



Convergence Towards the Vlasov–Poisson Equation from the N -Fermionic Schrödinger Equation

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Abstract. We consider the quantum dynamics of N interacting fermions in the large N limit. The particles in the system interact with each other via repulsive interaction that is regularized Coulomb potential with a polynomial cutoff with respect to N . From the quantum system, we derive the Vlasov–Poisson system by simultaneously estimating the semiclassical and mean-field residues in terms of the Husimi measure.

1. Introduction

In this study, we consider a system of N identical spinless fermions characterized by the wave function $\psi_N : \mathbb{R}^{3N} \rightarrow \mathbb{C}$ in $L_a^2(\mathbb{R}^{3N})$ with $\|\psi_N\|_{L^2}^2 = 1$. The antisymmetric space $L_a^2(\mathbb{R}^{3N})$, which is a subspace of $L^2(\mathbb{R}^{3N})$, is given by

$$\begin{aligned} L_a^2(\mathbb{R}^{3N}) &:= \{ \psi_N \in L^2(\mathbb{R}^{3N}) : \psi_N(x_{\pi(1)}, \dots, x_{\pi(N)}) \\ &= \varepsilon_\pi \psi_N(x_1, \dots, x_N), \text{ for all } \pi \in S_N \}, \end{aligned} \quad (1.1)$$

where S_N is the odd-permutation group and ε_π is the sign of the permutation π .

The antisymmetric space considered above is a reflection of fermions obeying the Pauli exclusion principle, i.e., no two identical fermions simultaneously occupy the same single quantum state. It is observed that when N fermions are initially trapped in a volume of order one, their kinetic energy is at least of order $N^{5/3}$. This implies that the coupling constant should be chosen as $N^{-1/3}$ to balance the order of the potential energy and the kinetic energy in the Hamiltonian. Thus, the mean-field Hamiltonian acting on $L_a^2(\mathbb{R}^{3N})$ is given by

$$H_N = -\frac{1}{2} \sum_{j=1}^N \Delta_{x_j} + \frac{1}{2N^{1/3}} \sum_{i \neq j}^N V_N(x_i - x_j),$$

where Δ_{x_j} is the Laplacian acting on particle x_j and V_N is the interaction potential given by the regularized Coulomb potential defined as follows:

Definition 1.1. For any $x \in \mathbb{R}^3$ and let $V(x) = |x|^{-1}$, then we call the following V_N to be the regularized Coulomb potential:

$$V_N(x) = (V * \mathcal{G}_{\beta_N})(x), \quad (1.2)$$

where $\mathcal{G}_{\beta_N}(x) := \frac{1}{(2\pi\beta_N^2)^{3/2}} e^{-(x/\beta_N)^2}$.

The regularized Coulomb potential defined in (1.2) can be understood as an interaction potential between spherical particles with a vanishing radius $\beta_N \rightarrow 0$ as $N \rightarrow \infty$. This method of using the regularized Coulomb potential depending on $N \rightarrow \infty$ has been applied in many works, for example, in [35, 43] for the derivation of the Vlasov–Poisson dynamics from N -body classical dynamics. In [17], such a regularized potential was considered for the bosonic case.

Observe that, the time-dependent Schrödinger equation is given by

$$i\partial_\tau \psi_{N,\tau} = H_N \psi_{N,\tau},$$

for all $\psi_{N,\tau} \in L_a^2(\mathbb{R}^{3N})$ and $\tau \geq 0$. Since the average kinetic energy for each fermionic particle is of order $N^{2/3}$, then its average velocity is of order $N^{1/3}$. Therefore, in the mean-field regime, the time evolution of the fermion system is expected to be of order $N^{-1/3}$. Rescaling the time variable $t = N^{1/3}\tau$, one obtains the following Schrödinger equation for N fermions:

$$N^{1/3}i\partial_t \psi_{N,t} = H_N \psi_{N,t}. \quad (1.3)$$

As suggested in Thomas–Fermi theory in [44, 46], we set $\hbar = N^{-1/3}$ as the semiclassical scale. Then, multiplying both sides of (1.3) by \hbar^2 , we obtain the time-dependent Schrödinger equation as follows:¹

$$\begin{cases} i\hbar\partial_t \psi_{N,t} = \left[-\frac{\hbar^2}{2} \sum_{j=1}^N \Delta_{x_j} + \frac{1}{2N} \sum_{i \neq j}^N V_N(x_i - x_j) \right] \psi_{N,t}, \\ \psi_{N,0} = \psi_N, \end{cases} \quad (1.4)$$

where ψ_N is the initial data in $L_a^2(\mathbb{R}^{3N})$. The choice of other coupling constants for different scenarios is summarized in [6].

Solving numerically the Schrödinger equation in (1.4) with a large particle number and analyzing the behavior of its solution is hard even for $N = 1000$. An efficient way to analyze and solve the behavior of a large quantum system is to derive its corresponding effective evolution equations. Therefore, we consider the density matrix operator instead of the wave function $\psi_{N,t}$. Namely, for

¹Note that \hbar here can be interpreted as the effective Planck's constant.

$t \geq 0$, we define the 1-particle reduced density matrix $\gamma_{N,t}$, a positive semidefinite trace class operator in $L_a^2(\mathbb{R}^{3N})$, with trace equal to N . Specifically, for pure states, it is an operator with the corresponding kernel given by

$$\gamma_{N,t}^{(1)}(x; y) := N \int \cdots \int dx_2 \cdots dx_N \overline{\psi_{N,t}(y, x_2, \dots, x_N)} \psi_{N,t}(x, x_2, \dots, x_N),$$

for any normalized $\psi_{N,t} \in L_a^2(\mathbb{R}^{3N})$. It can be easily shown that the trace of the 1-density particle is given by $\text{Tr} \gamma_{N,t}^{(1)} = N$. Furthermore, for indistinguishable fermions, we can analyze the quantum dynamics by density matrices depending on a small number of particles, $1 \leq k \ll N$. Denoting $\text{Tr}^{(k)}$ as the k -partial trace, we define the k -particle reduced density matrix as

$$\gamma_{N,t}^{(k)} = \frac{N!}{(N-k)!} \text{Tr}^{(k)} \gamma_{N,t}, \quad (1.5)$$

where its corresponding integral kernel is given by

$$\begin{aligned} \gamma_{N,t}^{(k)}(x_1, \dots, x_k; y_1, \dots, y_N) \\ = \frac{N!}{(N-k)!} \int \cdots \int dx_{k+1} \cdots dx_N \gamma_N(x_1, \dots, x_k, x_{k+1}, \dots, x_N; y_1, \dots, \\ y_k, x_{k+1}, \dots, x_N). \end{aligned}$$

We denote the inner-product of $L_a^2(\mathbb{R}^{3N})$ as $\langle \psi, \phi \rangle = \int dx \overline{\psi(x)} \phi(x)$. Given any N and time t , the expectation of the physical observable associated with a self-adjoint operator O is given as

$$\langle \psi_{N,t}, O \psi_{N,t} \rangle = \int \cdots \int dx_1 \cdots dx_N \overline{\psi_{N,t}(x_1, \dots, x_N)} (O \psi_{N,t})(x_1, \dots, x_N).$$

Equivalently, we can write the expectation of an observable O with

$$\text{Tr} O \gamma_{N,t} = \langle \psi_{N,t}, O \psi_{N,t} \rangle, \quad (1.6)$$

and the expectation of any k -observables $O^{(k)}$ is

$$\text{Tr}(O^{(k)} \otimes \mathbf{1}^{(N-k)}) \gamma_{N,t} = \frac{N!}{(N-k)!} \text{Tr} O^{(k)} \gamma_{N,t}^{(k)}.$$

Therefore, the k -particle reduced density matrix $\gamma_{N,t}^{(k)}$ is also a positive semidefinite trace class operator with trace

$$\text{Tr} \gamma_{N,t}^{(k)} = \frac{N!}{(N-k)!}.$$

With a k -particle density matrix, we can avoid analyzing the complicated case with N -particles by finding an approximating effective equation that describes the system. In the fermionic case, we let $\gamma_{N,0}^{(1)} \equiv \omega_N$, a 1-particle density matrix associated with initial state ψ_N , be a Slater determinant defined as

$$\psi_N^{\text{Slater}}(x_1, \dots, x_N) = (N!)^{-1/2} \det\{e_i(x_j)\}_{i,j=1}^N, \quad (1.7)$$

for any family of orthonormal bases $\{e_j\}_{j=1}^N \subset L^2(\mathbb{R}^3)$. In particular, we have

$$\omega_N = \sum_{j=1}^N |e_j\rangle\langle e_j|, \quad (1.8)$$

which corresponds to the 1-particle reduced density matrix with an integral kernel of $\omega_N(x; y) = \sum_{j=1}^N \overline{e_j(y)}e_j(x)$. In [57], the mean-field approximation of the Schrödinger equation is given by the following Hartree–Fock equation:

$$\begin{cases} i\hbar\partial_t\omega_{N,t} = [-\hbar^2\Delta + (|\cdot|^{-1} * \rho_{N,t}) - X_t, \omega_{N,t}], \\ \omega_{N,t}|_{t=0} = \omega_N, \end{cases} \quad (1.9)$$

where $\rho_{N,t}$ has the integral kernel $\frac{1}{N}\omega_{N,t}(x; x)$, X_t is the exchange operator with the integral kernel $\frac{1}{N}|x - y|^{-1}\omega_{N,t}(x; y)$, and the commutator is denoted as $[A, B] := AB - BA$ for any bounded operators A and B .

The mean-field limit from the Schrödinger equation to the Hartree–Fock equation has been studied extensively. In [23], where the Slater determinant constitutes the initial data and a regular interaction is assumed, the convergence is obtained by the use of the Bogoliubov–Born–Green–Kirkwood–Yvon (BBGKY) hierarchy method for short times. In [10], the rates of convergence in both the trace norm and Hilbert–Schmidt norm for pure states are obtained for an arbitrary time and more general potential in the framework of second quantization. The extension to mixed states has been considered in [8] for a positive temperature and for the relativistic case in [11]. Furthermore, by utilizing the Fefferman–de la Llave decomposition presented in [6, 25, 34], the rate of convergence, with more assumptions on the initial data is obtained in [57] for Coulomb potential and in [59] for inverse power law potential. Further literature on the mean-field limit for fermionic cases can be found in [27, 53–55].

The semiclassical limit from the Hartree–Fock equation to the Vlasov equation has also been extensively studied. In [47], this is achieved by using the Wigner–Weyl transformation of the density matrix. In [9], the authors compared the inverse Wigner transform of the Vlasov solution and the solution of the Hartree–Fock equation and obtained the rate of convergence in the trace norm as well as the Hilbert–Schmidt norm with regular assumptions on the initial data. In fact, [9, 60] utilized the k -particle Wigner measure as follows:

$$\begin{aligned} & W_{N,t}^{(k)}(x_1, p_1, \dots, x_k, p_k) \\ &= \binom{N}{k}^{-1} \int \cdots \int (dy)^{\otimes k} \gamma_{N,t}^{(k)} \left(x_1 + \frac{\hbar}{2}y_1, \dots, x_k + \frac{\hbar}{2}y_k; x_1 \right. \\ & \quad \left. - \frac{\hbar}{2}y_1, \dots, x_k - \frac{\hbar}{2}y_k \right) e^{-i\sum_{i=1}^k p_i \cdot y_i}, \end{aligned} \quad (1.10)$$

where $\gamma_{N,t}^{(k)}$ is the kernel of the k -particle reduced density defined in (1.5).

The works in this direction have also been extended for the inverse power law potential in [61], rate of convergence in the Schatten norm in [42], Coulomb potential and mixed states in [60], and convergence in the Wasserstein distance in [40, 41]. The convergence of relativistic Hartree dynamic to the relativistic

Vlasov equation was considered in [21]. Further analysis of the semiclassical limit from the Hartree–Fock equation to the Vlasov equation can be found in [2, 3, 5, 28, 51].

We can combine both mean-field and semiclassical limits and directly obtain the convergence from the Schrödinger equation to the Vlasov equation. The notable pioneers in this direction are Narnhofer and Sewell in [52] and Spohn in [64]. They proved the limit from the Schrödinger equation to Vlasov, in which the interaction potential V was assumed to be analytic in [52] and C^2 in [64]. The rate of convergence of the combined limit in terms of the Wasserstein (pseudo)distance was obtained in [31–33]. In fact, the authors studied the rate of convergence in terms of the Wasserstein distance by treating the Vlasov equation as a transport equation and applying the Dobrushin estimate with appropriately chosen initial data. Then, the result for the Husimi measure was obtained by transforming its Wigner measure similar to (1.11) with a specifically chosen coherent state. In this study, we instead consider a more generalized coherent state. Recently, the combined limit for the singular potential case was obtained in [18]. They provided a derivation of the Vlasov equation using the weighted Schatten norm with a higher moment, and more conditions on the initial data were assumed.

Nevertheless, it is known that the Wigner measure defined in (1.10) is not a true probability density, as it may be negative in a certain phase space. This is shown numerically in [39] for chosen Fock states. Moreover, in [38], a *vis-à-vis* comparison of the classical and quantum systems of a nonlinear Duffing resonator shows that the classical system develops a probability density in the traditional sense, while the quantum system yields a negative region in phase space corresponding to the Wigner measure. In fact, it is proven in [37, 50, 63] that the Wigner measure is nonnegative if and only if the pure quantum states are Gaussian. Additionally, in [13], it is stated that the Wigner measure is nonnegative if the state is a convex combination of coherent states. The issue of incompatibility between the quantum Wigner and classical regimes remains an open question [14].

Nevertheless, it has been shown that we can obtain a nonnegative probability measure by taking the convolution of the Wigner measure with a Gaussian function as a mollifier; this is known as the Husimi measure [19, 26, 66]. In particular, from [26, p.21], given a specific Gaussian coherent state, the relation between the Husimi measure and Wigner measure is given by the following convolution: for any $1 \leq k \leq N$,

$$\mathfrak{m}_{N,t}^{(k)} = \frac{N(N-1) \cdots (N-k+1)}{N^k} W_{N,t}^{(k)} * \mathcal{G}^{\hbar}, \quad (1.11)$$

where $\mathfrak{m}_{N,t}^{(k)}$ is the k -particle Husimi measure and

$$\mathcal{G}^{\hbar} := (\pi\hbar)^{-3k} \exp \left(-\hbar^{-1} \left(\sum_{j=1}^k |q_j|^2 + |p_j|^2 \right) \right).$$

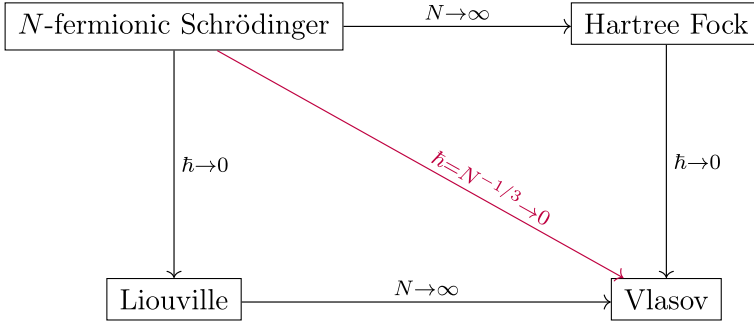


FIGURE 1. Relations of N -fermionic Schrödinger systems to other mean-field equations [15, 30, 31]

The smoothing of the Wigner measure presented in (1.11) motivates the objective of our study: to directly obtain the Vlasov–Poisson equation from the Schrödinger equation in terms of the Husimi measure.² In fact, we have explored the direct method in [15] with the use of the BBGKY hierarchy method, under the assumption that $V \in W^{2,\infty}(\mathbb{R}^3)$. The main contribution of the current work is that by using the generalized version of Husimi measure defined later in (2.10), we are able to write N -fermionic Schrödinger equation directly into Vlasov type of equation in (3.4) and obtain a convergence in combined limit without the use of BBGKY hierarchy method. Furthermore, compared to [15], the new remainder terms in (3.4) obtained in this paper allow us to handle the regularized Coulomb potential defined in (1.2).

Note that the case for bosons has been extensively studied. In fact, there are more studies on bosonic cases than on fermionic cases. As bosons are not the main concern of this paper, we mention only a selected few of these studies in passing. In particular, [24] proved that the Schrödinger equation for bosons converges to the nonlinear Hartree equation for the Coulomb potential. In addition, the convergence for the aforementioned equation is obtained in [58] with a rate of $N^{1/2}$ for the Coulomb potential. The convergence rate of N^{-1} has been optimized in [17] for the Coulomb potential, as well as for more singular potentials in [16].

This article is organized as follows. Brief introductions to the second quantization and Husimi measure are presented in Sects. 2.1 and 2.2, respectively. This is followed by the statement of our main theorem and proof strategy in Sect. 3. Then, uniform estimates are given in Sect. 3.2, followed by the proof of the main theorem in Sect. 3.3. The estimates for the residual terms are covered in Sect. 4.

²See Fig. 1.

2. Preliminaries

2.1. Second Quantization

In the study of large particle systems, we expect the operators to interact with different Hilbert spaces of the N -particle system by creating and annihilating particles. Therefore, to analyze a large particle system, it is convenient for us to build a ‘larger’ Hilbert space that accompanies the aforementioned interactions, equipped with the norm $\|\cdot\|$. In particular, for a large fermionic system, we consider the Fock space for fermions as

$$\mathcal{F}_a := \mathbb{C} \bigoplus_{n \geq 1} L_a^2(\mathbb{R}^{3n}, (dx)^{\otimes n}),$$

where $L_a^2(\mathbb{R}^{3n}; (dx)^{\otimes n})$ represents the n -fold antisymmetric tensor product of $L^2(\mathbb{R}^3)$. Moreover, the vacuum state is denoted as $\Omega = 1 \oplus 0 \oplus 0 \oplus \cdots \in \mathcal{F}_a$.

For $f \in L^2(\mathbb{R}^3)$, the annihilation operator $a(f)$ and creation operator $a^*(f)$ acting on $\Psi = \bigoplus_{n \geq 0} \psi^{(n)} \in \mathcal{F}_a$ are defined by

$$\begin{aligned} (a(f)\Psi)^{(n)} &:= \sqrt{n+1} \int dx f(x) \psi^{(n+1)}(x, x_1, \dots, x_n), \\ (a^*(f)\Psi)^{(n)} &:= \frac{1}{\sqrt{n}} \sum_{j=1}^n f(x_j) \psi^{(n-1)}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n). \end{aligned}$$

Here, $\Psi^{(n)}$ denotes the n -th particle sector of $\Psi \in \mathcal{F}_a$. Following the notations from [10], we will use the operator valued distributions a_x^* and a_x , to represent the creation and annihilation operators:

$$a^*(f) = \int dx f(x) a_x^*, \quad a(f) = \int dx \overline{f(x)} a_x. \quad (2.1)$$

Note that the operator-valued distribution a_x^* formally creates a particle at position $x \in \mathbb{R}^3$, while the operator-valued distribution a_x annihilates a particle at x .

Furthermore, by the corresponding canonical anticommutation relations (CAR) in the fermionic system, we have that for any $f, g \in L^2(\mathbb{R}^3)$

$$\{a(f), a^*(g)\} = \langle f, g \rangle, \quad \{a^*(f), a^*(g)\} = \{a(f), a(g)\} = 0, \quad (2.2)$$

where $\{A, B\} := AB + BA$ is the anticommutator. Following from (2.2), the CAR for operator kernels holds as follows:

$$\{a_x, a_y^*\} = \delta_{x=y}, \quad \{a_x^*, a_y^*\} = \{a_x, a_y\} = 0. \quad (2.3)$$

For any normalized $\Psi_{N,t} \in \mathcal{F}_a$, it is straightforward to show that

$$\|a(f)\Psi_{N,t}\|^2 \leq \|f\|_{L^2}^2, \quad \|a^*(f)\| = \|a(f)\| \quad (2.4)$$

for any $f \in L^2(\mathbb{R}^3)$.³

³See Theorem 3.52 in [20] for a more pedagogical approach to the annihilation and creation operator for the fermionic case.

We extend the Hamilton operator appeared in (1.4) acting on $L_a^2(\mathbb{R}^{3N})$ to an operator acting on the Fock space \mathcal{F}_a by $(\mathcal{H}_N \Psi)^{(n)} = \mathcal{H}_N^{(n)} \psi^{(n)}$ with

$$\mathcal{H}_N^{(n)} = \sum_{j=1}^n -\frac{\hbar^2}{2} \Delta_{x_j} + \frac{1}{2N} \sum_{i \neq j}^n V(x_i - x_j).$$

Then, we can write the Hamiltonian \mathcal{H}_N in terms of the operator-valued distributions a_x and a_x^* by

$$\mathcal{H}_N = \frac{\hbar^2}{2} \int dx \nabla_x a_x^* \nabla_x a_x + \frac{1}{2N} \iint dx dy V_N(x-y) a_x^* a_y^* a_y a_x. \quad (2.5)$$

In this article, we will consider only the following Schrödinger equation in Fock space:

$$\begin{cases} i\hbar \partial_t \Psi_{N,t} = \mathcal{H}_N \Psi_{N,t}, \\ \Psi_{N,0} = \Psi_N, \end{cases} \quad (2.6)$$

for all $\Psi_{N,t} \in \mathcal{F}_a$ and $\|\Psi_{N,t}\| = 1$ for $t \in [0, T]$.

Next, we denote the number of particles operator and kinetic energy operator as

$$\mathcal{N} = \int dx a_x^* a_x \quad \text{and} \quad \mathcal{K} = \frac{\hbar^2}{2} \int dx \nabla_x a_x^* \nabla_x a_x, \quad (2.7)$$

respectively.

For any given $\Psi \in \mathcal{F}_a$ in the n -th sector, we can interpret the number of particles operator as

$$(\mathcal{N} \Psi)^{(n)} = n \psi^{(n)}, \quad (2.8)$$

where $\psi^{(n)} \in L_a^2(\mathbb{R}^{3n})$ for any $n \geq 1$. In the vacuum state, we have $\mathcal{N} \Omega = 0$. It is therefore straightforward to show that for $k \geq 1$,

$$\langle \Psi_{N,t}, \mathcal{N}^k \Psi_{N,t} \rangle = N(N-1) \cdots (N-k+1),$$

for any normalized $\Psi_{N,t} \in \mathcal{F}_a$ and $t \geq 0$. Clearly, the relation between the number of particles operator and the 1-particle reduced density matrix is given as

$$\langle \Psi_{N,t}, \mathcal{N} \Psi_{N,t} \rangle = \int dw \langle \Psi_{N,t}, a_w^* a_w \Psi_{N,t} \rangle = \gamma_{N,t}^{(1)}(w; w),$$

and observe $\text{Tr} \gamma_{N,t}^{(1)} = N$.

2.2. The Husimi Measure

We use the definition of the Husimi measure given in [26]. Let f be any real-valued normalized function in Hilbert space; then, the coherent state is defined as

$$f_{q,p}^{\hbar}(y) := \hbar^{-\frac{3}{4}} f\left(\frac{y-q}{\sqrt{\hbar}}\right) e^{\frac{i}{\hbar} p \cdot y}. \quad (2.9)$$

Then, the projection of coherent state is given by

$$\frac{1}{(2\pi\hbar)^3} \iint dqdp |f_{q,p}^{\hbar}\rangle \langle f_{q,p}^{\hbar}| = \mathbb{1}.$$

For any $\Psi_{N,t} \in \mathcal{F}_a$, $1 \leq k \leq N$ and $t \geq 0$, the k -particle Husimi measure is defined as

$$\begin{aligned} & m_{N,t}^{(k)}(q_1, p_1, \dots, q_k, p_k) \\ & := \int \cdots \int (dwdu)^{\otimes k} \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k} \langle \Psi_{N,t}, a_{w_1}^* \cdots a_{w_k}^* a_{u_1} \cdots a_{u_k} \Psi_{N,t} \rangle \\ & = \int \cdots \int (dwdu)^{\otimes k} \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k} \gamma_{N,t}^{(k)}(u_1, \dots, u_k; w_1, \dots, w_k), \end{aligned} \quad (2.10)$$

where we use the short notations

$$\begin{aligned} (dwdu)^{\otimes k} & := dw_1 du_1 \cdots dw_k du_k, \text{ and } \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes k} \\ & := \prod_{j=1}^k f_{q_j, p_j}^{\hbar}(w_j) \overline{f_{q_j, p_j}^{\hbar}(u_j)}. \end{aligned}$$

The Husimi measure defined in (2.10) measures how many particles, in particular fermions, are in the k -semiclassical boxes with a length scale of $\sqrt{\hbar}$ centered in its respective phase-space pairs, $(q_1, p_1), \dots, (q_k, p_k)$.

Remark 2.1. The Husimi measure (2.10) is a more generalized version of (1.11). If f is given by a Gaussian function, then the definitions of $m_{N,t}^{(k)}$ and $\mathfrak{m}_{N,t}^{(k)}$ coincide.

Then, we observe that by using the operator kernels defined in (2.1), the Husimi measure can be expressed by

The relation between the Husimi measure and the number of particles operator can be expressed as follows, for the 1-particle Husimi measure $m_{N,t} := m_{N,t}^{(1)}$,

$$\begin{aligned} & \iint dqdp m_{N,t}(q, p) \\ & = \iint dqdp \iint dw_1 du_1 f_{q,p}^{\hbar}(w_1) \gamma_{N,t}^{(1)}(w_1; u_1) \overline{f_{q,p}^{\hbar}(u_1)} \\ & = \hbar^{-\frac{3}{2}} \int dq \iint dw_1 du_1 f\left(\frac{w_1 - q_1}{\sqrt{\hbar}}\right) f\left(\frac{u_1 - q_1}{\sqrt{\hbar}}\right) \left(\int dp e^{\frac{i}{\hbar} p \cdot (w_1 - u_1)} \right) \gamma_{N,t}^{(1)}(w_1; u_1) \\ & = (2\pi\hbar)^3 \hbar^{-\frac{3}{2}} \iint dq_1 dw_1 \left| f\left(\frac{w_1 - q_1}{\sqrt{\hbar}}\right) \right|^2 \gamma_{N,t}^{(1)}(w_1; w_1) \\ & = (2\pi\hbar)^3 \int d\tilde{q} |f(\tilde{q})|^2 \int dw_1 \gamma_{N,t}^{(1)}(w_1; w_1) \\ & = (2\pi)^3, \end{aligned}$$

where we use the Dirac-delta $\delta_x(y) := (2\pi\hbar)^{-3} \int e^{\frac{i}{\hbar} p \cdot (x-y)} dp$. Further properties of the Husimi measure are covered in Lemma 3.1. Observe that if the initial data is described by Slater determinant as in (1.8), then the Husimi

measure at initial time is

$$m_N^{\text{Slater}}(q, p) = \sum_{j=1}^N \iint dw_1 du_1 f_{q,p}^{\hbar}(w_1) \overline{e_j(w_1)} e_j(u_1) \overline{f_{q,p}^{\hbar}(u_1)}. \quad (2.11)$$

We are now ready to state the main theorem.

3. Main Result

In this section, we provide our main result, proof strategies, and the *a priori* estimates. The complete proof will be presented in Sect. 3.3. In the following, we denote $\nabla_q f$ and $\nabla_p f$ to be the gradients of f with respect to the position and momentum variables, respectively.

Theorem 3.1. *Suppose that V_N is the regularized Coulomb potential given in (1.2) with $\beta_N := N^{-\epsilon}$ and $0 < \epsilon < \frac{1}{24}$ hold. For any fixed $T > 0$, let $\Psi_{N,t} \in \mathcal{F}_a$, $t \in [0, T]$, be the solution to the Schrödinger equation (2.6) with the Slater determinant as the initial data. Let $m_{N,t}$ be the 1-particle Husimi measure defined in (2.10), where f is a compact supported positive-valued function in $H^1(\mathbb{R}^3)$ with $\|f\|_{L^2} = 1$. Moreover, let m_N^{Slater} be the initial 1-particle Husimi measure with its L^1 -weak limit m_0 and there exists a constant $C > 0$ independent of N such that*

$$\iint dq dp (|p|^2 + |q|) m_N(q, p) \leq C. \quad (3.1)$$

Then, $m_{N,t}$ has a weak- \star convergent subsequence in $L^\infty((0, T]; L^1(\mathbb{R}^3 \times \mathbb{R}^3))$ with limit m_t , and m_t is the solution of the Vlasov–Poisson equation with repulsive Coulomb potential,

$$\begin{cases} \partial_t m_t(q, p) + p \cdot \nabla_q m_t(q, p) = \nabla_q (|\cdot|^{-1} * \varrho_t)(q) \cdot \nabla_p m_t(q, p), \\ m_t(q, p)|_{t=0} = m_0(q, p), \end{cases} \quad (3.2)$$

in the sense of distribution where $\varrho_t(q) := \int dp m_t(q, p)$.

Remark 3.1. Since the total energy is conserved in this problem, the assumption of repulsive interacting potential is important to give uniform estimates both for kinetic energy and potential energy.⁴ In fact, the result in Theorem 3.1 holds also for attractive singular potential if the kinetic energy can be bounded uniformly in N .

Remark 3.2. It is proven in Proposition 3.1 that the first moment of the Husimi measure $m_{N,t}$ is uniformly bounded. Therefore, by Theorem 7.12 in [65], the convergence stated in theorem also holds in terms of the 1-Wasserstein metric.⁵

⁴See Lemma 3.2 below.

⁵The 1-Wasserstein metric is defined as $W_1(\mu, \nu) := \max_{\pi \in \Pi(\mu, \nu)} \int |x - y| d\pi(x, y)$, where μ and ν are probability measures and $\Pi(\mu, \nu)$ the set of all probability measures with marginals μ and ν [65].

Remark 3.3. In [31], the rate of convergence from Schrödinger to the Vlasov equation in the pseudometric is obtained for the interaction potential $V \in C^{1,1}$. In addition, the authors commented that their result can be extended for the truncated Coulomb interaction, but with order higher than $C/\sqrt{\ln N}$ for some constant $C > 0$. In Theorem 3.1, the mollification of the Coulomb interaction can be handled with polynomial truncation.

Remark 3.4. The global existence of classical solution to the Vlasov–Poisson equation in 3-dimension is proven in [48, 56] for a general class of initial data. The uniqueness of the solution is proven in [48] for initial datum with strong moment conditions and integrability. In [49], the uniqueness of the solution is also proven for bounded macroscopic density. Furthermore, the global existence of weak solutions is provided in [4] for bounded initial data and kinetic energy. The result is then relaxed to only L^p -bound for $p > 1$ in [29]. Result on existence with symmetric initial data is proven in [7, 22, 62]. For other results, we refer to the works given in [1, 12, 36] to list a few.

3.1. Proof Strategies

From [15, Proposition 2.1], we obtain the following equation from the Schrödinger equation given (1.4), i.e.,

$$\begin{aligned} & \partial_t m_{N,t}(q, p) + p \cdot \nabla_q m_{N,t}(q, p) - \nabla_q \cdot (\hbar \Im \langle \nabla_q a(f_{q,p}^{\hbar}) \psi_{N,t}, a(f_{q,p}^{\hbar}) \psi_{N,t} \rangle) \\ &= \frac{1}{(2\pi)^3} \nabla_p \cdot \iint dw_1 du_1 \iint dw_2 du_2 \iint dq_2 dp_2 \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes 2} \\ & \quad \int_0^1 ds \nabla V_N(su_1 + (1-s)w_1 - w_2) \gamma_{N,t}^{(2)}(u_1, u_2; w_1, w_2), \end{aligned} \quad (3.3)$$

where we denote

$$\left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes 2} := f_{q,p}^{\hbar}(w_1) \overline{f_{q,p}^{\hbar}(u_1)} f_{q_2,p_2}^{\hbar}(w_2) \overline{f_{q_2,p_2}^{\hbar}(u_2)}.$$

In particular, this can be rewritten into the Vlasov equation with remainder terms, i.e.,

$$\begin{aligned} & \partial_t m_{N,t}(q, p) + p \cdot \nabla_q m_{N,t}(q, p) \\ &= \frac{1}{(2\pi)^3} \nabla_p \cdot \int dq_2 \nabla V_N(q - q_2) \varrho_{N,t}(q_2) m_{N,t}(q, p) + \nabla_q \cdot \widetilde{\mathcal{R}} + \nabla_p \cdot \mathcal{R}, \end{aligned} \quad (3.4)$$

where $\varrho_{N,t}(q) := \int dp m_{N,t}(q, p)$, $\widetilde{\mathcal{R}}$ and $\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2$ are given by

$$\begin{aligned} \widetilde{\mathcal{R}} &:= \hbar \Im \langle \nabla_q a(f_{q,p}^{\hbar}) \psi_{N,t}, a(f_{q,p}^{\hbar}) \psi_{N,t} \rangle, \\ \mathcal{R}_1 &:= \frac{1}{(2\pi)^3} \iint dw_1 du_1 \iint dw_2 du_2 \iint dq_2 dp_2 \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes 2} \\ & \quad \left[\int_0^1 ds \nabla V_N(su_1 + (1-s)w_1 - w_2) - \nabla V_N(q - q_2) \right] \gamma_{N,t}^{(2)}(u_1, u_2; w_1, w_2), \\ \mathcal{R}_2 &:= \frac{1}{(2\pi)^3} \iint dw_1 du_1 \iint dw_2 du_2 \iint dq_2 dp_2 \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes 2} \\ & \quad \nabla V_N(q - q_2) \left[\gamma_{N,t}^{(2)}(u_1, u_2; w_1, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1) \gamma_{N,t}^{(1)}(u_2; w_2) \right]. \end{aligned} \quad (3.5)$$

The main contribution of this article is to rigorously prove the limit $N \rightarrow \infty$ from (3.4) to the Vlasov–Poisson equation (3.2) in the sense of distribution.

First, from the uniform estimate of the kinetic energy shown in Lemma 3.2, we prove in Proposition 3.1 the uniform estimate for the moments of Husimi measure. Additionally, because the Husimi measure belongs to $L^\infty([0, T]; L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3))$ (see Lemma 3.1), we obtain directly the weak compactness of the two linear terms on the left-hand side of (3.4) by the Dunford–Pettis theorem.⁶

For the quadratic term on the right-hand side of (3.4), the classical Thomas–Fermi theory gives that $\varrho_{N,t} \in L^\infty([0, T]; L^{5/3}(\mathbb{R}^3))$. With the *a priori* estimate obtained in Sect. 3.2, the Aubin–Lions compact embedding theorem shows the strong compactness of $\nabla V_N * \varrho_{N,t}$.

The estimate for the remainder term $\tilde{\mathcal{R}}$ is provided in [15, Proposition 2.4]. Thus, the main work of this paper is dealing with the challenging term \mathcal{R} . Unlike the BBGKY hierarchy used in [15], where the remainder term contains only the difference between the 2-particle density matrices, we write the term \mathcal{R} as a combination of the semiclassical and mean-field terms as \mathcal{R}_1 and \mathcal{R}_2 , respectively.⁷ Thus, the factorization effect can be directly obtained from \mathcal{R}_2 instead of using the method of the BBGKY hierarchy.

The estimates for \mathcal{R}_1 and \mathcal{R}_2 are shown in Proposition 4.2 and Proposition 4.3, respectively, in which we utilize the estimates of the ‘cutoff’ number operator and momentum oscillation presented in Lemma 4.1 and Lemma 4.2, to control the growth of the Lipschitz constant V_N , which is of order β_N^{-2} .

3.2. A priori Estimates

We present in this subsection a sequence of estimates that is used repeatedly in the proof.

First, we cite the following properties of k -particle Husimi measures from (or [15, Lemma 2.2] for the time dependent version).

Lemma 3.1. *Suppose that $\Psi_{N,t} \in \mathcal{F}_a$ is normalized for any $t \geq 0$. Then, the following properties hold true for $m_{N,t}^{(k)}$,*

1. $m_{N,t}^{(k)}(q, p, \dots, q_k, p_k)$ is symmetric,
2. $\frac{1}{(2\pi)^{3k}} \int \cdot \int (dq dp)^{\otimes k} m_{N,t}^{(k)}(q, p, \dots, q_k, p_k) = \frac{N(N-1)\cdots(N-k+1)}{N^k}$,
3. $\frac{1}{(2\pi\hbar)^3} \iint dq_k dp_k m_{N,t}^{(k)}(q, p, \dots, q_k, p_k) = (N-k+1) m_{N,t}^{(k-1)}(q, p, \dots, q_{k-1}, p_{k-1})$,
4. $0 \leq m_{N,t}^{(k)}(q, p, \dots, q_k, p_k) \leq 1$ a.e.,

where $1 \leq k \leq N$.

Then, due to the conservation of energy and the repulsive effect of the Coulomb force, we obtain the following estimate for the kinetic energy.

⁶See Proposition 3.2.

⁷See (4.5) for the full structure.

Lemma 3.2. *Assuming that $V_N(x) \geq 0$ and the initial total energy is bounded in the sense that $\frac{1}{N}\langle \Psi_N, \mathcal{H}_N \Psi_N \rangle \leq C$, then there exists a constant $C > 0$ independent of N such that*

$$\left\langle \Psi_{N,t}, \frac{\mathcal{K}}{N} \Psi_{N,t} \right\rangle \leq C. \quad (3.6)$$

Proof. We define the operator

$$\mathcal{V}_N := \frac{1}{N} \iint dx dy V_N(x-y) a_x^* a_y^* a_y a_x.$$

Since $V_N \geq 0$, we have $\langle \Psi_{N,t}, \mathcal{V}_N \Psi_{N,t} \rangle \geq 0$. Then

$$\langle \Psi_{N,t}, \mathcal{H}_N \Psi_{N,t} \rangle = \langle \Psi_{N,t}, \mathcal{K} \Psi_{N,t} \rangle + \langle \Psi_{N,t}, \mathcal{V}_N \Psi_{N,t} \rangle,$$

implies

$$0 \leq \langle \Psi_{N,t}, \mathcal{K} \Psi_{N,t} \rangle \leq \langle \Psi_{N,t}, \mathcal{H}_N \Psi_{N,t} \rangle.$$

Hence,

$$\frac{1}{N} \langle \Psi_{N,t}, \mathcal{K} \Psi_{N,t} \rangle \leq \frac{1}{N} \langle \Psi_{N,t}, \mathcal{H}_N \Psi_{N,t} \rangle = \frac{1}{N} \langle \Psi_N, \mathcal{H}_N \Psi_N \rangle \leq C.$$

□

Consequently, the moment estimate of the Husimi measure is obtained directly from the uniform bound in Lemma 3.2.

Proposition 3.1. *For $t \geq 0$, we have the following finite moments:*

$$\iint dq dp (|q| + |p|^2) m_{N,t}(q, p) \leq C(1+t), \quad (3.7)$$

where $C > 0$ is a constant that depends on initial data $\iint dq dp (|q| + |p|^2) m_N(q, p)$.

Proof. First, from equation (2.16) in [15], we obtain that

$$\left\langle \psi_{N,t}, \frac{\mathcal{K}}{N} \psi_{N,t} \right\rangle = \frac{1}{(2\pi)^3} \iint dq dp |p|^2 m_{N,t}(q, p) + \hbar \int dq |\nabla f(q)|^2, \quad (3.8)$$

which implies that

$$\frac{1}{(2\pi)^3} \iint dq dp |p|^2 m_{N,t}(q, p) \leq \left\langle \psi_{N,t}, \frac{\mathcal{K}}{N} \psi_{N,t} \right\rangle \leq C, \quad (3.9)$$

where we use Lemma 3.2 in the last inequality.

Then, for the moment with respect to q , we obtain from (3.4) that

$$\begin{aligned} \frac{d}{dt} \iint dq dp |q| m_{N,t}(q, p) &= \iint dq dp |q| \partial_t m_{N,t}(q, p) \\ &= \iint dq dp |q| \left(-p \cdot \nabla_q m_{N,t}(q, p) + \frac{1}{(2\pi)^3} \nabla_p \right. \\ &\quad \cdot \iint dw_1 du_1 \iint dw_2 du_2 \iint dq_2 dp_2 \int_0^1 ds \end{aligned}$$

$$\begin{aligned} & \nabla V (su_2 + (1-s)w_1 - w_2) f_{q,p}^{\hbar}(w_1) \overline{f_{q,p}^{\hbar}(u_1)} f_{q_2,p_2}^{\hbar}(w_2) \overline{f_{q_2,p_2}^{\hbar}(u_2)} \\ & \langle a_{w_2} a_{w_1} \Psi_{N,t}, a_{u_2} a_{u_1} \Psi_{N,t} \rangle + \nabla_q \cdot \tilde{\mathcal{R}} \Big). \end{aligned} \quad (3.10)$$

By applying the divergence theorem first with respect to p and then with respect to q in (3.10), we obtain

$$\begin{aligned} \frac{d}{dt} \iint dq dp |q| m_{N,t}(q, p) &= \iint dq dp \frac{q}{|q|} \cdot p m_{N,t}(q, p) \\ &\leq \iint dq dp (1 + |p|^2) \cdot m_{N,t}(q, p), \end{aligned}$$

where we use Young's product inequality. Finally, taking the integral over t , we obtain the desired result. \square

3.3. Proof of Theorem 3.1

First, denoting $\varrho_{N,t}(q) := \int m_{N,t}(q, p) dp$, recall the Vlasov equation

$$\begin{aligned} & \partial_t m_{N,t}(q, p) + p \cdot \nabla_q m_{N,t}(q, p) \\ &= \frac{1}{(2\pi)^3} \nabla_p \cdot \int dq_2 \nabla V_N(q - q_2) \varrho_{N,t}(q_2) m_{N,t}(q, p) + \nabla_q \cdot \tilde{\mathcal{R}} + \nabla_p \cdot \mathcal{R} \\ &= \frac{1}{(2\pi)^3} (\nabla V_N * \varrho_{N,t})(q) \cdot \nabla_p m_{N,t}(q, p) + \nabla_q \cdot \tilde{\mathcal{R}} + \nabla_p \cdot \mathcal{R}, \end{aligned} \quad (3.11)$$

with

$$\begin{aligned} \tilde{\mathcal{R}} &:= \hbar \Im \langle \nabla_q a(f_{q,p}^{\hbar}) \Psi_{N,t}, a(f_{q,p}^{\hbar}) \Psi_{N,t} \rangle, \\ \mathcal{R} &:= (2\pi)^3 \iint dw_1 du_1 \iint dw_2 du_2 \iint dq_2 dp_2 \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes 2} \\ & \quad \left[\int_0^1 ds \nabla V_N(su_1 + (1-s)w_1 - w_2) \gamma_{N,t}^{(2)}(u_1, u_2; w_1, w_2) \right. \\ & \quad \left. - \nabla V_N(q - q_2) \gamma_{N,t}^{(1)}(u_1; w_1) \gamma_{N,t}^{(1)}(u_2; w_2) \right]. \end{aligned} \quad (3.12)$$

The main task is now reduced to taking limits in (3.12). In fact, Sect. 4 is devoted to deriving the estimates for the residuals. As a summary, it is proven in Sect. 4 that for $\varphi, \phi \in C_0^\infty(\mathbb{R}^3)$, there exists a positive constant K such that

$$\begin{aligned} \left| \iint dq dp \varphi(q) \phi(p) \nabla_q \cdot \tilde{\mathcal{R}}(q, p) \right| &\leq K \hbar^{\frac{1}{2} - \delta}, \\ \left| \iint dq dp \varphi(q) \phi(p) \nabla_p \cdot \mathcal{R}(q, p) \right| &\leq K \left(\hbar^{\frac{1}{4}(6\alpha_1 - 5) - 2\delta} + \hbar^{\frac{3}{2}(\alpha_2 - \frac{1}{2}) - 2\delta} \right), \end{aligned} \quad (3.13)$$

where $\frac{5}{6} < \alpha_1 < 1$, $\frac{1}{2} < \alpha_2 < 1$ and $0 < \delta \ll 1$. The estimates in (3.13) show that the residual terms converge to zero in the sense of distribution.

Next, we have the following result on weak convergent in L^1 :

Proposition 3.2 (Proposition 2.7 of [15]). *Let $\{m_{N,t}\}_{N \in \mathbb{N}}$ be the 1-particle Husimi measure; then, there exists a subsequence $\{m_{N_j,t}\}_{j \in \mathbb{N}}$ that converges weakly in $L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ to a function $(2\pi)^3 m_t$; i.e., for all $\Phi \in L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$, it holds that*

$$\frac{1}{(2\pi)^3} \iint dq dp m_{N_j,t}(q,p) \Phi(q,p) \rightarrow \iint dq dp m_t(q,p) \Phi(q,p),$$

as $j \rightarrow \infty$.

Remark 3.5. The proof for Lemma 3.2 is obtained by proving its uniform integrability and employing the Dunford–Pettis theorem for L^1 compactness.

Furthermore, to prove the convergence of the nonlinear term $(\nabla V_N * \rho_N) \cdot \nabla_p m_{N,t}$, we first show the strong convergence of $\nabla V_N * \rho_N$.

Lemma 3.3. *Let V_N be defined as (1.2). Then, for $t \in [0, \infty)$, there exists constant $C > 0$ independent on N such that*

$$\|\nabla V_N * \varrho_{N,t}\|_{L^\infty([0, \infty); W^{1, \frac{5}{3}}(\mathbb{R}^3))} \leq C, \quad (3.14)$$

$$\|\partial_t(\nabla V_N * \varrho_{N,t})\|_{L^\infty([0, \infty); W^{-1, \frac{15}{7}}(\mathbb{R}^3))} \leq C. \quad (3.15)$$

Proof. From Lemma 3.1 and Proposition 3.1, one finds that $m_{N,t}$ is uniformly bounded in $L^\infty([0, \infty); L^1(\mathbb{R}^3 \times \mathbb{R}^3)) \cap L^\infty([0, \infty); L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$ and $|p|^2 m_{N,t}(q,p)$ uniformly in $L^\infty([0, \infty); L^1(\mathbb{R}^3 \times \mathbb{R}^3))$, respectively. As a consequence, it holds that

$$\|\varrho_{N,t}\|_{L^\infty([0, \infty); L^{\frac{5}{3}}(\mathbb{R}^3))} \leq C.$$

Thus, $V_N * \varrho_{N,t} = V * \mathcal{G}_{\beta_N} * \varrho_{N,t}$ is uniformly bounded in $L^\infty([0, \infty); W^{2, \frac{5}{3}}(\mathbb{R}^3))$ due to the fact that V is the fundamental solution of the Poisson equation and

$$\|\mathcal{G}_{\beta_N} * \varrho_{N,t}\|_{L^\infty([0, \infty); L^{\frac{5}{3}}(\mathbb{R}^3))} \leq \|\mathcal{G}_{\beta_N}\|_{L^1(\mathbb{R}^3)} \cdot \|\varrho_{N,t}\|_{L^\infty([0, \infty); L^{\frac{5}{3}}(\mathbb{R}^3))}.$$

This implies the result (3.14) directly.

To prove (3.15), recall again the transport equation for $m_{N,t}$

$$\partial_t m_{N,t} + p \cdot \nabla_q m_{N,t} - \frac{1}{(2\pi)^3} (\nabla V_N * \varrho_{N,t}) \cdot \nabla_p m_{N,t} = \nabla_q \cdot \tilde{\mathcal{R}} + \nabla_p \cdot \mathcal{R}, \quad (3.16)$$

where $\varrho_{N,t}(q) := \int m_{N,t}(q,p) dp$. Taking the integral with respect to p ,

$$\partial_t \int dp m_{N,t}(q,p) + \nabla_q \cdot \int dp p m_{N,t}(q,p) = \nabla_q \cdot \int dp \tilde{\mathcal{R}}.$$

Next, by taking the convolution with ∇V_N , we obtain

$$\partial_t(\nabla V_N * \varrho_{N,t}) + \nabla_q \cdot (\nabla V_N \otimes_* J_{N,t}) = \nabla_q \cdot \left(\nabla V_N \otimes_* \int dp \tilde{\mathcal{R}} \right), \quad (3.17)$$

where $J_{N,t}(q) := \int dp p m_{N,t}(q, p)$, $(u \otimes_* v)_{ij} = u_i * v_j$ for $u, v \in \mathbb{R}^3$. Then, we observe that

$$\begin{aligned} & \left| \int dp p m_{N,t}(q, p) \right| \\ & \leq \left[\int dp |p|^2 m_{N,t} \right]^{\frac{1}{2}} \left[\int dp m_{N,t} \right]^{\frac{1}{2}} = \left[\int dp |p|^2 m_{N,t} \right]^{\frac{1}{2}} \varrho_{N,t}^{\frac{1}{2}}. \end{aligned} \quad (3.18)$$

Therefore, we have

$$\begin{aligned} \int dq |J_{N,t}(q)|^{\frac{5}{4}} &= \int dq \left| \int dp p m_{N,t}(q, p) \right|^{\frac{5}{4}} \\ &\leq \left[\iint dq dp |p|^2 m_{N,t} \right]^{\frac{5}{8}} \left[\int dq \varrho_{N,t}^{\frac{5}{8}} \right]^{\frac{5}{8}} \leq C, \end{aligned}$$

where we use Proposition 3.1 in the last inequality, yielding that $J_{N,t}$ is uniformly bounded in $L^\infty([0, \infty); L^{\frac{5}{4}}(\mathbb{R}^3))$. Then, for any test function $\varphi \in L^{\frac{15}{8}}(\mathbb{R}^3)$, we obtain for a.e. $t \geq 0$ that

$$\begin{aligned} & \left| \int dq |\varphi(q)| \left| \int dq_2 \nabla V_N(q - q_2) J_{N,t}(q_2) \right| \right| \\ & \leq \int dq |\varphi(q)| \left| \int dq_2 \nabla V(q - q_2) \mathcal{G}_{\beta_N} * J_{N,t}(q_2) \right| \\ & \leq \iint dq dq_2 |\varphi(q)| |\nabla V(q - q_2)| |\mathcal{G}_{\beta_N} * J_{N,t}(q_2)| \leq C \|\varphi\|_{L^{\frac{15}{8}}} \|\mathcal{G}_{\beta_N} * J_{N,t}\|_{L^{\frac{5}{4}}} \\ & \leq C \|\varphi\|_{L^{\frac{15}{8}}} \|\mathcal{G}_{\beta_N}\|_{L^1} \|J_{N,t}\|_{L^{\frac{5}{4}}} \leq C \|\varphi\|_{L^{\frac{15}{8}}}, \end{aligned}$$

where we use the Hardy–Littlewood–Sobolev inequality in the third inequality. This implies that, by using dual formulation of L^p norms and taking the supremum in $\varphi \in L^{\frac{15}{8}}$, one obtains that $\nabla V_N * J_{N,t}$ is uniformly bounded in $L^\infty([0, \infty); L^{\frac{15}{7}}(\mathbb{R}^3))$.

Therefore, focusing on the estimate of $\tilde{\mathcal{R}}$, we have

$$\begin{aligned} \left| \int dp \tilde{\mathcal{R}} \right| &\leq \hbar \int dp \left| \langle \nabla_q a(f_{q,p}^{\hbar}) \Psi_{N,t}, a(f_{q,p}^{\hbar}) \Psi_{N,t} \rangle \right| \\ &\leq \hbar \int dp \left| |\nabla_q a(f_{q,p}^{\hbar}) \Psi_{N,t}| \right| \left| |a(f_{q,p}^{\hbar}) \Psi_{N,t}| \right| \\ &\leq \left[\hbar^2 \int dp \langle \nabla_q a(f_{q,p}^{\hbar}) \Psi_{N,t}, \nabla_q a(f_{q,p}^{\hbar}) \Psi_{N,t} \rangle \right]^{\frac{1}{2}} \left[\int dp m_{N,t}(q, p) \right]^{\frac{1}{2}} \\ &= \left[\hbar^2 \int dp \langle \nabla_q a(f_{q,p}^{\hbar}) \Psi_{N,t}, \nabla_q a(f_{q,p}^{\hbar}) \Psi_{N,t} \rangle \right]^{\frac{1}{2}} \varrho_{N,t}^{\frac{1}{2}}. \end{aligned}$$

Note that since it holds that

$$\begin{aligned} & \hbar^2 \iint dq dp \langle \nabla_q a(f_{q,p}^{\hbar}) \Psi_{N,t}, \nabla_q a(f_{q,p}^{\hbar}) \Psi_{N,t} \rangle \\ &= \hbar^{\frac{1}{2}} \iint dq dp \iint dw du \nabla_q f \left(\frac{w - q}{\sqrt{\hbar}} \right) \nabla_q f \left(\frac{u - q}{\sqrt{\hbar}} \right) e^{\frac{1}{\hbar} p \cdot (w - u)} \langle \Psi_{N,t}, a_w^* a_u \Psi_{N,t} \rangle \end{aligned}$$

$$\begin{aligned}
 &= \hbar^{\frac{1}{2}+3} \iint dqdw \hbar^{-1} \left| \nabla f \left(\frac{w-q}{\sqrt{\hbar}} \right) \right|^2 \langle \Psi_{N,t}, a_w^* a_w \Psi_{N,t} \rangle \\
 &= \hbar^4 \int d\tilde{q} |\nabla f(\tilde{q})|^2 \langle \Psi_{N,t}, \mathcal{N} \Psi_{N,t} \rangle \\
 &\leq \hbar \|\nabla f\|_2^2,
 \end{aligned}$$

this implies

$$\begin{aligned}
 &\int dq \left| \int dp \tilde{\mathcal{R}} \right|^{\frac{5}{4}} \\
 &\leq \left[\hbar^2 \iint dqdp \langle \nabla_q a(f_{q,p}^{\hbar}) \Psi_{N,t}, \nabla_q a(f_{q,p}^{\hbar}) \Psi_{N,t} \rangle \right]^{\frac{5}{8}} \left(\int dq \varrho_{N,t}^{\frac{5}{3}} \right)^{\frac{3}{8}} \leq \hbar^{\frac{5}{4}} C.
 \end{aligned}$$

Repeating the calculation in (3.18), we have that $\nabla V_N * \int dp \tilde{\mathcal{R}}$ is uniformly bounded in $L^\infty([0, \infty); L^{\frac{15}{7}}(\mathbb{R}^3))$, which implies that $\nabla V_N * \int dp (\nabla_q \cdot \tilde{\mathcal{R}} + \nabla_p \cdot \mathcal{R})$ is uniformly bounded in $L^\infty([0, \infty); L^{\frac{15}{7}}(\mathbb{R}^3))$. Thus, from (3.17), we have that there exists a C which is independent of N such that

$$\|\partial_t(\nabla V_N * \varrho_{N,t})\|_{L^\infty([0, \infty); W^{-1, \frac{15}{7}}(\mathbb{R}^3))} \leq C.$$

This completes the proof for Lemma 3.3. \square

Finally, we conclude the proof of main theorem with the following compactness argument.

3.3.1. Compactness Argument. As in Sect. 3.1, the weak convergence of the linear terms in the Vlasov equation is obtained from Proposition 3.2. The following discussion is focused on the nonlinear term. Without loss of generality, assume that $\Phi(q, p) = \varphi(q)\phi(p)$ for any test functions $\varphi, \phi \in C_0^\infty(\mathbb{R}^3)$, and let the sphere B_ℓ with radius $\ell > 0$ be the support of φ . Due to the Sobolev's embedding theorem, we have

$$W^{1, \frac{5}{3}}(B_\ell) \hookrightarrow L^r(B_\ell) \hookrightarrow W^{-1, \frac{15}{7}}(B_\ell),$$

where $\frac{5}{4} \leq r < \frac{15}{4}$ and \hookrightarrow means the compact embedding. Recall the results in Lemma 3.3, we have

$$\|(\nabla V_N * \varrho_{N,t})\|_{L^\infty([0, \infty); W^{1, \frac{5}{3}}(\mathbb{R}^3))} + \|\partial_t(\nabla V_N * \varrho_{N,t})\|_{L^\infty([0, \infty); W^{-1, \frac{15}{7}}(\mathbb{R}^3))} \leq C.$$

Then, by Aubin–Lions lemma, we obtain that there exists a subsequence denoted also by $(\nabla V_N * \varrho_{N,t})_{N \in \mathbb{N}}$, and $h \in L^\infty([0, T]; L^r(B_\ell))$ such that, as $N \rightarrow \infty$, we have

$$\nabla V_N * \varrho_{N,t} \rightharpoonup h \quad \text{in } L^\infty([0, T]; L^r(\mathbb{R}^3)), \quad (3.19)$$

where $\frac{5}{4} \leq r < \frac{15}{4}$. The weak star convergence of $\varrho_{N,t} \rightharpoonup^* \varrho_t$ in $L^\infty([0, T]; L^{\frac{5}{3}}(\mathbb{R}^3))$, where $\varrho_t(q) := \int m_t(q, p) dp$, and the definition of V_N in (1.2) imply that the limit function h coincides with $\nabla V * \varrho_t$ a.e. in B_ℓ .

Now, to show the convergence to the Vlasov–Poisson equation, we first compute

$$\left| \int_0^T dt \iint dqdp \varphi(q) \nabla_p \phi(p) \cdot [(\nabla V_N * \varrho_{N,t})(q) m_{N,t}(q, p) - (\nabla V * \varrho_t) m_t(q, p)] \right|$$

$$\begin{aligned}
 &= \left| \int_0^T dt \iint dq dp \varphi(q) \nabla_p \phi(p) \cdot [(\nabla V_N * \varrho_{N,t})(q) - (\nabla V * \varrho_t)(q)] m_{N,t}(q, p) \right. \\
 &\quad \left. + \int_0^T dt \iint dq dp \varphi(q) \nabla_p \phi(p) \cdot (\nabla V * \varrho_t)(q) [m_{N,t}(q, p) - m_t(q, p)] \right| \\
 &\leq \left| \int_0^T dt \iint dq dp \varphi(q) \nabla_p \phi(p) \cdot [(\nabla V_N * \varrho_{N,t})(q) - (\nabla V * \varrho_t)(q)] m_{N,t}(q, p) \right| \\
 &\quad + \left| \int_0^T dt \iint dq dp \varphi(q) \nabla_p \phi(p) \cdot (\nabla V * \varrho_t)(q) [m_{N,t}(q, p) - m_t(q, p)] \right| \\
 &=: \mathcal{A}_1 + \mathcal{A}_2.
 \end{aligned}$$

Let us focus on the first term.

$$\begin{aligned}
 \mathcal{A}_1 &= \left| \int_0^T dt \int_{B_\ell} dq \varphi(q) [(\nabla V_N * \varrho_{N,t})(q) - (\nabla V * \varrho_t)(q)] \cdot \int dp \nabla_p \phi(p) m_{N,t}(q, p) \right| \\
 &\leq T \sup_{t \in [0, T]} \|(\nabla V_N * \varrho_{N,t}) - (\nabla V * \varrho_t)\|_{L^r(B_\ell)} \left\| \varphi \int dp \nabla_p \phi(p) m_{N,t}(\cdot, p) \right\|_{L^{r'}(B_\ell)} \\
 &\leq T \sup_{t \in [0, T]} \|(\nabla V_N * \varrho_{N,t}) - (\nabla V * \varrho_t)\|_{L^r(B_\ell)} \|\varphi\|_{L^{r'}(B_\ell)} \|\nabla_p \phi\|_{L^1(\mathbb{R}^3)} \\
 &\leq C_T \|(\nabla V_N * \varrho_{N,t}) - (\nabla V * \varrho_t)\|_{L^\infty([0, T]; L^r(B_\ell))},
 \end{aligned}$$

where we use the fact that $0 \leq m_{N,t} \leq 1$ almost everywhere. Taking the limit $N \rightarrow \infty$ on both sides, then we have

$$\lim_{N \rightarrow \infty} \mathcal{A}_1 = 0.$$

We focus now on \mathcal{A}_2 . We observe that since $\|m_{N,t}\|_{L^\infty}$ is uniformly bounded, it is implied that there is a subsequence still denoted by $(m_{N,t})_{N \in \mathbb{N}}$ such that $m_{N,t} \rightharpoonup^* m_t$ in $L^\infty((0, T); L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$ as $N \rightarrow \infty$. Since $\varphi(q) \nabla_p \phi(p) \cdot (\nabla V * \varrho_t)(q) \in L^1((0, T); L^1(\mathbb{R}^3 \times \mathbb{R}^3))$, we have $\lim_{N \rightarrow \infty} \mathcal{A}_2 = 0$. This completes the proof of Theorem 3.1. \square

4. Estimates of Residuals

To estimate the residuals outlined in (3.5), we first present the following important facts, which are used frequently in the proof: the \hbar -weighted Fourier transformation is given as

$$\int dy G(y) F(y) = \int dy G(y) \frac{1}{(2\pi\hbar)^3} \iint dp_2 dv \widehat{F}(v) e^{\frac{i}{\hbar} p_2 \cdot (y-v)}, \quad (4.1)$$

for any given function $F, G \in L^2(\mathbb{R}^3)$.

The results in [15] for the localized number operator and oscillation estimates are

Lemma 4.1. (Lemma 2.4 of [15]) *For $t \geq 0$, let $\Psi_{N,t} \in \mathcal{F}_a$ with $\|\Psi_{N,t}\| = 1$ and $R_1 > 0$ be the radius of a ball such that the volume is 1. Then, for all $1 \leq k \leq N$, we have*

$$\int \cdots \int (dq dx)^{\otimes k} \left(\prod_{n=1}^k \chi_{|x_n - q_n| \leq \sqrt{\hbar} R_1} \right) \gamma_{N,t}^{(k)}(x_1, \dots, x_k; x_1, \dots, x_k) \leq \hbar^{-\frac{3}{2}k},$$

(4.2)

where χ is a characteristic function.

Lemma 4.2. (Lemma 2.5 of [15]) For $g \in C_0^\infty(\mathbb{R}^3)$ and

$$\Omega_{\hbar} := \{x \in \mathbb{R}^3; \max_{1 \leq j \leq 3} |x_j| \leq \hbar^\alpha\}, \quad (4.3)$$

it holds that for every $\alpha \in (0, 1)$, $s \in \mathbb{N}$, and $x \in \mathbb{R}^3 \setminus \Omega_{\hbar}$,

$$\left| \int_{\mathbb{R}^3} dp e^{\frac{i}{\hbar} p \cdot x} g(p) \right| \leq c_1 \hbar^{(1-\alpha)s}, \quad (4.4)$$

where the constant c_1 depends on the compact support and the $W^{s,\infty}$ -norm of the test function g .

The estimate for the residual term $\widetilde{\mathcal{R}}$ given in (3.5) is obtained exactly as shown in [15], i.e.,

Proposition 4.1 (Proposition 2.4 of [15]). Suppose that $f \in H^1(\mathbb{R}^3)$, $\|f\|_{L^2} = 1$ and has compact support; then, we have the following bound for $\widetilde{\mathcal{R}}$ in (3.4); i.e., for an arbitrarily small $\delta > 0$, there exists $s(\delta) > 0$ such that the following estimate holds for any test function $\varphi, \phi \in C_0^\infty(\mathbb{R}^3)$

$$\left| \iint dq dp \varphi(q) \phi(p) \nabla_q \cdot \widetilde{\mathcal{R}}(q, p) \right| \leq c_2 \hbar^{\frac{1}{2}-\delta},$$

where the constant c_2 depends on $\|\nabla\varphi\|_{L^\infty}$ and $\|\phi\|_{W^{s,\infty}}$.

For the residual term \mathcal{R} , we insert the terms

$$\pm \nabla V_N(q - q_2) \gamma_{N,t}^{(2)}(u_1, u_2; w_1, w_2),$$

and write into a sum $\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2$, where

$$\begin{aligned} \mathcal{R}_1 &:= (2\pi)^3 \iint dw_1 du_1 \iint dw_2 du_2 \iint dq_2 dp_2 \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes 2} \\ &\quad \left[\int_0^1 ds \nabla V_N(su_1 + (1-s)w_1 - w_2) - \nabla V_N(q - q_2) \right] \gamma_{N,t}^{(2)}(u_1, u_2; w_1, w_2), \\ \mathcal{R}_2 &:= (2\pi)^3 \iint dw_1 du_1 \iint dw_2 du_2 \iint dq_2 dp_2 \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes 2} \nabla V_N(q - q_2) \\ &\quad \left[\gamma_{N,t}^{(2)}(u_1, u_2; w_1, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1) \gamma_{N,t}^{(1)}(u_2; w_2) \right]. \end{aligned} \quad (4.5)$$

Note that in (4.5), \mathcal{R}_1 represents the semiclassical limit part, and \mathcal{R}_2 represents the mean-field limit.

As a preparation for the estimates of the residual term \mathcal{R} , we present the following estimate for regularized Coulomb potential:

Lemma 4.3. Let V_N be the regularized Coulomb potential given in (1.2), then it holds that

$$\|\nabla V_N\|_{L^\infty} \leq C \beta_N^{-2}. \quad (4.6)$$

Proof. Let \widehat{V}_N be the Fourier transform of V_N . Recall that for $V_N = |\cdot|^{-1} * \mathcal{G}_{\beta_N}$, [45, Theorem 5.9] yields

$$\widehat{V}_N(p) = \frac{C}{|p|^2} e^{-\left(p \frac{\beta_N}{2}\right)^2},$$

for some positive constant C . Then, by inverse Fourier transform, we get

$$\begin{aligned} |\nabla V_N(x)| &= \frac{1}{(2\pi)^{3/2}} \left| \nabla_x \int dp e^{ip \cdot x} \widehat{V}_N(p) \right| \\ &\leq C \int dp |p| |\widehat{V}_N(p)| \\ &= C \int dp \frac{1}{|p|} e^{-\left(p \frac{\beta_N}{2}\right)^2} \\ &\leq C \beta_N^{-2}, \end{aligned}$$

where we use the spherical coordination in the last inequality. \square

In the following, we treat the semiclassical and mean-field residual terms, i.e., \mathcal{R}_1 and \mathcal{R}_2 , by using the truncated radius β_N , the oscillation estimate, the cutoff number operator and the kinetic operator estimates outlined in Sect. 3.2.

4.1. Estimate for the Semiclassical Residual Term \mathcal{R}_1

In this subsection, we present in full detail the estimate for the semiclassical residue.

Proposition 4.2. *Let $\varphi, \phi \in C_0^\infty(\mathbb{R}^3)$. Then, for $\frac{5}{6} < \alpha_1 < 1$, $0 < \delta < \frac{1}{8}(6\alpha_1 - 5)$, and $s = \left\lceil \frac{3(2\alpha_1 + 1)}{4(1 - \alpha_1)} \right\rceil$, we have*

$$\left| \iint dq dp \varphi(q) \phi(p) \nabla_p \cdot \mathcal{R}_1(q, p) \right| \leq \widetilde{C} \hbar^{\frac{1}{4}(6\alpha_1 - 5) - 2\delta}, \quad (4.7)$$

where the constant \widetilde{C} depends on $\|\varphi\|_{W^{1,\infty}}$, $\|\nabla \phi\|_{L^1 \cap W^{s,\infty}}$, $\text{supp } \phi$, $\|f\|_{L^\infty \cap H^1}$, and $\text{supp } f$.

Proof. Recall from (4.5) that we have

$$\begin{aligned} \mathcal{R}_1 &:= (2\pi)^3 \iint dw_1 du_1 \iint dw_2 du_2 \iint dq_2 dp_2 \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes 2} \\ &\quad \left[\int_0^1 ds \nabla V_N(su_1 + (1-s)w_1 - w_2) - \nabla V_N(q - q_2) \right] \gamma_{N,t}^{(2)}(u_1, u_2; w_1, w_2). \end{aligned} \quad (4.8)$$

For $\varphi, \phi \in C_0^\infty(\mathbb{R}^3)$, we have

$$\begin{aligned} &\left| \iint dq dp \varphi(q) \phi(p) \nabla_p \cdot \mathcal{R}_1(q, p) \right| \\ &= (2\pi)^3 \left| \int \cdot \int (dq dp)^{\otimes 2} \varphi(q) \nabla_p \phi(p) \cdot \iint dw_1 du_1 \iint dw_2 du_2 \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes 2} \right. \\ &\quad \left. \left[\int_0^1 ds \nabla V_N(su_1 + (1-s)w_1 - w_2) - \nabla V_N(q - q_2) \right] \gamma_{N,t}^{(2)}(u_1, u_2; w_1, w_2) \right| \end{aligned}$$

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$$\begin{aligned}
 &= (2\pi)^6 \left| \iint (dq)^{\otimes 2} dp \iiint dw_1 du_1 dw_2 \varphi(q) \nabla \phi(p) \cdot f\left(\frac{w_1 - q}{\sqrt{\hbar}}\right) f\left(\frac{u_1 - q}{\sqrt{\hbar}}\right) \right. \\
 &\quad \left. e^{\frac{i}{\hbar} p \cdot (w_1 - u_1)} \left| f\left(\frac{w_2 - q_2}{\sqrt{\hbar}}\right) \right|^2 \left[\int_0^1 ds \nabla V_N(su_1 + (1-s)w_1 - w_2) - \nabla V_N(q - q_2) \right] \right. \\
 &\quad \left. \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) \right|,
 \end{aligned}$$

where we apply the fact that $(2\pi\hbar)^3 \delta_x(y) = \int e^{\frac{i}{\hbar} p \cdot (x-y)} dp$. Then, inserting $\pm \nabla V_N(q - w_2)$, by the triangle inequality, we have

$$\begin{aligned}
 &\leq (2\pi)^6 \left| \iint (dq)^{\otimes 2} dp \iiint dw_1 du_1 dw_2 \varphi(q) \nabla \phi(p) \cdot f\left(\frac{w_1 - q}{\sqrt{\hbar}}\right) f\left(\frac{u_1 - q}{\sqrt{\hbar}}\right) \right. \\
 &\quad \left. e^{\frac{i}{\hbar} p \cdot (w_1 - u_1)} \left| f\left(\frac{w_2 - q_2}{\sqrt{\hbar}}\right) \right|^2 \left[\nabla_{w_2} \int_0^1 ds V_N(su_1 + (1-s)w_1 - w_2) - V_N(q - w_2) \right] \right. \\
 &\quad \left. \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) \right| \\
 &+ (2\pi)^6 \left| \iint (dq)^{\otimes 2} dp \iiint dw_1 du_1 dw_2 \varphi(q) \nabla \phi(p) \cdot f\left(\frac{w_1 - q}{\sqrt{\hbar}}\right) f\left(\frac{u_1 - q}{\sqrt{\hbar}}\right) \right. \\
 &\quad \left. e^{\frac{i}{\hbar} p \cdot (w_1 - u_1)} \left| f\left(\frac{w_2 - q_2}{\sqrt{\hbar}}\right) \right|^2 \nabla_q [V_N(q - w_2) - V_N(q - q_2)] \right. \\
 &\quad \left. \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) \right| \\
 &= (2\pi)^6 \left| \iint (dq)^{\otimes 2} dp \iiint dw_1 du_1 dw_2 \varphi(q) \nabla \phi(p) \cdot f\left(\frac{w_1 - q}{\sqrt{\hbar}}\right) f\left(\frac{u_1 - q}{\sqrt{\hbar}}\right) \right. \\
 &\quad \left. e^{\frac{i}{\hbar} p \cdot (w_1 - u_1)} \left| f\left(\frac{w_2 - q_2}{\sqrt{\hbar}}\right) \right|^2 \left[\int_0^1 ds V_N(su_1 + (1-s)w_1 - w_2) - V_N(q - w_2) \right] \right. \\
 &\quad \left. \nabla_{w_2} \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) \right| \\
 &+ (2\pi)^6 \left| \iint (dq)^{\otimes 2} dp \iiint dw_1 du_1 dw_2 \varphi(q) \nabla \phi(p) \cdot f\left(\frac{w_1 - q}{\sqrt{\hbar}}\right) f\left(\frac{u_1 - q}{\sqrt{\hbar}}\right) \right. \\
 &\quad \left. e^{\frac{i}{\hbar} p \cdot (w_1 - u_1)} \nabla_{w_2} \left| f\left(\frac{w_2 - q_2}{\sqrt{\hbar}}\right) \right|^2 \left[\int_0^1 ds V_N(su_1 + (1-s)w_1 - w_2) - V_N(q - w_2) \right] \right. \\
 &\quad \left. \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) \right| \\
 &+ (2\pi)^6 \left| \iint (dq)^{\otimes 2} dp \iiint dw_1 du_1 dw_2 \nabla \phi(p) \cdot \nabla_q \left(\varphi(q) f\left(\frac{w_1 - q}{\sqrt{\hbar}}\right) f\left(\frac{u_1 - q}{\sqrt{\hbar}}\right) \right) \right. \\
 &\quad \left. e^{\frac{i}{\hbar} p \cdot (w_1 - u_1)} \left| f\left(\frac{w_2 - q_2}{\sqrt{\hbar}}\right) \right|^2 [V_N(q - w_2) - V_N(q - q_2)] \right. \\
 &\quad \left. \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) \right|, \\
 &=: I_1 + J_1 + K_1
 \end{aligned}$$

where we use integration by parts in the second to last equality.

Before advancing, we observe that by splitting the integral with respect to momentum space Ω_{\hbar} and Ω_{\hbar}^c as defined in (4.3), for constant C_1 depending

on $\|\nabla\phi\|_{L^1\cap W^{s,\infty}}$ and $\text{supp}\phi$, we have

$$\begin{aligned} \left| \int dp \nabla\phi(p) e^{\frac{i}{\hbar}P\cdot(w-u)} \right| &= \left| \int dp (\chi_{(w_1-u_1)\in\Omega_{\hbar}} + \chi_{(w_1-u_1)\in\Omega_{\hbar}^c}) \phi(p) e^{\frac{i}{\hbar}P\cdot(w-u)} \right| \\ &\leq C_1 \left(\chi_{(w_1-u_1)\in\Omega_{\hbar}} + \hbar^{(1-\alpha_1)s} \right), \end{aligned} \quad (4.9)$$

where we use (4.4) in the last inequality.

Now, we want to separately estimate the terms I_1 and J_1 . We begin by estimating I_1 . Recall that

$$\begin{aligned} I_1 &= (2\pi)^6 \hbar^{\frac{3}{2}} \left| \iint dq dp \iiint dw_1 du_1 dw_2 \varphi(q) \nabla\phi(p) \cdot f\left(\frac{w_1-q}{\sqrt{\hbar}}\right) f\left(\frac{u_1-q}{\sqrt{\hbar}}\right) \right. \\ &\quad \left. e^{\frac{i}{\hbar}P\cdot(w_1-u_1)} \left(\int d\tilde{q}_2 |f(\tilde{q}_2)|^2 \right) \left[\int ds V_N(su_1 + (1-s)w_1 - w_2) - V_N(q - w_2) \right] \right. \\ &\quad \left. \nabla_{w_2} \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) \right|. \end{aligned}$$

By using (4.9) we have,

$$\begin{aligned} I_1 &\leq \|\nabla V_N\|_{L^\infty} C_1 \hbar^{\frac{3}{2}} \int dq |\varphi(q)| \iint dw_1 du_1 dw_2 (\chi_{(w_1-u_1)\in\Omega_{\hbar}} \\ &\quad + \hbar^{(1-\alpha_1)s}) \left| f\left(\frac{w_1-q}{\sqrt{\hbar}}\right) f\left(\frac{u_1-q}{\sqrt{\hbar}}\right) \right| \\ &\quad (|u_1-q| + |w_1-q|) |\nabla_{w_2} \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2)| \\ &\leq \|\nabla V_N\|_{L^\infty} C_1 \hbar^{\frac{3}{2}} \int dq |\varphi(q)| \iint dw_1 du_1 (\chi_{(w_1-u_1)\in\Omega_{\hbar}} \\ &\quad + \hbar^{(1-\alpha_1)s}) \left| f\left(\frac{w_1-q}{\sqrt{\hbar}}\right) f\left(\frac{u_1-q}{\sqrt{\hbar}}\right) \right| \\ &\quad (|u_1-q| + |w_1-q|) \int dw_2 |\nabla_{w_2} \gamma_{N,t}^{(2)}(w_1, w_2; u_1, w_2)| \\ &= \|\nabla V_N\|_{L^\infty} C_1 \hbar^{\frac{3}{2}} \int dq |\varphi(q)| \iint dw_1 du_1 (\chi_{(w_1-u_1)\in\Omega_{\hbar}} \\ &\quad + \hbar^{(1-\alpha_1)s}) \left| f\left(\frac{w_1-q}{\sqrt{\hbar}}\right) f\left(\frac{u_1-q}{\sqrt{\hbar}}\right) \right| \\ &\quad (|u_1-q| + |w_1-q|) \int dw_2 |\nabla_{w_2} \langle a_{w_2} a_{w_1} \Psi_{N,t}, a_{w_2} a_{u_1} \Psi_{N,t} \rangle| \\ &\leq \|\nabla V_N\|_{L^\infty} C_1 \hbar^{\frac{3}{2}} \int dq |\varphi(q)| \iint dw_1 du_1 (\chi_{(w_1-u_1)\in\Omega_{\hbar}} \\ &\quad + \hbar^{(1-\alpha_1)s}) \left| f\left(\frac{w_1-q}{\sqrt{\hbar}}\right) f\left(\frac{u_1-q}{\sqrt{\hbar}}\right) \right| \\ &\quad (|u_1-q| + |w_1-q|) \int dw_2 \left[\|\nabla_{w_2} a_{w_2} a_{w_1} \Psi_{N,t}\| \|a_{w_2} a_{u_1} \Psi_{N,t}\| \right. \\ &\quad \left. + \|a_{w_2} a_{w_1} \Psi_{N,t}\| \|\nabla_{w_2} a_{w_2} a_{u_1} \Psi_{N,t}\| \right] \\ &\leq \|\nabla V_N\|_{L^\infty} C_1 \hbar^{\frac{3}{2}} \int dq |\varphi(q)| \iint dw_1 du_1 (\chi_{(w_1-u_1)\in\Omega_{\hbar}} \\ &\quad + \hbar^{(1-\alpha_1)s}) \left| f\left(\frac{w_1-q}{\sqrt{\hbar}}\right) f\left(\frac{u_1-q}{\sqrt{\hbar}}\right) \right| \end{aligned}$$

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$$\begin{aligned}
 & (|u_1 - q| + |w_1 - q|) \left[\left(\int dw_2 \|\nabla_{w_2} a_{w_2} a_{w_1} \Psi_{N,t}\|^2 \right)^{\frac{1}{2}} \left(\int dw_2 \|a_{w_2} a_{u_1} \Psi_{N,t}^2\| \right)^{\frac{1}{2}} \right. \\
 & \left. + \left(\int dw_2 \|a_{w_2} a_{w_1} \Psi_{N,t}\|^2 \right)^{\frac{1}{2}} \left(\int dw_2 \|\nabla_{w_2} a_{w_2} a_{u_1} \Psi_{N,t}\|^2 \right)^{\frac{1}{2}} \right] \\
 & =: C_1 \|\nabla V_N\|_{L^\infty} \left[i_{1,1} + i_{1,2} \right],
 \end{aligned}$$

Before we continue, we observe that from the definition of the kinetic energy operator \mathcal{K} and number operator \mathcal{N} , we have

$$\begin{aligned}
 & \int dq |\varphi(q)| \left[\iint dw_1 du_1 \chi_{|w_1 - q| \leq R_1 \sqrt{\hbar}} \chi_{|u_1 - q| \leq R_1 \sqrt{\hbar}} \right. \\
 & \left. \left(\int dw_2 \|\nabla_{w_2} a_{w_2} a_{w_1} \Psi_{N,t}\|^2 \right) \left(\int dw_2 \|a_{w_2} a_{u_1} \Psi_{N,t}\|^2 \right) \right]^{\frac{1}{2}} \\
 & = 2\hbar^{-1} \int dq |\varphi(q)| \left[\iint dw_1 du_1 \chi_{|w_1 - q| \leq R_1 \sqrt{\hbar}} \chi_{|u_1 - q| \leq R_1 \sqrt{\hbar}} \right. \\
 & \left. \langle \Psi_{N,t}, a_{w_1}^* \mathcal{K} a_{w_1} \Psi_{N,t} \rangle \langle \Psi_{N,t}, a_{u_1}^* \mathcal{N} a_{u_1} \Psi_{N,t} \rangle \right]^{\frac{1}{2}} \\
 & = 2\hbar^{-1} \int dq |\varphi(q)| \left[\int dw_1 \chi_{|w_1 - q| \leq R_1 \sqrt{\hbar}} \langle \Psi_{N,t}, \mathcal{K}(a_{w_1}^* a_{w_1} - 1) \Psi_{N,t} \rangle \right. \\
 & \left. \int du_1 \chi_{|u_1 - q| \leq R_1 \sqrt{\hbar}} \langle \Psi_{N,t}, (\mathcal{N} - 1) a_{u_1}^* a_{u_1} \Psi_{N,t} \rangle \right]^{\frac{1}{2}} \\
 & \leq 2\hbar^{-1} \left(\iint dq dw_1 |\varphi(q)| \chi_{|w_1 - q| \leq R_1 \sqrt{\hbar}} \langle \Psi_{N,t}, \mathcal{K} a_{w_1}^* a_{w_1} \Psi_{N,t} \rangle \right)^{\frac{1}{2}} \\
 & \quad \left(\iint dq du_1 |\varphi(q)| \chi_{|u_1 - q| \leq R_1 \sqrt{\hbar}} \langle \Psi_{N,t}, \mathcal{N} a_{u_1}^* a_{u_1} \Psi_{N,t} \rangle \right)^{\frac{1}{2}} \\
 & \leq C_2 \hbar^{-4 - \frac{3}{2}},
 \end{aligned} \tag{4.10}$$

where in the last step we use a direct outcome of (4.2) and (3.2), i.e.,

$$\begin{aligned}
 & \iint dq dx \chi_{|x - q| \leq R_1 \sqrt{\hbar}} |\varphi(q)| \langle \Psi_{N,t}, a_x^* \mathcal{K} a_x \Psi_{N,t} \rangle \\
 & = \iint dq dx \chi_{|x - q| \leq R_1 \sqrt{\hbar}} |\varphi(q)| \langle \Psi_{N,t}, \mathcal{K}(a_x^* a_x - 1) \Psi_{N,t} \rangle \\
 & \leq \iint dq dx \chi_{|x - q| \leq R_1 \sqrt{\hbar}} |\varphi(q)| \langle \Psi_{N,t}, \mathcal{K} a_x^* a_x \Psi_{N,t} \rangle \\
 & = \left\langle \Psi_{N,t}, \mathcal{K} \iint dq dx \chi_{|x - q| \leq R_1 \sqrt{\hbar}} |\varphi(q)| a_x^* a_x \Psi_{N,t} \right\rangle \\
 & \leq C_2 \hbar^{-\frac{3}{2}} \langle \Psi_{N,t}, \mathcal{K} \Psi_{N,t} \rangle \\
 & \leq C_2 \hbar^{-\frac{3}{2} - 3}.
 \end{aligned} \tag{4.11}$$

In the above estimate, C_2 is a constant depends on $\|\varphi\|_{L^\infty}$ and $\text{supp } f$.

To continue, we apply the Hölder inequality to $i_{1,1}$ with respect to the terms w_1 and u_1 ,

$$\begin{aligned}
i_{1,1} &\leq \hbar^{\frac{3}{2}} \int dq |\varphi(q)| \left[\iint dw_1 du_1 \chi_{(w_1-u_1) \in \Omega_h} |f \right. \\
&\quad \left. \left(\frac{w_1-q}{\sqrt{\hbar}} \right) f \left(\frac{u_1-q}{\sqrt{\hbar}} \right) \right|^2 (|u_1-q| + |w_1-q|)^2 \Big]^{\frac{1}{2}} \\
&\quad \left[\iint dw_1 du_1 \chi_{|w_1-q| \leq R_1 \sqrt{\hbar}} \chi_{|u_1-q| \leq R_1 \sqrt{\hbar}} \right. \\
&\quad \left. \left(\int dw_2 \|\nabla_{w_2} a_{w_2} a_{w_1} \Psi_{N,t}\|^2 \right) \left(\int dw_2 \|a_{w_2} a_{u_1} \Psi_{N,t}\|^2 \right) \right. \\
&\quad \left. + \iint dw_1 du_1 \chi_{|w_1-q| \leq R_1 \sqrt{\hbar}} \chi_{|u_1-q| \leq R_1 \sqrt{\hbar}} \right. \\
&\quad \left. \left(\int dw_2 \|a_{w_2} a_{w_1} \Psi_{N,t}\|^2 \right) \left(\int dw_2 \|\nabla_{w_2} a_{w_2} a_{u_1} \Psi_{N,t}\|^2 \right) \right]^{\frac{1}{2}} \\
&= \hbar^{\frac{3}{2}} \left[\hbar^3 \iint d\tilde{w}_1 d\tilde{u}_1 \chi_{|\tilde{w}_1-\tilde{u}_1| \leq \hbar^{\alpha_1+\frac{1}{2}}} |f(\tilde{w}_1) f(\tilde{u})|^2 \hbar (|\tilde{u}_1| + |\tilde{w}_1|)^2 \right]^{\frac{1}{2}} \\
&\quad \int dq |\varphi(q)| \left[2 \iint dw_1 du_1 \chi_{|w_1-q| \leq R_1 \sqrt{\hbar}} \chi_{|u_1-q| \leq R_1 \sqrt{\hbar}} \right. \\
&\quad \left. \left(\int dw_2 \|\nabla_{w_2} a_{w_2} a_{w_1} \Psi_{N,t}\|^2 \right) \left(\int dw_2 \|a_{w_2} a_{u_1} \Psi_{N,t}\|^2 \right) \right]^{\frac{1}{2}}.
\end{aligned}$$

Then by using (4.10), the estimate goes further

$$\begin{aligned}
&\leq C_2 \hbar^{3-4-\frac{3}{2}+\frac{1}{2}} \left[\iint d\tilde{w}_1 d\tilde{u}_1 \chi_{|\tilde{w}_1-\tilde{u}_1| \leq R_1 \hbar^{\alpha_1+\frac{1}{2}}} |f(\tilde{w}_1) f(\tilde{u})|^2 (|\tilde{u}_1| + |\tilde{w}_1|)^2 \right]^{\frac{1}{2}} \\
&\leq C_2 \hbar^{-2} \left[\iint d\tilde{w}_1 d\tilde{u}_1 \chi_{|\tilde{w}_1-\tilde{u}_1| \leq R_1 \hbar^{\alpha_1+\frac{1}{2}}} |f(\tilde{w}_1) f(\tilde{u})|^2 |\tilde{u}_1 + \tilde{w}_1|^2 \right]^{\frac{1}{2}} \\
&\leq C_3 \hbar^{-2+\frac{3}{2}(\alpha_1+\frac{1}{2})} = C_3 \hbar^{\frac{6\alpha_1-5}{4}},
\end{aligned}$$

where C_3 depends on $\|\varphi\|_{L^\infty}$, $\|f\|_{L^\infty \cap L^2}$, $\text{supp } f$ and we use the following estimate in the last inequality above:

$$\begin{aligned}
&\int dw |f(w)|^2 \int du \chi_{|w-u| \leq R_1 \hbar^{\alpha_1+\frac{1}{2}}} |f(u)|^2 \\
&\leq \sup_u |f(u)|^2 \int dw |f(w)|^2 \int du \chi_{|w-u| \leq R_1 \hbar^{\alpha_1+\frac{1}{2}}} \quad (4.12) \\
&\leq \|f\|_{L^\infty} \|f\|_{L^2} \hbar^{3(\alpha_1+\frac{1}{2})},
\end{aligned}$$

where the fixed radius R_1 arises from the compactness assumption of f .

With steps similar to those for $i_{1,1}$, we have

$$\begin{aligned} i_{1,2} &\leq C_3 \hbar^{-2} \hbar^{(1-\alpha_1)s} \left[\iint d\tilde{w}_1 d\tilde{u}_1 |f(\tilde{w}_1) f(\tilde{u})|^2 (|\tilde{u}_1| + |\tilde{w}_1|)^2 \right]^{\frac{1}{2}} \\ &\leq C_4 \hbar^{-2+(1-\alpha_1)s}, \end{aligned}$$

where the constant C_4 depends on $\|\varphi\|_\infty$, $\|f\|_{L^\infty \cap L^2}$, and $\text{supp } f$. To balance the order between $i_{1,1}$ and $i_{1,2}$, s is chosen to be

$$s = \left\lfloor \frac{3(2\alpha_1 + 1)}{4(1 - \alpha_1)} \right\rfloor,$$

for $\alpha_1 \in (\frac{5}{6}, 1)$. Therefore, we have

$$I_1 \leq \tilde{C} \|\nabla V_N\|_{L^\infty} \hbar^{\frac{6\alpha_1-5}{4}}. \quad (4.13)$$

To estimate J_1 , we compute

$$\begin{aligned} J_1 &= (2\pi)^6 \left| \iint dq dp \iiint dw_1 du_1 dw_2 \varphi(q) \nabla \phi(p) \cdot f \right. \\ &\quad \left. \left(\frac{w_1 - q}{\sqrt{\hbar}} \right) f \left(\frac{u_1 - q}{\sqrt{\hbar}} \right) e^{\frac{i}{\hbar} p \cdot (w_1 - u_1)} \left[2\hbar \int d\tilde{q}_2 f(\tilde{q}_2) \nabla f(\tilde{q}_2) \right] \right. \\ &\quad \left. \left[\int_0^1 ds V_N(su_1 + (1-s)w_1 - w_2) - V_N(q - w_2) \right] \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) \right| \\ &\leq (2\pi)^6 \hbar \int dq |\varphi(q)| \iiint dw_1 du_1 dw_2 \left| \int dp (\chi_{(w_1 - u_1) \in \Omega_h} \right. \\ &\quad \left. + \chi_{(w_1 - u_1) \in \Omega_h^c}) \nabla \phi(p) \cdot e^{\frac{i}{\hbar} p \cdot (w_1 - u_1)} \right| \left| \int d\tilde{q}_2 |\nabla f(\tilde{q}_2)| \right| \left| f \left(\frac{w_1 - q}{\sqrt{\hbar}} \right) f \right. \\ &\quad \left. \left(\frac{u_1 - q}{\sqrt{\hbar}} \right) \left[\int_0^1 ds V_N(su_1 + (1-s)w_1 - w_2) - V_N(q - w_2) \right] \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) \right| \\ &\leq C_1 \hbar \int dq |\varphi(q)| \iiint dw_1 du_1 dw_2 \left(\chi_{(w_1 - u_1) \in \Omega_h} + \hbar^{(1-\alpha_1)s} \right) \left| \right. \\ &\quad \left. f \left(\frac{w_1 - q}{\sqrt{\hbar}} \right) f \left(\frac{u_1 - q}{\sqrt{\hbar}} \right) \left[\int_0^1 ds |V_N(su_1 + (1-s)w_1 - w_2) - V_N(q - w_2)| \right] \right| \\ &\quad \left| \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) \right| \\ &\leq C_1 \|\nabla V_N\|_{L^\infty} \hbar \int dq |\varphi(q)| \iint dw_1 du_1 (\chi_{(w_1 - u_1) \in \Omega_h} \\ &\quad + \hbar^{(1-\alpha_1)s}) \left| f \left(\frac{w_1 - q}{\sqrt{\hbar}} \right) f \left(\frac{u_1 - q}{\sqrt{\hbar}} \right) \right| \\ &\quad (|u_1 - q| + |w_1 - q|) \int dw_2 \|a_{w_2} a_{w_1} \Psi_{N,t}\|_2 \|a_{w_2} a_{u_1} \Psi_{N,t}\|_2 \\ &\leq C_1 \|\nabla V_N\|_{L^\infty} \hbar \int dq |\varphi(q)| \iint dw_1 du_1 (\chi_{(w_1 - u_1) \in \Omega_h} \\ &\quad + \hbar^{(1-\alpha_1)s}) \left| f \left(\frac{w_1 - q}{\sqrt{\hbar}} \right) f \left(\frac{u_1 - q}{\sqrt{\hbar}} \right) \right| \\ &\quad (|u_1 - q| + |w_1 - q|) \left(\int dw_2 \|a_{w_2} a_{w_1} \Psi_{N,t}\|_2^2 \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} & \left(\int dw_2 \|a_{w_2} a_{u_1} \Psi_{N,t}\|_2^2 \right)^{\frac{1}{2}} \\ & =: C_1 \|\nabla V_N\|_{L^\infty} [j_{1,1} + j_{1,2}]. \end{aligned} \quad (4.14)$$

As in part I_1 , we separately analyze $j_{1,1}$ and $j_{1,2}$.

$$\begin{aligned} j_{1,1} &= \hbar \int dq |\varphi(q)| \iint dw_1 du_1 \chi_{(w_1 - u_1) \in \Omega_{\hbar}} \left| f \left(\frac{w_1 - q}{\sqrt{\hbar}} \right) f \left(\frac{u_1 - q}{\sqrt{\hbar}} \right) \right| (|u_1 - q| + |w_1 - q|) \\ & \quad \chi_{|u_1 - q| \leq R_1 \sqrt{\hbar}} \chi_{|w_1 - q| \leq R_1 \sqrt{\hbar}} \left(\int dw_2 \langle \Psi_{N,t}, a_{w_1}^* a_{w_2}^* a_{w_2} a_{w_1} \Psi_{N,t} \rangle \right)^{\frac{1}{2}} \\ & \quad \left(\int dw_2 \langle \Psi_{N,t}, a_{u_1}^* a_{w_2}^* a_{w_2} a_{u_1} \Psi_{N,t} \rangle \right)^{\frac{1}{2}} \\ & \leq \hbar \int dq |\varphi(q)| \left(\iint dw_1 du_1 \chi_{(w_1 - u_1) \in \Omega_{\hbar}} \left| f \left(\frac{w_1 - q}{\sqrt{\hbar}} \right) f \left(\frac{u_1 - q}{\sqrt{\hbar}} \right) \right|^2 (|u_1 - q| + |w_1 - q|)^2 \right)^{\frac{1}{2}} \\ & \quad \left(\iint dw_1 dw_2 \chi_{|w_1 - q| \leq R_1 \sqrt{\hbar}} \langle \Psi_{N,t}, a_{w_1}^* a_{w_2}^* a_{w_2} a_{w_1} \Psi_{N,t} \rangle \right) \\ & = \hbar \left(\hbar^3 \iint d\tilde{w}_1 d\tilde{u}_1 \chi_{(\tilde{w}_1 - \tilde{u}_1) \in \Omega_{\hbar}} |f(\tilde{w}) f(\tilde{u})|^2 \hbar \right. \\ & \quad \left. (|\tilde{u}| + |\tilde{w}|)^2 \right)^{\frac{1}{2}} \iint dq dw_1 \chi_{|w_1 - q| \leq R_1 \sqrt{\hbar}} |\varphi(q)| \\ & \quad \langle \Psi_{N,t}, a_{w_1}^* \mathcal{N} a_{w_1} \Psi_{N,t} \rangle \\ & \leq \|\varphi\|_{L^\infty} \hbar^{1+2-3-\frac{3}{2}} \left(\iint d\tilde{w}_1 d\tilde{u}_1 \chi_{(\tilde{w}_1 - \tilde{u}_1) \in \Omega_{\hbar}} |f(\tilde{w}) f(\tilde{u})|^2 (|\tilde{u}| + |\tilde{w}|)^2 \right)^{\frac{1}{2}} \\ & \leq C_4 \hbar^{\frac{3}{2}(\alpha_1 - \frac{1}{2})}, \end{aligned}$$

where we use (4.12) in the last inequality.

On the other hand, from the definition of $j_{1,2}$ in (4.14), we get

$$\begin{aligned} j_{1,2} &= \hbar \int dq |\varphi(q)| \iint dw_1 du_1 \hbar^{(1-\alpha_1)s} \left| f \left(\frac{w_1 - q}{\sqrt{\hbar}} \right) f \left(\frac{u_1 - q}{\sqrt{\hbar}} \right) \right| (|u_1 - q| + |w_1 - q|) \\ & \quad \chi_{|u_1 - q| \leq R_1 \sqrt{\hbar}} \chi_{|w_1 - q| \leq R_1 \sqrt{\hbar}} \left(\int dw_2 \|a_{w_2} a_{w_1} \Psi_{N,t}\|_2^2 \right)^{\frac{1}{2}} \left(\int dw_2 \|a_{w_2} a_{u_1} \Psi_{N,t}\|_2^2 \right)^{\frac{1}{2}} \\ & \leq \hbar^{1+(1-\alpha_1)s} \left(\iint dw_1 du_1 \left| f \left(\frac{w_1 - q}{\sqrt{\hbar}} \right) f \left(\frac{u_1 - q}{\sqrt{\hbar}} \right) \right|^2 (|u_1 - q| + |w_1 - q|)^2 \right)^{\frac{1}{2}} \\ & \quad \int dq |\varphi(q)| \iint dw_1 dw_2 \chi_{|w_1 - q| \leq R_1 \sqrt{\hbar}} \langle \Psi_{N,t}, a_{w_1}^* a_{w_2}^* a_{w_2} a_{w_1} \Psi_{N,t} \rangle \end{aligned}$$

$$\begin{aligned} &\leq \hbar^{1+(1-\alpha_1)s-3-\frac{3}{2}} \left(\hbar^4 \iint d\tilde{w}_1 d\tilde{u}_1 \chi_{|\tilde{w}_1-\tilde{u}_1| \leq \hbar^{\alpha_1+\frac{1}{2}}} |f(\tilde{w}_1)f(\tilde{u})|^2 (|\tilde{u}_1| + |\tilde{w}_1|)^2 \right)^{\frac{1}{2}} \\ &\leq C_4 \hbar^{(1-\alpha_1)s-\frac{3}{2}}. \end{aligned}$$

To obtain the same order for $j_{1,1}$ and $j_{1,2}$, we can choose

$$s = \left\lceil \frac{3(2\alpha_1 + 1)}{4(1 - \alpha_1)} \right\rceil.$$

Thus, for $\alpha_1 \in (\frac{1}{2}, 1)$, we have

$$J_1 \leq \tilde{C} \|\nabla V_N\|_{L^\infty} \hbar^{\frac{3}{2}(\alpha_1 - \frac{1}{2})}. \quad (4.15)$$

Now, we want to estimate K_1 .

$$\begin{aligned} K_1 &= (2\pi)^6 \left| \iiint (dq)^{\otimes 2} dp \iiint dw_1 du_1 dw_2 \nabla \phi(p) \cdot \nabla_q \left(\varphi(q) f\left(\frac{w_1 - q}{\sqrt{\hbar}}\right) f\left(\frac{u_1 - q}{\sqrt{\hbar}}\right) \right) \right. \\ &\quad \left. e^{\frac{i}{\hbar} p \cdot (w_1 - u_1)} \left| f\left(\frac{w_2 - q_2}{\sqrt{\hbar}}\right) \right|^2 [V_N(q - w_2) - V_N(q - q_2)] \right. \\ &\quad \left. \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) \right| \\ &= (2\pi)^6 \left| \iiint (dq)^{\otimes 2} dp \iiint dw_1 du_1 dw_2 \nabla \phi(p) \cdot \left[\nabla \varphi(q) f\left(\frac{w_1 - q}{\sqrt{\hbar}}\right) f\left(\frac{u_1 - q}{\sqrt{\hbar}}\right) \right. \right. \\ &\quad \left. \left. - \hbar^{-\frac{1}{2}} \varphi(q) \nabla f\left(\frac{w_1 - q}{\sqrt{\hbar}}\right) f\left(\frac{u_1 - q}{\sqrt{\hbar}}\right) - \hbar^{-\frac{1}{2}} \varphi(q) f\left(\frac{w_1 - q}{\sqrt{\hbar}}\right) \nabla f\left(\frac{u_1 - q}{\sqrt{\hbar}}\right) \right] \right. \\ &\quad \left. e^{\frac{i}{\hbar} p \cdot (w_1 - u_1)} \left| f\left(\frac{w_2 - q_2}{\sqrt{\hbar}}\right) \right|^2 [V_N(q - w_2) - V_N(q - q_2)] \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) \right| \\ &=: k_{1,1} + k_{1,2} + k_{1,3}. \end{aligned}$$

Note that, for any $\varphi \in C_0^\infty$ and $f \in W_0^{1,2}$, the term $k_{1,1}$ is $\sqrt{\hbar}$ -order higher than $k_{1,2}$ and $k_{1,3}$. Moreover, the estimate of the terms $k_{1,2}$ and $k_{1,3}$ is the same when doing change of variables in the final steps. Therefore, we focus only on the term $k_{1,2}$.

$$\begin{aligned} k_{1,2} &\leq (2\pi)^6 \hbar^{-\frac{1}{2}} \left| \iint (dq)^{\otimes 2} \iiint dw_1 du_1 dw_2 \left| \int dp \nabla \phi(p) e^{\frac{i}{\hbar} p \cdot (w_1 - u_1)} \right| \right. \\ &\quad \left| \varphi(q) \nabla f\left(\frac{w_1 - q}{\sqrt{\hbar}}\right) f\left(\frac{u_1 - q}{\sqrt{\hbar}}\right) \right| \\ &\quad \left| f\left(\frac{w_2 - q_2}{\sqrt{\hbar}}\right) \right|^2 |V_N(q - w_2) - V_N(q - q_2)| |\gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2)| \right. \\ &= (2\pi)^6 \hbar^{-\frac{1}{2}} \int dq \iiint dw_1 du_1 dw_2 \left| \int dp (\chi_{(w_1 - u_1) \in \Omega_\hbar} \right. \\ &\quad \left. + \chi_{(w_1 - u_1) \in \Omega_\hbar^c}) \nabla \phi(p) \cdot e^{\frac{i}{\hbar} p \cdot (w_1 - u_1)} \right| \\ &\quad \left| \varphi(q) \nabla f\left(\frac{w_1 - q}{\sqrt{\hbar}}\right) f\left(\frac{u_1 - q}{\sqrt{\hbar}}\right) \right| \left| \int \hbar^{\frac{3}{2}} d\tilde{q}_2 \right. \\ &\quad \left. |f(\tilde{q}_2)|^2 |V_N(q - w_2) - V_N(q - \sqrt{\hbar}\tilde{q}_2 - w_2)| \right. \\ &\quad \left. |\gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2)| \right| \end{aligned}$$

$$\begin{aligned}
&\leq C_1 \|\nabla V_N\|_{L^\infty} \hbar^{1+\frac{1}{2}} \int dq \iint dw_1 du_1 \\
&\quad \left| \chi_{(w_1-u_1) \in \Omega_h} + \hbar^{(1-\alpha_1)s} \right| \\
&\quad \left| \varphi(q) \nabla f \left(\frac{w_1-q}{\sqrt{\hbar}} \right) f \left(\frac{u_1-q}{\sqrt{\hbar}} \right) \right| \\
&\quad \left(\int d\tilde{q}_2 |\tilde{q}_2| |f(\tilde{q}_2)|^2 \right) \int dw_2 |\gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2)| \\
&\leq C_1 \|\nabla V_N\|_{L^\infty} \hbar^{1+\frac{1}{2}} \int dq |\varphi(q)| \iint dw_1 du_1 (\chi_{(w_1-u_1) \leq \hbar^{\alpha_1}} \\
&\quad + \hbar^{(1-\alpha_1)s}) \left| \nabla f \left(\frac{w_1-q}{\sqrt{\hbar}} \right) f \left(\frac{u_1-q}{\sqrt{\hbar}} \right) \right| \\
&\quad \chi_{|w_1-q| \leq R_1 \sqrt{\hbar}} \chi_{|u_1-q| \leq R_1 \sqrt{\hbar}} \\
&\quad \left(\int dw_2 \|a_{w_2} a_{w_1} \Psi_{N,t}\|^2 \right)^{\frac{1}{2}} \left(\int dw_2 \|a_{w_2} a_{u_1} \Psi_{N,t}\|^2 \right)^{\frac{1}{2}} \\
&=: C_1 \|\nabla V_N\|_{L^\infty} [\tilde{k}_1 + \tilde{k}_2,].
\end{aligned}$$

Using the Hölder inequality with respect to w_1 and u_1 , we obtain that

$$\begin{aligned}
\tilde{k}_1 &\leq \hbar^{1+\frac{1}{2}} \int dq |\varphi(q)| \left[\iint dw_1 du_1 \chi_{(w_1-u_1) \leq \hbar^{\alpha_1}} \left| \nabla f \left(\frac{w_1-q}{\sqrt{\hbar}} \right) f \left(\frac{u_1-q}{\sqrt{\hbar}} \right) \right|^2 \right]^{\frac{1}{2}} \\
&\quad \left[\iint dw_1 du_1 \chi_{|w_1-q| \leq R_1 \sqrt{\hbar}} \chi_{|u_1-q| \leq R_1 \sqrt{\hbar}} \right. \\
&\quad \left. \left(\int dw_2 \|a_{w_2} a_{w_1} \Psi_{N,t}\|^2 \right) \left(\int dw_2 \|a_{w_2} a_{u_1} \Psi_{N,t}\|^2 \right) \right]^{\frac{1}{2}} \\
&= \hbar^{\frac{3}{2}} \left[\hbar^3 \iint d\tilde{w}_1 d\tilde{u}_1 \chi_{(\tilde{w}_1-\tilde{u}_1) \leq \hbar^{\alpha_1+\frac{1}{2}}} \left| \nabla f(\tilde{w}_1) f(\tilde{u}_1) \right|^2 \right]^{\frac{1}{2}} \\
&\quad \int dq |\varphi(q)| \left[\int dw_1 \chi_{|w_1-q| \leq R_1 \sqrt{\hbar}} \langle \Psi_{N,t}, a_{w_1}^* \mathcal{N} a_{w_1} \Psi_{N,t} \rangle \right] \\
&\leq C_5 \hbar^{\frac{3}{2}(\alpha_1+\frac{1}{2})-\frac{3}{2}} = C_5 \hbar^{\frac{3(2\alpha_1-1)}{4}},
\end{aligned}$$

where we use (4.2) in the last inequality and C_5 depending on $\|f\|_{L^\infty}$, $\|\nabla f\|_{L^2}$, $\text{supp } f$, and $\|\varphi\|_{L^\infty}$.

Similarly, to calculate $j_{1,2}$,

$$\begin{aligned}
\tilde{k}_2 &\leq \hbar^{1+\frac{1}{2}} \int dq |\varphi(q)| \iint dw_1 du_1 \hbar^{(1-\alpha_1)s} \left| \nabla f \left(\frac{w_1-q}{\sqrt{\hbar}} \right) f \left(\frac{u_1-q}{\sqrt{\hbar}} \right) \right| \\
&\quad \chi_{|w_1-q| \leq R_1 \sqrt{\hbar}} \chi_{|u_1-q| \leq R_1 \sqrt{\hbar}} \left(\int dw_2 \|a_{w_2} a_{w_1} \Psi_{N,t}\|^2 \right)^{\frac{1}{2}} \\
&\quad \left(\int dw_2 \|a_{w_2} a_{u_1} \Psi_{N,t}\|^2 \right)^{\frac{1}{2}} \\
&\leq C_5 \hbar^{(1-\alpha_1)s-\frac{3}{2}},
\end{aligned}$$

where s is chosen as

$$s = \left\lceil \frac{3(2\alpha_1 + 1)}{4(1 - \alpha_1)} \right\rceil,$$

for $\alpha_1 \in (\frac{1}{2}, 1)$. Thus,

$$K_1 \leq \tilde{C} \|\nabla V_N\|_{L^\infty} \hbar^{\frac{3(2\alpha_1 - 1)}{4}}, \quad (4.16)$$

where we recall that the constant \tilde{C} depends on $\|\varphi\|_{W^{1,\infty}}$, $\|\nabla\phi\|_{L^1 \cap W^{s,\infty}}$, $\text{supp } \phi$, $\|f\|_{L^\infty \cap H^1}$, and $\text{supp } f$.

Therefore, in summary, we have

$$\left| \iint dq dp \varphi(q) \phi(p) \nabla_p \cdot \mathcal{R}_1(q, p) \right| \leq \tilde{C} \|\nabla V_N\|_{L^\infty} \hbar^{\frac{1}{4}(6\alpha_1 - 5)} \leq \tilde{C} \beta_N^{-2} \hbar^{\frac{1}{4}(6\alpha_1 - 5)},$$

where we use (4.6) in the second inequality.

Setting $\beta_N = \hbar^\delta$ for $0 < \delta < \frac{1}{8}(6\alpha_1 - 5)$, we obtain the desired result. \square

4.2. Estimate for the Mean-Field Residual Term \mathcal{R}_2

Proposition 4.3. *Let $\varphi, \phi \in C_0^\infty(\mathbb{R}^3)$. Then, for $\frac{1}{2} < \alpha_2 < 1$, $0 < \delta < \frac{3}{4}(\alpha_2 - \frac{1}{2})$, and $s = \left\lceil \frac{3(2\alpha_2 + 1)}{4(1 - \alpha_2)} \right\rceil$, we have*

$$\left| \iint dq dp \varphi(q) \phi(p) \nabla_p \cdot \mathcal{R}_2(q, p) \right| \leq \tilde{C} \hbar^{\frac{3}{2}(\alpha_2 - \frac{1}{2}) - 2\delta} \quad (4.17)$$

where the constant \tilde{C} depends on $\|\varphi\|_\infty$, $\|\nabla\phi\|_{L^1 \cap W^{s,\infty}}$, $\|f\|_{L^\infty \cap H^1}$, $\text{supp } f$, and $\text{supp } \phi$.

Proof. Recall that from (4.5), we have

$$\begin{aligned} \mathcal{R}_2 := & (2\pi)^3 \iint dw_1 du_1 \iint dw_2 du_2 \iint dq_2 dp_2 \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes 2} \nabla V_N(q - q_2) \\ & \left[\gamma_{N,t}^{(2)}(u_1, u_2; w_1, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1) \gamma_{N,t}^{(1)}(u_2; w_2) \right]. \end{aligned} \quad (4.18)$$

Then, we have

$$\begin{aligned} & \left| \iint dq dp \varphi(q) \phi(p) \nabla_p \cdot \mathcal{R}_2 \right| \\ &= \left| \int \dots \int (dq dp)^{\otimes 2} (dw du)^{\otimes 2} \varphi(q) \nabla \phi(p) \cdot \left(f_{q,p}^{\hbar}(w) \overline{f_{q,p}^{\hbar}(u)} \right)^{\otimes 2} \nabla V_N(q - q_2) \right. \\ & \quad \left. \left[\gamma_{N,t}^{(2)}(u_1, u_2; w_1, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1) \gamma_{N,t}^{(1)}(u_2; w_2) \right] \right| \\ &= \hbar^{-3} \left| \int \dots \int (dq dp)^{\otimes 2} (dw du)^{\otimes 2} \varphi(q) \nabla \phi(p) \cdot \right. \\ & \quad \left(f \left(\frac{w - q}{\sqrt{\hbar}} \right) f \left(\frac{u - q}{\sqrt{\hbar}} \right) e^{\frac{i}{\hbar} p \cdot (w - u)} \right)^{\otimes 2} \nabla V_N(q - q_2) \\ & \quad \left. \left[\gamma_{N,t}^{(2)}(u_1, u_2; w_1, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1) \gamma_{N,t}^{(1)}(u_2; w_2) \right] \right| \end{aligned}$$

$$\begin{aligned}
&= \hbar^{-3} \left| \int \cdots \int (dq dw du)^{\otimes 2} \left(f \left(\frac{w-q}{\sqrt{\hbar}} \right) f \right. \right. \\
&\quad \left. \left. \left(\frac{u-q}{\sqrt{\hbar}} \right) \right)^{\otimes 2} \left(\int dp \varphi(q) \nabla \phi(p) \cdot e^{\frac{i}{\hbar} p \cdot (w_1 - u_1)} \right) \right. \\
&\quad \left. \nabla V_N(q - q_2) \left(\int dp_2 e^{\frac{i}{\hbar} p_2 \cdot (w_2 - u_2)} \right) \right. \\
&\quad \left. \left[\gamma_{N,t}^{(2)}(u_1, u_2; w_1, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1) \gamma_{N,t}^{(1)}(u_2; w_2) \right] \right| \\
&= (2\pi)^3 \left| \int \cdots \int (dq)^{\otimes 2} dw_1 du_1 dw_2 f \left(\frac{w_1 - q}{\sqrt{\hbar}} \right) f \left(\frac{u_1 - q}{\sqrt{\hbar}} \right) \left| f \left(\frac{w_2 - q_2}{\sqrt{\hbar}} \right) \right|^2 \right. \\
&\quad \left. \left(\int dp \varphi(q) \nabla \phi(p) \cdot e^{\frac{i}{\hbar} p \cdot (w_1 - u_1)} \right) \right. \\
&\quad \left. \nabla V_N(q - q_2) \left[\gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1) \gamma_{N,t}^{(1)}(w_2; w_2) \right] \right|,
\end{aligned}$$

where we use the weighted Dirac-delta function in the last equality; i.e.,

$$\frac{1}{(2\pi\hbar)^3} \int dp_2 e^{\frac{i}{\hbar} p_2 \cdot (w_2 - u_2)} = \delta_{w_2}(u_2).$$

Now, splitting the domains of w_1 and u_1 into two, namely with the characteristic functions $\chi_{(w_1 - u_1) \in \Omega_\hbar}$ and $\chi_{(w_1 - u_1) \in \Omega_\hbar^c}$ as defined in (4.3), we have

$$\begin{aligned}
&\leq (2\pi)^3 \left| \iint (dq)^{\otimes 2} \varphi(q) \iiint dw_1 du_1 dw_2 f \right. \\
&\quad \left. \left(\frac{w_1 - q}{\sqrt{\hbar}} \right) f \left(\frac{u_1 - q}{\sqrt{\hbar}} \right) \left| f \left(\frac{w_2 - q_2}{\sqrt{\hbar}} \right) \right|^2 \right. \\
&\quad \left. \left(\int dp \chi_{(w_1 - u_1) \in \Omega_\hbar} e^{\frac{i}{\hbar} p \cdot (w_1 - u_1)} \nabla \phi(p) \right) \cdot \nabla V_N(q - q_2) \right. \\
&\quad \left. \left[\gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1) \gamma_{N,t}^{(1)}(w_2; w_2) \right] \right| \\
&+ (2\pi)^3 \left| \iint (dq)^{\otimes 2} \varphi(q) \iiint dw_1 du_1 dw_2 f \left(\frac{w_1 - q}{\sqrt{\hbar}} \right) f \right. \\
&\quad \left. \left(\frac{u_1 - q}{\sqrt{\hbar}} \right) \left| f \left(\frac{w_2 - q_2}{\sqrt{\hbar}} \right) \right|^2 \right. \\
&\quad \left. \left(\int dp \chi_{(w_1 - u_1) \in \Omega_\hbar^c} e^{\frac{i}{\hbar} p \cdot (w_1 - u_1)} \nabla \phi(p) \right) \cdot \nabla V_N(q - q_2) \right. \\
&\quad \left. \left[\gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1) \gamma_{N,t}^{(1)}(w_2; w_2) \right] \right| \\
&=: \mathbf{I}_2 + \mathbf{J}_2.
\end{aligned}$$

Without the loss of generality, we let $\Phi(q, p) = \varphi(q)\phi(p)$. First, considering the term J_2 ,

$$\begin{aligned} J_2 &= (2\pi)^3 \left| \iint dq dq_2 \varphi(q) \iiint dw_1 du_1 dw_2 f \right. \\ &\quad \left. \left(\frac{w_1 - q}{\sqrt{\hbar}} \right) f \left(\frac{u_1 - q}{\sqrt{\hbar}} \right) \left| f \left(\frac{w_2 - q_2}{\sqrt{\hbar}} \right) \right|^2 \right. \\ &\quad \left. \left(\int dp \chi_{(w_1 - u_1) \in \Omega_{\hbar}^c} \nabla \phi(p) e^{\frac{i}{\hbar} p \cdot (w_1 - u_1)} \right) \cdot \nabla V_N(q - q_2) \right. \\ &\quad \left. \left(\gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1) \gamma_{N,t}^{(1)}(w_2; w_2) \right) \right|. \end{aligned}$$

By the change of variable $\sqrt{\hbar} \tilde{q}_2 = w_2 - q_2$, we obtain

$$\begin{aligned} &= (2\pi)^3 \left| \int dq \varphi(q) \iiint dw_1 du_1 dw_2 f \left(\frac{w_1 - q}{\sqrt{\hbar}} \right) f \right. \\ &\quad \left. \left(\frac{u_1 - q}{\sqrt{\hbar}} \right) \left(\hbar^{\frac{3}{2}} \int d\tilde{q}_2 |f(\tilde{q}_2)|^2 \right) \nabla V_N(q - w_2 + \sqrt{\hbar} \tilde{q}_2) \right. \\ &\quad \left. \left(\int dp \chi_{(w_1 - u_1) \in \Omega_{\hbar}^c} \nabla \phi(p) e^{\frac{i}{\hbar} p \cdot (w_1 - u_1)} \right) \right. \\ &\quad \left. \left(\gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1) \gamma_{N,t}^{(1)}(w_2; w_2) \right) \right| \\ &\leq C \|\nabla V_N\|_{L^\infty} \hbar^{\frac{3}{2}} \int dq |\varphi(q)| \iiint dw_1 du_1 dw_2 |f \\ &\quad \left(\frac{w_1 - q}{\sqrt{\hbar}} \right) f \left(\frac{u_1 - q}{\sqrt{\hbar}} \right) \left| \int dp \chi_{(w_1 - u_1) \in \Omega_{\hbar}^c} \nabla \phi(p) e^{\frac{i}{\hbar} p \cdot (w_1 - u_1)} \right| \\ &\quad \left. \left(\gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1) \gamma_{N,t}^{(1)}(w_2; w_2) \right) \right| \\ &\leq C \|\nabla V_N\|_{L^\infty} \hbar^{\frac{3}{2}} \int dq |\varphi(q)| \iint dw_1 du_1 \left| f \left(\frac{w_1 - q}{\sqrt{\hbar}} \right) \right. \\ &\quad \left. f \left(\frac{u_1 - q}{\sqrt{\hbar}} \right) \chi_{|w_1 - u_1| \leq 2R_1 \sqrt{\hbar}} \right. \\ &\quad \left. \int dw_2 \left| \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1) \gamma_{N,t}^{(1)}(w_2; w_2) \right| \right. \\ &\quad \left. \left| \int dp \chi_{(w_1 - u_1) \in \Omega_{\hbar}^c} \nabla \phi(p) e^{\frac{i}{\hbar} p \cdot (w_1 - u_1)} \right| \right|. \end{aligned}$$

Recall again from Lemma 4.2 that we have

$$\begin{aligned} &\left| \int dp \chi_{(w_1 - u_1) \in \Omega_{\hbar}^c} e^{\frac{i}{\hbar} p \cdot (w_1 - u_1)} \nabla \phi(p) \right| \\ &\leq \|\nabla \phi\|_{W^{s, \infty}} \hbar^{(1 - \alpha_2)s}, \end{aligned}$$

for s to be chosen later. Then, we obtain

$$\begin{aligned} J_2 &\leq C \|\nabla\phi\|_{W^{s,\infty}} \|\nabla V_N\|_{L^\infty} \hbar^{\frac{3}{2}+(1-\alpha_2)s} \int dq |\varphi(q)| \iint dw_1 du_1 \left| f \left(\frac{w_1 - q}{\sqrt{\hbar}} \right) f \left(\frac{u_1 - q}{\sqrt{\hbar}} \right) \right| \\ &\quad \text{Tr}^{(1)} \left| \gamma_{N,t}^{(2)} - \gamma_{N,t}^{(1)} \otimes \gamma_{N,t}^{(1)} \right| (u_1; w_1) \chi_{|w_1 - u_1| \leq 2R_1 \sqrt{\hbar}}, \end{aligned}$$

The Hölder inequality yields

$$\begin{aligned} J_2 &\leq C \|\nabla\phi\|_{W^{s,\infty}} \|\nabla V_N\|_{L^\infty} \hbar^{\frac{3}{2}+(1-\alpha_2)s} \int dq |\varphi(q)| \\ &\quad \left(\iint dw_1 du_1 \chi_{|w_1 - u_1| \leq 2R_1 \sqrt{\hbar}} \left| f \left(\frac{w_1 - q}{\sqrt{\hbar}} \right) f \left(\frac{u_1 - q}{\sqrt{\hbar}} \right) \right|^2 \right)^{\frac{1}{2}} \\ &\quad \left(\iint dw_1 du_1 \left[\text{Tr}^{(1)} \left| \gamma_{N,t}^{(2)} - \gamma_{N,t}^{(1)} \right. \right. \right. \\ &\quad \left. \left. \left. \otimes \gamma_{N,t}^{(1)} \right| (u_1; w_1) \right]^2 \right)^{\frac{1}{2}} \\ &= C \|\varphi\|_{L^\infty} \|\nabla\phi\|_{W^{s,\infty}} \|\nabla V_N\|_{L^\infty} \hbar^{\frac{3}{2}+(1-\alpha_2)s} \\ &\quad \left(\hbar^3 \iint d\tilde{w}_1 d\tilde{u}_1 \chi_{|\tilde{w}_1 - \tilde{u}_1| \leq 2R_1} |f(\tilde{w}) f(\tilde{u})|^2 \right)^{\frac{1}{2}} \\ &\quad \int dq \left(\iint dw_1 du_1 \left[\text{Tr}^{(1)} \left| \gamma_{N,t}^{(2)} - \gamma_{N,t}^{(1)} \right. \right. \right. \\ &\quad \left. \left. \left. \otimes \gamma_{N,t}^{(1)} \right| (u_1; w_1) \right]^2 \chi_{|w_1 - q| \leq R_1 \sqrt{\hbar}} \right)^{\frac{1}{2}} \\ &\leq C \|\varphi\|_{L^\infty} \|\nabla\phi\|_{W^{s,\infty}} \|\nabla V_N\|_{L^\infty} \hbar^{3+(1-\alpha_2)s} \\ &\quad \left(\iint d\tilde{w}_1 d\tilde{u}_1 |f(\tilde{w}) f(\tilde{u})|^2 \right)^{\frac{1}{2}} \\ &\quad \hbar^{\frac{3}{2}} \int d\tilde{q}_1 \chi_{|\tilde{q}_1| \leq R_1} \left(\iint dw_1 du_1 \left[\text{Tr}^{(1)} \left| \gamma_{N,t}^{(2)} - \gamma_{N,t}^{(1)} \right. \right. \right. \\ &\quad \left. \left. \left. \otimes \gamma_{N,t}^{(1)} \right| (u_1; w_1) \right]^2 \right)^{\frac{1}{2}} \\ &\leq \tilde{C} \|\nabla V_N\|_{L^\infty} \hbar^{3+(1-\alpha_2)s+\frac{3}{2}} \\ &\quad \left(\iint dw_1 du_1 \left[\text{Tr}^{(1)} \left| \gamma_{N,t}^{(2)} - \gamma_{N,t}^{(1)} \right. \otimes \gamma_{N,t}^{(1)} \right| (u_1; w_1) \right]^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where we denote

$$\text{Tr}^{(1)} \left| \gamma_{N,t}^{(2)} - \gamma_{N,t}^{(1)} \otimes \gamma_{N,t}^{(1)} \right| (u_1; w_1)$$

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$$= \int dw_2 |\gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1) \gamma_{N,t}^{(1)}(w_2; w_2)|.$$

Thus, we have

$$\begin{aligned} J_2 &\leq \tilde{C} \|\nabla V_N\|_{L^\infty} \hbar^{3+(1-\alpha_2)s+\frac{3}{2}} \left(\iint dw_1 du_1 \right. \\ &\quad \left. \left[\text{Tr}^{(1)} \left| \gamma_{N,t}^{(2)} - \gamma_{N,t}^{(1)} \otimes \gamma_{N,t}^{(1)} \right| (u_1; w_1) \right]^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Now, we focus on $I_{2,1}$

$$\begin{aligned} I_{2,1} &= (2\pi)^3 \left| \iint (dq)^{\otimes 2} \varphi(q) \iiint dw_1 du_1 dw_2 f\left(\frac{w_1 - q}{\sqrt{\hbar}}\right) f\left(\frac{u_1 - q}{\sqrt{\hbar}}\right) \left| f\left(\frac{w_2 - q_2}{\sqrt{\hbar}}\right) \right|^2 \right. \\ &\quad \left. \left(\int dp \chi_{(w_1 - u_1) \in \Omega_{\hbar}} e^{\frac{i}{\hbar} p \cdot (w_1 - u_1)} \nabla \phi(p) \right) \cdot \nabla V_N(q - q_2) \right. \\ &\quad \left. \left[\gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1) \gamma_{N,t}^{(1)}(w_2; w_2) \right] \right|. \end{aligned}$$

We observe that

$$\left| \int dp e^{\frac{i}{\hbar} p \cdot (w_1 - u_1)} \nabla \phi(p) \right| \leq \|\nabla \phi\|_{L^1}.$$

Then, we obtain the following estimate:

$$\begin{aligned} I_2 &\leq C \|\nabla \phi\|_{L^1} \|\nabla V_N\|_{L^\infty} \int dq |\varphi(q)| \left(\iint dw_1 du_1 \chi_{|w_1 - u_1| \leq \hbar^{\alpha_2}} \chi_{|w_1 - u_1| \leq 2R_1 \sqrt{\hbar}} \right. \\ &\quad \left. \left| f\left(\frac{w_1 - q}{\sqrt{\hbar}}\right) f\left(\frac{u_1 - q}{\sqrt{\hbar}}\right) \right|^2 \right)^{\frac{1}{2}} \\ &\leq \hbar^{\frac{3}{2}} \int d\tilde{q}_2 |f(\tilde{q}_2)|^2 \left(\iint dw_1 du_1 \left[\text{Tr}^{(1)} \right. \right. \\ &\quad \left. \left. \left| \gamma_{N,t}^{(2)} - \gamma_{N,t}^{(1)} \otimes \gamma_{N,t}^{(1)} \right| (u_1; w_1) \right]^2 \chi_{|w_1 - q| \leq R_1 \sqrt{\hbar}} \right)^{\frac{1}{2}} \\ &\leq C \|\varphi\|_{L^\infty} \|\nabla \phi\|_{L^1} \|\nabla V_N\|_{L^\infty} \hbar^{\frac{3}{2}} \\ &\quad \left(\hbar^3 \iint d\tilde{w}_1 d\tilde{u}_1 \chi_{|\tilde{w}_1 - \tilde{u}_1| \leq \hbar^{\alpha_2 + \frac{1}{2}}} \chi_{|\tilde{w}_1 - \tilde{u}_1| \leq 2R_1} \right. \\ &\quad \left. |f(\tilde{w}_1) f(\tilde{u}_1)|^2 \right)^{\frac{1}{2}} \\ &\leq \tilde{C} \|\nabla V_N\|_{L^\infty} \hbar^3 \int dq \left(\iint dw_1 du_1 \left[\text{Tr}^{(1)} \left| \gamma_{N,t}^{(2)} - \gamma_{N,t}^{(1)} \otimes \gamma_{N,t}^{(1)} \right| (u_1; w_1) \right]^2 \chi_{|w_1 - q| \leq R_1 \sqrt{\hbar}} \right)^{\frac{1}{2}} \\ &\leq \tilde{C} \|\nabla V_N\|_{L^\infty} \hbar^3 \end{aligned}$$

$$\left(\iint d\tilde{w}_1 d\tilde{u}_1 \chi_{|\tilde{w}_1 - \tilde{u}_1| \leq \hbar^{\alpha_2 + \frac{1}{2}}} |f(\tilde{w}_1) f(\tilde{u}_1)|^2 \right)^{\frac{1}{2}} \hbar^{\frac{3}{2}} \int d\tilde{q}_1 \chi_{|\tilde{q}_1| \leq R_1} \left(\iint dw_1 du_1 \left[\text{Tr}^{(1)} \left| \gamma_{N,t}^{(2)} - \gamma_{N,t}^{(1)} \otimes \gamma_{N,t}^{(1)} \right| (u_1; w_1) \right]^2 \right)^{\frac{1}{2}}.$$

From (4.12), we have

$$I_2 \leq \tilde{C} \|\nabla V_N\|_{L^\infty} \hbar^{3 + \frac{3}{2}(\alpha_2 + \frac{1}{2}) + \frac{3}{2}} \left(\iint dw_1 du_1 \left[\text{Tr}^{(1)} \left| \gamma_{N,t}^{(2)} - \gamma_{N,t}^{(1)} \otimes \gamma_{N,t}^{(1)} \right| (u_1; w_1) \right]^2 \right)^{\frac{1}{2}}.$$

To balance the order between I_2 and J_2 , s is chosen to be

$$s = \left\lceil \frac{3(2\alpha_2 + 1)}{4(1 - \alpha_2)} \right\rceil,$$

for $\alpha_2 \in [0, 1)$. Therefore, we have

$$\begin{aligned} & \left| \iint dq dp \varphi(q) \phi(p) \nabla_p \cdot \mathcal{R}_2 \right| \leq I_2 + J_2 \\ & \leq \tilde{C} \|\nabla V_N\|_{L^\infty} \hbar^{3 + \frac{3}{2}(\alpha_2 + \frac{1}{2}) + \frac{3}{2}} \left(\iint dw_1 du_1 \left[\text{Tr}^{(1)} \left| \gamma_{N,t}^{(2)} - \gamma_{N,t}^{(1)} \otimes \gamma_{N,t}^{(1)} \right| (u_1; w_1) \right]^2 \right)^{\frac{1}{2}} \\ & \leq \tilde{C} \beta_N^{-2} \hbar^{3 + \frac{3}{2}(\alpha_2 + \frac{1}{2}) + \frac{3}{2}} N^2. \end{aligned}$$

Setting $\beta_N = \hbar^\delta$ for $0 < \delta < \frac{1}{8}(6\alpha_1 - 5)$, we have the desired inequality. □

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