JOURNAL OF TIME SERIES ANALYSIS
J. Time Ser. Anal. (2021)
Published online in Wiley Online Library
(wileyonlinelibrary.com) DOI: 10.1111/jtsa.12612

NOTES AND COMMENTS

AUTOREGRESSIVE SPECTRAL ESTIMATES UNDER IGNORED CHANGES IN THE MEAN

MATEI DEMETRESCU[a] AND MEHDI HOSSEINKOUCHACK[b*]

[a] *Institute for Statistics and Econometrics, Christian-Albrechts-University of Kiel, Kiel, Germany*
[b] *Department of Economics, University of Mannheim, Mannheim, Germany*

Periodogram-based estimators of the spectral density are known to exhibit distorted behavior in neighborhoods of the origin in case of so-called low frequency contamination, mimicking long-range dependence. This note quantifies the behavior of the estimator based on autoregressive approximations of order increasing with the sample size. Not surprisingly, the autoregressive spectral estimator is not consistent at the origin under ignored changes in the mean, but turns out to be consistent at non-zero frequencies. We furthermore show how a specific trimming of the fitted long autoregression can be used to restore consistency in the vicinity of the origin.

Received 17 November 2020; Accepted 05 July 2021

Keywords: Spectral density; long autoregression; spurious pole

JEL. C22.

MOS subject classification: 62M10.

1. MOTIVATION

The standard estimator of the spectral density of a stochastic process (assumed to be weakly stationary) relies on smoothing the periodogram. It is however known since at least Künsch (1986) that trends, when not accounted for, change the behavior of the periodogram in neighborhoods of the origin, inducing a spurious pole and thus mimicking long-range dependence. More recently, Diebold and Inoue (2001), Granger and Hyung (2004), Haldrup and Nielsen (2007), or Davidson and Sibbertsen (2005) discuss random mean components with infrequent changes, while Perron and Qu (2007, 2010) analyze the periodogram in a two-component model with random level shifts and long memory. Iacone (2010) and Qu (2011) work with slowly varying trend models, while McCloskey and Perron (2013) assume both random level shifts and trends. Relatedly, Qu and Perron (2007) discuss methods of break detection. All findings concur that changes in the location of a process, and more generally low-frequency contamination, decisively affect the behavior of the periodogram near the origin.

The spectral density of a process may however be estimated using a number of alternative methods, for instance by plugging in (quasi-) maximum likelihood estimators into a parametric expression of the spectral density, or by semi-parametric approaches. (Also, there are alternative ways of reducing the variability of the raw periodogram, for instance Bartlett's method of averaged periodograms, 1950.) The semi-parametric autoregressive spectral estimator analyzed under linearity by Berk (1974) is computationally convenient as it only requires fitting a so-called long autoregression (i.e., of order going to infinity with the sample size); see also the recent analysis of Wang and Politis (2021) under weaker assumptions.

We discuss here the behavior of the autoregressive spectral estimator when the process of interest has time-varying mean. Building on results of Demetrescu and Hassler (2016), we quantify the behavior of the

* Correspondence to: Mehdi Hosseinkouchack, Department of Economics, University of Mannheim, L7, 3-5, D-68161 Mannheim, Germany. E-mail: hosseinkouchack@uni-mannheim.de

long-autoregression based spectral density estimator. Under ignored changes in the mean, the autoregressive spectral estimator is not consistent at the origin, yet consistency is given at non-zero frequencies. To deal with the inconsistency at the origin, we suggest a trimming-like procedure consisting in leaving out some of the estimated autoregressive coefficients when computing the autoregressive spectral estimate, and show that such trimming leads to consistency at all frequencies. Finally, we use a Monte Carlo simulation to illustrate our theoretical results with regard to a series with a break in the mean that is ignored in the estimation. While, in general, the autoregressive spectral density estimator does not exhibit the known jagged behavior of the periodogram when shifts in the mean are ignored, the trimmed version of the autoregressive spectral density estimator works more reliably also in the vicinity of the origin.

2. SETUP AND RESULT

Let the series y_t be given by the usual component model

$$y_t = m_t + x_t, \quad t = 1, \ldots, T,$$

where x_t, $t \in \mathbb{Z}$, is zero-mean stationary, and the mean m_t is time-varying. We make the following technical assumptions about this data generating process.

Assumption 1. Let $x_t = \sum_{j\geq 0} b_j \varepsilon_{t-j}$ be an invertible linear process with 1-summable coefficients, $\sum_{j\geq 0} j\left|b_j\right| < \infty$, and zero-mean i.i.d. errors with finite fourth-order moments.

The process x_t thus possesses an infinite-order representation given by $x_t = \sum_{j\geq 1} a_j x_{t-j} + \varepsilon_t$, where the AR coefficients a_j are known to be 1-summable (Brillinger, 1975, p. 79); we note that $\sum_{j\geq 1} a_j < 1$ and that the autocovariances of x_t, $\gamma_h = \mathrm{Cov}\left[x_t, x_{t-h}\right] = \mathrm{E}\left[x_t x_{t-h}\right]$, are also 1-summable. The assumption imposes short-range dependence and allows x_t to be, for example, a usual ARMA process. The mean component is taken to be piecewise slowly varying in the following sense.

Assumption 2. The mean m_t is given by $m_t = m_{t,T} = \nu({}^t/_T)$, where $\nu(\cdot)$ is piecewise Lipschitz continuous on $[0, 1]$ such that the discontinuities are interior points of $[0, 1]$. That is, for any interval $(a, b) \subset [0, 1]$ which does not include a discontinuity of $\nu(\cdot)$, we have $\left|\nu\left(\tau_2\right) - \nu\left(\tau_1\right)\right| \leq C\left|\tau_2 - \tau_1\right|$ for some constant C and $\tau_1, \tau_2 \in (a, b)$.

This assumption allows for local smooth trends as well as sudden breaks, both of which are known to mimic persistence if ignored. In fact, m_t may even be stochastic, and independence of x_t is not required; the essential assumption is the piecewise smooth variation of $\nu(\cdot)$.

Consider now the long-autoregression based estimator of the spectral density $f(\lambda) = \frac{1}{2\pi}\sum_{h\in\mathbb{Z}} \gamma_h e^{-\mathrm{i}h\lambda}$ ignoring the potentially time-varying nature of the mean function m_t, given by

$$\hat{f}(\lambda) = \frac{\hat{\sigma}^2}{2\pi\left|1 - \sum_{j=1}^{h_T} \hat{a}_{j,h_T} e^{\mathrm{i}j\lambda}\right|^2},$$

where $\hat{a}_{j,h_T}$ are the coefficients from a misspecified least-squares autoregression of order h_T ($h_T \to \infty$ under suitable rate restrictions) with intercept only, and $\hat{\sigma}^2 = \frac{1}{T-h_T}\sum_{t=h_T+1}^{T} \hat{u}^2_{t,h_T}$ is the usual residual variance estimator, where

$$\hat{u}_{t,h_T} = y_t - \hat{c} - \sum_{j=1}^{h_T} \hat{a}_{j,h_T} y_{t-j}.$$

Journal of Time Series Analysis published by John Wiley & Sons Ltd. DOI: 10.1111/jtsa.12612

Let $\boldsymbol{\Sigma}_{h_T} = \left\{\gamma_{i-j}\right\}_{i,j=1,\dots,h_T}$ and $\boldsymbol{\Gamma}_{h_T} = \left(\gamma_1, \dots, \gamma_{h_T}\right)'$ with $\left\{\gamma_h\right\}_{h\in\mathbb{Z}}$ the autocovariance sequence of x_t, and note that the eigenvalues Λ_j of $\boldsymbol{\Sigma}_{h_T}$ are uniformly bounded and bounded away from zero (see the fundamental theorem of Grenander and Szegö given, for instance, in Brockwell and Davis, 1991, Theorem 4.5.3).

Denote by $\bar{\boldsymbol{a}}_{h_T}$ the coefficients of the best h_T-order autoregressive projection of x_t, where $\bar{\boldsymbol{a}}_{h_T} = \boldsymbol{\Sigma}_{h_T}^{-1}\boldsymbol{\Gamma}_{h_T}$. Let furthermore $\boldsymbol{a}_{h_T} = \left(a_1, \dots, a_{h_T}\right)'$ stack the true autoregressive coefficients, and note that $\left\|\boldsymbol{a}_{h_T} - \bar{\boldsymbol{a}}_{h_T}\right\| \to 0$. In fact, it follows from Poskitt (2007, Corollary 1 and Theorem 5) that $\left\|\boldsymbol{a}_{h_T} - \bar{\boldsymbol{a}}_{h_T}\right\| = o\left(h_T^{-1}\right)$ under our assumptions.

Demetrescu and Hassler (2016) study the properties of the estimated autoregressive coefficients $\hat{a}_{j,h_T}$ under ignored changes in the mean. They find that ignoring the mean changes induces a particular form of second-order bias. To quantify this bias, let

$$\tilde{\boldsymbol{a}}_{h_T} = \bar{\boldsymbol{a}}_{h_T} + \frac{\bar{\mu}^2}{1 + \bar{\mu}^2 \boldsymbol{\iota}_{h_T}' \boldsymbol{\Sigma}_{h_T}^{-1} \boldsymbol{\iota}_{h_T}} \Sigma_{h_T}^{-1} \boldsymbol{\iota}_{h_T} \left(1 - \boldsymbol{\iota}_{h_T}' \bar{\boldsymbol{a}}_{h_T}\right),$$

where $\bar{\mu}^2 = \int_0^1 v^2(s)\,\mathrm{d}s - \left(\int_0^1 v(s)\,\mathrm{d}s\right)^2$ and $\boldsymbol{\iota}$ is an h_T-vector of ones. Note that $\bar{\mu}^2 = 0$ whenever there are no changes in the mean, such that $\tilde{\boldsymbol{a}}_{h_T} = \bar{\boldsymbol{a}}_{h_T}$ in this case. Then, Proposition 3 of Demetrescu and Hassler (2016) indicates that, as $T, h_T \to \infty$ with $h_T = O(T^\kappa)$ for $0 < \kappa < 1/4$ and $\bar{\mu}^2 \neq 0$,

$$\left\|\hat{\boldsymbol{a}}_{h_T} - \tilde{\boldsymbol{a}}_{h_T}\right\| = o_p\left(h_T^{-1/2}\right) \quad \text{but} \quad \left\|\tilde{\boldsymbol{a}}_{h_T} - \bar{\boldsymbol{a}}_{h_T}\right\| = \Theta\left(h_T^{-1/2}\right),$$

where $b_T = \Theta(c_T)$ signifies that $b_T = O(c_T)$ and $c_T = O(b_T)$ simultaneously. Therefore, the fitted coefficients are closer in a sense to $\tilde{\boldsymbol{a}}_{h_T}$ than to $\bar{\boldsymbol{a}}_{h_T}$, and this affects $\hat{f}$ as follows.

Proposition 1. Under Assumptions 1 and 2, the following hold true for $h_T^{-1} + h_T/T^\kappa \to 0$ for some $\kappa \in (0, 1/4)$ and $\bar{\mu}^2 \neq 0$.

(i) For $\lambda \in [\varepsilon, \pi)$, $\forall\, \varepsilon \in (0, \pi)$, $\hat{f}(\lambda) \xrightarrow{p} f(\lambda)$ uniformly in λ.

(ii) For $\lambda_T \to 0$ such that $\lambda_T = o\left(h_T^{-1}\right)$, $h_T^{-2}\hat{f}(\lambda_T) \xrightarrow{p} \frac{\bar{\mu}^4}{2\pi\omega^2} > 0$ where $\omega^2 = 2\pi f(0)$ is the long-run variance of x_t.

A long autoregression therefore leads to the correct limit for frequencies bounded away from zero. This is however not the case in neighborhoods of the origin, where $\hat{f}$ diverges, spuriously indicating long-range dependence. While item (i) is, to the best of our knowledge, new, the finding in (ii) is not very surprising and reflects the well-documented dominance of low-frequency contamination on the behavior of the periodogram in the vicinity of the origin.

The divergence rate of $\hat{f}$ in the vicinity of the origin depends on the variation $\bar{\mu}^2$ of the mean process, on the signal-to-noise ratio $\bar{\mu}^2/\omega^2$, and on the order h_T of the autoregressive approximation. In contrast, the behavior of the autoregressive spectral density estimator for a genuine unit root process is different at the origin, where it diverges at rate T^2 (this follows from the T-consistency of the estimator of the unit root irrespective of the short-run dynamics). It is worth stressing that the finding in (ii) parallels the quantification of the divergence rate at the origin for the periodogram, given uniformly for all non-zero Fourier frequencies under different forms of level shifts; see for example, the discussion in McCloskey (2013), McCloskey and Perron (2013) and also Christensen and Varneskov (2017, Lemma 1). However, the rate of divergence of the periodogram-based estimator at Fourier frequencies close to zero depends rather on the Fourier frequency (see also Perron and Qu, 2007, 2010).

It is of course unfortunate that no consistency is given at the zero frequency. Iacone (2010) modifies the average periodogram estimator by trimming the lowest frequencies, which, for suitable choices of the trimming and bandwidth parameters, leads to consistent estimation of the spectrum at the zero frequency as well. Along the same lines, McCloskey (2013) allows for low-frequency contamination when estimating long memory stochastic volatility models, McCloskey and Hill (2017) analyze frequency-domain maximum likelihood estimation of ARMA and GARCH models in the presence of changes in the mean, and Christensen and Varneskov (2017) study fractional cointegration in a similar setup.

We take this as a motivation to suggest a trimming procedure in the time domain: rather than using all h_T estimated coefficients to construct an estimate of the spectral density, we suggest to restrict ourselves to the first $\ell_T < h_T$ coefficients, and therefore compute

$$\hat{f}_{tr}(\lambda) = \frac{\hat{\sigma}^2}{2\pi \left|1 - \sum_{j=1}^{\ell_T} \hat{a}_{j,h_T} e^{\mathrm{i}j\lambda}\right|^2},$$

where $\ell_T \to \infty$, but at a slower rate than h_T. Since essentially less autoregressive estimates are considered, the impact of the ignored change in the mean on the sum of $\ell_T = o(h_T)$ terms $\hat{a}_{j,h_T} e^{\mathrm{i}j\lambda}$ will diminish, such that consistent estimation at the zero frequency becomes possible:

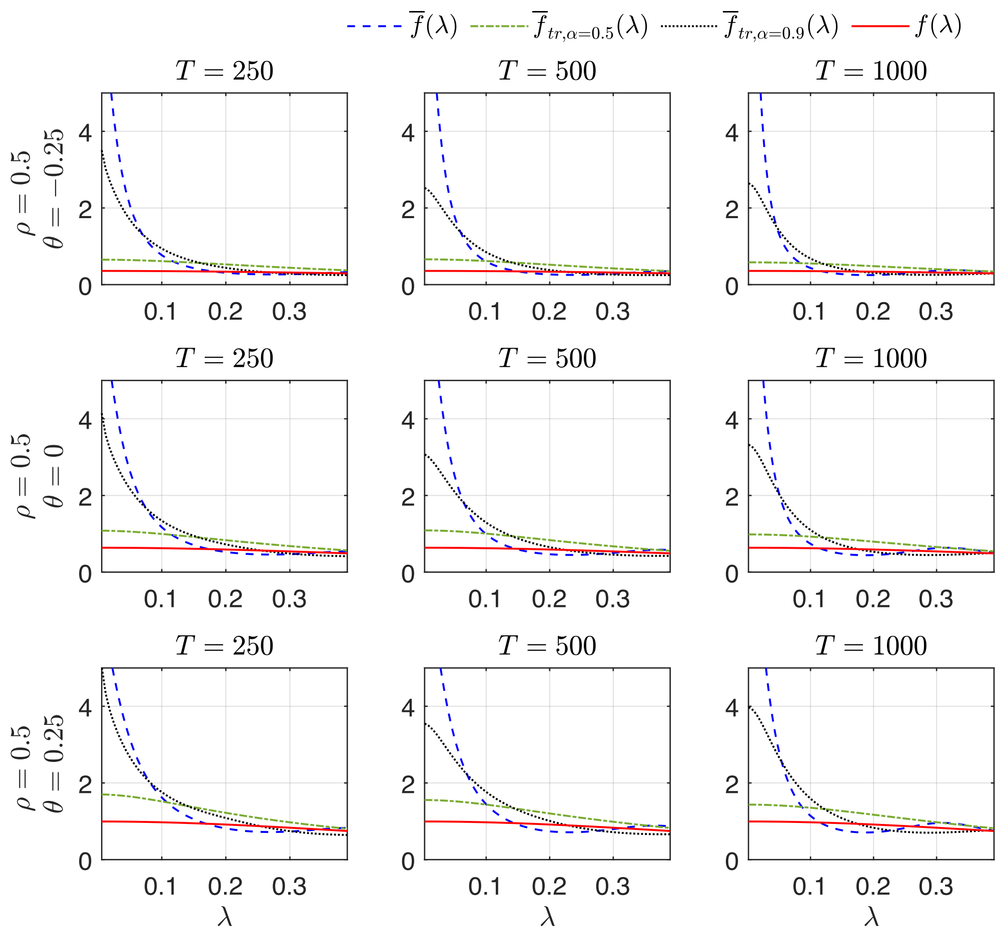

Figure 1. Average behavior of the autoregressive spectral density estimator over 10^4 Monte Carlo replications. The data are generated using $y_t = m_t + x_t$, $t = 1, 2, \ldots, T$ where $m_t = 1.5 + 1.5\mathbf{I}_{\{t>T/2\}}$, $x_t = \rho x_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1}$ where $\rho = 0.5$, $\theta \in \{-0.25, 0, 0.25\}$ and $\varepsilon_t \sim$ i.i.d.$\mathcal{N}(0, 1)$. Here, $f(\lambda)$ is the theoretical spectral density of x_t, $\overline{f}(\lambda)$ is the Monte Carlo average of the spectral density estimator of x_t resulting from an AR(h_T) regression of y_t with $h_T = \lfloor 12\,(T/100)^{1/4} \rfloor$ while $\overline{f}_{tr,\alpha}(\lambda)$ is its trimmed version with $\ell_T = \lfloor h_T^{\alpha} \rfloor$

J. Time Ser. Anal. (2021)
DOI: 10.1111/jtsa.12612

Corollary 1. Under the assumptions of Proposition 1, the following hold true.

(i) For $\lambda \in [\epsilon, \pi), \forall\, \epsilon \in (0, \pi), \hat{f}_{tr}(\lambda) \xrightarrow{p} f(\lambda)$ uniformly in λ.
(ii) For $\lambda_T \to 0$ such that $\lambda_T = o\left(h_T^{-1}\right)$ and as $\ell_T^{-1} + \ell_T/h_T \to 0, \hat{f}_{tr}(\lambda_T) \xrightarrow{p} f(0)$.

Finally, given the connection between autoregressive spectrum estimation, long-run variance estimation, and unit root testing, we note that these results could have important implications for implementation of unit root tests, which are known to be biased under structural breaks, for example, Perron (1989, 1990); see Haldrup *et al.* (2013) for a review.

3. MONTE CARLO EVIDENCE

The results in Proposition 1 and Corollary 1 are now illustrated in a small Monte Carlo experiment. Figure 1 compares the Monte Carlo average of three autoregressive spectral density estimators. Here, we consider the initial estimator $\hat{f}$ together with two variants of the trimmed autoregressive spectral estimator $\hat{f}_{tr}$, given by

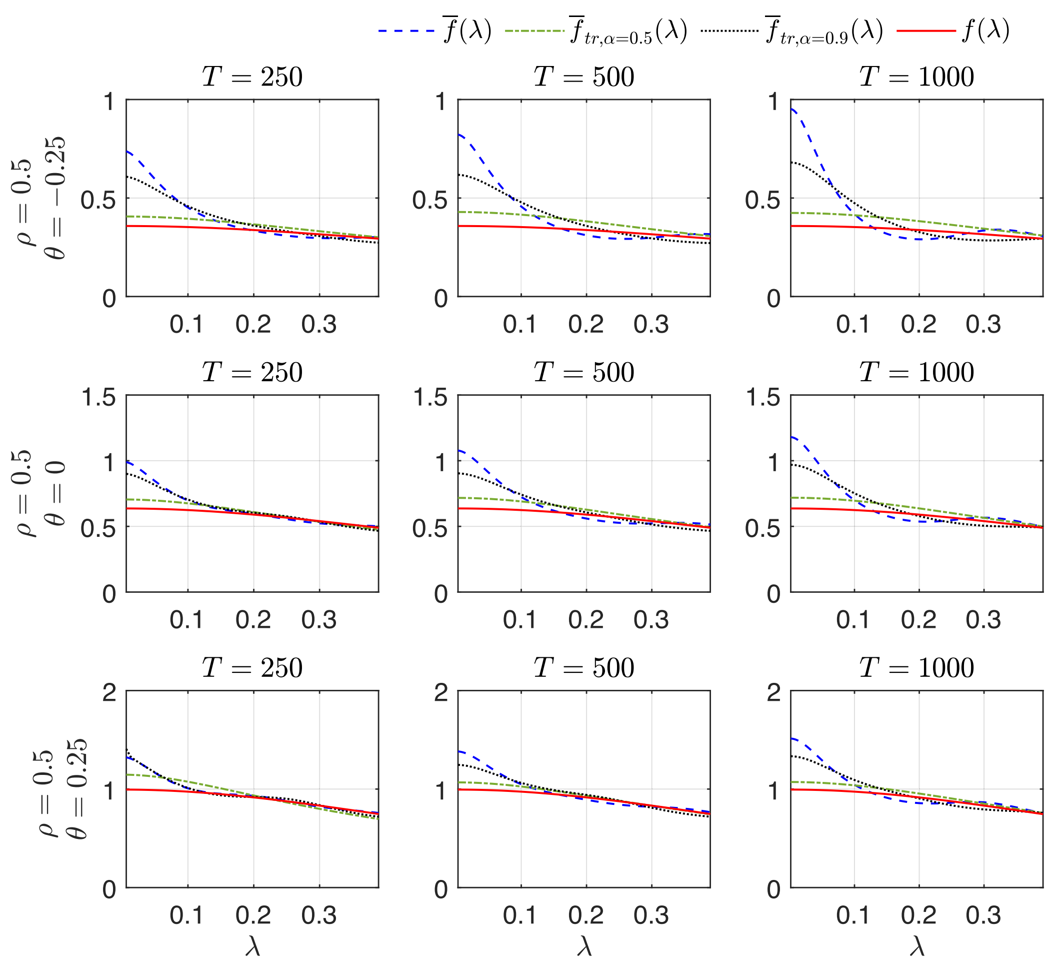

Figure 2. Average behavior of the autoregressive spectral density estimator over 10^4 Monte Carlo replications. The data are generated using $y_t = m_t + x_t$, $t = 1, 2, \dots, T$ where $m_t = 1.5 + 0.5\mathbf{I}_{\{t>T/2\}}$, $x_t = \rho x_{t-1} + \epsilon_t + \theta\epsilon_{t-1}$ where $\rho = 0.5$, $\theta \in \{-0.25, 0, 0.25\}$ and $\epsilon_t \sim$ i.i.d.$\mathcal{N}(0, 1)$. See Figure 1 for further details

$\ell_T = \lfloor h_T^{0.5} \rfloor$ and $\ell_T = \lfloor h_T^{0.9} \rfloor$. The data generating process is ARMA(1,1) with various parameter values and unity error variance, and exhibits a break in the mean, where the mean shift is of magnitude 1.5. We note how $\hat{f}$ tends to increase on average at low frequencies as T increases. As the persistence of x_t increases, the distortions due to the break in the mean become relatively less relevant. As suggested by Corollary 1, Figure 1 further shows that, upon trimming with $\ell_T = \lfloor h_T^{0.5} \rfloor$, $\hat{f}_{tr}(\lambda)$ is close to recovering $f(\lambda)$ on average not only for λ away from the origin but also for λ close to the origin. The estimation improves, though slowly as could perhaps be expected, when T increases.

Figure 2 repeats the exercise with a smaller mean shift of magnitude 0.5. Here, the impact of the break is smaller, and already the initial autoregressive spectral estimator $\hat{f}$ works reasonably well in the vicinity of the origin. The trimmed estimators behave even better, where trimming with $\ell_T = \lfloor h_T^{0.5} \rfloor$ is again of advantage, albeit less so than in the case of a larger break.

4. CONCLUDING REMARKS

We discuss how an unattended mean shift induces a spurious pole in the estimated spectral density using a long-autoregression of order h_T. We furthermore establish that this pole diverges at the rate of h_T^2. At non-zero frequencies, consistency is however given. Then, we propose a trimmed version of the spectral density estimator which is consistent in the vicinity of the origin as well. Finally, a small Monte Carlo experiment suggests that the trimmed estimator performs well.

ACKNOWLEDGEMENTS

The authors thank the handling co-editor (Morten Nielsen) and three anonymous referees, as well as Uwe Hassler, for very helpful comments. Open Access funding enabled and organized by Projekt DEAL.

DATA AVAILABILITY STATEMENT

The simulated data used in this paper are available on request from the authors.

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DOI: 10.1111/jtsa.12612

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APPENDIX PROOFS

Throughout, let $\boldsymbol{\iota}$ be an h_T-vector of ones and $\boldsymbol{e}_{\lambda,h_T} = \left(e^{\mathrm{i}\lambda}, \dots, e^{\mathrm{i}h_T\lambda}\right)'$, while C denotes a generic constant whose value may change from occurrence to occurrence. We first state a useful result.

Lemma A1. Under the assumptions of Proposition 1, it holds true that

(i) [(i)] $\sup_{\lambda\in[\varepsilon;\pi)} \left|\boldsymbol{\iota}'\boldsymbol{\Sigma}_{h_T}^{-1}\boldsymbol{e}_{\lambda,h_T}\right| = O(1)$ uniformly on $[\varepsilon;\pi)$ for any $\varepsilon \in (0;\pi)$;
(ii) $\max_{j=1,\dots,h_T} \left|q_j\right|$ is bounded, where q_j is the jth element of $\Sigma_{h_T}^{-1}\boldsymbol{\iota}$.

Proof of Lemma A1. (i) Define the $h_T \times h_T$ circulant matrix $\boldsymbol{\Sigma}_{h_T}^{(s)}$,

$$\boldsymbol{\Sigma}_{h_T}^{(s)} = \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_2 & \gamma_1 \\ \gamma_1 & \gamma_0 & \gamma_1 & \cdots & \gamma_3 & \gamma_2 \\ \gamma_2 & \gamma_1 & \gamma_0 & \cdots & \gamma_4 & \gamma_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma_2 & \gamma_3 & \gamma_4 & \cdots & \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_1 & \gamma_0 \end{bmatrix}.$$

Consider the case when h_T is odd. The eigenvector matrix, $\mathbf{P}_{h_T}$, of $\boldsymbol{\Sigma}_{h_T}^{(s)}$ can then be written as (see chapter 4 of Brockwell and Davis, 1991)

$$\mathbf{P}_{h_T} = \left[\, \boldsymbol{c}_0' \ \boldsymbol{c}_1' \ \boldsymbol{s}_1' \ \cdots \ \boldsymbol{c}_{[h_T/2]}' \ \boldsymbol{s}_{[h_T/2]}' \,\right]',$$

where $\boldsymbol{c}_0 = \frac{1}{\sqrt{h_T}}\boldsymbol{\iota}_{h_T}'$, and, with $\omega_j = \frac{2\pi j}{h_T}$ for $j = 1, 2, \dots, [h_T/2]$,

$$\boldsymbol{c}_j = \frac{\sqrt{2}}{\sqrt{h_T}} \left[1, \cos\omega_j, \cos 2\omega_j, \dots, \cos(h_T - 1)\omega_j\right],$$

$$\boldsymbol{s}_j = \frac{\sqrt{2}}{\sqrt{h_T}} \left[0, \sin\omega_j, \sin 2\omega_j, \dots, \sin(h_T - 1)\omega_j\right].$$

DOI: 10.1111/jtsa.12612

Furthermore, with

$$\mathbf{D}_{h_T} = \operatorname{diag}\left\{f(0), f\left(\omega_1\right), f\left(\omega_1\right), \dots, f\left(\omega_{[h_T/2]}\right), f\left(\omega_{[h_T/2]}\right)\right\},$$

we have using Brockwell and Davis (1991, Proposition 4.5.2) that the components of $\mathbf{R}_{h_T} = \mathbf{P}_{h_T}\mathbf{\Sigma}_{h_T}\mathbf{P}'_{h_T} - 2\pi\mathbf{D}_{h_T}$ converge to zero uniformly. Write then

$$\mathbf{\Sigma}_{h_T} = 2\pi\mathbf{P}'_{h_T}\mathbf{D}_{h_T}\mathbf{P}_{h_T} + \mathbf{P}'_{h_T}\mathbf{R}_{h_T}\mathbf{P}_{h_T}.$$

From the proof of Proposition 4.5.2 in Brockwell and Davis (1991) we learn that the components of $\mathbf{R}_{h_T}$ are bounded by $\frac{C}{h_T}\sum_{j=1}^{[h_T/2]} j|\gamma_j|$ and $C\sum_{j=[h_T/2]}^{\infty}|\gamma_j|$ for some constant $C > 0$. Using the 1-summability of the autocovariances, both of these terms turn out to be uniformly bounded by Ch_T^{-1}. This in turn implies that the components of $\mathbf{P}_{h_T}\mathbf{R}_{h_T}\mathbf{P}'_{h_T}$ are uniformly bounded by Ch_T^{-1}. Since the eigenvalues of $\mathbf{\Sigma}_{h_T}$ are bounded and bounded away from zero, we have that

$$\boldsymbol{\iota}'\mathbf{\Sigma}_{h_T}^{-1}\boldsymbol{e}_{\lambda,h_T} = \frac{1}{2\pi}\boldsymbol{\iota}'\mathbf{P}'_{h_T}\mathbf{D}_{h_T}^{-1}\mathbf{P}_{h_T}\boldsymbol{e}_{\lambda,h_T} + \boldsymbol{\iota}'\boldsymbol{e}_{\lambda,h_T}O\left(h_T^{-1}\right).$$

For λ away from 0 and π it hence holds that $\boldsymbol{\iota}'\boldsymbol{e}_{\lambda,h_T}O\left(h_T^{-1}\right) = O(1)$; furthermore, since $\boldsymbol{\iota}'\boldsymbol{c}_j = 0$ and $\boldsymbol{\iota}'\boldsymbol{s}_j = \frac{\sqrt{2}}{\sqrt{h_T}}$ we obtain

$$\begin{aligned}
\frac{1}{2\pi}\boldsymbol{\iota}'\mathbf{P}'_{h_T}\mathbf{D}_{h_T}^{-1}\mathbf{P}_{h_T}\boldsymbol{e}_{\lambda,h_T} &= \frac{1}{2\pi f(0)}\frac{e^{\mathrm{i}\lambda}\left(1 - e^{\mathrm{i}h_T\lambda}\right)}{1 - e^{\mathrm{i}\lambda}} \\
&+ \frac{2\mathrm{i}}{2\pi h_T}\sum_{j=1}^{[h_T/2]}\frac{1}{4f\left(\omega_j\right)}\csc\left(\frac{\pi j}{h_T} - \frac{\lambda}{2}\right)\csc\left(\frac{\pi j}{h_T} + \frac{\lambda}{2}\right) \\
&\times\left(\sin\left(\frac{2\pi(h_T-1)j}{h_T}\right)\sin((h_T+1)\lambda)\right. \\
&\left.+ \sin(\lambda)\left(\sin\left(\frac{2\pi j}{h_T}\right) - 2\cos\left(\frac{2\pi j}{h_T}\right)\right) + \sin(2\lambda)\right) \\
&+ \frac{2}{2\pi h_T}\sum_{j=1}^{[h_T/2]}\frac{1}{4f\left(\omega_j\right)}\csc\left(\frac{\pi j}{h_T} - \frac{\lambda}{2}\right)\csc\left(\frac{\pi j}{h_T} + \frac{\lambda}{2}\right) \\
&\times\left(\cos(\lambda)\left(\sin\left(\frac{2\pi j}{h_T}\right) - 2\cos\left(\frac{2\pi j}{h_T}\right) + 2\cos(\lambda)\right)\right. \\
&\left.+ \sin\left(\frac{2\pi(h_T-1)j}{h_T}\right)\cos((h_T+1)\lambda)\right).
\end{aligned}$$

Given the restriction on λ, $|\csc\left(\frac{\pi j}{h_T} - \frac{\lambda}{2}\right)\csc\left(\frac{\pi j}{h_T} + \frac{\lambda}{2}\right)| > \epsilon > 0$. Therefore $\frac{1}{2\pi}\boldsymbol{\iota}'\mathbf{P}'_{h_T}\mathbf{D}_{h_T}^{-1}\mathbf{P}_{h_T}\boldsymbol{e}_{\lambda,h_T} = O(1)$. The result follows analogously for the case of even h_T.

(ii) Making use of $\mathbf{\Sigma}_{h_T} = 2\pi\mathbf{P}'_{h_T}\mathbf{D}_{h_T}\mathbf{P}_{h_T} + \mathbf{P}'_{h_T}\mathbf{R}_{h_T}\mathbf{P}_{h_T}$ and noting again that each element of $\mathbf{R}_{h_T}$ is uniformly bounded by Ch_T^{-1}, we shall just analyze the behavior of the jth row sum of $\mathbf{P}'_{h_T}\mathbf{D}_{h_T}^{-1}\mathbf{P}_{h_T}$, say $q_{1,j}$. Since $\boldsymbol{c}_j'\boldsymbol{\iota} = 0$

and $\boldsymbol{s}_j'\boldsymbol{\iota} = \frac{\sqrt{2}}{\sqrt{h_T}}$ we obtain

$$\begin{aligned} |q_{1,j}| &= \left| \frac{1}{h_T f(0)} + \frac{2}{h_T} \sum_{k=1}^{[h_T/2]} \frac{\cos\left((j-1)\frac{2\pi k}{h_T}\right)}{f(\omega_k)} \right| \\ &\leq \frac{1}{h_T f(0)} + \frac{2C}{h_T} \sum_{k=1}^{[h_T/2]} \left| \cos\left((j-1)\frac{2\pi k}{h_T}\right) \right| \\ &= O(1) \end{aligned}$$

uniformly in j since $\int_0^{1/2} |\cos(2\pi(j-1)x)| \mathrm{d}x \leq \frac{1}{2}$.

□

Proof of Proposition 1. Note first that $\hat{\sigma}^2 \overset{p}{\to} \sigma^2$ follows immediately by extending Proposition 2 in Demetrescu and Hassler (2016) to AR infinity models; we omit the details to save space. Then, examine the denominator of the fitted transfer function,

$$1 - \sum_{j=1}^{h_T} \hat{a}_{j,h_T} e^{\mathrm{i}j\lambda} = 1 - \bar{\boldsymbol{a}}_{h_T}' \boldsymbol{e}_{\lambda,h_T} - \left(\hat{\boldsymbol{a}}_{h_T} - \tilde{\boldsymbol{a}}_{h_T}\right)' \boldsymbol{e}_{\lambda,h_T} - \left(\tilde{\boldsymbol{a}}_{h_T} - \bar{\boldsymbol{a}}_{h_T}\right)' \boldsymbol{e}_{\lambda,h_T}.$$

Irrespective of λ, we have $\|\boldsymbol{e}_{\lambda,h_T}\| = h_T^{1/2}$ and therefore

$$\left| \left(\hat{\boldsymbol{a}}_{h_T} - \tilde{\boldsymbol{a}}_{h_T}\right)' \boldsymbol{e}_{\lambda,h_T} \right| \leq \|\hat{\boldsymbol{a}}_{h_T} - \tilde{\boldsymbol{a}}_{h_T}\| \|\boldsymbol{e}_{\lambda,h_T}\| = o_p(1).$$

(i) For any $\lambda \in [\varepsilon, \pi)$, we may write

$$\left(\tilde{\boldsymbol{a}}_{h_T} - \bar{\boldsymbol{a}}_{h_T}\right)' \boldsymbol{e}_{\lambda,h_T} = \frac{\bar{\mu}^2 \left(1 - \boldsymbol{\iota}_{h_T}' \bar{\boldsymbol{a}}_{h_T}\right)}{1 + \bar{\mu}^2 \boldsymbol{\iota}_{h_T}' \boldsymbol{\Sigma}_{h_T}^{-1} \boldsymbol{\iota}_{h_T}} \boldsymbol{\iota}_{h_T}' \boldsymbol{\Sigma}_{h_T}^{-1} \boldsymbol{e}_{\lambda,h_T},$$

where we know from Lemma A1(i) that $\boldsymbol{\iota}'\boldsymbol{\Sigma}_{h_T}^{-1}\boldsymbol{e}_{\lambda,h_T} = O(1)$ uniformly for $\lambda \in [\varepsilon, \pi)$ for any $\varepsilon \in (0, \pi)$. The expression $\frac{\bar{\mu}^2\left(1-\boldsymbol{\iota}'\bar{\boldsymbol{a}}_{h_T}\right)}{1+\bar{\mu}^2\boldsymbol{\iota}'\boldsymbol{\Sigma}_{h_T}^{-1}\boldsymbol{\iota}}$ does not depend on λ, and is $O\left(h_T^{-1}\right)$ since $\left|\boldsymbol{\iota}'\bar{\boldsymbol{a}}_{h_T}\right| \leq \left|\boldsymbol{\iota}'\boldsymbol{a}_{h_T}\right| + h_T^{1/2}\left\|\boldsymbol{a}_{h_T} - \bar{\boldsymbol{a}}_{h_T}\right\| = O(1)$, while $\boldsymbol{\iota}'\boldsymbol{\Sigma}_{h_T}^{-1}\boldsymbol{\iota} \geq h_T \min_{1\leq j\leq h_T} \Lambda_j^{-1}$ with the eigenvalues Λ_j of $\boldsymbol{\Sigma}_{h_T}$ being positive, leading to

$$\left| \left(\tilde{\boldsymbol{a}}_{h_T} - \boldsymbol{a}_{h_T}\right)' \boldsymbol{e}_{\lambda,h_T} \right| \leq C \frac{\bar{\mu}^2 \left|1 - \boldsymbol{\iota}'\boldsymbol{a}_{h_T}\right|}{1 + \bar{\mu}^2 \boldsymbol{\iota}'\boldsymbol{\Sigma}_{h_T}^{-1}\boldsymbol{\iota}} = o(1).$$

Summing up, we have uniformly on $\lambda \in [\varepsilon, \pi)$

$$1 - \sum_{j=1}^{h_T} \hat{a}_{j,h_T} e^{\mathrm{i}j\lambda} = 1 - \bar{\boldsymbol{a}}_{h_T}' \boldsymbol{e}_{\lambda,h_T} + o_p(1).$$

The desired denominator of the transfer function is given by

$$1-\sum_{j\geq 0} a_j e^{\mathrm{i}j\lambda} = 1-\bar{\boldsymbol{a}}'_{h_T}\boldsymbol{e}_{\lambda,h_T} + \sum_{j>h_T} a_j e^{\mathrm{i}j\lambda} + \left(\bar{\boldsymbol{a}}_{h_T}-\boldsymbol{a}_{h_T}\right)'\boldsymbol{e}_{\lambda,h_T}.$$

Since the series $\sum a_j$ of autoregressive coefficients is 1-summable,

$$\sup_{\lambda\in[0,1]}\left|\sum_{j>h_T} a_j e^{\mathrm{i}j\lambda}\right| \leq \sup_{\lambda\in[0,1],j\in\mathbb{Z}}\left|e^{\mathrm{i}j\lambda}\right|\sum_{j>h_T}\left|a_j\right| \to 0,$$

and the desired result follows given that $\|\boldsymbol{a}_{h_T}-\bar{\boldsymbol{a}}_{h_T}\| = o\left(h_T^{-1/2}\right)$ and $\|\boldsymbol{e}_{\lambda,h_T}\| = O\left(h_T^{1/2}\right)$.

(ii) Moving on to $\lambda_T\to 0$, it is more convenient to write

$$1-\sum_{j=1}^{h_T}\hat{a}_{j,h_T}e^{\mathrm{i}j\lambda_T} = 1-\tilde{\boldsymbol{a}}'_{h_T}\boldsymbol{e}_{\lambda_T,h_T} - \left(\hat{\boldsymbol{a}}_{h_T}-\tilde{\boldsymbol{a}}_{h_T}\right)'\boldsymbol{e}_{\lambda_T,h_T}$$

and we only have to discuss the behavior of $1-\tilde{\boldsymbol{a}}'_{h_T}\boldsymbol{e}_{\lambda_T,h_T}$. To this end, write the exponentials as

$$\begin{aligned} e^{\mathrm{i}j\lambda_T} &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\mathrm{i}j\lambda_T\right)^k = 1+\mathrm{i}j\lambda_T\left(\sum_{\ell=0}^{\infty}\frac{1}{(\ell+1)!}\left(\mathrm{i}j\lambda_T\right)^{\ell}\right)\\ &= 1+\mathrm{i}j\lambda_T\xi_{j\lambda}\end{aligned}$$

where

$$\left|\xi_{j\lambda}\right| \leq \sum_{\ell=0}^{\infty}\frac{1}{\ell!}\left|\left(\mathrm{i}j\lambda_T\right)^{\ell}\right| = \exp\left(\left|j\lambda_T\right|\right) = O(1),$$

uniformly in $j\leq h_T$ for $\lambda_T = o\left(h_T^{-1}\right)$. Then,

$$1-\tilde{\boldsymbol{a}}'_{h_T}\boldsymbol{e}_{\lambda_T,h_T} = 1-\tilde{\boldsymbol{a}}'_{h_T}\boldsymbol{\iota}+\mathrm{i}\lambda_T\sum_{j=1}^{h_T}j\xi_{j\lambda}\tilde{a}_{j,h_T},$$

where

$$\begin{aligned}\left|\sum_{j=1}^{h_T}j\xi_{j\lambda}\tilde{a}_{j,h_T}\right| &\leq \sum_{j=1}^{h_T}j\left|a_j\right| + \left|\sum_{j=1}^{h_T}j\left(\tilde{a}_{j,h_T}-a_j\right)\right|\\ &\leq h_T\sum_{j=1}^{h_T}\left|a_j\right| + \left\|\left(1,\dots,h_T\right)'\right\|\left\|\tilde{\boldsymbol{a}}_{h_T}-\boldsymbol{a}_{h_T}\right\|\\ &= O\left(h_T\right),\end{aligned}$$

since a_j are 1-summable and $\left\|\left(1,\dots,h_T\right)'\right\| = O\left(h_T^{3/2}\right)$. Now

$$\left\|\tilde{\boldsymbol{a}}_{h_T}-\boldsymbol{a}_{h_T}\right\| \leq \left\|\tilde{\boldsymbol{a}}_{h_T}-\bar{\boldsymbol{a}}_{h_T}\right\| + \left\|\bar{\boldsymbol{a}}_{h_T}-\boldsymbol{a}_{h_T}\right\| = O_p\left(h_T^{-1/2}\right)+o_p\left(h_T^{-1/2}\right).$$

DOI: 10.1111/jtsa.12612

Therefore, $\mathrm{i}\lambda \sum_{j=1}^{h_T} j\xi_{j\lambda}\tilde{a}_{j,h_T} = O\left(h_T\lambda_T\right)$, such that, for $\lambda_T = o\left(h_T^{-1}\right)$,

$$1 - \tilde{\boldsymbol{a}}_{h_T}' \boldsymbol{e}_{\lambda_T,h_T} = 1 - \tilde{\boldsymbol{a}}_{h_T}' \boldsymbol{\iota} + o(1).$$

Analyzing $1 - \tilde{\boldsymbol{a}}_{h_T}' \boldsymbol{\iota}_{h_T}$, we have that

$$\begin{aligned} 1 - \tilde{\boldsymbol{a}}_{h_T}' \boldsymbol{\iota} &= \left(1 - \boldsymbol{\iota}'\bar{\boldsymbol{a}}_{h_T}\right)\left(1 - \frac{\bar{\mu}^2}{1 + \bar{\mu}^2\boldsymbol{\iota}'\boldsymbol{\Sigma}_{h_T}^{-1}\boldsymbol{\iota}}\boldsymbol{\iota}'\Sigma_{h_T}^{-1}\boldsymbol{\iota}\right) \\ &= \frac{1 - \boldsymbol{\iota}'\bar{\boldsymbol{a}}_{h_T}}{1 + \bar{\mu}^2\boldsymbol{\iota}'\boldsymbol{\Sigma}_{h_T}^{-1}\boldsymbol{\iota}}, \end{aligned}$$

where $1 - \boldsymbol{\iota}'\bar{\boldsymbol{a}}_{h_T} = 1 - \boldsymbol{\iota}'\boldsymbol{a}_{h_T} + o(1)$ is bounded away from zero and $\boldsymbol{\iota}'\boldsymbol{\Sigma}_{h_T}^{-1}\boldsymbol{\iota} \geq h_T \min_{1\leq j\leq h_T} \Lambda_j^{-1}$. Therefore, we have for $\hat{f}$ in a neighborhood of the origin

$$\frac{1}{h_T^2}\hat{f}(\lambda_T) \overset{p}{\to} \frac{\sigma^2\bar{\mu}^4}{2\pi\left(1 - \sum_{j\geq 1} a_j\right)^2}\left(\lim h_T^{-1}\boldsymbol{\iota}'\boldsymbol{\Sigma}_{h_T}^{-1}\boldsymbol{\iota}\right)^2.$$

Given that $\lim h_T^{-1}\boldsymbol{\iota}'\boldsymbol{\Sigma}_{h_T}^{-1}\boldsymbol{\iota} = \frac{1}{2\pi f(0)}$ and $f(0) = \frac{\sigma^2}{2\pi\left(1-\sum_{j\geq 1} a_j\right)^2}$, the limit of $\frac{1}{h_T^2}\hat{f}(\lambda)$ reads $\frac{\sigma^2\bar{\mu}^4}{2\pi\left(\frac{\sigma^2}{2\pi f(0)}\right)}\left(\frac{1}{2\pi f(0)}\right)^2 = \frac{\bar{\mu}^4}{4\pi^2 f(0)}$ as required. □

Proof of Corollary 1. (i) Write

$$1 - \sum_{j=1}^{\ell_T} \hat{a}_{j,h_T} e^{\mathrm{i}j\lambda} = 1 - \sum_{j=1}^{h_T} \hat{a}_{j,h_T} e^{\mathrm{i}j\lambda} + \sum_{j=\ell_T+1}^{h_T} \hat{a}_{j,h_T} e^{\mathrm{i}j\lambda},$$

such that it suffices to show that $\sum_{j=\ell_T+1}^{h_T} \hat{a}_{j,h_T} e^{\mathrm{i}j\lambda} = o_p(1)$. To this end, write

$$\begin{aligned} \sum_{j=\ell_T+1}^{h_T} \hat{a}_{j,h_T} e^{\mathrm{i}j\lambda} &= \sum_{j=\ell_T+1}^{h_T} a_j e^{\mathrm{i}j\lambda} + \sum_{j=\ell_T+1}^{h_T} \left(\hat{a}_{j,h_T} - \tilde{a}_{j,h_T}\right) e^{\mathrm{i}j\lambda} \\ &\quad + \sum_{j=\ell_T+1}^{h_T} \left(\bar{a}_{j,h_T} - a_j\right) e^{\mathrm{i}j\lambda} + \sum_{j=\ell_T+1}^{h_T} \left(\tilde{a}_{j,h_T} - \bar{a}_{j,h_T}\right) e^{\mathrm{i}j\lambda}, \end{aligned}$$

where the first summand on the r.h.s. vanishes by 1-summability of the true autoregressive coefficients, and the absolute values of the second and the third are bounded by terms $\|\hat{\boldsymbol{a}}_{h_T} - \tilde{\boldsymbol{a}}_{h_T}\|\|\boldsymbol{e}_{\lambda,h_T}\|$ and $\|\bar{\boldsymbol{a}}_{h_T} - \boldsymbol{a}_{h_T}\|\|\boldsymbol{e}_{\lambda,h_T}\|$ respectively, which are shown to vanish in the proof of Proposition 1(i). Then,

$$\sum_{j=\ell_T+1}^{h_T} \left(\tilde{a}_{j,h_T} - \bar{a}_{j,h_T}\right) e^{\mathrm{i}j\lambda} = \left(\tilde{\boldsymbol{a}}_{h_T} - \bar{\boldsymbol{a}}_{h_T}\right)' \boldsymbol{e}_{\lambda,h_T} - \sum_{j=1}^{\ell_T} \left(\tilde{a}_{j,h_T} - \bar{a}_{j,h_T}\right) e^{\mathrm{i}j\lambda},$$

where the first summand on the r.h.s. is shown to vanish in the proof of Proposition 1(i). To conclude, note that

$$\tilde{a}_{j,h_T} = \bar{a}_{j,h_T} + \frac{\bar{\mu}^2\left(1 - \boldsymbol{\iota}'\bar{\boldsymbol{a}}_{h_T}\right)}{1 + \bar{\mu}^2\boldsymbol{\iota}'\boldsymbol{\Sigma}_{h_T}^{-1}\boldsymbol{\iota}} q_j,$$

where q_j is the inner product of the jth row of $\Sigma_{h_T}^{-1}$ and $\boldsymbol{\iota}$, and $\max_{j=1,\dots,h_T}\left|q_j\right|$ is bounded by Lemma A1(ii). Therefore, uniformly in j,

$$\tilde{a}_{j,h_T} - \bar{a}_{j,h_T} = O(h_T^{-1})$$

such that $\sum_{j=1}^{\ell_T}\left(\tilde{a}_{j,h_T} - \bar{a}_{j,h_T}\right)e^{\mathrm{i}j\lambda} = O\left(\ell_T/h_T\right) = o(1)$ as required.

(ii) Write analogously to the proof of Proposition 1(ii)

$$1 - \sum_{j=1}^{\ell_T}\hat{a}_{j,h_T}e^{\mathrm{i}j\lambda_T} = 1 - \sum_{j=1}^{\ell_T}\tilde{a}_{j,h_T}e^{\mathrm{i}j\lambda_T} - \sum_{j=1}^{\ell_T}\left(\hat{a}_{j,h_T} - \tilde{a}_{j,h_T}\right)e^{\mathrm{i}j\lambda_T},$$

where

$$\left|\sum_{j=1}^{\ell_T}\left(\hat{a}_{j,h_T} - \tilde{a}_{j,h_T}\right)e^{\mathrm{i}j\lambda_T}\right| \le \sqrt{\sum_{j=1}^{\ell_T}\left(\hat{a}_{j,h_T} - \tilde{a}_{j,h_T}\right)^2\sum_{j=1}^{\ell_T}|e^{\mathrm{i}j\lambda_T}|^2} \le \|\hat{\boldsymbol{a}} - \tilde{\boldsymbol{a}}\|\left\|\boldsymbol{e}_{\lambda_T,h_T}\right\| = o_p(1).$$

Using the same approximation for the exponential, $e^{\mathrm{i}j\lambda_T} = 1 + \mathrm{i}j\lambda_T\xi_{j\lambda}$ with $\xi_{j\lambda} = O(1)$, we obtain similarly

$$1 - \sum_{j=1}^{\ell_T}\tilde{a}_{j,h_T}e^{\mathrm{i}j\lambda_T} = 1 - \sum_{j=1}^{\ell_T}\tilde{a}_{j,h_T} + o(1).$$

To conclude, recall from the proof of item (i) of this corollary that

$$\tilde{a}_{j,h_T} - \bar{a}_{j,h_T} = O(h_T^{-1}),$$

such that $\sum_{j=1}^{\ell_T}\tilde{a}_{j,h_T} = \sum_{j=1}^{\ell_T}\bar{a}_{j,h_T} + O\left(\ell_T/h_T\right)$ and

$$1 - \sum_{j=1}^{\ell_T}\hat{a}_{j,h_T}e^{\mathrm{i}j\lambda} = 1 - \sum_{j=1}^{\ell_T}\bar{a}_{j,h_T} + o_p(1),$$

as required.

□

Journal of Time Series Analysis published by John Wiley & Sons Ltd.

DOI: 10.1111/jtsa.12612