towards average heading of neighbours;

3. Cohesion - steer towards average position of neighbours (long range attraction).

There are parallels with the shoaling behavior of fish, the swarming behavior of insects, and herd behavior of land animals. Mathematically, Cucker-Smale [31, 30] in 2007 postulated a model for the flocking of birds with the following system of ODEs:

$$\begin{cases} \dfrac{d}{dt} x_i = v_i, \\ \dfrac{d}{dt} v_i = \displaystyle\sum_{j=1}^{N} m_j (v_j - v_i) \varphi(|x_j - x_i|), \end{cases}$$

where N is the number of the particles while $x_i(t)$, $v_i(t)$ and m_i denote respectively the position and velocity of the i-th particle at the time t with its mass. The function $\varphi : [0, \infty) \to [0, \infty)$ is referred to as the communication weight, which is non-negative and non-increasing. The corresponding Vlasov-type equation

$$\partial_t f + v \cdot \nabla f + \mathrm{div}_v (F(f) f) = 0, \quad x \in \mathbb{R}^3, v \in \mathbb{R}^3,$$

$$F(f)(t, x, v) := \iint_{\mathbb{R}^6} \varphi(|y - x|)(w - v) f(t, y, w) \, dw dy.$$

The literature on aggregation models associated with Vlasov-type equations is rich thus we mention only a few examples of the most popular branches of the field. Here we find analysis of time asymptotics (see e.g. [57]) and pattern formation (see e.g. [56, 115]) or analysis of the models with additional forces that simulate various natural factors (see e.g. [20, 42]-deterministic forces or [29] - stochastic forces). The other variations of the model include forcing particles to avoid collisions (see e.g. [27]) or to aggregate under the leadership of certain individuals (see e.g. [28]). A well rounded analysis of a model that includes effects of attraction, repulsion and alignment is presented in [14]. The story of the CS model should probably begin with [120] by Vicsek et al., where a model of flocking with nonlocal interactions was introduced and it is widely recognized to be up to some degree an inspiration for [31] . Since 2007 the CS model with a regular communication weight of the form

$$\varphi_{cs}(s) = \frac{K}{(1 + s^2)^{\frac{\beta}{2}}}, \quad \beta \geq 0, K > 0$$

was extensively studied in the directions similar to those of more general aggregation models (i.e. collision avoiding, flocking under leadership, asymptotics and pattern formation as well as additional deterministic or stochastic forces - see [2, 19, 55, 59, 93, 108]). Particularly interest- ing from our point of view is the case of passage from the particle system to the kinetic equation, which in case of the regular communication weight was done for example in [61] or [62]. For a more general overview of the passage from microscopic to mesoscopic and macroscopic descriptions in aggregation models, we refer to [15, 34, 36]. In the paper [61] from 2009 the authors considered the CS model with the singular weight $\varphi(s) = s^{-\alpha}, s > 0, \alpha > 0$ obtaining asymptotics for the particle system but even the basic question of existence of solutions remained open till later years. It turned out that system (1.2.6) possesses drastically different qualitative properties depending on whether $\alpha \in (0,1)$ or $\alpha \in [1,\infty)$. More precisely in [1] the authors observed that for $\alpha \geq 1$ the trajectories of the particles exhibit a tendency to avoid collisions, which they used to prove conditional existence and uniqueness of smooth solutions to the particle system. On the other hand in [96] the author proved existence of so called piecewise weak solutions to the particle system with $\alpha \in (0,1)$ and gave an example of solution that experienced not only collisions of the trajectories but also sticking (i.e. two different trajectories could start to coincide at some point). This dichotomy is an effect of integrability (or of the lack of thereof) of φ in a neighborhood of 0. It is also the reason why the approach to the CS model should vary depending on α [89]. One of the latest contributions to this topic is [16] where the authors showed local in-time well posedness for the kinetic equation with a singular communication weight $\varphi(s) = s^{-\alpha}, s > 0, \alpha > 0$ and with an optional nonlinear dependence on the velocity in the definition of $F(f)$. They also presented a thorough analysis of the asymptotics for this model. The other more recent addition is [97], where the author proved existence and uniqueness of $W^{1,1}$ strong solutions to the particle system with a singular weight and $\alpha \in (0,1)$. In the model with regular weight its purpose is to suppress the distant interactions between particles. However from the modeling point of view it is often convenient to also amplify the local interactions, which was done for example in [88] by introducing a different nonsymmetric CS-type model known as the Mosch-Tadmor model. Singular communication weight in the CS model can also be viewed as a less effective yet easier to analyze way to emphasize the local interactions between particles.

Chapter 2

Mean Field Kinetic Equation with non-Lipschitz Force

In this chapter, we investigate a two-dimensional kinetic mean field equation with position $x \in \mathbb{R}^2$ and velocity $v \in \mathbb{R}^2$

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot [(F * f)f] + \nabla_v \cdot (Gf) = 0. \tag{2.0.1}$$

This kind of equation has long been used to characterize social phenomena (or so-called social fields [83] or social forces [66]) and industrial production, for example pedestrian flow model and material flow model. These models, due to a great increase in interest over the last few years, have been developed and investigated from a numerical and theoretical point of view, see for example [6, ?, 90] for a general overview. Highly inspired by fluid dynamics, pedestrian models or material flow models can also be further extended to the applications of other behavioral models including group dynamics [7], opinion formation [116], minimal travel times [38, 67] or evacuation scenarios [100, 118]. On the other hand, model hierarchies for pedestrian models or material flow models have been introduced in [33, 44, 52, 53]. Therein, macroscopic equations are formally derived from a microscopic pedestrian Newtonian system. Depending on the closure assumption, different non-local continuum models can occur, cf. [25]. However, from an analytical point of view, there are still several open problems that need to be thoroughly investigated as for instance the detailed derivation from the N-particle pedestrian Newtonian system to its mean field limit or Vlasov equation, see [22]. Instead of the formal derivation with the help of the BBGKY hierarchy [44, 111], the kinetic description has been rigorously derived by a probabilistic method [8, 11, 63, 65, 99, 112].

In this chapter, we now aim to prove the global existence of the weak solution to the mean field kinetic equation mentioned above, namely (2.0.1). In our equation, $F(x,v)$ denotes the total interaction force and has the similar structure as $\dfrac{x}{|x|}$, i.e.,

$$F(x,v) = \nabla_x V(|x|,v) = \partial_r V(r,v)\frac{x}{|x|},$$

where $V(|x|,v)$ is some (regular) potential. More precisely, $F(x,v)$ can be a composition of the interaction force $F_{int}(x)$ and the dissipative force $F_{diss}(x,v)$, i.e.,

$$F(x,v) = (F_{int}(x) + F_{diss}(x,v))\mathcal{H}(x,v) \tag{2.0.2}$$

and $\mathcal{H}(x,v) := \mathcal{H}_{2R}(|x|) \cdot \widetilde{\mathcal{H}}_{2\widetilde{R}}(|v|)$, where $\mathcal{H}_{2R}(|x|)$ and $\widetilde{\mathcal{H}}_{2\widetilde{R}}(|v|)$ are smooth functions with compact support such that

$$\mathcal{H}_{2R}(|x|) = \begin{cases} 0, & |x| > 2R, \\ 1, & |x| < R, \end{cases} \quad \text{and} \quad \widetilde{\mathcal{H}}_{2\widetilde{R}}(|v|) = \begin{cases} 0, & |v| > 2\widetilde{R}, \\ 1, & |v| < \widetilde{R}. \end{cases}$$

In order to cover a realistic behavior of pedestrians or material flow, the functions $\mathcal{H}_{2R}(|x|)$ and $\widetilde{\mathcal{H}}_{2\widetilde{R}}(|v|)$ are used to express that the interaction force and the pedestrian velocity are of finite range. So the total force is considered on a bounded domain.

The other term $G(x,v)$ in the equation represents the desired velocity and direction acceleration and can be further written as

$$G(x,v) = g(x) - v, \tag{2.0.3}$$

where $\|g\|_{L^\infty}$ is bounded by some constant.

Apparently, the proposed model equation involves a singularity comparable to the Coulomb potential in 2-d, resulting from the total interaction force. That means this singularity, or in other words the non-local term, needs extra care in the final limiting process. For more information about the Coulomb potential and the Vlasov-Poisson system we refer to [98, 104, 107].

We now briefly explain our approach. In order to obtain the existence of the weak solution, we consider an approximate problem (kinetic equation with cut-off) as a starting point and show that the approximate problem has a weak solution, where the mean field characteristic flow is of great importance. Unlike the 3-d Vlasov-Poisson equation [41, 84], the non-local operator in (3.1.2) cannot be decoupled into an elliptic equation. Hence, the Calderón-Zygmund continuity theorem [51] for second order elliptic equations is not applicable in this case and we have to find an alternative way to fix the desired compactness arguments. The idea is now to use the Aubin-Lions lemma [23, 109] and to argue that due to that compact embedding theorem, we are able to pass the limit especially in the non-local term. We also remark that the result obtained in the present paper plays a crucial role in the proof of the rigorous derivation of the mean field equation in [22].

2.1 Method of Characteristics

In this section, we first recall some basic knowledge from transport equation. As we all know, the following transport equation

$$\partial_t f + v \cdot \nabla_x f = 0, \quad x \in \mathbb{R}^n, t > 0$$

with initial data $f(0,x) = f_0(x)$ has the explicit solution formula

$$f(t,x) = f_0(x - tv),$$

which is a unique C^1-solution if the initial data is C^1-function. This solution formula is attained by the characteristics with respect to the transport operator $\partial_t + v \cdot \nabla_x$.
Similarly, we consider the following mean field partial differential equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \nabla_v \cdot [(F * f)f] = 0, \quad x \in \mathbb{R}^n, v \in \mathbb{R}^n, t > 0, \\ f(0,x,v) = f_0(x,v) \end{cases}$$

Here the vector field F can be, in some cases, time-dependent or non time-dependent, can also depends on both the position variable x and velocity v or only depends on x. First we need the following definition and theorem, which are typical and standard. One can further refer to [51].

Definition 2.1.1. *Let $(x(t), v(t))$ be the solution of the ordinary differential system*

$$\begin{cases} \dfrac{dx}{dt} = v \\ \dfrac{dv}{dt} = \displaystyle\iint_{\mathbb{R}^n \times \mathbb{R}^n} F(t, x-y, v-w) f(t,y,w)\, dydw \\ (x(0), v(0)) = (v_0, \iint_{\mathbb{R}^n \times \mathbb{R}^n} F(0, x-y, v-w) f(0,y,w)\, dydw). \end{cases}$$

The set

$$\{(t, x(t), v(t)) \,|\, t \in [0,T]\}$$

is called the characteristic curve of the ordinary differential system passing through

$$\left(v_0, \iint_{\mathbb{R}^n \times \mathbb{R}^n} F(0, x-y, v-w) f(0,y,w)\, dydw\right)$$

at time $t = 0$.

After defining the flow associated to the ordinary differential system of characteristic curves, we are then able to use the flow to solve the mean field equation.

Theorem 2.1.1. *Assume that the vector field F satisfies the condition*

$$(H1) \quad F \in C([0,T] \times \mathbb{R}^n, \mathbb{R}^n) \quad \textit{and} \quad \nabla_x F, \nabla_v F \in C([0,T] \times \mathbb{R}^n, M_n(\mathbb{R}))$$

$$(H2) \quad \textit{there exists}\, k > 0 \textit{such that}$$

$$|F(t,x,v)| \leq k(1 + |x| + |v|), \quad \textit{for all}\, (t,x,v) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$$

Then for each $t \in [0,T]$ and each $(x,v) \in \times\mathbb{R}^n \times \mathbb{R}^n$, the ordinary differential system

$$\begin{cases} \dfrac{dx}{dt} = v \\ \dfrac{dv}{dt} = \displaystyle\iint_{\mathbb{R}^n \times \mathbb{R}^n} F(t, x-y, v-w) f(t,y,w)\, dydw \\ (x(0), v(0)) = (v_0, \iint_{\mathbb{R}^n \times \mathbb{R}^n} F(0, x-y, v-w) f(0,y,w)\, dydw) \end{cases}$$

has a unique solution of class C^1 on $[0,T]$.

The theorem can be easily proven by the standard argument of the Cauchy-Lipschitz theorem, the assumption of which is guaranteed by the regular condition $(H1)$. And the condition $(H2)$ is equally important to assure that the solution is global. We also mention that if we impose further regularity on the vector field F, say

$$(H3)\, F \in C^k([0,T] \times \mathbb{R}^n, \mathbb{R}^n) \quad \text{and} \quad \nabla_x F, \nabla_v F \in C^k([0,T] \times \mathbb{R}^n, M_n(\mathbb{R})), \text{for some}\, k > 1,$$

the solution will then have a high regularity of class C^{k+1} on $[0,T]$.

Definition 2.1.2. *Let (X_1, Σ_1) and (X_2, Σ_2) be measurable spaces (meaning that Σ_1 and Σ_2 are σ-algebras of the subsets of X_1 and X_2, respectively). Let $T : X_1 \to X_2$ be a (Σ_1, Σ_2)-measurable map and μ be a positive measure on (X_1, Σ_1). Then, the formula*

$$\nu(B) := \mu(T^{-1}(B)), \quad \forall\, B \in \Sigma_2$$

defines a positive measure on (X_2, Σ_2), denoted by

$$\nu =: T\#\mu,$$

and is referred to as the push-forward of the measure μ under the map T.

Due to the property of the transport equation, we know that solving the equation (2.2.6) is equivalent to investigating the corresponding characteristic system, i.e.,

$$\begin{cases} \dfrac{d}{dt} Z(t, z_0, \mu_0) = \displaystyle\int_{\mathbb{R}^{2n}} K\left(Z(t, z_0), z'\right) \mu(t, dz'), \\ Z(0, z_0, \mu_0) = z_0, \end{cases} \tag{2.1.1}$$

where

$$K^N(z, z') = K^N(x, v, x', v') := \big(v, F^N(x - x', v - v') + G^N(x, v)\big)$$

and $\mu(t, \cdot)$ is the push-forward of the measure μ_0. Here, for the sake of convenience, we use $z = (x, v)$ and Z as the four-dimensional vector.

The solvability of the cut-off problem can be then obtained via the standard argument using Banach Fixed-Point Theorem. For completeness, we present the proof in the following proposition and theorem.

We denote $\mathcal{P}_p(\mathbb{R}^n \times \mathbb{R}^n)$ as the set of Borel probability measures on $\mathbb{R}^n \times \mathbb{R}^n$ and $\mathcal{P}_p(\mathbb{R}^n \times \mathbb{R}^n)$ is defined for each $p > 0$ by

$$\mathcal{P}_p(\mathbb{R}^n \times \mathbb{R}^n) := \Big\{\mu \in \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n) \,\Big|\, \int_{\mathbb{R}^n \times \mathbb{R}^n} (|x|^p + |v|^p) \mu(dx, dv) < \infty \Big\}.$$

Proposition 2.1.1. *Assume that the interaction kernel $K(z, z') \in C(\mathbb{R}^{2n} \times \mathbb{R}^{2n}; \mathbb{R}^{2n})$ is Lipschitz continuous in z, uniformly in z' (and conversely), i.e., there exists a constant $L > 0$ such that*

$$\sup_{z' \in \mathbb{R}^{2n}} |K(z_1, z') - K(z_2, z')| \le L|z_1 - z_2|,$$
$$\sup_{z \in \mathbb{R}^{2n}} |K(z, z_1) - K(z, z_2)| \le L|z_1 - z_2|.$$

For any given $z_0 = (x_0, v_0) \in \mathbb{R}^n \times \mathbb{R}^n$ and Borel probability measure $\mu_0 \in \mathcal{P}_1(\mathbb{R}^{2n})$, there exists a unique C^1-solution, denoted by

$$\mathbb{R}_+ \ni t \mapsto Z(t, z_0, \mu_0) \in \mathbb{R}^{2n},$$

to the problem

$$\begin{cases} \dfrac{d}{dt} Z(t, z_0, \mu_0) = \displaystyle\int_{\mathbb{R}^{2n}} K\left(Z(t, z_0), z'\right) \mu(t, dz'), \\ Z(0, z_0, \mu_0) = z_0, \end{cases} \tag{2.1.2}$$

where $\mu(t, \cdot)$ is the push-forward of the measure μ_0, i.e., $\mu(t, \cdot) = Z(t, \cdot, \mu_0)\#\mu_0$.

Proof. Let $\mu_0 \in \mathcal{P}_1(\mathbb{R}^4)$ and denote

$$\kappa := \int_{\mathbb{R}^{2n}} |v| \mu_0(dx, dv).$$

For $t^* := \dfrac{1}{2L(2+\kappa)}$, let

$$\mathcal{X} := \left\{ Z(t,z) \in C([0,t^*]; C(\mathbb{R}^{2n}; \mathbb{R}^{2n})) \,\Big|\, \sup_{0 \le t \le t^*} \sup_{z=(x,v) \in \mathbb{R}^4} \frac{|Z(t,z)|}{1+|v|} < \infty \right\}$$

be a Banach space equipped with the norm

$$\|Z\|_{\mathcal{X}} := \sup_{0 \le t \le t^*} \sup_{z=(x,v) \in \mathbb{R}^4} \frac{|Z(t,z)|}{1+|v|}.$$

The assumption for the Lipschitz continuity of the kernel $K(z, z')$ actually implies that K grows at most linearly at infinity, i.e.,

$$|K(z,z')| \le L(|z| + |z'|), \quad z, z' \in \mathbb{R}^{2n}.$$

The map $\mathcal{T} : \mathcal{X} \to \mathcal{X}$, defined by

$$\mathcal{T} Z(t,z) := z + \int_0^t \int_{\mathbb{R}^{2n}} K\left(Z(s,z), Z(s,\zeta)\right) \mu_0(d\zeta) ds,$$

constitutes a contraction which can be seen from the following estimates. For each $Z, \hat{Z} \in \mathcal{X}$, we have for $0 \le s \le t^*$

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^{2n}} K\left(Z(s,z), Z(s,\zeta)\right) \mu_0(d\zeta) - \int_{\mathbb{R}^4} K\left(\hat{Z}(s,z), \hat{Z}(s,\zeta)\right) \mu_0(d\zeta) \Big| \\
\le\ & L \int_{\mathbb{R}^{2n}} \left(|Z(s,z) - \hat{Z}(s,z)| + |Z(s,\zeta) - \hat{Z}(s,\zeta)| \right) \mu_0(d\zeta) \\
\le\ & L(1 + |v| + 1 + \kappa) \sup_{z=(x,v) \in \mathbb{R}^4} \frac{|Z(s,z) - \hat{Z}(s,z)|}{1+|v|}.
\end{aligned}$$

Consequently, we get

$$
\begin{aligned}
& \|\mathcal{T}Z(t,\cdot) - \mathcal{T}\hat{Z}(t,\cdot)\|_{\mathcal{X}} \\
= \;& \Big\| \int_0^t \int_{\mathbb{R}^{2n}} K\left(Z(s,z), Z(s,\zeta)\right) \mu_0(d\zeta) ds - \int_0^t \int_{\mathbb{R}^{2n}} K\left(\hat{Z}(s,z), \hat{Z}(s,\zeta)\right) \mu_0(d\zeta) ds \Big\|_{\mathcal{X}} \\
\leq \;& L\|Z(t,\cdot) - \hat{Z}(t,\cdot)\|_{\mathcal{X}}(2+\kappa)t^* \\
\leq \;& \frac{1}{2}\|Z(t,\cdot) - \hat{Z}(t,\cdot)\|_{\mathcal{X}}.
\end{aligned}
$$

Then, by Banach Fixed-Point Theorem, there exists a unique $Z^* \in \mathcal{X}$ such that $\mathcal{T}Z^* = Z^*$, i.e.,

$$
Z^*(t,z) = z + \int_0^t \int_{\mathbb{R}^4} K\left(Z^*(s,z), Z^*(s,\zeta)\right) \mu_0(d\zeta) ds. \tag{2.1.3}
$$

The interaction kernel is globally Lipschitz continuous, i.e., t^* is a fixed constant, which implies that the solution can be easily extended to all time t. Since $Z^* \in C(\mathbb{R}_+; C(\mathbb{R}^{2n}; \mathbb{R}^{2n}))$, $K \in C(\mathbb{R}^{2n} \times \mathbb{R}^4; \mathbb{R}^{2n})$ and $\mu_0 \in \mathcal{P}_1(\mathbb{R}^{2n})$, the function

$$
s \mapsto \int_{\mathbb{R}^4} K\left(Z^*(s,z_0), Z^*(s,\zeta)\right) \mu_0(d\zeta)
$$

is continuous on $\mathbb{R}_+$ for all $z_0 \in \mathbb{R}^{2n}$. Exploiting the integral equation (2.1.3) shows that the function $t \mapsto Z(t,z_0)$ is C^1 in t and satisfies

$$
\begin{cases}
\dfrac{d}{dt}Z^*(t,z_0) = \displaystyle\int_{\mathbb{R}^{2n}} K\left(Z^*(t,z), Z^*(t,\zeta)\right) \mu_0(d\zeta), \\
Z(0,z_0) = z_0.
\end{cases}
$$

Substituting $z' = Z^*(t,\zeta)$ in the integral above leads to

$$
\int_{\mathbb{R}^{2n}} K\left(Z^*(t,z_0), Z^*(t,\zeta)\right) \mu_0(d\zeta) = \int_{\mathbb{R}^{2n}} K(Z^*(t,z_0), z') Z^*(t,\cdot)\#\mu_0(dz'),
$$

which means that the function Z^* is the solution to the problem (2.2.8). ∎

Theorem 2.1.2. *Assume that the vector field $F = F(t,x,v) \in \mathbb{R}^n$ satisfies assumptions $(H1)$, $(H2)$ and $(H3)$ with $k=1$. Let $f_0 \in C^1(\mathbb{R}^n \times \mathbb{R}^n)$. Then the mean field differential equation*

$$
\begin{cases}
\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot [(F * f) f] = 0, \quad x \in \mathbb{R}^n, v \in \mathbb{R}^n, t > 0, \\
f(0,x,v) = f_0(x,v)
\end{cases}
$$

has a unique solution $f \in C([0,T]\times\mathbb{R}^n\times\mathbb{R}^n)$. *The solution is given by the formula*

$$f(t,x,v) = f_0(Z(t,\cdot,\cdot)^{-1}(x,v))J(0,t,x,v), \quad \forall\, t\in[0,T], (x,v)\in\mathbb{R}^n\times\mathbb{R}^n,$$

where $J(0,t,x,v)$ *is the Jacobian, i.e.,*

$$J(0,t,x,v) = \exp\left(\int_t^0 div_v\left(F * f(s,Z(s,x,v))\right)\, ds\right).$$

In particular, $f(t,\cdot,\cdot)\in L^1(\mathbb{R}^n\times\mathbb{R}^n)$ *for all* $t\in[0,T]$ *if* $f_0\in L^1(\mathbb{R}^n\times\mathbb{R}^n)$, *and one has*

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} f(t,x,v)\,dxdv = \iint_{\mathbb{R}^n\times\mathbb{R}^n} f_0(x,v)\,dxdv.$$

The result for the regular vector field is quite easy to understand and needs not so much technical care. The following sections are devoted the non-Lipschitz force case, where detailed handling will be step by step illustrated.

2.2 Mean Field Equation with Cut-off

We start with the definition of weak solution to the mean field equation (2.0.1).

Definition 2.2.1. *Let* $f_0(x,v)\in L^1(\mathbb{R}^2\times\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2\times\mathbb{R}^2)$. *A function* $f=f(t,x,v)$ *is said to be a weak solution to the kinetic mean field equation* (3.1.2) *with initial data* f_0, *if there holds*

$$\begin{aligned}\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)\varphi(x,v)\,dxdv &= \iint_{\mathbb{R}^2\times\mathbb{R}^2} f_0(x,v)\varphi(x,v)\,dxdv\\ &+\int_0^t\iint_{\mathbb{R}^2\times\mathbb{R}^2} vf(s,x,v)\cdot\nabla_x\varphi(x,v)\,dxdvds\\ &+\int_0^t\iint_{\mathbb{R}^2\times\mathbb{R}^2}\left(F(x,v)*f(s,x,v)\right)f(s,x,v)\cdot\nabla_v\varphi(x,v)\,dxdvds\\ &+\int_0^t\iint_{\mathbb{R}^2\times\mathbb{R}^2} G(x,v)f(s,x,v)\cdot\nabla_v\varphi(x,v)\,dxdvds\end{aligned}\tag{2.2.1}$$

for all $\varphi(x,v)\in C_0^\infty(\mathbb{R}^2\times\mathbb{R}^2)$ *and* $t\in\mathbb{R}_+$.

Now, we present the main theorem of this paper. In the following, $G(x,v)$ is given by (2.0.3) while $F(x,v)$ is defined by (2.0.2).

Theorem 2.2.1. *For* $F(x,v)=\nabla_x V(|x|,v)=\partial_r V(r,v)\dfrac{x}{|x|}$ *and* $G(x,v)=g(x)-v$, *assume that* $\partial_r V(r,v), \nabla_v\partial_r V(r,v)\in L^\infty(\mathbb{R}^2\times\mathbb{R}^2)$ *and* $g\in L^\infty(\mathbb{R}^2\times\mathbb{R}^2)$. *Let* $f_0(x,v)$ *be a nonnegative* *function in* $L^1(\mathbb{R}^2 \times \mathbb{R}^2) \cap L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$, $|x|^2 f_0(x,v) \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$, *and*

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{2}|v|^2 f_0(x,v)\,dxdv =: \mathcal{E}_0 < \infty.$$

Then, there exists a weak solution $f \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^2 \times \mathbb{R}^2))$ *to the mean field equation* (3.1.2) *with initial data* f_0*. Moreover this solution satisfies*

$$0 \le f(t,x,v) \le \|f_0\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} e^{Ct}, \quad \textit{for a.e.}\,(x,v) \in \mathbb{R}^2 \times \mathbb{R}^2,\, t \ge 0 \tag{2.2.2}$$

together with the mass conservation

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)\,dxdv = \iint_{\mathbb{R}^2\times\mathbb{R}^2} f_0(x,v)\,dxdv =: \mathcal{M}_0 \tag{2.2.3}$$

and the kinetic energy bound

$$\mathcal{E}(t) := \iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{2}|v|^2 f(t,x,v)\,dxdv \le C, \quad \forall\, t \ge 0, \tag{2.2.4}$$

where the constant C *is independent of* t*.*

Under the assumptions above, the interaction force is bounded but not Lipschitz continuous in x, we need to use the standard cut-off to over come this difficulty. Another difficulty in this context is that the interaction force $F(x,v)$ not only depends on the position x but also on the velocity v. This leads to a totally different structure compared to the Vlasov-Poisson equation, where the $W^{2,p}$ theory for Poisson equations is generally used. The proof of Theorem 2.2.1 is therefore not as straightforward und intuitive as one might expect and needs to be dedicately handled step by step within the next sections. On the other hand, the self-generating force (or desired velocity and direction acceleration) $G(x,v)$ is not Lipschitz continuous, which requires an additional work of mollification.

We briefly recall essential assumptions and properties, cf. [22], that are necessary for the existence proof.

2.2.1 Notations and Preliminary Work

We consider the flow with cut-off of order $N^{-\theta}$ with arbitrary positive θ, i.e.,

$$F^N(x,v) = \begin{cases} V'(|x|,v)\dfrac{x}{|x|}\mathcal{H}(x,v), & |x| \ge N^{-\theta}, \\ N^\theta V'(|x|,v) x \mathcal{H}(x,v), & |x| < N^{-\theta}. \end{cases} \tag{2.2.5}$$

Then, the mean field cut-off equation becomes

$$\partial_t f^N + v \cdot \nabla_x f^N + \nabla_v \cdot [(F^N * f^N) f^N] + \nabla_v \cdot (G^N f^N) = 0, \tag{2.2.6}$$

where we also take the cut-off of $G(x, v)$ into consideration, i.e.,

$$G^N(x, v) = j_{\frac{1}{N}} * g(x) - v$$

with $j_{\frac{1}{N}}(x)$ being the standard mollifier.
We also point out several properties for the interaction force $F^N(x, v)$ and the acceleration $G^N(x, v)$, namely

(a) $F^N(x, v)$ is bounded, i.e., $|F^N(x, v)| \leq C$.

(b) $F^N(x, v)$ satisfies

$$|F^N(x, v) - F^N(y, v)| \leq q^N(x, v)|x - y|,$$

where q^N has compact support in $B_{2R} \times B_{2\widetilde{R}}$ with

$$q^N(x, v) := \begin{cases} C \cdot \dfrac{1}{|x|} + C, & |x| \geq N^{-\theta}, \\ C \cdot N^{\theta}, & |x| < N^{-\theta}. \end{cases}$$

(c) $\nabla_v F^N(x, v)$ is uniformly bounded in N.

(d) $|G^N(x, v) - G^N(y, v)| \leq C \cdot N \cdot |x - y|$.

Here, we use C as a universal constant that might depend on all the given constants $k_n, R, \widetilde{R}, \gamma_n, \gamma_t$. Furthermore, if there is a singularity in the velocity v in the interaction potential similar to property (b), it can be treated by using the same method as above and the results also apply.

2.2.2 Mean Field Characteristic Flow with Cut-off

Before we start to prove the existence of the unique weak solution to the equation (2.2.6), we need first the following definition.

Definition 2.2.2. *Let (X_1, Σ_1) and (X_2, Σ_2) be measurable spaces (meaning that Σ_1 and Σ_2 are σ-algebras of the subsets of X_1 and X_2, respectively). Let $T : X_1 \to X_2$ be a (Σ_1, Σ_2)-measurable map and μ be a positive measure on (X_1, Σ_1). Then, the formula*

$$\nu(B) := \mu(T^{-1}(B)), \quad \forall\, B \in \Sigma_2$$

defines a positive measure on (X_2, Σ_2)*, denoted by*

$$\nu =: T\#\mu,$$

and is referred to as the push-forward of the measure μ *under the map* T*.*

The definition is often used when it comes to solving mean field characteristic flow. For more detailed information, we refer to [51]. Due to the property of the transport equation, we know that solving the equation (2.2.6) is equivalent to investigating the corresponding characteristic system, i.e.,

$$\begin{cases} \dfrac{d}{dt} Z(t, z_0, \mu_0) = \displaystyle\int_{\mathbb{R}^4} K\left(Z(t, z_0), z'\right) \mu(t, dz'), \\ Z(0, z_0, \mu_0) = z_0, \end{cases} \tag{2.2.7}$$

where

$$K^N(z, z') = K^N(x, v, x', v') := \big(v, F^N(x - x', v - v') + G^N(x, v)\big)$$

and $\mu(t, \cdot)$ is the push-forward of the measure μ_0. Here, for the sake of convenience, we use $z = (x, v)$ and Z as the four-dimensional vector.

We denote $\mathcal{P}(\mathbb{R}^4)$ as the set of Borel probability measures on $\mathbb{R}^4$ and $\mathcal{P}_1(\mathbb{R}^4)$ is defined by

$$\mathcal{P}_1(\mathbb{R}^4) := \Big\{ \mu \in \mathcal{P}(\mathbb{R}^4) \,\Big|\, \int_{\mathbb{R}^4} |v| \mu(dx, dv) < \infty \Big\}.$$

Proposition 2.2.1. *Assume that the interaction kernel* $K(z, z') \in C(\mathbb{R}^4 \times \mathbb{R}^4; \mathbb{R}^4)$ *is Lipschitz continuous in* z*, uniformly in* z' *(and conversely), i.e., there exists a constant* $L > 0$ *such that*

$$\sup_{z' \in \mathbb{R}^4} |K(z_1, z') - K(z_2, z')| \leq L|z_1 - z_2|,$$
$$\sup_{z \in \mathbb{R}^4} |K(z, z_1) - K(z, z_2)| \leq L|z_1 - z_2|.$$

For any given $z_0 = (x_0, v_0) \in \mathbb{R}^2 \times \mathbb{R}^2$ *and Borel probability measure* $\mu_0 \in \mathcal{P}_1(\mathbb{R}^4)$*, there exists a unique* C^1*-solution, denoted by*

$$\mathbb{R}_+ \ni t \mapsto Z(t, z_0, \mu_0) \in \mathbb{R}^4,$$

to the problem

$$\begin{cases} \dfrac{d}{dt} Z(t, z_0, \mu_0) = \displaystyle\int_{\mathbb{R}^4} K\left(Z(t, z_0), z'\right) \mu(t, dz'), \\ Z(0, z_0, \mu_0) = z_0, \end{cases} \tag{2.2.8}$$

where $\mu(t,\cdot)$ is the push-forward of the measure μ_0, i.e., $\mu(t,\cdot) = Z(t,\cdot,\mu_0)\#\mu_0$.

This proposition is typically obtained via the standard argument using Banach Fixed-Point Theorem, see [51].

With Proposition 2.2.1, we are now able to prove that there exists a unique weak solution to the Vlasov equation with cut-off (2.2.6).

Theorem 2.2.2. *Let F and G satisfy the same assumptions as in theorem 2.2.1 and f_0^N be a nonnegative compactly supported function in $L^1(\mathbb{R}^2\times\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2\times\mathbb{R}^2)$ satisfying*

$$\|f_0^N\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)} = \mathcal{M}_0 \quad \text{and} \quad f_0^N(x,v) \le \|f_0\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)},$$

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{2}|v|^2 f_0^N(x,v)\,dxdv \le \mathcal{E}_0 < \infty,$$

and

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{2}|x|^2 f_0^N(x,v)\,dxdv \le \mathcal{M}_2 < \infty.$$

Then, there exists a unique weak solution $f^N \in C^1(\mathbb{R}_+; L^1(\mathbb{R}^2\times\mathbb{R}^2))$ to the mean field cut-off equation (2.2.6) with initial data f_0^N, i.e., $f^N(t,x,v)$ satisfies

$$\begin{aligned}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \partial_t f^N(t,x,v)\varphi(x,v)\,dxdv &= \iint_{\mathbb{R}^2\times\mathbb{R}^2} v f^N(t,x,v)\cdot\nabla_x\varphi(x,v)\,dxdv \\ &+ \iint_{\mathbb{R}^2\times\mathbb{R}^2} \left(F^N(x,v) * f^N(t,x,v)\right) f^N(s,x,v)\cdot\nabla_v\varphi(x,v)\,dxdv \\ &+ \iint_{\mathbb{R}^2\times\mathbb{R}^2} G^N(x,v) f^N(t,x,v)\cdot\nabla_v\varphi(x,v)\,dxdv \end{aligned} \tag{2.2.9}$$

for all $\varphi(x,v)\in C_0^\infty(\mathbb{R}^2\times\mathbb{R}^2)$. Moreover this solution satisfies

$$\lim_{t\to 0} f^N(t,x,v) = f_0^N(x,v), \quad \text{for a.e. } (x,v)\in\mathbb{R}^2\times\mathbb{R}^2,$$

$$0 \le f^N(t,x,v) \le \|f_0^N\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} e^{Ct}, \quad \text{for a.e. } (x,v)\in\mathbb{R}^2\times\mathbb{R}^2,\ t\ge 0 \tag{2.2.10}$$

together with the mass conservation

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} f^N(t,x,v)\,dxdv = \iint_{\mathbb{R}^2\times\mathbb{R}^2} f_0^N(x,v)\,dxdv =: \mathcal{M}_0, \tag{2.2.11}$$

the kinetic energy bound

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{2}|v|^2 f^N(t,x,v)\,dxdv \le C, \quad \forall\, t\ge 0, \tag{2.2.12}$$

and the bound of second moment

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{2}|x|^2 f^N(t,x,v)\,dxdv \le \mathcal{M}_2 e^{Ct}, \quad \forall\, t\ge 0, \tag{2.2.13}$$

where the constant C is independent of N and t.

Proof. Without loss of generality, we assume that $\mathcal{M}_0 = 1$. If we choose the interaction kernel K as

$$K^N(z,z') = K^N(x,v,x',v') := \left(v, F^N(x-x',v-v') + G^N(x,v)\right),$$

the mean field cut-off equation (2.2.6) can be put into the form

$$\partial_t f^N(t,z) + \mathrm{div}_z\Big(f^N(t,z)\iint_{\mathbb{R}^2\times\mathbb{R}^2} K^N(z,z')f^N(t,z')dz'\Big) = 0.$$

Notice that the non-linear non-local dynamical system that appears in Proposition 2.2.1 is exactly the equation of characteristics for the mean field kinetic equation with cut-off (2.2.6), which we refer to as the mean field characteristic flow (with cut-off). The existence and uniqueness of the solution to (2.2.6) are therefore achieved as a direct result of the construction of the mean field characteristic flow. By Proposition 2.2.1, there exists a unique map

$$\mathbb{R}_+ \times \mathbb{R}^4 \times \mathcal{P}_1(\mathbb{R}^4) \ni (t,z_0,\mu_0) \mapsto Z^N(t,z_0,\mu_0) \in \mathbb{R}^4$$

such that $t \mapsto Z^N(t,z_0,\mu_0)$ is the integral curve of the vector field

$$z \mapsto \iint_{\mathbb{R}^2\times\mathbb{R}^2} K^N(z,z')\mu^N(t,dz')$$

passing through z_0 at time $t=0$, where $\mu^N(t) := Z^N(t,\cdot,\mu_0)\#\mu_0$. For the given initial data f_0^N, letting $d\mu_0 = f_0^N dz$ results in

$$f^N(t,z) := f_0^N\left(Z^N(t,\cdot)^{-1}(z)\right) J(0,t,z), \quad \forall\, t\ge 0,$$

where $J(0,t,z)$ is the Jacobian, i.e.,

$$J(0,t,z) = \exp\Bigg(\int_t^0 \mathrm{div}_v\left(F^N * f^N(s,Z^N(s,z)) + G^N(Z^N(s,z))\right)\,ds\Bigg).$$

Then we have

$$\begin{aligned}
|f^N(t,z)| &\le |f_0^N\left(Z^N(t,\cdot)^{-1}(z)\right) J(0,t,z)| \\
&\le \|f_0^N\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \exp\left(\int_0^t \|\nabla_v F^N * f^N\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}\, ds + Ct\right) \\
&\le \|f_0^N\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \exp\left(\int_0^t \|\nabla_v F^N\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \|f^N\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)}\, ds + Ct\right) \\
&\le \|f_0^N\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} e^{Ct},
\end{aligned}$$

where we have used the property of the acceleration $G^N(x,v)$, i.e., $G^N(x,v) = j_{\frac{1}{N}} * g(x) - v$, where $j_{\frac{1}{N}} * g(x)$ is a L^∞-function. From the equation, (2.2.11) are straightforward. Property (2.2.12) is left to be proven. For the kinetic energy estimate, we will again use the property of the acceleration $G^N(x,v)$ and remark that v in $G^N(x,v)$ is critical in the estimate because it serves as a damping term. We now choose $\{\varphi_\eta(x)\phi_\eta(v)\}$ to be a smooth function which satisfies

$$\varphi_\eta(x) = \begin{cases} 0, & |x| > \frac{1}{\eta}, \\ 1, & |x| < \frac{1}{2\eta}, \end{cases} \quad \text{and} \quad \phi_\eta(v) = \begin{cases} 0, & |v| > \frac{1}{\eta}, \\ 1, & |v| < \frac{1}{2\eta}, \end{cases}$$

and

$$\left|\nabla_z\Big(\varphi_\eta(x)\phi_\eta(v)\Big)\right| \le \eta\Big|\varphi_\eta(x)\phi_\eta(v)\Big|.$$

Since $\varphi_\eta(x)\phi_\eta(v)$ is monotone and converges to one for almost all x and v as η goes to zero, we have

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} v^2 f^N(t,x,v)\varphi_\eta(x)\phi_\eta(v)\, dxdv \to \iint_{\mathbb{R}^2\times\mathbb{R}^2} v^2 f^N(t,x,v)\, dxdv, \quad \text{as}\, \eta \to 0.$$

The compact support of f_0^N implies that $f^N(t,x,v)$ has compact support in (x,v) for any fixed time t. By the definition of weak solution for test functions $v^2\varphi_\eta(x)\phi_\eta(v)$, we have

$$\begin{aligned}
& \frac{d}{dt}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{2} v^2 f^N(t,x,v)\varphi_\eta(x)\phi_\eta(v)\, dxdv \\
= \; & \frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} v f^N(t,x,v)\cdot\nabla_x\left(v^2\varphi_\eta(x)\phi_\eta(v)\right)\, dxdv \\
& + \frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \left(F^N(x,v) * f^N(t,x,v)\right) f^N(s,x,v)\cdot\nabla_v\left(v^2\varphi_\eta(x)\phi_\eta(v)\right)\, dxdv \\
& + \frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} G^N(x,v) f^N(t,x,v)\cdot\nabla_v\left(v^2\varphi_\eta(x)\phi_\eta(v)\right)\, dxdv
\end{aligned}$$

$$
\begin{aligned}
= \;& \frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} v^2 f^N(t,x,v)\phi_\eta(v)v\cdot\nabla_x\big(\varphi_\eta(x)\big)\,dxdv \\
&+\iint_{\mathbb{R}^2\times\mathbb{R}^2} v\left(F^N(x,v)*f^N(t,x,v)\right) f^N(t,x,v)\varphi_\eta(x)\phi_\eta(v)\,dxdv \\
&+\frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} v^2\left(F^N(x,v)*f^N(t,x,v)\right) f^N(s,x,v)\cdot\nabla_v\left(\varphi_\eta(x)\phi_\eta(v)\right)\,dxdv \\
&+\iint_{\mathbb{R}^2\times\mathbb{R}^2} v\cdot G^N(x,v)f^N(t,x,v)\varphi_\eta(x)\phi_\eta(v)\,dxdv \\
&+\frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} v^2 G^N(x,v)f^N(t,x,v)\cdot\nabla_v\left(\varphi_\eta(x)\phi_\eta(v)\right)\,dxdv \\
=: \;& \sum_{j=1}^{5} I_j.
\end{aligned}
$$

Next, we estimate the expressions $I_j, j=1,\ldots,5$ individually. It is easy to see

$$
\begin{aligned}
|I_1| \;&\leq\; \frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\left|v^2 f^N(t,x,v)\phi_\eta(v)v\cdot\nabla_x\big(\varphi_\eta(x)\big)\right|dxdv \\
&\leq\; \frac{1}{2}\eta\iint_{\mathbb{R}^2\times\mathbb{R}^2}|v|^3 f^N(t,x,v)|\phi_\eta(v)\varphi_\eta(x)|\,dxdv.
\end{aligned}
$$

Due to the fact that f_0^N is compactly supported, i.e., f^N has also compact support for any finite time t, I_1 converges to zero as $\eta\to 0$ for fixed N. The same argument holds for I_3 and I_5, i.e., I_3 and I_5 converge to zero as $\eta\to 0$:

$$
\begin{aligned}
|I_3| \;&\leq\; \frac{1}{2}\cdot C\eta\|F^N*f^N\|_{L^\infty}\iint_{\mathbb{R}^2\times\mathbb{R}^2} v^2 f^N(t,x,v)\varphi_\eta(x)\phi_\eta(v)\,dxdv \\
&\leq\; \frac{1}{2}\cdot C\eta\|F^N\|_{L^\infty}\|f^N\|_{L^1}\iint_{\mathbb{R}^2\times\mathbb{R}^2} v^2 f^N(t,x,v)\varphi_\eta(x)\phi_\eta(v)\,dxdv \\
I_5 \;&\leq\; \frac{1}{2}\cdot\eta\|j_{\frac{1}{N}}*g\|_{L^\infty}\iint_{\mathbb{R}^2\times\mathbb{R}^2} v^2 f^N(t,x,v)\varphi_\eta(x)\phi_\eta(v)\,dxdv \\
&\quad -\frac{1}{2}\eta\iint_{\mathbb{R}^2\times\mathbb{R}^2}|v|^3 f^N(t,x,v)\phi_\eta(v)\varphi_\eta(x)\,dxdv.
\end{aligned}
$$

However, for the other integral estimates, we need some extra calculations. Using the properties of the desired velocity and direction acceleration $G^N(x,v)$, we arrive at

$$
\begin{aligned}
I_2 &\le \|F^N * f^N\|_{L^\infty} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \left(\frac{1}{4\varepsilon}+\varepsilon v^2\right) f^N(t,x,v)\varphi_\eta(x)\phi_\eta(v)\,dxdv \\
&\le \|F^N\|_{L^\infty}\|f^N\|_{L^1} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \left(\frac{1}{4\varepsilon}+\varepsilon v^2\right) f^N(t,x,v)\varphi_\eta(x)\phi_\eta(v)\,dxdv \\
I_4 &\le \|j_{\frac{1}{N}} * g\|_{L^\infty} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \left(\frac{1}{4\varepsilon}+\varepsilon v^2\right) f^N(t,x,v)\varphi_\eta(x)\phi_\eta(v)\,dxdv \\
&\quad - \iint_{\mathbb{R}^2\times\mathbb{R}^2} v^2 f^N(t,x,v)\varphi_\eta(x)\phi_\eta(v)\,dxdv
\end{aligned}
$$

Combining all the five terms, taking η to zero in the inequality above and setting ε small enough such that

$$
\varepsilon < \frac{1}{2(\|F^N\|_{L^\infty}\|f^N\|_{L^1}+\|g\|_{L^\infty})},
$$

where the fact that $\|j_{\frac{1}{N}} * g\|_{L^\infty} \le \|g\|_{L^\infty}$ has been used, we end up with

$$
\frac{d}{dt}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{2}v^2 f^N(t,x,v)\,dxdv \le C - \iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{2}v^2 f^N(t,x,v)\,dxdv,
$$

where C does not depend on N. A direct computation shows that the kinetic energy is bounded uniformly in t and N. The estimate for the second moment follows from

$$
\begin{aligned}
\frac{d}{dt}\iint_{\mathbb{R}^2\times\mathbb{R}^2} |x|^2 f^N(t,x,v)\,dxdv &= \iint_{\mathbb{R}^2\times\mathbb{R}^2} |x|^2\partial_t f^N(t,x,v)\,dxdv \\
&= \iint_{\mathbb{R}^2\times\mathbb{R}^2} x\cdot v f^N(t,x,v)\,dxdv \\
&\le \iint_{\mathbb{R}^2\times\mathbb{R}^2} (|x|^2+|v|^2) f^N(t,x,v)\,dxdv \\
&\le \iint_{\mathbb{R}^2\times\mathbb{R}^2} |x|^2 f^N(t,x,v)\,dxdv + C.
\end{aligned}
$$

■

2.3 Compactness Arguments

In this section, we aim to achieve all the compactness arguments that are needed to pass the limit and to obtain the desired weak formulation of the non-cut-off kinetic equation, namely to prove the main result Theorem 2.2.1.

For given initial data f_0, let f_0^N be a sequence of functions with compact support which are w.l.o.g. assumed to be in B_N, i.e., a ball of radius N centered at origin. Furthermore f_0^N

satisfies

$$\|f_0^N - f_0\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \to 0, \text{ as } N\to\infty.$$

Let $f^N(t,x,v)$ be the solution obtained from Theorem 2.2.2 with initial data $f_0^N(x,v)$. Then, we know

$$0 \le f^N(t,x,v) \le \|f_0\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} e^{Ct}, \quad \text{for a.e. } (x,v)\in\mathbb{R}^2\times\mathbb{R}^2,\ t\ge 0,$$

and for any fixed $T>0$, there exists a subsequence of f^N, still denoted by f^N for simplicity, such that

$$f^N \overset{*}{\rightharpoonup} f \quad \text{in } L^\infty((0,T); L^\infty(\mathbb{R}^2\times\mathbb{R}^2)).$$

Due to the tightness in the variable x and v of the sequence f^N, implied from (2.2.12) and (2.2.13), we conclude that $f\in L^1(\mathbb{R}^2\times\mathbb{R}^2)$. Moreover, we notice that the total mass is preserved, i.e.,

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)\,dxdv = \iint_{\mathbb{R}^2\times\mathbb{R}^2} f_0^N(x,v)\,dxdv =: \mathcal{M}_0.$$

By the definition of weak* convergence for characteristic functions $\chi_{|x|+|v|\le r}\in L^1(\mathbb{R}^2\times\mathbb{R}^2)$, we have for each $a<b\in\mathbb{R}_+$

$$\begin{aligned} &\int_a^b\iint_{\mathbb{R}^2\times\mathbb{R}^2} \chi_{|x|+|v|\le r} f(t,x,v)\,dxdvdt \\ = &\lim_{N\to\infty}\int_a^b\iint_{\mathbb{R}^2\times\mathbb{R}^2} \chi_{|x|+|v|\le r} f^N(t,x,v)\,dxdvdt \\ \le &\lim_{N\to\infty}\int_a^b\iint_{\mathbb{R}^2\times\mathbb{R}^2} f^N(t,x,v)\,dxdvdt = \mathcal{M}_0(b-a). \end{aligned}$$

Letting $r\to\infty$ and applying Fatou's lemma yields

$$\begin{aligned} &\int_a^b\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)\,dxdvdt \\ \le &\varliminf_{r\to\infty}\int_a^b\iint_{\mathbb{R}^2\times\mathbb{R}^2} \chi_{|x|+|v|\le r} f(t,x,v)\,dxdvdt \\ \le &\lim_{N\to\infty}\int_a^b\iint_{\mathbb{R}^2\times\mathbb{R}^2} f^N(t,x,v)\,dxdvdt = \mathcal{M}_0(b-a). \end{aligned}$$

By a similar argument for test functions of type $\chi_{|x|+|v|\le r}|v|^2$, we can show that

$$\int_a^b\iint_{\mathbb{R}^2\times\mathbb{R}^2} |v|^2 f(t,x,v)\,dxdvdt \le C(b-a)$$

by using

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{2}|v|^2 f^N(t,x,v)\,dxdv \le C(b-a), \quad \forall\, t\ge 0.$$

Since the above two inequalities hold for all $a<b\in\mathbb{R}_+$, they also hold for a.e. $t\in\mathbb{R}_+$.

Using all the estimates presented in Theorem 2.2.2, we are now ready to pass the limit in (2.2.6) to the desired weak formulation of the non-cut-off kinetic equation

$$\partial_t f + v\cdot\nabla_x f + \nabla_v\cdot[(F*f)f] + \nabla_v\cdot(Gf) = 0.$$

However, we need to take special care on the non-linear term, i.e., the consideration of the function $F^N * f^N$. In the following, we use the notation $L^p(L^q)$ to denote $L^p([0,T];L^q(\mathbb{R}^2\times\mathbb{R}^2)), 1\le p,q\le\infty$. It is obvious to see that

$$\begin{aligned}
&\|F^N * f^N\|_{L^\infty(L^1)}\\
=\ & \Big\|\iint_{\mathbb{R}^2\times\mathbb{R}^2}\Big(\iint_{\mathbb{R}^2\times\mathbb{R}^2} F^N(x-y,v-w)f^N(t,y,w)\,dydw\Big)\,dxdv\Big\|_{L^\infty([0,T])}\\
=\ & \Big\|\iint_{\mathbb{R}^2\times\mathbb{R}^2} f^N(t,y,w)\Big(\iint_{\mathbb{R}^2\times\mathbb{R}^2} F^N(x-y,v-w)\,dxdv\Big)\,dydw\Big\|_{L^\infty([0,T])}\\
\le\ & C\left(\|F\|_{L^1},\mathcal{M}_0,\bar{R}\right)
\end{aligned}$$

and

$$\begin{aligned}
\|F^N * f^N\|_{L^\infty(L^\infty)} &= \Big\|\iint_{\mathbb{R}^2\times\mathbb{R}^2} F^N(x-y,v-w)f^N(t,y,w)\,dydw\Big\|_{L^\infty(L^\infty)}\\
&\le C\left(\|F\|_{L^\infty},\mathcal{M}_0\right).
\end{aligned}$$

Since $\nabla_v F^N$ is bounded uniformly in N, we get

$$\begin{aligned}
&\|\nabla_v\left(F^N * f^N\right)\|_{L^\infty(L^1)}\\
=\ & \Big\|\iint_{\mathbb{R}^2\times\mathbb{R}^2}\Big(\iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla_v F^N(x-y,v-w)f^N(t,y,w)\,dydw\Big)\,dxdv\Big\|_{L^\infty(\mathbb{R}_+)}\\
=\ & \Big\|\iint_{\mathbb{R}^2\times\mathbb{R}^2} f^N(t,y,w)\Big(\iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla_v F^N(x-y,v-w)\,dxdv\Big)\,dydw\Big\|_{L^\infty(\mathbb{R}_+)}\\
\le\ & C\left(\|\nabla_v F\|_{L^1},\mathcal{M}_0,\bar{R}\right)
\end{aligned}$$

and

$$\begin{aligned}
\|\nabla_v\left(F^N * f^N\right)\|_{L^\infty(L^\infty)} &= \Big\|\iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla_v F^N(x-y,v-w)f^N(t,y,w)\,dydw\Big\|_{L^\infty(L^\infty)}\\
&\le C\left(\|\nabla_v F\|_{L^\infty},\mathcal{M}_0\right).
\end{aligned}$$

So far, we can conclude by interpolation that $F^N * f^N$ and $\nabla_v F^N * f^N$ are in $L^\infty(L^2)$. Furthermore, it holds

$$\|\nabla_x \left(F^N * f^N\right)\|_{L^\infty(L^2)} \leq C \cdot \left\| \left(\chi_{\bar{R}} \cdot \frac{1}{|x|}\right) * f^N \right\|_{L^\infty(L^2)} \leq \|f^N\|_{L^\infty(L^p)}, \quad \forall\, p > 1,$$

where $\chi_{\bar{R}} \cdot \frac{1}{|x|} \in L^r, \forall\, 1 < r < 2$, and Young's inequality have been used. Hence, we conclude that $F^N * f^N$ then belongs to $L^\infty(\mathbb{R}_+; W^{1,2}(\mathbb{R}^2 \times \mathbb{R}^2))$. Since

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} \left(v f^N(t,x,v)\right)^2 \, dxdv \leq \|f^N\|_{L^\infty} \|v^2 f^N\|_{L^\infty(L^1)} \leq C(T),$$

we can get for every $\varphi \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ that

$$\begin{aligned} & \left\| \iint_{\mathbb{R}^2\times\mathbb{R}^2} v f^N(t,x,v) \nabla_x \varphi(x,v)\, dxdv \right\|_{L^\infty(\mathbb{R}_+)} \\ \leq\ & \|f^N\|^{\frac{1}{2}}_{L^\infty(L^\infty)} \cdot \|v^2 f^N\|^{\frac{1}{2}}_{L^\infty(L^1)} \cdot \|\nabla_x \varphi\|_{L^2} \\ \leq\ & C(T) \|\nabla_x \varphi\|_{L^2}. \end{aligned} \tag{2.3.1}$$

Moreover, we have

$$\begin{aligned} & \left\| \iint_{\mathbb{R}^2\times\mathbb{R}^2} G^N(x,v) f^N(t,x,v) \nabla_v \varphi(x,v)\, dxdv \right\|_{L^\infty(\mathbb{R}_+)} \\ \leq\ & \|j_{\frac{1}{N}} * g\|_{L^\infty} \cdot \|f^N\|^{\frac{1}{2}}_{L^\infty(L^\infty)} \cdot \|f^N\|^{\frac{1}{2}}_{L^\infty(L^1)} \cdot \|\nabla_v \varphi\|_{L^2} \\ & + \|f^N\|^{\frac{1}{2}}_{L^\infty(L^\infty)} \cdot \|v^2 f^N\|^{\frac{1}{2}}_{L^\infty(L^1)} \cdot \|\nabla_v \varphi\|_{L^2} \\ \leq\ & \|g\|_{L^\infty} \cdot \|f^N\|^{\frac{1}{2}}_{L^\infty(L^\infty)} \cdot \|f^N\|^{\frac{1}{2}}_{L^\infty(L^1)} \cdot \|\nabla_v \varphi\|_{L^2} \\ & + \|f^N\|^{\frac{1}{2}}_{L^\infty(L^\infty)} \cdot \|v^2 f^N\|^{\frac{1}{2}}_{L^\infty(L^1)} \cdot \|\nabla_v \varphi\|_{L^2} \\ \leq\ & C(T) \|\nabla_v \varphi\|_{L^2}. \end{aligned} \tag{2.3.2}$$

On the other hand, we know

$$\begin{aligned} & \left\| \iint_{\mathbb{R}^2\times\mathbb{R}^2} \left(F^N * f^N\right)(t,x,v) \cdot f^N(t,x,v) \nabla_v \varphi(x,v)\, dxdv \right\|_{L^\infty(\mathbb{R}_+)} \\ \leq\ & \|F^N * f^N\|_{L^\infty(L^\infty)} \cdot \|f^N\|_{L^\infty(L^2)} \cdot \|\nabla_v \varphi\|_{L^2} \\ \leq\ & C \|\nabla_v \varphi\|_{L^2}. \end{aligned} \tag{2.3.3}$$

Combining (2.3.1)-(2.3.3), it holds for every $\varphi \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ that

$$\begin{aligned}
& \Big\| \iint_{\mathbb{R}^2\times\mathbb{R}^2} \partial_t f^N(t,x,v)\varphi(x,v)\,dxdv \Big\|_{L^\infty(\mathbb{R}_+)} \\
\le\ & \Big\| \iint_{\mathbb{R}^2\times\mathbb{R}^2} v f^N(t,x,v)\nabla_x\varphi(x,v)\,dxdv \Big\|_{L^\infty(\mathbb{R}_+)} \\
& + \Big\| \iint_{\mathbb{R}^2\times\mathbb{R}^2} \left(F^N * f^N\right)(t,x,v)\cdot f^N(t,x,v)\nabla_v\varphi(x,v)\,dxdv \Big\|_{L^\infty(\mathbb{R}_+)} \\
& + \Big\| \iint_{\mathbb{R}^2\times\mathbb{R}^2} G^N(x,v) f^N(t,x,v)\nabla_v\varphi(x,v)\,dxdv \Big\|_{L^\infty(\mathbb{R}_+)} \\
\le\ & C\|\varphi\|_{W^{1,2}},
\end{aligned}$$

which implies

$$\begin{aligned}
& \Big\| \iint_{\mathbb{R}^2\times\mathbb{R}^2} \partial_t\Big((F^N * f^N)(t,x,v)\Big)\varphi(x,v)\,dxdv \Big\|_{L^\infty(\mathbb{R}_+)} \\
=\ & \Big\| \iint_{\mathbb{R}^2\times\mathbb{R}^2} \partial_t f^N(t,x,v)(F^N * \varphi)(x,v)\,dxdv \Big\|_{L^\infty(\mathbb{R}_+)} \\
\le\ & C\|F^N * \varphi\|_{W^{1,2}} \\
=\ & C\Big\| \iint_{\mathbb{R}^2\times\mathbb{R}^2} F^N(y,w)\varphi(x-y,v-w)\,dydw \Big\|_{W^{1,2}} \\
\le\ & C\|F^N\|_{L^\infty}\|\varphi\|_{W^{1,2}} \\
\le\ & C\|F\|_{L^\infty}\|\varphi\|_{W^{1,2}}
\end{aligned}$$

or, in other words,

$$\|\partial_t(F^N * f^N)\|_{L^\infty(W^{-1,2})} = \|F^N * \partial_t f^N\|_{L^\infty(W^{-1,2})} \le C.$$

We then get $\forall\, \varphi \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$

$$F^N * f^N \in L^\infty([0,T]; W^{1,2}(\Omega)), \quad \partial_t(F^N * f^N) \in L^\infty([0,T]; W^{-1,2}(\Omega)),$$

where $\Omega = \mathrm{supp}\varphi$. According to Aubin-Lions compact embedding theorem, e.g. [109], [23], there exists a subsequence and $h \in L^\infty([0,T]; L^2(\Omega))$ such that

$$F^N * f^N \to h \quad \text{in}\, L^\infty([0,T]; L^2(\Omega)).$$

It is not difficult to check that $h = F * f$. Therefore we obtain the following estimates:

$$
\begin{aligned}
& \Big| \int_0^t \iint_{\mathbb{R}^2\times\mathbb{R}^2} \Big(\big((F^N * f^N) f^N\big)(s,x,v) \nabla_v \varphi(x,v) - \big((F*f)f\big)(s,x,v)\nabla_v\varphi(x,v) \Big)\, dxdvds \Big| \\
= \; & \Big| \int_0^t \iint_{\Omega} \Big(\big((F^N * f^N) f^N\big)(s,x,v) \nabla_v \varphi(x,v) - \big((F*f)f^N\big)(s,x,v)\nabla_v\varphi(x,v) \\
& \quad + \big((F*f)f^N\big)(s,x,v)\nabla_v\varphi(x,v) - \big((F*f)f\big)(s,x,v)\nabla_v\varphi(x,v) \Big)\, dxdvds \Big| \\
\leq \; & \Big| \int_0^t \iint_{\Omega} \Big(\big((F^N * f^N) f^N\big)(s,x,v) \nabla_v \varphi(x,v) - \big((F*f)f^N\big)(s,x,v)\nabla_v\varphi(x,v) \Big)\, dxdvds \Big| \\
& \quad + \Big| \int_0^t \iint_{\Omega} \Big(\big((F*f)f^N\big)(s,x,v)\nabla_v\varphi(x,v) - \big((F*f)f\big)(s,x,v)\nabla_v\varphi(x,v) \Big)\, dxdvds \Big| \\
=: \; & J_1 + J_2.
\end{aligned}
$$

For the first term J_1, we have

$$
\lim_{N\to\infty} J_1 \leq \lim_{N\to\infty} \|F^N * f^N - F * f\|_{L^\infty(L^2(\Omega))} \|f^N\|_{L^\infty(L^\infty)} \|\nabla_v \varphi\|_{L^2} = 0,
$$

while for the second term J_2 we use the fact that $f^N \overset{*}{\rightharpoonup} f$ in $L^\infty(\mathbb{R}_+; L^\infty(\mathbb{R}^2\times\mathbb{R}^2))$ for $F * f \cdot \nabla_v\varphi \in L^1(L^1)$, namely

$$
\lim_{N\to\infty} J_2 = 0.
$$

Finally, we have to examine the initial data. Since f^N is the weak solution to the cut-off mean field equation (2.2.6), it obviously satisfies

$$
\begin{aligned}
\iint_{\mathbb{R}^2\times\mathbb{R}^2} f^N(t,x,v)\varphi(x,v)\, dxdv = & \iint_{\mathbb{R}^2\times\mathbb{R}^2} f_0^N(x,v)\varphi(x,v)\, dxdv \\
& + \int_0^t \iint_{\mathbb{R}^2\times\mathbb{R}^2} v f^N(s,x,v)\cdot\nabla_x\varphi(x,v)\, dxdvds \\
& + \int_0^t \iint_{\mathbb{R}^2\times\mathbb{R}^2} \big(F^N(x,v) * f^N(s,x,v)\big) f^N(s,x,v)\cdot\nabla_v\varphi(x,v)\, dxdvds \\
& + \int_0^t \iint_{\mathbb{R}^2\times\mathbb{R}^2} G^N(x,v) f^N(s,x,v)\cdot\nabla_v\varphi(x,v)\, dxdvds
\end{aligned}
$$

for any test function $\varphi(x,v) \in C_0^\infty(\mathbb{R}^2\times\mathbb{R}^2)$. We recall

$$
\|f_0^N - f_0\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \to 0, \text{ as } N\to\infty,
$$

and that terms on the right (second till last) hand side are uniformly continuous in time t. Then, taking limit $t\to 0^+$ on both sides of the above equation verifies the initial data.

Chapter 3

Pedestrian Flow Model

The notable interest of pedestrian flow models can be dated back to four decades ago with a considerable increase in interest since about year 2000. For a general recent overview we refer to [6, 7, 25, 26, 33, 66, 67, 100] and the references therein. Pedestrian models share striking analogies in classical physics such as gases and fluids, but are also applied to the description of opinion formation [116], group dynamics or other social phenomena [90]. Pedestrian flow models are an ideal starting point for the derivation of other or more general quantitative behavioral models, since the relevant quantities of pedestrian motions are easily measured so that corresponding models are comparable with empirical data [66]. The modelling presented here is based on the idea that behavioral changes are guided by so-called social fields or social forces, which have been suggested by Lewin [83]. Numerical simulations have been recently carried out in [44] on the microscopic and macroscopic level using the finite particle method (FPM). Some interesting spatiotemporal patterns are observed.

This chapter provides the detailed derivation from the N-particle (pedestrian) Newtonian system to its mean field limit or Vlasov equation. Instead of the formal derivation with the help of the BBGKY hierarchy, which can be found in [44, 111], we will rigorously derive the kinetic description by a probabilistic method, which is inspired by Boers and Pickl [8], Hauray and Jabin [63, 65], Philipowski [99] and Sznitman [112] and all the references therein.

However, the proposed pedestrian model involves a singularity, which comes from the albeit bounded interaction force and is similar to the one generated by the Coulomb potential in 2-d. While the authors in [8] do not tackle the direct Coulomb potential in 3-d, i.e. they consider the singularity that is a little weaker than for the Coulomb potential, we are capable to deal with the singularity directly due to the compact support of the considered interaction force. Another difficulty lies in the treatment of the dissipative terms since the interaction force depends not only on the position x but also on the velocity v. This will lead to extra work on the estimates and is up to our knowledge rarely done before.

We now briefly explain our approach. In order to obtain the convergence between the exact and the mean field dynamics, we mainly split the proof into two parts: Using the Newtonian system with cut-off as a starting point, we show that the Newtonian and the intermediate system (Vlasov flow with cut-off) are close to each other for N being large enough. The next step is to show the intermediate system converges to the Vlasov flow without cut-off. Inbetween we use characteristics as a bridge to connect the Newtonian system and the mean field dynamics.

Additionally, assuming stochastic initial data offers a way to rule out those deterministic dynamics that do not fit into the proper configuration of the Vlasov equation in the sense that those particles have small probability to appear. In doing so, we obtain the convergence in (probability) measure between the exact and the mean field dynamics. As a direct implication of the convergence, we prove the propagation of chaos in terms of bounded Lipschitz distance.

This chapter is organized as follows: we start with the introduction of the pedestrian flow model in Section 3.1. Then, in Section 3.2 some notations and preliminary work will be introduced. In Section 3.3 we state the main results and present the corresponding proofs. Section 3.4 is devoted to the propagation of chaos. At this point, we also refer to [112] for other classical results with bounded Lipschitz continuity. Finally, we summarize our results.

3.1 Modeling of Pedestrian Flow

Following the pedestrian flow model originally introduced in [44], we consider a two-dimensional interacting particle system with position $x_i \in \mathbb{R}^2$ and velocity $v_i \in \mathbb{R}^2, i = 1, \ldots, N$. The equations of motion read

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \\ \dfrac{dv_i}{dt} = \dfrac{1}{N-1} \displaystyle\sum_{i \neq j} F(x_i - x_j, v_i - v_j) + G(x_i, v_i), \end{cases} \tag{3.1.1}$$

where $F(x, v)$ denotes the total interaction force and $G(x, v)$ the desired velocity and direction acceleration. More precisely, $F(x, v)$ consists of the interaction force $F_{int}(x)$ and the dissipative force $F_{diss}(x, v)$, i.e.,

$$F(x, v) = (F_{int}(x) + F_{diss}(x, v))\mathcal{H}(x, v)$$

with

$$F_{int}(x) = k_n \frac{x}{|x|}(2R - |x|) = 2Rk_n \frac{x}{|x|} - k_n x,$$

$$\begin{aligned}
F_{diss}(x,v) &= F^n_{diss}(x,v) + F^t_{diss}(x,v) \\
&= -\gamma_n \frac{\langle v,x\rangle}{|x|^2}x - \gamma_t\left(v - \frac{\langle v,x\rangle}{|x|^2}x\right) \\
&= \frac{\langle v,x\rangle}{|x|^2}(\gamma_t-\gamma_n)x - \gamma_t v
\end{aligned}$$

and

$$\mathcal{H}(x,v) := \mathcal{H}_{2R}(|x|)\cdot \widetilde{\mathcal{H}}_{2\widetilde{R}}(|v|),$$

where $\mathcal{H}_{2R}(|x|)$ and $\widetilde{\mathcal{H}}_{2\widetilde{R}}(|v|)$ are smooth functions with compact support that satisfy

$$\mathcal{H}_{2R}(|x|) = \begin{cases} 0, & |x| > 2R, \\ 1, & |x| < R, \end{cases} \quad \text{and} \quad \widetilde{\mathcal{H}}_{2\widetilde{R}}(|v|) = \begin{cases} 0, & |v| > 2\widetilde{R}, \\ 1, & |v| < \widetilde{R}. \end{cases}$$

Here, $F^n_{diss}(x,v)$ and $F^t_{diss}(x,v)$ are the normal dissipative force and the tangential friction force, respectively. Moreover, k_n is the interaction constant and γ_n, γ_t are suitable positive friction constants.

Remark 3.1.1. *To obtain a realistic behavior of pedestrians, the functions $\mathcal{H}_{2R}(|x|)$ and $\widetilde{\mathcal{H}}_{2\widetilde{R}}(|v|)$ are used to express that the interaction force and the pedestrian velocity are of finite range. Mathematically, the total force is considered on a bounded domain.*

The desired velocity and direction acceleration is given by

$$G(x,v) := G(x,v,\rho) = \frac{1}{T}\left(-U(\rho)\frac{\nabla\Phi(x)}{|\nabla\Phi(x)|} - v\right),$$

where

$$\rho = \rho(x) = \frac{1}{N^R_{\max}} \sum_{j,|x-x_j|<R} 1.$$

N^R_{max} depends on the time t via the coupling to the positions x_j. For a fixed time t, N^R_{max} describes the maximal number of particles in a ball of radius R and is used here as a normalization parameter. This means, we only scale the number of particles in this region in the sense how compressed they are. Φ is given by the solution of the eikonal equation

$$U(\rho(x))|\nabla\Phi| - 1 = 0,$$

where $U : [0,1] \to [0, U_{\max}]$ is a density-dependent velocity function. The reaction time T might also depend on the density ρ.

The kinetic equation associated with this particle system describes the evolution of the (effective one particle) density $f(t,x,v)$ as

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot [(F * f)f] + \nabla_v \cdot (Gf) = 0. \tag{3.1.2}$$

See [44] for more details and the derivation of macroscopic models for different moment closures.

3.2 Notations and Preliminary Work

Now, we consider the pedestrian flow model (3.1.1) with cut-off of order $N^{-\theta}$ with $0 < \theta < \frac{1}{4}$, i.e.,

$$F^N(x,v) = \begin{cases} \left(2Rk_n \dfrac{x}{|x|} - k_n x + \dfrac{\langle v, x\rangle}{|x|^2}(\gamma_t - \gamma_n)x - \gamma_t v\right)\mathcal{H}(x,v), & |x| \geq N^{-\theta}, \\ \left((2Rk_n N^{\theta} - k_n)x + N^{2\theta}\langle v, x\rangle(\gamma_t - \gamma_n)x - \gamma_t v\right)\mathcal{H}(x,v), & |x| < N^{-\theta}. \end{cases}$$

In order to present the analytical results in Section 3.3 in a concise and clear manner, we restrict to the following notations.

Definition 3.2.1. *1. Let (X_t^N, V_t^N) be the trajectory on $\mathbb{R}^{4N}$ which evolves according to the Newtonian equation of motion with cut-off, i.e.,*

$$\begin{cases} \dfrac{d}{dt} X_t^N = V_t^N, \\ \\ \dfrac{d}{dt} V_t^N = \Psi^N(X_t^N, V_t^N) + \Gamma(X_t^N, V_t^N), \end{cases} \tag{3.2.1}$$

where $\Psi^N(X_t^N, V_t^N)$ denotes the total interaction force with

$$\left(\Psi^N(X_t^N, V_t^N)\right)_i = \frac{1}{N-1}\sum_{i \neq j} F^N(x_i^N - x_j^N, v_i^N - v_j^N),$$

while $\Gamma(X_t^N, V_t^N)$ stands for the desired velocity and direction acceleration with

$$\left(\Gamma(X_t^N, V_t^N)\right)_i = G(x_i^N, v_i^N).$$

2. Let $(\overline{X}_t^N, \overline{V}_t^N)$ be the trajectory on $\mathbb{R}^{4N}$ which evolves according to the Vlasov equation

$$\partial_t f^N + v \cdot \nabla_x f^N + \nabla_v \cdot [(F^N * f^N)f^N] + \nabla_v \cdot (Gf^N) = 0, \tag{3.2.2}$$

i.e.,

$$\begin{cases} \dfrac{d}{dt}\overline{X}_t^N = \overline{V}_t^N, \\ \\ \dfrac{d}{dt}\overline{V}_t^N = \overline{\Psi}^N(\overline{X}_t^N, \overline{V}_t^N) + \Gamma(\overline{X}_t^N, \overline{V}_t^N), \end{cases} \tag{3.2.3}$$

where $\left(\overline{\Psi}^N(\overline{X}_t^N, \overline{V}_t^N)\right)_i = \iint F^N(\overline{x}_i^N - y, \overline{v}_i^N - w) f^N(t,y,w)\, dydw$ *and* $\left(\Gamma(\overline{X}_t^N, \overline{V}_t^N)\right)_i = G(\overline{x}_i^N, \overline{v}_i^N)$ *represent the total interaction force and the desired velocity and direction acceleration, respectively.*

If N is removed from the superscript, then (X_t, V_t) and $(\overline{X}_t, \overline{V}_t)$ denote the particle configurations driven by the force without cut-off. Analogically, if t is removed from the subscript, (X, V) and $(\overline{X}, \overline{V})$ represent the stochastic initial data, which are independent and identically distributed. Note that we always consider the same initial data for both systems, that means $(X, V) = (\overline{X}, \overline{V})$.

Remark 3.2.1. *We also point out several facts for the interaction force $F^N(x,v)$ with cut-off and the acceleration $G(x,v)$. All the properties can be checked by direct computations.*

(a) $F^N(x,v)$ *is bounded, i.e.,* $|F^N(x,v)| \le C$.

(b) $F^N(x,v)$ *satisfies the following property*

$$|F^N(x,v) - F^N(y,v)| \le q^N(x,v)|x-y|,$$

where q^N has compact support in $B_{2R} \times B_{2\widetilde{R}}$ with

$$q^N(x,v) := \begin{cases} C \cdot \dfrac{1}{|x|} + C, & |x| \ge N^{-\theta}, \\ \\ C \cdot N^{\theta}, & |x| < N^{-\theta}. \end{cases}$$

(c) $F^N(x,v)$ *is Lipschitz continuous in* v.

(d) $G(x,v)$ *is bounded, i.e.,* $|G(x,v)| \le C$.

In this context, we use C as a universal constant that might depend on $k_n, R, \widetilde{R}, \gamma_n, \gamma_t$.

Furthermore, if there is a singularity in the velocity v in the interaction potential similar to Remark 3.2.1(b), i.e.,

$$|F^N(x,v) - F^N(x,w)| \le \widetilde{q}^N(x,v)|v-w|,$$

where $\widetilde{q}^N(x,v)$ has compact support in $B_{2R} \times B_{2\widetilde{R}}$ with

$$\widetilde{q}^N(x,v) := \begin{cases} C \cdot \dfrac{1}{|v|} + C, & |v| \ge N^{-\theta}, \\ \\ C \cdot N^{\theta}, & |v| < N^{-\theta}, \end{cases}$$

it can be treated by using the same method as above and the results also apply.

3.3 Mean Field Limit

In this section, we present our key results in full detail. To show the desired convergence, our method can be summarized as follows. First, we start from the Newtonian system with carefully chosen cut-off and meanwhile introduce an intermediate system which involves convolution-type interaction with cut-off, namely (3.2.2) and (3.2.3). Then, we show the convergence of the intermediate system to the final mean field limit, where the law of large number comes into play. The crucial point of this method is that we apply stochastic initial data or in other words we consider a stochastic process. It enables us to use the tools from probability theory, which helps to better understand the mean field process. The overall procedure can be summarized as follows:

$$\begin{array}{ccc}
\begin{cases} \dfrac{dx_i}{dt} = v_i \\ \dfrac{dv_i}{dt} = \dfrac{1}{N-1}\displaystyle\sum_{i \ne j} F(x_i - x_j, v_i - v_j) + G(x_i, v_i) \end{cases} & \xrightarrow{\text{cut-off}} & \begin{cases} \dfrac{dx_i^N}{dt} = v_i^N \\ \dfrac{dv_i^N}{dt} = \dfrac{1}{N-1}\displaystyle\sum_{i \ne j} F^N(x_i^N - x_j^N, v_i^N - v_j^N) + G(x_i^N, v_i^N) \end{cases} \\
 & & \Big\downarrow{\scriptstyle N \to \infty} \\
\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot [(F * f) f] + \nabla_v \cdot (Gf) = 0 & \xleftarrow{\text{without cut-off}} & \partial_t f^N + v \cdot \nabla_x f^N + \nabla_v \cdot [(F^N * f^N) f^N] + \nabla_v \cdot (G f^N) = 0 \\
\Big\downarrow{\scriptstyle \text{characteristics}} & & \Big\downarrow{\scriptstyle \text{characteristics}} \\
\begin{cases} \dfrac{d\overline{x}_i}{dt} = \overline{v}_i \\ \dfrac{d\overline{v}_i}{dt} = \displaystyle\iint F(\overline{x}_i - y, \overline{v}_i - w) f(t,y,w)\, dy dw + G(\overline{x}_i, \overline{v}_i) \end{cases} & & \begin{cases} \dfrac{d\overline{x}_i^N}{dt} = \overline{v}_i^N \\ \dfrac{d\overline{v}_i^N}{dt} = \displaystyle\iint F^N(\overline{x}_i^N - y, \overline{v}_i^N - w) f^N(t,y,w)\, dy dw + G(\overline{x}_i^N, \overline{v}_i^N) \end{cases}
\end{array}$$

The following assumptions are used throughout this section.

Assumption 3.3.1. *We assume that*

(a) there exists a time $t > 0$ and a constant C such that the solution $f(t,x,v)$ of the Vlasov

equation (3.1.2) *satisfies*

$$\sup_{0\le s\le t}\left\|\iint \frac{1}{|x-y|} f(s,y,v)\,dydv\right\|_\infty \le C,$$

(b) the function $G(x,v)$ is Lipschitz continuous both in x and v, i.e., there exists a constant L such that

$$|G(x,v)-G(x',v')| \le L\,(|x-x'|+|v-v'|),\quad \forall\,(x,v),(x',v')\in\mathbb{R}^{4N}.$$

Definition 3.3.1. *Let $\alpha\in(0,\frac{1}{5})$ and $S_t:\mathbb{R}^{4N}\times\mathbb{R}\to\mathbb{R}$ be the stochastic process given by*

$$S_t=\min\Big\{1, N^\alpha \sup_{0\le s\le t}\Big|(X_s^N,V_s^N)-(\overline{X}_s^N,\overline{V}_s^N)\Big|_\infty\Big\}.$$

The set, where $|S_t|=1$, is defined as $\mathcal{N}_\alpha$, i.e.,

$$\mathcal{N}_\alpha := \left\{(X,V):\sup_{0\le s\le t}\Big|(X_s^N,V_s^N)-(\overline{X}_s^N,\overline{V}_s^N)\Big|_\infty > N^{-\alpha}\right\}. \tag{3.3.1}$$

Here and in the following we use $|\cdot|_\infty$ as the supremum norm on $\mathbb{R}^{4N}$. Note that

$$\mathbb{E}_0(S_{t+dt}-S_t\,|\,\mathcal{N}_\alpha)\le 0,$$

since S_t takes the value of one for $(X,V)\in\mathcal{N}_\alpha$.

Theorem 3.3.1. *Let $\theta\in(0,\frac{1}{4})$, $\alpha\in(0,\frac{1}{5})$, $\beta\in(\alpha,\frac{1-\alpha}{4})$, $\gamma\in(0,\frac{1-\alpha}{4}-\theta)$ and $f^N(t,x,v)$ be the solution to the Vlasov equation (3.2.2). Suppose that $f^N(t,x,v)$ satisfies Assumption 4.1(a) and Assumption 4.1(b) holds for $G(x,v)$. Then there exists a constant C such that*

$$\mathbb{P}_0\left(\sup_{0\le s\le t}\Big|(X_s^N,V_s^N)-(\overline{X}_s^N,\overline{V}_s^N)\Big|_\infty > N^{-\alpha}\right)\le e^{Ct}\cdot N^{-n},$$

where $n=\min\{1-\alpha-4\beta, 1-\alpha-4\theta-4\gamma, \beta-\alpha\}$. Furthermore, if $f^N(t,x,v)\in L^\infty((0,\infty);L^1(\mathbb{R}^2\times\mathbb{R}^2))\cap L^\infty((0,\infty);L^\infty(\mathbb{R}^2\times\mathbb{R}^2))$, it holds with a θ-independent convergence rate that

$$\mathbb{P}_0\left(\sup_{0\le s\le t}\Big|(X_s^N,V_s^N)-(\overline{X}_s^N,\overline{V}_s^N)\Big|_\infty > N^{-\alpha}\right)\le e^{Ct}\cdot r(N),$$

where the convergence rate $r(N)=\max\{N^{-(1-\alpha-4\beta)}, N^{\alpha-\beta}, N^{-(1-\alpha-4\gamma)}\ln^2 N\}$.

We remark that $f^N(t,x,v)\in L^1(\mathbb{R}^2\times\mathbb{R}^2)$ is automatically satisfied due to the mass conservation.

With additional assumption on the initial condition f_0 for the equations (3.1.2), (3.2.2) and on the solution of the Vlasov equation without cut-off, we further extend our result to

Theorem 3.3.2. *Let $f(t,x,v)$ and $f^N(t,x,v)$ be the solution to the Vlasov equation* (3.1.2) *and* (3.2.2) *respectively with the same initial data f_0. Suppose that Assumption 4.1(b) is satisfied. Moreover, ∇f_0 is integrable and $f(t,x,v) \in L^\infty((0,\infty); L^\infty(\mathbb{R}^2 \times \mathbb{R}^2))$. Then there holds*

$$\lim_{N\to\infty} \mathbb{P}_0 \left(\sup_{0\le s\le t} \left| (X_s^N, V_s^N) - (\overline{X}_s, \overline{V}_s) \right|_\infty > N^{-\alpha} \right) = 0.$$

The proofs of both theorems will be presented at the end of this section.

The additional requirement on $f(t,x,v)$ stems from the existence and uniqueness of the solution to the Vlasov equation, which will be shown in another independent work in the near future.

Definition 3.3.2. *Let $\beta \in (\alpha, \frac{1-\alpha}{4}), \gamma \in (0, \frac{1-\alpha}{4} - \theta)$. The sets $\mathcal{N}_\beta$ and $\mathcal{N}_\gamma$ are characterized by*

$$\mathcal{N}_\beta := \left\{ (X,V) : \left| \Psi^N(\overline{X}_t^N, \overline{V}_t^N) - \overline{\Psi}^N(\overline{X}_t^N, \overline{V}_t^N) \right|_\infty > N^{-\beta} \right\}, \tag{3.3.2}$$

$$\mathcal{N}_\gamma := \left\{ (X,V) : \left| Q^N(\overline{X}_t^N, \overline{V}_t^N) - \overline{Q}^N(\overline{X}_t^N, \overline{V}_t^N) \right|_\infty > N^{-\gamma} \right\}, \tag{3.3.3}$$

where $Q^N(\overline{X}_t^N, \overline{V}_t^N)$ and $\overline{Q}^N(\overline{X}_t^N, \overline{V}_t^N)$ are understood in the sense of

$$(Q^N(\overline{X}_t^N, \overline{V}_t^N))_i := \frac{1}{N-1} \sum_{i\neq j} q^N(\overline{x}_i^N - \overline{x}_j^N, \overline{v}_i^N - \overline{v}_j^N)$$

and correspondingly

$$(\overline{Q}^N(\overline{X}_t^N, \overline{V}_t^N))_i := \iint q^N(\overline{x}_i^N - y, \overline{v}_i^N - w) f^N(t,y,w)\, dydw.$$

Next, we will see that the measures of both sets $\mathcal{N}_\beta$ and $\mathcal{N}_\gamma$ can be arbitrarily small, i.e., the probability of each set tends to 0 as N goes to infinity. We prove the following two lemmas:

Lemma 3.3.1. *There exists a constant $C < \infty$ such that*

$$\mathbb{P}_0(\mathcal{N}_\beta) \le C N^{-(1-4\beta)}.$$

Proof. First, we let the set $\mathcal{N}_\beta$ evolve along the characteristics of the Vlasov equation

$$\mathcal{N}_{\beta,t} := \left\{ (\overline{X}_t^N, \overline{V}_t^N) : \left| N^\beta \Psi^N(\overline{X}_t^N, \overline{V}_t^N) - N^\beta \overline{\Psi}^N(\overline{X}_t^N, \overline{V}_t^N) \right|_\infty > 1 \right\}$$

and consider the following fact

$$\mathcal{N}_{\beta,t} \subseteq \bigoplus_{i=1}^{N} \mathcal{N}_{\beta,t}^{i},$$

where

$$\mathcal{N}_{\beta,t}^{i} := \left\{ (\overline{x}_i^N, \overline{v}_i^N) : \left| N^{\beta} \cdot \frac{1}{N-1} \sum_{i \neq j} F^N(\overline{x}_i^N - \overline{x}_j^N, \overline{v}_i^N - \overline{v}_j^N) - N^{\beta}(F^N * f^N)(t, \overline{x}_i^N, \overline{v}_i^N) \right|_{\infty} > 1 \right\}.$$

We therefore get

$$\mathbb{P}_t(\mathcal{N}_{\beta,t}) \leq \sum_{i=1}^{N} \mathbb{P}_t(\mathcal{N}_{\beta,t}^{i}) = N\mathbb{P}_t(\mathcal{N}_{\beta,t}^{1}),$$

where in the last step we use the symmetry argument in exchanging any two coordinates. Using Markov inequality gives

$$\begin{aligned} \mathbb{P}_t(\mathcal{N}_{\beta,t}^{1}) &\leq \mathbb{E}_t \left[\left(N^{\beta} \cdot \frac{1}{N-1} \sum_{j=2}^{N} F^N(\overline{x}_1^N - \overline{x}_j^N, \overline{v}_1^N - \overline{v}_j^N) - N^{\beta}(F^N * f^N)(t, \overline{x}_1^N, \overline{v}_1^N) \right)^4 \right] \\ &= \left(\frac{N^{\beta}}{N-1} \right)^4 \mathbb{E}_t \left[\left(\sum_{j=2}^{N} F^N(\overline{x}_1^N - \overline{x}_j^N, \overline{v}_1^N - \overline{v}_j^N) - (N-1)(F^N * f^N)(t, \overline{x}_1^N, \overline{v}_1^N) \right)^4 \right]. \end{aligned} \tag{3.3.4}$$

Let $h_j := F^N(\overline{x}_1^N - \overline{x}_j^N, \overline{v}_1^N - \overline{v}_j^N) - \iint F^N(\overline{x}_1^N - y, \overline{v}_1^N - w) f^N(t, y, w)\, dydw$. Then, each term in the expectation (3.3.4) takes the form of $\prod_{j=2}^{N} h_j^{k_j}$ with $\sum_{j=1}^{N} k_j = 4$, and more importantly, the expectation assumes the value of zero whenever there exists a j such that $k_j = 1$. This can be easily verified by integrating over the j-th variable first or, in other words, by acknowledging the fact that $\forall\, j = 2, \ldots, N$, there holds

$$\mathbb{E}_t \left[F^N(\overline{x}_1^N - \overline{x}_j^N, \overline{v}_1^N - \overline{v}_j^N) - \iint F^N(\overline{x}_1^N - y, \overline{v}_1^N - w) f^N(t, y, w)\, dydw \right] = 0.$$

Then, we can simplify the estimate (3.3.4) to

$$\mathbb{P}_t(\mathcal{N}_{\beta,t}^{1}) \leq \left(\frac{N^{\beta}}{N-1} \right)^4 \mathbb{E}_t \left[\sum_{j=2}^{N} h_j^4 + \sum_{2 \leq m < n}^{N} \binom{4}{2} h_m^2 h_n^2 \right].$$

Since F^N is bounded and $||f^N||_1 = 1$, we thus have for any fixed j

$$|h_j| \leq |F^N(\overline{x}_1^N - \overline{x}_j^N, \overline{v}_1^N - \overline{v}_j^N)| + \iint |F^N(\overline{x}_1^N - y, \overline{v}_1^N - w)| f^N(t, y, w)\, dydw \leq C.$$

Therefore $|h_j|$ is bounded to any power and we obtian

$$\mathbb{E}_t\left[h_m^2 h_n^2\right] \le C \quad \text{and} \quad \mathbb{E}_t\left[h_j^4\right] \le C$$

and consequently

$$\mathbb{P}_t(\mathcal{N}_{\beta,t}^1) \le \left(\frac{N^\beta}{N-1}\right)^4 \cdot \left(C\cdot(N-1) + C\cdot\frac{(N-1)(N-2)}{2}\right) \le C\cdot N^{-(2-4\beta)}.$$

By noticing the fact that

$$\mathbb{P}_0(\mathcal{N}_\beta) = \mathbb{P}_t\left(\mathcal{N}_{\beta,t}\right) \le N\mathbb{P}_t(\mathcal{N}_{\beta,t}^1) \le N\cdot C\cdot N^{-(2-4\beta)} = C\cdot N^{-(1-4\beta)},$$

we obtain the desired result. ■

In fact, this result holds for any β if we change accordingly the power in the proof to be another even number (depending on β) greater than four.

Due to the singularity of $\nabla_x F$, which is also the motivation for the cut-off, we exploit a slightly different technique as in Lemma 3.3.1 to prove

Lemma 3.3.2. *There exists a constant $C < \infty$ such that*

$$\mathbb{P}_0(\mathcal{N}_\gamma) \le C\cdot\widetilde{r}(N),$$

where $\widetilde{r}(N)$ is the convergence rate, which is $N^{-(1-4\gamma)}\ln^2 N$ if $f^N \in L^\infty(\mathbb{R}^2\times\mathbb{R}^2)$ or $N^{-(1-4\theta-4\gamma)}$ otherwise.

Proof. Let the set $\mathcal{N}_\gamma$ evolve along the characteristics of the Vlasov equation

$$\mathcal{N}_{\gamma,t} := \left\{(\overline{X}_t^N, \overline{V}_t^N) : \left|N^\gamma Q^N((\overline{X}_t^N, \overline{V}_t^N) - N^\gamma \overline{Q}^N((\overline{X}_t^N, \overline{V}_t^N)\right|_\infty > 1\right\}$$

and consider the fact

$$\mathcal{N}_{\gamma,t} \subseteq \bigoplus_{i=1}^N \mathcal{N}_{\gamma,t}^i,$$

where

$$\mathcal{N}_{\gamma,t}^i := \left\{(\overline{x}_i^N, \overline{v}_i^N) : \left|N^\gamma\cdot\frac{1}{N-1}\sum_{i\ne j} q^N(\overline{x}_i^N - \overline{x}_j^N, \overline{v}_i^N - \overline{v}_j^N) - N^\gamma(q^N * f^N)(t, \overline{x}_i^N, \overline{v}_i^N)\right|_\infty > 1\right\}.$$

Due to the symmetry in exchanging any two coordinates, we get

$$\mathbb{P}_t(\mathcal{N}_{\gamma,t}) \leq \sum_{i=1}^{N} \mathbb{P}_t(\mathcal{N}_{\gamma,t}^i) = N\mathbb{P}_t(\mathcal{N}_{\gamma,t}^1).$$

Using Markov inequality gives

$$\begin{aligned}
\mathbb{P}_t(\mathcal{N}_{\gamma,t}^1) &\leq \mathbb{E}_t\left[\left(N^\gamma \cdot \frac{1}{N-1}\sum_{j=2}^{N} q^N(\overline{x}_1^N - \overline{x}_j^N, \overline{v}_1^N - \overline{v}_j^N) - N^\gamma (q^N * f^N)(t, \overline{x}_1^N, \overline{v}_1^N)\right)^4\right] \\
&= \left(\frac{N^\gamma}{N-1}\right)^4 \mathbb{E}_t\left[\left(\sum_{j=2}^{N} q^N(\overline{x}_1^N - \overline{x}_j^N, \overline{v}_1^N - \overline{v}_j^N) - (N-1)(q^N * f^N)(t, \overline{x}_1^N, \overline{v}_1^N)\right)^4\right].
\end{aligned} \tag{3.3.5}$$

In order to avoid redundant complexity, we borrow the notation from Lemma 4.1 and also define $h_j := q^N(\overline{x}_1^N - \overline{x}_j^N, \overline{v}_1^N - \overline{v}_j^N) - \iint q^N(\overline{x}_1^N - y, \overline{v}_1^N - w) f^N(t, y, w)\, dydw$. With the same argument as in Lemma 3.3.1, we simplify the estimate (3.3.5) to

$$\mathbb{P}_t(\mathcal{N}_{\gamma,t}^1) \leq \left(\frac{N^\gamma}{N-1}\right)^4 \mathbb{E}_t\left[\sum_{j=2}^{N} h_j^4 + \sum_{2\leq m<n}^{N} 6h_m^2 h_n^2\right].$$

On one hand, due to the cut-off, it is clear that

$$||q^N||_\infty \leq C \cdot N^\theta.$$

On the other hand, by taking out the L^∞-norm of q^N and using the integrability of f^N, we achieve

$$\left|\iint q^N(\overline{x}_1^N - y, \overline{v}_1^N - w) f^N(t, y, w)\, dydw\right| \leq C \cdot N^\theta.$$

Therefore $|h_j|$ is bounded by $C \cdot N^\theta$ and it is now obvious to see that

$$\mathbb{E}_t\left[h_j^4\right] \leq C \cdot N^{4\theta} \quad \text{and} \quad \mathbb{E}_t\left[h_m^2 h_n^2\right] \leq C \cdot N^{4\theta}$$

and consequently

$$\mathbb{P}_t(\mathcal{N}_{\gamma,t}^1) \leq C \cdot \left(\frac{N^\gamma}{N-1}\right)^4 \left(N^{4\theta} \cdot (N-1) + N^{4\theta} \cdot \frac{(N-1)(N-2)}{2}\right) \leq C \cdot N^{-(2-4\theta-4\gamma)}.$$

By noticing the fact that

$$\mathbb{P}_0(\mathcal{N}_\gamma) = \mathbb{P}_t\left(\mathcal{N}_{\gamma,t}\right) \leq N\mathbb{P}_t(\mathcal{N}^1_{\gamma,t}) \leq C \cdot N^{-(1-4\theta-4\gamma)},$$

we complete the first part of the lemma.

Furthermore , if $f^N(t,x,v) \in L^\infty((0,\infty); L^1(\mathbb{R}^2 \times \mathbb{R}^2)) \cap L^\infty((0,\infty); L^\infty(\mathbb{R}^2 \times \mathbb{R}^2))$, by applying the inequality $\mathbb{E}\left[(X - \mathbb{E}[X])^2\right] \leq \mathbb{E}[X^2]$ for any random variable X, we have for any fixed j

$$\begin{aligned}
&\mathbb{E}_t\left[\left(q^N(\overline{x}^N_1 - \overline{x}^N_j, \overline{v}^N_1 - \overline{v}^N_j) - \iint q^N(\overline{x}^N_1 - y, \overline{v}^N_1 - w) f^N(t,y,w)\, dydw\right)^2\right] \\
\leq\;& \mathbb{E}_t\left[\left(q^N(\overline{x}^N_1 - \overline{x}^N_j, \overline{v}^N_1 - \overline{v}^N_j)\right)^2\right] \\
\leq\;& \iint \left(\iint_{|z-y|<N^{-\theta}} \left(C \cdot N^\theta + C\right)^2 f^N(t,y,w)\, dydw\right) f^N(t,z,u)\, dzdu \\
&+ \iint \left(\iint_{|z-y|\geq N^{-\theta}} \left(C \cdot \frac{1}{|z-y|}\right)^2 f^N(t,y,w)\, dydw\right) f^N(t,z,u)\, dzdu.
\end{aligned}$$

We take out the L^∞-norm of f^N in both terms. The integral left inside the first term is bounded by a constant while in the second term the integral can be estimated by

$$\iint_{|z-y|\geq N^{-\theta}} \left(C \cdot \frac{1}{|z-y|}\right)^2 f^N(t,y,w)\, dydw \leq C + 2\pi\theta \ln N \leq C \cdot \ln N,$$

where we use that q^N has compact support. Therefore for any fixed j

$$\mathbb{E}_t\left[h_j^4\right] \leq ||h_j||^2_\infty \mathbb{E}_t\left[h_j^2\right] \leq C \cdot N^{2\theta} \ln N,$$

$$\mathbb{E}_t\left[h_m^2 h_n^2\right] \leq C \cdot \ln^2 N.$$

Consequently

$$\begin{aligned}
\mathbb{P}_t(\mathcal{N}^1_{\gamma,t}) \;&\leq\; C \cdot \left(\frac{N^\gamma}{N-1}\right)^4 \left(N^{2\theta} \ln N \cdot (N-1) + \ln^2 N \cdot \frac{(N-1)(N-2)}{2}\right) \\
&\leq\; C \cdot N^{-(2-4\gamma)} \ln^2 N.
\end{aligned}$$

Thus it holds that

$$\mathbb{P}_0(\mathcal{N}_\gamma) = \mathbb{P}_t\left(\mathcal{N}_{\gamma,t}\right) \leq N\mathbb{P}_t(\mathcal{N}^1_{\gamma,t}) \leq C \cdot N^{-(1-4\gamma)} \ln^2 N.$$

∎

Lemma 3.3.3. *Let $\mathcal{N}_\alpha$, $\mathcal{N}_\beta$, $\mathcal{N}_\gamma$ be defined as in (3.3.1)-(3.3.3). Suppose that $f^N(t,x,v)$ satisfies Assumption 3.3.1(a) and Assumption 3.3.1(b) holds for $G(x,v)$. Then there exists a constant $C < \infty$ such that*

$$\left|\left(V_t^N, \Psi^N(X_t^N, V_t^N) + \Gamma(X_t^N, V_t^N)\right) - \left(\overline{V}_t^N, \overline{\Psi}^N(\overline{X}_t^N, \overline{V}_t^N) + \Gamma(\overline{X}_t^N, \overline{V}_t^N)\right)\right|_\infty \leq CS_t(X,V)N^{-\alpha}+N^{-\beta}$$

for all initial data $(X,V) \in (\mathcal{N}_\alpha \cup \mathcal{N}_\beta \cup \mathcal{N}_\gamma)^c$.

Proof. Applying triangle inequality gives

$$\begin{aligned}
&\left|\left(V_t^N, \Psi^N(X_t^N, V_t^N) + \Gamma(X_t^N, V_t^N)\right) - \left(\overline{V}_t^N, \overline{\Psi}^N(\overline{X}_t^N, \overline{V}_t^N) + \Gamma(\overline{X}_t^N, \overline{V}_t^N)\right)\right|_\infty \\
\leq\ & \left|V_t^N - \overline{V}_t^N\right|_\infty + \left|\Psi^N(X_t^N, V_t^N) - \overline{\Psi}^N(\overline{X}_t^N, \overline{V}_t^N)\right|_\infty + \left|\Gamma(X_t^N, V_t^N) - \Gamma(\overline{X}_t^N, \overline{V}_t^N)\right|_\infty \\
\leq\ & \left|V_t^N - \overline{V}_t^N\right|_\infty + \left|\Psi^N(X_t^N, V_t^N) - \Psi^N(\overline{X}_t^N, \overline{V}_t^N)\right|_\infty \\
& + \left|\Psi^N(\overline{X}_t^N, \overline{V}_t^N) - \overline{\Psi}^N(\overline{X}_t^N, \overline{V}_t^N)\right|_\infty + \left|\Gamma(X_t^N, V_t^N) - \Gamma(\overline{X}_t^N, \overline{V}_t^N)\right|_\infty \\
=:\ & |I_1| + |I_2| + |I_3| + |I_4|.
\end{aligned}$$

Next, we estimate term by term.

- Since $(X,V) \notin \mathcal{N}_\alpha$,
$$|I_1| := \left|V_t^N - \overline{V}_t^N\right|_\infty \leq S_t(X,V)N^{-\alpha}.$$

- With the help of q^N which is defined in Remark 3.2.1 and the fact that F^N is Lipschitz continuous in v, we obtain
$$\begin{aligned}
&\left|\frac{1}{N-1}\sum_{i\neq j} F^N(x_i^N - x_j^N, v_i^N - v_j^N) - \frac{1}{N-1}\sum_{i\neq j} F^N(\overline{x}_i^N - \overline{x}_j^N, \overline{v}_i^N - \overline{v}_j^N)\right| \\
\leq\ & \frac{1}{N-1}\sum_{i\neq j}\left|q^N(\overline{x}_i^N - \overline{x}_j^N, \overline{v}_i^N - \overline{v}_j^N)\right|\left(2|x_i^N - \overline{x}_i^N| + 2|v_i^N - \overline{v}_i^N|\right). \qquad (3.3.6)
\end{aligned}$$

  Since $(X,V) \notin \mathcal{N}_\alpha$, it follows in particular for any $1 \leq i \leq N$ that
$$|x_i^N - \overline{x}_i^N| \leq N^{-\alpha} \quad \text{and} \quad |v_i^N - \overline{v}_i^N| \leq N^{-\alpha}.$$

  So together with (3.3.6), we have
$$\left|\left(\Psi^N(X_t^N, V_t^N)\right)_i - \left(\Psi^N(\overline{X}_t^N, \overline{V}_t^N)\right)_i\right| \leq 4\left|\left(Q^N(\overline{X}_t^N, \overline{V}_t^N)\right)_i\right| N^{-\alpha}.$$

On the other hand, because $(X, V) \notin \mathcal{N}_\gamma$, it follows

$$\left| \left(Q^N(\overline{X}_t^N, \overline{V}_t^N)\right)_i \right| \leq ||q^N * f^N||_\infty + N^{-\gamma} < C$$

and thus

$$|I_2| := \left| \Psi^N(X_t^N, V_t^N) - \Psi^N(\overline{X}_t^N, \overline{V}_t^N) \right|_\infty \leq C S_t(X,V) N^{-\alpha}.$$

- Since $(X, V) \notin \mathcal{N}_\beta$, it follows directly

$$|I_3| := \left| \Psi^N(\overline{X}_t^N, \overline{V}_t^N) - \overline{\Psi}^N(\overline{X}_t^N, \overline{V}_t^N) \right|_\infty \leq N^{-\beta}.$$

- Since $G(x, v)$ under Assumption 3.3.1(b) is Lipschitz continuous, we have for each $1 \leq i \leq N$ and $(x_i^N, v_i^N) = \left((X_t^N, V_t^N)\right)_i$, $(\overline{x}_i^N, \overline{v}_i^N) = \left((\overline{X}_t^N, \overline{V}_t^N)\right)_i$

$$\left| G(x_i^N, v_i^N) - G(\overline{x}_i^N, \overline{v}_i^N) \right| \leq L \left| (x_i^N, v_i^N) - (\overline{x}_i^N, \overline{v}_i^N) \right|.$$

Together with the fact that $(X, V) \notin \mathcal{N}_\alpha$, there holds

$$|I_4| := \left| \Gamma(X_t^N, V_t^N) - \Gamma(\overline{X}_t^N, \overline{V}_t^N) \right|_\infty \leq L S_t(X,V) N^{-\alpha}.$$

Combining all the four terms, we end up with

$$\left| \left(V_t^N, \Psi^N(X_t^N, V_t^N) + \Gamma(X_t^N, V_t^N)\right) - \left(\overline{V}_t^N, \overline{\Psi}^N(\overline{X}_t^N, \overline{V}_t^N) + \Gamma(\overline{X}_t^N, \overline{V}_t^N)\right) \right|_\infty \leq C S_t(X,V) N^{-\alpha} + N^{-\beta}$$

for all $(X, V) \in (\mathcal{N}_\alpha \cup \mathcal{N}_\beta \cup \mathcal{N}_\gamma)^c$. ∎

Using Lemmas 3.3.1 - 3.3.3 we can now prove Theorem 4.1 and Theorem 4.2:

Proof of Theorem 3.3.1

From the definition of the Newtonian flow (3.2.1) and the characteristics of the Vlasov equation (3.2.3), we know

$$\begin{aligned}
(X_{t+dt}^N, V_{t+dt}^N) &= (X_t^N, V_t^N) + (V_t^N, \Psi^N(X_t^N, V_t^N) + \Gamma(X_t^N, V_t^N))dt + o(dt), \\
(\overline{X}_{t+dt}^N, \overline{V}_{t+dt}^N) &= (\overline{X}_t^N, \overline{V}_t^N) + (\overline{V}_t^N, \overline{\Psi}^N(\overline{X}_t^N, \overline{V}_t^N) + \Gamma(\overline{X}_t^N, \overline{V}_t^N))dt + o(dt).
\end{aligned}$$

Thus

$$\begin{aligned}
&\left| (X_{t+dt}^N, V_{t+dt}^N) - (\overline{X}_{t+dt}^N, \overline{V}_{t+dt}^N) \right|_\infty \leq \left| (X_t^N, V_t^N) - (\overline{X}_t^N, \overline{V}_t^N) \right|_\infty \\
&+ \left| \left(V_t^N, \Psi^N(X_t^N, V_t^N) + \Gamma(X_t^N, V_t^N)\right) - \left(\overline{V}_t^N, \overline{\Psi}^N(\overline{X}_t^N, \overline{V}_t^N) + \Gamma(\overline{X}_t^N, \overline{V}_t^N)\right) \right|_\infty dt + o(dt),
\end{aligned}$$

i.e.,

$$S_{t+dt}-S_t \leq \left|\Big(V_t^N, \Psi^N(X_t^N,V_t^N)+\Gamma(X_t^N,V_t^N)\Big)-\Big(\overline{V}_t^N, \overline{\Psi}^N(\overline{X}_t^N,\overline{V}_t^N)+\Gamma(\overline{X}_t^N,\overline{V}_t^N)\Big)\right|_\infty N^\alpha dt+o(dt)$$

Taking the expectation over both sides yields

$$\begin{aligned}
\mathbb{E}_0\left[\, S_{t+dt}-S_t\right] &= \mathbb{E}_0\left[S_{t+dt}-S_t \,|\, \mathcal{N}_\alpha\,\right]+\mathbb{E}_0\left[\, S_{t+dt}-S_t \,|\, \mathcal{N}_\alpha^c\,\right] \\
&\leq \mathbb{E}_0\left[\, S_{t+dt}-S_t \,|\, (\mathcal{N}_\beta\cup\mathcal{N}_\gamma)\setminus\mathcal{N}_\alpha\,\right]+\mathbb{E}_0\left[\, S_{t+dt}-S_t \,|\, (\mathcal{N}_\alpha\cup\mathcal{N}_\beta\cup\mathcal{N}_\gamma)^c\,\right] \\
&\leq \mathbb{E}_0\left[\,\left|V_t^N-\overline{V}_t^N\right|_\infty \,\Big|\, (\mathcal{N}_\beta\cup\mathcal{N}_\gamma)\setminus\mathcal{N}_\alpha\right] N^\alpha dt \\
&\quad + \mathbb{E}_0\left[\,\left|\Psi^N(X_t^N,V_t^N)-\overline{\Psi}^N(\overline{X}_t^N,\overline{V}_t^N)\right|_\infty \,\Big|\, (\mathcal{N}_\beta\cup\mathcal{N}_\gamma)\setminus\mathcal{N}_\alpha\right] N^\alpha dt \\
&\quad + \mathbb{E}_0\left[\,\left|\Gamma(X_t^N,V_t^N)-\Gamma(\overline{X}_t^N,\overline{V}_t^N)\right|_\infty \,\Big|\, (\mathcal{N}_\beta\cup\mathcal{N}_\gamma)\setminus\mathcal{N}_\alpha\right] N^\alpha dt \\
&\quad + \mathbb{E}_0\left[\, S_{t+dt}-S_t \,|\, (\mathcal{N}_\alpha\cup\mathcal{N}_\beta\cup\mathcal{N}_\gamma)^c\,\right]+o(dt) \\
&=: J_1+J_2+J_3+J_4+o(dt),
\end{aligned}$$

where in the second step we use $\mathbb{E}_0(S_{t+dt}-S_t\,|\,\mathcal{N}_\alpha)\leq 0$ and decompose the set $\mathcal{N}_\alpha^c$ into $(\mathcal{N}_\beta\cup\mathcal{N}_\gamma)\setminus\mathcal{N}_\alpha$ and $(\mathcal{N}_\alpha\cup\mathcal{N}_\beta\cup\mathcal{N}_\gamma)^c$.

Since $(X,V)\notin\mathcal{N}_\alpha$, it follows

$$\begin{aligned}
J_1 &= \mathbb{E}_0\left[\,\left|V_t^N-\overline{V}_t^N\right|_\infty \,\Big|\, (\mathcal{N}_\beta\cup\mathcal{N}_\gamma)\setminus\mathcal{N}_\alpha\right] N^\alpha dt \\
&\leq \big(\mathbb{P}_0(\mathcal{N}_\beta)+\mathbb{P}_0(\mathcal{N}_\gamma)\big)dt.
\end{aligned}$$

Due to the definition of Ψ^N, $\overline{\Psi}^N$, Γ as well as the boundedness of F^N, we obtain

$$J_2 \leq \big(||F^N||_\infty+||F^N*f||_\infty\big)\big(\mathbb{P}_0(\mathcal{N}_\beta)+\mathbb{P}_0(\mathcal{N}_\gamma)\big)N^\alpha dt,$$

$$J_3 \leq 2||G||_\infty\big(\mathbb{P}_0(\mathcal{N}_\beta)+\mathbb{P}_0(\mathcal{N}_\gamma)\big)N^\alpha dt.$$

Thanks to Lemma 3.3.1 and Lemma 3.3.2, we get

$$\begin{aligned}
J_1+J_2+J_3 &= \left[N^{-\alpha}+C\,\right]\big(\mathbb{P}_0(\mathcal{N}_\beta)+\mathbb{P}_0(\mathcal{N}_\gamma)\big)N^\alpha dt \\
&\leq C\cdot\max\{\widetilde{r}(N), N^{-(1-4\beta)}\}N^\alpha dt
\end{aligned}$$

where $\widetilde{r}(N)$ is the convergence rate, which is $N^{-(1-4\gamma)}\ln^2 N$ if $f^N\in L^\infty(\mathbb{R}^2\times\mathbb{R}^2)$ or $N^{-(1-4\theta-4\gamma)}$

otherwise. On the other hand, Lemma 3.3.3 states that

$$\begin{aligned} J_4 &= \mathbb{E}_0 \left[S_{t+dt} - S_t \,|\, (\mathcal{N}_\alpha \cup \mathcal{N}_\beta \cup \mathcal{N}_\gamma)^c \right] \\ &\le (C \cdot \mathbb{E}_0 \left[S_t\right] N^{-\alpha} + N^{-\beta}) \cdot N^\alpha dt + o(dt) \\ &= C \cdot \mathbb{E}_0 \left[S_t\right] dt + N^{\alpha-\beta} dt + o(dt). \end{aligned}$$

Therefore, we can determine the estimate

$$\begin{aligned} \mathbb{E}_0 \left[S_{t+dt}\right] - \mathbb{E}_0[S_t] &\le \mathbb{E}_0 \left[S_{t+dt} - S_t\right] \\ &\le C \cdot \mathbb{E}_0 \left[S_t\right] dt + C \cdot \max\{\widetilde{r}(N) N^\alpha, N^{-(1-\alpha-4\beta)}), N^{\alpha-\beta}\} dt + o(dt). \end{aligned}$$

Equivalently, we have

$$\frac{d}{dt} \mathbb{E}_0[S_t] \le C \cdot \mathbb{E}_0[S_t] + C \cdot \max\{\widetilde{r}(N) N^\alpha, N^{-(1-\alpha-4\beta)}), N^{\alpha-\beta}\}.$$

Gronwall's inequality yields

$$\mathbb{E}_0 \left[S_t\right] \le e^{Ct} \cdot \max\{\widetilde{r}(N) N^\alpha, N^{-(1-\alpha-4\beta)}), N^{\alpha-\beta}\}.$$

The proof is completed by the following Markov inequality

$$\mathbb{P}_0 \left(\sup_{0 \le s \le t} \left| (X_s^N, V_s^N) - (\overline{X}_s^N, \overline{V}_s^N) \right|_\infty > N^{-\alpha} \right) = \mathbb{P}_0(S_t = 1) \le \mathbb{E}_0 \left[S_t\right].$$

■

Proof of Theorem 3.3.2

Let $N \in \mathbb{N}$ and

$$W_t := \sup_{(X,V) \in \mathbb{R}^{4N}} \left| (\overline{X}_t^N, \overline{V}_t^N) - (\overline{X}_t, \overline{V}_t) \right|.$$

With the same argument as in the proof of Theorem 3.3.1, it is not difficult to deduce

$$W_{t+dt} - W_t \le \underbrace{\left| \left(\overline{V}_t^N, \overline{\Psi}^N(\overline{X}_t^N, \overline{V}_t^N) + \Gamma(\overline{X}_t^N, \overline{V}_t^N)\right) - \left(\overline{V}_t, \overline{\Psi}(\overline{X}_t, \overline{V}_t) + \Gamma(\overline{X}_t, \overline{V}_t)\right) \right|_\infty}_{=:D} dt + o(dt).$$

Furthermore, with the Lipschitz continuity of $G(x,v)$, we get

$$\begin{aligned} D &\le W_t + \left|\overline{\Psi}^N(\overline{X}_t^N, \overline{V}_t^N) - \overline{\Psi}(\overline{X}_t, \overline{V}_t)\right|_\infty + \left|\Gamma(\overline{X}_t^N, \overline{V}_t^N) - \Gamma(\overline{X}_t, \overline{V}_t)\right|_\infty \\ &\le C \cdot W_t + \left|\overline{\Psi}^N(\overline{X}_t^N, \overline{V}_t^N) - \overline{\Psi}(\overline{X}_t, \overline{V}_t)\right|_\infty \\ &\le C \cdot W_t + \sup_{1\le i\le N} \left|F^N * f^N(\overline{x}_i^N, \overline{v}_i^N) - F * f(\overline{x}_i, \overline{v}_i)\right| \\ &\le C \cdot W_t + \sup_{1\le i\le N} \left|F^N * f^N(\overline{x}_i^N, \overline{v}_i^N) - F^N * f^N(\overline{x}_i, \overline{v}_i)\right| \\ &\quad + \sup_{1\le i\le N} \left|F^N * f^N(\overline{x}_i, \overline{v}_i) - F^N * f(\overline{x}_i, \overline{v}_i)\right| \\ &\quad + \sup_{1\le i\le N} \left|F^N * f(\overline{x}_i, \overline{v}_i) - F * f(\overline{x}_i, \overline{v}_i)\right|. \end{aligned}$$

By using the integrability of ∇F^N , we estimate the second term by

$$\sup_{1\le i\le N} \left|F^N * f^N(\overline{x}_i^N, \overline{v}_i^N) - F^N * f^N(\overline{x}_i, \overline{v}_i)\right| \le ||\nabla F^N||_1 ||f^N||_\infty W_t \le C \cdot W_t.$$

Due to the integrability of ∇f_0, the third term can be controlled by

$$\sup_{1\le i\le N} \left|F^N * f^N(\overline{x}_i, \overline{v}_i) - F^N * f(\overline{x}_i, \overline{v}_i)\right| \le ||F^N||_\infty ||f^N - f||_1 \le C||\nabla f_0||_1 W_t.$$

In the estimates above, the reversibility of both particle trajectories is used. The last term is straightforward to estimate

$$\sup_{1\le i\le N} \left|F^N * f(\overline{x}_i, \overline{v}_i) - F * f(\overline{x}_i, \overline{v}_i)\right| \le ||f||_\infty ||F^N - F||_1 \le C \cdot N^{-\theta}.$$

Therefore we arrive at

$$W_{t+dt} - W_t \le \big(C \cdot W_t + C \cdot N^{-\theta}\big)dt + o(dt),$$

or equivalently

$$\frac{d}{dt} W_t \le C \cdot W_t + C \cdot N^{-\theta}.$$

Gronwall's inequality gives

$$W_t \le C \cdot N^{-\theta}.$$

Together with Theorem 3.3.1, we complete the proof. ■

3.4 Propagation of Chaos

We can clearly see as the direct byproduct of the results stated above that chaos indeed propagates, which means the convergence of the one particle marginals of the N-particle system to the solution of the Vlasov equation in the sense of bounded Lipschitz distance. We illustrate the propagation of chaos also in two steps by using the Vlasov flow with cut-off as an intermediate tool. We present the result in full detail under the conditions of Theorem 3.3.1.

Definition 3.4.1. *For any two probability densities μ, $\nu : \mathbb{R}^4 \to \mathbb{R}^+$, the bounded Lipschitz distance is defined by*

$$d_L(\mu,\nu) := \sup_{g\in\mathcal{L}} \left| \int \big(\mu(x,v) - \nu(x,v)\big) g(x,v)\, dxdv \right|,$$

where $\mathcal{L} := \{g : ||g||_\infty = ||g||_L = 1\}$ and $||g||_L$ denotes the global Lipschitz constant of g.

In order to simplify the notation, we also introduce hereafter $(x_{i,-t}, v_{i,-t})$ and $(\overline{x}_{i,-t}, \overline{v}_{i,-t})$ to be the position and velocity of the i-th particle at initial time, which evolves according to the Newtonian and Vlasov flow with cut-off starting from (x_i, v_i) at time t, respectively.

Theorem 3.4.1. *Let $f_t^N : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}^+$ be the solution to* (3.2.2)*, $\mu_t : \mathbb{R} \times \mathbb{R}^{4N} \to \mathbb{R}^+$ be the N-particle density of the Newtonian flow and the one-particle marginals $\mu_t^{(1)}$ be given by*

$$\mu_t^{(1)}(x_1,v_1) := \int \mu_t(x_1,v_1,\cdots,x_N,v_N)\, dx_2dv_2\ldots dx_Ndv_N,$$

where

$$\mu_t(x_1,v_1,\cdots,x_N,v_N) := \mu_0(x_{1,-t},v_{1,-t},\cdots,x_{N,-t},v_{N,-t}).$$

Assume that initially the one particle marginals converges to the initial probability density f_0^N in the sense of bounded Lipschitz distance, i.e.,

$$\lim_{N\to\infty} d_L(\mu_0^{(1)}, f_0^N) = 0.$$

Then under the conditions of Theorem 3.3.1, there holds

$$\lim_{N\to\infty} d_L(\mu_t^{(1)}, f_t^N) = 0.$$

Proof. By definition, we have

$$\begin{aligned}
d_L(\mu_t^{(1)}, f_t^N) &= \sup_{g\in\mathcal{L}} \left| \int \big(\mu_t^{(1)}(x_1,v_1) - f_t^N(x_1,v_1)\big) g(x_1,v_1)\, dx_1 dv_1 \right| \\
&= \sup_{g\in\mathcal{L}} \bigg| \int \big(\mu_t(x_1,v_1,\cdots,x_N,v_N) - \\
&\qquad \prod_{i=1}^N f_t^N(x_i,v_i)\big) g(x_1,v_1)\, dx_1 dv_1 dx_2 dv_2 \dots dx_N dv_N \bigg|. \qquad (3.4.1)
\end{aligned}$$

Since both the Newtonian and Vlasov flow leave the measure invariant, then

$$\begin{aligned}
(3.4.1) &= \sup_{g\in\mathcal{L}} \bigg| \int \mu_0(x_1,v_1,\cdots,x_N,v_N) g(x_{1,-t},v_{1,-t})\, dx_1 dv_1 \dots dx_N dv_N \\
&\qquad - \int \prod_{i=1}^N f_0^N(x_i,v_i) g(\overline{x}_{1,-t},\overline{v}_{1,-t})\, dx_1 dv_1 \dots dx_N dv_N \bigg| \\
&\le \sup_{g\in\mathcal{L}} \left| \int \mu_0(x_1,v_1,\cdots,x_N,v_N)\big(g(x_{1,-t},v_{1,-t}) - g(\overline{x}_{1,-t},\overline{v}_{1,-t})\big)\, dx_1 dv_1 \dots dx_N dv_N \right| \\
&\quad + \sup_{g\in\mathcal{L}} \left| \int \Big(\mu_0(x_1,v_1,\cdots,x_N,v_N) - \prod_{i=1}^N f_0^N(x_i,v_i)\Big) g(\overline{x}_{1,-t},\overline{v}_{1,-t})\, dx_1 dv_1 \dots dx_N dv_N \right| \\
&=: M_1 + M_2.
\end{aligned}$$

Further we decompose M_1 into $M_{11} + M_{12}$, where

$$M_{11} = M_1 \Big|_{\left\{\sup_{0\le s\le t} \left|(X_s^N, V_s^N) - (\overline{X}_s^N, \overline{V}_s^N)\right|_\infty > N^{-\alpha}\right\}}$$

and

$$M_{12} = M_1 \Big|_{\left\{\sup_{0\le s\le t} \left|(X_s^N, V_s^N) - (\overline{X}_s^N, \overline{V}_s^N)\right|_\infty \le N^{-\alpha}\right\}}.$$

Under Theorem 3.3.1, we know

$$\lim_{N\to\infty} \mathbb{P}_0 \left(\sup_{0\le s\le t} \left|(X_s^N, V_s^N) - (\overline{X}_s^N, \overline{V}_s^N)\right|_\infty > N^{-\alpha} \right) = 0.$$

By using the fact that $||g||_\infty = 1$, we thus obtain

$$\begin{aligned} M_{11} &< 2\int \mu_0(x_1, v_1, \cdots, x_N, v_N)\, dx_1 dv_1 \dots dx_N dv_N \\ &< 2\mathbb{P}_0\left(\sup_{0\le s\le t}\left|(X_s^N, V_s^N) - (\overline{X}_s^N, \overline{V}_s^N)\right|_\infty > N^{-\alpha}\right) \\ &\to 0, \quad \text{as } N\to\infty. \end{aligned}$$

On the other hand, due to the reversibility of both particle trajectories and $||g||_L = 1$, we have

$$\begin{aligned} M_{12} &< \int \mu_0(x_1, v_1, \cdots, x_N, v_N)\left|(x_{1,-t}, v_{1,-t}) - (\overline{x}_{1,-t}, \overline{v}_{1,-t})\right| dx_1 dv_1 \dots dx_N dv_N \\ &= \mathbb{E}_0\left[\left|(X_{1,-t}, V_{1,-t}) - (\overline{X}_{1,-t}, \overline{V}_{1,-t})\right|\right] \\ &< \mathbb{E}_0\left[\sup_{0\le s\le t}\left|(X_s^N, V_s^N) - (\overline{X}_s^N, \overline{V}_s^N)\right|_\infty\right] \\ &\to 0, \quad \text{as } N\to\infty. \end{aligned}$$

In summary, M_1 converges to zero as N goes to infinity. Meanwhile it is also clear that M_2 tends to zero as $N\to\infty$ due to the assumption on the initial probability density. Combining all the terms completes the proof. ∎

Theorem 3.4.2. *Let $f_t : \mathbb{R}\times\mathbb{R}^4 \to \mathbb{R}^+$ be the solution to* (3.1.2)*, $\mu_t : \mathbb{R}\times\mathbb{R}^{4N} \to \mathbb{R}^+$ be the N-particle density of the Newtonian flow and the one-particle marginals $\mu_t^{(1)}$ be given by*

$$\mu_t^{(1)}(x_1, v_1) := \int \mu_t(x_1, v_1, \cdots, x_N, v_N)\, dx_2 dv_2 \dots dx_N dv_N,$$

where

$$\mu_t(x_1, v_1, \cdots, x_N, v_N) := \mu_0(x_{1,-t}, v_{1,-t}, \cdots, x_{N,-t}, v_{N,-t}).$$

Assume that initially the one particle marginals converges to the initial probability density f_0 in the sense of bounded Lipschitz distance, i.e.,

$$\lim_{N\to\infty} d_L(\mu_0^{(1)}, f_0) = 0.$$

Then under the conditions of Theorem 3.3.2, there holds

$$\lim_{N\to\infty} d_L(\mu_t^{(1)}, f_t) = 0.$$

Proof. By replacing f_t^N with f_t in the proof of Theorem 3.4.1 and using the conditions of Theorem 3.3.2, one will directly get the desired result. But we emphasize that Theorem 3.4.2 actually implies the convergence of the solution of (3.2.2) to the solution of (3.1.2) in the sense of bounded Lipschitz distance. ■

Note that if the initial one particle marginals converges in a certain rate to the initial probability density in both theorems above, we can also achieve the convergence rate for any fixed time t.

Chapter 4

From Vlasov-Maxwell to Vlasov-Poisson

The time evolution of plasmas is a very important topic of physics. In many cases, for example when considering the plasma in a nuclear fusion reactor, the temperature of the particles forming the plasma is sufficiently high to neglect quantum effects. Given that the number of particles forming the plasma is very high, also a mean-field approximation for the internal electromagnetic forces of the system can be argued [50], thus the system is in good approximation given by the relativistic Vlasov-Maxwell equations:

$$\begin{cases} \partial_t f_m + \hat{v} \cdot \nabla_x f_m + (E_m + c^{-1}\hat{v} \times B_m) \cdot \nabla_v f_m = 0, \\ \partial_t E_m = c\nabla \times B_m - j_m, \quad \nabla \cdot E_m = \rho_m, \\ \partial_t B_m = -c\nabla \times E_m, \quad \nabla \cdot B_m = 0, \end{cases} \tag{4.0.1}$$

where $\hat{v} = \dfrac{v}{\sqrt{1+c^{-2}v^2}}$, $\rho_m(t,x) = \displaystyle\int_{\mathbb{R}^3} f_m(t,x,v)\,dv$ and $j_m(t,x) = \displaystyle\int_{\mathbb{R}^3} \hat{v} f_m(t,x,v)\,dv$. The parameter c is the speed of light, (E_m, B_m) is the electro-magnetic field, and the distribution function $f_m(t,x,v) \geq 0$ describes the density of particles with position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$. Assuming that there are no external electromagnetic fields, the initial data

$$\begin{cases} f_m(0,x,v) = f_0(x,v), \\ E_m(0,x) = E_0(x), \\ B_m(0,x) = B_0(x), \end{cases} \tag{4.0.2}$$

satisfy the compatibility conditions $\nabla \cdot E_0(x) = \rho_0(x) = \displaystyle\int_{\mathbb{R}^3} f_0(x,v)\,dv$, $\nabla \cdot B_0(x) = 0$.
Local existence and uniqueness of classical solutions to this initial value problem for smooth and compactly supported data was established in [47]. These solutions can be extended globally in time provided the momentum support can be controlled assuming certain conditions on the initial data, e.g. smallness [48] closeness to neutrality [46] or closeness to spherical symmetry

[102]. It is worth to mention that different approaches to the results in [47] were given in [9, 73]. In order to obtain global existence of solutions, DiPerna and Lions restricted the solution concept to weak solutions. We refer to [39].

As was shown in [106] using an integral representation for the electric and magnetic field due to Glassey and Strauss [47], the solutions of relativistic Vlasov-Maxwell system converge in the pointwise sense to solutions of the non-relativistic Vlasov-Poisson system (below) at the rate of $1/c$ as c tends to infinity. The Vlasov-Poisson system reads

$$\begin{cases} \partial_t f_p + v \cdot \nabla_x f_p + E_p \cdot \nabla_v f_p = 0, \\ E_p(t,x) = \dfrac{1}{4\pi} \displaystyle\int_{\mathbb{R}^3} \rho_p(t,y) \frac{x-y}{|x-y|^3} \, dy, \\ \rho_p(t,y) = \displaystyle\int_{\mathbb{R}^3} f_p(t,y,v) \, dv, \end{cases} \tag{4.0.3}$$

with the initial data $f_p(0,x,v) = f_0(x,v)$. Here the indexes m and p in (4.0.1) and (4.0.3) stand for Maxwell and Poisson respectively. We note that there are global existence results for classical solutions of the Vlasov-Poisson system [84, 98, 107].

However a more interesting and challenging question is to consider what the corresponding particle model of relativistic Vlasov-Maxwell equation might be and whether we can prove the validity of the mean field description rigorously in the limit $N \to \infty$. Up to our knowledge and at the time of this writing, taking both the mean filed limit and the non-relativistic limit (or classical limit) of Vlasov-Maxwell system together into account is rare in the literatures. Concerning the mean field limit, Braun and Hepp [11] and Dobrushin [41] have proposed rigorous derivations of a system analogous to the Vlasov-Poisson system with a twice differentiable mollification of the Coulomb potential. Hauray and Jabin [63] have succeeded in treating the case of singular potentials, but not including the Coulomb singularity yet. Until recently, Lazarovici and Pickl [81] gave a probabilisitic proof of the validity of the mean field limit and propagation of chaos N-particle systems in three dimensions with Coulomb potential with N-dependent cutoff, which provides us with a very constructive idea of method. Lazarovici generalized this result including electromagnetic fields, proving the validity of the relativistic Vlasov-Maxwell equation [80] considering charges of radius $N^{-\delta}$ with $\delta < 1/12$. In this chapter, similar ideas are used in handling the mean field limit $N \to \infty$.

Writing down the corresponding N-particle model of the non-relativistic Vlasov-Maxwell system is a perplexing task because one needs to find a suitable description for the electromagnetic self-interaction within the theory of classical electrodynamics [43, 70, 110]. The problem of deriving a regularized version of the Vlasov-Maxwell system from a particle model was explicitly mentioned by Kiessling in [72]. Only after several years did Golse [50] establish the mean field limit of a N-particle system towards a regularized version of the relativistic Vlasov-Maxwell system with the help of [43] by Elsken, Kiessling and Ricci.

In the present chapter, we want to combine the mean field limit and non-relativistic limit of the regularized relativistic Vlasov-Maxwell particle model to Vlasov-Poisson equation. The method we apply here is more or less along the line of [47, 50, 81] using a mollifier for regularization removes the difficulties caused by the electromagnetic self-interaction forces. Unlike regularizing the Coulomb potential in the mean field limit established in [11, 41], the regularization of the self-interaction force in the Vlasov-Maxwell system is more difficult since the electromagnetic field involves both a scalar and vector potentials [50]. The solutions of the relativistic Vlasov-Maxwell system, as was discussed by Glassey and Strauss in [47], are closely related to the wave equation. This connection uses Kirchhoff's formula, which we also used in this paper. We would like to mention that there are other representations of the solutions of the relativistic Vlasov-Maxwell system, for example [9, 10], but they are all in fact equivalent.

This chapter is organized as follows: in Section 4.2, we prepare the regularization-procedure of both, the relativistic Vlasov-Maxwell and the Vlasov-Poisson system, and provide estimates between the solutions of these two systems. In Section 4.3, we introduce the particle model of the relativistic Vlasov-Maxwell system and apply the probabilisitic method to carry out the estimates between the characteristic equation of the particle model and that of the relativistic Vlasov-Maxwell system. We summarize our results in Section 4.4.

4.1 Regularization of the Vlasov-Maxwell and the Vlasov-Poisson Systems

Let $\chi \in C_0^\infty$ satisfy

$$\chi(x) = \chi(-x) \geq 0, \quad supp(\chi) \subset B_1(0), \quad \int_{\mathbb{R}^3} \chi(x)\, dx = 1,$$

and define the regularizing sequence

$$\chi^N(x) = N^{3\theta} \chi\left(N^\theta x\right).$$

The regularized version of the Vlasov-Maxwell System (VMN) with unknown (f_m^N, B_m^N, E_m^N) is given by:

$$\begin{cases} \partial_t f_m^N + \hat{v} \cdot \nabla_x f_m^N + \left(E_m^N + c^{-1}\hat{v} \times B_m^N\right) \cdot \nabla_v f_m^N = 0, \\ \partial_t E_m^N = c\nabla \times B_m^N - \chi^N *_x \chi^N *_x j_m^N, \qquad \nabla \cdot E_m^N = \chi^N *_x \chi^N *_x \rho_m^N, \\ \partial_t B_m^N = -c\nabla \times E_m^N, \qquad \nabla \cdot B_m^N = 0, \end{cases} \tag{4.1.1}$$

and initial data (IVMN)

$$\begin{cases} f_m^N(0,x,v) = f_0(x,v), \\ E_m^N(0,x) = \chi^N *_x \chi^N *_x E_0(x), \\ B_m^N(0,x) = \chi^N *_x \chi^N *_x B_0(x), \end{cases} \tag{4.1.2}$$

The regularized version of the Vlasov-Poisson System (VPN) with unknown (f_p^N, E_p^N) is given by:

$$\begin{cases} \partial_t f_p^N + v \cdot \nabla_x f_p^N + E_p^N \cdot \nabla_v f_p^N = 0, \\ E_p^N(t,x) = \dfrac{1}{4\pi} \displaystyle\iiint_{\mathbb{R}^9} \dfrac{x-y}{|x-y|^3} \chi^N(p)\chi^N(z)\rho_p^N(t, y-p-z)\, dydpdz, \\ \rho_p^N(t,y) = \displaystyle\int_{\mathbb{R}^3} f_p^N(t,y,v)\, dv, \end{cases} \tag{4.1.3}$$

with the initial data $f_p^N(0,x,v) = f_0(x,v)$.

Theorem 4.1.1. *Let f_0 be a nonnegative C^1-function with compact support in $\mathbb{R}^6$ and B_0 be in $C_0^2(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3) \cap W^{2,1}(\mathbb{R}^3)$. Assume further that*

$$\|\nabla_x B_0\|_{L^\infty(\mathbb{R}^3)} + \|B_0\|_{L^\infty(\mathbb{R}^3)} \leq \frac{1}{c^2},$$

and

$$E_0(x) = \frac{1}{4\pi} \iint_{\mathbb{R}^6} \frac{x-y}{|x-y|^3} f_0(y,v)\, dvdy.$$

Then

1. *There exists a $\overline{T} > 0$ such that (VPN) with initial data $f_p^N(0,x,v) = f_0(x,v)$ admits a unique C^1-solution (f_p^N, E_p^N) on the time interval $[0,\overline{T})$.*

2. *There exists a $T^* > 0$ (independent of c) such that for $c \geq 1$, (VMN) with the initial condition (IVMN) has a unique C^1- solution (f_m^N, B_m^N, E_m^N) on the time interval $[0,T^*)$. Furthermore there exists nondecreasing functions (independent of c and N) $q(t) : [0,T^*) \to \mathbb{R}$ and $H(t) : [0,T^*) \to \mathbb{R}$ such that*

$$f_m^N = 0, \quad \text{if} \quad |v| \geq q(t),$$

$$\|E_m^N(t,x)\|_{L^\infty([0,T^*)\times\mathbb{R}^3)} + \|B_m^N(t,x)\|_{L^\infty([0,T^*)\times\mathbb{R}^3)} \leq H(t).$$

3. *Let $\widetilde{T} = \min(\overline{T}, T^*)$, then for every $T \in [0,\widetilde{T})$ there exists a constant M (depending on T and the initial data, but not on c) such that for $c \geq 1$*

$$\|f_m^N - f_p^N\|_{L^\infty([0,T)\times\mathbb{R}^3\times\mathbb{R}^3)} + \|E_m^N - E_p^N\|_{L^\infty([0,T)\times\mathbb{R}^3)} + \|B_m^N\|_{L^\infty([0,T)\times\mathbb{R}^3)} \leq \frac{M}{c}.$$

The proof of the Theorem involves many long and tedious calculations since it includes many cut-offs and mollifications, however, no technical difficulties other than presented in the paper [106] appear. Therefore we deliver the proof in the appendix at the end of the paper for those readers who want to take a closer look at the details.

Remark 4.1.1. *In the current setting, i.e., repulsive particle interactions, both $\overline{T}$ and T^* can be global. But in the attractive case, there might be lack of global existence of solutions. Therefore both existence results we give are locally in time. The limits $N \to \infty$ and $c \to \infty$ in our paper are taken in the time interval where both solutions exist.*

We assume in Theorem 4.1.1 that f_0 has compact support, so let

$$q_0 = \sup\{|v| : \text{there exists}\, x \in \mathbb{R}^3 \,\text{such that}\, f_0(x, v) \neq 0\}.$$

Further, we define the characteristic curves $(x(t, x_0, v_0, t_0), v(t, x_0, v_0, t_0))$ (or in short $(x(t), v(t))$) by

$$\begin{cases} \dfrac{dx}{dt} = \hat{v}, \\ \dfrac{dv}{dt} = E_m^N + c^{-1}\hat{v} \times B_m^N. \end{cases} \tag{4.1.4}$$

Therefore f_m^N remains non-negative if f_0 is non-negative and that

$$\sup\{f_m^N(t, x, v) : x \in \mathbb{R}^3, v \in \mathbb{R}^3, t \in [0, T^*)\} = \|f_0\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)}.$$

We also define

$$p_0 = \sup\{|x| : \text{there exists}\, v \in \mathbb{R}^3 \,\text{such that}\, f_0(x, v) \neq 0\}.$$

Hence $f_m^N(t, x, v) = 0$, if $|x| \geq p_0 + tq(t)$.

Before we prove the Theorem, we write the second order form of Maxwell's equation:

$$\begin{cases} \partial_{tt} E_m^N - c^2 \Delta E_m^N = -\chi^N *_x \chi^N *_x (c^2 \nabla_x \rho_m^N + \partial_t j_m^N), \\ \partial_{tt} B_m^N - c^2 \Delta B_m^N = c\chi^N *_x \chi^N *_x \nabla \times j_m^N, \\ E_m^N(0, x) = \chi^N *_x \chi^N *_x E_0, \\ B_m^N(0, x)) = \chi^N *_x \chi^N *_x B_0, \\ \partial_t E_m^N(0, x) = c\nabla \times B_m^N(0, x) - \chi^N *_x \chi^N *_x j_m^N(0, x), \\ \partial_t B_m^N(0, x) = -c\nabla \times E_m^N(0, x). \end{cases} \tag{4.1.5}$$

Proposition 4.1.1. *Let $Y(t,x) \in \mathcal{D}'(\mathbb{R}\times\mathbb{R}^3)$ satisfy*

$$\begin{cases} \partial_{tt}Y - c^2\Delta Y = \delta_{(t,x)=(0,0)}, \\ supp\, Y \subset \{(t,x)\in\mathbb{R}_+\times\mathbb{R}^3, \quad |x|\le ct\}, \end{cases} \tag{4.1.6}$$

then $Y(t,x) = \dfrac{\mathbb{1}_{t>0}}{4\pi c|x|}\delta(|x|-ct)$.

The proof of this Proposition is standard. $Y(t,x)$ is called the fundamental solution of the wave equation. Set $Y^N = \chi^N *_x \chi^N *_x Y$, then the solutions of (4.1.5) are given in terms of

$$\begin{cases} E_m^N = \partial_t Y^N *_x E_0 + Y^N *_x (c\nabla\times B_0 - j_m^N(0,\cdot)) - Y^N *_{t,x} (c^2\nabla\rho_m^N + \partial_t j_m^N), \\ B_m^N = \partial_t Y^N *_x B_0 - cY^N *_x \nabla\times E_0 + cY^N *_{t,x} \nabla\times j_m^N. \end{cases} \tag{4.1.7}$$

Since it is not difficult to show that

$$\int_{|y-x|\le ct} h(ct-|y-x|,y)\,dy = c^2\int_0^t\int_{|\omega|=1}(t-\tau)^2 h(c\tau, x+c(t-\tau)\omega)\,d\omega d(c\tau),$$

we can also write the solutions of (4.1.5) in the form

$$\begin{cases} E_m^N = \mathbb{E}_0 - \dfrac{1}{4\pi c^2}\displaystyle\iiint_{\mathbb{R}^9} dpdzdy\chi^N(p)\chi^N(z)\frac{(c^2\nabla_y\rho_m^N + \partial_t j_m^N)(t-c^{-1}|x-y|, y-p-z)}{|x-y|}, \\ B_m^N = \mathbb{B}_0 + \dfrac{1}{4\pi c}\displaystyle\iiint_{\mathbb{R}^9} dpdzdy\chi^N(p)\chi^N(z)\frac{\nabla_y\times j_m^N(y-p-z, t-c^{-1}|x-y|)}{|x-y|}, \end{cases} \tag{4.1.8}$$

where

$$\begin{cases} \mathbb{E}_0 = \partial_t\displaystyle\int_{|\omega|=1}\frac{t}{4\pi}E_m^N(0,x+ct\omega)\,d\omega + \frac{t}{4\pi}\int_{|\omega|=1}\partial_t E_m^N(0,x+ct\omega)\,d\omega, \\ \mathbb{B}_0 = \partial_t\displaystyle\int_{|\omega|=1}\frac{t}{4\pi}B_m^N(0,x+ct\omega)\,d\nu + \frac{t}{4\pi}\int_{|\omega|=1}\partial_t B_m^N(0,x+ct\omega)\,d\omega. \end{cases} \tag{4.1.9}$$

4.2 Combined Mean Field Limit and Non-relativistic Limit

4.2.1 Regularized Vlasov-Maxwell Particle System

The regularized Vlasov-Maxwell system is given by

$$\begin{cases} \partial_t f_m^N + \hat{v}\cdot\nabla_x f_m^N + \left(E_m^N + c^{-1}\hat{v}\times B_m^N\right)\cdot\nabla_v f_m^N = 0, \\ \partial_t E_m^N = c\nabla\times B_m^N - \chi^N *_x \chi^N *_x j_m^N, \qquad \nabla\cdot E_m^N = \chi^N *_x \chi^N *_x \rho_m^N, \\ \partial_t B_m^N = -c\nabla\times E_m^N, \qquad\qquad \nabla\cdot B_m^N = 0, \end{cases} \tag{4.2.1}$$

where $\hat{v} = \dfrac{v}{\sqrt{1+c^{-2}v^2}}$, $\rho_m^N(t,x) = \int_{\mathbb{R}^3} f_m^N(t,x,v)\,dv$, $j_m^N(t,x) = \int_{\mathbb{R}^3} \hat{v} f_m^N(t,x,v)\,dv$ and the initial data

$$\begin{cases} f_m^N(0,x,v) = f_0(x,v), \\ E_m^N(0,x) = \chi^N *_x \chi^N *_x E_0(x), \\ B_m^N(0,x) = \chi^N *_x \chi^N *_x B_0(x), \end{cases} \tag{4.2.2}$$

satisfy the compatibility conditions $\nabla \cdot E_0(x) = \rho_m^N(0,x) = \rho_0(x)$, $\quad \nabla \cdot B_0(x) = 0$.
We consider the corresponding interacting particle system with position $x_i \in \mathbb{R}^3$ and velocity $v_i \in \mathbb{R}^3, i = 1, \ldots, N$. The equations of the characteristics read

$$\begin{cases} \dfrac{d}{dt} x_i = \hat{v}(v_i) = \dfrac{v_i}{\sqrt{1+c^{-2}v_i^2}}, \\ \dfrac{d}{dt} v_i = E_m^N(t,x_i) + c^{-1}\hat{v}(v_i) \times B_m^N(t,x_i), \end{cases} \tag{4.2.3}$$

where

$$\begin{cases} E_m^N = \partial_t Y^N *_x E_0 + Y^N *_x (c\nabla \times B_0 - j_m^N(0,.)) - Y^N *_{t,x} (c^2 \nabla \rho_m^N + \partial_t j_m^N), \\ B_m^N = \partial_t Y^N *_x B_0 - cY^N *_x \nabla \times E_0 + cY^N *_{t,x} \nabla \times j_m^N. \end{cases} \tag{4.2.4}$$

In order to present the analytical results in Section 4.3 in a concise and clear manner, we restrict to the following notations.

Definition 4.2.1. *1. For any $1 \le i \le N$ (labeling the particle with position $x_m^{i,N} \in \mathbb{R}^3$ and velocity $v_m^{i,N} \in \mathbb{R}^3$) we denote the pair-interaction force by*

$$F_m^{1,N}(t, x_m^{i,N}) = -\frac{1}{N-1} \sum_{j=1, j\neq i}^{N} \int_0^t (\hat{v}(v_m^{j,N}(s))\partial_t + c^2 \nabla_x) Y^N(t-s, x_m^{i,N}(t) - x_m^{j,N}(s))\,ds,$$

$$F_m^{2,N}(t, x_m^{i,N}, v_m^{i,N}) = -\frac{1}{N-1} \sum_{j=1, j\neq i}^{N} \int_0^t \hat{v}(v_m^{i,N}(t)) \times \Big(\hat{v}(v_m^{j,N}(s)) \times \nabla_x Y^N(t-s, x_m^{i,N}(t) - x_m^{j,N}(s))\Big)\,ds,$$

and the mean-field force of the Vlasov system by

$$\begin{aligned} F_m^{3,N}(t, x_m^{i,N}, v_m^{i,N}) \;=\; & E_0 *_x \partial_t Y^N(t, x_m^{i,N}) + (c\nabla \times B_0 - j_m^N(0,\cdot)) *_x Y^N(t, x_m^{i,N}) \\ & + c^{-1}\hat{v}(v_m^{i,N}) \times B_0 *_x \partial_t Y^N(t, x_m^{i,N}) \\ & - \hat{v}(v_m^{i,N}) \times (\nabla \times E_0) *_x Y^N(t, x_m^{i,N}). \end{aligned}$$

Here we point out that $F_m^{1,N}$ and $F_m^{2,N}$ are indeed two interacting forces, which means they *actually also depend on all the other particles.*

2. *Let* $(X_m^N(t), V_m^N(t))$ *be the trajectory on* $\mathbb{R}^{6N}$ *which evolves according to the Newtonian equation of motion for the regularized Vlasov-Maxwell system, i.e.,*

$$\begin{cases} \dfrac{d}{dt} X_m^N(t) = \hat{V}(V_m^N(t)), \\ \\ \dfrac{d}{dt} V_m^N(t) = \Psi_m^{1,N}(t, X_m^N(t), V_m^N(t)) + \Psi_m^{2,N}(t, X_m^N(t), V_m^N(t)) + \Gamma_m^N(t, X_m^N(t), V_m^N(t)), \end{cases} \tag{4.2.5}$$

where $\Psi_m^{1,N}(t, X_m^N(t), V_m^N(t))$ *and* $\Psi_m^{2,N}(t, X_m^N(t), V_m^N(t))$ *denote the total interaction force with*

$$\begin{aligned} & \left(\Psi_m^{1,N}(t, X_m^N(t), V_m^N(t))\right)_i = F_m^{1,N}(t, x_m^{i,N}) \\ = \;& -\frac{1}{N-1} \sum_{j=1, j\neq i}^{N} \int_0^t (\hat{v}(v_m^{j,N}(s))\partial_t + c^2 \nabla_x) Y^N(t-s, x_m^{i,N}(t) - x_m^{j,N}(s))\, ds, \\ & \left(\Psi_m^{2,N}(t, X_m^N(t), V_m^N(t))\right)_i = F_m^{2,N}(t, x_m^{i,N}, v_m^{i,N}) \\ = \;& -\frac{1}{N-1} \sum_{j=1, j\neq i}^{N} \int_0^t \hat{v}(v_m^{i,N}(t)) \times \Big(\hat{v}(v_m^{j,N}(s)) \times \nabla_x Y^N(t-s, x_m^{i,N}(t) - x_m^{j,N}(s))\Big)\, ds \end{aligned}$$

while $\Gamma_m^N(t, X_m^N(t), V_m^N(t))$ *stands for the self-driven force with*

$$\begin{aligned} & \left(\Gamma_m^N(t, X_m^N(t), V_m^N(t))\right)_i \\ = \;& F_m^{3,N}(t, x_m^{i,N}, v_m^{i,N}) \\ = \;& E_0^N *_x \partial_t Y^N(t, x_m^{i,N}) + (c\nabla \times B_0^N - j_m^N(0,\cdot)) *_x Y^N(t, x_m^{i,N}) \\ & + c^{-1}\hat{v}(v_m^{i,N}) \times B_0^N *_x \partial_t Y^N(t, x_m^{i,N}) - \hat{v}(v_m^{i,N}) \times (\nabla \times E_0^N) *_x Y^N(t, x_m^{i,N}). \end{aligned}$$

3. *Let* $(\overline{X}_m^N(t), \overline{V}_m^N(t))$ *be the trajectory on* $\mathbb{R}^{6N}$ *which evolves according to the regularized Vlasov-Maxwell equation*

$$\partial_t f_m^N + \hat{v} \cdot \nabla_x f_m^N + \left(E_m^N + c^{-1}\hat{v} \times B_m^N\right) \cdot \nabla_v f_m^N = 0, \tag{4.2.6}$$

i.e.,

$$\begin{cases} \dfrac{d}{dt}\overline{X}^N_m(t) = \hat{V}(\overline{V}^N_m(t)), \\ \\ \dfrac{d}{dt}\overline{V}^N_m(t) = \overline{\Psi}^{1,N}_m(t,\overline{X}^N_m(t)) + \overline{\Psi}^{2,N}_m(t,\overline{X}^N_m(t),\overline{V}^N_m(t)) + \Gamma^N_m(t,\overline{X}^N_m(t),\overline{V}^N_m(t)), \end{cases} \tag{4.2.7}$$

where

$$\begin{aligned} \big(\overline{\Psi}^{1,N}_m(t,\overline{X}^N_m(t))\big)_i &= \overline{F}^{1,N}_m(t,\overline{x}^{i,N}_m) = -(c^2\nabla\rho^N_m + \partial_t j^N_m) *_{t,x} Y^N(t,\overline{x}^{i,N}_m) \qquad (4.2.8)\\ &= -\iint_{\mathbb{R}^6}\int_0^t dsdydv \\ &\qquad (c^2\nabla + \hat{v}(v)\partial_s) f^N_m(s,\overline{x}^{i,N}_m - y, v) Y^N(t-s,y), \\ \big(\overline{\Psi}^{2,N}_m(t,\overline{X}^N_m(t),\overline{V}^N_m(t))\big)_i &= \overline{F}^{2,N}_m(t,\overline{x}^{i,N}_m,\overline{v}^{i,N}_m) \\ &= -\iint_{\mathbb{R}^6}\int_0^t dsdydv \\ &\qquad \hat{v}(\overline{v}^{i,N}_m) \times \hat{v}(v) \times \nabla_x f^N_m(s,\overline{x}^{i,N}_m - y, v) Y^N(t-s,y) (4.2.9)\\ \big(\Gamma^N_m(t,\overline{X}^N_m(t),\overline{V}^N_m(t))\big)_i &= \overline{F}^{3,N}_m(t,\overline{x}^{i,N}_m,\overline{v}^{i,N}_m) \\ &= E^N_0 *_x \partial_t Y^N(t,\overline{x}^{i,N}_m) + (c\nabla \times B^N_0 - j^N_m(0,\cdot)) *_x Y^N(t,\overline{x}^{i,N}_m) \\ &\quad + c^{-1}\hat{v}(\overline{v}^{i,N}_m) \times B^N_0 *_x \partial_t Y^N(t,\overline{x}^{i,N}_m) \\ &\quad - \hat{v}(\overline{v}^{i,N}_m) \times (\nabla \times E^N_0) *_x Y^N(t,\overline{x}^{i,N}_m). \end{aligned}$$

represent the total interaction forces and the self-driven force, respectively.

$(X(t),V(t))$ and $(\overline{X}(t),\overline{V}(t))$ without superscript N denote the particle configurations driven by the force without cut-off. (X,V) and $(\overline{X},\overline{V})$, without the argument t, stand for the stochastic initial data, which are independent and identically distributed. Note that we always consider the same initial data for both systems, that means $(X,V) = (\overline{X},\overline{V})$. The following lemma gives us and estimates on the interaction forces, which will be used in the limiting procedure.

Lemma 4.2.1. *Let* $\overline{F}^{1,N}_m(t,x)$ *and* $\overline{F}^{2,N}_m(t,x,v)$ *be defined as in* (4.2.8) *and* (4.2.9). *Then there exists a constant* M *such that*

$$\|\overline{F}^{1,N}_m(t,x)\|_{L^\infty([0,T]\times\mathbb{R}^3)} + \|\overline{F}^{2,N}_m(t,x,v)\|_{L^\infty([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)} \le cM.$$

Proof. By definition, we know that

$$\begin{aligned}
& \|\overline{F}^{1,N}_m(t,x)\|_{L^\infty([0,T]\times\mathbb{R}^3)} \\
= & \| -(c^2\nabla\rho^N_m + \partial_t j^N_m) *_{t,x} Y^N(t,x)\|_{L^\infty([0,T]\times\mathbb{R}^3)} \\
= & \Big\| \iint_{\mathbb{R}^6}\int_0^t (c^2\nabla_x + \hat{v}(v)\partial_s) f^N_m(s,x-y,v) Y^N(t-s,y)\, dsdydv \Big\|_{L^\infty([0,T]\times\mathbb{R}^3)}, \quad (4.2.10)
\end{aligned}$$

where for $s<t$

$$\begin{aligned}
Y^N(t-s,y) &= \int_{\mathbb{R}^3}\int_{|z|\le c(t-s)} \frac{1}{4\pi c|z|}\delta(|z|-c(t-s))\chi^N(y-p-z)\chi^N(p)\, dzdp \\
&= \int_{\mathbb{R}^3}\int_{|\omega|=1}\int_0^{c(t-s)} \frac{\tau}{4\pi c}\delta(\tau - c(t-s))\chi^N(y-p-\tau\omega)\chi^N(p)\, d\tau d\omega dp \\
&= \int_{\mathbb{R}^3}\int_{|\omega|=1} \frac{t-s}{4\pi}\chi^N(y-p-c(t-s)\omega)\chi^N(p)\, d\omega dp,
\end{aligned}$$

and

$$\|Y^N(t-s,y)\|_{L^\infty([0,T]\times\mathbb{R}^3)} \le \frac{M}{4\pi c}\int_{\mathbb{R}^3}\chi^N(p)\, dp = \frac{M}{4\pi c}.$$

So

$$\begin{aligned}
& (4.2.10) \\
= & \Big\| \frac{1}{4\pi}\iiint_{\mathbb{R}^9} dpdydv \int_0^t ds \int_{|\omega|=1} d\omega \\
& \qquad (t-s)(c^2\nabla_x + \hat{v}(v)\partial_s) f^N_m(s,x-y,v)\chi^N(y-p-c(t-s)\omega)\chi^N(p) \Big\|_{L^\infty([0,T]\times\mathbb{R}^3)} \\
\le & cM\Big(\sup_{0\le t\le T}\|\partial_t f^N_m(t,\cdot,\cdot)\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)} + \sup_{0\le t\le T}\|\nabla_x f^N_m(t,\cdot,\cdot)\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)}\Big).
\end{aligned}$$

And similarly we have

$$\begin{aligned}
& \|\overline{F}^{2,N}_m(t,x,v)\|_{L^\infty([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)} \\
= & \Big\| -\iint_{\mathbb{R}^6}\int_0^t \hat{v}(v)\times\hat{v}(z)\times\nabla_x f^N_m(s,x-y,z) Y^N(t-s,y)\, dsdydz \Big\|_{L^\infty([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)} \\
\le & cM\Big(\sup_{0\le t\le T}\|\nabla_x f^N_m(t,\cdot,\cdot)\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)}\Big).
\end{aligned}$$

∎

4.2.2 Regularized Vlasov-Poisson Particle Model

In this section, we consider the Vlasov-Poisson particle model and deduce estimates of the distance between the solutions of Vlasov-Maxwell and Vlasov-Poisson

The regularization of the Vlasov-Poisson System (VPN) with unknown (f_p^N, E_p^N) is given by

$$\begin{cases} \partial_t f_p^N + v \cdot \nabla_x f_p^N + E_p^N \cdot \nabla_v f_p^N = 0, \\ E_p^N(t,x) = \dfrac{1}{4\pi} \displaystyle\iiint_{\mathbb{R}^9} dydpdz \frac{x-y}{|x-y|^3} \chi^N(p)\chi^N(z)\rho_p^N(t, y-p-z), \\ \rho_p^N(t,y) = \displaystyle\int_{\mathbb{R}^3} f_p^N(t,y,v)\, dv, \end{cases} \tag{4.2.11}$$

with the initial data $f_p^N(x,v,0) = f_0(x,v)$. Thus the corresponding Vlasov-Poisson equations of characteristics read

$$\begin{cases} \dfrac{d}{dt}\overline{x}_p^N = \overline{v}_p^N, \\ \dfrac{d}{dt}\overline{v}_p^N = E_p^N(t, \overline{x}_p^N) = \dfrac{1}{4\pi} \displaystyle\iiint_{\mathbb{R}^9} dydpdz \frac{\overline{x}_p^N - y}{|\overline{x}_p^N - y|^3} \chi^N(p)\chi^N(z)\rho_p^N(t, y-p-z). \end{cases} \tag{4.2.12}$$

Now we can compare the Vlasov-Maxwell and Vlasov-Poisson equations (or in other words solutions) with respect to their characteristic curves, which requires a more detailed estimates on these two solutions, namely f_m^N and f_p^N. Using the results in section 4.3, we know that

$$\begin{aligned} \|E_m^N + c^{-1}\hat{v} \times B_m^N - E_p^N\|_{L^\infty(\mathbb{R}^3)} &\le \|E_m^N - E_p^N\|_{L^\infty(\mathbb{R}^3)} + \|c^{-1}\hat{v} \times B_m^N\|_{L^\infty(\mathbb{R}^3)} \\ &\le c^{-1}M. \end{aligned} \tag{4.2.13}$$

We will next compare the N-particle Vlasov-Maxwell equation with the Vlasov-Poisson equation. Since the N-body system is subject to a regularized force it is most natural to introduce that regularization also for the Vlasov-Poisson system. The translation of the one-body Vlasov-Poisson system to an N-body dynamics is straight forward: each particle moves with the same flow given by the Vlasov-Poisson equation. This allows now comparison with the N-body characteristics coming from the N-particle Vlasov-Maxwell equation.

Definition 4.2.2. *Let $(\overline{X}_p^N(t), \overline{V}_p^N(t))$ be the trajectory on $\mathbb{R}^{6N}$ which evolves according to the regularized Vlasov-Poisson equation*

$$\partial_t f_p^N + v \cdot \nabla_x f_p^N + E_p^N \cdot \nabla_v f_p^N = 0, \tag{4.2.14}$$

i.e.,

$$\begin{cases} \frac{d}{dt}\overline{X}_p^N(t) = \overline{V}_p^N(t), \\ \\ \frac{d}{dt}\overline{V}_p^N(t) = \overline{\Psi}_p^N(t, \overline{X}_p^N(t)), \end{cases} \tag{4.2.15}$$

where

$$\begin{aligned} \left(\overline{\Psi}_p^N(t, \overline{X}_p^N(t))\right)_i &= \overline{F}_p^N(t, \overline{x}_p^{i,N}) = E_p^N(t, \overline{x}_p^{i,N}) \\ &= \frac{1}{4\pi}\iiint_{\mathbb{R}^9} dydpdz \frac{\overline{x}_p^{i,N} - y}{|\overline{x}_p^{i,N} - y|^3}\chi^N(p)\chi^N(z)\rho_p^N(t, y-p-z). \end{aligned}$$

4.2.3 Estimates for the Mean Field Limit

In this section, we present our key results in full detail. To show the desired convergence, our method can be summarized as follows. First, we start from the Newtonian system with carefully chosen cut-off and meanwhile introduce an intermediate system which involves convolution-type interaction with cut-off. Then, we show the convergence of the intermediate system to the final mean field limit, where the law of large number comes into play. The crucial point of this method is that we apply stochastic initial data or in other words we consider a stochastic process. This enables us to use tools from probability theory, which helps to better understand the mean field process.

The following assumptions are used throughout this section.

Assumption 4.2.1. *We assume that*

(a) E_0 *and* B_0 *are all Lipschitz continuous functions.*

(b) $\alpha \in \left(0, \frac{1}{8}\right)$, $\beta \in \left(\alpha, \frac{1-\alpha}{4}\right)$ *and* $\theta \in \left(0, \frac{1-\alpha-4\beta}{16}\right)$.

Definition 4.2.3. *Let* $S_t : \mathbb{R}^{6N} \times \mathbb{R} \to \mathbb{R}$ *be the stochastic process given by*

$$S_t = \min\left\{1, N^\alpha \sup_{0\le s\le t}\left|(X_m^N(s), V_m^N(s)) - (\overline{X}_m^N(s), \overline{V}_m^N(s))\right|_\infty\right\}.$$

The set, where $|S_t| = 1$*, is defined as* $\mathcal{N}_\alpha$*, i.e.,*

$$\mathcal{N}_\alpha := \left\{(X,V) : \sup_{0\le s\le t}\left|(X_m^N(s), V_m^N(s)) - (\overline{X}_m^N(s), \overline{V}_m^N(s))\right|_\infty > N^{-\alpha}\right\}. \tag{4.2.16}$$

Here and in the following we use $|\cdot|_\infty$ as the supremum norm on $\mathbb{R}^{6N}$. Note that

$$\mathbb{E}_0(S_{t+dt} - S_t \,|\, \mathcal{N}_\alpha) \leq 0,$$

since S_t takes the value of one for $(X,V) \in \mathcal{N}_\alpha$.

Theorem 4.2.1. *Let $f_m^N(t,x,v)$ be a solution of the regularized Vlasov-Maxwell equation* (4.2.6). *Suppose that Assumptions 4.2.1 are satisfied. Then there exists a constant M such that*

$$\mathbb{P}_0\left(\sup_{0\leq s\leq t}\left|(X_m^N(s), V_m^N(s)) - (\overline{X}_m^N(s), \overline{V}_m^N(s))\right|_\infty > N^{-\alpha}\right) \leq e^{Mt}\cdot c^4 N^{-(1-\alpha-4\beta-16\theta)}.$$

The proof of the theorem will be presented later in this section.

Definition 4.2.4. *The sets $\mathcal{N}_\beta$ and $\mathcal{N}_\gamma$ are characterized by*

$$\mathcal{N}_\beta := \left\{(X_m, V_m) : \left|\Psi_m^{1,N}\left(\overline{X}_m^N(t), \overline{V}_m^N(t)\right) - \overline{\Psi}_m^{1,N}\left(\overline{X}_m^N(t), \overline{V}_m^N(t)\right)\right|_\infty > N^{-\beta}\right\}, \quad (4.2.17)$$

$$\mathcal{N}_\gamma := \left\{(X_m, V_m) : \left|\Psi_m^{2,N}\left(\overline{X}_m^N(t), \overline{V}_m^N(t)\right) - \overline{\Psi}_m^{2,N}\left(\overline{X}_m^N(t), \overline{V}_m^N(t)\right)\right|_\infty > N^{-\gamma}\right\}. \quad (4.2.18)$$

Next, we will see that the probability of both sets $\mathcal{N}_\beta$ and $\mathcal{N}_\gamma$ can be arbitrarily small, i.e., the probability of each set tends to 0 as N goes to infinity. We prove the following two lemmas:

Lemma 4.2.2. *There exists a constant $M < \infty$ such that*

$$\mathbb{P}_0(\mathcal{N}_\beta) \leq Mc^4 \cdot N^{-(1-4\beta-16\theta)}.$$

Proof. First, we let the set $\mathcal{N}_\beta$ evolve along the characteristics of the regularized Vlasov-Maxwell equation

$$\mathcal{N}_{\beta,t} := \left\{(\overline{X}_m^N(t), \overline{V}_m^N(t)) : \left|N^\beta\Psi_m^{2,N}\left(\overline{X}_m^N(t), \overline{V}_m^N(t)\right) - N^\beta\overline{\Psi}_m^{1,N}\left(\overline{X}_m^N(t), \overline{V}_m^N(t)\right)\right|_\infty > 1\right\}$$

and consider the following fact

$$\mathcal{N}_{\beta,t} \subseteq \bigoplus_{i=1}^N \mathcal{N}_{\beta,t}^i,$$

where

$$\mathcal{N}^i_{\beta,t} := \left\{ \quad (\overline{x}^{i,N}_m, \overline{v}^{i,N}_m) : \right| \\ N^\beta \cdot \frac{1}{N-1} \sum_{j=1, j\neq i}^{N} \int_0^t (\hat{v}(v^{j,N}_m(s))\partial_t + c^2\nabla_x) Y^N(t-s, x^{i,N}_m(t) - x^{j,N}_m(s))\, ds \\ -N^\beta \iint_{\mathbb{R}^6} \int_0^t (c^2\nabla + \hat{v}(v)\partial_s) f^N_m(s, \overline{x}^{j,N}_m - y, v) Y^N(t-s, y)\, dsdydv \Big|_\infty > 1 \left. \right\}.$$

We therefore get

$$\mathbb{P}_t(\mathcal{N}_{\beta,t}) \leq \sum_{i=1}^{N} \mathbb{P}_t(\mathcal{N}^i_{\beta,t}) = N\mathbb{P}_t(\mathcal{N}^1_{\beta,t}),$$

where in the last step we use symmetry in exchanging any two coordinates.

Using Markov inequality gives

$$\begin{aligned} \mathbb{P}_t(\mathcal{N}^1_{\beta,t}) &\leq \mathbb{E}_t \left[\left(N^\beta \cdot \frac{1}{N-1} \sum_{j=2}^{N} F^N_1(t, \overline{x}^{1,N}_m - \overline{x}^{j,N}_m) - N^\beta \overline{F}^N_1(t, \overline{x}^{1,N}_m) \right)^4 \right] \\ &= \left(\frac{N^\beta}{N-1} \right)^4 \mathbb{E}_t \left[\left(\sum_{j=2}^{N} F^N_1(t, \overline{x}^{1,N}_m - \overline{x}^{j,N}_m) - (N-1)\overline{F}^N_1(t, \overline{x}^{1,N}_m) \right)^4 \right]. \end{aligned} \tag{4.2.19}$$

Let $h_j := F^N_1(t, \overline{x}^{1,N}_m - \overline{x}^{j,N}_m) - \overline{F}^N_1(t, \overline{x}^{1,N}_m)$. Then, each term in the expectation (4.2.19) takes the form of $\prod_{j=2}^N h_j^{k_j}$ with $\sum_{j=1}^N k_j = 4$, and more importantly, the expectation assumes the value of zero whenever there exists a j such that $k_j = 1$. This can be easily verified by integrating over the j-th variable first or, in other words, by acknowledging the fact that $\forall\, j = 2, \ldots, N$, there holds

$$\mathbb{E}_t \left[F^N_1(t, \overline{x}^{1,N}_m - \overline{x}^{j,N}_m) - \overline{F}^N_1(t, \overline{x}^{1,N}_m) \right] = 0.$$

Then, we can simplify the estimate (4.2.19) to

$$\mathbb{P}_t(\mathcal{N}^1_{\beta,t}) \leq \left(\frac{N^\beta}{N-1} \right)^4 \mathbb{E}_t \left[\sum_{j=2}^{N} h_j^4 + \sum_{2\leq m<n}^{N} \binom{4}{2} h_m^2 h_n^2 \right].$$

Since

$$\|F^{1,N}_m\|_{L^\infty([0,T]\times\mathbb{R}^3)} \leq Mc^2 N^{4\theta},$$

and

$$\|\overline{F}_m^{1,N}(t,x)\|_{L^\infty([0,T]\times\mathbb{R}^3)} \le cM\Big(\sup_{0\le t\le T}\|f_m^N(t,\cdot,\cdot)\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)} + \sup_{0\le t\le T}\|\nabla_x f_m^N(t,\cdot,\cdot)\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)}\Big)$$

we thus have for any fixed j

$$\begin{aligned} |h_j| &\le |F_1^N(t,\overline{x}_m^{1,N}-\overline{x}_m^{j,N})| + |\overline{F}_1^N(t,\overline{x}_m^{1,N})| \\ &\le cM\left(N^{4\theta} + \sup_{0\le t\le T}\|f_m^N(t,\cdot,\cdot)\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)} + \sup_{0\le t\le T}\|\nabla_x f_m^N(t,\cdot,\cdot)\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)}\right). \end{aligned}$$

Therefore $|h_j|$ is bounded to any power and we obtian

$$\mathbb{E}_t\left[h_m^2h_n^2\right] \le Mc^4N^{16\theta} \quad \text{and} \quad \mathbb{E}_t\left[h_j^4\right] \le Mc^4N^{16\theta}$$

and consequently

$$\begin{aligned} \mathbb{P}_t(\mathcal{N}_{\beta,t}^1) &\le \left(\frac{N^\beta}{N-1}\right)^4 \cdot \left(Mc^4\cdot(N-1) + Mc^4N^{16\theta}\cdot\frac{(N-1)(N-2)}{2}\right) \\ &\le Mc^4N^{16\theta}\cdot N^{-(2-4\beta-16\theta)}. \end{aligned}$$

By noticing the fact that

$$\begin{aligned} \mathbb{P}_0(\mathcal{N}_\beta) &= \mathbb{P}_t\left(\mathcal{N}_{\beta,t}\right) \le N\mathbb{P}_t(\mathcal{N}_{\beta,t}^1) \\ &\le N\cdot Mc^4\cdot N^{-(2-4\beta-16\theta)} = Mc^4\cdot N^{-(1-4\beta-16\theta)}, \end{aligned}$$

we obtain the desired result. ■

In fact, this result holds for any β if we change accordingly the power in the proof to be another even number (depending on β) greater than four. So with similar estimates we get

Lemma 4.2.3. *There exists a constant $M<\infty$ such that*

$$\mathbb{P}_0(\mathcal{N}_\gamma) \le Mc^4\cdot N^{-(1-4\gamma-16\theta)}.$$

Lemma 4.2.4. *Let $\mathcal{N}_\alpha$, $\mathcal{N}_\beta$, $\mathcal{N}_\gamma$ be defined as in (4.2.16)-(4.2.18). Suppose that $f_m^N(t,x,v)$ is a solution of the regularized Vlasov-Maxwell equation and Assumption 4.2.1 is satisfied. Then* *there exists a constant $M < \infty$ such that*

$$\Big|\Big(\hat{V}(V_m^N(t)), \Psi_m^{1,N}(t, X_m^N(t), V_m^N(t)) + \Psi_m^{2,N}(t, X_m^N(t), V_m^N(t)) + \Gamma_m^N(t, X_m^N(t), V_m^N(t))\Big) - \Big(\hat{V}(\overline{V}_m^N(t)), \overline{\Psi}_m^{1,N}(t, \overline{X}_m^N(t), \overline{V}_m^N(t)) + \overline{\Psi}_m^{2,N}(t, \overline{X}_m^N(t), \overline{V}_m^N(t)) + \Gamma_m^N(t, \overline{X}_m^N(t), \overline{V}_m^N(t))\Big)\Big|_\infty \leq M S_t(X, V) N^{-\alpha} + N^{-\beta}$$

for all initial data $(X, V) \in (\mathcal{N}_\alpha \cup \mathcal{N}_\beta \cup \mathcal{N}_\gamma)^c$.

Proof. Applying triangle inequality gives

$$\begin{aligned}
&\Big|\Big(\hat{V}(V_m^N(t)), \Psi_m^{1,N}(t, X_m^N(t), V_m^N(t)) + \Psi_m^{2,N}(t, X_m^N(t), V_m^N(t)) + \Gamma_m^N(t, X_m^N(t), V_m^N(t))\Big) \\
&\quad - \Big(\hat{V}(\overline{V}_m^N(t)), \overline{\Psi}_m^{1,N}(t, \overline{X}_m^N(t), \overline{V}_m^N(t)) + \overline{\Psi}_m^{2,N}(t, \overline{X}_m^N(t), \overline{V}_m^N(t)) + \Gamma_m^N(t, \overline{X}_m^N(t), \overline{V}_m^N(t))\Big)\Big|_\infty \\
\leq\ & \Big|\hat{V}(V_m^N(t)) - \hat{V}(\overline{V}_m^N(t))\Big|_\infty + \Big|\Psi_m^{1,N}(t, X_m^N(t), V_m^N(t)) - \overline{\Psi}_m^{1,N}(t, \overline{X}_m^N(t), \overline{V}_m^N(t))\Big|_\infty \\
& + \Big|\Psi_m^{2,N}(t, X_m^N(t), V_m^N(t)) - \overline{\Psi}_m^{2,N}(t, \overline{X}_m^N(t), \overline{V}_m^N(t))\Big|_\infty \\
& + \Big|\Gamma_m^N(t, X_m^N(t), V_m^N(t)) - \Gamma_m^N(t, \overline{X}_m^N(t), \overline{V}_m^N(t))\Big|_\infty \\
\leq\ & \Big|\hat{V}(V_m^N(t)) - \hat{V}(\overline{V}_m^N(t))\Big|_\infty + \Big|\Psi_m^{1,N}(t, X_m^N(t), V_m^N(t)) - \Psi_m^{1,N}(t, \overline{X}_m^N(t), \overline{V}_m^N(t))\Big|_\infty \\
& + \Big|\Psi_m^{1,N}(t, \overline{X}_m^N(t), \overline{V}_m^N(t)) - \overline{\Psi}_m^{1,N}(t, \overline{X}_m^N(t), \overline{V}_m^N(t))\Big|_\infty \\
& + \Big|\Psi_m^{2,N}(t, X_m^N(t), V_m^N(t)) - \Psi_m^{2,N}(t, \overline{X}_m^N(t), \overline{V}_m^N(t))\Big|_\infty \\
& + \Big|\Psi_m^{2,N}(t, \overline{X}_m^N(t), \overline{V}_m^N(t)) - \overline{\Psi}_m^{2,N}(t, \overline{X}_m^N(t), \overline{V}_m^N(t))\Big|_\infty \\
& + \Big|\Gamma_m^N(t, X_m^N(t), V_m^N(t)) - \Gamma_m^N(t, \overline{X}_m^N(t), \overline{V}_m^N(t))\Big|_\infty \\
=:\ & |I_1| + |I_2| + |I_3| + |I_4| + |I_5| + |I_6|.
\end{aligned}$$

Next, we estimate term by term.

- Since $(X, V) \notin \mathcal{N}_\alpha$,

$$|I_1| := \Big|\hat{V}(V_m^N(t)) - \hat{V}(\overline{V}_m^N(t))\Big|_\infty \leq M S_t(X, V) N^{-\alpha}.$$

- With the fact that $F_m^{1,N}$ is Lipschitz continuous in x, we denote L as the global Lipschitz constant for all the Lipschitz continuous functions in this paper. Thus we obtain

$$\begin{aligned} & \left| \frac{1}{N-1} \sum_{i \neq j} F_m^{1,N}(t, x_m^{i,N}) - \frac{1}{N-1} \sum_{i \neq j} F_m^{1,N}(t, \overline{x}_m^{i,N}) \right| \\ \leq \quad & \frac{1}{N-1} \sum_{i \neq j} L \cdot 2 |x_m^{i,N} - \overline{x}_m^{i,N}|. \end{aligned} \tag{4.2.20}$$

Since $(X, V) \notin \mathcal{N}_\alpha$, it follows in particular for any $1 \leq i \leq N$ that

$$|x_m^{i,N} - \overline{x}_m^{i,N}| \leq N^{-\alpha}.$$

So together with (4.2.20), we have

$$\left| \Psi_m^{1,N}(t, X_m^N(t), V_m^N(t)) - \Psi_m^{1,N}(t, \overline{X}_m^N(t), \overline{V}_m^N(t)) \right|_\infty \leq 2LN^{-\alpha},$$

and thus

$$|I_2| := \left| \Psi_m^{1,N}(t, X_m^N(t), V_m^N(t)) - \Psi_m^{1,N}(t, \overline{X}_m^N(t), \overline{V}_m^N(t)) \right|_\infty \leq MS_t(X, V)N^{-\alpha}.$$

Similarly

$$|I_4| := \left| \Psi_m^{2,N}(t, X_m^N(t), V_m^N(t)) - \Psi_m^{2,N}(t, \overline{X}_m^N(t), \overline{V}_m^N(t)) \right|_\infty \leq MS_t(X, V)N^{-\alpha}.$$

- Since $(X, V) \notin \mathcal{N}_\beta$, it follows directly

$$|I_3| := \left| \Psi_m^{1,N}(t, \overline{X}_m^N(t), \overline{V}_m^N(t)) - \overline{\Psi}_m^{1,N}(t, \overline{X}_m^N(t), \overline{V}_m^N(t)) \right|_\infty \leq N^{-\beta}.$$

- Since $(X, V) \notin \mathcal{N}_\gamma$, it follows directly

$$|I_5| := \left| \Psi_m^{2,N}(t, \overline{X}_m^N(t), \overline{V}_m^N(t)) - \overline{\Psi}_m^{2,N}(t, \overline{X}_m^N(t), \overline{V}_m^N(t)) \right|_\infty \leq N^{-\beta}.$$

- Since E_0 and B_0 under Assumption 4.1(a) are Lipschitz continuous, we have for each $1 \leq i \leq N$, $(x_m^{i,N}, v_m^{i,N}) = \big((X_m^N(t), V_m^N(t))\big)_i$, $(\overline{x}_m^{i,N}, \overline{v}_m^{i,N}) = \big((\overline{X}_m^N(t), \overline{V}_m^N(t))\big)_i$ and together with the fact that $(X, V) \notin \mathcal{N}_\alpha$, there holds

$$|I_6| := \left| \Gamma_m^N(t, X_m^N(t), V_m^N(t)) - \Gamma_m^N(t, \overline{X}_m^N(t), \overline{V}_m^N(t)) \right|_\infty \leq LS_t(X, V)N^{-\alpha}.$$

Combining all the six terms, we end up with

$$\begin{aligned}&\Big|\Big(\hat{V}(V_m^N(t)), \Psi_m^{1,N}(t, X_m^N(t), V_m^N(t)) + \Psi_m^{2,N}(t, X_m^N(t), V_m^N(t)) + \Gamma_m^N(t, X_m^N(t), V_m^N(t))\Big)\\ &\quad -\Big(\hat{V}(\overline{V}_m^N(t)), \overline{\Psi}_m^{1,N}(t, \overline{X}_m^N(t), \overline{V}_m^N(t)) + \overline{\Psi}_m^{2,N}(t, \overline{X}_m^N(t), \overline{V}_m^N(t)) + \Gamma_m^N(t, \overline{X}_m^N(t), \overline{V}_m^N(t))\Big)\Big|_\infty\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \le MS_t(X,V)N^{-\alpha} + N^{-\beta}\end{aligned}$$

for all $(X, V) \in (\mathcal{N}_\alpha \cup \mathcal{N}_\beta \cup \mathcal{N}_\gamma)^c$. ■

Using all the Lemmas above we can now prove Theorem 4.2.1:

Proof of Theorem 4.2.1

From the definition of the Newtonian flow (4.2.5) and the characteristics of the Vlasov equation (4.2.7), we know that

$$\begin{aligned}&(X_m^N(t+dt), V_m^N(t+dt))\\ =\ &(X_m^N(t), V_m^N(t))\\ &+\Big(\hat{V}(V_m^N(t)), \Psi_m^{1,N}(t, X_m^N(t), V_m^N(t)) + \Psi_m^{2,N}(t, X_m^N(t), V_m^N(t)) + \Gamma_m^N(t, X_m^N(t), V_m^N(t))\Big)dt + o(dt),\end{aligned}$$

and

$$\begin{aligned}&(\overline{X}_m^N(t+dt), \overline{V}_m^N(t+dt))\\ =\ &(\overline{X}_m^N(t), \overline{V}_m^N(t))\\ &+\Big(\hat{V}(\overline{V}_m^N(t)), \overline{\Psi}_m^{1,N}(t, \overline{X}_m^N(t)) + \overline{\Psi}_m^{2,N}(t, \overline{X}_m^N(t), \overline{V}_m^N(t)) + \Gamma_m^N(t, \overline{X}_m^N(t), \overline{V}_m^N(t))\Big)dt + o(dt).\end{aligned}$$

Thus

$$\begin{aligned}&\Big|(X_m^N(t+dt), V_m^N(t+dt)) - (\overline{X}_m^N(t+dt), \overline{V}_m^N(t+dt))\Big|_\infty \le \Big|(X_m^N(t), V_m^N(t)) - (\overline{X}_m^N(t), \overline{V}_m^N(t))\Big|_\infty\\ &+\Big|\Big(\hat{V}(V_m^N(t)), \Psi_m^{1,N}(t, X_m^N(t), V_m^N(t)) + \Psi_m^{2,N}(t, X_m^N(t), V_m^N(t)) + \Gamma_m^N(t, X_m^N(t), V_m^N(t))\Big)\\ &\quad -\Big(\hat{V}(\overline{V}_m^N(t)), \overline{\Psi}_m^{1,N}(t, \overline{X}_m^N(t)) + \overline{\Psi}_m^{2,N}(t, \overline{X}_m^N(t), \overline{V}_m^N(t)) + \Gamma_m^N(t, \overline{X}_m^N(t), \overline{V}_m^N(t))\Big)\Big|_\infty dt + o(dt),\end{aligned}$$

i.e.,

$$\begin{aligned}&S_{t+dt} - S_t\\ \le\ &\Big|\Big(\hat{V}(V_m^N(t)), \Psi_m^{1,N}(t, X_m^N(t), V_m^N(t)) + \Psi_m^{2,N}(t, X_m^N(t), V_m^N(t)) + \Gamma_m^N(t, X_m^N(t), V_m^N(t))\Big)\\ &\quad -\Big(\hat{V}(\overline{V}_m^N(t)), \overline{\Psi}_m^{1,N}(t, \overline{X}_m^N(t)) + \overline{\Psi}_m^{2,N}(t, \overline{X}_m^N(t), \overline{V}_m^N(t)) + \Gamma_m^N(t, \overline{X}_m^N(t), \overline{V}_m^N(t))\Big)\Big|_\infty dt + o(dt),\end{aligned}$$

Taking the expectation over both sides yields

$$
\begin{aligned}
& \mathbb{E}_0\left[\, S_{t+dt} - S_t\right] \\
= \;& \mathbb{E}_0\left[S_{t+dt} - S_t \,|\, \mathcal{N}_\alpha\,\right] + \mathbb{E}_0\left[\, S_{t+dt} - S_t \,|\, \mathcal{N}_\alpha^c\,\right] \\
\leq \;& \mathbb{E}_0\left[\, S_{t+dt} - S_t \,|\, (\mathcal{N}_\beta \cup \mathcal{N}_\gamma) \setminus \mathcal{N}_\alpha\,\right] + \mathbb{E}_0\left[\, S_{t+dt} - S_t \,|\, (\mathcal{N}_\alpha \cup \mathcal{N}_\beta \cup \mathcal{N}_\gamma)^c\,\right] \\
\leq \;& \mathbb{E}_0\left[\left|\hat{V}(V_m^N(t)) - \hat{V}(\overline{V}_m^N(t))\right|_\infty \,\Big|\, (\mathcal{N}_\beta \cup \mathcal{N}_\gamma) \setminus \mathcal{N}_\alpha\right] N^\alpha dt \\
& + \mathbb{E}_0\left[\left|\Psi_m^{1,N}(t, X_m^N(t), V_m^N(t)) - \overline{\Psi}_m^{1,N}(t, \overline{X}_m^N(t))\right|_\infty \,\Big|\, (\mathcal{N}_\beta \cup \mathcal{N}_\gamma) \setminus \mathcal{N}_\alpha\right] N^\alpha dt \\
& + \mathbb{E}_0\left[\left|\Psi_m^{2,N}(t, X_m^N(t), V_m^N(t)) - \overline{\Psi}_m^{2,N}(t, \overline{X}_m^N(t), \overline{V}_m^N(t))\right|_\infty \,\Big|\, (\mathcal{N}_\beta \cup \mathcal{N}_\gamma) \setminus \mathcal{N}_\alpha\right] N^\alpha dt \\
& + \mathbb{E}_0\left[\left|\Gamma_m^N(t, X_m^N(t), V_m^N(t)) - \Gamma_m^N(t, \overline{X}_m^N(t), \overline{V}_m^N(t))\right|_\infty \,\Big|\, (\mathcal{N}_\beta \cup \mathcal{N}_\gamma) \setminus \mathcal{N}_\alpha\right] N^\alpha dt \\
& + \mathbb{E}_0\left[\, S_{t+dt} - S_t \,|\, (\mathcal{N}_\alpha \cup \mathcal{N}_\beta \cup \mathcal{N}_\gamma)^c\,\right] + o(dt) \\
=: \;& J_1 + J_2 + J_3 + J_4 + J_5 + o(dt),
\end{aligned}
$$

where in the second step we use $\mathbb{E}_0(S_{t+dt} - S_t \,|\, \mathcal{N}_\alpha) \leq 0$ and decompose the set $\mathcal{N}_\alpha^c$ into $(\mathcal{N}_\beta \cup \mathcal{N}_\gamma) \setminus \mathcal{N}_\alpha$ and $(\mathcal{N}_\alpha \cup \mathcal{N}_\beta \cup \mathcal{N}_\gamma)^c$.

Since $(X, V) \notin \mathcal{N}_\alpha$, it follows

$$
\begin{aligned}
J_1 &= \mathbb{E}_0\left[\left|\hat{V}(V_m^N(t)) - \hat{V}(\overline{V}_m^N(t))\right|_\infty \,\Big|\, (\mathcal{N}_\beta \cup \mathcal{N}_\gamma) \setminus \mathcal{N}_\alpha\right] N^\alpha dt \\
&\leq L\big(\mathbb{P}_0(\mathcal{N}_\beta) + \mathbb{P}_0(\mathcal{N}_\gamma)\big) dt.
\end{aligned}
$$

Due to the definition of $\Psi_m^{1,N}$, $\overline{\Psi}_m^{1,N}$, $\Psi_m^{2,N}$, $\overline{\Psi}_m^{2,N}$ and Γ_m^N, we obtain

$$
J_2 + J_3 + J_4 \leq M\big(\mathbb{P}_0(\mathcal{N}_\beta) + \mathbb{P}_0(\mathcal{N}_\gamma)\big) N^\alpha dt.
$$

Thanks to Lemma 4.2.2 and Lemma 4.2.3, we get

$$
\begin{aligned}
J_1 + J_2 + J_3 + J_4 &\leq M\big(\mathbb{P}_0(\mathcal{N}_\beta) + \mathbb{P}_0(\mathcal{N}_\gamma)\big) N^\alpha dt \\
&\leq M c^4 \cdot N^{-(1-4\beta-16\theta)} N^\alpha dt.
\end{aligned}
$$

On the other hand, Lemma 4.2.4 states that

$$
\begin{aligned}
J_5 &= \mathbb{E}_0\left[\, S_{t+dt} - S_t \,|\, (\mathcal{N}_\alpha \cup \mathcal{N}_\beta \cup \mathcal{N}_\gamma)^c\,\right] \\
&\leq (M \cdot \mathbb{E}_0\left[S_t\right] N^{-\alpha} + N^{-\beta}) \cdot N^\alpha dt + o(dt) \\
&= M \cdot \mathbb{E}_0\left[S_t\right] dt + N^{\alpha-\beta} dt + o(dt).
\end{aligned}
$$

Therefore, we can determine the estimate

$$\begin{aligned}\mathbb{E}_0\left[S_{t+dt}\right]-\mathbb{E}_0[S_t] &\le \mathbb{E}_0\left[S_{t+dt}-S_t\right]\\ &\le M\cdot\mathbb{E}_0\left[S_t\right]dt+M\cdot c^4\cdot N^{-(1-\alpha-4\beta-16\theta)}dt+o(dt).\end{aligned}$$

Equivalently, we have

$$\frac{d}{dt}\,\mathbb{E}_0[S_t]\le M\cdot\mathbb{E}_0[S_t]+M\cdot c^4\cdot N^{-(1-\alpha-4\beta-16\theta)}.$$

Gronwall's inequality yields

$$\mathbb{E}_0\left[S_t\right]\le e^{Mt}\cdot c^4\cdot N^{-(1-\alpha-4\beta-16\theta)}.$$

The proof is completed by the following Markov inequality

$$\mathbb{P}_0\left(\sup_{0\le s\le t}\left|(X_m^N(s),V_m^N(s))-(\overline{X}_m^N(s),\overline{V}_m^N(s))\right|_\infty>N^{-\alpha}\right)=\mathbb{P}_0(S_t=1)\le\mathbb{E}_0\left[S_t\right].$$

■

4.2.4 Estimates for the Non-relativistic Limit

Due to the key estimate (4.2.13), it is easy to repeat the whole procedure in the previous subsection to obtian

Theorem 4.2.2. *Let $f_m^N(t,x,v)$ and $f_p^N(t,x,v)$ be the solutions to the regularized Vlasov-Maxwell equation* (4.2.6) *and* (4.2.14) *respectively with the same initial data f_0. Suppose that Assumptions 4.2.1 are satisfied. Then there holds*

$$\mathbb{P}_0\left(\sup_{0\le s\le t}\left|(\overline{X}_m^N(s),\overline{V}_m^N(s))-(\overline{X}_p^N(s),\overline{V}_p^N(s))\right|_\infty>N^{-\alpha}\right)\le e^{Mt}\frac{M}{c}.$$

Remark 4.2.1. *We point out that the proof is straightforward when we use the flows of* (4.2.7) *and* (4.2.15).

4.2.5 Combined Limit

Now with all the estimates we achieved above, we take $c=N^\eta$, $\eta\in\left(0,\frac{1-\alpha-4\beta-16\theta}{4}\right)$. Then

Theorem 4.2.3. *Let $f_m^N(t,x,v)$ and $f_p^N(t,x,v)$ be the solutions to the regularized Vlasov-Maxwell equation* (4.2.6) *and* (4.2.14) *respectively with the same initial data f_0. Suppose that*

Assumption 4.2.1 is satisfied. Then there holds

$$\lim_{N\to\infty,c\to\infty}\mathbb{P}_0\left(\sup_{0\le s\le t}\left|(X_m^N(s),V_m^N(s))-(\overline{X}_p^N(s),\overline{V}_p^N(s))\right|_\infty > N^{-\alpha}\right)=0.$$

Chapter 5

Appendix

Proof. 1. Using the same method as Kurth, R. in [79], it is easy to prove that (VPN) has a unique C^1-solution (f_p^N, E_p^N) on the time interval $[0, \overline{T} > 0)$.

2. The proof of existence of solutions of (VMN) is similar to Glassey, R., Strauss, W [47], while the proof of existence of functions $q(t)$ and $F(t)$ with the respective properties follows the ideas of Jack Schaeffer as given in [106]. Therefore we omit the proof in this manuscript.

3. Next we begin to prove the third part of the theorem. Similar to (A13) and (A14) in the Appendix of [106], we use the convenient notation $\nu = \dfrac{y-x}{|y-x|}$, $x, y \in \mathbb{R}^3$. Then we obtain

$$
\begin{aligned}
& E_m^N(t,x) \\
= \;& \mathbb{E}_0 - \frac{1}{4\pi ct}\iiint_{\mathbb{R}^9} dvdpdz \int_{|x-y|=ct} dS_y \chi^N(p)\chi^N(z) f_0(y-p-z,v)\frac{\nu - c^{-2}\hat{v}\cdot\nu\hat{v}}{(1+c^{-1}\hat{v}\cdot\nu)} \\
& -\frac{1}{4\pi}\iiint_{\mathbb{R}^9} dvdpdz \int_{|x-y|<ct} dy \\
& \qquad\qquad \chi^N(p)\chi^N(z)\frac{f_m^N(t-c^{-1}|x-y|, y-p-z, v)}{|x-y|^2}\frac{(1-c^{-2}|\hat{v}|^2)(\nu + c^{-1}\hat{v})}{(1+c^{-1}\hat{v}\cdot\nu)^2} \\
& -\frac{1}{4\pi c^2}\iiint_{\mathbb{R}^9} dvdpdz \int_{|x-y|<ct} dy \chi^N(p)\chi^N(z)\frac{f_m^N(t-c^{-1}|x-y|, y-p-z, v)}{|x-y|(1+c^{-1}\hat{v}\cdot\nu)^2(1+c^{-2}|v|^2)^{\frac{1}{2}}} \\
& \times\Big[(1+c^{-1}\hat{v}\cdot\nu)(E_m^N + c^{-1}\hat{v}\times B_m^N) + c^{-2}(\hat{v}\cdot\nu\nu - \hat{v})\hat{v}\cdot E_m^N \\
& \qquad\qquad\qquad -(\nu + c^{-1}\hat{v})\nu\cdot(E_m^N + c^{-1}\hat{v}\times B_m^N)\Big]\Big|_{(t-c^{-1}|x-y|, y-p-z)} \\
= \;& \mathbb{E}_0 - \mathbb{E}_1 - \mathbb{E}_2 - \mathbb{E}_3,
\end{aligned}
$$

and

$$\begin{aligned}
& B_m^N \\
= \; & \mathbb{B}_0 + \frac{1}{4\pi ct}\iiint_{\mathbb{R}^9} dvdpdz \int_{|x-y|=ct} dS_y \chi^N(p)\chi^N(z) f_0(y-p-z,v)\frac{(\nu\times c^{-1}\hat{v})}{(1+c^{-1}\hat{v}\cdot\nu)} \\
& + \frac{1}{4\pi c}\iiint_{\mathbb{R}^9} dvdpdz \int_{|x-y|<ct} dy \\
& \qquad \chi^N(p)\chi^N(z)\frac{f_m^N(t-c^{-1}|x-y|, y-p-z, v)}{|x-y|^2}\frac{(1-c^{-2}|\hat{v}|^2)(\nu\times\hat{v})}{(1+c^{-1}\hat{v}\cdot\nu)^2} \\
& + \frac{1}{4\pi c^2}\iiint_{\mathbb{R}^9} dvdpdz \int_{|x-y|<ct} dy \chi^N(p)\chi^N(z)\frac{f_m^N(t-c^{-1}|x-y|, y-p-z, v)}{|x-y|(1+c^{-1}\hat{v}\cdot\nu)^2(1+c^{-2}|v|^2)^{\frac{1}{2}}} \\
& \times\Big[(1+c^{-1}\hat{v}\cdot\nu)\nu\times(E_m^N + c^{-1}\hat{v}\times B_m^N) \\
& \qquad\qquad -c^{-2}(\nu\times\hat{v})(\hat{v}+c\nu)\cdot(E_m^N + c^{-1}\hat{v}\times B_m^N)\Big]\Big|_{(t-c^{-1}|x-y|, y-p-z)} \\
= \; & \mathbb{B}_0 + \mathbb{B}_1 + \mathbb{B}_2 + \mathbb{B}_3,
\end{aligned}$$

where $\Big|_{(t-c^{-1}|x-y|,y-p-z)}$ means $E_m^N(t-c^{-1}|x-y|, y-p-z)$ and $B_m^N(t-c^{-1}|x-y|, y-p-z)$. In order to prove Theorem 4.1.1, we note that the core of the proof consists in comparing the integral representation of (E_m^N, B_m^N) given above with the one of E_p^N given in (VPN) that is

$$E_p^N(t,x) = \frac{1}{4\pi}\iiiint_{\mathbb{R}^{12}} dvdydpdz \chi^N(p)\chi^N(z) f_p^N(t, y-p-z)\frac{x-y}{|x-y|^3}.$$

To obtain uniform convergence, we will thoroughly calculate E_m^N and B_m^N. First, we consider E_m^N.

Lemma 5.0.1. *([106], Lemma 1) Let g be a continuous function of compact support on $\mathbb{R}^3$, then there exists a constant $M>0$ such that*

$$r\int_{|\omega|=1} |g(x+r\omega)|\, d\omega \le M.$$

for all $r>0$.

Note that for $|v|\le q(t)$, with $q(t)\ge 1$,

$$|\hat{v}| \le \frac{q(t)}{(1+c^{-2}|v|^2)^{\frac{1}{2}}} \le q(t),$$

and

$$\frac{1}{1+c^{-1}\hat{v}\cdot\nu} \le 2c^{-2}(c^2+q^2(t)) \le 4q^2(t).$$

From the proposition and the above two inequalities, we get $\forall x\in\mathbb{R}^3$, $t\in[0,T]$

$$\begin{aligned}
&\Big|\frac{1}{4\pi ct}\iiint_{\mathbb{R}^9} dvdpdz\int_{|x-y|=ct} dS_y\chi^N(p)\chi^N(z)f_0(y-p-z,v)\frac{c^{-2}\hat{v}\cdot\nu\hat{v}}{(1+c^{-1}\hat{v}\cdot\nu)}\Big|\\
\le\ & \frac{M}{c^2}tq^4(t)\iiint_{\mathbb{R}^9} dvdpdz\chi^N(p)\chi^N(z)\int_{|\omega|=1} d\omega ctf_0(x-p-z+ct\omega,v)\\
\le\ & Mq^4(t)c^{-2}=O(c^{-2})
\end{aligned}$$

and

$$\begin{aligned}
&\Big|\frac{1}{4\pi ct}\iiint_{\mathbb{R}^9} dvdpdz\int_{|x-y|=ct} dS_y\chi^N(p)\chi^N(z)f_0(y-p-z,v)\frac{c^{-1}\hat{v}\cdot\nu\nu}{(1+c^{-1}\hat{v}\cdot\nu)}\Big|\\
\le\ & \frac{M}{c}tq^3(t)\iiint_{\mathbb{R}^9} dvdpdz\chi^N(p)\chi^N(z)\int_{|\omega|=1} d\omega ctf_0(x-p-z+ct\omega,v)\\
\le\ & Mq^3(t)c^{-2}=O(c^{-1}).
\end{aligned}$$

Hence $\forall x\in\mathbb{R}^3$, $t\in[0,T]$

$$\mathbb{E}_1(t,x)=\frac{1}{4\pi ct}\iiint_{\mathbb{R}^9} dvdpdz\int_{|x-y|=ct} dS_y\chi^N(p)\chi^N(z)f_0(y-p-z,v)\nu+O(c^{-1}).$$

As

$$\begin{aligned}
&\Big|\frac{1}{4\pi}\iiint_{\mathbb{R}^9} dvdpdz\int_{|x-y|<ct} dy\\
&\qquad\qquad \chi^N(p)\chi^N(z)\frac{f_m^N(t-c^{-1}|x-y|,y-p-z,v)}{|x-y|^2}\frac{|\hat{v}|^2(\nu+c^{-1}\hat{v})}{(1+c^{-1}\hat{v}\cdot\nu)^2c^2}\Big|\\
\le\ & \frac{1}{4\pi c^2}\int_{|y|<P_0+tq(t)}\int_{|v|<q(t)}(4q^2(t))^2q^2(t)(1+c^{-1}q(t))\frac{\|f_0\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)}}{|x-y|^2}\,dvdy\\
\le\ & \frac{M}{c^2}q^6(t)(1+c^{-1}q(t))q^3(t)\int_{|y|<P_0+tq(t)}\frac{1}{|x-y|^2}\,dy\\
\le\ & \frac{M}{c^2},
\end{aligned}$$

where we have in the last step used the fact that

$$\sup_x\int_{|y|<P_0+tq(t)}\frac{1}{|x-y|^2}\,dy < M(P_0,q(t)). \tag{5.0.1}$$

In the same way, we obtain

$$\Big|\frac{1}{4\pi}\iiint_{\mathbb{R}^9} dvdpdz\int_{|x-y|<ct} dy \\ \chi^N(p)\chi^N(z)\frac{f_m^N(t-c^{-1}|x-y|,y-p-z,v)}{|x-y|^2}\frac{\hat{v}}{(1+c^{-1}\hat{v}\cdot\nu)^2 c}\Big| \leq \frac{M}{c},$$

so we have

$$\begin{aligned} & \mathbb{E}_2(t,x) \\ = & \frac{1}{4\pi}\iiint_{\mathbb{R}^9} dvdpdz\int_{|x-y|<ct} dy \\ & \qquad \chi^N(p)\chi^N(z)\frac{f_m^N(t-c^{-1}|x-y|,y-p-z,v)}{|x-y|^2}\frac{\nu}{(1+c^{-1}\hat{v}\cdot\nu)^2}+O(c^{-1}) \\ = & \frac{1}{4\pi}\iiint_{\mathbb{R}^9} dvdpdz\int_{|x-y|<ct} dy \\ & \qquad \chi^N(p)\chi^N(z)\frac{f_m^N(t-c^{-1}|x-y|,y-p-z,v)}{|x-y|^2}\nu+O(c^{-1}), \end{aligned}$$

where the following estimate has been used

$$\Big|\frac{1}{(1+c^{-1}\hat{v}\cdot\nu)^2}-1\Big| = \frac{|2c^{-1}\hat{v}\cdot\nu+c^{-2}(\hat{v}\cdot\nu)^2|}{(1+c^{-1}\hat{v}\cdot\nu)^2} \leq \frac{M}{c}q^4(t)\left(q(t)+c^{-1}q^2(t)\right) \leq \frac{M}{c}.$$

Recalling Theorem 4.1.1 and $|\hat{v}|<c$, we get

$$\begin{aligned} |\mathbb{E}_3| \leq & \frac{1}{4\pi c^2}\iiint_{\mathbb{R}^9} dvdpdz\int_{|x-y|<ct} dy \\ & \left(4q^2(t)\right)^2\chi^N(p)\chi^N(z)6H(t-c^{-1}|x-y|)\frac{f_m^N(t-c^{-1}|x-y|,y-p-z,v)}{|x-y|} \\ \leq & \frac{M}{c^2}\int_{|y|<P_0+tq(t)}\frac{1}{|x-y|}\,dy\int_{|v|<q(t)}\|f_0\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)}\,dv \leq \frac{M}{c}. \end{aligned}$$

Lemma 5.0.2. *([106], Lemma 2) Let $g\in C^2(\mathbb{R}^3)$. Assume that Δg has compact support for $c>0$ and $t\geq 0$,*

$$\partial_t\left(t\int_{|\omega|=1} g(x+ct\omega)\,d\omega\right) = -\int_{|x-y|>ct}\frac{\Delta g(y)}{|x-y|}\,dy.$$

Now using this lemma, we estimate $\mathbb{E}_0$. We know

$$\begin{aligned}\mathbb{E}_0 &= \partial_t \int_{|\omega|=1} \frac{t}{4\pi} E_m^N(0, x + ct\omega)\, d\omega + \frac{t}{4\pi} \iint_{\mathbb{R}^6} dpdz \int_{|\omega|=1} d\omega \\ &\quad \chi^N(p)\chi^N(z) \left(c\nabla \times B_0(x - p - z + ct\omega) - \int_{\mathbb{R}^3} \hat{v} f_0(x - p - z + ct\omega, v)\, dv \right).\end{aligned}$$

From Lemma 5.0.1, we get

$$\frac{t}{4\pi} \left| \iint_{\mathbb{R}^6} dpdz \int_{|\omega|=1} d\omega \chi^N(p)\chi^N(z)(c\nabla \times B_0(x - p - z + ct\omega) \right| \leq \frac{M}{c}$$

and by Lemma 5.0.2, we obtain

$$\begin{aligned}&\frac{t}{4\pi} \left| \iint_{\mathbb{R}^6} dpdz \int_{|\omega|=1} d\omega \chi^N(p)\chi^N(z) \int_{\mathbb{R}^3} \hat{v} f_0(x - p - z + ct\omega, v)\, dv \right| \\ &= \frac{1}{4\pi c} \left| \iint_{\mathbb{R}^6} dpdz \int_{|\omega|=1} d\omega \chi^N(p)\chi^N(z) \int_{\mathbb{R}^3} \hat{v} ct f_0(x - p - z + ct\omega, v) dv \right| \leq \frac{M}{c},\end{aligned}$$

thus

$$\mathbb{E}_0 = \partial_t \int_{|\omega|=1} \frac{t}{4\pi} E_m^N(0, x + ct\omega)\, d\omega + O(c^{-1}).$$

Now in order to further calculate $\mathbb{E}_0$, we set

$$g(x) := \frac{1}{4\pi} \iiiint_{\mathbb{R}^{12}} dvdydpdz \chi^N(p)\chi^N(z) \frac{f_0(y - p - z, v)}{|x - y|}.$$

Note that $\nabla g(x) = -E_m^N(0, x)$ and $\Delta g(x) = \iiint_{\mathbb{R}^9} dvdpdz \chi^N(p)\chi^N(z) f_0(x - p - z, v)$. Using Lemma 5.0.2, we get

$$\begin{aligned}&\partial_t \int_{|\omega|=1} \frac{t}{4\pi} E_m^N(0, x + ct\omega)\, d\omega \\ &= -\partial_t \int_{|\omega|=1} \frac{t}{4\pi} \nabla g(x + ct\omega)\, d\omega \\ &= -\frac{1}{4\pi} \nabla \int_{|x-y|>ct} \frac{\Delta g(y)}{|x - y|}\, dy \\ &= -\frac{1}{4\pi} \nabla \iiint_{\mathbb{R}^9} dvdpdz \int_{|x-y|>ct} dy \chi^N(p)\chi^N(z) \frac{f_0(y - p - z, v)}{|x - y|} \\ &= -\frac{1}{4\pi} \iiint_{\mathbb{R}^9} dvdpdz \int_{|x-y|>ct} dy \chi^N(p)\chi^N(z) \frac{\nabla_y f_0(y - p - z, v)}{|x - y|}.\end{aligned}$$

Recall that f_0 has compact support, so by the divergence theorem, we have

$$\begin{aligned}
&-\iiint_{\mathbb{R}^9} dvdpdz \int_{|x-y|=ct} dS_y \chi^N(p)\chi^N(z)\frac{f_0(y-p-z,v)\nu}{|x-y|}\\
=\;& \iiint_{\mathbb{R}^9} dvdpdz \int_{|x-y|>ct} dy \nabla_y \left(\chi^N(p)\chi^N(z)\frac{f_0(y-p-z,v)}{|x-y|}\right)\\
=\;& \iiint_{\mathbb{R}^9} dvdpdz \int_{|x-y|>ct} dy \chi^N(p)\chi^N(z)\frac{\nabla_y f_0(y-p-z,v)}{|x-y|}\\
&-\iiint_{\mathbb{R}^9} dvdpdz \int_{|x-y|>ct} dy \chi^N(p)\chi^N(z)\frac{f_0(y-p-z,v)\nu}{|x-y|^2}
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{E}_0 \;=\;& \frac{1}{4\pi}\iiint_{\mathbb{R}^9} dvdpdz \int_{|x-y|=ct} dS_y \chi^N(p)\chi^N(z)\frac{f_0(y-p-z,v)\nu}{|x-y|}\\
&-\frac{1}{4\pi}\iiint_{\mathbb{R}^9} dvdpdz \int_{|x-y|>ct} dy \chi^N(p)\chi^N(z)\frac{f_0(y-p-z,v)\nu}{|x-y|^2} + O(c^{-1}).
\end{aligned}$$

Therefore

$$\begin{aligned}
& E_m^N(t,x)\\
=\;& -\frac{1}{4\pi}\iiint_{\mathbb{R}^9} dvdpdz \int_{|x-y|<ct} dy \chi^N(p)\chi^N(z)\frac{f_m^N(t-c^{-1}|x-y|,y-p-z,v)\nu}{|x-y|^2}\\
& -\frac{1}{4\pi}\iiint_{\mathbb{R}^9} dvdpdz \int_{|x-y|>ct} dy \chi^N(p)\chi^N(z)\frac{f_0(y-p-z,v)\nu}{|x-y|^2} + O(c^{-1})\\
=\;& -\frac{1}{4\pi}\iiiint_{\mathbb{R}^{12}} dvdpdzdy\\
& \qquad \chi^N(p)\chi^N(z)\frac{f_m^N(\max\{0,t-c^{-1}|x-y|\},y-p-z,v)\nu}{|x-y|^2} + O(c^{-1}).
\end{aligned}$$

From the representation of $E_p^N(t,x)$ from (VPN), we have

$$\begin{aligned}
& |E_m^N(t,x) - E_p^N(t,x)| \\
= & \frac{1}{4\pi}\Big| \iiiint_{\mathbb{R}^{12}} dvdpdzdy \chi^N(p)\chi^N(z)\frac{\nu}{|x-y|^2} \\
& \quad \times \left(f_m^N(\max\{0, t-c^{-1}|x-y|\}, y-p-z, v) - f_p^N(t, y-p-z, v)\right)\Big| + O(c^{-1}) \\
\leq & \frac{M}{c} + \frac{1}{4\pi}\iiiint_{\mathbb{R}^{12}} dvdpdzdy \\
& \quad \chi^N(p)\chi^N(z)\frac{|f_m^N - f_p^N|(\max\{0, t-c^{-1}|x-y|\}, y-p-z, v)}{|x-y|^2} \\
& + \frac{1}{4\pi}\iiiint_{\mathbb{R}^{12}} dvdpdzdy \\
& \quad \chi^N(p)\chi^N(z)\frac{|f_p^N(\max\{0, t-c^{-1}|x-y|\}, y-p-z, v) - f_p^N(t, y-p-z, v)|}{|x-y|^2}.
\end{aligned}$$

Recall that (f_p^N, E_p^N) is a C^1-solution of (VPN). Now since E_p^N is C^1 and f_0 has compact support, it follows that

$$q_p^N = \sup\{|v| : \exists\, x \in \mathbb{R}^3,\ \tau \in [0,t] \text{ s.t. } f_p^N(\tau, x, v) \neq 0\}$$

is finite on $[0,T]$. Also $\partial_t f_p^N$ is bounded on $[0,T] \times \mathbb{R}^6$. Let

$$Q := \max\{q(T), q_p^N(T)\}$$

and

$$G(t) := \sup\left\{|f_m^N(\tau,x,v) - f_p^N(\tau,x,v)| : x \in \mathbb{R}^3, v \in \mathbb{R}^3 \text{ and } \tau \in [0,t]\right\}.$$

Then

$$\begin{aligned}
& |E_m^N(t,x) - E_p^N(t,x)| \\
\leq & \int_{|y|<P_0+TQ}\int_{|v|<Q} \frac{G(\max\{0, t-c^{-1}|x-y|\})}{|x-y|^2}\, dvdy \\
& + \iiiint_{\mathbb{R}^{12}} dvdpdzdy \\
& \quad \chi^N(p)\chi^N(z)\frac{1}{|x-y|^2}\int_{\max\{0, t-c^{-1}|x-y|\}}^{t} |\partial_t f_p^N(\tau, y-p-z, v)|\, d\tau + \frac{M}{c} \\
\leq & G(t) M Q^3 \int_{|y|<P_0+TQ} \frac{1}{|x-y|^2}\, dy + MQ^3\int_{|y|<P_0+TQ} \frac{c^{-1}|x-y|}{|x-y|^2}\, dy + \frac{M}{c} \\
\leq & MG(t) + \frac{M}{c},
\end{aligned}$$

where we have used (5.0.1). Now we begin to estimate B_m^N. Using Lemma 5.0.2, we get for the first term $\mathbb{B}_1$

$$\begin{aligned}
|\mathbb{B}_1| =& \frac{1}{4\pi ct}\left|\iiint_{\mathbb{R}^9} dvdpdz \int_{|x-y|=ct} dS_y \chi^N(p)\chi^N(z) f_0(y-p-z,v)\frac{(\nu\times c^{-1}\hat{v})}{(1+c^{-1}\hat{v}\cdot\nu)}\right| \\
\le& \frac{1}{4\pi}\iiint_{\mathbb{R}^9} dvdpdz \chi^N(p)\chi^N(z) 4q^2(t)c^{-1}q(t)\int_{|\omega|=1} ct f_0(x-p-z+ct\omega,v)\,d\omega \\
\le& \frac{M}{c}\iint_{\mathbb{R}^6}\chi^N(p)\chi^N(z)\,dpdz = \frac{M}{c}.
\end{aligned}$$

Secondly we look into $\mathbb{B}_2$.

$$\begin{aligned}
|\mathbb{B}_2| \;=\;& \frac{1}{4\pi c}\Big|\iiint_{\mathbb{R}^9} dvdpdz\int_{|x-y|<ct} dy \\
& \qquad \chi^N(p)\chi^N(z)\frac{f_m^N(t-c^{-1}|x-y|,y-p-z,v)}{|x-y|^2}\frac{(1-c^{-2}|\hat{v}|^2)(\nu\times\hat{v})}{(1+c^{-1}\hat{v}\cdot\nu)^2}\Big| \\
\le\;& \frac{1}{4\pi c}\iiint_{\mathbb{R}^9} dvdpdz\int_{|x-y|<ct} dy\chi^N(p)\chi^N(z)\frac{\|f_0\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)}(4q^2(t))^2 2q(t)}{|x-y|^2} \\
\le\;& \frac{M}{c}\int_{|y|<P_0+TQ}\int_{|v|<Q}\frac{1}{|x-y|^2}\,dvdy \le \frac{M}{c},
\end{aligned}$$

where (5.0.1) has been used again. The last term $\mathbb{B}_3$ can be shown to be $O(c^{-2})$ in the same way as $\mathbb{E}_3$. Now what is left is $\mathbb{B}_0$. It is easy to calculate that $\partial_t B_0 = -c\nabla\times E_0 = 0$. Using Lemma 5.0.2 and Theorem 4.1.1, we get

$$\begin{aligned}
& \left|\partial_t\int_{|\omega|=1}\frac{t}{4\pi}B_m^N(0,x+ct\omega)\,d\omega\right| \\
=\;& \left|\partial_t\iint_{\mathbb{R}^6} dpdz\int_{|\omega|=1} d\omega\frac{t}{4\pi}\chi^N(p)\chi^N(z)B_0(x-p-z+ct\omega)\right| \\
\le\;& \iint_{\mathbb{R}^6} dpdz\frac{1}{4\pi ct}\chi^N(p)\chi^N(z)\int_{|\omega|=1} ct|B_0(x-p-z+ct\omega)|\,d\omega \\
& +\iint_{\mathbb{R}^6} dpdz\frac{1}{4\pi}\chi^N(p)\chi^N(z)\int_{|\omega|=1} ct|\nabla B_0(x-p-z+ct\omega)|\,d\omega \\
\le\;& M\iint_{\mathbb{R}^6}\frac{1}{4\pi ct}\chi^N(p)\chi^N(z)\,dpdz + \iint_{\mathbb{R}^6} dpdz\int_{|\omega|=1}\frac{1}{4\pi}\chi^N(p)\chi^N(z)ct\frac{1}{c^2}\,d\omega \\
\le\;& \frac{M}{c}.
\end{aligned}$$

Hence

$$B_m^N = \mathbb{B}_0+\mathbb{B}_1+\mathbb{B}_2+\mathbb{B}_3 = O(c^{-1}). \tag{5.0.2}$$

Combing (5.0.2) and (5.0.2), we know that

$$|E_p^N - E_m^N - c^{-1}\hat{v} \times B_m^N| \le MG(t) + \frac{M}{c}, \quad t < T, \tag{5.0.3}$$

for $|\hat{v}| < c$.

It remains to estimate $f_m^N - f_p^N$. For ease of notation, we define $g = f_m^N - f_p^N$. It is not difficult to calculate that

$$\begin{aligned} &\partial_t g + \hat{v} \cdot \nabla_x g + (E_m^N + c^{-1}\hat{v} \times B_m^N) \cdot \nabla_v g \\ &= (v - \hat{v}) \cdot \nabla_x f_p^N + (E_p^N - E_m^N - c^{-1}\hat{v} \times B_m^N) \cdot \nabla_v f_p^N \\ &= \frac{|v|^2 \hat{v}}{c^2(1 + \sqrt{1 + c^{-2}|v|^2})} \cdot \nabla_x f_p^N + (E_p^N - E_m^N - c^{-1}\hat{v} \times B_m^N) \cdot \nabla_v f_p^N \end{aligned} \tag{5.0.4}$$

Note that both $|\nabla_x f_p^N|$ and $|\nabla_v f_p^N|$ are bounded on $[0, T] \times \mathbb{R}^6$ and $\nabla_x f_p^N(t, x, v) = 0$ if $|v| > q_p^N(t)$. Hence

$$\begin{aligned} &|\partial_t g + \hat{v} \cdot \nabla_x g + (E_m^N + c^{-1}\hat{v} \times B_m^N) \cdot \nabla_v g| \\ &\le \frac{M}{c^2} + M|E_p^N - E_m^N - c^{-1}\hat{v} \times B_m^N| \\ &\le \frac{M}{c} + MG(t), \quad 0 \le t \le T. \end{aligned} \tag{5.0.5}$$

For any $x \in \mathbb{R}^3, \ v \in \mathbb{R}^3, t \in [0, T]$, we define $(x(t), v(t))$ as in (4.1.4) and calculate

$$\left|\frac{d}{dt} g(t, x(t), v(t))\right| = |\partial_t g + \hat{v} \cdot \nabla_x g + (E_m^N + c^{-1}\hat{v} \times B_m^N) \cdot \nabla_v g| \tag{5.0.6}$$

$$\le \frac{M}{c} + MG(t), \quad 0 \le t \le T. \tag{5.0.7}$$

Note that $g(t, x(t), v(t))|_{t=0} = 0$, so $\forall x, v, t$, let $(x(0), v(0))$ be the corresponding initial data of (4.1.4). Then

$$\begin{aligned} |g(t, x, v)| &= |g(t, x(t), v(t)) - g(0, x(0), v(0))| \\ &= \left| \int_0^t \frac{d}{ds} g(s, x(s), v(s)) \, ds \right| \\ &\le \int_0^t \left(\frac{M}{c} + MG(s) \right) ds \\ &\le \frac{Mt}{c} + \int_0^t MG(s) \, ds, \quad 0 \le t \le T. \end{aligned}$$

By the definition of g and $G(t)$, we get

$$G(t) \leq \frac{M}{c} + M \int_0^t G(s)\, ds, \quad 0 \leq t \leq T.$$

Using the Gronwall's inequality, we get

$$G(t) \leq \frac{M}{c} \exp(Mt) \leq \frac{M}{c}, \quad 0 \leq t \leq T.$$

Therefore

$$\|f_m^N - f_p^N\|_{L^\infty([0,T)\times\mathbb{R}^3\times\mathbb{R}^3)} + \|E_m^N - E_p^N\|_{L^\infty([0,T)\times\mathbb{R}^3)} + \|B_m^N\|_{L^\infty([0,T)\times\mathbb{R}^3)} \leq \frac{M}{c}.$$

for all $c \geq 1$. This completes the proof of Theorem.

$\blacksquare$

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