Geometry of the sets of Nash equilibria in mixed extensions of finite games

Inauguraldissertation
zur Erlangung des akademischen Grades
eines Doktors der Naturwissenschaften
der Universität Mannheim

vorgelegt von

Matija Vujić
aus Mannheim

Mannheim, 2022

Dekan:	Dr. Bernd Lübcke,	Universität Mannheim
Referent:	Prof. Dr. Claus Hertling,	Universität Mannheim
Korreferent:	Prof. Dr. Bernhard von Stengel,	London School of Economics

Tag der mündlichen Prüfung: 23. März 2023

Abstract

The theory of strategic games in normal form is a part of game theory. The most important solution concept for them is the notion of a Nash equilibrium. Nash defined it and proved that the mixed extension of any finite game has Nash equilibria. Here the space in which the Nash equilibria live is a product of simplices, namely a product of spaces of probability distributions, each over a finite set of pure strategies. The existence leads to questions on the shape of the set of all Nash equilibria for a given game.

In this thesis we concentrate on generic games. There it is well-known that the number of Nash equilibria is finite and odd. It is interesting to think about the maximal number of Nash equilibria in generic games with fixed number of players and fixed finite sets of pure strategies. In general, the precise number is unknown. But in the case of 2 players, there are good upper and lower bounds, which are not so far apart. In the case of $m \geq 3$ players, up to now only an upper bound was known. In the case of m players each of whom has exactly two pure strategies, we present a lower bound, which is surprisingly close to the known upper bound. It is more than half of the upper bound.

This result was the outcome of a mixture of conceptual and calculational steps. We present more calculational results for such games. We also study with computer a certain 2-person game where each player has six pure strategies.

One chapter recalls a good part of the history of the problem. The penultimate chapter works out an old foundational result on the union of the sets of mixed Nash equilibria for all games with fixed player set and fixed finite sets of pure strategies. The second chapter presents a stronger result on generic games than can be found in the literature. The product of simplices embeds naturally into a product of real projective spaces. The equalities and inequalities for Nash equilibria make sense in this bigger space. In the case of generic games all involved hypersurfaces are smooth and maximally transversal in this bigger space.

Zusammenfassung

Die Theorie der strategischen Spiele in Normalform ist ein Teil der Spieltheorie. Das wichtigste Lösungsbegriff for diese ist der Begriff des Nash-Gleichgewichts. Nash definierte es und bewies, dass die gemischte Erweiterung jedes endlichen Spiels Nash-Gleichgewichte hat. Der Raum, in welchem Nash-Gleichgewichte liegen, ist ein Produkt von Simplizes, und zwar ein Produktraum von Wahrscheinlichkeitsverteilungen. Die Existenz führt zu Fragen über die Gestalt der Menge aller Nash-Gleichgewichte für ein gegebenes Spiel.

Wir konzentrieren uns in dieser Dissertation auf generische Spiele. Dort ist wohlbekannt, dass die Anzahl der Nash-Gleichgewichte endlich und ungerade ist. Es ist interessant, sich über die maximale Anzahl der Nash-Gleichgewichte bei generischen Spielen mit fester Spieleranzahl und festen endlichen reinen Strategiemengen Gedanken zu machen. Im Allgemeinen ist die genaue Anzahl unbekannt, aber im Fall von 2 Spielern gibt es gute obere und untere Schranken. Im Fall $m \geq 3$ war bis jetzt nur eine obere Schranke bekannt. Im Fall mit m Spielern, wobei jeder Spieler genau zwei reine Strategien hat, beweisen wir eine untere Schranke, welche überraschend nah an der bekannten oberen Schranke ist. Sie ist größer als die Hälfte der oberen Schranke.

Das Resultat war die Folge einer Mischung von konzeptionellen und rechnerischen Schritten. Wie stellen mehrere rechnerische Resultate für solche Spiele dar. Außerdem betrachten wir mithilfe eines Computers ein bestimmtes 2-Personenspiel, wobei jeder Spieler sechs reine Strategien hat.

Ein Kapitel betrachtet ein gutes Stück der Geschichte der Aufgabenstellung. Das vorletzte Kapitel betrachtet ein altes und grundlegendes Resultat über die Vereinigung der Mengen der gemischten Nash-Gleichgewichte für alle Spiele mit fester Spieleranzahl und festen endlichen Strategiemengen. Das zweite Kapitel präsentiert ein stärkeres Resultat über generische Spiele, als in der Literatur gefunden werden kann. Das Produkt der Simplizes bettet sich in natürlicher Weise in ein Produkt reeller projektiver Räume ein. Die Gleichungen und Ungleichungen für Nash-Gleichgewichte ergeben in diesem größeren Raum Sinn. In dem Fall von generischen Spielen sind alle beteiligten Hyperflächen glatt und maximal transversal in dem größeren Raum.

Contents

List of Figures

List of Tables

1 Introduction

This dissertation extends my master's thesis [Vuj19]. It is a culmination of over three years of hard work.

A good part of game theory is about strategic games in normal form. There each player i in a fixed finite set $\mathcal{A} = \{1, ..., m\}$ of players has a fixed space S^i of possible strategies. A tuple $s = (s^1, ..., s^m) \in S^1 \times \cdots \times S^m =: S$ of strategies $s^i \in S^i$ comes equipped with a utility value $U^i(s) \in \mathbb{R}$ for player i. These values yield a utility map $U = (U^1, ..., U^m) : S \to \mathbb{R}^m$. The players have to choose strategies simultaneously. Each of them would like to maximize his utility. But they can't talk with each other (or at least they can't trust each another) before the simultaneous decisions.

The most important solution concept for this situation is due to Nash [Nas51]. A *Nash equilibrium* $s = (s^1, ..., s^m) \in S^1 \times \ldots S^m = S$ is a strategy combination such that no player can increase his utility if he alone deviates from his strategy s^i.

Not every game has Nash equilibria. Especially, if the game is *finite*, i.e. if all the sets S^i are finite, often the game has no Nash equilibria. But a finite game has a *mixed extension*: The mixed strategies $g^i \in G^i$ are probability distributions over S^i, so one embeds S^i as the standard basis into $\mathbb{R}^{|S^i|}$ and identifies G^i with its convex hull. The utility map is extended multilinearly to the product $\prod_{i \in \mathcal{A}} \mathbb{R}^{|S^i|}$ and then restricted to $G = G^1 \times \cdots \times G^m$. We call the new utility map $V = (V^1, \ldots, V^m) : G \to \mathbb{R}^m$.

Nash proved that the mixed extension $(\mathcal{A}, G, V)$ of any finite game $(\mathcal{A}, S, U)$ has Nash equilibria. So, in this important case, existence is not a problem. Though often there are many Nash equilibria. Then the problem arises which to choose. On the one hand, this leads to refinements of the concept of a Nash equilibrium. On the other hand, it also leads to the question how the set of Nash equilibria in the mixed extension of a finite game can look like. It is this question to which this thesis is devoted. It is amenable to a treatment with tools from combinatorics, (differential) topology and real algebraic geometry. It is not so close to applications in economics, but it is a clean and interesting mathematical problem.

A good starting point is to fix $\mathcal{A}$ and S and thus G, but to consider all possible utility maps $U \in (\mathbb{R}^m)^S =: \mathcal{U}$ and their extensions $V : G \to \mathbb{R}^m$. Wilson [Wil71], Rosenmüller [Ros71] and Harsanyi [Har73] proved that in the case of a *generic* game the set $\mathcal{N}(U)$ of mixed Nash equilibria is a finite and odd set. In the case of 2 players, this was shown before by Lemke and Howson [LH64]. For non-generic U, the set of Nash equilibria is a semi-algebraic set which can become very complicated.

Though the maximal number of mixed Nash equilibria for generic games with fixed sets $\mathcal{A}$ and S is in most cases unknown. Upper and lower bounds were found only in the late 90s. These studies help to understand the complexity in the determination of the sets of Nash equilibria. The best studied case (and probably also the most important case) is the case of 2 players, $m = 2$. In this case the multilinearity of the utility map leads one into the domain of polytopes. This was observed already in the 60s. But the best use of it was made by Keiding and von Stengel, both using McMullen's upper bound theorem for simple (or simplicial) polytopes. They considered the maximal number $\mu(n)$ of Nash equilibria for generic 2-person games with $n = |S^1| = |S^2|$. Keiding [Kei97] found an upper bound, von Stengel [Ste97][Ste99] found a lower bound. For large even n they are roughly

$$0.95 \cdot 2.4^n/\sqrt{n} < \mu(n) < 0.92 \cdot 2.6^n/\sqrt{n}$$

For arbitrary m, McKelvey and McLennan [MM97] restricted attention to the totally mixed Nash equilibria, i.e. the Nash equilibria in the interior of G, because for them one has to deal only with equalities, not with inequalities. Using a result of Bernstein, Khovanskii and Kushnirenko on the number of zeros of a tuple of Laurent polynomials, they could determine the precise maximal number of totally mixed Nash equilibria of generic games with fixed sets $\mathcal{A}$ and S. It is a number of *block derangements* (see Chapter 3.3 for details). Vidunas [Vid17] used it for a rough upper bound of the maximal number of all Nash equilibria for a generic game with fixed sets $\mathcal{A}$ and S. In the case $m = 2$, this bound has the asymptotics $4^n/\sqrt{n}$, so there it is much coarser than Keiding's bound. He simply used that any

Nash equilibrium is a totally mixed Nash equilibrium of the mixed extension of a suitable subgame of the pure game, so essentially he ignored the inequalities in the definition of Nash equilibria.

The best result in this thesis concerns the case with arbitrary $\mathcal{A}$, but $|S^1| = \cdots = |S^m| = 2$. We call such games *pre-tropical*. We call Vidunas' upper bound for such games $\mathcal{V}(m) \in \mathbb{N}$. Surprisingly, in this case the upper bound is not so coarse. We find the lower bound $\frac{1}{2}(!m + \mathcal{V}(m))$, which is more than half of the upper bound (Theorems 4.3.4 and 5.5.1). Here $!m$ is the number of derangements, i.e. the number of permutations in S_m which have no fixed points. Within a family of special pre-tropical games, which are not generic and which we call *inner tropical*, we find games which have $\frac{1}{2}(!m + \mathcal{V}(m))$ many Nash equilibria. The Nash equilibria are regular, so small deformations of these games are generic and have the same number of Nash equilibria. The inner tropical games are amenable to a careful analysis of the combinatorics in the set of defining inequalities for Nash equilibria.

The first proof of this result and our way to find it are described in Chapter 5. We had rephrased the problem of existence of our maximal inner tropical games in a system of linear equations which we could control in several steps, first for small m, later in general.

The second and shorter proof is given in Chapter 4. It controls the combinatorics in a more direct way. We intend to publish the result with this proof in a separate paper.

The Chapters 4, 5 and 6 are all devoted to pre-tropical and inner tropical games. At an early stage, we went beyond McKelvey and McLennan and studied the Nash equilibria whose strategies are pure for 2 players and totally mixed (in the interior of G^i) for all other players. Our results with computer on the possible numbers of these Nash equilibria in the cases $m = 4, 5, 6$ are documented in Chapter 6 and in the Appendices A and B.

In Chapter 7, we turn to an explicit 2-player game with $|S^1| = |S^2| = 6$ whose mixed extension has 75 Nash equilibria and which was proposed by von Stengel [Ste97]. Here 75 is his lower bound for the number of maximal Nash equilibria for generic games. Here $G = G^1 \times G^2$ can be embedded naturally into the product $\mathbb{P}^5\mathbb{R} \times \mathbb{P}^5\mathbb{R}$ of two five dimensional real projective spaces. In these spaces 923 points satisfy the homogeneous versions of the equalities which Nash equilibria satisfy. We propose in Chapter 7 three types of genericity conditions and study with computer all 923 points with respect to these genericity conditions. It turns out that the game is not completely generic, but in a harmless way. All points are regular in a precise way, so that a small deformation of U leads to a generic game with the same number 75 of Nash equilibria.

The other Chapters 2, 3 and 9 present concepts and theory. A good part is not new. Chapter 3 goes through the details of the argument of Wilson and Rosenmüller for the oddness of the number of Nash equilibria, and it describes in some detail the mentioned results of McKelvey-McLennan, Keiding and von Stengel. It does not contain new results.

Chapter 2 presents a stronger result about *generic* games than can be found in the literature. The space $G = \prod_{i\in\mathcal{A}} G^i$ embeds naturally into a product $\prod_{i\in\mathcal{A}} \mathbb{P}^{|S^i|-1}\mathbb{R}$ of real projective spaces. We show that there is a real semi-algebraic codimension 1 subset $\mathcal{D} \subset \mathcal{U} = \mathbb{R}^U$ such that for $U \in \mathcal{U} \setminus \mathcal{D}$ all hypersurfaces which come from homogeneous versions of the equalities and inequalities which define the Nash equilibria are smooth in $\prod_{i\in\mathcal{A}} \mathbb{P}^{|S^i|-1}$ and are maximally transversal (Theorem 2.2.1). Our proof follows the idea of a proof of Khovanskii [Kho77] for generic systems of complex Laurent polynomials and associated toric varieties.

We hoped to make use of it in order to close the gap between upper and lower bound for the maximal number of Nash equilibria for generic pre-tropical games, but we did not succeed. Still we hope that it will be useful in the future. We intend to publish it in a separate paper.

Finally, Chapter 8 takes up a beautiful result of Kohlberg and Mertens [KM86]. They considered for fixed sets $\mathcal{A}$ and S the union $\bigcup_{U\in\mathcal{U}} \mathcal{N}(U)$ of sets of mixed Nash equilibria and showed that it is a topological manifold. More precisely, they gave a piecewise algebraic parametrization of it. We worked their very short presentation out in more detail and prepared it by parametrizations of best reply graphs. The case of a single player can be considered as a kind of real blow up of families of points in $\mathbb{R}^N$ to families of simplices such that the total space is still a piecewise linear manifold.

We hoped to use it for a study of non-generic games and their Nash equilibria, controlling them via the generic games in their neighborhood, but we did not come to it within this thesis.

We use the following notations in this thesis. $\mathbb{F}_2$ is the unique field with two elements up to isomorphism with 0 as the neutral element of the addition and 1 as the neutral element of the multiplication. The usual construction of $\mathbb{F}_p$ is the quotient of $\mathbb{Z}/(p\mathbb{Z})$, but this makes an embedding from $\mathbb{F}_2 \to \mathbb{Z}$ with $\mathbb{F}_2 \ni 0 \mapsto 0 \in \mathbb{Z}$ and $\mathbb{F}_2 \ni 1 \mapsto 1 \in \mathbb{Z}$ necessary. Thus we shall identify the set $\{0, 1\} \subset \mathbb{Z}$ as $\mathbb{F}_2$. We have $1 + 1 = 2$ and $1 +_2 1 = 0$.

Let K be a field. The tuples $(k_1, \ldots, k_n) \in K^n$ are identified with the column vector space. The row vector $(k_1 \ldots k_n) \in K^{1\times n}$ contains no commas. It is

$$(k_1, \ldots, k_n) = \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} = (k_1 \ldots k_n)^T.$$

We call $e_i^{(n)}$ the i-th standard basis vector of K^n.

We identify (A, B) as a bimatrix game where the row indicates the strategy of player 1 and the column indicates the strategy of player 2. However unlike other literature we transpose the second matrix twice. Thus a bimatrix game is written as (A, B^T) where both A and B^T have the same amount of rows and columns. We use this notation to save on the transpose sign in the utility calculation, where space for additional subscripts is scarce. The utility of player 2 is thus given by the product $(\underline{\gamma}^2)^T B \underline{\gamma}^1$.

2 Generic games

The first section introduces the central notions, a finite game, its mixed extension, best reply maps, Nash equilibria. The set of pure strategies $S^i = \{s_0^i, ..., s_{n_i}^i\}$ of a player i in the player set $\mathcal{A}$ has $n_i + 1 \in \mathbb{N}$ elements. The set $G^i \supset S^i$ of mixed strategies is a simplex with the pure strategies as vertices.

The other three sections are devoted to a discussion of generic games. This notion had been treated before [LH64] [Har73] [GPS93] [Rit94] But our version is stronger than the versions in the literature. The space G^i embeds naturally into a real projective space $\mathbb{P}^{n_i}\mathbb{R}$ of dimension n_i, and all the equations which are relevant for the best reply maps and Nash equilibria, make sense in the product $\prod_{i\in\mathcal{A}} \mathbb{P}^{n_i}\mathbb{R}$. We will show that for a given player set $\mathcal{A}$ and given sets S^i of pure strategies, there is a semialgebraic subset $\mathcal{D} \subset \mathcal{U}$ of the affine space of all possible utility maps U such that for any game with $U \in \mathcal{U} \setminus \mathcal{D}$ each equation induces a smooth hypersurface in $\prod_{i\in\mathcal{A}} \mathbb{P}^{n_i}\mathbb{R}$ and that these hypersurfaces are maximally transversal (Theorem 2.2.1). Our proof uses Sard's theorem as well as the semialgebraic character of the setting. The second section gives the result, the third gives background material, the fourth gives the proof. The proof is inspired by a result and proof of Khovanskii for systems of Laurent polynomials and toric varieties [Kho77, §2].

2.1 The mixed extension of a finite game

Definition 2.1.1. (a) $(\mathcal{A}, S, U)$ denotes a finite game.
Here $m \in \mathbb{N} = \{1, 2, 3, \dots\}$, $\mathcal{A} := \{1, \dots, m\}$ is the set of players,
$S^i = \{s^i_0, \dots, s^i_{n_i}\}$ with $n_i \in \mathbb{N}$ is the set of pure strategies of player $i \in \mathcal{A}$,
$S = S^1 \times \dots \times S^m$ is the set of pure strategy combinations,
$U^i : S \to \mathbb{R}$ is the utility function of player i,
and $U = U^1 \times \dots \times U^m : S \to \mathbb{R}^m$.
We denote $N^i := \{1, \dots, n_i\}$, $N^i_0 := \{0\} \cup N^i$ and $J := \prod_{i=1}^m N^i_0$.
The pure strategy combinations are given as tuples $(s^1_{j_1}, \dots, s^m_{j_m}) \in S$ with $\underline{j} = (j_1, \dots, j_m) \in J$.

(b) $(\mathcal{A}, G, V)$ denotes the *mixed extension* of the finite game in (a). Here

$$W^i := \bigoplus_{j=0}^{n_i} \mathbb{R} \cdot s^i_j, \qquad W := W^1 \times \dots \times W^m,$$

$$A^i := \{\sum_{j=0}^{n_i} \gamma^i_j s^i_j \in W^i \mid \sum_{j=0}^{n_i} \gamma^i_j = 1\}, \qquad A := A^1 \times \dots \times A^m \subset W,$$

$$G^i := \{\sum_{j=0}^{n_i} \gamma^i_j s^i_j \in A^i \mid \gamma^i_j \in [0,1]\}, \qquad G := G^1 \times \dots \times G^m \subset A.$$

So, W^i and W are real vector spaces, $A^i \subset W^i$ and $A \subset W$ are affine linear subspaces of codimension 1 respectively m, $G^i \subset A^i$ is a simplex in A^i of the same dimension n_i as A^i, and $G \subset A$ is a product of simplices, so especially a convex polytope, and it has the same dimension $\sum_{i=1}^m n_i$ as A.
The map $V^i_W : W \to \mathbb{R}^m$ is the multilinear extension of U^i,

$$V^i_W(g) := \sum_{(j_1,\dots,j_m) \in J} \Big(\prod_{k=1}^m \gamma^k_{j_k} \Big) \cdot U^i(s^1_{j_1}, \dots, s^m_{j_m}),$$

$$\text{where} \quad g = (g^1, \dots, g^m) \in W \text{ with } g^k = \sum_{j=0}^{n_k} \gamma^k_j s^k_j,$$

$V^i_A : A \to \mathbb{R}$ is the restriction of V^i_W to A, and $V^i : G \to \mathbb{R}^m$ is the restriction of V^i_W to G. Then $V = (V^1, \dots, V^m) : G \to \mathbb{R}^m$.
An element $a \in A$ is called a *virtual mixed strategy combination*.
An element $g \in G$ is called a *mixed strategy combination*.
The support of an element $g^i = \sum_{j=0}^{n_i} \gamma^i_j s^i_j \in W^i$ is the set

$$\operatorname{supp}(g^i) := \{j \in N^i_0 \,|\, \gamma^i_j \neq 0\}.$$

We also denote $G^{-i} := G^1 \times \cdots \times G^{i-1} \times G^{i+1} \times \cdots \times G^m$, and its elements $g^{-i} := (g^1, \dots, g^{i-1}, g^{i+1}, \dots, g^m) \in G^{-i}$. We follow the standard convention and we identify $G^i \times G^{-i}$ with G and (g^i, g^{-i}) with g by abuse of notation.

(c) Fix $i \in \mathcal{A}$. The *best reply map* $r^i : G^{-i} \to \mathcal{P}(G^i)$ associates to each element $g^{-i} \in G^{-i}$ the set of its best replies in G^i,

$$r^i(g^{-i}) := \{g^i \in G^i \,|\, V^i(g^i, g^{-i}) \geq V^i(\widetilde{g}^i, g^{-i}) \text{ for any } \widetilde{g}^i \in G^i\}.$$

Its graph is the set

$$\mathrm{Gr}(r^i) := \bigcup_{g^{-i} \in G^{-i}} r^i(g^{-i}) \times \{g^{-i}\} \subset G^i \times G^{-i} = G.$$

A *Nash equilibrium* is an element of the set $\mathcal{N} := \bigcap_{i \in \mathcal{A}} \mathrm{Gr}(r^i)$. The set $\mathcal{N}$ is the set of Nash equilibria.

(d) Write $\underline{\gamma}^i = (\gamma^i_1, \dots, \gamma^i_{n_i})$,

$$\begin{aligned} \underline{\gamma} &:= (\underline{\gamma}^1; \dots; \underline{\gamma}^m) = (\gamma^1_1, \dots, \gamma^1_{n_1}; \dots; \gamma^m_1, \dots, \gamma^m_{n_m}) \quad \text{and} \\ \underline{\gamma}^{-i} &:= (\underline{\gamma}^1; \dots; \underline{\gamma}^{i-1}; \underline{\gamma}^{i+1}; \dots, \underline{\gamma}^m). \end{aligned}$$

Fix a subset $K \subset \mathcal{A}$. The monomials $\prod_{i \in K} \gamma^i_{j_i}$ for $(j_i \mid i \in K) \in \prod_{i \in K} N^i_0$ in $\mathbb{R}[\gamma^i_0, \underline{\gamma}^i \mid i \in K]$ (with $\prod_{i \in \emptyset}(\dots) := 1$) are called *$K$-multilinear.* A polynomial which is a real linear combination of K-multilinear monomials is also called K-multilinear. The space of K-multilinear polynomials is called P^K. Especially $V^i_W \in P^{\mathcal{A}}$. A polynomial in $\mathbb{R}[\gamma^i_0, \underline{\gamma}^i \mid i \in \mathcal{A}]$ is *multi affine linear* if each monomial in it with nonvanishing coefficient is K-multilinear for a suitable set $K \subset \mathcal{A}$.

The set $\mathcal{N}$ of Nash equilibria is not empty. This was first proved by Nash [Nas51]. The following lemma is rather trivial, but worth to be noted.

Lemma 2.1.2. *Let $(\mathcal{A}, G, V)$ be the mixed extension of a finite game $(\mathcal{A}, S, U)$.*

(a) The tuple $\underline{\gamma}^i$ is a tuple of (affine linear) coordinates on A^i, because in A^i we have $\gamma^i_0 = 1 - \sum_{j=1}^{n_i} \gamma^i_j$. The tuple $\underline{\gamma} = (\underline{\gamma}^1; \dots; \underline{\gamma}^m)$ is a tuple of (affine linear) coordinates on A. The map V^i_A is a multi affine linear

polynomial in $\underline{\gamma}$. *It has the shape*

$$V_A^i(g) = \kappa^i(\underline{\gamma}^{-i}) + \sum_{j=1}^{n_i} \gamma_j^i \cdot \lambda_j^i(\underline{\gamma}^{-i}) \quad \text{for} \quad g \in A, \tag{2.1}$$

where κ^i *and all* λ_j^i *are unique multi affine linear polynomials in* $\underline{\gamma}^{-i}$. *Define additionally*

$$\lambda_0^i := 0 \quad \textit{for } i \in \mathcal{A}.$$

(b) An element $g = (g^i, g^{-i}) \in G$ *is in* $\mathrm{Gr}(r^i)$ *if and only if the following holds.*

$$\lambda_j^i(\underline{\gamma}^{-i}) - \lambda_k^i(\underline{\gamma}^{-i}) = 0 \ , \ \textit{if } j, k \in \operatorname{supp}(g^i), \tag{2.2}$$

$$\lambda_j^i(\underline{\gamma}^{-i}) - \lambda_k^i(\underline{\gamma}^{-i}) \geq 0 \ , \ \textit{if } j \in \operatorname{supp}(g^i), k \notin \operatorname{supp}(g^i). \tag{2.3}$$

Proof. (a) Part (a) holds because A^i is the affine hyperplane in W^i defined by $\gamma_0^i = 1 - \sum_{j=1}^{n_i} \gamma_j^i$.

(b) A change of $g^i = \sum_{j=0}^{n_i} \gamma_j^i s_j^i$ may not increase $V_A^i(g)$ in (2.1). Therefore $\lambda_j^i(\underline{\gamma}^{-i})$ for $j \in \operatorname{supp}(g^i)$ must be the maximum of all $\lambda_k^i(\underline{\gamma}^{-i})$. This includes the case $j = 0$ if $0 \in \operatorname{supp}(g^i)$, and it includes the case $k = 0$. □

Definition 2.1.3. An element g, for which (2.2), but not necessarily (2.3) holds, is called an *equilibrium candidate.* If $g \in A$ then the equilibrium candidate is virtual.

2.2 Compactification of A and generic games

Consider as in Section 2.1 a finite set $\mathcal{A} = \{1, \dots, m\}$ of players and for each player $i \in \mathcal{A}$ a finite set $S^i = \{s^i_0, \dots, s^i_{n_i}\}$ with $n_i \in \mathbb{N}$ of pure strategies. Let $\mathcal{U}^i = \mathbb{R}^S$ be the set of all possible utility functions $U^i : S \to \mathbb{R}$. The set of all possible utility maps $U = (U^1, \dots, U^m)$ is then $\mathcal{U} := \prod_{i=1}^m \mathcal{U}^i \cong (\mathbb{R}^S)^m$.

Consider a fixed map U, the tuple of all hyperplanes in A which bound $G \subset A$ and the subvarieties $(\lambda^i_j - \lambda^i_k)^{-1}(0) \subset A$ for $i \in \mathcal{A}$ and $j, k \in N^i_0$ with $j < k$. By Lemma 2.1.2 (b), the graphs $\mathrm{Gr}(r^i)$ of the best reply maps and the set $\mathcal{N}$ of Nash equilibria are determined by the geometry of these hyperplanes and these subvarieties.

The hyperplanes which bound G are smooth and transversal. Theorem 2.2.1 below will imply that for generic $U \in \mathcal{U}$ also the subvarieties $(\lambda^i_j - \lambda^i_k)^{-1}(0)$ are smooth hypersurfaces in A and that they and the hyperplanes in A which bound G are as transversal as possible.

Theorem 2.2.1 is probably known. But we are not aware of a reference. Therefore we formulate it below, and we prove it in Section 2.4. The proof is an application of Sard's theorem or of a version of it in the semialgebraic setting.

The fact that V^i_W is $\mathcal{A}$-multilinear motivates to consider the natural compactification of A^i to the real projective space $\mathbb{P}W^i$, which is the set $(W^i \setminus \{0\})/\mathbb{R}^*$ of lines through 0 in W^i, and the natural compactification of A to the product of real projective spaces

$$\mathbb{P}^{\mathcal{A}}W := \prod_{i=1}^{m} \mathbb{P}W^i.$$

We denote $\mathbb{P}^{-i}W := \prod_{j \in \mathcal{A} \setminus \{i\}} \mathbb{P}W^i$, and we identify (following the slightly incorrect convention in Definition 2.1.1 (b)) $\mathbb{P}W^i \times \mathbb{P}^{-i}W$ with $\mathbb{P}^{\mathcal{A}}W$. Under the projection

$$pr_W : \prod_{i=1}^{m} (W^i \setminus \{0\}) \to \mathbb{P}^{\mathcal{A}}W,$$

the affine linear space $A \subset W$ embeds as a Zariski open subset into $\mathbb{P}^{\mathcal{A}}W$. The complement is the union of m hyperplanes

$$H^{i,\infty} := (\mathbb{P}W^i \setminus A^i) \times \mathbb{P}^{-i}W \subset \mathbb{P}W^i \times \mathbb{P}^{-i}W = \mathbb{P}^{\mathcal{A}}W \qquad (2.4)$$

for $i \in \mathcal{A}$. For a subset $B^i \subset A^i$, let $\overline{B^i}^{Zar}$ denote its Zariski closure in $\mathbb{P}W^i$, that is the smallest real algebraic subvariety in $\mathbb{P}W^i$ which contains B^i. The Zariski closure in $\mathbb{P}^{\mathcal{A}}W$ of a subset $B \subset \mathbb{P}^{\mathcal{A}}W$ is denoted by $\overline{B}^{Zar}$. For $i \in \mathcal{A}$ and $j \in N_0^i$ denote by

$$H^{i,j} := \overline{\{g^i \in A^i \,|\, \gamma_j^i = 0\}}^{Zar} \times \mathbb{P}^{-i}W \subset \mathbb{P}W^i \times \mathbb{P}^{-i}W = \mathbb{P}^{\mathcal{A}}W \qquad (2.5)$$

the Zariski closures in $\mathbb{P}^{\mathcal{A}}W$ of the hyperplanes in A which bound G. Define

$$N_0^{i,2} := \{(j,k) \in N_0^i \times N_0^i \,|\, j < k\}.$$

For $i \in \mathcal{A}$ and $(j,k) \in N_0^{i,2}$ consider the difference $\lambda_j^i - \lambda_k^i$ as a function on A (so lift it as a function from $\prod_{l \neq i} A^l$ to A), and consider the Zariski closure in $\mathbb{P}^{\mathcal{A}}W$

$$H^{i,(j,k)} := \overline{(\lambda_j^i - \lambda_k^i)^{-1}(0)}^{Zar} \subset \mathbb{P}^{\mathcal{A}}W. \qquad (2.6)$$

The notion *everywhere transversal* in Theorem 2.2.1 is defined in Definition 2.3.4 (b). In Theorem 2.2.1, a subset of the set of all hyperplanes in (2.4) and (2.5) and all subvarieties in (2.6) is considered. Such a subset is characterized by its set of indices, namely sets $T^i \subset N_0^i \cup \{\infty\}$ and sets $R^i \subset N_0^{i,2}$ for $i \in \mathcal{A}$ define a subset

$$\bigcup_{i \in \mathcal{A}} \Big(\{H^{i,j} \,|\, j \in T^i\} \cup \{H^{i,(j,k)} \,|\, (j,k) \in R^i\} \Big). \qquad (2.7)$$

A set $R^i \subset N_0^{i,2}$ defines a graph with vertex set N_0^i and set R^i of edges. The set in (2.7) is called *good* if the graph (N_0^i, R^i) is a union of trees.

The reason for the introduction of this notion of a *good* set is that in the case of $|\operatorname{supp}(g^i)| \geq 3$ there is some redundancy in the equations (2.2). Equality for all pairs (j,k) with $j,k \in \operatorname{supp}(g^i)$ is implied by equality for a set of pairs (j,k) such that the graph with vertex set $\operatorname{supp}(g^i)$ and edge set this set of pairs is a tree.

Theorem 2.2.1. *Let $\mathcal{A} = \{1,\dots,m\}$ and S be as in Section 2.1 and as above. There is semialgebraic subset $\mathcal{D} \subset \mathcal{U}$ of codimension at least 1 (equivalently, it is of Lebesgue measure 0) such that for any tuple $U \in \mathcal{U} \backslash \mathcal{D}$ of utility functions the following holds. The hyperplanes $H^{i,j}, i \in \mathcal{A}, j \in$*

$N_0^i \cup \{\infty\}$, and the subvarieties $H^{i,(j,k)}, i \in \mathcal{A}, (j,k) \in {N_0^i}^2$, are smooth hypersurfaces in $\mathbb{P}^{\mathcal{A}}W$. Any good subset of them is everywhere transversal.

The proof of Theorem 2.2.1 will be given in Section 2.4. Before, Section 2.3 will recall some basic notions and facts from differential topology.

2.3 Transversality of submanifolds

Definition 2.3.1. Let M and N be C^∞-manifolds, and let $F : M \to N$ be a C^∞-map. A point $p \in M$ is a *regular point* of F if the linear map $dF|_{T_pM} : T_pM \to T_{F(p)}N$ is surjective. A point $p \in M$ is a *critical point* if it is not a regular point. A point $q \in N$ is a *regular value* if either $F^{-1}(q) = \emptyset$ or every point $p \in F^{-1}(q)$ is a regular point. A point $q \in N$ is a *critical value* if it is not a regular value.

The following theorem is famous. It is also crucial in the proof of Theorem 2.2.1.

Theorem 2.3.2. *(Sard's theorem [Sar42], e.g. [BL75, Theorem 2.1]) Let $F : M \to N$ be a C^∞-map between C^∞-manifolds. The subset of N of critical values of F has Lebesgue measure 0.*

In our situation, the setting is semialgebraic. Therefore the subset of critical values is then a semialgebraic subset of N of Lebesgue measure 0. This means that it has everywhere smaller dimension than N. A variant of Sard's theorem in the semialgebraic setting says precisely this [BCR98, ch. 9.5] [BR90, 2.5.12].

The implicit function theorem says how a map $F : M \to N$ looks near a regular point $p \in M$. It is a trivial fibration with smooth fibers.

Theorem 2.3.3. *(Implicit function theorem, e.g. [BL75, Theorem 1.3]) Let $F : M \to N$ be a C^∞-map between C^∞-manifolds, and let $p \in M$ be a regular point of F. Then $\dim M \geq \dim N$, and there are open neighborhoods $U_1 \subset M$ of p and $U_2 \subset N$ of $F(p)$ with $U_2 \supset F(U_1)$, open balls $B_1 \subset \mathbb{R}^{\dim M}$ around 0 and $B_2 \subset \mathbb{R}^{\dim N}$ around 0 and C^∞-diffeomorphisms $\varphi_1 : B_1 \to U_1$ and $\varphi_2 : B_2 \to U_2$ with the following property. The map $\varphi_2^{-1} \circ F \circ \varphi_1 : B_1 \to B_2$ is the standard projection in* (2.8),

$$\begin{tikzcd} U_1 \arrow[r, "F"] & U_2 \\ B_1 \arrow[u, "\varphi_1"] \arrow[r, "\varphi_2^{-1}\circ F\circ\varphi_1"] & B_2 \arrow[u, "\varphi_2"'] \end{tikzcd} \tag{2.8}$$
$$(x_1, \ldots, x_{\dim M}) \longmapsto (x_1, \ldots, x_{\dim N})$$

Definition 2.3.4. (a) Let M be a C^∞-manifold, let $p \in M$ and let $H \subset M$ be a subset with $p \in H$ and which is in a neighborhood of p a C^∞-submanifold of M. A *defining map* F for the pair (H, p) is a function $F : U \to \mathbb{R}^n$ with $U \subset M$ an open neigborhood of p such that F is regular on each point of U and $H \cap U = F^{-1}(F(p))$. Then $n = \dim M - \dim H \cap U$.

(b) Let M be a C^∞-manifold, and let $H_1, \dots, H_a$ be C^∞-submanifolds. Now it will be defined when they are *transversal at a point* $p \in M$.

(i) They are transversal at $p \in M \setminus (\cup_{i=1}^a H_i)$.

(ii) They are transversal at $p \in \cap_{i=1}^a H_i$ if for some (or for any, that is equivalent) tuple of defining maps $F_i : U \to \mathbb{R}^{n_i}$ of (H_i, p) ($i \in \{1, \dots, a\}$) with joint definition domain U the map $(F_1, \dots, F_a) : U \to \mathbb{R}^{n_1+\dots+n_a}$ is regular at p.

(iii) They are transversal at $p \in \cup_{i=1}^a H_i$ if the subset $\{H_i \,|\, p \in H_i\}$ of the set of manifolds $H_1, \dots, H_a$ is transversal at p.

Finally, they are *transversal* (or *everywhere transversal*) if they are transversal at each point of M.

Remark 2.3.5. Defining maps for (H, p) exist and are related by local diffeomorphisms.

Part (a) of the following lemma states an obvious consequence on the intersection of a family of transversal submanifolds. Part (b) gives a useful criterion for proving transversality of a family of submanifolds.

Lemma 2.3.6. *Let M be a C^∞-manifold, and let $H_1, \dots, H_a$ be C^∞-submanifolds.*

(a) If they are transversal then either $\cap_{i=1}^a H_i = \emptyset$ or this intersection is a submanifold of M of dimension $\dim M - \sum_{i=1}^a \operatorname{codim} H_i$. Especially, in the second case, this number is non-negative.

(b) Suppose that for some $b \in \{1, .., a-1\}$ the submanifolds $H_1, \dots, H_b$ are transversal at a point $p \in \cap_{i=1}^a H_i$. The intersection $L := \cap_{i=1}^b H_i$ is by part (a) in a suitable open neighborhood of p a submanifold of M. The following two conditions are equivalent:

(i) $H_1, \dots, H_a$ are transversal at p.

(ii) For $j \in \{b+1, \dots, a\}$ there are defining maps $F_j : U_j \to \mathbb{R}^{n_j}$ on open neighborhoods $U_j \subset M$ of p such that for $U := \bigcap_{j=b+1}^{a} U_j$ the intersection $U \cap L$ is a submanifold of U and the map $(F_{b+1}, \dots, F_a)|_{U\cap L} : U \cap L \to \mathbb{R}^{n_{b+1}+\dots+n_a}$ is regular at p.

Proof. (a) Part (a) follows immediately from the definition of transversality and the implicit function theorem.

(b) By the implicit function theorem we can choose an open neighborhood U of p in M, coordinates $\underline{x} = (x_1, \dots, x_m)$ on U with $\underline{x}(p) = 0$ and defining maps $F_i := U \to \mathbb{R}^{n_i}$ of (H_i, p) for $i \in \{1, \dots, a\}$ such that $(F_1, \dots, F_b) = (x_1, \dots, x_{\sum_{i=1}^{b} n_i})$. Condition (i) is equivalent to

$$\operatorname{rank}\left(\frac{\partial F_i}{\partial x_j}(p)\right)_{\substack{i\,\in\,\{1,\dots,a\}\\ j\,\in\,\{1,\dots,\dim M\}}} = \sum_{i=1}^{a} n_i. \tag{2.9}$$

Condition (ii) is equivalent to

$$\operatorname{rank}\left(\frac{\partial F_i|_{U\cap L}}{\partial x_j}(p)\right)_{\substack{i\,\in\,\{b+1,\dots,a\}\\ j\,\in\,\{1+\sum_{i=1}^{b} n_i,\dots,\dim M\}}} = \sum_{i=b+1}^{a} n_i. \tag{2.10}$$

The matrix in (2.9) has a block triangular shape $\left(\begin{array}{c|c} \mathbf{1} & 0 \\ \hline * & * \end{array}\right)$ with the matrix in (2.10) in the lower right place. Therefore (2.9) and (2.10) are equivalent.

□

Remark 2.3.7. (i) In condition (ii) in Lemma 2.3.6 (b), one can replace the existence of the defining maps by demanding that the condition holds for all choices of defining maps.

(ii) In the proof in Section 2.4 of the transversality in Theorem 2.2.1, Lemma 2.3.6 (b) will be useful. The hyperplanes $H^{i,j}$ in a given set in (2.7) take the role of the submanifolds $H_1, \dots, H_b$. The reason is that they are fixed if one moves $U \in \mathcal{U}$, while the other subvarieties $H^{i,(j,k)}$ move if one moves $U \in \mathcal{U}$.

(iii) In Lemma 2.3.6 (b) one cannot replace condition (ii) by the simpler condition (ii)': $L \cap H_{b+1}, \dots, L \cap H_a$ *are submanifolds and are transversal at* p. The reason is that $L \cap H_j$ for some $j \in \{b+1, \dots, a\}$ can be a submanifold of the correct dimension although L and H_j are not transversal.

2.4 Proof of Theorem 2.2.1

This section is devoted to the proof of Theorem 2.2.1.

Proof. First we discuss affine charts on $\mathbb{P}W^i$ and on $\mathbb{P}^{\mathcal{A}}W$. The linear coordinates $(\gamma_0^i, \ldots, \gamma_{n_i}^i)$ on W^i induce $n_i + 1$ affine charts of the projective space $\mathbb{P}W^i$. But none of them contains the whole set G^i. The following linear coordinates $\underline{\widetilde{\gamma}}^i := (\widetilde{\gamma}_0^i, \ldots, \widetilde{\gamma}_{n_i}^i)$ on W^i are equally natural, and one of the affine charts which they induce on $\mathbb{P}W^i$ will turn out to be A^i,

$$\widetilde{\gamma}_j^i := \gamma_j^i \text{ for } j \in N^i, \quad \widetilde{\gamma}_0^i := \sum_{j=0}^{n_i} \gamma_j^i.$$

From now on we use the induced homogeneous coordinates $(\widetilde{\gamma}^{i_0} : \cdots : \widetilde{\gamma}_{n_i}^i)$ on $\mathbb{P}W^i$ if not said otherwise. The $n_i + 1$ induced affine charts of $\mathbb{P}W^i$ are

$$A_j^i := \{(\widetilde{\gamma}_0^i : \cdots : \widetilde{\gamma}_{n_i}^i) \in \mathbb{P}W^i \mid \underline{\widetilde{\gamma}}^i \in W^i, \widetilde{\gamma}_j^i = 1\} \subset \mathbb{P}W^i \quad \text{for } j \in N_0^i.$$

A_j^i comes equipped with natural coordinates
$\underline{\gamma}^{i,j} = (\gamma_0^{i,j}, \ldots, \gamma_{j-1}^{i,j}, \gamma_{j+1}^{i,j}, \ldots, \gamma_{n_i}^{i,j})$ from the isomorphism

$$\mathbb{R}^{n_i} \to A_j^i, \quad \underline{\gamma}^{i,j} \mapsto (\gamma_0^{i,j} : \cdots : \gamma_{j-1}^{i,j} : 1 : \gamma_{j+1}^{i,j} : \cdots : \gamma_{n_i}^{i,j}),$$

and we have a natural embedding

$$\alpha^{i,j} : A_j^i \hookrightarrow W^i, \quad \underline{\gamma}^{i,j} \mapsto (\gamma_0^{i,j}, \ldots, \gamma_{j-1}^{i,j}, 1, \gamma_{j+1}^{i,j}, \ldots, \gamma_{n_i}^{i,j}),$$

with $\mathrm{pr}_W^i \circ \alpha^{i,j} = \mathrm{id}$ where $\mathrm{pr}_W^i : W^i \setminus \{0\} \to \mathbb{P}W^i$ is the natural projection. The image $\alpha^{i,0}(A_0^i) \subset W^i$ coincides with the set $A^i \subset W^i$ in Definition 2.1.1 (b), and the identification $\underline{\gamma}^{i,0} = \underline{\gamma}^i$ identifies A_0^i with A^i.

All possible products over $i \in \mathcal{A}$ of these charts give the affine charts

$$A_{\underline{j}} := \prod_{i=1}^m A_{j_i}^i \cong \mathbb{R}^{\sum_{i=1}^m n_i} \quad \text{for } \underline{j} = (j_1, \ldots, j_m) \in J = \prod_{i=1}^m N_0^i.$$

of $\mathbb{P}^{\mathcal{A}}W$ with coordinates $\underline{\gamma}^{\underline{j}} = (\underline{\gamma}^{1,j_1}; \ldots; \underline{\gamma}^{m,j_m})$. The embeddings $\alpha^{i,j_i} : A_{j_i}^i \hookrightarrow W^i$ combine to an embedding

$$\alpha^{\underline{j}} : A_{\underline{j}} \hookrightarrow W$$

with $\alpha^{\underline{j}} \circ \mathrm{pr}_W = \mathrm{id}$ on $A_{\underline{j}}$. The image $\alpha^{\underline{0}}(A_{\underline{0}}) \subset W$ coincides with the set $A \subset W$ in Definition 2.1.1 (b), and the identification $\underline{\gamma}^{\underline{0}} = \underline{\gamma}$ identifies $A_{\underline{0}}$ with A.

Now we discuss the hyperplanes $H^{i,j}$ and the subvarieties $H^{i,(j,k)}$ from Section 2.2 with respect to the new linear coordinates $\underline{\widetilde{\gamma}} = (\underline{\widetilde{\gamma}}^1; \ldots; \underline{\widetilde{\gamma}}^m)$ on W. We define for $i \in \mathcal{A}$

$$\widetilde{H}^{i,j} := H^{i,j} \text{ for } j \in N^i, \quad \widetilde{H}^{i,0} := H^{i,\infty}, \quad \widetilde{H}^{i,\infty} := H^{i,0}.$$

We have for $i \in \mathcal{A}$

$$\begin{aligned}
\widetilde{H}^{i,j} &= H^{i,j} = \mathrm{pr}_W\Big(\{\underline{\widetilde{\gamma}} \in \prod_{k=1}^m (W^k \setminus \{0\}) \mid \widetilde{\gamma}^i_j = 0\}\Big) \quad \text{for } j \in N^i, \\
\widetilde{H}^{i,0} &= H^{i,\infty} = \mathrm{pr}_W\Big(\{\underline{\widetilde{\gamma}} \in \prod_{k=1}^m (W^k \setminus \{0\}) \mid \widetilde{\gamma}^i_0 = 0\}\Big), \\
\widetilde{H}^{i,\infty} &= H^{i,0} = \mathrm{pr}_W\Big(\{\underline{\widetilde{\gamma}} \in \prod_{k=1}^m (W^k \setminus \{0\}) \mid \widetilde{\gamma}^i_0 - \sum_{j=1}^{n_i} \widetilde{\gamma}^i_j = 0\}\Big).
\end{aligned}$$

Obviously for each chart $A_{\underline{j}}$, the complement is

$$\mathbb{P}^{\mathcal{A}} W \setminus A_{\underline{j}} = \bigcup_{i=1}^m \widetilde{H}^{i,j_i}. \tag{2.11}$$

For each of the hyperplanes $\widetilde{H}^{i,j} \cap A_{\underline{j}}$ with $j \in N^i_0 \setminus \{j_i\}$, in the chart $A_{\underline{j}}$ a defining map (in the sense of Definition 2.3.4 (a)) is $\widetilde{\gamma}^i_j \circ \alpha^{\underline{j}}$. For each of the hyperplanes $\widetilde{H}^{i,\infty} \cap A_{\underline{j}}$ in the chart $A_{\underline{j}}$ a defining map is $\Big(\widetilde{\gamma}^i_0 - \sum_{j=1}^{n_i} \widetilde{\gamma}^i_j\Big) \circ \alpha^{\underline{j}}$.

Obviously, each subset of the set $\{\widetilde{H}^{i,j} \mid i \in \mathcal{A}, j \in N^i_0 \cup \{\infty\}\}$ of all these hyperplanes is transversal everywhere in $\mathbb{P}^{\mathcal{A}} W$. The map $V^i_W : W \to \mathbb{R}$ is an $\mathcal{A}$-multilinear map also in the new linear coordinates $\underline{\widetilde{\gamma}}$. Write it as a sum

$$V^i_W(\underline{\widetilde{\gamma}}) = \widetilde{\gamma}^i_0 \cdot K^i(\underline{\widetilde{\gamma}}^{-i}) + \sum_{j=1}^{n_i} \widetilde{\gamma}^i_j \cdot \Lambda^i_j(\underline{\widetilde{\gamma}}^{-i}) \quad \text{for} \quad \underline{\widetilde{\gamma}} = (\underline{\widetilde{\gamma}}^1, \ldots, \underline{\widetilde{\gamma}}^m) \in W.$$

Here $K^i(\underline{\widetilde{\gamma}}^{-i})$ and $\Lambda^i_j(\underline{\widetilde{\gamma}}^{-i})$ are $\mathcal{A}\setminus\{i\}$-multilinear in $\underline{\widetilde{\gamma}}^{-i}$. Define additionally

$$\Lambda^i_0 := 0.$$

We have for $i \in \mathcal{A}$ and $(j,k) \in N_0^{i,2}$

$$H^{i,(j,k)} = \mathrm{pr}_W\Big(\{\underline{\widetilde{\gamma}} \in \prod_{k=1}^{m}(W^k \setminus \{0\}) \mid (\Lambda^i_j - \Lambda^i_k)(\underline{\widetilde{\gamma}}) = 0\}\Big). \tag{2.12}$$

The intersection $H^{i,(j,k)} \cap A_{\underline{l}}$ of the subvariety $H^{i,(j,k)} \subset \mathbb{P}^{\mathcal{A}}W$ with the affine chart $A_{\underline{l}}$ for $\underline{l} = (l_1, \ldots, l_m) \in J$ is the zero set of the function $(\Lambda^i_j - \Lambda^i_k) \circ \alpha^{\underline{l}}$.

Especially, the identifications $\underline{\gamma}^{\underline{0}} = \underline{\gamma}$ and $A_{\underline{0}} = A$ yield the identification $\Lambda^i_j \circ \alpha^{\underline{0}} = \lambda^i_j$. This connects the description of $H^{i,(j,k)}$ in (2.12) with the one in (2.6).

Now the main point will be to prove that there is a subset $\mathcal{D} \subset \mathcal{U}$ of Lebesgue measure 0 with the properties in Theorem 2.2.1. Later we will argue that it is semialgebraic. Then $\mathcal{D}$ having Lebesgue measure 0 is equivalent to $\mathcal{D}$ having codimension at least 1 in $\mathcal{U}$.

Choose some $\underline{l} \in J$ and consider the chart $A_{\underline{l}}$ of $\mathbb{P}^{\mathcal{A}}W$. Choose for each $i \in \mathcal{A}$ a set $T^i \subset N_0^i \cup \{\infty\}$ with $l_i \notin T^i$ (we exclude the case l_i because $\widetilde{H}^{i,l_i} \subset \mathbb{P}^{\mathcal{A}}W \setminus A_{\underline{l}}$ by (2.11)) and a set $R^i \subset N_0^{i,2}$ such that (N_0^i, R^i) is a union of trees and such that (for all i together) $\bigcup_{i=1}^m R^i \neq \emptyset$. Though $T^i = \emptyset$ (for some or all $i \in \mathcal{A}$) and $R^i = \emptyset$ (for some, but not all $i \in \mathcal{A}$) are allowed. Now we go into Lemma 2.3.6 (b) with

$$M = A_{\underline{l}} \quad \text{and} \quad \{H_1, \ldots, H_b\} = \{\widetilde{H}^{i,j} \cap A_{\underline{l}} \,|\, i \in \mathcal{A}, j \in T^i\}$$

(the second set is empty if all $T^i = \emptyset$) and

$$L = \bigcap_{i \in \mathcal{A}, j \in T^i} \widetilde{H}^{i,j} \cap A_{\underline{l}},$$

which is an affine linear subspace of $A_{\underline{l}}$ of codimension $\sum_{i=1}^m |T^i|$. We can choose a part of the coordinates $\underline{\gamma}^{\underline{l}}$ on $A_{\underline{l}}$ which forms affine linear coordinates on L. We choose such a part and call it $\underline{\gamma}^{\underline{l},L}$.

For a moment, fix one map $U \in \mathcal{U}$ and consider the map

$$\begin{aligned} F^{(U)} &: L \to N \quad \text{with } N := \mathbb{R}^{\sum_{i \in \mathcal{A}} |R^i|} \\ &\underline{\gamma}^{\underline{l}} \mapsto \Big(((\Lambda^i_j - \Lambda^i_k) \circ \alpha^{\underline{l}})(\underline{\gamma}^{\underline{l}}) \mid i \in \mathcal{A}, (j,k) \in R^i\Big). \end{aligned}$$

Each entry $(\Lambda^i_j - \Lambda^i_k) \circ \alpha^{\underline{l}})|_L$ of this map $F^{(U)}$ is a multi affine linear function in those coordinates of $\underline{\gamma}^{\underline{l},L}$ which are not in $\underline{\gamma}^{i,\underline{l}}$.

By Sard's theorem (Theorem 2.3.2) the set of critical values of $F^{(U)}$ has Lebesgue measure 0 in N.

Now we consider simultaneously all $U \in \mathcal{U}$. We claim that the set of $U \in \mathcal{U}$ such that $0 \in N$ is a critical value of $F^{(U)}$ has Lebesgue measure 0 in $\mathcal{U}$. The argument for this will connect some natural coordinate system on $\mathcal{U}$ with the coefficients of the monomials in the entries of $F^{(U)}$.

The space of possible maps $F^{(U)}$ can be identified with the product

$$\mathcal{U}^{(I)} := \prod_{i \in \mathcal{A}, (j,k) \in R^i} \mathbb{R}[\underline{\gamma}^{\underline{l},L} \backslash \underline{\gamma}^{i,\underline{l}}]_{\mathbf{mal}}$$

where $\mathbb{R}[\underline{\gamma}^{\underline{l},L} \backslash \underline{\gamma}^{i,\underline{l}}]_{\mathbf{mal}}$ denotes the space of multi affine linear polynomials in those coordinates in $\underline{\gamma}^{\underline{l},L}$ which are not in $\underline{\gamma}^{i,\underline{l}}$ (here **mal** stands for ***m***ulti ***a***ffine ***l***inear). The space of possible constant summands in these tuples $F^{(U)}$ can be identified with the space $N = \mathbb{R}^{\sum_{i \in \mathcal{A}} |R^i|}$.

There are natural linear maps $p_{(I)}$ and $p_{(II)}$ defined by

$$\begin{aligned} p_{(I)} : \mathcal{U} \to \mathcal{U}^{(I)}, \quad & p_{(I)}(U) = F^{(U)}, \\ p_{(II)} : \mathcal{U}^{(I)} \to N, \quad & p_{(II)}(F) = F(0). \end{aligned}$$

Because of the hypothesis that (N_0^i, R^i) is for any $i \in \mathcal{A}$ a union of trees, the maps $p_{(I)}$ and $p_{(II)}$ are surjective. To see this, recall that the $\mathcal{A}$-multilinear maps V_W^i are sums of all monomials in the variables $\widetilde{\gamma}_j^i$ with coefficients, which form a natural linear coordinate system of the $\mathbb{R}$-vector space $\mathcal{U}$. For example, the coefficient of the monomial $\widetilde{\gamma}_j^i \prod_{a \in \mathcal{A} - \{i\}} \widetilde{\gamma}_{l_a}^a$ in V_W^i is the coefficient of the constant term in the multi affine linear map $\Lambda_j^i \circ \alpha^{\underline{l}}$. The entries of $F^{(U)}$ for a fixed U are the multi affine linear maps $((\Lambda_j^i - \Lambda_k^i) \circ \alpha^{\underline{l}})|_L$ for $i \in \mathcal{A}, (j,k) \in R^i$. The condition that (N_0^i, R^i) is a union of trees takes care that sufficiently many of the coefficients of the monomials in the maps V_W^i turn up in the maps $p_{(I)}$ and $p_{(II)}$, so that these maps are surjective.

For $F \in \ker(p_{(II)}) \subset \mathcal{U}^{(I)}$ denote by $C(F) \subset N$ the set of critical values of $F : L \to N$. It has Lebesgue measure 0 in N by Sard's theorem (Theorem 2.3.2). For each map $F - c : L \to N$ with $c \in N$ the value $0 \in N$ is a critical value only if $c \in C(F)$. Therefore 0 is a critical value of $F^{(U)} : L \to N$ for $U \in \mathcal{U}$ only if

$$U \in \bigcup_{F \in \ker(p_{(II)})} \bigcup_{c \in C(F)} p_{(I)}^{-1}(F - c),$$

which is a set of Lebesgue measure 0 in $\mathcal{U}$ (see e.g. Theorem 2.8 (Fubini) in [BL75]).

There are only finitely many charts $A_{\underline{l}}$ and finitely many choices of sets T^i and R^i as above. Each leads only to a set of Lebesgue measure 0 in $\mathcal{U}$. Also their union $\mathcal{D}$ has Lebesgue measure 0 in $\mathcal{U}$. If $U \in \mathcal{U} \setminus \mathcal{D}$, then for each map $F^{(U)} : L \to N$ as above, the value $0 \in N$ is a regular value.

For an arbitrary chart $A_{\underline{l}}$, the special choice $T^a = \emptyset$ for all $a \in \mathcal{A}$, $R^i = \{(j,k)\}$ for one $i \in \mathcal{A}$, and $R^a = \emptyset$ for all $a \neq i$ shows that $H^{i,(j,k)} \cap A_{\underline{l}}$ is a smooth hypersurface for $U \in \mathcal{U} \setminus \mathcal{D}$. It also shows that the map $(\Lambda^i_j - \Lambda^i_k) \circ \alpha_{\underline{l}}$ is a defining map for the pair $(H^{i,(j,k)}, p)$ at each point $p \in H^{i,(j,k)} \cap A_{\underline{l}}$ (in the sense of Definition 2.3.4 (a)).

For an arbitrary chart $A_{\underline{l}}$ and all choices of the sets T^i and R^i, the construction of $\mathcal{D}$ above together with Lemma 2.3.6 (b) and Definition 2.3.4 (b) show that for $U \in \mathcal{U} \setminus \mathcal{D}$ any good subset of the smooth hypersurfaces $H^{i,j}$ and $H^{i,(j,k)}$ is transversal on $A_{\underline{l}}$. Considering all charts together, we obtain all statements of Theorem 2.2.1 except that $\mathcal{D}$ is semialgebraic.

$\mathcal{D}$ is semialgebraic because of the following. For a chart $A_{\underline{l}}$ and sets T^i and R^i as above, the maps $F^{(U)} : L \to N$ unite to an algebraic map

$$\mathcal{F} : \mathcal{U} \times L \to \mathcal{U} \times N, \quad (U, x) \mapsto (U, F^{(U)}(x)).$$

The set of its critical points in $\mathcal{U} \times L$ with critical value $(U, 0) \in \mathcal{U} \times \{0\} \subset \mathcal{U} \times N$ is an algebraic subset of $\mathcal{U} \times L$. Its image under the projection to $\mathcal{U}$ is semialgebraic. Also the union $\mathcal{D}$ of these sets over all choices of charts $A_{\underline{l}}$ and sets T^i and R^i is semialgebraic. □

Remark 2.4.1. This proof of Theorem 2.2.1 is inspired by the proof of Khovanskii of a theorem on generic systems of Laurent polynomials with fixed Newton polyhedra. He considers complex coefficients, we consider real coefficients. But apart from that, our situation and proof can be seen as a special case of his situation and proof. It is the main theorem in §2 in [Kho77]. Though, he uses, but does not formulate the Lemma 2.3.6. The analogue of the toric compactification which he has to construct is in our situation the space $\mathbb{P}^{\mathcal{A}}W$.

The proof of Theorem 2.2.1 in this Section 2.4 implies the following corollary. The regularity which it expresses is most useful in the case of points (e.g. Nash equilibria) in the standard affine chart $A_{\underline{0}}$ and written with the standard defining maps in this chart.

Corollary 2.4.2. *Consider the situation in Theorem 2.2.1. Let U be the utility map of a game with $U \in \mathcal{U} \setminus \mathcal{D}$ (so it is generic). Let $A_{\underline{l}} \subset \mathbb{P}^{\mathcal{A}}W$ be any one of the affine charts of $\mathbb{P}^{\mathcal{A}}W$. Let $\underline{\gamma}^{\underline{l}}$ be any point in $A_{\underline{l}}$. Consider any set of smooth hypersurfaces as in* (2.7) *which is good and which contains the point $\underline{\gamma}^{\underline{l}}$. Then the Jacobian of the defining maps in this chart (which were used in the proof above) for these hypersurfaces is nondegenerate.*

Proof. By Theorem 2.2.1 the subvarieties in a set as in (2.7) are smooth hypersurfaces and everywhere transversal. The transversality in any affine chart $A_{\underline{l}}$ was shown by proving that the point $\underline{\gamma}^{\underline{l}}$ is a regular point of the tuple of defining maps in this chart of the hypersurfaces. $\square$

3 Applications of Chapter 2 and diverse Remarks

Here we consider the same data as in Chapter 2, a finite game $(\mathcal{A}, S, U)$, its mixed extension $(\mathcal{A}, G, V)$ and all other in Chapter 2 associated data. In the following $\mathcal{A}$ and S will be fixed and U will be chosen in $\mathcal{U} \setminus \mathcal{D}$, where $\mathcal{U}$ and $\mathcal{D}$ are as in Theorem 2.2.1, so U is generic.

Theorem 2.2.1 stated that for such U the hyperplanes $H^{i,j}$, $i \in \mathcal{A}$, $j \in N_0^i \cup \{\infty\}$, and the subvarieties $H^{i,(j,k)}$, $i \in \mathcal{A}$, $(j,k) \in N_0^{i,2}$ are smooth hypersurfaces in the product $\mathbb{P}^{\mathcal{A}}W = \prod_{i \in \mathcal{A}} \mathbb{P}W^i$ of projective spaces and that any good subset of them is everywhere transversal.

This chapter collects some applications of this. It tells about known results which fit to the research in other chapters of this thesis. This chapter does not contain own research.

3.1 Finite set of Nash equilibria in generic m-person games

By Definition 2.1.1 (c) and Lemma 2.1.2 (b), an element g of the affine chart $A = A_{\underline{0}}$ of $\mathbb{P}^{\mathcal{A}}W$ is a Nash equilibrium if and only if the following conditions hold for any $i \in \mathcal{A}$,

$$g \in H^{i,(j,k)} \quad \text{for } j,k \in \operatorname{supp}(g^i), \tag{3.1}$$

$$\lambda^i_j(\underline{\gamma}^{-i}) - \lambda^i_j(\underline{\gamma}^{-i}) \geq 0 \quad \text{for } j \in \operatorname{supp}(g^i), k \notin \operatorname{supp}(g^i), \tag{3.2}$$

$$\gamma^i_j > 0 \quad \text{for } j \in \operatorname{supp}(g^i). \tag{3.3}$$

In order to get a complete system of equalities and inequalities, we rewrite the conditions in the definition of the supports,

$$g \in H^{i,j} \quad \text{for } j \in N^i_0 \setminus \operatorname{supp}(g^i). \tag{3.4}$$

There is some redundancy in the equations (3.1). In order to get rid of it, define for $g^i \in A^i$ the set $T^i := \operatorname{supp}(g^i) \subset N^i_0$ and its smallest element

$$j_{min}(T^i) := \min(k \in T^i). \tag{3.5}$$

Then (3.1) can be reduced to the set of equations

$$g \in H^{i,(j_{min}(T^i),k)} \qquad \text{for } k \in T^i \setminus \{j_{min}(T^i)\}. \tag{3.6}$$

The set of hypersurfaces in the conditions (3.4) and (3.6) is

$$\bigcup_{i\in\mathcal{A}} \Big(\{H^{i,j} \mid j \in N^i_0 \setminus T^i\} \cup \{H^{i,(j_{min}(T^i),k)} \mid k \in T^i \setminus \{j_{min}(T^i)\}\} \Big) \tag{3.7}$$

It is a good set of hypersurfaces. By Theorem 2.2.1 it is everywhere transversal. The number of hypersurfaces in this set ist $\sum_{i\in\mathcal{A}} n_i = \dim \mathbb{P}^{\mathcal{A}}W$. Therefore their intersection is a finite set of points in $\mathbb{P}^{\mathcal{A}}W$, which depends (besides the choice of U) only on the tuple $\underline{T} := (T^i)_{i\in\mathcal{A}}$ of supports. The union over all tuples of possible supports is also a finite subset of $\mathbb{P}^{\mathcal{A}}W$. It contains the set $\mathcal{N}(U)$ of all Nash equilibria of the game $(\mathcal{A}, G, V)$. This reproves the following classical result.

Theorem 3.1.1. *$U \in \mathcal{U} \setminus \mathcal{D}$, the set $\mathcal{N}(U)$ of Nash equilibria is finite.*

The transversality of the involved hypersurfaces, respectively the equations which define them and which satisfy the regularity which leads to the transversality, says also that all Nash equilibria are in a strong sense *regular*. This transversality is our way to define the regularity of a Nash equilibrium. Therefore for $U \in \mathcal{U} \setminus \mathcal{D}$ all Nash equilibria are regular.

Similar statements were proved before, of course. Lemke and Howson [LH64] proved that for generic games with only two players the set of Nash equilibria is finite and odd. Their argument that sufficiently generic games exist, is elementary and cannot easily be translated to games with $m \geq 3$ players.

Wilson [Wil71] and Rosenmüller [Ros71] generalized the construction of Lemke and Howson which leads to the oddness of the number of Nash equilibria. We will rewrite this argument in the next section. But both did not give a proof of existence of sufficiently generic games. Wilson points to the construction in [LH64] (which is elementary, but cannot easily be generalized to $m \geq 3$). Rosenmüller says no word about the existence of sufficiently generic games.

Harsanyi [Har73] gave another proof that for generic games the set of Nash equilibria is finite and odd and that they are regular (with his definition of regularity). He used (in the proof of his Theorem 3) Sard's theorem.

Other proofs that the set of Nash equilibria is finite and odd and that they are regular (always with their own definitions of regularity) were given by Gül, Pearce and Stacchetti [GPS93, Theorem 3] and by Ritzberger [Rit94, Theorem 2]. In [GPS93] a variant due to Stacchetti and Reinoza of Sard's theorem is used, in [Rit94] a parametric transversality theorem is used.

3.2 Generic m-person games have an odd number of Nash equilibria

Theorem 3.2.1. *For $U \in \mathcal{U} \setminus \mathcal{D}$, the set $\mathcal{N}(U)$ of Nash equilibria is finite and odd.*

This result was first proved in the case of 2-person games by Lemke and Howson [LH64]. Their argument for the oddness of the number of Nash equilibria was generalized by Wilson [Wil71] and Rosenmüller [Ros71]. But Wilson and Rosenmüller both lacked to prove existence of sufficiently generic games. Our Theorem 2.1.2 provides it.

This section is devoted to recalling the argument of Wilson and Rosenmüller for the oddness of the number of Nash equilibria.

Their construction starts with the choice of a player $i_0 \in \mathcal{A}$ and a strategy $s^{i_0}_{j_0} \in S^{i_0}$. The conditions (3.2)–(3.6) for Nash equilibria are relaxed with respect to the pair (i_0, j_0) so that obtains a set of curves.

We give the final argument first: It will turn out that one obtains a finite set of disjoint curves each of which is homeomorphic to the circle S^1 or to the interval $[0, 1]$ and that the endpoints of the intervals are either in the set $\mathcal{N}_2$ or in the set $\mathcal{N}_3$, where $\mathcal{N}_2$ and $\mathcal{N}_3$ are defined as follows.

$$\begin{aligned}
\widetilde{S} &:= S^1 \times \dots \times S^{i_0-1} \times \{s^{i_0}_{j_0}\} \times S^{i_0+1} \times \dots \times S^m \subset S,\\
(\mathcal{A}, \widetilde{G}, \widetilde{V}) &:= \text{the mixed extension of the game } (\mathcal{A}, \widetilde{S}, U|_{\widetilde{S}}),\\
\mathcal{N}_1 &:= \mathcal{N}(\mathcal{A}, \widetilde{G}, \widetilde{V}) \cap \mathcal{N}(U),\\
\mathcal{N}_2 &:= \mathcal{N}(\mathcal{A}, \widetilde{G}, \widetilde{V}) \setminus \mathcal{N}_1,\\
\mathcal{N}_3 &:= \mathcal{N}(U) \setminus \mathcal{N}_1, \qquad \text{so that}\\
\mathcal{N}(\mathcal{A}, \widetilde{G}, \widetilde{V}) &= \mathcal{N}_1 \dot\cup \mathcal{N}_2, \quad \mathcal{N}(U) = \mathcal{N}_1 \dot\cup \mathcal{N}_3.
\end{aligned} \tag{3.8}$$

As the set of endpoints of the intervals is even, the number $|\mathcal{N}_2| + |\mathcal{N}_3|$ is even. By induction the number $|\mathcal{N}(A, \widetilde{G}, \widetilde{V})| = |\mathcal{N}_1| + |\mathcal{N}_2|$ is odd. Therefore the number

$$|\mathcal{N}(U)| = |\mathcal{N}_1| + |\mathcal{N}_3| = \Big(|\mathcal{N}_1| + |\mathcal{N}_2|\Big) + \Big(|\mathcal{N}_2 + \mathcal{N}_3|\Big) - 2|\mathcal{N}_2|$$

is odd. This finishes the final part of the argument.

Now we come to the main part. First choose a tuple $\underline{T} = (T^1, \dots, T^m)$

of subsets $T^i \subset N_0^i$ with

$$\emptyset \neq T^i \subset N_0^i \quad \text{if } i \neq i_0, \qquad \emptyset \neq T^{i_0} \subset N_0^{i_0} \setminus \{j_0\}.$$

Recall the definition of the smallest index $j_{min}(T^i)$ in T^i in (3.5).

Instead of the set of hypersurfaces in (3.7), we consider the by one element smaller set of hypersurfaces

$$\bigcup_{i\in\mathcal{A}} \Big(\{H^{i,j} \mid , j \in N_0^i \setminus T^i, j \neq j_0 \text{ if } i = i_0\} \cup \\ \{H^{i,(j_{min}(T^i),k)} \mid k \in T^i \setminus \{j_{min}(T^i)\}\} \Big). \tag{3.9}$$

It is a good set of hypersurfaces. By Theorem 2.2.1 it is everywhere transversal. The number of hypersurfaces in this set is $-1 + \sum_{i\in\mathcal{A}} n_i = \dim \mathbb{P}^{\mathcal{A}}W - 1$. Therefore their intersection is a finite union of disjoint smooth real algebraic curves in $\mathbb{P}^{\mathcal{A}}W$. We call this intersection $\mathcal{K}_1(\underline{T})$.

Furthermore, we propose the following inequalities for $i \in \mathcal{A}$,

$$\lambda^i_{j_{i,min(T^i)}}(\underline{\gamma}^{-i}) - \lambda^i_k(\underline{\gamma}^{-i}) \geq 0 \quad \text{for } k \in N_0^i \setminus T^i, \tag{3.10}$$

$$\gamma^i_j \geq 0 \quad \text{for } j \in T^i \text{ or } j = j_0 \text{ if } i = i_0. \tag{3.11}$$

We call the set of $g \in \mathcal{K}_1(\underline{T})$ which satisfy (3.10) and (3.11) $\mathcal{K}_2(\underline{T})$. It is a union of disjoint semialgebraic curves which are topologically intervals or circles. The set $\mathcal{K}_2(\underline{T})$ as well as the set $\mathcal{K}_1(\underline{T})$ might be empty.

The union $\mathcal{K}_2 := \bigcup_{\underline{T}} \mathcal{K}_2(\underline{T})$ is much better behaved than one might expect at first sight.

Claim: *The circles and intervals in all the sets $\mathcal{K}_2(\underline{T})$ do not share any interior points (so each circle is disjoint from the rest of the set $\mathcal{K}_2$). A boundary point of one interval may be boundary point of at most one other interval. Besides the circles, this leads to chains of intervals, which are themselves topologically circles or intervals. The boundary points of the resulting intervals are the points in $\mathcal{N}_2 \dot\cup \mathcal{N}_3$.*

It remains to prove this claim. An inequality in (3.10) or (3.11) becomes *bounding*, i.e. an equality, if and only if g is in one of the following hypersurfaces,

$$H^{i,(j_{min}(T^i),k)} \quad \text{for } k \in N_0^i \setminus T^i, \text{ i.e. in the case (3.10),} \tag{3.12}$$

$$H^{i,j} \quad \text{for } j \in T^i \text{ or } j = j_0 \text{ if } i = i_0, \text{ i.e. in the case (3.11).} \tag{3.13}$$

If one adds one hypersurface in (3.12) or (3.13) to the hypersurfaces in (3.9), the new set is still a good set. So by Theorem 2.2.1 each hypersurface in (3.12) or (3.13) intersects the union of smooth curves in $\mathcal{K}_1(\underline{T})$ for each $\underline{T}$ transversely. Therefore each boundary point of $\mathcal{K}_2(\underline{T})$ is in precisely one of the hypersurfaces in (3.12) or (3.13), and no other point of $\mathcal{K}_2(\underline{T})$ is in any of the hypersurfaces in (3.12) or (3.13). Especially, the circles in $\mathcal{K}_2(\underline{T})$ are the circles in $\mathcal{K}_1(\underline{T})$ which do not meet any of the hypersurfaces in (3.12) and (3.13).

Now we have to discuss the different possibilities for the boundary points.

1st case, $g \in \partial\mathcal{K}_2(\underline{T})$ with $g \in H^{i,(j_{min}(T^i),k)}$ with $k \in N_0^i \setminus T^i$ and $k \neq j_0$: Then g is also the boundary point of an interval in $\mathcal{K}_2(\underline{\widetilde{T}})$ with $\widetilde{T}^a = T^a$ for $a \neq i$ and $\widetilde{T}^i = T^i \cup \{k\}$ and with bounding hypersurface $H^{i,k}$.

2nd case, $g \in \partial\mathcal{K}_2(\underline{T})$ with $g \in H^{i,(j_{min}(T^i),k)}$ with $i = i_0$ and $k = j_0$: Then g is not a boundary point of an interval in $\mathcal{K}_2(\underline{\widetilde{T}})$ for any $\underline{\widetilde{T}} \neq \underline{T}$. It is a Nash equilibrium of $(\mathcal{A}, G, V)$. It is not in $\widetilde{G}$, because $\gamma^{i_0}_{j_{min}(T^{i_0})} > 0$. Therefore $g \in \mathcal{N}_3$.

3rd case, $g \in \partial\mathcal{K}_2(\underline{T})$ with $g \in H^{i_0,j_0}$: Then g is not a boundary point of an interval in $\mathcal{K}_2(\underline{\widetilde{T}})$ for any $\underline{\widetilde{T}} \neq \underline{T}$. It is a Nash equilibrium of$(\mathcal{A}, G, V)$. It is not in $\widetilde{G}$, because $\gamma^{i_0}_{j_0} = 0$. Therefore $g \in \mathcal{N}_3$.

4th case, $g \in \partial\mathcal{K}_2(\underline{T})$ with $g \in H^{i,k}$ with $k \in T^i$ and $|T^i| \geq 2$: Then g is also the boundary point of an interval in $\mathcal{K}_2(\underline{\widetilde{T}})$ with $\widetilde{T}^a = T^a$ for $a \neq i$ and $\widetilde{T}^i = T^i \setminus \{k\}$ and with bounding hypersurface $H^{i,(j_{min}(T^i),k)}$.

5th case, $g \in \partial\mathcal{K}_2(\underline{T})$ with $g \in H^{i,k}$ with $T^i = \{k\}$: Then $\gamma^{i_0}_{j_0} = 1$. Then g is not a boundary point of an interval in $\mathcal{K}_2(\underline{\widetilde{T}})$ for any $\underline{\widetilde{T}} \neq \underline{T}$. It is not a Nash equilibrium of $(\mathcal{A}, G, V)$ because $\gamma^{i_0}_{j_0} = 1$, but $\lambda^{i_0}_k(\underline{\gamma}^{-i_0}) - \lambda^{i_0}_{j_0}(\underline{\gamma}^{-i_0}) > 0$. It is a Nash equilibrium of the smaller game $(\mathcal{A}, \widetilde{G}, \widetilde{V})$. Therefore $g \in \mathcal{N}_2$.

Vice versa, any $g \in \mathcal{N}_3$ turns up as boundary point in the 2nd or 3rd case, and any $g \in \mathcal{N}_2$ turns up as a boundary point in the 5th case. This finishes the presentation of the arguments in [Wil71] and [Ros71]. $\square$

3.3 Possible sizes of the set of TMNE for generic m-person games

Theorem 3.2.1 says that for a generic m-person game the number of Nash equilibria of its mixed extension is finite and odd. The next obvious question is, which size the set of Nash equilibria can have and how one could sort the Nash equilibria into different classes. Here a result of McKelvey and McLennan [MM97] will be reported.

Fix $\mathcal{A} = \{1, \ldots, m\}$ and $S = \prod_{i \in \mathcal{A}} S^i$ with $S^i = \{s_0^i, \ldots, s_{n_i}^i\}$. We will consider varying $U \in \mathcal{U}$, usually $U \in \mathcal{U} \setminus \mathcal{D}$. For any $U \in \mathcal{U}$, $\mathrm{int}(G)$ denotes the interior of the set G of the mixed extension $(\mathcal{A}, G, V)$ of the finite game $(\mathcal{A}, S, U)$ as a subset of the affine space A (see Definition 2.1.1).

Definition 3.3.1. Choose any $U \in \mathcal{U}$. (a) A mixed strategy combination $g \in G$ is *totally mixed* if $g \in \mathrm{int}(G)$, i.e. $g^i = \sum_j \gamma_j^i s_j^i$ with all $\gamma_j^i > 0$.

(b) A *block derangement* is a partition of the set $\bigcup_{i \in \mathcal{A}} (S^i \setminus \{s_0^i\})$ into sets $B^1, \ldots, B^m$ with

$$\dot{\bigcup}_{i \in \mathcal{A}} (S^i \setminus \{s_0^i\}) = \dot{\bigcup}_{i \in \mathcal{A}} B^i \quad \text{and} \quad S^i \cap B^i = \emptyset \quad \text{and} \quad |B^i| = |S^i \setminus \{s_0^i\}|.$$

The number of all possible block derangements is called $E(n_1, \ldots, n_m)$.

One can interpret a block derangement as a shuffling of the cards in $\bigcup_{i \in \mathcal{A}} (S^i \setminus \{s_0^i\})$ where player i owns the cards in $S^i \setminus \{s_0^i\}$, though after the shuffling obtains no own cards, but n_i cards of other players.

It is easy to see [Vid17] that the number $E(n_1, \ldots, n_m)$ is the coefficient of the monomial $x_1^{n_1} \ldots x_m^{n_m}$ in the polynomial

$$\prod_{i \in \mathcal{A}} \Big(-x_i + \sum_{j \in \mathcal{A}} x_j \Big)^{n_i}. \tag{3.14}$$

Namely, one puts the linear factors of each power $\Big(-x_i + \sum_{j \in \mathcal{A}} x_j\Big)^{n_i}$ in (3.14) into a fixed order. A choice of the variable x_j in the k-th linear factor ($k \in \{1, \ldots, n_i\}$) of the power $\Big(-x_i + \sum_{j \in \mathcal{A}} x_j\Big)^{n_i}$ means that player i gets the card s_k^j. Then any choice of one variable in each linear factor in (3.14) such that their product is $x_1^{n_1} \ldots x_m^{n_m}$ is a shuffling which gives a block derangement, and all block derangements are obtained in this way.

Vidunas [Vid17] applies MacMahon's master theorem to give a generating function for all numbers $E(n_1,\ldots,n_m)$. He mentions a relation with Laguerre polynomials, he discusses special cases, he gives recurrence relations and he gives asymptotics. The relevance for us of the numbers $E(n_1,\ldots,n_m)$ is due to the following result.

Theorem 3.3.2. *(McKelvey and McLennan [MM97]) The maximal number of totally mixed Nash equilibria for any generic game* $(\mathcal{A},S,U)$ *is* $E(n_1,\ldots,n_m)$. *In formulas:*

$$\max\Big(|\mathcal{N}(U)\cap\mathrm{int}(G)|\;\Big|\;U\in\mathcal{U}\setminus\mathcal{D}\Big)=E(n_1,\ldots,n_m).$$

In [MM97, ch. 4] they construct explicitly a game with $U\in\mathcal{U}$ and $|\mathcal{N}(U)\cap\mathrm{int}(G)|=E(n_1,\ldots,n_m)$. Our notion of inner tropical games in Chapter 4 is a special case of their construction.

In [MM97, ch. 3] they show that $E(n_1,\ldots,n_m)$ is an upper bound for $|\mathcal{N}(U)\cap\mathrm{int}(G)|$ for any $U\in\mathcal{U}\setminus\mathcal{D}$. They use the BKK-bound for this, a bound for the numbers of zeros in $(\mathbb{C}^*)^N$ of a system of N Laurent polynomials such that each of them has a certain support and is generic within all Laurent polynomials with this support. Then the BKK-bound expresses the maximal number of zeros as the mixed volume of the Newton polytopes which stem from the supports of the Laurent polynomials. It is due to Bernstein [Ber75], Khovanskii and Kushnirenko [Kus76].

Vidunas proposes a simpler way to see that $E(n_1,\ldots,n_m)$ is an upper bound. Define $W^i_\mathbb{C}:=W^i\otimes_\mathbb{R}\mathbb{C}$ as complexifications of the real vector space W^i, and consider the product $\mathbb{P}^\mathcal{A}W_\mathbb{C}:=\prod_{i\in\mathcal{A}}\mathbb{P}W^i_\mathbb{C}$ of complex projective spaces. The *multihomogeneous Bézout theorem* states that for a generic system of n multihomogeneous polynomials of certain multi-degrees, the number of joint zeros is finite and is equal to the *multihomogeneous Bézout bound.* Additionally, for any non-generic system, the number of components of zeros is bounded from above by the Bézout bound. The multihomogeneous Bézout theorem is cited in [Vid17, Theorem 2.1]. The statement seems to be old. A detailed proof is given in [MSW95]. In our situation, the hypersurfaces $H^{i,(0,k)}$ with $i\in\mathcal{A},k\in N^i=\{1,\ldots,n_i\}$, are the zero sets in $\mathbb{P}^\mathcal{A}W$ of multihomogeneous polynomials with multi-degrees $(1,\ldots,1)-e_i^{(m)}$. They form a good set. By Theorem 2.2.1 their intersection is finite. By (3.6) this intersection contains the set $\mathcal{N}(U)\cap\mathrm{int}(G)$ of totally mixed Nash equilibria. Therefore the number $|\mathcal{N}(U) \cap \mathrm{int}(G)|$ is bounded from above by the multihomogeneous Bézout bound, which is here (for the given multi-degrees) just $E(n_1, \ldots, n_m)$.

Remark 3.3.3. (i) In the case of $n_1 = \cdots = n_m = 1$, a block derangement is simply a permutation $\sigma \in S_m$ without fixed points. Such permutations are called *derangements*. The number of derangements $E(\underbrace{1, \ldots, 1}_{m \text{ times}})$ is also called $!m$. See Lemma 4.2.5 for the first values of $!m$ and two recurrence formulas and a closed formula.

(ii) Table 1.1 in [MM97] gives the values of $E(\underbrace{k, \ldots, k}_{m \text{ times}})$ for small values of m and k. In order to calculate this table, they embedded the values $E(n_1, \ldots, n_m)$ into a bigger series of natural numbers, for which they established a recurrence relation.

(iii) Consider the condition

$$\text{for each } i \in \mathcal{A} \quad n_i \leq \sum_{j \in \mathcal{A} \setminus \{i\}} n_j. \tag{3.15}$$

Then obviously

$$E(n_1, \ldots, n_m) \geq \quad 1 \iff \quad (3.15) \text{ holds}, \tag{3.16}$$

and in the extreme case with $n_i = \sum_{j \in \mathcal{A} \setminus \{i\}} n_j$ for some $i \in \mathcal{A}$,

$$E(n_1, \ldots, n_m) = \frac{n_i!}{n_1! \cdot \cdots \cdot n_{i-1}! \cdot n_{i+1}! \cdot \cdots \cdot n_m!}. \tag{3.17}$$

(iv) (3.16) fits well to results of Kreps [Kre81] and Chin, Parthasarathy and Raghavan [CPR74]. Kreps showed that for given $\mathcal{A}$ and S and a given totally mixed strategy combination $g \in \mathrm{int}(G)$

$$U \text{ with } \mathcal{N}(U) = \{g\} \text{ exists} \quad \iff \; (3.15) \text{ holds}. \tag{3.18}$$

Chin, Parthasarathy and Raghavan studied mainly games with $m = 3$ players. The corollary after Theorem 3 in [CPR74] says for $m = 3$:

$$U \text{ with } \mathcal{N}(U) \subset \mathrm{int}(G) \text{ exists} \quad \iff \quad (3.15)\text{holds}. \tag{3.19}$$

Of course, $\Leftarrow$ in (3.19) follows from Kreps' result (3.18). And $\Rightarrow$ in (3.19) for $U \in \mathcal{U} \backslash \mathcal{D}$ follows from Theorem 3.3.2 above and from (3.16). But (3.19) is proved in [CPR74] for all games, not only for generic games. Theorem 6 in [CPR74] says for $m = 3$:

$$n_1 = n_2 = n_3 = 1 \text{ and } \mathcal{N}(U) \subset \text{int}(G) \implies +|\mathcal{N}(U)| = 1. \tag{3.20}$$

This follows for $U \in \mathcal{U} \setminus \mathcal{D}$ from $E(1, 1, 1) = 2$, from Theorem 3.2.1 and from Theorem 3.3.2. But (3.20) is proved in [CPR74] for all games, not only for generic games.

3.4 Possible sizes of the set of all NE for generic m-person games

Vidunas [Vid17] used Theorem 3.3.2 for an estimate from above of the number $|\mathcal{N}(U)|$ of mixed Nash equilibria for generic games with fixed $\mathcal{A}$ and S. He used that each Nash equilibrium $g \in \mathcal{N}(U)$ is also a totally mixed Nash equilibrium of the mixed extension of the subgame $(\mathcal{A}, \widetilde{S}, \widetilde{U})$ with $\widetilde{U} = U|_{\widetilde{G}}$ and $\widetilde{S} = \prod_{i\in\mathcal{A}}\{s_j^i \mid j \in \operatorname{supp}(g^i)\}$. He applied the bound in Theorem 3.3.2 to each possible tuple $\underline{T} = (T^1, \dots, T^m)$ of supports $T^i \subset N_0^i$ (with $T^i \neq \emptyset$). He obtained for any $U \in \mathcal{U} \setminus \mathcal{D}$ the upper bound

$$\begin{aligned} |\mathcal{N}(U)| &\leq B(n_1+1, \dots, n_m+1) \\ &:= \sum_{\underline{T}} E(|T^1|-1, \dots, |T^m|-1) \\ &= \sum_{(k_1,\dots,k_m)\in\prod_{i\in\mathcal{A}} N_0^i} \binom{n_1+1}{k_1+1} \cdots \binom{n_m+1}{k_m+1} \end{aligned} \tag{3.21}$$

He also proved the simpler formula [Vid17, Theorem 5.2]

$$B(n_1+1, \dots, n_m+1) = \sum_{(k_1,\dots,k_m)\in\prod_{i\in\mathcal{A}} N_0^i} \frac{(k_1+\dots+k_m)!}{k_1! \cdot \dots \cdot k_m!}. \tag{3.22}$$

In general, this upper bound is coarse. In the case $m = 2$, $E(k_1, k_2) = 0$ if $k_1 \neq k_2$ and $E(k_1, k_2) = 1$ if $k_1 = k_2$, so in the case $m = 2$

$$\begin{aligned} B(n_1+1, n_2+1) &= \sum_{k=1}^{\min(n_1,n_2)+1} \binom{n_1+1}{k}\binom{n_2+1}{k} \\ &\overset{(3.22)}{=} \binom{n_1+n_2+2}{n_1+1} - 1. \end{aligned} \tag{3.23}$$

($\overset{(3.22)}{=}$ is also easy to see directly.) In the following $f(n) \sim_{n\to\infty} g(n)$ for two sequences $(f(n))_{n\in\mathbb{N}}$ and $(g(n))_{n\in\mathbb{N}}$ of numbers means $f(n)/g(n) \to_{n\to\infty} 1$. With Stirling's formula

$$n! \sim_{n\to\infty} \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \tag{3.24}$$

the growth of $B(n+1, n+1)$ can be estimated by

$$B(n+1, n+1) \sim_{n\to\infty} \frac{1}{\sqrt{\pi n}} 4^n. \tag{3.25}$$

This is much coarser than Keiding's upper bound (3.29), which will be discussed below.

On the other hand, in the cases with $m \geq 2$ arbitrary and $n_1 = \cdots = n_m = 1$, the upper bound (3.21) for $|\mathcal{N}(U)|$ is surprisingly good. Then it is

$$\begin{aligned} B(\underbrace{2,\ldots,2}_{m \text{ times}}) &= \sum_{(k_1,\ldots,k_m)\in\{0;1\}^m} 2^{m-\sum_i k_i} \cdot E(k_1,\ldots,k_m) \\ &= \sum_{(k_1,\ldots,k_m)\in\{0;1\}^m} 2^{m-\sum_i k_i} \cdot !\Big(\sum_i k_i\Big) \\ &= \sum_{l=0}^{m} \binom{m}{l} \cdot 2^l \cdot !(m-l) &\text{(3.26)} \\ &\overset{(3.22)}{=} \sum_{l=0}^{m} \binom{m}{l} \cdot (m-l)! = \sum_{l=0}^{m} \frac{m!}{l!}. &\text{(3.27)} \end{aligned}$$

Here $\overset{(3.22)}{=}$ follows easily with the formula $l! = \sum_{a=0}^{l} \binom{l}{a} \cdot !a$. Here the upper bound $|\mathcal{N}(U)| \leq B(\underbrace{2,\ldots,2}_{m \text{ times}})$ is good, because we will complement it in Theorem 4.3.4 with the lower bound

$$\max\Big(|\mathcal{N}(U)| \;\Big|\; U \in \mathcal{U} \setminus \mathcal{D}\Big) \geq \frac{1}{2}\Big(B(\underbrace{2,\ldots,2}_{m \text{ times}}) + !m\Big), \tag{3.28}$$

which is very close to the upper bound. The first values are as follows,

m	2	3	4	5
$B(\underbrace{2,\ldots,2}_{m \text{ times}})$	5	16	65	326
$\frac{1}{2}(B(\underbrace{2,\ldots,2}_{m \text{ times}}) + !m)$	3	9	37	187

Table 3.1: Lower and upper bound comparison for pre-tropical games

We can imagine that the lower bound is sharp, so that equality holds in (3.28), but we do not know as of now. At least, the lower bound is always odd, because by (3.26) it is modulo 2 equal to

$$\frac{1}{2}\Big(!m+\sum_{l=0}^{1}\binom{m}{l}\cdot 2^l\cdot!(m-l)\Big)=m\cdot!(m-1)+!m\overset{(4.5)}{=}2\cdot!m+(-1)^{m+1}.$$

Remark 3.4.1. (i) An invariant of a regular Nash equilibrium, which we will touch only in these remarks, is its *index*. Fix $\mathcal{A}$ and S and a generic $U\in\mathcal{U}\setminus\mathcal{D}$. Then it turns out that the best reply graphs $\mathrm{Gr}(r^i)\subset G$ are indeed topological manifolds with boundary and are unions of semialgebraic manifolds with boundary. Because of Theorem 2.2.1, any subset of these manifolds intersects transversally everywhere. Especially, all of them intersect transversally at all Nash equilibria. A choice of orientations of G and of the best reply graphs $\mathrm{Gr}(r^i)$ equips each Nash equilibrium $g\in\mathcal{N}(U)$ with an index $\mathrm{ind}(g)\in\{\pm 1\}$. It is well known that (with the right choice of all orientations) $\sum_{g\in\mathcal{N}(U)}\mathrm{ind}(g)=1$. Working out these well known facts gives another proof of the oddness of $|\mathcal{N}(U)|$, so of Theorem 3.2.1.

(ii) It is also known that each pure Nash equilibrium $g\in S$ has index $\mathrm{ind}(g)=1$. Gül, Pearce and Stacchetti [GPS93] proved this and used it to show that in the presence of k pure Nash equibria one has at least $k-1$ other Nash equilibria. Ritzberger [Rit94] had his own way to define regularity of Nash equilibria and their indices. He rederived the oddness of the number of Nash equilibria and also the result of Gül, Pearce and Stacchetti.

(iii) The indices of Nash equilibria for generic games lead also to indices for components of the set $\mathcal{N}(U)$ of Nash equilibria for non-generic U. This is interesting, because if one deforms a non-generic U to a generic one, a component in $\mathcal{N}(U)$ with index $\neq 0$ cannot vanish completely, but it will reduce to a non-empty set of regular Nash equilibria. This is one approach to the notion of *stability* of components of $\mathcal{N}(U)$ for non-generic U.

(iv) In the case of $m=2$, Balthasar [Bal09] has worked out elegant approaches to indices of components of $\mathcal{N}(U)$ and to stability notions. Other references for these subjects are [Wil92] and [Sch05].

3.5 Generic 2-person games

Now we restrict to the case $m = 2$, i.e. to the mixed extensions of generic 2-player games with $S^1 = \{s^1_0, \dots, s^1_{n_1}\}$, $S^2 = \{s^2_0, \dots, s^2_{n_2}\}$. We are interested in the maximal number

$$N(n_1, n_2) := \max\Big(|\mathcal{N}(U)| \;\Big|\; U \in \mathcal{U} \setminus \mathcal{D}\Big) \in \mathbb{N}.$$

of Nash equilibria in mixed extensions of generic 2-player games with S as above. It is finite and odd by Theorem 3.2.1. We will also use the notations

$$\phi(2l+1, a) := 2\binom{a-l-1}{l} \qquad \text{for } l, a \in \mathbb{N}_0 \text{ with } 2l+1 < a,$$

$$\phi(2l, a) := \frac{a}{l}\binom{a-l-1}{l-1} \qquad \text{for } l, a \in \mathbb{N}_0 \text{ with } 2l < a,$$

$$\sigma(l) := \sum_{k=0}^{a}\binom{l+k}{k}\binom{l}{k} \qquad \text{for } l \in \mathbb{N}_0,$$

so

l	0	1	2	3	4	5	6
$\sigma(l)$	1	3	13	63	321	1683	8989

.

Table 3.2 gives the (to our knowledge) best known upper and lower bounds for the number $N(n_1, n_2)$ for many cases $(n_1, n_2) \in \mathbb{N}_0^2$.

lower bound	$N(n_1, n_2)$	upper bound
1	$= N(n_1, 0) =$	1
3	$= N(1, 1) =$	3
7	$= N(2, 2) =$	7
15	$= N(3, 3) =$	15
31	$\leq N(4, 4) \leq$	41
75	$\leq N(5, 5) \leq$	111
127	$\leq N(6, 6) \leq$	239
383	$\leq N(7, 7) \leq$	659
$\sigma(l) + \sigma(l-1) - 1$	$\leq N(2l-1, 2l-1) \leq$	$\phi(2l, 4l) - 1$
$\max(2^{2l+1}, 2\sigma(l)) - 1$	$\leq N(2l, 2l) \leq$	$\phi(2l+1, 4l+2) - 1$
?	$\leq N(n_1, n_2) \leq$	$\phi(\min(n_1, n_2) + 1,$ $n_1 + n_2 + 2) - 1$

Table 3.2: Lower and upper bounds for $N(n_1, n_2)$

The number $N(n_1, n_2)$ is in most cases unknown. The upper bound is due to Keiding [Kei97], the lower bound is due to von Stengel [Ste97][Ste99]. We will discuss some aspects of their works below.

In the case of $N(3,3)$, the upper bound 15 is better than the general upper bound in this case, $\phi(4,8) - 1 = 19$. The proof in [Kei97] of the better upper bound 15 in [Kei97] uses one more argument (which we will describe below) than the general bound. A different proof of it was given by McLennan and Park [MP99].

With Stirling's formula (3.24) it is easy to estimate the upper bound [Ste97][Ste99]

$$\begin{aligned} \phi(2l, 4l) = 4\binom{3l-1}{l-1} = 2\binom{3l-1}{l} &\sim_{l\to\infty} \frac{2}{\sqrt{3\pi l}}\left(\frac{27}{4}\right)^l, \\ \phi(2l+1, 4l+2) = 2\binom{3l+1}{l} &\sim_{l\to\infty} \frac{3}{2}\sqrt{\frac{3}{\pi l}}\left(\frac{27}{4}\right)^l, \\ \phi(n_1+1, 2n_1+2) \sim_{n_1\to\infty} &\begin{cases} \frac{0.921}{\sqrt{n_1+1}} \cdot 2.5981^{n_1+1} & \text{if } n_1 \text{ is odd,} \\ \frac{0.798}{\sqrt{n_1+1}} \cdot 2.5981^{n_1+1} & \text{if } n_1 \text{ is even.} \end{cases} \end{aligned} \tag{3.29}$$

The factor $\sqrt{\frac{27}{4}}^{n_1+1} \approx 2.5981^{n_1+1}$ is much better than the factor 4^{n_1+1} in the upper bound $B(n_1+1, n_1+1)$ in (3.25).

It is much more difficult to estimate the lower bound. von Stengel succeeded and obtained the following [Ste97][Ste99],

$$\begin{aligned} (\text{lower bound}) \sim_{n_1\to\infty} &\begin{cases} \sqrt{\frac{2\sqrt{2}}{\pi(n_1+1)}}(1+\sqrt{2})^{n_1+1} & \text{if } n_1 \text{ is odd,} \\ \sqrt{\frac{\sqrt{2}}{\pi(n_1+1)}}(1+\sqrt{2})^{n_1+1} & \text{if } n_1 \text{ is even.} \end{cases} \\ \approx &\begin{cases} \frac{0.949}{\sqrt{n_1+1}} \cdot 2.414^{n_1+1} & \text{if } n_1 \text{ is odd,} \\ \frac{0.671}{\sqrt{n_1+1}} \cdot 2.414^{n_1+1} & \text{if } n_1 \text{ is even.} \end{cases} \end{aligned} \tag{3.30}$$

This is much better than an older lower bound $2^{n_1+1} - 1$ for $N(n_1, n_1)$ which was established by Quint and Shubik [QS97]. They had even conjectured $N(n_1, n_1) = 2^{n_1+1} - 1$, which was refuted by von Stengel's lower bound.

The gap between the upper and lower bounds in (3.29) and (3.30) is not so very large. Still one could hope to make it smaller. The constructions of both bounds use pairs of simple polytopes. If one could control better some property of simple polytopes, this might allow to diminish the upper bound substantially. We did not succeed, but find this idea interesting enough to explain it and thus also explain the construction of both bounds.

We start with a pair $U = (U^1, U^2)$ of utility functions with the property

$$\max_{j\in N_0^1} U^1(s_j^1, s_k^1) > 0 \text{ for any } k \in N_0^2,$$
$$\max_{k\in N_0^2} U^2(s_j^1, s_k^1) > 0 \text{ for any } j \in N_0^1.$$

This assumption is harmless as one can add an arbitrary constant to U^i without changing the game essentially. The assumption implies that the two functions $u^1 : G^2 \to \mathbb{R}$ and $u^2 : G^1 \to \mathbb{R}$ with

$$u^1(g^2) := \max_{j\in N_0^1} \lambda_j^1(g^2), \quad u^2(g^1) := \max_{k\in N_0^2} \lambda_k^2(g^1)$$

take values in $\mathbb{R}_{>0}$. They are piecewise linear, because the functions λ_j^1 and λ_k^2 are affine linear in the case $m = 2$. We obtain two convex non-compact polyhedra $P^2 \subset A^2 \times \mathbb{R}$ and $P^1 \subset A^1 \times \mathbb{R}$ by

$$P^2 := \{(g^2, r) \in G^2 \times \mathbb{R}_{>0} \mid u^1(g^2) \le r\},$$
$$P^1 := \{(g^1, r) \in G^1 \times \mathbb{R}_{>0} \mid u^2(g^1) \le r\}$$

([MP99] gives a picture). We compactify them with ∞,

$$\overline{P}^2 := P^2 \cup \{\infty\}, \quad \overline{P}^1 := P^1 \cup \{\infty\}.$$

We define subsets of their boundaries $\partial\overline{P}^2$ and $\partial\overline{P}^1$ as follows,

$$\overline{P}^2(s_k^2) := \{(g^2, r) \in \partial P^2 \mid \gamma_k^2 = 0\} \cup \{\infty\} \quad \text{for } s_k^2 \in S^2,$$
$$\overline{P}^2(s_j^1) := \{(g^1, r) \in \partial P^2 \mid \lambda_j^1(g^2) = u^1(g^2)\} \quad \text{for } s_j^1 \in S^1,$$
$$\overline{P}^1(s_j^1) := \{(g^1, r) \in \partial P^1 \mid \gamma_j^1 = 0\} \cup \{\infty\} \quad \text{for } s_j^1 \in S^1,$$
$$\overline{P}^1(s_k^2) := \{(g^2, r) \in \partial P^1 \mid \lambda_k^2(g^1) = u^2(g^1)\} \quad \text{for } s_k^2 \in S^2.$$

The sets $\overline{P}^2(s_k^2)$ and $\overline{P}^1(s_j^1)$ are clearly facets of the polyhedra $\overline{P}^2$ respectively $\overline{P}^1$. Because the maps u^2 and u^1 are piecewise linear, the sets $\overline{P}^2(s_j^1)$ and $\overline{P}^1(s_k^2)$ are empty or polytopes of some dimension between 0 and n_2 respectively 0 and n_1. The projections $pr_i : A^i \times \mathbb{R}$ to A^i map the facets

$\overline{P}^2(s^2_k) \setminus \{\infty\}$ and $\overline{P}^1(s^1_j) \setminus \{\infty\}$ to the facets of G^2 and G^1. They map the non-empty of the polytopes $\overline{P}^2(s^1_j)$ and $\overline{P}^1(s^2_k)$ isomorphically to polytopes in G^2 respectively G^1, which together fill G^2 respectively G^1.

We define markings $\mu^2 : \partial\overline{P}^2 \to \mathcal{P}(S^1 \cup S^2)$ and $\mu^1 : \partial\overline{P}^1 \to \mathcal{P}(S^1 \cup S^2)$ of the points in the boundaries of $\overline{P}^2$ and $\overline{P}^1$ by subsets of $S^1 \cup S^2$ in the following way,

$$\mu^2(p) := \{s \in S^1 \cup S^2 \mid p \in \overline{P}^2(s)\} \quad (\text{so that } \mu^2(\infty) := S^2),$$
$$\mu^1(p) := \{s \in S^1 \cup S^2 \mid p \in \overline{P}^1(s)\}, \quad (\text{so that } \mu^1(\infty) := S^1).$$

In fact, these markings on P^2 and P^1 do not depend on r, so they induce markings $\mu^2_G : G^2 \to \mathcal{P}(S^1 \cup S^2)$ and $\mu^1_G : G^1 \to \mathcal{R}(S^1 \cup S^2)$. Together they characterize the Nash equilibria, namely one sees easily

$$(g^1, g^2) \in \mathcal{N}(U) \iff \mu^2_G(g^2) \cup \mu^1_G(g^1) = S^1 \cup S^2.$$

As the maps u^1 and u^2 are piecewise linear, for any non-empty set $T^i \subset S^1 \cup S^2$, the set $\{g^i \in G^i \mid \mu^i(g^i) \supset T^i\}$ is empty or a convex polytope of some dimension in $\{0, 1, \ldots, n_i\}$. Therefore the set $\mathcal{N}(U)$ of all Nash equilibria is a finite union of products of polytopes,

$$\bigcup_{\substack{(T^1,T^2)\subset(S^1\times S^2)^2:\\ T^1\dot\cup T^2=S^1\cup S^2}} \{g^1 \in G^1 \mid \mu^1(g^1) \supset T^1\} \times \{g^2 \in G^2 \mid \mu^2(g^2) \supset T^2\}.$$

From now on we restrict to the case of a generic game, so with $U = (U^1, U^2) \in \mathcal{U} \setminus \mathcal{D}$. There the introduction of the polyhedra P^1 and P^2 pays off. First, they can be mapped to polytopes as follows. The maps

$$\pi^1 : \overline{P}^1 \to W^1, \quad (g^1, r) \mapsto g^1/r, \quad \infty \mapsto 0,$$
$$\pi^2 : \overline{P}^2 \to W^2, \quad (g^2, r) \mapsto g^2/r, \quad \infty \mapsto 0,$$

are *projective transformations*. They are bijective (a picture of them is given in [Ste97]). The markings μ^2 and μ^1 induce markings on the polytopes $\pi^2(\overline{P}^2)$ and $\pi^1(\overline{P}^1)$ which we denote by μ^2_W and μ^1_W.

Theorem 2.2.1 and $U \in \mathcal{U} \setminus \mathcal{D}$ imply the following.

Lemma 3.5.1. *The polytopes $\pi^1(\overline{P}^1)$ and $\pi^2(\overline{P}^2)$ are simple (i.e. their facets have generic positions), and there are non-empty subsets $B^i \subset S^i$*

such that the facets of the polytopes are as follows.

$$\begin{aligned}
&\text{facets of } \pi^2(\overline{P}^2): \quad \overline{P}^2(s^2_k),\ s^2_k \in S^2; \quad \overline{P}^2(s^1_j),\ s^1_j \in B^1,\\
&\text{facets of } \pi^1(\overline{P}^1): \quad \overline{P}^1(s^1_j),\ s^1_j \in S^1; \quad \overline{P}^1(s^2_k),\ s^2_k \in B^2.
\end{aligned}$$

If one marks each facet by the corresponding strategy, then these markings induce the markings μ^2_W and μ^1_W of the points of the polytopes. The Nash equilibria are the preimages under (π^1, π^2) of those pairs $(p^1, p^2) \in \overline{P}^1 \times \overline{P}^2 \setminus \{(0,0)\}$ of vertices such that p^1 and p^2 have complementary markings, i.e. $\mu^1_W(p^1) \dot\cup \mu^2_W(p^2) = S^1 \cup S^2$.

In [Ste97][Ste99] the papers of Vorob'ev [Vor58], Kuhn [Kuh61] and Mangasarian [Man64] are cited for this approach via polytopes to Nash equilibria in 2-person games. Von Stengel even showed that a converse to Lemma 3.5.1 holds, namely one can go from suitable polytopes with markings to games.

Lemma 3.5.2. *[Ste97, Proposition 2.4][Ste99, Proposition 2.1] Let Q^1 and Q^2 be simple polytopes of dimension n_1+1 respectively n_2+1 with markings μ^1_Q in $S^1 \cup B^2$ respectively μ^2_Q in $S^2 \cup B^1$ of their facets, where $\emptyset \neq B^2 \subset S^2$ and $\emptyset \neq B^1 \subset S^1$, such a pair $(q^1_\infty, q^2_\infty) \in Q^1 \times Q^2$ of vertices with markings $\mu^1_Q(q^1_\infty) = S^1$ and $\mu^2_Q(q^2_\infty) = S^2$ exists.*

(a) Then there are affine linear isomorphisms $Q^1 \to \pi^1(\overline{P}^1)$ and $Q^2 \to \pi^2(\overline{P}^2)$ to the polytopes of a game with $U \in \mathcal{U} \setminus \mathcal{D}$ which respect the markings.

(b) The preimages in $Q^1 \times Q^2$ of the Nash equilibria of this game are the pairs $(q^1, q^2) \in Q^1 \times Q^2 \setminus \{(q^1_\infty, q^2_\infty)\}$ of vertices with complementary markings, i.e. $\mu^1_Q(q^1) \cup \mu^2_Q(q^2) = S^1 \cup S^2$.

Lemma 3.5.2 reduces the study of the size and shape of $\mathcal{N}(U)$ for $U \in \mathcal{U} \setminus \mathcal{D}$ to the purely combinatorial study of pairs (Q^1, Q^2) of polytopes with markings of their facets as in the Lemma.

Keiding's upper bound and von Stengel's lower bound for $N(n_1, n_2)$ build on Lemma 3.5.2. Both work with a very distinguished family of simple polytopes, namely the duals $C_d(N)^\Delta$ of the cyclic polytopes $C_d(N)$, where d is its dimension and N its number of vertices [McM70][Zie95]. Then d is also the dimension of $C_d(N)^\Delta$, and N is the number of facets of $C_d(N)$. The number of vertices of $C_d(N)^\Delta$ is $\phi(d, N)$ (which was defined above).

The upper bound theorem [McM70] (see also [Zie95]) states that no convex polytope of dimension d and with N facets has more than $\phi(d, N)$ vertices.

Together with Lemma 3.5.2, this establishes the upper bound $\phi(\min(n_1, n_2) + 1, n_1 + n_2 + 2) - 1$ for $N(n_1, n_2)$ of Keiding [Kei97].

Von Stengel [Ste97][Ste99] worked in the case $n_1 = n_2$ with $Q^1 = Q^2 = C_{n_1+1}(2n_1+2)^\Delta$. He found markings of the facets of Q^1 and of Q^2 such that the number of vertex pairs with complementary markings is $\sigma(l) + \sigma(l-1)$ if $n_1 + 1 = 2l$ and is $2\sigma(l)$ if $n_1 + 1 = 2l + 1$. Together with Lemma 3.5.2, this establishes these numbers as lower bounds. For even $n_1 \in \{2, 4, 6\}$ the other lower bound $2^{n_1+1} - 1$ of Quint and Shubik [QS97] is better. They work with $Q^1 = Q^2 =$ (the hypercube of dimension $n_1 + 1$). There markings exist such that for each vertex in Q^1 a partner vertex in Q^2 with complementary marking exists. The hypercube has 2^{n_1+1} vertices.

The better upper bound 15 in the case of $N(3, 3)$ is proved in [Kei97] using an obstruction to existence of vertex pairs with complementary markings, which is formulated in the following lemma.

Lemma 3.5.3. *In the situation of Lemma 3.5.2, denote by Q^1_V the set of vertices of Q^1 for which a vertex in Q^2 with complementary marking exists, and similarly Q^2_V. Then for each triangle in Q^1 or in Q^2, at most two of its three vertices are in Q^1_V respectively Q^2_V.*

The lemma follows from the easy fact that the three vertices of one triangle cannot have three partners with complementary markings.

The combinatorial types of 4-dimensional simple polytopes with 8 facets are listed in [GS67]. They have between 14 and 20 vertices. Keiding considered those polytopes which have $16 + k$ vertices where $k \in \{1, 2, 3, 4\}$. He found in each of them k disjoint triangles. Together with Lemma 3.5.3 and Lemma 3.5.2 this establishes the upper bound 15 for $N(3, 3)$.

We hope that also in higher dimensions simple polytopes of dimension $n_1 + 1$ with $2n_1 + 2$ facets and more vertices than the 2^{n_1+1} vertices of the $n_1 + 1$ dimensional hypercube have many triangles so that Lemma 3.5.3 leads to many obstructions. This should decrease the upper bound. Unfortunately, we were not successful trying to control these triangles. We finish this section with two examples and one reference.

Remark 3.5.4. (i) In the case $n_1 = n_2 = 5$ von Stengel and Keiding worked with the polytope $C_6(12)^\Delta$. It is not hard to see the following.

$C_6(12)^\Delta$ has 12 tetrahedra and 48 triangles which are not part of tetrahedra. Each vertex of $C_6(12)^\Delta$ is part of at most two triangles or tetrahedra.

The set of vertices of this polytope has 112 elements. Let U be a subset which contains at least one vertex of each triangle. Then it contains at least two vertices of each tetrahedron. Altogether it contains at least $\frac{1}{2}(48 + 2 \cdot 12) = 36$ vertices. Therefore for any marking of $Q^1 = C_6(12)^\Delta$ and any marking of another simple polytope Q^2, the set $(C_6(12)^\Delta)_V$ can have at most $112 - 36 = 76$ elements. Therefore a pair (Q^1, Q^2) with $Q^1 = C_6(12)^\Delta$ leads to at most 75 Nash equilibria. Unfortunately, it does not look feasible to carry out similar arguments for all other simple polytopes of dimension 6 with 12 facets. We also did not succeed to find more conceptual arguments.

(ii) In the case $n_1 = n_2 = 8$ von Stengel and Keiding worked with the polytope $C_8(16)^\Delta$. It is not hard to see the following.

$C_8(16)^\Delta$ has 660 vertices. It has 336 triangles, 96 tetrahedra and 16 four-dimensional simplices. The situation at the 660 vertices is as follows. Here V_k denotes a k-dimensional simplex.

vertices of some type	40	80	16	32	440	48	4
simplices at such vertex	$4V_3$	$3V_3$	$1V_5, 2V_3$	$1V_4, 2V_3$	$1V_k, 1V_l$	$1V_k$	no V_k

Let U be a subset of the set of all vertices which contains at least one vertex of each triangle. The information above is too coarse to determine $\min_U |U|$. We cannot confirm or disprove our conjecture $\min_U |U| = 660 - 384 = 276$.

(iii) The g-theorem in the theory of simplicial and simple polytopes gives complete information on the possible f-vectors of simplicial or simple polytopes. But there are almost no results about finer information like existence and numbers of triangles in simple polytopes. The only reference which we found is the paper [BB90] of G. Blind and R. Blind.

Theorem 1 in [BB90] states that any d-dimensional polytope without triangles has in any dimension at least as many faces as the d-dimensional hypercube. Additionally, if in at least one dimension (smaller than d) equality holds, then the polytope is combinatorially equivalent to the hypercube.

In our situation it implies that an n_1+1-dimensional simple polytope Q^1 with $2n_1+2$ facets (and more vertices than the hypercube) must contain at least one triangle. This is a very weak obstruction. Together with Lemma 3.5.3, it improves Keiding's upper bound just by 1.

4 Inner tropical games

A large part of this thesis studies m-player games where each player has only two pure strategies. The mixed extensions of such games are called *pre-tropical games* below. Building on the precise maximal number for totally mixed Nash equilibria for generic games by McKelvey and McLennan [MM97], Vidunas [Vid17] gave an upper bound for the maximal number of Nash equilibria for generic games. The main result in this chapter is a construction of special pre-tropical games where the number of Nash equilibria is surprisingly close to this bound, namely more than half of it (Theorem 4.3.4).

For this we consider in the second section a subfamily of pre-tropical games, the *inner tropical games* and study closely their combinatorics. For them we find in the Section 4.2 an a priori upper bound for the maximal number of Nash equilibria. In Section 4.3 we show that there are inner tropical games which realize this bound. The combinatorics in the proof are intricate. In fact, the first proof which we found, is different. The first proof and our way to it are presented in Chapter 5.

Section 4.1 establishes group theoretic material which is useful for the control of the inner tropical games. The main point is that an inner tropical game comes equipped with a characteristic tuple $((v_1, \sigma_1), ..., (v_m, \sigma_m)) \in (\mathbb{F}_2 \times S_m)^m$. In Section 4.2 we need this data together with some group action on them. This is all prepared in the following section.

4.1 Wreath product

The following structure is useful to study symmetries introduced in Chapter 6 of the inner tropical games.

Definition 4.1.1. [BMMN98] Let C be an abstract group and D a group acting on a set Δ. Define

$$K := C^{\Delta} = \{f \mid f : \Delta \to C\}$$

Then K is a group under pointwise multiplication defined as follows. Given $f_1, f_2 \in K$ and $\delta \in \Delta$ we have

$$(f_1 f_2)(\delta) := f_1(\delta) f_2(\delta).$$

Let us define a right action of D on K which takes $f \in K$ to $f^d \in K$ for $d \in D$, by specifying that

$$f^d(\delta) := f(\delta d^{-1}).$$

The map

$$\theta_d := \begin{cases} K \to K \\ f \mapsto f^d \end{cases}$$

is an automorphism of K. The map $\theta : D \to \mathrm{Aut}(K)$ is a homomorphism because

$$\begin{aligned}(f\theta_{d_1 d_2})(\delta) &= f^{d_1 d_2}(\delta) = f(\delta(d_1 d_2)^{-1}) = f(\delta d_2^{-1} d_1^{-1}) \\ &= f^{d_1}(\delta d_2^{-1}) = (f^{d_1})^{d_2}(\delta) = (f\theta_{d_1}\theta_{d_2})(\delta)\end{aligned}$$

holds for all $f \in K$ and all $d_1, d_2 \in D$.

Definition 4.1.2. [BMMN98] The (external) direct product $A \times B$ of groups A and B is defined on the cartesian product set with multiplication defined componentwise. If A and B are subgroups of G with $A \trianglelefteq G$, $B \trianglelefteq G$, $G = AB$ and $A \cap B = \{e\}$. then $A \times B \simeq G$ is called the (internal) direct product of A and B.

Let two groups A and B be given, and a right action of B on A, so $(a \mapsto a^b)$ is a group automorphism of A for any $b \in B$, and $a^{(b_1 b_2)} = (a^{b_1})^{b_2}$.

Then the (external) semidirect product $A \rtimes B$ of A by B is also defined on the cartesian product set. Then the multiplication is

$$(a_1, b_1)(a_2, b_2) = (a_1 a_2^{b_1^{-1}}, b_1 b_2).$$

A group is called an (internal) semidirect product, also denoted by $A \rtimes B$, if $A \trianglelefteq G$, $B \leq G$, $G = AB$ and $A \cap B = \{e\}$.

Definition 4.1.3. [BMMN98] The wreath product W of C by D is defined to be the (external) semidirect product of K by D. That is,

$$W := K \rtimes D = C^{\Delta} \rtimes D$$

is defined on the set $\{(f, d) \mid f \in K, d \in D\}$ with multiplication defined by

$$(f_1, d_1)(f_2, d_2) = (f_1 f_2^{d_1^{-1}}, d_1 d_2).$$

We write $W := C \wr_{\Delta} D$ or $W := C \wr D$,when it is clear what Δ is, to denote that W is a wreath product of C by D. D is called the top group, C is called the bottom group and K is called the base group. The factors ($\simeq C$) of K are called the co-ordinate subgroups of K indexed by Δ.

The following lemma will allow us to identify elements of the wreath product by the elements of the base group. It is important once we introduce partial densities.

Lemma 4.1.4. *[BMMN98] The wreath product can be identified as an (internal) semidirect product.*

Proof. We identify K and D with the subsets $\{(f, e_D) \mid f \in K\} \subset W$ and $\{(e_K, d) \mid d \in D\} \subset W$. Obviously $K \cap D = \{(e_K, e_D)\}$. We need to show that K is a normal subgroup of W.

$$(f, d)^{-1} = ((f^{-1})^d, d^{-1}) = ((f^d)^{-1}, d^{-1}),$$

then

$$(f, d)(f, d)^{-1} = (f, d)((f^{-1})^d, d^{-1}) = (f((f^{-1})^d)^{d^{-1}}, dd^{-1}) = (e_K, e_D)$$

Similarily $(f,d)^{-1}(f,d) = (e_K, e_D)$. Now

$$\begin{aligned}(f,d)(g,e_D)(f,d)^{-1} &= (f,d)(g,e_D)((f^d)^{-1}, d^{-1})\\ &= (f,d)(gf^{d^{-1}}, d^{-1}) = (fg^{d^{-1}}f^{-1}, e_D).\end{aligned}$$

□

Definition 4.1.5. Use Definition 4.1.3. Set
$\Delta(n) = \{1,\ldots,n\}$, $C_0^{(n)} = \mathbb{F}_2 \times S_n$, $C_1^{(n)} = \mathbb{F}_2$, $C_2^{(n)} = S_n$, $D(n) = S_n$, $K_i^{(n)} = (C_i^{(n)})^{\Delta(n)}$ and $W_i^{(n)} = C_i^{(n)} \wr D(n) \simeq K_i^{(n)} D(n)$ for $i \in \{0,1,2\}$.

Remark 4.1.6. The following diagram commutes

$$\begin{tikzcd} K_1^{(n)} \arrow[r, "\iota_1"] & W_1^{(n)} \\ K_0^{(n)} \arrow[u, "\mathrm{pr}_K^1"] \arrow[r, "\iota_0"] \arrow[d, "pr_K^2"] & W_0^{(n)} \arrow[u, "\mathrm{pr}_W^1"] \arrow[d, "\mathrm{pr}_W^2"] \\ K_2^{(n)} \arrow[r, "\iota_2"] & W_2^{(n)}. \end{tikzcd}$$

Define for $i \in \{0,1,2\}$ the canonical embedding

$$\iota_i : \begin{cases} K_i^{(n)} & \to W_i^{(n)} \\ k & \mapsto (k, \mathrm{id}) \end{cases} \tag{4.1}$$

Let $k \in K_0^{(n)}$ with $w = \iota_0(k)$, then define $\underline{v} := \mathrm{pr}_K^1(k) = (v_1,\ldots,v_n)$ and $\underline{\sigma} := \mathrm{pr}_K^2(k) = (\sigma_1,\ldots,\sigma_n)$.

4.2 Pre-tropical games

In this section we fix $m = n$, then we have $\mathcal{A} = \{1, \dots, m\} = \{1, \dots, n\} = \Delta(n)$.

Definition 4.2.1. (a) A *pre-tropical game* is a mixed extension $(\mathcal{A}, G, V)$ of a finite game with $n_1 = ... = n_m = 1$. Then we write γ^i instead of γ^i_1 and λ^i instead of λ^i_1.

(b) For $L \subset \mathcal{A}$, the complementary set is $L^{\complement} := \mathcal{A} \setminus L$. Let $|L| \in \{0, 1, ..., m\}$ be the number of elements of L.

(c) Consider a pre-tropical game. It will be useful to consider the following subsets of A and G for $L_0, L_1 \subset \mathcal{A}$ with $L_0 \cap L_1 = \emptyset$, for $L \subset \mathcal{A}$, and for $d \in \{0, 1, ..., m\}$.

$$
\begin{aligned}
C^{\mathbb{R}}_{L_0,L_1} &:= \{g \in A \mid \gamma^i = 0 \text{ for } i \in L_0, \gamma^i = 1 \text{ for } i \in L_1, \\
&\qquad\qquad\qquad \gamma^i \in \mathbb{R} \setminus \{0,1\} \text{ for } i \in (L_0 \cup L_1)^{\complement}\}, \\
C^{\mathbb{R}}_L &:= \{g \in A \mid \gamma^i \in \{0,1\} \text{ for } i \in L, \gamma^i \in \mathbb{R} \setminus \{0,1\} \text{ for } i \in L^{\complement}\}, \\
C^{\mathbb{R}}_d &:= \bigcup_{L \subset \mathcal{A}:\, |L| = d} C^{\mathbb{R}}_L, \\
C_{L_0,L_1} &:= C^{\mathbb{R}}_{L_0,L_1} \cap G, \ C_L := C^{\mathbb{R}}_L \cap G, \\
C^P_d &:= C^{\mathbb{R}}_d \cap G, \qquad C^M_d := C^P_{m-d}.
\end{aligned}
$$

Note that the letter P (or M) in C^P_d (or C^M_d) is not a variable, rather it is a a fixed part of the formula. Its meaning is pure (or mixed). Then $(C^{\mathbb{R}}_L)_{L \subset \mathcal{A}}$ and $(C^{\mathbb{R}}_d)_{d \in \{0,1,\dots,m\}}$ are two stratifications for A, and $(C_L)_{L \subset \mathcal{A}}$, $(C^P_d)_{d \in \{0,1,\dots,m\}}$ and $(C^M_d)_{d \in \{0,1,\dots,m\}}$ are two stratifications for G.

(d) For a pre-tropical game and an element $g \in A$ write $L_0(g) := \{i \in \mathcal{A} \mid \gamma^i = 0\}$, $L_1(g) := \{i \in \mathcal{A} \mid \gamma^i = 1\}$, $L(g) := L_0(g) \cup L_1(g)$, $d(g) := |L(g)|$.

The notion of an inner tropical game in part (a) of the next definition is similar to games which were considered by McKelvey and McLennan [MM97, ch. 4] in their proof of the upper bound for the number of totally mixed Nash equilibria in generic games. Our situation is more special, as we suppose $n_1 = \cdots = n_m = 1$. But we want to control all Nash equilibria. Therefore we need more notions around inner tropical games. They are developed in the other parts of the next definition.

Definition 4.2.2. (a) A pre-tropical game $(\mathcal{A}, G, V)$ is *inner tropical*, if each polynomial λ^i has the shape

$$\lambda^i(\underline{\gamma}^{-i}) = (-1)^{v_i} \cdot \prod_{j \in \mathcal{A} \setminus \{i\}} (\gamma^j - a^i_j) \tag{4.2}$$

with suitable $a^i_j \in (0,1)$ and a suitable sign tuple $\underline{v} = (v_1, \dots, v_m) \in \mathbb{F}_2^{\mathcal{A}}$, where $a^{i_1}_j \neq a^{i_2}_j$ for $j \in \mathcal{A}$ and $i_1, i_2 \in \mathcal{A} \setminus \{j\}$ with $i_1 \neq i_2$ is demanded.

(b) Let $(\mathcal{A}, G, V)$ be an inner tropical game. The unique permutation $\sigma_j \in S_m$ with

$$\sigma_j(j) = j, \tag{4.3}$$
$$1 > a_j^{(\sigma_j)^{-1}(1)} > \dots > a_j^{(\sigma_j)^{-1}(j-1)} > a_j^{(\sigma_j)^{-1}(j+1)} > \dots > a_j^{(\sigma_j)^{-1}(m)} > 0,$$

is called the j-th associated permutation to the inner tropical game. The characteristic tuple of the inner tropical game $(\mathcal{A}, G, V)$ is $k \in K_0^{(m)}$ with $\mathrm{pr}^1_K(k) = \underline{v}$ and $\mathrm{pr}^2_K(k) = \underline{\sigma} = (\sigma_1, ..., \sigma_m)$.

(c) Recall Definition 4.1.5 and Remark 4.1.6. Let $k \in K_0^{(n)}$ with $\mathrm{pr}^1_K(k) \equiv \underline{v}$ and $\mathrm{pr}^2_K(k) \equiv \underline{\sigma}$, $v_i := \underline{v}(i) = \Big(\mathrm{pr}^1_K(k)\Big)(i) \in \mathbb{F}_2$ and $\sigma_i := \underline{\sigma}(i) = \Big(\mathrm{pr}^2_K(k)\Big)(i) \in S_n$ for all $i \in \Delta(n)(= \mathcal{A})$. The set of characteristic tuples of inner tropical games is

$$\mathcal{K}^{(n)} := \Big\{ k \in K_0^{(n)} \ \Big| \ \sigma_i(i) = i \ \ \forall i \in \Delta(n)(= \mathcal{A}) \Big\}.$$

The unique element $k \in \mathcal{K}^{(n)}$ can be identified with the tuple $(\underline{v}, \underline{\sigma}) = (\mathrm{pr}^1_K(k), \mathrm{pr}^2_K(k)) \in \mathbb{F}_2^n \times (S_n)^n$, even though it is formally an $(\mathbb{F}_2 \times S_n)^n$ tuple.

(d) (Introduction to an algebraic structure) The set of representations of inner tropical games is given by the canonical embedding $w = \iota_0(k)$ given by (4.1)

$$\mathcal{W}^{(n)} := \iota_0(\mathcal{K}^{(n)}) = \Big\{ (k, \beta) \in W_0^{(n)} \ \Big| \ k \in \mathcal{K}^{(n)} \text{ and } \beta = \mathrm{id} \Big\}. \tag{4.4}$$

(e) Recall that $W_0^{(n)}$ is a group. The neutral n-player representation is the neutral element of $W_0^{(n)}$ and is denoted by $0^{(n)}_{\mathcal{W}}$. It is $0^{(n)}_{\mathcal{W}} = (k, \mathrm{id}) \in \mathcal{W}^{(n)}$ with $\mathrm{pr}^1_K(k) = (0, ..., 0)$ and $\mathrm{pr}^2_K(k) = (\mathrm{id}, ..., \mathrm{id})$.

Remark 4.2.3. (i) For arbitrary $a^i_j \in (0,1)$ with $a^{i_1}_j \neq a^{i_2}_j$ for $j \in \mathcal{A}$ and

$i_1 \neq i_2$ and an arbitrary sign tuple $\underline{v} = (v_1, \dots, v_m)$, an inner tropical game $(\mathcal{A}, G, V)$ with polynomials $\lambda^i(\underline{\gamma}^{-i})$ as in (4.2) exists: Any tuple $(V_A^1, ..., V_A^m)$ of multi affine linear polynomials in $\underline{\gamma}$ as in Lemma 2.1.2 is realized by a suitable game $(\mathcal{A}, G, V)$.

(ii) Inner tropical games with the same characteristic tuple k have similar sets of equilibrium candidates and equilibria, see below Theorem 4.2.6.

(iii) For a given permutation $\sigma_j \in S_m$ with $\sigma_j(j) = j$, one tuple $(a_j^1, \dots, a_j^{j-1}, a_j^{j+1}, \dots, a_j^m)$ with (4.3) is the tuple with $a_j^{(\sigma_j)^{-1}(i)} := \frac{1}{i+1}$ for $i \in \mathcal{A} \setminus \{j\}$.

(iv) Definition 4.2.2(d) will be useful in Chapter 6. It will allow us to use a tight notation but it is not useful for this chapter. In preparation for Chapter 6 we will already use the representations in Chapter 5 to have clean definitions.

For all inner tropical games with fixed $\mathcal{A}$ and S, the sets of equilibrium candidates are finite and have the same structure. But the sets of equilibria depend strongly on the characteristic tuples k. Both statements are subject of Theorem 4.2.6. The latter is prepared by Definition 4.2.4 and Lemma 4.2.5, which recall a standard notion in the theory of permutation groups.

Definition 4.2.4. (a) A permutation $\pi \in S_n$ (for some $n \in \mathbb{N}$) is a *derangement* if $\pi(i) \neq i$ for any $i \in \{1, \dots, n\}$. The set of derangements in S_n is called Der_n. The *subfactorial* $!n$ is the number $|\,\mathrm{Der}_n\,|$ of derangements in S_n.

(b) For $\pi \in S_m$, denote $L(\pi) := \{i \in \mathcal{A} \mid \pi(i) = i\} \subset \mathcal{A}$. Then π restricted to $L(\pi)^{\complement}$ is a derangement on the set $L(\pi)^{\complement}$.

Lemma 4.2.5. *(E.g. [MM97, Proposition 5.4]) The subfactorials satisfy the following recursions* (4.5) *and* (4.6) *the closed formula* (4.7)*. The table* (4.8) *gives the first subfactorials and (for comparison) the first factorials.*

$$!n = n{\cdot}!(n-1) + (-1)^n \quad \text{for } n \geq 1, \tag{4.5}$$

$$!n = (n-1)(!(n-1) + !(n-2)) \quad \text{for } n \geq 2, \tag{4.6}$$

$$!n = n! \cdot \sum_{j=0}^{n} \frac{(-1)^j}{j!} = n! \cdot \sum_{j=2}^{n} \frac{(-1)^j}{j!} \tag{4.7}$$

n	0	1	2	3	4	5	6	7
$!n$	1	0	1	2	9	44	265	1854
$n!$	1	1	2	6	24	120	720	5040

(4.8)

Table 4.1: Number of derangements for $n \in \{0, 1, \dots, 7\}$

Proof. The recursion (4.5) implies the recursion (4.6) because of the following calculation,

$$\begin{aligned} !n &\overset{(4.5)}{=} (n-1)\cdot!(n-1) + (-1)^n + !(n-1) \\ &\overset{(4.5)}{=} (n-1)\cdot!(n-1) + (-1)^n + (n-1)\cdot!(n-2) + (-1)^{n-1} \\ &= (n-1)(!(n-1) + !(n-2)). \end{aligned}$$

The same calculation backwards can be interpreted as the induction step in a proof that the recursion (4.5) holds if the recursion (4.6) holds. The beginning of the induction is given by $!1 = 0 = 1\cdot!0 + (-1)^1$. Therefore the two recursions are equivalent.

The recursion (4.6) will be proved now. The set Der_n splits into the $n-1$ sets $\{\pi \in \mathrm{Der}_n \mid \pi(1) = i\}$ for $i \in \{2, \dots, n\}$. Each of these sets splits into the two sets $\{\pi \in \mathrm{Der}_n \mid \pi(1) = i, \pi(i) = 1\}$ and $\{\pi \in \mathrm{Der}_n \mid \pi(1) = i, \pi(i) \neq 1\}$, while the latter set is bijective to the set $\mathrm{Der}_n(\{2, \dots, n\})$ with the following bijection: $\pi \mapsto \widetilde{\pi}$, such that

$$\widetilde{\pi}(j) := \begin{cases} i & \text{if } \pi(j) = 1, \\ \pi(j) & \text{if } \pi(j) \neq 1. \end{cases}$$

The former set has $!(n-2)$ elements and the latter set has $!(n-1)$ elements. Formula (4.7) follows with induction from (4.5) and $!0 = 1$. □

Theorem 4.2.6. *Let $(\mathcal{A}, G, V)$ be an inner tropical game with characteristic tuple k and numbers $a_j^i \in (0,1)$ as in Definition 4.2.2.*

(a) The set of equilibrium candidates is the set

$$\begin{aligned} EC &:= \bigcup_{\pi \in S_m} EC(\pi) \quad \text{with} \\ EC(\pi) &:= \{g \in G \,|\, \gamma^i \in \{0,1\} \text{ for } i \in L(\pi), \\ &\qquad \gamma^j = a_j^{\pi(j)} \text{ for j} \in L(\pi)^{\complement}\}. \end{aligned}$$

We have $|EC(\pi)| = 2^{|L(\pi)|}$ *and* $|EC| = \sum_{l=0}^{m} \binom{m}{l} \cdot !(m-l) \cdot 2^l$.

(b) Consider $\pi \in \mathrm{Der}_m$, *i.e.* $\pi \in S_m$ *with* $|L(\pi)| = 0$. *Then* $EC(\pi)$ *has only one element, and this is an equilibrium.*

(c) Consider $\pi \in S_m$ *with* $L(\pi) \neq \emptyset$, *and consider an equilibrium candidate* $g \in EC(\pi)$. *Its* increment map $\mathrm{Inc}(g) : L(\pi) \to \{0,1\}$ *is defined by*

$$\mathrm{Inc}(g,i) := \Big(1 + \gamma^i + v_i + \Big|L_0(g) \setminus \{i\}\Big| + \sum_{j \in L(\pi)^{\complement}} \chi(\sigma_j(\pi(j)), \sigma_j(i))\Big) \mod 2, \tag{4.9}$$

where χ *is defined by*

$$\chi : \mathbb{R} \times \mathbb{R} \to \{0,1\} \text{ with } \chi(a,b) = 1 \iff a \geq b. \tag{4.10}$$

The values $\mathrm{Inc}(g,i) \in \{0,1\} (= \mathbb{F}_2)$ *are called* increments. *The following holds.*

(i) *g is an equilibrium if and only if* $\mathrm{Inc}(g,i) = 0$ *for all* $i \in L(\pi)$.

(ii) *Let* $g(\pi) \in EC(\pi)$ *be the equilibrium candidate in* $EC(\pi)$ *with* $\gamma^i = 1$ *for all* $i \in L(\pi)$. *The map* $\mathrm{Inc} : EC(\pi) \to \{0,1\}^{L(\pi)}$ *has only two values on* $EC(\pi)$, *the map* $\mathrm{Inc}(g(\pi))$ *and the opposite map which takes at each* $i \in L(\pi)$ *the opposite value* $1 + \mathrm{Inc}(g(\pi), i) \mod 2$. *The map* $\mathrm{Inc}(g)$ *coincides with the map* $\mathrm{Inc}(g(\pi))$ *if and only if g and* $g(\pi)$ *differ in an even number of coefficients* γ^i *for* $i \in L(\pi)$ *(i.e.* $|L_0(g)|$ *is even).*

(iii) *Either* $EC(\pi)$ *contains no equilibrium or half of its elements are equilibria.*

(d) The only permutation π *with* $|L(\pi)| = m$ *is* $\pi = \mathrm{id}$. *In the case* $\pi = \mathrm{id}$, *half of the elements of* $EC(\pi)$ *are equilibria if and only if* $\underline{v} = (0, \dots, 0)$ *or* $\underline{v} = (1, \dots, 1)$.

(e) There is no permutation $\pi \in S_m$ *with* $|L(\pi)| = m-1$.

Proof. (a) Let $g \in G$ be an equilibrium candidate. Recall $L_0(g), L_1(g)$ and $L(g) = L_0(g) \cup L_1(g)$ from Definition 4.2.1 (d). For $i \in L_0(g)$ $\gamma^i = 0$, for

$i \in L_1(g)$ $\gamma^i = 1$, and

$$\text{for} \quad i \in L(g)^{\complement} \qquad 0 = \lambda^i(\underline{\gamma}^{-i}) = \prod_{j \neq i} (\gamma^j - a^i_j).$$

Therefore there is a map

$$\begin{aligned} \widetilde{\pi} : \mathcal{A} \to \mathcal{A} \quad & \text{with } \widetilde{\pi}(i) = i \text{ for } i \in L(g), \\ & \text{and with } \widetilde{\pi}(i) \neq i \text{ and } \gamma^{\widetilde{\pi}(i)} = a^i_{\widetilde{\pi}(i)} \text{ for } i \in L(g)^{\complement}. \end{aligned}$$

Because for each $j \in \mathcal{A}$ the coefficients a^i_j for $i \in \mathcal{A} \setminus \{j\}$ are pairwise different and not in $\{0,1\}$, the map $\widetilde{\pi}$ is a permutation with $L(\widetilde{\pi}) = L(g)$, and it is unique. Write $\pi := \widetilde{\pi}^{-1}$. Also π is a permutation with $L(\pi) = L(g)$. Therefore the condition $\gamma^{\widetilde{\pi}(i)} = a^i_{\widetilde{\pi}(i)}$ for $i \in L(g)^{\complement}$ can also be written as

$$\gamma^j = a^{\pi(j)}_j \quad \text{for} \quad j \in L(\pi)^{\complement}. \tag{4.11}$$

Vice versa, for any permutation $\pi \in S_m$, the element g with (4.11) and $g^i \in \{0,1\}$ for $i \in L(\pi)$ is an equilibrium candidate. Obviously $|EC(\pi)| = 2^{|L(\pi)|}$. The number of subsets $L \subset \mathcal{A}$ with $|L| = l$ is $\binom{m}{l}$. The number of derangements on a set $L^{\complement}$ with $|L| = l$ is $!(m-l)$. Therefore $|EC|$ is as claimed.

(b) In the case $\pi \in \mathrm{Der}_m$, the set $EC(\pi)$ has only one element which is now called g. All its coefficients γ^i are in $(0,1)$. Applying Lemma 2.1.2(b) gives the result.

(c) Now consider $\pi \in S_m$ with $L(\pi) \neq \emptyset$. An equilibrium candidate $g \in EC(\pi)$ is by Lemma 2.1.2(b) an equilibrium if $\lambda^i(\underline{\gamma}^{-i}) \geq 0$ for $\gamma^i = 1$ and $\lambda^i(\underline{\gamma}^{-i}) \leq 0$ for $\gamma^i = 0$. But $\lambda^i(\underline{\gamma}^{-i}) \in \mathbb{R}^*$ for $i \in L(\pi) = L(g)$ because it is

$$\begin{aligned} &\lambda^i(\underline{\gamma}^{-i}) = (-1)^{v_i} \prod_{j \neq i} (\gamma^j - a^i_j) = \\ &(-1)^{v_i} \Big(\prod_{j \in L_1(g) \setminus \{i\}} (1 - a^i_j) \Big) \Big(\prod_{j \in L_0(g) \setminus \{i\}} (0 - a^i_j) \Big) \Big(\prod_{j \in L(\pi)^{\complement}} (a^{\pi(j)}_j - a^i_j) \Big), \end{aligned}$$

and all $a^i_j \in (0,1)$, and furthermore $a^{\pi(j)}_j \neq a^i_j$ for $j \in L(\pi)^{\complement}$ because $\pi(j) \in L(\pi)^{\complement}$ and $i \in L(\pi)$. Therefore $g \in EC(\pi)$ is an equilibrium if and only if $\mathrm{sgn}(\lambda^i(\underline{\gamma}^{-i})) = (-1)^{1+\gamma^i}$ for any $i \in L(\pi)$.

The sign of $\lambda^i(\underline{\gamma}^{-i})$ is

$$\begin{aligned}\operatorname{sgn}(\lambda^i(\underline{\gamma}^{-i})) &= (-1)^{v_i}(-1)^{|L_0(g)\setminus\{i\}|}\cdot \prod_{j\in L(\pi)^\complement} \operatorname{sgn}\Big(a_j^{\pi(j)} - a_j^i\Big) \\ &= (-1)^{v_i}(-1)^{|L_0(g)\setminus\{i\}|}\cdot \prod_{j\in L(\pi)^\complement} (-1)^{\chi(\sigma_j(\pi(j)),\sigma_j(i))}.\end{aligned}$$

Here $\operatorname{sgn}\Big(a_j^{\pi(j)} - a_j^i\Big) = (-1)^{\chi(\sigma_j(\pi(j)),\sigma_j(i))}$ because of (4.3) (and (4.10)). The condition $\operatorname{sgn}(\lambda^i(\underline{\gamma}^{-i})) = (-1)^{1+\gamma^i}$ for $i\in L(\pi)$ is equivalent to the condition $\operatorname{Inc}(g,i)=0$. This proves part (i).

For part (ii), consider an equilibrium candidate $\widetilde{g}\in EC(\pi)$ which differs from g only in one coordinate, so $\widetilde{\gamma}^j = \gamma^j = a_j^{\pi(j)}$ for $j\in L(\pi)^\complement$, $\widetilde{\gamma}^i=\gamma^i$ for $i\in L(\pi)\setminus\{i_0\}$ for one $i_0\in L(\pi)$, and $\widetilde{\gamma}^{i_0}\equiv 1+\gamma^{i_0} \mod 2$. Only the part

$$\Big(\gamma^i + |L_0(g)\setminus\{i\}|\Big) \mod 2$$

of $\operatorname{Inc}(g,i)$ depends on g. For $i\in L(\pi)\setminus\{i_0\}$, we have $\widetilde{\gamma}^i=\gamma^i$ and $|L_0(\widetilde{g})\setminus\{i\}|\equiv 1+|L_0(g)\setminus\{i\}| \mod 2$, so $\operatorname{Inc}(\widetilde{g},i)\equiv 1+\operatorname{Inc}(g,i) \mod 2$. For $i=i_0$, we have $\widetilde{\gamma}^i\equiv 1+\gamma^i \mod 2$ and $|L_0(\widetilde{g})\setminus\{i\}|\equiv|L_0(g)\setminus\{i\}| \mod 2$, so $\operatorname{Inc}(\widetilde{g},i)\equiv 1+\operatorname{Inc}(g,i) \mod 2$. This proves part (ii).

Part (iii) follows immediately from the parts (i) and (ii).

(d) In the case $\pi=\mathrm{id}$, we have $\operatorname{Inc}(g(\pi),i)\equiv 1+1+v_i+0\equiv v_i \mod 2$. So $g(\pi)$ is an equilibrium if and only if $v_i=0$ for all $i\in\mathcal{A}$, and an equilibrium candidate in $EC(\pi)$ which differs in an odd number of coefficients from $g(\pi)$ is an equilibrium if and only if $v_i=1$ for all $i\in\mathcal{A}$.

(e) This follows from $!1=0$ or $\mathrm{Der}_1=\emptyset$. □

Definition 4.2.7. Let $(\mathcal{A},G,V)$ be an inner tropical game with characteristic tuple k. A permutation $\pi\in S_m$ with $L(\pi)\neq\emptyset$ is an *equilibrium permutation* if half of the equilibrium candidates in $EC(\pi)$ are equilibria.

The next corollary is an immediate consequence of Theorem 4.2.6 (c) (i) and (ii).

Corollary 4.2.8. *In the situation of Definition 4.2.7, a permutation $\pi\in S_m$ with $L(\pi)\neq\emptyset$ is an equilibrium permutation if and only if* $\operatorname{Inc}(g(\pi))$ *has either only value 0 or only value 1.* □

4.3 Maximal games

Due to Theorem 4.2.6 (a), the set EC of equilibrium candidates of an inner tropical game is finite and is a union $EC = \bigcup_{\pi \in S_m} EC(\pi)$ of sets $EC(\pi)$ with $2^{|L(\pi)|}$ elements. Due to Theorem 4.2.6 (c) (ii), for an inner tropical game and any permutation π, either half of the equilibrium candidates in $EC(\pi)$ or none can be equilibria.

Definition 4.3.1. An inner tropical game $(\mathcal{A}, G, V)$ with the characteristic tuple k is maximal if for any permutation $\pi \in S_m \setminus \mathrm{Der}_m$ half of the equilibrium candidates in $EC(\pi)$ are equilibria.

The following lemma motivates Theorem 4.3.4. It will show that there are maximal games for all $m \in \mathbb{N}$. The permutations in Lemma 4.3.2 are part of the characteristic tuple of a maximal game.

Lemma 4.3.2. *Fix $m \in \mathbb{N}$ and $\mathcal{A} = \{1, ..., m\}$. For $i \in \mathcal{A}$ define the following three permutations α_i, β_i and $\delta_i \in S_m$. (Recall the definition of $\chi : \mathbb{R} \times \mathbb{R} \to \{0, 1\}$ in (4.10)).*

$$\begin{aligned}
\alpha_i &:= \begin{pmatrix} j & \mapsto & j & \text{if} & 1 \leq j \leq i-1, \\ i & \mapsto & m & & \\ j & \mapsto & j-1 & \text{if} & i+1 \leq j \leq m. \end{pmatrix} \\
&= \begin{pmatrix} 1 & \ldots & i-1 & i & i+1 & \ldots & m \\ 1 & \ldots & i-1 & m & i & \ldots & m-1 \end{pmatrix} = (m\ m-1\ \ldots\ i+1\ i), \\
\beta_i &:= \begin{pmatrix} j & \mapsto & m-i+j & \text{if} & 1 \leq j \leq i-1, \\ j & \mapsto & j-i+1 & \text{if} & i \leq j \leq m-1, \\ m & \mapsto & m & & \end{pmatrix} \\
&= \begin{pmatrix} 1 & \ldots & i-1 & i & \ldots & m-1 & m \\ m-i+1 & \ldots & m-1 & 1 & \ldots & m-i & m \end{pmatrix}, \\
\delta_i &:= (\alpha_i)^{-1} \circ \beta_i \circ \alpha_i.
\end{aligned} \tag{4.12}$$

δ_i is the permutation

$$\delta_i = \begin{pmatrix} j & \mapsto & m-i+j+\chi(m-i+j, i) & \text{if} & 1 \leq j \leq i-1, \\ i & \mapsto & i & & \\ j & \mapsto & j-i+\chi(j-i, i) & \text{if} & i+1 \leq j \leq m. \end{pmatrix}. \tag{4.13}$$

δ_i is the unique permutation in S_m with $\delta_i(i) = i$ and with the following property: $\delta_i(j_1) > \delta_i(j_2)$ for $j_1, j_2 \in \mathcal{A} \setminus \{i\}$ with $j_1 < j_2$ if and only if $j_1 \in \{1, ..., i-1\}$ and $j_2 \in \{i+1, ..., m\}$.

Proof. Let us compute $\delta_i(j)$. First case we consider $j < i$

$$\begin{aligned}\delta_i(j) &= (\alpha_i)^{-1}\Big(\beta_i\big(\alpha_i(j)\big)\Big)\\ &= (\alpha_i)^{-1}\big(\beta_i(j)\big)\\ &= (\alpha_i)^{-1}(m-i+j) \overset{?}{=} m-i+j+\chi(m-i+j,i).\end{aligned}$$

Now consider either $m-i+j < i$ or $m-i+j \geq i$. In the first case we have $\alpha_i(m-i+j+\chi(m-i+j,i)) = \alpha_i(m-i+j) \overset{m-i+j\leq i-1}{=} m-i+j$ and in the second case we have $\alpha_i(m-i+j+\chi(m-i+j,i)) = \alpha_i(m-i+j+1) \overset{m-i+j+1\geq i+1}{=} m-i+j+1-1 = m-i+j$.
The second case $j = i$ is trivial. We have

$$\delta_i(i) = (\alpha_i)^{-1}\Big(\beta_i\big(\alpha_i(i)\big)\Big) = (\alpha_i)^{-1}\big(\beta_i(m)\big) = (\alpha_i)^{-1}(m) = i.$$

The final case $j > i$ is similar to the first case. We have

$$\begin{aligned}\delta_i(j) &= (\alpha_i)^{-1}\Big(\beta_i\big(\alpha_i(j)\big)\Big)\\ &= (\alpha_i)^{-1}\big(\beta_i(j-1)\big)\\ &= (\alpha_i)^{-1}(j-i+1) \overset{?}{=} j-i+\chi(j-i,i).\end{aligned}$$

Let us consider $j-i < i$ and $j-i \geq i$ seperately. In the first case we have $j-i \leq i-1$ and $\chi(j-i,i) = 0$, so $\alpha_i(j-i+\chi(j-i,i)) = \alpha_i(j-i) = i-i$ and in the second case we have $j-i \geq i$ and $\chi(j-i,i) = 1$, thus $\alpha_i(j-i+\chi(j-i,i)) = \alpha_i(j-i+1) \overset{j-i+1\geq i+1}{=} j-i+1-1 = j-i$. □

Lemma 4.3.3. *For $i \in L(\pi)$ and $j \in L(\pi)^\complement$*

$$\chi(\delta_j(\pi(j)), \delta_j(i)) = \begin{cases} 1 & \text{if } j < i \text{ and } (\pi(j) < j \text{ or } \pi(j) > i), \\ 0 & \text{if } j < i \text{ and } j < \pi(j) < i, \\ 0 & \text{if } j > i \text{ and } (\pi(j) < i \text{ or } \pi(j) > j), \\ 1 & \text{if } j > i \text{ and } i < \pi(j) < j. \end{cases} \tag{4.14}$$

Proof. Recall that $\chi(\delta_j(\pi(j)), \delta_j(i)) = 1$ if and only if $\delta_j(\pi(j)) > \delta_j(i)$. Recall the characterization of δ_j at the end of Lemma 4.3.2.

First consider the case $j \in L(\pi)^\complement$ with $j < i$. If $\pi(j) > i$ then also $\delta_j(\pi(j)) > \delta_j(i)$. If $\pi(j) \in \{j+1, \ldots, i-1\}$ then $j < \pi(j) < i$ and also $\delta_j(\pi(j)) < \delta_j(i)$. If $\pi(j) < j$ then $\delta_j(\pi(j)) > \delta_j(i)$.

Now consider the case $j \in L(\pi)^\complement$ with $j > i$. If $\pi(j) > j$ then $\delta_j(\pi(j)) < \delta_j(i)$. If $\pi(j) \in \{i+1, ..., j-1\}$ then $\pi(j) > i$ and also $\delta_j(\pi(j)) > \delta_j(i)$. If $\pi(j) < i$ then also $\delta_j(\pi(j)) < \delta_j(i)$. □

Theorem 4.3.4. *(Fundamental theorem of inner tropical games)*
Fix $m \in \mathbb{N}$ and $\mathcal{A} = \{1, ..., m\}$. Any inner tropical game $(\mathcal{A}, G, V)$ with the characteristic tuple $(v_1, \ldots, v_m) = (0, \ldots, 0)$ and $(\sigma_1, \ldots, \sigma_m) = (\delta_1, \ldots, \delta_m)$ is maximal.

Proof. Consider an inner tropical game with the characteristic tuple $(v_1, \ldots, v_m) = (0, \ldots, 0)$ and $(\sigma_1, \ldots, \sigma_m) = (\delta_1, \ldots, \delta_m)$. Because of Corollary 4.2.8, it is sufficient to show that the increment map $\mathrm{Inc}(g(\pi)) \in \mathbb{F}_2^{L(\pi)}$ has for any permutation $\pi \in S_m \setminus \mathrm{Der}_m$ either only value 0 or only value 1. Fix a permutation $\pi \in S_m \setminus \mathrm{Der}_m$ and fix an element $i \in L(\pi)$. Then

$$\mathrm{Inc}(g(\pi), i) \equiv \Big(\sum_{j \in L(\pi)^\complement} \chi(\delta_j(\pi(j)), \delta_j(i)) \Big) \mod 2 \quad \text{for} \quad i \in L(\pi).$$

If $|L(\pi)| = 1$, nothing has to be shown as then the definition domain $L(\pi)$ of $\mathrm{Inc}(g(\pi))$ has only one element. So suppose $|L(\pi)| \geq 2$, and fix two elements $i_1, i_2 \in L(\pi)$ with $i_1 < i_2$.

(4.14) shows which $j \in L(\pi)^\complement$ give the same contributions to $\mathrm{Inc}(g(\pi), i_1)$ and to $\mathrm{Inc}(g(\pi), i_2)$, and which give different contributions. The splitting into the following 9 cases is natural. In the following table, $\chi\Big(\delta_j(\pi(j), \delta_j(i_1))\Big)$ and $\chi\Big(\delta_j(\pi(j), \delta_j(i_2))\Big)$ are abbreviated as $\chi[i_1]$ and $\chi[i_2]$. Only the cases (3), (6) and (9) give different contributions to $\mathrm{Inc}(g(\pi), i_1)$

$$\begin{array}{l|l|l|l|l}
\textit{Case} & & & \chi[i_1] & \chi[i_2] \\
\hline
(1) & j < i_1 & \pi(j) < j \text{ or } \pi(j) > i_2 & 1 & 1 \\
(2) & j < i_1 & j < \pi(j) < i_1 & 0 & 0 \\
(3) & j < i_1 & i_1 < \pi(j) < i_2 & 1 & 0 \\
(4) & j > i_2 & \pi(j) < i_1 \text{ or } \pi(j) > j & 0 & 0 \\
(5) & j > i_2 & i_2 < \pi(j) < j & 1 & 1 \\
(6) & j > i_2 & i_1 < \pi(j) < i_2 & 1 & 0 \\
(7) & i_1 < j < i_2 & i_1 < \pi(j) < j & 1 & 1 \\
(8) & i_1 < j < i_2 & j < \pi(j) < i_2 & 0 & 0 \\
(9) & i_1 < j < i_2 & \pi(j) < i_1 \text{ or } \pi(j) > i_2 & 0 & 1
\end{array} \tag{4.15}$$

Table 4.2: Different cases for j, i_1 and i_2 for the maximal game

and to $\operatorname{Inc}(g(\pi), i_2)$. The number of the j in the cases (3) and (6) together is the same as the number of the j in the case (9), as π is a bijection. Therefore the number of different contributions is even. This shows $\operatorname{Inc}(g(\pi), i_1) = \operatorname{Inc}(g(\pi), i_2)$. Therefore the increment map $\operatorname{Inc}(g(\pi))$ has either only value 0 or only value 1. □

5 Linear equation system and the increment equations

In this chapter we will develop the original proof of Theorem 4.3.4, the **fundamental theorem of inner tropical games**. We will also explain the way how we came to it. This way is involved. We had some early attempts to disprove the theorem. The crucial tool to control the situation and prove the theorem was to cast the conditions for existence of maximal inner tropical games into a linear equation system. This made it feasible to attack the problem of existence with computer and computations.

The proof in the last chapter is shorter. But it does not show how one could find and how we found the theorem. The linear equation system in this chapter allows to reduce many questions down to computable mathematics. The author admits that he would not have been able to find maximal games without systems of linear equation. Without the progress in this chapter, none of the results in the previous Section 4.3 would have been possible.

An important intermediate step was the **small fundamental theorem of inner tropical games**. It says that if a maximal inner tropical game with m players exists, then also a maximal inner tropical game with $m-1$ players exists. Having this result, the question is how large is the maximal number m_0 such that maximal inner tropical games exist. First we found $m_0 \geq 4$, then $m_0 \geq 5$, then $m_0 \geq 6$, then $m_0 \geq 7$; with three different methods. Only then did we see the general construction which leads to the proof of the fundamental theorem of inner tropical games, so to $m_0 = \infty$.

Theorem 5.3.8 states that maximal games with m players induce maximal games of less than m players. Its Corollary 5.3.9 deserves the name **small fundamental theorem of inner tropical games** because it sets a clear goal, that is to improve upper and lower bounds for the maximal amount of players m_0. We now know that $m_0 = \infty$, but the actual progression was

$m_0 \geq 4$, then $m_0 \geq 5$ with similar but slightly different methods.

The first one was proven by calculating the partial densities of the subset $\Sigma(4,0) \subset \mathcal{W}^{(4)}$. This is brute forcable and took few minutes.

The second one was proven by taking a maximal game $w \in \mathcal{W}^{(4)}$ and inducing a maximal game with Theorem 5.3.8. We took an inversion vector of a maximal game $u_1^{(4)}$ and we calculated maximality of all games with vector $u_1^{(5)}$ such that $R_u\left(u_1^{(5)}\right) = u_1^{(4)}$ holds. This is also brute forcable and also took few minutes.

$m_0 \geq 6$ and $m_0 \geq 7$ were shown as follows. Let $b_\rho = 0$ be the desired vector for the maximal game (unclear if it exists yet). Solve a related linear equation system derived from (5.7).

$$A_{12}^{(n)} u^{(n)} = b_0^{(n)},$$

where $A_{12}^{(n)}$ is a matrix with $x(n)$ rows and $y_1(n) + y_2(n)$ columns and $u^{(n)}$ is a column vector with $y_1(n) + y_2(n)$ entries.

$$A_{12}^{(n)} = \left(A_1^{(n)} \,\middle|\, A_2^{(n)}\right), \; u^{(n)} = \begin{pmatrix} u_1^{(n)} \\ \hline u_2^{(n)} \end{pmatrix}$$

This gives a solution set

$$\left\{ z \in \mathbb{F}_2^{y_1(n)+y_2(n)} \,\middle|\, A_{12}^{(n)} z = b_0^{(n)} \right\}.$$

We can discard the dummy variables which gives a set of vectors which induces a set of inversions. Now we can iterate over this set with Lemma 5.2.5 to check if the vector is admissible. If it is admissible then the maximum game exists. This computation took again a few minutes. A computation for $m > 7$ was not attempted since the linear equation systems grow faster than exponential growth.

The author has found $m_0 = \infty$ by sheer luck. The original ansatz was to calculate all maximal games for $m = 3$ and $m = 4$ and find a link between those games. This was more complicated than originally thought. The solution set of maximal games was devoid of nearly all structure. Both the sets for $m = 3$ and $m = 4$ could not be extended to higher m trivially at first. They also had a lot more maximal games than expected, which made finding links near impossible. The final Hail Mary attempt was the most naïve approach. Split the vector $u_1^{(n)}$ such that the sets Z_i it induces by (5.8) are as simple as possible. We found the simple inequality chain (5.41) that induces the permutations (4.13) by applying Lemma 5.2.5. The final piece was the parametrization of the increment vector of the neutral game. $A^{(n)}$ and $u^{(n)}$ are easily parametrizable but the increment vector was trickier. In the above mentioned computations for $m_0 \geq 6$ and $m_0 \geq 7$ it was computed by calculating the increment of the neutral game for every $(\pi, i) \in X^{(n)}$ with $n = 6$ or $n = 7$. But for the general proof idea a concise parametrisation is necessary to work with.

While trying to prove a conjecture it is very useful to study the contraposition and try to find a counterexample or disprove it otherwise. We observe that Corollary 5.4.9 was the only chance we have gotten so far to find a counterexample. This corollary was found by studying partial densities which will be introduced in Chapter 6. This started a series of lengthy computations which took weeks of CPU-time and resulted in (5.34) A counterexample is impossible to find, but the calculations were helpful nonetheless. They resulted in Lemma 5.4.11, which upon further study resulted in Lemma 5.2.2. This completed the parametrization of all necessary data to find maximal games. We can parametrise all parts of the linear equation system. Theorem 5.5.1 follows now. The proof is a reformulated proof of Theorem 4.3.4 and the proof idea is identical.

This gives a rough outline of the chronological events that took place. It is also found in Figure 5.1.

5.1 Binary relations

This defines only the necessary parts of binary relations for completeness sake. A good reference is [Sch10]. We keep this section as short as possible.

Definition 5.1.1. (a) Let $X \neq \emptyset$ and $R \subset X \times X$ be a binary relation. It is called

- asymmetric, if $\forall a, b \in X: \ (a, b) \in R \Rightarrow (b, a) \notin R$,
- transitive, if $\forall a, b, c \in X: \ (a, b) \in R \wedge (b, c) \in R \Rightarrow (a, c) \in R$,
- complete, if $\forall a, b \in X: a \neq b \Rightarrow ((a, b) \in R \vee (b, a) \in R)$.

(b) The transitive closure R^+ is the smallest transitive relation over X containing R.

Remark 5.1.2. A complete binary relation is also called semi-connex.

Lemma 5.1.3. *Let $(X, <)$ be a strictly totally ordered finite set and let R be an asymmetric relation on X.*

(a) If R^+ is asymmetric, then there exists a $f \in \mathrm{Bij}(X, X)$ with $(f(a), f(b)) \in R^+ \Longrightarrow a < b$ for all $a, b \in X$.

(b) If R^+ is not asymmetric, then there exists no $f \in \mathrm{Bij}(X, X)$ with $(f(a), f(b)) \in R^+ \Longrightarrow a < b$ for all $a, b \in X$.

(c) If R^+ is asymmetric and complete, then the $f \in \mathrm{Bij}(X, X)$ is unique.

(d) If $\widetilde{R} \subset R$ with R^+ asymmetric, then we have that $\widetilde{R}^+$ is also asymmetric

Proof. The lemma is trivial. A proof idea is found in [Sch10, chapter 5]. $\square$

5.2 Linearization of the increment equations

Definition 5.2.1. (Linearization of the increment equation) (a) Let $n \in \mathbb{N}$ and set

$$\begin{aligned} x(n) &:= \sum_{i=0}^{n} (n-i)\cdot !i \cdot \binom{n}{n-i} \\ y_1(n) &:= \frac{n(n-1)(n-2)}{2}, \\ y_2(n) &:= n! - !n, \\ y_3(n) &:= n, \\ y(n) &:= y_1(n) + y_2(n) + y_3(n). \end{aligned}$$

We build an $\mathbb{F}_2$-linear equation system with $x(n)$ equations and $y(n)$ variables as follows. Define the set of variables from the following sets

$$\begin{aligned} Y_1^{(n)} &:= \{(i, j_1, j_2) \in \{1, \dots, n\}^3 \mid i \neq j_1 < j_2 \neq i\}, \\ Y_2^{(n)} &:= S_n \setminus \mathrm{Der}_n, \\ Y_3^{(n)} &:= \{1, \dots, n\}. \end{aligned}$$

Y_1 is ordered lexicographically by the order $<_l$. For $Y_2^{(n)}$ we use the lexicographic permutation order $<_p$, i.e.

$$\pi_1 <_p \pi_2 \iff (\pi_1(1), \dots, \pi_1(n)) <_l (\pi_2(1), \dots, \pi_2(n)). \tag{5.1}$$

Y_3 is the natural order on the integers $<$. Instead of writing $<_p$ and $<_l$ as in (5.1) we will use $<$ by slight abuse of notation. Now we define the set

$$X^{(n)} := \{(\pi, i) \in S_n \times \{1, \dots, n\} \mid \pi(i) = i\}.$$

Similarly this set is also ordered lexicographically.

(b) $A_k^{(n)}$ is an $\mathbb{F}_2$-matrix with $x(n)$ rows and $y_k(n)$ columns for $k \in \{1, 2, 3\}$. We are indexing the matrices over the sets $X^{(n)}$ and $Y_k^{(n)}$. We define the matrix $A_1^{(n)} = a_{(\pi^0, i^0),(i^1, j_1^1, j_2^1)}$, where $(\pi^0, i^0) \in X^{(n)}$ and

$(i^1, j_1^1, j_2^1) \in Y_1^{(n)}$ by the entries

$$a_{(\pi^0,i^0),(i^1,j_1^1,j_2^1)} := \begin{cases} 1 & \text{, if } i^0 = j_\alpha^1,\ \pi^0(i^1) = j_\beta^1 :\ \alpha, \beta \in \{1,2\},\ \alpha \neq \beta, \\ 0 & \text{, else.} \end{cases} \tag{5.2}$$

We define similarly for $A_2^{(n)}$ and $\pi^2 \in Y_2^{(n)}$

$$a_{(\pi^0,i^0),\pi^2} := \begin{cases} 1 & \text{, if } \pi^0 = \pi^2, \\ 0 & \text{, else,} \end{cases} \tag{5.3}$$

and for $A_3^{(n)}$ and $i^3 \in Y_3^{(n)}$

$$a_{(\pi^0,i^0),i^3} := \begin{cases} 1 & \text{, if } i^0 = i^3, \\ 0 & \text{, else.} \end{cases}$$

(c) Let us recall Theorem (4.2.6)(c)(ii). It states that the image of $\mathrm{Inc}(EC(\pi))$ has two elements for $L(\pi) \neq \emptyset$. Let $w \in \mathcal{W}^{(n)}$ be a representation of an inner tropical game. Define the set of candidate choosing maps

$$F^{(n)} := \left\{ f : Y_2^{(n)} \to \mathbb{F}_2 \right\}.$$

The pair $\rho := (w, f) \in \mathcal{W}^{(n)} \times F^{(n)}$ is called a *representation with choice* of an inner tropical game. Choose for any $(\pi, f) \in Y_2^{(n)} \times F^{(n)}$ an element $g_{(\pi,f)} \in EC(\pi)$ with $|L_0(g_{(\pi,f)})| \equiv f(\pi) \mod 2$. Define $b_\rho^{(n)}$ as an increment (column) vector with $x(n)$ entries where

$$(b_\rho^{(n)})_{(\pi,i)} := \mathrm{Inc}(g_{(\pi,f)}, i) \qquad \text{for all } (\pi, i) \in X^{(n)}.$$

$b_\rho^{(n)}$ is independent of the choice of $g_{(\pi,f)} \in EC(\pi)$.

(e) Recall Definition 4.2.2 (e). The pair $(0_{\mathcal{W}}^{(n)}, 0) \in \mathcal{W}^{(n)} \times F^{(n)}$ where $0_{\mathcal{W}}^{(n)}$ has the characteristic tuple with $\underline{v} = (0, \ldots, 0)$ and $\underline{\sigma} = (\mathrm{id}, \ldots, \mathrm{id})$ is called *the neutral representation with choice* of an inner tropical game. The increment vector $b_0^{(n)} := b_{(0_{\mathcal{W}}^{(n)},0)}^{(n)}$ is called the neutral increment vector.

Lemma 5.2.2.

$$(b_0^{(n)})_{(\pi,i)} = \left| \{ j \in L(\pi)^\complement \mid j > i \} \right| \mod 2. \tag{5.4}$$

Proof. Recall (4.9), which is repeated here for convenience

$$\begin{aligned}\mathrm{Inc}(g,i) = \Big(&1+\gamma^i+v_i+|L_0(g)\setminus\{i\}| \\ &+\sum_{j\in L(\pi)^\complement}\chi(\sigma_j(\pi(j)),\sigma_j(i))\Big) \mod 2.\end{aligned}$$

We can choose $g := g_{(\pi,f)} \in EC(\pi)$ such that $L_0(g)=\emptyset$, $L_1(g)=L(g)$. For the following equations all summations are considered as operations in $\mathbb{F}_2$. Clearly $\gamma^i = 1$, $v_i = 0$, $L_0(g)\setminus\{i\}=\emptyset \Longrightarrow |L_0(g)\setminus\{i\}| = 0$ gives

$$(b_0^{(n)})_{(\pi,i)} = \mathrm{Inc}(g,i) = 1+1+0+0+\sum_{j\in L(\pi)^\complement}\chi(\pi(j),i). \tag{5.5}$$

We can simplify (5.5) into

$$\sum_{j\in L(\pi)^\complement}\chi(\pi(j),i) = \sum_{\pi(j)\in L(\pi)^\complement}\chi(\pi(j),i) = \sum_{\substack{\pi(j)\in L(\pi)^\complement\\ \pi(j)>i}} 1 = \sum_{\substack{j\in L(\pi)^\complement\\ j>i}} 1,$$

which clearly evaluates to (5.4). $\square$

The idea of this linearization becomes clear if we try to calculate the "difference" of two games. The idea is to understand minimal changes of representations with choice of inner tropical games. Clearly a sign change or a candidate class change is minimal, but for the permutation vector it is not so apparent what that means. A minimal change for the permutation vector implies that at most one permutation in that vector changes, but how exactly that change looks is not a priori apparent. We want to change the game in a way that the inner sum in the increment calculation changes as little as possible. If we can find two games such that just one pair $\chi(\sigma(j),i)$ changes we are done. First we need to define the vectorization of an inner tropical game.

Definition 5.2.3. (a) (Vectorization of the inner tropical game) Let $\rho = (w,f) \in \mathcal{W}^{(n)}\times F^{(n)}$ be a representation with choice of an inner tropical game. Define

$$(u_1^\rho)_{(i,j_1,j_2)} := \begin{cases} 1 & \text{, if } \sigma_i(j_1) > \sigma_i(j_2) \\ 0 & \text{, if } \sigma_i(j_1) < \sigma_i(j_2), \end{cases} \tag{5.6}$$

$(u_2^\rho)_\pi := f(\pi)$ and $(u_3^\rho)_i := v_i$ with $u^\rho := (u_1^\rho, u_2^\rho, u_3^\rho)$.

(b) A vector triple $u^{(n)} = (u_1^{(n)}, u_2^{(n)}, u_3^{(n)})$ is called **admissible** if there exists an inner tropical game such that $u^{(n)} = u^\rho$.

The following Theorem 5.2.4 connects those concepts and shows that they coincide and that they calculate essentially the same.

Theorem 5.2.4. *($\mathbb{F}_2$-linearization of the increment equation) Let $\rho \in \mathcal{W}^{(n)} \times F^{(n)}$ be a representation with choice of an inner tropical game. Then we have*

$$A_1^{(n)} \cdot u_1^\rho + A_2^{(n)} \cdot u_2^\rho + A_3^{(n)} \cdot u_3^\rho + b_0^{(n)} = b_\rho^{(n)}. \tag{5.7}$$

Proof. The proof is trivial. The matrices $A_k^{(n)}$ for $k \in \{1,2,3\}$ are defined in such a way that they linearize the increment equation. □

Lemma 5.2.5. *(a) Let $u^{(n)}$ be an arbitrary but fixed vector triple. Define n sets $Z_i \subset (Y_3^{(n)} \setminus \{i\})^2$ of ordered pairs with the following property. Let $a < b$ for the following equation*

$$\begin{aligned} (a,b) \in Z_i &\Longleftrightarrow (u_1^{(n)})_{(i,a,b)} = 0, \\ (b,a) \in Z_i &\Longleftrightarrow (u_1^{(n)})_{(i,a,b)} = 1. \end{aligned} \tag{5.8}$$

$u^{(n)}$ is admissible if and only if Z_i^+ is asymmetric for all $i \in Y_3^{(n)}$.

(b) The resulting game is unique.

Proof. (a) Obviously $u_2^{(n)}$ and $u_3^{(n)}$ determine a map $f \in F^{(n)}$ and a sign tuple $\underline{v} \in \mathbb{F}_2^n$ by $f(\pi) = (u_2^{(n)})_\pi$ and $\underline{v} = u_3^{(n)}$.

In fact, $u_1^{(n)}$ determines a unique tuple $\underline{\sigma} = (\sigma_1, \ldots, \sigma_n)$ of permutations with $\sigma_i(i) = i$ if and only if Z_i^+ is asymmetric. In order to see this, let us take an arbitrary $u_1^{(n)}$ as well as the sets Z_i in (5.8). Assume for all $i \in Y_3^{(n)}$ that Z_i^+ is asymmetric. By applying Lemma 5.1.3(a) we get a $f_i \in \mathrm{Bij}(Y_3^{(n)} \setminus \{i\}, Y_3^{(n)} \setminus \{i\})$ which respects the induced ordering. Thus we can embed f_i into σ_i by setting

$$\sigma_i(j) = \begin{cases} f_i(j) & ,\text{if } i \neq j \\ i & ,\text{if } i = j. \end{cases}$$

If there exists an i such that Z_i^+ is not asymmetric, then we get a, b such that $(a, b) \in Z_i^+$ and $(b, a) \in Z_i^+$. By the transitive closure we thus get $(a, a) \in Z_i^+$ which implies $f(a) < f(a)$, which is a contradiction. This is also the proof of Lemma 5.1.3(b). No circular strict inequality chains exist. Every strict ordering is irreflexive. Thus no order preserving map exists.

(b) Clearly Z_i is complete by (5.8), thus we can again apply Lemma 5.1.3(c). □

Example 5.2.6. Here is the order of the set $Y_1^{(3)}$

$$(1, 2, 3) < (2, 1, 3) < (3, 1, 2)$$

Here is the order of the set $Y_2^{(3)}$. Note that those are permutation written as product of cycles with disjoint support. Note that there are no derangements in $Y_2^{(n)}$.

$$\mathrm{id} < (2\ 3) < (1\ 2) < (1\ 3)$$

Here is the order of the set $Y_3^{(3)}$

$$1 < 2 < 3.$$

Here is the order of the set $X^{(3)}$

$$(\mathrm{id}, 1) < (\mathrm{id}, 2) < (\mathrm{id}, 3) < ((2\ 3), 1) < ((1\ 2), 3) < ((1\ 3), 2).$$

5.3 Small fundamental theorem of inner tropical games

Definition 5.3.1. Let us define the following sets. Let (n, q) with $0 \leq q \leq n$. Then

$$\Sigma(n,q) := \{w \in \mathcal{W}^{(n)} \mid \sum_{i=0}^{n} v_i = q\}$$

represents all games where exactly q players have a negative sign in their λ-polynomial.

Definition 5.3.2. Two representations with choice $\rho \in \mathcal{W}^{(n)} \times F^{(n)}$ and $\widetilde{\rho} \in \mathcal{W}^{(n)} \times F^{(n)}$ of inner tropical games are called strongly equivalent if and only if

$$b_{\rho}^{(n)} = b_{\widetilde{\rho}}^{(n)}.$$

Lemma 5.3.3. *Let $\rho = (w, f)$ and $\widetilde{\rho} = (\widetilde{w}, 1-f)$ be two representations with choice of inner tropical games, where $\underline{\sigma} = \underline{\widetilde{\sigma}}$ and $v_i \neq \widetilde{v}_i$ for all $i \in Y_3^{(n)}$. Then ρ and $\widetilde{\rho}$ are strongly equivalent.*

Proof. Applying (5.7) gives

$$A_1^{(n)} u_1^{\rho} + A_2^{(n)} u_2^{\rho} + A_3^{(n)} u_3^{\rho} + b_0^{(n)} = b_{\rho}^{(n)} \tag{5.9}$$

$$A_1^{(n)} u_1^{\widetilde{\rho}} + A_2^{(n)} u_2^{\widetilde{\rho}} + A_3^{(n)} u_3^{\widetilde{\rho}} + b_0^{(n)} = b_{\widetilde{\rho}}^{(n)} \tag{5.10}$$

$v + v = 0$ in a vector space of characteristic 2 and $u_1^{\rho} = u_1^{\widetilde{\rho}}$ since $\underline{\sigma} = \underline{\widetilde{\sigma}}$. Adding (5.9) and (5.10) gives

$$A_2^{(n)}(u_2^{\rho} + u_2^{\widetilde{\rho}}) + A_3^{(n)}(u_3^{\rho} + u_3^{\widetilde{\rho}}) = b_{\rho}^{(n)} + b_{\widetilde{\rho}}^{(n)}. \tag{5.11}$$

We have $(u_2^{\rho} + u_2^{\widetilde{\rho}})_{\pi} = f(\pi) + (1 - f(\pi)) = 1$, together with (5.3) gives

$$A_2^{(n)}(1 \ \ldots \ 1)^t = (1 \ \ldots \ 1)^t \in \mathbb{F}_2^{x(n)}. \tag{5.12}$$

Similarly $(u_3^{\rho} + \widetilde{u_3}^{\rho})_i = v_i + \widetilde{v}_i = 1$ gives

$$A_3^{(n)}(1 \ \ldots \ 1)^t = (1 \ \ldots \ 1)^t \in \mathbb{F}_2^{x(n)}. \tag{5.13}$$

Substituting (5.12) and (5.13) in (5.11) gives

$$0 = b_{\rho}^{(n)} + b_{\widetilde{\rho}}^{(n)},$$

which implies $b_{\rho}^{(n)} = b_{\widetilde{\rho}}^{(n)}$. □

Corollary 5.3.4. *There exists a bijection $h : \Sigma(n,q) \to \Sigma(n, n-q)$ such that for all w and any f the induced representations with choice $\rho = (w, f)$ and $\widetilde{\rho} = (h(w), 1-f)$ of inner tropical games are strongly equivalent.* □

Corollary 5.3.4 motivates the following Definition 5.3.5. It is used extensively in the section where C_2^m-equilibrium classes are calculated. It is also used in the chapter where densities are calculated.

Definition 5.3.5. Let $n \in \mathbb{N}$, define the set $I(n) := \mathbb{Z} \cap [0, \frac{n}{2}]$.

Example 5.3.6. $I(4) = I(5) = \{0, 1, 2\}$ and $I(6) = \{0, 1, 2, 3\}$.

We will now state the (chronologically) first important fact of the theory of inner tropical games. For the sake of argument let us forget that maximal inner tropical games exist for all $m \in \mathbb{N}$. Our goal is to find maximal games, but there needs to be additional structure to pursue this avenue. Why is that the case? A priori it might be possible that maximal games exist for m_1 and m_3 players, but not for m_2 players, where $m_1 < m_2 < m_3$. We will define chains of games, where we sequentially remove the last player.

Definition 5.3.7. Let $n \in \mathbb{N}$. Define removal sets

$$\begin{aligned}
RY_1^{(n)} :=& \{(a,b,c) \in Y_1^{(n)} \mid \{a,b,c\} \cap \{n\} \neq \emptyset\}, \\
RY_2^{(n)} :=& \{\pi \in Y_2^{(n)} \mid \pi(n) \neq n\} \\
& \cup \ \{\pi \in Y_2^{(n)} \mid \pi(n) = n, \pi|_{\{1,\ldots,n-1\}} \in \mathrm{Der}_{n-1}\}, \\
RY_3^{(n)} :=& \{i \in Y_3^{(n)} \mid i = n\} = \{n\}, \\
RX^{(n)} :=& \{(\pi, i) \in X^{(n)} \mid \pi(n) \neq n \vee i = n\}.
\end{aligned}$$

(a-reduction) Define three reduction maps

$$R_a : \begin{cases} \bigcup_{k=1}^{3} \bigcup_{\substack{n\in\mathbb{N} \\ n\geq 2}} \mathbb{F}_2^{x(n)\times y_k(n)} & \to \bigcup_{k=1}^{3} \bigcup_{n\in\mathbb{N}} \mathbb{F}_2^{x(n)\times y_k(n)} \\ A_k^{(n)} & \mapsto R_a(A_k^{(n)}), \end{cases} \tag{5.14}$$

where $R_a(A_k^{n+1})$ removes the row with index $x \in X^{(n+1)}$ if and only if $x \in RX^{(n+1)}$ and also removes the column with index $y \in Y_k^{(n+1)}$ if and only if $y \in RY_k^{(n+1)}$.

(u-reduction)

$$R_u : \begin{cases} \bigcup_{k=1}^{3} \bigcup_{\substack{n\in\mathbb{N} \\ n\geq 2}} \mathbb{F}_2^{y_k(n)} & \to \bigcup_{k=1}^{3} \bigcup_{n\in\mathbb{N}} \mathbb{F}_2^{y_k(n)} \\ u_k^{(n)} & \mapsto R_u(u_k^{(n)}), \end{cases}$$

takes a vector $u_k^{(n+1)}$ and removes the entry with index $y \in Y_k^{(n+1)}$ if and only if $y \in RY_k^{(n+1)}$;

(b-reduction)

$$R_b : \begin{cases} \bigcup_{\substack{n\in\mathbb{N} \\ n\geq 2}} \mathbb{F}_2^{x(n)} & \to \bigcup_{n\in\mathbb{N}} \mathbb{F}_2^{x(n)} \\ b_\rho^{(n)} & \mapsto R_b(b_\rho^{(n)}), \end{cases}$$

takes a vector $b_\rho^{(n+1)}$ and removes the entry with index $x \in X^{(n+1)}$ if and only if $x \in RX^{(n+1)}$.

Now we are able to formulate

Theorem 5.3.8. *Let $\rho \in \mathcal{W}^{(n+1)} \times F^{(n+1)}$ be a representation with choice of a maximal inner tropical game. Then there exists a $\widetilde{\rho} \in \mathcal{W}^{(n)} \times F^{(n)}$, which is a representation with choice of a maximal inner tropical game with one fewer player.*

Proof. Define $u^\rho = (u_1^\rho, u_2^\rho, u_3^\rho)$ according to Defintion (5.2.3). By our hypothesis we have (5.7)

$$A_1^{(n+1)} \cdot u_1^\rho + A_2^{(n+1)} \cdot u_2^\rho + A_3^{(n+1)} \cdot u_3^\rho + b_0^{(n)} = b_\rho^{(n+1)} = 0.$$

An admissible vector $u^{(n)} = (u_1^{(n)}, u_2^{(n)}, u_3^{(n)})$ that solves

$$A_1^{(n)} \cdot u_1^{(n)} + A_2^{(n)} \cdot u_2^{(n)} + A_3^{(n)} \cdot u_3^{(n)} + b_0^{(n)} = 0 \tag{5.15}$$

shows that a maximal game exists.

Our claim is that $u^{(n)} = \Big(R_u(u_1^\rho), R_u(u_2^\rho), R_u(u_3^\rho)\Big)$ is admissible and that it solves (5.15). Let us show the following two equations

$$R_b(A_k^{(n+1)} \cdot u_k^\rho) = R_a(A_k^{(n+1)}) R_u(u_k^\rho) \tag{5.16}$$
$$R_a\Big(A_k^{(n+1)}\Big) = A_k^{(n)}, \tag{5.17}$$

for $k \in \{1,2,3\}$. Clearly $R_u(u_k^\rho) \in \mathbb{F}_2^{y_k(n)}$, and $R_b(A_k^{(n+1)} \cdot u_k^\rho) \in \mathbb{F}_2^{x(n)}$, so the vector is well-defined. We now show that if we change the value of the product we remove the entry. Let $y \in RY_k^{(n+1)}$. The reduction scheme is clearly order preserving, so if we show that no changed values appear we are done.

$k = 1$: $y \in RY_1^{(n+1)}$ means that y is one of the triples $(n+1, b, c), (a, n+1, c)$ or $(a, b, n+1)$. Recall (5.2). Let $x = (\pi, i) \in X^{(n+1)}$. We want to show

$$(A_1^{(n+1)} u_1^{(n+1)})_{xy} = 1 \Longrightarrow x \in RX^{(n+1)}$$

We first get that $(A_1^{(n+1)})_{xy} = 1$. A simple comparison of coefficients shows either

$$i = n+1 \wedge (\pi(a) = b \vee \pi(a) = c), \tag{5.18}$$

or

$$\pi(n+1) = b \wedge i = c \tag{5.19}$$

or

$$\pi(n+1) = c \wedge i = b \tag{5.20}$$

or

$$\pi(a) = n+1 \wedge (i = b \vee i = c) \tag{5.21}$$

$x \in RX^{(n+1)}$ in all cases. For (5.18) we have $i = n+1$, for (5.19) and (5.20) we have $\pi(n+1) \in \{b, c\} \not\ni n+1 \Rightarrow \pi(n+1) \neq n+1$ and for (5.21) we have $n+1 \neq a = \pi^{-1}(n+1) \Rightarrow \pi(n+1) \neq n+1$.

$k = 2$ and $k = 3$ are analogous. This shows (5.16)

Clearly $R_a(A_k^{(n+1)}) = A_k^{(n)}$ because it is an order preserving canonical projection of matrices. This implies (5.17).

We need to show $R_b(b_0^{(n+1)}) = b_0^{(n)}$. Since it is order preserving we have for

$$x = (\pi, i) \in X^{(n+1)} \setminus RX^{(n+1)} \Longrightarrow \pi(n+1) = n+1 \notin L(\pi)^{\complement}. \tag{5.22}$$

We have $\mathrm{pr}(x) := (\pi|_{\{1,\dots,n\}}, i) \in X^{(n)}$

$$\begin{aligned}(R_b(b_0^{(n+1)}))_{\mathrm{pr}(x)} = (b_0^{(n+1)})_x &\overset{(5.4)}{=} \left|\{j \in L(\pi)^{\complement} \mid j > i\}\right| \mod 2 \\ &\overset{(5.22)}{=} \left|\{j \in L(\pi|_{\{1,\dots,n\}})^{\complement} \mid j > i\}\right| \mod 2 \overset{(5.4)}{=} (b_0^{(n)})_{\mathrm{pr}(x)}\end{aligned} \tag{5.23}$$

Combining (5.16),(5.17) and (5.23) gives

$$\begin{aligned}R_b(0) \overset{\substack{\text{maximal}\\\text{game}}}{=} R_b(b_\rho^{(n+1)}) &= R_b\left(\left(\sum_{k=1}^{3} A_k^{(n+1)} u_k^\rho\right) + b_0^{(n+1)}\right) \\ &= \left(\sum_{k=1}^{3} A_k^{(n)} R_u(u_k^\rho)\right) + b_0^{(n)} = 0\end{aligned}$$

All that is left to show is that $u^{(n)} = \left(R_u(u_1^\rho), R_u(u_2^\rho), R_u(u_3^\rho)\right)$ is admissible. Apply Lemma 5.2.5 and Lemma 5.1.3(d) on $u^{(n)}$. □

Corollary 5.3.9. *(Small fundamental theorem of inner tropical games) There exists a unique $m_0 \in \mathbb{N} \cup \{\infty\}$ such that m-player maximal games exist if and only if $m \leq m_0$.* □

5.4 C_2^P-candidates

As we already know that this chapter is not chronological we formulate the first version of the linear equation system. In this one we have no dummy variables. Those were the first non-trivial deduction. As such we can only compute a very special set of equilibrium candidates, namely the C_2^P-candidates. Recall that a C_2^P-candidate is an element $g \in G$ with $\gamma^i \in \mathbb{F}_2$ for two players i and $\gamma^i \in (0,1)$ for all other players i. These candidates have very special properties and are very interesting. We devote Chapter 6, which is motivated by the results of this section, towards them.

Let us recall Corollary 5.3.9. While the proof of it was extensive and done with the machinery of linear equations, it has an elegant, although not very rigorous argument. The author, as such, would denote the proof as "Trivial, just delete the last player.", while the reader might be confused by such verbiage. Furthermore, in our chronological travel we assume full knowledge of Corollary 5.3.9, but only partial knowledge about the linear equation system machinery with dummy variables. The development of this chapter is thanks to the easier linear equation system, which we formulate soon. It follows easily from this chapter. However, it also follows from the increment equation directly. It is helpful to keep this dichotomy in mind, since the actual progression is the following. We will now give the initial ansatz, which is easily derived from the original linear equation system.

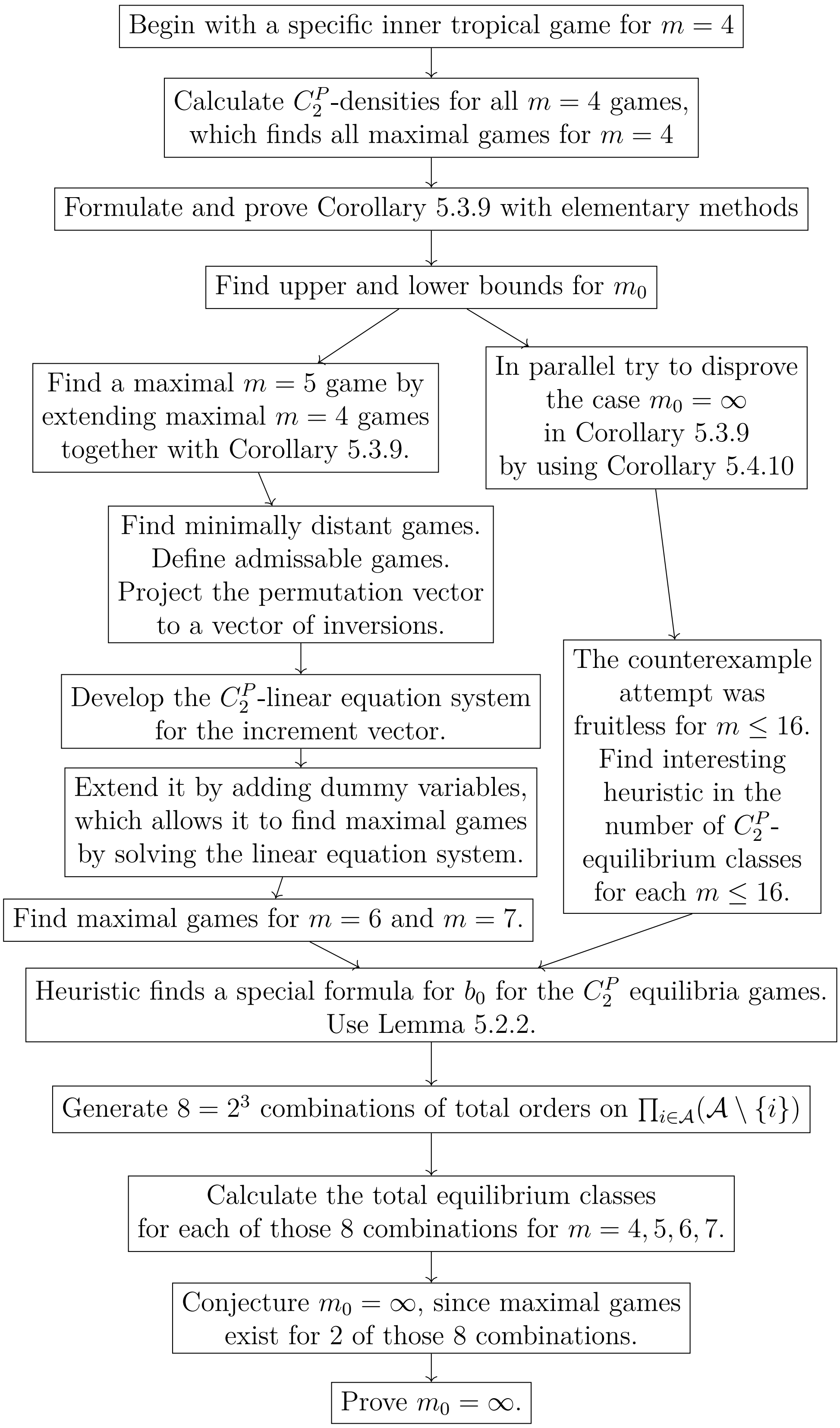


Figure 5.1: Chronological development

Definition 5.4.1. (C_2^P-counting function) Let $w \in \mathcal{W}^{(n)}$ be a representation of an inner tropical game. Define the map $t_n : \mathcal{W}^{(n)} \to \mathbb{Z}$ where

$$t_n(w) := \left|\left\{\pi \in S_n \mid L(\pi) = 2 \text{ and } EC(\pi) \text{ is an equilibrium class of } w.\right\}\right| \tag{5.24}$$

is the number of C_2^P-equilibrium classes.

Definition 5.4.2. (a) Recall Definition 5.2.1. Let $n \in \mathbb{N}$, $n \geq 4$ and set

$$\begin{aligned}
\widetilde{x}(n) &:= !(n-2) \cdot \frac{n(n-1)}{2} \\
\widetilde{y}_1(n) &:= \frac{n(n-1)(n-2)}{2} = y_1(n), \\
\widetilde{y}_2(n) &:= 0, \\
\widetilde{y}_3(n) &:= n = y_3(n), \\
\widetilde{y}(n) &:= \widetilde{y}_1(n) + \widetilde{y}_2(n) + \widetilde{y}_3(n).
\end{aligned}$$

We set

$$\begin{aligned}
\widetilde{Y}_1^{(n)} &:= Y_1^{(n)}, \\
\widetilde{Y}_2^{(n)} &:= \emptyset, \\
\widetilde{Y}_3^{(n)} &:= Y_3^{(n)},
\end{aligned}$$

and

$$\widetilde{X}^{(n)} := \{\pi \in S_n \mid |L(\pi)| = 2\}.$$

(b) Let $k \in \{1, 3\}$ and $\pi \in S_n$ with $|L(\pi)| = 2$, we set

$$\left(\widetilde{A}_k^{(n)}\right)_\pi = \sum_{i \in L(\pi)} \left(A_k{}^{(n)}\right)_{(\pi,i)}$$

(c) Similarly for $\pi \in S_n$ with $|L(\pi)| = 2$, set

$$(\widetilde{b}_w^{(n)})_\pi = \sum_{i \in L(\pi)} (b_\rho^{(n)})_{(\pi,i)}$$

where $\rho = (w, 0)$ is the representation w with choice $f = 0 \in F^{(n)}$ of an inner tropical game.

Lemma 5.4.3.

$$(\widetilde{b}_w^{(n)})_\pi = \begin{cases} 0 & \text{, if } EC(\pi) \text{ contains an equilibrium,} \\ 1 & \text{, else.} \end{cases}$$

Proof.

$$(\widetilde{b}_w^{(n)})_\pi = \sum_{i \in L(\pi)} (b_\rho^{(n)})_{(\pi,i)} = \sum_{i \in L(\pi)} \mathrm{Inc}(g_{(\pi,f)}, i)$$

for any $g_{(\pi,f)} \in EC(\pi)$. By Theorem 4.2.6 (c)(ii) we have that $g_{(\pi,f)} \in EC(\pi)$ is an equilibrium if all increments are equal. Clearly then either $0+0 = 1+1 = 0 \in \mathbb{F}_2$ If it is not an equilibrium then the increments differ. Since there are only two possible increments we get $0+1 = 1+0 = 1 \in \mathbb{F}_2$, which finishes the proof. □

Corollary 5.4.4. *Let $u^{(n)} = (u_1^{(n)}, u_2^{(n)}, u_3^{(n)})$ be an admissible triple of vectors. Then*

$$\widetilde{A}_1^{(n)} \cdot u_1^{(n)} + \widetilde{A}_3^{(n)} \cdot u_3^{(n)} + \widetilde{b}_0^{(n)} = \widetilde{b}_w^{(n)}.$$

Proof. Use Theorem 5.2.4 and apply Lemma 5.4.3. Note that $A_2^{(n)} u_2^{(n)}$ cancels out, because both rows of $A_2^{(n)}$ for $i \in L(\pi)$ are equal $0 + 0 = 1 + 1 = 0 \in \mathbb{F}_2$. □

Definition 5.4.5. Let $v \in \mathbb{F}_2^n$, define the taxicab norm

$$\|v\|_1 := \sum_{i=1}^{n} |v_i| = \Big|\{i \in \{1, \dots, n\} \mid v_i \neq 0\}\Big|$$

where we have the obvious valuation $\mathbb{F}_2 \to \{0, 1\}$ on $\mathbb{F}_2$.

Lemma 5.4.6. *Let $w \in \mathcal{W}^{(n)}$ be a representation of an inner tropical game. Then*

$$\|\widetilde{b}_w^{(n)}\|_1 + t_n(w) = \widetilde{x}(n)$$

Proof. This follows from Lemma 5.4.3. □

Lemma 5.4.7. *Let $v_1, v_2 \in \mathbb{F}_2^n$, then*

$$\|v_1 + v_2\|_1 \equiv \|v_1\|_1 + \|v_2\|_1 \mod 2.$$

Proof. Trivial. □

Lemma 5.4.8. *Let $w, \widetilde{w}$ be two representations of an inner tropical game with $\underline{v} = \underline{\widetilde{v}}$: We have*

$$t_n(w) \equiv t_n(\widetilde{w}) \mod 2.$$

Proof. We extend w and $\widetilde{w}$ to representations $(w, 0)$ and $(\widetilde{w}, 0)$ with trivial choice $f = 0 \in F^{(n)}$. Then we have $\widetilde{b}_w^{(n)} + \widetilde{b}_{\widetilde{w}}^{(n)} = \widetilde{A}_1^{(n)}(u_1^{(w,0)} + u_1^{(\widetilde{w},0)})$ by Corollary 5.4.4. Applying Lemma 5.4.7 and induction over the number of ones in $u_1^{(n)}$ we need to show that $\|\widetilde{A}_1^{(n)} e_k^{(\widetilde{x}(n))}\|_1$ is even for the canonical unit vectors $e_k^{(\widetilde{x}(n))}$. This, however, is equivalent to the number of ones in the k-th column of $\widetilde{A}_1^{(n)}$. We can now use a marvellous counting argument. The matrix $\widetilde{A}_1^{(n)}$ has precisely $\widetilde{x}(n) \cdot 2 \cdot (n-2)$ ones in it. This holds since it has $\widetilde{x}(n)$ rows and each row has $2 \cdot (n-2)$ ones in it. Also let us recall Definition 5.4.2. By symmetry all columns of $\widetilde{A}_1^{(n)}$ have the same number of ones in them. Therefore we have that $\|\widetilde{A}_1^{(n)} e_k^{(\widetilde{x}(n))}\|_1 = \frac{\widetilde{x}(n) \cdot 2 \cdot (n-2)}{\widetilde{y}_1(n)} = 2 \cdot !(n-2)$, which is even. Lemma 5.4.6 gives

$$t_n(w) + t_n(\widetilde{w}) = 2\widetilde{x}(n) - (\|\widetilde{b}_w^{(n)}\|_1 + \|\widetilde{b}_{\widetilde{w}}^{(n)}\|_1),$$

which implies

$$t_n(w) + t_n(\widetilde{w}) \equiv \|\widetilde{A}_1^{(n)} u_1^{(n)}\|_1 \equiv 0 \mod 2.$$

□

This gives us our only discovered non-trivial structure towards inner tropical games.

Corollary 5.4.9. *Let $w \in \Sigma(n, 0)$ be a representation of an inner tropical game. If*

$$t_n(w) \not\equiv \widetilde{x}(n) \mod 2, \tag{5.25}$$

then there exists no maximal n-player game.

Proof. Assume that a maximal game exists, then by Theorem 4.2.6 (d) we have either $\underline{v} = (0, \ldots, 0)$ or $\underline{v} = (1, \ldots, 1)$. We can assume $\underline{v} = (0, \ldots, 0)$ by Corollary 5.3.4. Therefore let $w_0 \in \Sigma(n, 0)$ be any representation of that maximal game. It is

$$t_n(w_0) = \widetilde{x}(n), \tag{5.26}$$

by definition but also

$$t_n(w_0) \equiv t_n(w) \mod 2 \tag{5.27}$$

by Lemma 5.4.8. (5.26) and (5.27) together imply

$$t_n(w) \equiv \tilde{x}(n) \mod 2$$

for any $w \in \Sigma(n, 0)$, which contradicts (5.25). □

Corollary 5.4.10. *If*

$$t_n(0_{\mathcal{W}}^{(n)}) \not\equiv \tilde{x}(n) \mod 2,$$

then there exists no maximal n-player game. □

Corollary 5.4.10 can be checked with a computer. The results of the calculation are found in (5.34). There the differences

$$\Delta_{xt}(n) := \tilde{x}(n) - t_n(0_{\mathcal{W}}^{(n)})$$

are given for $n \in \{4, 5, \ldots, 16\}$. Clearly the differences must all be even because maximal games exist for arbitrarily many players. However the calculation was fruitful nonetheless. Let us consider four sequences beginning at $n = 5$,

$$\begin{aligned}
a_1(n) &:= \frac{\Delta_{xt}(n)}{\Delta_{xt}(n-1)}, \\
a_2(n) &:= \frac{\Delta_{xt}(n)}{\Delta_{xt}(n-1)} - n = a_1(n) - n, \\
a_3(n) &:= n\left(\frac{\Delta_{xt}(n)}{\Delta_{xt}(n-1)} - n\right) = n \cdot a_2(n), \\
a_4(n) &= n^2\left(\frac{\Delta_{xt}(n)}{\Delta_{xt}(n-1)} - n\right) = n \cdot a_3(n)
\end{aligned}$$

Lemma 5.4.11.

$$\lim_{n\to\infty} a_1(n) = \infty \tag{5.28}$$

$$\lim_{n\to\infty} a_2(n) = 0 \tag{5.29}$$

$$\lim_{n\to\infty} a_3(2n) = 0 \tag{5.30}$$

$$\lim_{n\to\infty} a_3(2n+1) = 2 \tag{5.31}$$

$$\lim_{n\to\infty} a_4(2n) = 0 \tag{5.32}$$

$$\lim_{n\to\infty} a_4(2n+1) = \infty \tag{5.33}$$

Proof. Consider the neutral representation $0_{\mathcal{W}}^{(n)}$ of an inner tropical game and let $\pi \in S_n$ be arbitrary and fixed with $|L(\pi)| = 2$, where $\{l_1, l_2\} = L(\pi)$. By using Lemma 5.2.2 together with Definition 5.4.2 and Lemma 5.4.3 we get that $EC(\pi)$ is not an equilibrium class if and only if $l_1 \equiv l_2 \mod 2$. For the rest of the proof let us assume that n is sufficiently large. Let us calculate

$$\Delta_{xt}(n) = \begin{cases} !(n-2)\cdot m(m-1) & \text{if } n = 2m, \\ !(n-2)\cdot m^2 & \text{if } n = 2m+1. \end{cases}$$

Clearly (5.28) follows from (5.29), so let us show that. We get that

$$a_2(2n) = \frac{!(2n-2)n(n-1)}{!(2n-3)(n-1)^2} - 2n \approx (2n-2)\frac{n(n-1)}{(n-1)^2} - 2n = 0,$$

while

$$\begin{aligned} a_2(2n+1) &= \frac{!(2n-1)n^2}{!(2n-2)n(n-1)} - (2n+1) \\ &\approx (2n-1) + \frac{2(n-1)+1}{n-1} - (2n+1) \\ &= (2n+1) + \frac{1}{n-1} - (2n+1) = \frac{1}{n-1}, \end{aligned}$$

which both show (5.29). Now (5.30) and (5.32) follow since $a_2(2n)$ vanishes at least exponentially fast. Note that one can use (4.5) to convince oneself.

We get

$$a_3(2n+1) = (2n+1) \cdot a_2(2n+1) \approx \frac{2n+1}{n-1} = 2 + \frac{3}{n-1},$$

clearly this implies (5.31). Finally we can see that

$$a_4(2n+1) = (2n+1) \cdot a_3(2n+1) > n,$$

which shows (5.33). □

Here is the table of differences between C_2^P-equilibrium classes and total C_2^P-classes for the representation of the neutral inner tropical game.

n	$\Delta_{xt}(n) := \widetilde{x}(n) - t_n(0_{\mathcal{W}}^{(n)})$
4	2
5	8
6	54
7	396
8	3180
9	29664
10	296660
11	3337400
12	40048830
13	528644520
14	7401023322
15	112248853668
16	1795981658744

(5.34)

Table 5.1: Number of non-equilibria C_2^P-classes

Example 5.4.12. Let us now calculate all maximal games for $n = 4$. Theorem 4.2.6(d), Corollary 5.3.4 and Corollary 5.4.4 imply that

$$\widetilde{A}_1^{(n)} u_1^{(n)} = \widetilde{b}_0^{(n)}$$

The solution set is 7-dimensional with 48 admissible vectors.

$$\widetilde{A}_1^{(4)} = \begin{pmatrix} 0&0&0&0&0&0&0&1&1&0&1&1\\ 0&0&0&1&0&1&1&0&1&0&0&0\\ 0&0&0&0&1&1&0&0&0&1&0&1\\ 1&1&0&1&1&0&0&0&0&0&0&0\\ 1&0&1&0&0&0&1&1&0&0&0&0\\ 0&1&1&0&0&0&0&0&0&1&1&0 \end{pmatrix}, \ \widetilde{b}_0^{(4)} = \begin{pmatrix}0\\0\\1\\0\\1\\0\end{pmatrix}$$

with the following solution set L. It is $\widetilde{A}_1^{(4)} u_0 = \widetilde{b}_0^{(4)}$. And $\widetilde{A}_1^{(4)} u_i = 0$ for $i \in \{1, 2, \ldots, 7\}$.

$$\begin{aligned} L &= u_0 + \operatorname{span}(u_1, \ldots, u_7) \\ &= \begin{pmatrix}1\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} + \operatorname{span}\left(\begin{pmatrix}1\\1\\1\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\1\\1\\0\\0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\\0\\0\\1\\0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\\0\\0\\0\\1\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\1\\0\\0\\0\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\\0\\0\\0\\1\\0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\\1\\0\\0\\1\\0\\0\\0\\1\end{pmatrix} \right) \end{aligned}$$

One sees that $Z_2(u_0) = \{(1,3), (4,1), (3,4)\}$ by (5.8), thus $(1,1) \in Z_2^+(u_0)$. Lemma 5.2.5(a) shows that $u_0 \in L$ is not an admissible vector. $u_0 + u_5 = (1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0)^T$ is an admissible vector. Observe that

$$Z_1(u_0 + u_5) = \{(3,2), (4,2), (3,4)\} \stackrel{!}{=} Z_1^+(u_0 + u_5), \tag{5.35}$$

$$Z_2(u_0 + u_5) = \{(1,3), (1,4), (3,4)\} \stackrel{!}{=} Z_2^+(u_0 + u_5), \tag{5.36}$$

$$Z_3(u_0 + u_5) = \{(1,2), (1,4), (2,4)\} \stackrel{!}{=} Z_3^+(u_0 + u_5), \tag{5.37}$$

$$Z_4(u_0 + u_5) = \{(2,1), (1,3), (2,3)\} \stackrel{!}{=} Z_4^+(u_0 + u_5). \tag{5.38}$$

We get $\sigma_1(3) < \sigma_1(4) < \sigma_1(2)$, thus $\sigma_1 = (2\ 4\ 3)$,

$\sigma_2(1) < \sigma_2(3) < \sigma_2(4)$, thus $\sigma_2 = \mathrm{id}$, $\sigma_3(1) < \sigma_3(2) < \sigma_3(4)$, thus $\sigma_3 = \mathrm{id}$, and $\sigma_4(2) < \sigma_4(1) < \sigma_4(3)$, thus $\sigma_4 = (1\ 2)$.

Our goal in the next chapter is to calculate the partial densities. For $n \geq 7$ the amount of cases to check is far too great to be feasible. However with the results in this section we can at least calculate some values of the partial density given by the Definition 6.1.7 Recall Definition 5.4.1, then $L_n(a) = |t_n^{-1}(a) \cap \Sigma(n, 0)|$ for $a \in \{0, \widetilde{x}(n)\}$ counts the number of elements $w \in \Sigma(n, 0)$ that have either all C_2^P-equilibrium classes or none. $L_n(0) = \{y \in \mathbb{F}_2^{y_1(n)} \mid A_1^{(n)} \cdot y = b_0^{(n)}\}$ and $L_n(\widetilde{x}(n)) = \{y \in \mathbb{F}_2^{y_1(n)} \mid A_1^{(n)} \cdot y = b_0^{(n)} + (1\ 1\ \dots\ 1)^T\}$ The following calculation shows the dimension of the solution space.

Lemma 5.4.13. *The following table gives the dimensions of the solution spaces of the linear equation systems.*

n	4	5	6	7	8	9
$\dim(L_n(\widetilde{x}(n)))$	7	15	25	35	49	63
$\dim(L_n(0))$	7	15	$\emptyset$	35	49	63

Table 5.2: Dimension of C_2^P-linear equation systems for $4 \leq n \leq 9$

Proof. Search with a computer gives the results. □

Our goal was to calculate $|L_7(0)|$ and $|L_7(924)|$. The author managed to check 5000 vectors a second on his CPU. Thus a worst case runtime estimate is

$$2 \cdot 2^{35} \cdot \frac{s}{5000} \approx 160 \text{ days}.$$

The author unfortunately had no more time left to run the computation.

This section illustrates the usefulness of the linear equation systems. It allows a search for specific games somewhat efficiently. Densities are the main topic of Chapter 6, however there are some problems that are not easily solvable for $m = 7$, namely the computation of $|L_7(0)|$ and $|L_7(924)|$.

$m = 6$ is the upper bound for calculations in Chapter 6, which exploited symmetries of the wreath product extensively. $m = 7$ is the upper bound for calculations in this chapter. For $m > 7$ we need very specific algorithms, namely searches that halt once a specific solution is found. Exhaustive searches are out of reach for $m > 7$.

5.5 Fundamental theorem of inner tropical games

Here we will sketch parts of the original proof of the Theorem 4.3.4

Theorem 5.5.1. *(Fundamental theorem of inner tropical games, linear equation system variant) For every $n \in \mathbb{N}$ there exists an admissible $u^{(n)}$ such that*

$$A_1^{(n)} u_1^{(n)} + A_2^{(n)} u_2^{(n)} + A_3^{(n)} u_3^{(n)} + b_0^{(n)} = 0. \tag{5.39}$$

Proof. The case is trivial for $n < 3$, any inner tropical game is maximal. We may assume $n \geq 3$. Set

$$(u_1^*)_{(i,j_1,j_2)} = \begin{cases} 1 & , \text{ if } j_1 < i < j_2, \\ 0 & , \text{ else.} \end{cases} \tag{5.40}$$

and $u_3^* = 0$. It is easy to check that the i-th chain

$$i+1 <_{i*} i+2 <_{i*} \cdots <_{i*} n-1 <_{i*} n <_{i*} 1 <_{i*} 2 <_{i*} \cdots <_{i*} i-1 \tag{5.41}$$

comes from Z_i. The inequality chain (5.41) shows that Z_i^+ is asymmetric, as the chain depicts transitive closure. We may now apply Lemma 5.2.5 (a) to get that any vector tuple $u^{(n)}$ with $u_1^{(n)}$ as in (5.40) is admissible.

We now need to show the existence of a solution of the linear equation system (5.39). A maximal game exists if there exists a solution of the linear equation system

$$A_2^{(n)} u_2^{(n)} = A_1^{(n)} u_1^* + A_3^{(n)} u_3^* + b_0^{(n)},$$

Clearly this simplifies to

$$A_2^{(n)} u_2^{(n)} = A_1^{(n)} u_1^* + b_0^{(n)}. \tag{5.42}$$

where the left side of (5.42) is variable and the right side of (5.42) is fixed.

Let $\pi^* \in Y_2^{(n)}$ be arbitrary and fixed. We see that $L(\pi^*) \neq \emptyset$. Choose an arbitrary $i \in L(\pi^*)$. We have

$$\left(A_2^{(n)} u_2^{(n)}\right)_{(\pi^*,i)} \overset{(5.3)}{=} \sum_{\pi \in Y_2^{(n)}} \delta_{\pi^* \pi} (u_2^{(n)})_\pi = (u_2^{(n)})_{\pi^*},$$

By (5.42) we have

$$(u_2^{(n)})_{\pi^*} = \left(A_1^{(n)} u_1^* + b_0^{(n)}\right)_{(\pi^*,i)}$$

Clearly this holds for all π^* with $|L(\pi^*)| = 1$, now let us assume $|L(\pi^*)| > 1$.

(5.42) has a solution if and only if

$$(u_2^{(n)})_{\pi^*} = \left(A_1^{(n)} u_1^* + b_0^{(n)}\right)_{(\pi^*,i)}$$

holds for all $\pi^* \in Y_2^{(n)}$ and all $i \in L(\pi^*)^\complement$. It is enough to show that for arbitrary $i_1, i_2 \in L(\pi^*)$

$$\left(A_1^{(n)} u_1^* + b_0^{(n)}\right)_{(\pi^*,i_1)} = \left(A_1^{(n)} u_1^* + b_0^{(n)}\right)_{(\pi^*,i_2)}$$

holds. We need to show

$$\left(A_1^{(n)} u_1^*\right)_{(\pi^*,i_1)} + \left(A_1^{(n)} u_1^*\right)_{(\pi^*,i_2)} \overset{!}{=} (b_0^{(n)})_{(\pi^*,i_1)} + (b_0^{(n)})_{(\pi^*,i_2)}. \tag{5.43}$$

The definitions of $A_1^{(n)}$ and of u_1^* give

$$\begin{aligned}
(A_1^{(n)})_{(\pi^*,i_1)} = {} & |\{(i,j) \in (L(\pi^*)^\complement)^2 \mid i_1 < i < j, \pi^*(i) = j\}| \\
& \cup |\{(i,j) \in (L(\pi^*)^\complement)^2 \mid j < i < i_1, \pi^*(i) = j\}| \mod 2 \\
= {} & |\{i \in L(\pi^*)^\complement \mid i_1 < i < \pi^*(i)\}| \\
& \cup |\{i \in L(\pi^*)^\complement \mid \pi^*(i) < i < i_1, \}| \mod 2.
\end{aligned}$$

and similarly for i_2. Now suppose $i_1 < i_2$. Then

$$(A_1^{(n)})_{(\pi^*,i_1)} + (A_1^{(n)})_{(\pi^*,i_2)} = |\{i \in L(\pi)^\complement \mid i_1 < i < i_2\}| \mod 2$$

By Lemma 5.2.2 this also equals the right hand side of (5.43). Therefore (5.43) holds. The solution set is not empty. Let us call u_2^* such a solution. Then the vector triple (u_1^*, u_2^*, u_3^*) is admissible and solves the linear equation system for a maximal game. $\square$

6 Densities

When we studied the existence of maximal inner tropical games (and did not know the existence yet), one heuristical tool was to calculate with computer the number of C_2^P-equilibria, that is, the equilibria whose strategies are pure for two players and mixed for all other players. Our calculations were quite systematic. They are reported in this section. At the end, there are tables for the inner tropical games with $n \in \{4, 5, 6\}$ players. We studied separately the games with fixed number n of players and fixed number $q = \sum_{i=1}^{n} v_i \in [0, \frac{n}{2}]$ (the restriction to $q \leq \frac{n}{2}$ is justified by some symmetry between the cases q and $n - q$). We calculated for such games the fraction of those games which have x C_2^P-equilibria, where x ranges from 0 to the maximal number. This fraction is called *density* $\psi_{(n,q)} \in [0, 1] \cap \mathbb{Q}$.

One of the first, very naïve , but nonetheless interesting assumptions was that the increments in each C_2^P-class are independently distributed. While each individual increment for any random $\pi \in S_n$ is randomly distributed, the total collection is not. Our first observation was that some values are not hit. The second observation is that it follows a somewhat predictable density. Values in the middle of the interval $[0, \widetilde{x}(n)]$ are denser than outside of the middle. However some outliers exist, namely the maximal game outlier.

Our second chronological result, namely $m_0 \geq 4$, was proven by calculating all possible densities for $n = 4$. The author personally conjectured $m_0 = 5$, for the above reasons and is very happy to have been wrong. Reasons for this conjecture were fast growing dimensions and heuristic of a probability of a maximal game close to 0 for $n = 6$.

6.1 A group action on the set $\mathcal{W}^{(n)}$ of representations of inner tropical games

Let $w \in \mathcal{W}^{(n)}$ be a representation of an inner tropical game. For this section only $4 \leq n \leq 6$, but the following definitions are for general $n \in \mathbb{N}$.

Definition 6.1.1.

$$\mathcal{W}_1^{(n)} := \{w \in \mathcal{W}^{(n)} \mid w = ((v_1, \sigma_1), \dots, (v_n, \sigma_n), \mathrm{id}) \text{ with } \underline{\sigma} = (\mathrm{id}, \dots, \mathrm{id})\ \},$$
$$\mathcal{W}_2^{(n)} := \{w \in \mathcal{W}^{(n)} \mid w = ((v_1, \sigma_1), \dots, (v_n, \sigma_n), \mathrm{id}) \text{ with } \underline{v} = (0, \dots, 0)\ \}.$$

Remark 6.1.2. $\mathcal{W}_1^{(n)} \simeq K_1^{(n)}$, but $\mathcal{W}_2^{(n)}$ has fewer elements than $K_2^{(n)}$. There exists a canonical embedding $\iota : \mathcal{W}_2^{(n)} \to K_2^{(n)}$ such that $\mathcal{W}_2^{(n)} \simeq \iota(\mathcal{W}_2^{(n)}) \subset K_2^{(n)}$.

Definition 6.1.3. Recall Definition 4.2.2. Let $w \in \mathcal{W}^{(n)}$ be a representation of an inner tropical game. For every $\alpha \in S_n$ define an auxiliary $\beta_i^\alpha \in S_n$ by

$$\beta_i^\alpha(j) := \begin{cases} j + \mathrm{sgn}(\alpha^{-1}(i) - i) & ,\text{if } j \in \{i, \dots, \alpha^{-1}(i)\} \setminus \{\alpha^{-1}(i)\}, \\ i & ,\text{if } j = \alpha^{-1}(i), \\ j & ,\text{else} \end{cases}$$
$$\begin{aligned} \beta_i^\alpha &= \begin{pmatrix} i & i + \mathrm{sgn}(\alpha^{-1}(i) - i) & \dots & i + |\alpha^{-1}(i) - i| \, \mathrm{sgn}(\alpha^{-1}(i) - i) \end{pmatrix} \\ &= \begin{pmatrix} i & i + \mathrm{sgn}(i - \alpha^{-1}(i)) & \dots & \alpha^{-1}(i) \end{pmatrix}. \end{aligned}$$

Furthermore we define

$$l_\alpha := ((0, \beta_1^\alpha), \dots, (0, \beta_n^\alpha), \alpha) \in W_0(n)$$

and

$$r_\alpha := ((0, \alpha^{-1}), \dots, (0, \alpha^{-1}), \alpha^{-1}) \in W_0(n).$$

Let us define

$$h : \begin{cases} S_n \times W_0(n) & \to W_0(n) \\ (\alpha, w) & \mapsto l_\alpha \cdot w \cdot r_\alpha, \end{cases}$$

where we identify $h_\alpha : W_0(n) \to W_0(n)$ with $h_\alpha(w) = h(\alpha, w)$.

Lemma 6.1.4. *(a) h is a left group action of S_n on $W_0(n)$.*
(b) $h_\alpha(\mathcal{W}^{(n)}) = \mathcal{W}^{(n)}$ for every $\alpha \in S_n$.
(c) For every $w \in \mathcal{W}^{(n)}$ there exist unique $w_1 \in \mathcal{W}_1^{(n)}$ and $w_2 \in \mathcal{W}_2^{(n)}$ such that $w = w_1 \cdot w_2$.

Now let $w_1 \in \mathcal{W}_1^{(n)}$ and $w_2 \in \mathcal{W}_2^{(n)}$.
(d) $w_1 \cdot w_2 = w_2 \cdot w_1$,
(e)

$$h_\alpha(w_1 \cdot w_2) = h_\alpha(w_1) \cdot h_\alpha(w_2). \tag{6.1}$$

Proof. (a) Let $w \in W_0(n)$. Then $h(\mathrm{id}, w) = l_{\mathrm{id}} \cdot w \cdot r_{\mathrm{id}}$. Observe that $l_{\mathrm{id}} = r_{\mathrm{id}} = 0_{\mathcal{W}}^{(n)}$ which shows $h(\mathrm{id}, w) = w$.

Observe that

$$\begin{aligned}
\beta_i^\delta \circ \beta_{\delta^{-1}(i)}^\varepsilon &= \beta_i^\delta \circ \Big(\delta^{-1}(i) \quad \dots \quad \varepsilon^{-1}(\delta^{-1}(i))\Big) \\
&= \Big(i \quad \dots \quad \delta^{-1}(i)\Big) \circ \Big(\delta^{-1}(i) \quad \dots \quad (\delta \circ \varepsilon)^{-1}(i)\Big) \\
&= \Big(i \quad \dots \quad (\delta \circ \varepsilon)^{-1}(i)\Big) = \beta_i^{\delta\circ\varepsilon},
\end{aligned}$$

which shows

$$\begin{aligned}
l_\delta \cdot l_\varepsilon &= \Big((0, \beta_1^\delta), \dots, (0, \beta_n^\delta), \delta\Big) \cdot \Big((0, \beta_1^\varepsilon), \dots, (0, \beta_n^\varepsilon), \varepsilon\Big) \\
&= \Big((0, \beta_1^\delta \circ \beta_{\delta^{-1}(1)}^\varepsilon), \dots, (0, \beta_n^\delta \circ \beta_{\delta^{-1}(n)}^\varepsilon), \delta \circ \varepsilon\Big) \\
&= \Big((0, \beta_1^{\delta\circ\varepsilon}), \dots, (0, \beta_n^{\delta\circ\varepsilon}), \delta \circ \varepsilon\Big) = l_{\delta\circ\varepsilon}.
\end{aligned}$$

We also get

$$\begin{aligned}
r_\varepsilon \cdot r_\delta &= \Big((0, \varepsilon^{-1}), \dots, (0, \varepsilon^{-1}), \varepsilon^{-1}\Big) \cdot \Big((0, \delta^{-1}), \dots, (0, \delta^{-1}), \delta^{-1}\Big) \\
&= \Big((0, \varepsilon^{-1} \circ \delta^{-1}), \dots, (0, \varepsilon^{-1} \circ \delta^{-1}), \varepsilon^{-1} \circ \delta^{-1}\Big) \\
&= \Big((0, (\delta \circ \varepsilon)^{-1}), \dots, (0, (\delta \circ \varepsilon)^{-1}), (\delta \circ \varepsilon)^{-1}\Big) = r_{\delta\circ\varepsilon}.
\end{aligned}$$

Now

$$h_{\delta\circ\varepsilon}(w) = l_{\delta\circ\varepsilon} \cdot w \cdot r_{\delta\circ\varepsilon} = l_\delta(l_\varepsilon \cdot w \cdot r_\varepsilon) r_\delta = h_\delta(l_\varepsilon \cdot w \cdot r_\varepsilon) = h_\delta(h_\varepsilon(w)).$$

(b) Calculating

$$\begin{aligned}
h_\alpha(w) = l \cdot \Big((v_1,\sigma_1),\dots,(v_n,\sigma_n),\mathrm{id}\Big) \cdot r = \\
l \cdot \Big((v_1,\sigma_1),\dots,(v_n,\sigma_n),\mathrm{id}) \cdot ((0,\alpha^{-1}),\dots,(0,\alpha^{-1}),\alpha^{-1}\Big) = \\
l \cdot \Big((v_1,\sigma_1\circ\alpha^{-1}),\dots,(v_n,\sigma_n\circ\alpha^{-1}),\alpha^{-1}\Big) = \\
\Big((0,\beta_1^\alpha),\dots(0,\beta_n^\alpha),\alpha\Big)\cdot\Big((v_1,\sigma_1\circ\alpha^{-1}),\dots,(v_n,\sigma_n\circ\alpha^{-1}),\alpha^{-1}\Big) = \\
\Big((v_{\alpha^{-1}(1)},\beta_1^\alpha\circ\sigma_{\alpha^{-1}(1)}\circ\alpha^{-1}),\dots,(v_{\alpha^{-1}(n)},\beta_n^\alpha\circ\sigma_{\alpha^{-1}(n)}\circ\alpha^{-1}),\mathrm{id}\Big).
\end{aligned} \tag{6.2}$$

shows that $\beta = \mathrm{id}$ in (4.4). The new permutation tuple is given by $\underline{\widetilde{\sigma}}$ with

$$\widetilde{\sigma}_i = \beta_i^\alpha \circ \sigma_{\alpha^{-1}(i)} \circ \alpha^{-1}. \tag{6.3}$$

We need to show $\widetilde{\sigma}^i(i) = i$ for all $i \in \Delta(n)$. Let us calculate

$$\begin{aligned}
\widetilde{\sigma}_i(i) &= \beta_i^\alpha\Big(\sigma_{\alpha^{-1}(i)}\Big(\alpha^{-1}(i)\Big)\Big) \\
&= \beta_i^\alpha\Big(\alpha^{-1}(i)\Big) \\
&= \Big(i \quad \dots \quad \alpha^{-1}(i)\Big)\Big(\alpha^{-1}(i)\Big) \\
&= i.
\end{aligned}$$

This shows $h_\alpha(w) \in \mathcal{W}^{(n)}$ and thus $h_\alpha(\mathcal{W}^{(n)}) \subset \mathcal{W}^{(n)}$. All we need to see is that the restriction, which by abuse of notation we also call $h_\alpha : \mathcal{W}^{(n)} \to \mathcal{W}^{(n)}$, is surjective. Since $\mathcal{W}^{(n)}$ is a finite set we show that h_α is injective. Let $h_\alpha(w) = h_\alpha(\widetilde{w})$. Obviously $\underline{v} = \underline{\widetilde{v}}$ thus we only need to compare i-th entry of $\underline{\sigma}$ on both sides.

$$\beta_i^{\alpha^{-1}(i)} \circ \sigma_{\alpha^{-1}(i)} \circ \alpha^{-1} = \beta_i^{\alpha^{-1}(i)} \circ \widetilde{\sigma}_{\alpha^{-1}(i)} \circ \alpha^{-1}$$

shows $\underline{\sigma} = \underline{\widetilde{\sigma}}$ by two-sided cancellation property of a group.

(c) Trivial.

(d) Trivial.

(e) Trivial. □

The following will show the usefulness of Lemma 6.1.4.

Example 6.1.5. Consider the following inner tropical game $(\mathcal{A}, G, V)$ with $m = 5$.

$$
\begin{aligned}
\lambda^1 &= -(\gamma^2 - \frac{5}{9})(\gamma^3 - \frac{3}{4})(\gamma^4 - \frac{2}{3})(\gamma^5 - \frac{2}{3}),\\
\lambda^2 &= +(\gamma^1 - \frac{1}{4})(\gamma^3 - \frac{11}{15})(\gamma^4 - \frac{4}{5})(\gamma^5 - \frac{5}{9}),\\
\lambda^3 &= +(\gamma^1 - \frac{1}{9})(\gamma^2 - \frac{2}{7})(\gamma^4 - \frac{1}{2})(\gamma^5 - \frac{10}{17}),\\
\lambda^4 &= +(\gamma^1 - \frac{2}{7})(\gamma^2 - \frac{1}{6})(\gamma^3 - \frac{5}{9})(\gamma^5 - \frac{1}{2}),\\
\lambda^5 &= -(\gamma^1 - \frac{1}{11})(\gamma^2 - \frac{1}{7})(\gamma^3 - \frac{1}{13})(\gamma^4 - \frac{3}{4}).
\end{aligned}
$$

Then by (4.3) we get

$$
\begin{aligned}
1 > a_1^4 > a_1^2 > a_1^3 > a_1^5 > 0,\\
1 > a_2^1 > a_2^3 > a_2^4 > a_2^5 > 0,\\
1 > a_3^1 > a_3^2 > a_3^4 > a_3^5 > 0,\\
1 > a_4^2 > a_4^5 > a_4^1 > a_4^3 > 0,\\
1 > a_5^1 > a_5^3 > a_5^2 > a_5^4 > 0,
\end{aligned}
$$

which induces the associated permutations

$$
\begin{aligned}
\sigma_1 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 2 & 5 \end{pmatrix} = (2\ 3\ 4),\\
\sigma_2 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = \mathrm{id},\\
\sigma_3 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = \mathrm{id},\\
\sigma_4 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 4 & 2 \end{pmatrix} = (1\ 3\ 5\ 2),\\
\sigma_5 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 \end{pmatrix} = (2\ 3),
\end{aligned}
$$

as well as $\underline{v} = (1, 0, 0, 0, 1)$. We call the representation of this game w. Now we can define the equilibrium invariant permutation in the λ-space.

Let $\alpha = (1\ 5\ 3)(2\ 4)$. Define a new inner tropical game as follows. Write $\lambda^i = \widetilde{\lambda}^{\alpha(i)}$ and $\gamma^j = \widetilde{\gamma}^{\alpha(j)}$. This gives a new inner tropical game with

$$\begin{aligned}
\widetilde{\lambda}^5 &= -(\widetilde{\gamma}^4 - \frac{5}{9})(\widetilde{\gamma}^1 - \frac{3}{4})(\widetilde{\gamma}^2 - \frac{2}{3})(\widetilde{\gamma}^3 - \frac{2}{3}),\\
\widetilde{\lambda}^4 &= +(\widetilde{\gamma}^5 - \frac{1}{4})(\widetilde{\gamma}^1 - \frac{11}{15})(\widetilde{\gamma}^2 - \frac{4}{5})(\widetilde{\gamma}^3 - \frac{5}{9}),\\
\widetilde{\lambda}^1 &= +(\widetilde{\gamma}^5 - \frac{1}{9})(\widetilde{\gamma}^4 - \frac{2}{7})(\widetilde{\gamma}^2 - \frac{1}{2})(\widetilde{\gamma}^3 - \frac{10}{17}),\\
\widetilde{\lambda}^2 &= +(\widetilde{\gamma}^5 - \frac{2}{7})(\widetilde{\gamma}^4 - \frac{1}{6})(\widetilde{\gamma}^1 - \frac{5}{9})(\widetilde{\gamma}^3 - \frac{1}{2}),\\
\widetilde{\lambda}^3 &= -(\widetilde{\gamma}^5 - \frac{1}{11})(\widetilde{\gamma}^4 - \frac{1}{7})(\widetilde{\gamma}^1 - \frac{1}{13})(\widetilde{\gamma}^2 - \frac{3}{4}).
\end{aligned}$$

Using (4.3) again gives the following ordering

$$\begin{aligned}
1 > \widetilde{a}_5^2 > \widetilde{a}_5^4 > \widetilde{a}_5^1 > \widetilde{a}_5^3 > 0,\\
1 > \widetilde{a}_4^5 > \widetilde{a}_4^1 > \widetilde{a}_4^2 > \widetilde{a}_4^3 > 0,\\
1 > \widetilde{a}_1^5 > \widetilde{a}_1^4 > \widetilde{a}_1^2 > \widetilde{a}_1^3 > 0,\\
1 > \widetilde{a}_2^4 > \widetilde{a}_2^3 > \widetilde{a}_2^5 > \widetilde{a}_2^1 > 0,\\
1 > \widetilde{a}_3^5 > \widetilde{a}_3^1 > \widetilde{a}_3^4 > \widetilde{a}_3^2 > 0,
\end{aligned}$$

which induces the following associated permutations

$$\begin{aligned}
\widetilde{\sigma}^1 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 5 & 3 & 2 \end{pmatrix} = (2\ 4\ 3\ 5),\\
\widetilde{\sigma}^2 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 3 & 1 & 4 \end{pmatrix} = (1\ 5\ 4),\\
\widetilde{\sigma}^3 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 4 & 1 \end{pmatrix} = (1\ 2\ 5),\\
\widetilde{\sigma}^4 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix} = (1\ 2\ 3\ 5),\\
\widetilde{\sigma}^5 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 2 & 5 \end{pmatrix} = (1\ 3\ 4\ 2),
\end{aligned}$$

as well as $\widetilde{\underline{v}} = (0, 0, 1, 0, 1)$. We call the representation of this game $\widetilde{w}$. We claim $h_\alpha(w) = \widetilde{w}$. Let us observe that the sign tuple of $h_\alpha(w)$ is just the sign tuple of $\widetilde{w}$. Thus we only need to check (6.3). Let us now use (6.2) to calculate

$$h_\alpha(w) = \Big((v_{\alpha^{-1}(1)}, \beta_1^\alpha \circ \sigma_{\alpha^{-1}(1)} \circ \alpha^{-1}), \ldots, (v_{\alpha^{-1}(5)}, \beta_5^\alpha \circ \sigma_{\alpha^{-1}(5)} \circ \alpha^{-1}), \mathrm{id}\Big).$$

This is

$$\begin{aligned}
\tilde{\sigma}^1 &= \beta_1^\alpha \circ \sigma_3 \circ \alpha^{-1} = (1\ 2\ 3)\mathrm{id}(1\ 3\ 5)(2\ 4) = (2\ 4\ 3\ 5),\\
\tilde{\sigma}^2 &= \beta_2^\alpha \circ \sigma_4 \circ \alpha^{-1} = (2\ 3\ 4)(1\ 3\ 5\ 2)(1\ 3\ 5)(2\ 4) = (1\ 5\ 4),\\
\tilde{\sigma}^3 &= \beta_3^\alpha \circ \sigma_5 \circ \alpha^{-1} = (3\ 4\ 5)(2\ 3)(1\ 3\ 5)(2\ 4) = (1\ 2\ 5),\\
\tilde{\sigma}^4 &= \beta_4^\alpha \circ \sigma_2 \circ \alpha^{-1} = (4\ 3\ 2)\mathrm{id}(1\ 3\ 5)(2\ 4) = (1\ 2\ 3\ 5),\\
\tilde{\sigma}^5 &= \beta_5^\alpha \circ \sigma_1 \circ \alpha^{-1} = (5\ 4\ 3\ 2\ 1)(2\ 3\ 4)(1\ 3\ 5)(2\ 4) = (1\ 3\ 4\ 2),
\end{aligned}$$

which shows $h_\alpha(w) = \widetilde{w}$. But not only does it show that. It also shows that the equilibrium classes in all the strata C_d^P are invariant, which implies that

$$t_n(w) = t_n(\widetilde{w})$$

as in (5.24).

Fact 6.1.6. Suppose $(\lambda^1, \ldots, \lambda^m)$ with

$$\lambda^i = (-1)^{v_i} \cdot \prod_{j \in \mathcal{A} \setminus \{i\}} (\gamma^j - a_j^i) \tag{6.4}$$

is an inner tropical game. The following procedure creates an equivalent inner tropical game with

$$\widetilde{\lambda}^{\alpha(i)} = (-1)^{v_i} \cdot \prod_{j \in \mathcal{A} \setminus \{\alpha(i)\}} (\widetilde{\gamma}^{\alpha(j)} - a_j^i). \tag{6.5}$$

Let w be the representation of (6.4) and let $\widetilde{w}$ be the representation of (6.5). Then $h_\alpha(w) = \widetilde{w}$.

Proof. See Example 6.1.5 for a proof idea. $\square$

Definition 6.1.7. Recall Definition 5.4.1. Now we can define the partial density for $\emptyset \neq X \subset \mathcal{W}^{(n)}$

$$\psi_X : \begin{cases} \mathbb{N}_0 & \to [0,1] \\ y & \mapsto \frac{1}{|X|} \cdot \Big| X \cap t_n^{-1}(\{y\}) \Big|. \end{cases}$$

6.2 Symmetric reductions

The following will be of great use for calculating partial densities.

Definition 6.2.1. Recall Definition 5.3.5, $I(n) = \mathbb{Z} \cap [0, \frac{n}{2}]$. A tuple (n, q) with $n \geq 4$ and $q \in I(n)$ is called a player-invariance tuple.

Definition 6.2.2. Recall Definition 5.3.1, $\Sigma(n, q) = \{w \in \mathcal{W}^{(n)} \mid \sum_{i=1}^{n} v_i = q\}$. Let (n, q) be a player-invariance tuple. Identify $\psi_{(n,q)} \equiv \psi_{\Sigma(n,q)}$.

Definition 6.2.3. For $p \in S_n$ and $\underline{s} \in \mathbb{F}_2^n$ define

$$\Sigma(n, p, \underline{s}) := \{w \in \mathcal{W}^n \mid p = \sigma^1 \text{ and } s = \underline{v}\}$$

and

$$\Sigma(n, \underline{s}) = \bigcup_{\substack{p \in S_n \\ p(1)=1}} \Sigma(n, p, \underline{s})$$

We now prove some fairly trivial results.

Lemma 6.2.4. *(First symmetric reduction)*

$$\psi_{(n,q)} = \psi_{\Sigma(n,q)} = \psi_{\Sigma(n,n-q)}.$$

Proof. First equation is the definition, second equation follows from Corollary 5.3.4. □

The first symmetric reduction reduces the complexity of computations of densities by nearly a half.

Definition 6.2.5. (a) A representation of an inner tropical game w with (n, q)-invariance tuple is in V-normal form if

$$v_i = 1 \iff i \leq q.$$

(b) A representation of an inner tropical game w is in S-normal form if $\sigma_1 = \mathrm{id}$.

(c) A set is in V- (or S)-normal form if all its elements are in V- (or S)-normal form.

(d) $V_*^{(n)} := \{v \in \mathbb{F}_2^n \mid v_i > v_j \Rightarrow i < j\}$ and
$V^{(n)} := \{v \in V_*^{(n)} \mid v_i = 0 \text{ for } i \notin I(n)\}$.

Remark 6.2.6.

$$V_*^{(6)} = \Big\{(0,0,0,0,0,0),(1,0,0,0,0,0),(1,1,0,0,0,0),(1,1,1,0,0,0),\\ (1,1,1,1,0,0),(1,1,1,1,1,0),(1,1,1,1,1,1)\Big\}$$

and

$$V^{(6)} = \Big\{(0,0,0,0,0,0),(1,0,0,0,0,0),(1,1,0,0,0,0),(1,1,1,0,0,0)\Big\}.$$

Lemma 6.2.7. *Let $\underline{s} \in \mathbb{F}_2^n$, then $\Sigma(n,\mathrm{id},\underline{s})$ is in S-normal form.*

Let $\underline{v} \in V^{(n)}$ and $p \in S_n$ with $p(1) = 1$, then $\Sigma(n,p,\underline{v})$ is in V-normal form.

Proof. Trivial. □

Lemma 6.2.8. *(Second symmetric reduction) Let (n,q) be a player-invariance tuple, let $e_i^{(n)}$ be the i-th standard basis vector of $\mathbb{F}_2^n$ and let $\underline{v} = \sum_{i=1}^{q} e_i^{(n)} \in V^{(n)}$, then*

$$\psi_{(n,q)} = \psi_{\Sigma(n,\underline{v})}.$$

Proof. Consider an $\underline{s} \in \mathbb{F}_2^n$ with $\|s\|_1 = q$. Obviously $\Sigma(n,\underline{s}) \cap \mathcal{W}_1^{(n)} = \{w_1\}$ where $w_1 = ((s_1,\mathrm{id}),\ldots,(s_n,\mathrm{id}),\mathrm{id})$. Lemma 6.1.4 implies $\Sigma(n,\underline{s}) = \{w_1\} \times \mathcal{W}_2^{(n)}$. Choose an arbitrary $\alpha \in S_n$ such that the image of the set is

$$\alpha\Big(\{i \in \Delta(n) \mid s_i = 1\}\Big) = \{1,\ldots,q\}.$$

Then

$$h_\alpha(w) = h_\alpha(w_1 \cdot w_2) \overset{(6.1)}{=} h_\alpha(w_1) \cdot h_\alpha(w_2) \quad \text{for } w \in \Sigma(n,\underline{s}). \tag{6.6}$$

It is now evident that $h_\alpha(w_1)$ is in V-normal form. But then $h_\alpha(w)$ is also in V-normal form. We now have

$$h_\alpha\Big(\Sigma(n,\underline{s})\Big) = h_\alpha\Big(\{w_1\} \times \mathcal{W}_2^{(n)}\Big) \overset{(6.6)}{=} h_\alpha\Big(\{w_1\}\Big) \times h_\alpha\Big(\mathcal{W}_2^{(n)}\Big)$$

and $h_\alpha(\mathcal{W}_2^{(n)}) = \mathcal{W}_2^{(n)}$ by Lemma 6.1.4 (b). Thus

$$h_\alpha\Big(\Sigma(n,\underline{s})\Big) = \{h_\alpha(w_1)\} \times \mathcal{W}_2^{(n)} = \Sigma(n,\underline{v}),$$

where $\underline{v} = (1, \ldots, 1, 0 \ldots, 0) \in V^{(n)}$. $\square$

Lemma 6.2.9. *(Third symmetric reduction) Let* (n, q) *be a player-invariance tuple with* $q \geq 1$*. Define*

$$\mathcal{V}_q^{(n)} := \{\underline{v} \in \mathbb{F}_2^n \mid v_1 = 1 \text{ and } \|\underline{v}\|_1 = q\}.$$

Let $\underline{v} = \sum_{i=1}^{q} e_i^{(n)} \in V^{(n)} \cap \mathcal{V}_q^{(n)}$*, then*

$$\psi_{\Sigma(n,\underline{v})} = \frac{1}{|\mathcal{V}_q^{(n)}|} \sum_{s \in \mathcal{V}_q^{(n)}} \psi_{\Sigma(n,\mathrm{id},\underline{s})}. \tag{6.7}$$

Proof. Observe that

$$\Sigma(n, \underline{v}) = \bigcup_{\substack{p \in S_n \\ p(1)=1}} \Sigma(n, p, \underline{v}) \tag{6.8}$$

and consider a $\Sigma(n, p, \underline{v}) \ni w = w_1 \cdot w_2 \in \mathcal{W}_1^{(n)} \times \mathcal{W}_2^{(n)}$. Then

$$h_p(w) = h_p(w_1 \cdot w_2) \overset{(6.1)}{=} h_p(w_1) \cdot h_p(w_2)$$

But $h_p(w_2)$ is in S-normal form and so is $h_p(w)$. Now

$$\psi_{\Sigma(n,\underline{v})} \overset{(6.8)}{=} \frac{1}{(n-1)!} \sum_{\substack{p \in S_n \\ p(1)=1}} \psi_{\Sigma(n,p,\underline{v})} = \frac{1}{|\mathcal{V}_q^{(n)}|} \sum_{s \in \mathcal{V}_q^{(n)}} \psi_{\Sigma(n,\mathrm{id},\underline{s})}.$$

since the elements p take the element $\underline{v}$ on an arbitrary s with $s_1 = 1$. Since p is a permutation and $\|\underline{v}\|_1 = q$ then also $\|s\|_1 = q$. Thus $s \in \mathcal{V}_q^{(n)}$ and the multiplicities of each s are the same in the sum because of the symmetries. $\square$

6.3 Implementation of the symmetric reductions

Remark 6.3.1. We will briefly sketch how the partial densities $\psi(n, q)$ for $n = 6$ and $q \in I(6) = \{0, 1, 2, 3\}$ were calculated. Split the set $\mathcal{W}^{(6)} = \bigcup_{0 \leq q \leq 6} \Sigma(n, q)$ and apply the first symmetric reduction. Now let $(6, q)$ be a player-invariance tuple. Calculate $\psi_{(6,q)}$ as follows. Apply the second symmetric reduction to bring the set into V-normal form, split it into disjoint sets with respect to the permutation of the first player and apply the third symmetric reduction on each of those sets. This brings every set into S-normal form. The first algorithm defines the total routine. It loops over all possible representations of inner tropical games that are both in V- and S-normal form.

Algorithm 6.3.2: CalculateDensity $m = 6$

```
Output: A[k][l]: k × l matrix, k = 1 + Σ_{q=1}^{3} |V_q^(6)| = 17 and
        l = 1 + |{π ∈ S_n | L(π) = 2}| = 136
initialization
A: Matrix filled with 0
b ∈ ℤ^k: integer array of length k
begin
    foreach w ∈ W_2^(n) do
        for a from 1 to k do
            b[a] ← GetNumberOfEquilibria(w_2, a)
        end
        for c from 1 to k do
            for d from 0 to l − 1 do
                if d == b[c] then
                    A[c][d] ← A[c][d] + 1
                end
            end
        end
    end
end
```

Algorithm 6.3.3: GetNumberOfEquilibria

Input: $w \in \Sigma(6, \mathrm{id}, 0)$

Output: $b \in \mathbb{Z}^k$: array of $k = 17$ integers.

```
initialization
begin
    initialize b ← 0 ∈ F_2^k
    foreach π ∈ S_m with |L(π)| = 2 do
        initialize T, P, L(π), L(π)^∁, t_L, t_S
        for i from 1 to m do
            if i ∈ L(π) then
                L(t_L) ← i
                t_L ← t_L + 1
                P[i] ← 1
            else
                L(t_S) ← i
                t_S ← t_S + 1
                P[i] ← 0
            end
        end
    end
    Set boolean temporary quasi-increment t ← 0 ∈ F_2
    for i from 1 to 2 do
        l ← L[i]
        for j from 1 to 4 do
            s ← S[j]
            t ← t XOR χ(σ^s(π(s)), σ^s(l))
        end
    end
    T[1] ← t
    Hardcode XOR the other entries for T
    for a from 1 to k do
        b[a] ← b[a] + T[a]
    end
end
```

Remark 6.3.4. The second algorithm defines the subroutine. To a given representation w it counts for all possible sign tuples with $v_1 = 1 \iff q \geq 1$ the number of equilibria in parallel. In that algorithm we initialize special variables.

Let us first give the hardcoded entries of T.

$$\begin{aligned}
T[2] &\leftarrow T[1] \text{ XOR } P[1] \\
T[3] &\leftarrow T[2] \text{ XOR } P[2] \\
T[4] &\leftarrow T[2] \text{ XOR } P[3] \\
T[5] &\leftarrow T[2] \text{ XOR } P[4] \\
T[6] &\leftarrow T[2] \text{ XOR } P[5] \\
T[7] &\leftarrow T[2] \text{ XOR } P[6] \\
T[8] &\leftarrow T[3] \text{ XOR } P[3] \\
T[9] &\leftarrow T[3] \text{ XOR } P[4] \\
T[10] &\leftarrow T[3] \text{ XOR } P[5] \\
T[11] &\leftarrow T[3] \text{ XOR } P[6] \\
T[12] &\leftarrow T[4] \text{ XOR } P[4] \\
T[13] &\leftarrow T[4] \text{ XOR } P[5] \\
T[14] &\leftarrow T[4] \text{ XOR } P[6] \\
T[15] &\leftarrow T[5] \text{ XOR } P[5] \\
T[16] &\leftarrow T[5] \text{ XOR } P[6] \\
T[17] &\leftarrow T[6] \text{ XOR } P[6]
\end{aligned}$$

We can clearly see the parallelization of the calculation. $T \in \mathbb{F}_2^{17}$ is the temporary sub-increment boolean counter array and gets initialized to 0.
$P \in \mathbb{F}_2^6$ is the temporary negative sign increment boolean counter array and it also gets initialized to 0.
$L(\pi)$ is here an $\mathbb{F}_2$-array with 2 elements.
$L(\pi)^\complement$ is here an $\mathbb{F}_2$-array with 4 elements.
t_L and t_S are integers and are initialized to 1. They denote the current position of the array $L(\pi)$ and $L(\pi)^\complement$.

Remark 6.3.5. The number of C_2^P-equilibria in a maximal inner tropical game with n players is $\binom{n}{2}\cdot!(n-2)\cdot 2$. Any equilibrium class π with $|L(\pi)| = 2$ contains two C_2^P-equilibria. Therefore for such a game

$$t_n(w) = \binom{n}{2}\cdot!(n-2), \quad \text{for example} \quad \begin{array}{c|c|c|c|c} n & 3 & 4 & 5 & 6 \\ \hline \binom{n}{2}\cdot!(n-2) & 0 & 6 & 20 & 135 \end{array}$$

In the case $n = 3$ we have $t_n(w) = 0$ for any inner tropical game. In the cases $n \geq 4$, for any inner tropical game $t_n(w) \leq \binom{n}{2}\cdot!(n-2)$.

Remark 6.3.6. One observes in the tables for $n = 4, 5, 6$ that all inner tropical games with fixed number n of players and fixed value $|v|_1 = q \in I(n) = [0, \frac{n}{2}] \cap \mathbb{Z}$ either have an odd number of C_2^P equilibria, or they all have an even number of C_2^P-equilibria. Those are provable statements, which are very easy to prove. Therefore they are left as a fun exercise.

Definition 6.3.7. Let $g_n : \mathcal{P}(\mathcal{W}^{(n)}) \to \mathcal{P}(\mathbb{F}_2), X \mapsto g_n(X)$ be the map defined by $\mathbb{F}_2 \ni a \in g_n(X)$ if and only if there exists an $x \in X$ with $t_n(x) \equiv a \mod 2$.

A priori $g_n(\Sigma(n,q))$ for some $q \in I(n)$ could be one of the three sets $\{0\}$, $\{1\}$, and $\{0; 1\}$. But Remark 6.3.6 says that the third set is not realized. The following fact gives more precise statements. It is straightforward to prove. The ideas are similar to the proof of Lemma 5.4.8.

Fact 6.3.8. Let (n, q) be a player-invariance tuple. Then $g_n(\Sigma(n,q))$ is a set with exactly one element. We call this element $\mathcal{I}(n,q)$ the invariant of the player-invariance tuple. By definition it is

$$\mathcal{I}(n,q) = \begin{cases} 0, & \text{if } \psi_{(n,q)}(x) \neq 0 \implies x \text{ is even}, \\ 1, & \text{if } \psi_{(n,q)}(x) \neq 0 \implies x \text{ is odd}. \end{cases}$$

In fact, it is

$$\mathcal{I}(n,q) \equiv \frac{(n+1)(n+2q)}{2} \mod 2.$$

Remark 6.3.9. The observation of Fact 6.3.8 led to the development of the linear equation system. The proof idea is a straightforward application and the only observable structure. For C_d^P with $d > 2$ there is no such structure since there are too many degrees of freedom to control equilibrium and non-equilibrium conditions.

The following shows the density of the C_2^P equilibria for $n = 4$. Let Θ_n be the smallest positive integer such that $\Theta_n \cdot \psi_{n,q}$ only takes integer values. Define $A_q^n(x) = \Theta_q \cdot \psi_{(n,q)}(x)$

An easy calculation shows $\Theta_4 = 81$

x	0	1	2	3	4	5	6
$A_0^4(x)$	0	0	30	0	48	0	3

x	0	1	2	3	4	5	6
$A_1^4(x)$	0	15	0	51	0	15	0

x	0	1	2	3	4	5	6
$A_2^4(x)$	4	0	39	0	36	0	2

Table 6.1: Partial densities for $m = 4$

The following shows the C_2^P-density for $n = 5$. Here $A_q^5(x) := \Theta_5 \cdot \psi_{(5,q)}(x)$ with $\Theta_5 = 41472$.

x	0	2	4	6	8	10	12	14	16	18	20
$A_0^5(x)$	2	0	220	2060	8250	15068	11490	3940	420	20	2

x	0	2	4	6	8	10	12	14	16	18	20
$A_1^5(x)$	2	28	412	3004	9338	14628	10274	3412	356	16	4

x	0	2	4	6	8	10	12	14	16	18	20
$A_2^5(x)$	5	28	527	3484	9840	14436	9680	3124	331	16	1

Table 6.2: Partial densities for $m = 5$

The densities for $n = 6$ are in Appendix A.

6.4 Equivalent representations

Definition 6.4.1. Let $H := S_n$ act on $\mathcal{W}_2^{(n)}$ as in Definition 6.1.3, and let $w \in \mathcal{W}_2^{(n)}$. The orbit of w is

$$H \cdot w = \{h(\alpha, w) \mid \alpha \in H\},$$

the set of fixed points under $\alpha \in H$ is

$$H^\alpha = \{w \in \mathcal{W}_2^{(n)} \mid h(\alpha, w) = w\}$$

and the isotropy group of H with respect to w is

$$H_w = \{\alpha \in H \mid h(\alpha, w) = w\}.$$

The representations $w, \widetilde{w} \in \mathcal{W}_2^{(n)}$ are called essentially the same if the in the same orbit, otherwise they are called essentially different.

Lemma 6.4.2.

m	2	3	4	5
Number of essentially different representations	1	2	60	66360

Proof. This follows by applying Burnside's lemma [Rot95, Theorem 3.22]

$$|\mathcal{W}_2^{(n)}/H| = \frac{1}{|H|} \sum_{\alpha \in H} |\left(\mathcal{W}_2^{(n)}\right)^\alpha| = \frac{1}{|H|} \sum_{w \in \mathcal{W}_2^{(n)}} |H_w|$$

together with the calculations of Appendix B. □

Remark 6.4.3. Most isotropy groups are as expected trivial. Here are some examples of some representations with non-trivial isotropy group.

For $n = 2$ there are only non-trivial isotropy groups because there is only one representation invariant.

More interesting is the case $n = 3$. There are two essentially different representations, one of which has non-trivial isotropy group which is isomorphic to C_3.

σ_1	σ_2	σ_3
id	(1 3)	id
(2 3)	id	(1 2)

7 Perturbation theory and stability of equilibria

In [Ste97] von Stengel offered a concrete 2-person game with $|S^1| = |S^2| = 6$ which has 75 Nash equilibria, which is his lower bound for the maximal number of Nash equilibria of generic 2-person games with $|S^1| = |S^2| = 6$. It has 923 virtual equilibrium candidates, that is, points in $\mathbb{P}^5\mathbb{R} \times \mathbb{P}^5\mathbb{R}$ which satisfy the homogeneous versions of the equalities of Nash equilibria, but not necessarily the inequalities. This chapter is motivated by a study with computer of properties of all 923 points. The results are in Appendix C. Section 7.3 here explains some of them.

Section 7.2 defines three genericity conditions, which together are rather strong. They are all satisfied if $U \in \mathcal{D}$ (for $\mathcal{D}$ as in Theorem 2.2.1). It turns that not all 923 points satisfy all these conditions. Therefore von Stengel's game is not in $\mathcal{D}$. This is not harmful, as the Nash equilibria are regular and are thus preserved by a small deformation.

Section 7.1 gives general background results on deformations of matrices. They are used in Section 7.3 for a rough estimate how large a deformation may be so that the change keeps the genericity of the generic points of the 923 points.

7.1 Matrix norms and deformations of matrices

Please recall that we identify the tuples $(k_1, k_2, \dots, k_n) \in K^n$ as column vectors $(k_1 \; k_2 \; \dots \; k_n)^T \in K^{n\times 1}$. A good reference for this Section is [DH08].

Definition 7.1.1. [DH08, Definition 2.2] Let $V = \mathbb{R}^{m\times n}$ be the vector space of real $m \times n$ matrices, let $x \in \mathbb{R}^n$ and let $\|\cdot\|$ denote a norm on $\mathbb{R}^m$ as well as a norm on $\mathbb{R}^n$. Then this formula

$$\|A\| := \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\| \tag{7.1}$$

gives a matrix norm of A.

Lemma 7.1.2. *(i)* $\|Ax\| \leq \|A\|\|x\|$.
(ii) There exists an $x_0 \in \mathbb{R}^n \setminus \{0\}$ *with* $\|Ax_0\| = \|A\|\|x_0\|$.
(iii) $\|AB\| \leq \|A\|\|B\|$, *whenever* $m = n$.
(iv) $\|I_n\| = 1$.
(v) $\mathbb{R}^{m\times n}$ *is complete with any matrix norm.*

Proof. (i) It follows from the Definition.
(ii) The unit ball B of any norm of $\mathbb{R}^n$ is compact and the norm is continuous. Apply the extreme value theorem on $f : B \to \mathbb{R},\ x \mapsto \|x\|$.
(iii)

$$\|AB\| \overset{(7.1)}{=} \sup_{\|x\|=1} \|ABx\| \overset{\text{(i)}}{\leq} \sup_{\|x\|=1} \|A\|\|Bx\| = \|A\| \sup_{\|x\|=1} \|Bx\| \overset{(7.1)}{=} \|A\|\|B\|.$$

(iv) Trivial.
(v) All norms are equivalent on finite dimensional vector spaces. □

Definition 7.1.3. Let $x \in \mathbb{R}^n$, then

$$\|x\|_\infty := \max_{i=1,\dots,n} |x_i|$$

defines the maximum norm.

Fact 7.1.4. [DH08, Aufgabe 2.8 b)]

$$\|A\|_\infty = \max_{i=1,\dots,m} \sum_{j=1}^{n} |a_{ij}|.$$

Lemma 7.1.5. *(a) Let $C \in \mathbb{R}^{n\times n}$ and let $\|C\| < 1$ for some norm $\|\cdot\|$ on $\mathbb{R}^n$. Then $(1_n - C)$ is invertible with*

$$(1_n - C)^{-1} = \sum_{k=0}^{\infty} C^k = 1_n + C + C^2 + \dots .$$

(b) Let $A, C \in \mathbb{R}^{m\times n}$ with $\|C\| < \|A\|$ and let $x, \hat{x} \in \mathbb{R}^n$ be such that $Ax = b$ and $(A + C)\hat{x} = \hat{b}$. Consider an $\varepsilon > 0$ as well as $\|C\| < \frac{\varepsilon}{3\|x\|}$ if $x \neq 0$. Then $\delta = \frac{\varepsilon}{3\|A\|} > 0$ satisfies

$$\|\hat{x} - x\| < \delta \Longrightarrow \|\hat{b} - b\| < \varepsilon.$$

(c) (Variation in the matrix) Let $Ax = b$ be a linear equation system, let $A \in \mathbb{R}^{n\times n}$ be regular with the unique solution $y = A^{-1}b$. Consider any $C \in \mathbb{R}^{n\times n}$ with $\|C\| < \frac{1}{2\|A^{-1}\|}$. Then $A - C$ is invertible. Let $\hat{y}$ be the solution of the linear equation system $(A - C)x = b$. For all $\varepsilon > 0$ the value $\delta = \frac{\varepsilon}{2\|A^{-1}\|^2\|b\|} > 0$ satisfies

$$\|C\| < \delta \Longrightarrow \|y - \widehat{y}\| < \varepsilon. \tag{7.2}$$

Proof. (a) Let $D_i = \sum_{k=0}^{i} C^k$. Then $(D_i)_{i\in\mathbb{N}_0}$ is a Cauchy sequence. Let w.l.o.g. $j > i$, then

$$\|D_j - D_i\| \leq \sum_{k=i+1}^{j} \|C^k\| \leq \sum_{k=i+1}^{j} \|C\|^k \leq \|C\|^{i+1} \sum_{k=0}^{\infty} \|C\|^k \xrightarrow{i,j\to\infty} 0,$$

since $\|C\| < 1$. Thus $D_i \to D$ with $D \in \mathbb{R}^{n\times n}$. We need to show that $D = (1_n - C)$. But we have

$$D_i(1_n - C) = D_i - (D_{i+1} - 1_n) = 1_n + D_i - D_{i+1} \xrightarrow{i\to\infty} 1_n. \tag{7.3}$$

We also have

$$D_i(1_n - C) \xrightarrow{i\to\infty} D(1_n - C). \tag{7.4}$$

Combining (7.3) and (7.4) shows that $D(1_n - C) = 1_n$.

(b) Define $e := \hat{x} - x$. Clearly

$$\begin{aligned}
\hat{b} - b &= (A+C)\hat{x} - Ax = (A+C)(x+e) - Ax = Ae + Cx + Ce \\
\|\hat{b} - b\| &\leq \|Ae\| + \|Cx\| + \|Ce\| \leq \underbrace{\|C\|}_{<\frac{\varepsilon}{3\|x\|}} \|x\| + \underbrace{\|C\|}_{<\|A\|} \|e\| + \|A\|\|e\| \\
&< \frac{\varepsilon}{3} + 2\|A\|\|e\| \overset{(7.2)}{<} \frac{\varepsilon}{3} + 2\|A\|\frac{\varepsilon}{3\|A\|} = \varepsilon
\end{aligned}$$

(c) First we have that $(A - C) = A(1_n - A^{-1}C)$. We can apply (a) on the matrix $A^{-1}C$ because

$$\|A^{-1}C\| \leq \|A^{-1}\|\|C\| < \|A^{-1}\|\frac{1}{2\|A^{-1}\|} = \frac{1}{2} < 1.$$

Thus

$$(1_n - A^{-1}C)^{-1} = \sum_{k=0}^{\infty}(A^{-1}C)^k.$$

Clearly $y = A^{-1}b$ and $\hat{y} = (A - C)^{-1}b$. We assume w.l.o.g. $b \neq 0$ (equivalent: $\|b\| \neq 0$) since the case $b = 0$ is trivial. Now let us calculate

$$\begin{aligned}
\hat{y} &= (A - C)^{-1}b = (A(1_n - A^{-1}C))^{-1}b = (1_n - A^{-1}C)^{-1}A^{-1}b \\
&= \left(\sum_{k=0}^{\infty}(A^{-1}C)^k\right)A^{-1}b \\
\hat{y} - y &= \left(\sum_{k=1}^{\infty}(A^{-1}C)^k\right)A^{-1}b.
\end{aligned}$$

Repeated use of the triangle inequality gives

$$\|\hat{y} - y\| \leq \left(\sum_{k=1}^{\infty}\|(A^{-1}C)\|^k\right)\|A^{-1}b\| \leq \left(\sum_{k=1}^{\infty}\|A^{-1}\|^k\|C\|^k\right)\|A^{-1}b\|$$

which we can further estimate it by setting $d = \|A^{-1}\|\|C\|$, then we have

$$\begin{aligned}
&= \left(\sum_{k=1}^{\infty} d^k\right)\|A^{-1}b\| = \frac{d}{1-d}\|A^{-1}\|\|b\| < 2d\|A^{-1}\|\|b\| \\
&\leq 2\|C\|\|A^{-1}\|^2\|b\| = \varepsilon.
\end{aligned}$$

□

7.2 Bimatrix games

In this Section we have $m = 2$. We change Definition 2.1.1 for bimatrix games only. Now $S^1 = \{s_1^1, \ldots, s_{n_1}^1\}$ and $S^2 = \{s_1^2, \ldots, s_{n_2}^2\}$. $\widetilde{J} := N^1 \times N^2$ replaces J from Definition 2.1.1 and let $s = (s_{j_1}^1, s_{j_2}^2)$ for $\underline{j} = (j_1, j_2) \in \widetilde{J}$. This is done to equalize indices of strategies with indices of the rows of matrices. Then $|S^1| = n_1$ and $|S^2| = n_2$, also let $n := n_1 + n_2$.

The following Definition 7.2.1 formalizes all linear equations of an equilibrium candidate. We have the 2 equations $\sum_{j=1}^{n_i} \gamma_j^i = 1$ such that the equilibrium candidate $g \in W$ is in A, and for each player $i \in \mathcal{A} = \{1, 2\}$ we have $|T_i| - 1$ equalities of equal utility for played pure strategies and $|T_i^\complement|$ equalities for unplayed pure strategies (see Lemma 2.1.2(b)).

Definition 7.2.1. Linear equation system for candidates of a bimatrix game (Q, R^T). Player one has n_1 strategies, player two has n_2 strategies. Q and R^T are $n_1 \times n_2$ matrices and $\underline{\gamma}^1 = (\gamma_1^1, \ldots, \gamma_{n_1}^1)$, $\underline{\gamma}^2 = (\gamma_1^2, \ldots, \gamma_{n_2}^2)$. $g^1 \in \mathbb{R}^{S_1}$, $g^2 \in \mathbb{R}^{S_2}$ are mixed virtual strategies and $g = (g^1, g^2) \in A$ is the mixed virtual strategy combination.

$$\begin{aligned} V^1(g) &= (\underline{\gamma}^1)^T Q \underline{\gamma}^2 \\ V^2(g) &= (\underline{\gamma}^1)^T R^T \underline{\gamma}^2 = (\underline{\gamma}^2)^T R \underline{\gamma}^1 \end{aligned}$$

Let $\emptyset \neq \mathcal{T}_1 \subset N^1$ and $\emptyset \neq \mathcal{T}_2 \subset N^2$ and let $c = (c_1, c_2) := (\min \mathcal{T}_1, \min \mathcal{T}_2)$ be the pair of minimal indices.
Define $\mathcal{D}(n_1, n_2) := \mathbb{R}^{n_1 \times n_2} \times \mathbb{R}^{n_2 \times n_1} \times (\mathcal{P}(N^1) \setminus \{\emptyset\}) \times (\mathcal{P}(N^2) \setminus \{\emptyset\})$. The candidate generating pair (D, b) is generated by

$$D : \begin{cases} \mathcal{D}(n_1, n_2) & \to \mathbb{R}^{n \times n} \\ (Q, R, \mathcal{T}_1, \mathcal{T}_2) & \mapsto D(Q, R, \mathcal{T}_1, \mathcal{T}_2), \end{cases}$$

and

$$b : \begin{cases} \mathcal{P}(N^1) \times \mathcal{P}(N^2) & \to \mathbb{R}^n \\ (\mathcal{T}_1, \mathcal{T}_2) & \mapsto b(\mathcal{T}_1, \mathcal{T}_2), \end{cases}$$

where the matrix and the vector have following forms

$$D = \left(\begin{array}{c|c} D^{(1,1)} & D^{(1,2)} \\ \hline D^{(2,1)} & D^{(2,2)} \end{array}\right), \quad b = \left(\begin{array}{c} b^{(1)} \\ \hline b^{(2)} \end{array}\right) \tag{7.5}$$

Clearly $D^{(k,l)} \in \mathbb{R}^{n_k \times n_l}$ and $b^{(k)} \in \mathbb{R}^{n_k \times 1}$ for $k, l \in \{1, 2\}$. Fix $k \in \{1, 2\}$, then

$$\mathbb{R}^{1 \times n_k} \ni D_i^{(k,k)} := \begin{cases} \left(\sum\limits_{j=1}^{n_k} e_j^{(n_k)}\right)^T & \text{, if } i \in \mathcal{T}_k \text{ and } i = c_k = \min \mathcal{T}_k, \\ 0 & \text{, if } i \in \mathcal{T}_k \text{ and } i \neq c_k, \\ \left(e_i^{(n_k)}\right)^T & \text{, if } i \notin \mathcal{T}_k. \end{cases}$$

Now fix $k, l \in \{1, 2\}$ with $k \neq l$, then

$$\mathbb{R}^{1 \times n_l} \ni D_i^{(k,l)} := \begin{cases} Q_i - Q_{c_k} & \text{, if } k = 1 \text{ and } i \in \mathcal{T}_1, \\ R_i - R_{c_k} & \text{, if } k = 2 \text{ and } i \in \mathcal{T}_2, \\ 0 & \text{, else.} \end{cases}$$

$$b_i^{(k)} := \delta_{c_k i} = \begin{cases} 1 & \text{, if } i = c_k, \\ 0 & \text{, else.} \end{cases}$$

One easily sees that the set of the coefficients of virtual candidates (so all $g \in W$ which satisfy the equalities, but not necessary the inequalities, which define Nash equilibria) for a bimatrix game (Q, R^T) for some $\mathcal{T}_1, \mathcal{T}_2$ is given by

$$\mathcal{C}(\mathcal{T}_1, \mathcal{T}_2) := \{x \in \mathbb{R}^{n_1+n_2} \mid D(Q, R, \mathcal{T}_1, \mathcal{T}_2)x = b(\mathcal{T}_1, \mathcal{T}_2)\}. \tag{7.6}$$

Definition 7.2.2. Let (Q, R^T) be a bimatrix game and fix $(\mathcal{T}_1, \mathcal{T}_2)$. Let $\underline{\gamma} \in \mathcal{C}(\mathcal{T}_1, \mathcal{T}_2)$ be the coefficient vector of a fixed virtual candidate. Let (D, b) be the candidate generating pair as in (7.5).

(i) The game (Q, R^T) is type-0 generic for $(\mathcal{T}_1, \mathcal{T}_2)$ if D is invertible.

(ii) $\underline{\gamma}$ is type-1 generic for the player $i \in \{1, 2\}$ if for all $j_1, j_2 \in N^i$ with $j_1 \neq j_2$

$$V^i(s^i_{j_1}, g^{-i}) = V^i(s^i_{j_2}, g^{-i}) \Longrightarrow j_1 \in \mathcal{T}_i \text{ and } j_2 \in \mathcal{T}_i.$$

(iii) $\underline{\gamma}$ is type-1 generic if it is type-1 generic for both players.

(iv) $\underline{\gamma}$ is type-2 generic for the player $i \in \{1, 2\}$ if

$$\operatorname{supp}(\gamma^i) = \mathcal{T}_i$$

(v) $\underline{\gamma}$ is type-2 generic if it is type-2 generic for both players.

(vi) $\underline{\gamma}$ is generic if (i),(iii) and (v) hold.

(vii) The game (Q, R^T) is generic if for all pairs $(\mathcal{T}_1, \mathcal{T}_2)$ with $|\mathcal{T}_1| = |\mathcal{T}_2|$ if there exists a $\underline{\gamma} \in \mathcal{C}(\mathcal{T}_1, \mathcal{T}_2)$ such that (vi) holds.

Remark 7.2.3. Definition 7.2.2 (iv) is remarkably strong. It implies the notion of quasi-strong from Harsanyi [Har73, p.238].

If g is a virtual strategy combination with coefficients $\underline{\gamma}$ is type-1 generic with $(\mathcal{T}_1, \mathcal{T}_2) = (\operatorname{supp}(\gamma^1), \operatorname{supp}(\gamma^2))$ then it is also type-2 generic. This is the important part of the type-1 genericity. The additional conditions, that the values $V^i_A(s^i_j, g^{-i})$ are pairwise different for unplayed strategies s^i_j, is also a generic property, but not so important.

Definition 7.2.4. Let $O = (Q, R^T)$ be a bimatrix game and $\widetilde{O} = (\widetilde{Q}, \widetilde{R}^T)$ be a perturbed game. Let $H : [0, 1] \to \mathbb{R}^{n_1 \times n_2} \times \mathbb{R}^{n_2 \times n_1}$ be a continuous path with $H(0) = O$ and $H(1) = \widetilde{O}$. Fix $\mathcal{T}_1, \mathcal{T}_2$.

(i) $(O, \widetilde{O})$ is type-0 preserving if there exists a continuous path H such that $H(t)$ is type-0 generic for $\mathcal{T}_1, \mathcal{T}_2$ and all $t \in [0, 1]$.

(ii) $(O, \widetilde{O})$ is type-1 preserving if it is type-0 preserving with path H such that for all $t \in [0, 1]$ $H(t)$ is type-1 generic.

(iii) $(O, \widetilde{O})$ is type-2 preserving if it is type-0 preserving with path H such that for all $t \in [0, 1]$ $H(t)$ is type-2 generic.

(iv) $(O, \widetilde{O})$ is preserving if it is type-1 and type-2 preserving with the same path.

Lemma 7.2.5. *Let $(O, \widetilde{O})$ be type-0 preserving for $\mathcal{T}_1, \mathcal{T}_2$ and choose a continuous path H that fulfils Definition 7.2.4 (i). There exists a corresponding unique path of virtual candidates $\underline{\gamma}(t)$ that connects the virtual candidates of both games.*

Proof. The candidate generating matrix for $H(t)$ is invertible for all $t \in [0, 1]$ and the solution is unique. It is clearly continuous. □

Lemma 7.2.6. *Let $O = (Q, R^T)$ be a bimatrix game and fix $\mathcal{T}_1, \mathcal{T}_2$. Let $\underline{\gamma} \in \mathcal{C}(\mathcal{T}_1, \mathcal{T}_2)$ be generic. Then $|\mathcal{T}_1| = |\mathcal{T}_2|$.*

Proof. Assume $|\mathcal{T}_1| > |\mathcal{T}_2|$ Let $D^{(\cdot,i)} = \begin{pmatrix} D^{(1,i)} \\ \hline D^{(2,i)} \end{pmatrix}$ Clearly $\operatorname{rank}(D^{(\cdot,1)}) \leq \operatorname{rank}(D^{(1,1)}) + \operatorname{rank}(D^{(2,1)}) \leq (1 + |\mathcal{T}_1^{\complement}|) + (|\mathcal{T}_2| - 1) < (1 + |\mathcal{T}_1^{\complement}|) + (|\mathcal{T}_1| - 1) = n_1$ Thus $\operatorname{rank}(D) \leq \operatorname{rank}(D^{(\cdot,1)}) + \operatorname{rank}(D^{(\cdot,2)}) < n_1 + n_2 = n$. Thus D is not invertible. □

Lemma 7.2.7. *Let $O = (Q, R^T)$ be a bimatrix game and fix $\mathcal{T}_1, \mathcal{T}_2$. Let $\underline{\gamma} \in \mathcal{C}(\mathcal{T}_1, \mathcal{T}_2)$ be generic. Let $P_1 \in \mathbb{R}^{n_1 \times n_2}$ and $P_2 \in \mathbb{R}^{n_2 \times n_1}$ be two perturbation matrices with $\|P_1\|_\infty < \omega$ and $\|P_2\|_\infty < \omega$. Let D be the candidate generating matrix of the game (Q, R^T) and let $\widetilde{D}$ be the candidate generating matrix of the game $\widetilde{O} = (Q + P_1, (R + P_2)^T)$.*

(a) Then $\|\widetilde{D} - D\|_\infty \leq 2\omega$.

Recall from Definition 7.2.1 $c_i = \min \mathcal{T}_i$ and define

$$\mu_1 := \min_{\substack{(a,b) \in \mathcal{T}_1^{\complement} \cup \{c_1\} \\ a \neq b}} \left| (e_a^{(n_1)} - e_b^{(n_1)})^T Q \underline{\gamma}^2 \right|,$$

$$\mu_2 := \min_{\substack{(a,b) \in \mathcal{T}_2^{\complement} \cup \{c_2\} \\ a \neq b}} \left| (e_a^{(n_2)} - e_b^{(n_2)})^T R \underline{\gamma}^1 \right|,$$

as well as

$$\mu := \min\{\mu_1, \mu_2\}.$$

Let

$$\nu := \min_{\substack{i \in \{1,2\} \\ 1 \leq j \leq n_i}} |\gamma_j^i|$$

(b)

$$\omega \leq \min\left\{ \frac{1}{4\|D^{-1}\|_\infty}, \frac{\nu}{4\|D^{-1}\|_\infty^2}, \|Q\|_\infty, \|R\|_\infty, \frac{\mu}{6\|\underline{\gamma}\|_\infty}, \frac{\mu^1}{6\|Q\|_\infty}, \frac{\mu^2}{6\|R\|_\infty} \right\},$$

then the pair $(O, \widetilde{O})$ *is preserving.*

(c) If $|\mathcal{T}_1| = |\mathcal{T}_2| = 1$ *and if*

$$\omega \leq \frac{\mu}{2},$$

then the pair $(O, \widetilde{O})$ *is preserving.*

Proof. (a) Trivial.

(b) Because of Lemma 7.1.5(c) and $\omega \leq \frac{1}{\|D^{-1}\|_\infty}$, the pair $(O, \widetilde{O})$ is type-0 preserving.

Let $\underline{\widetilde{\gamma}}$ be the solution of $\widetilde{D}\underline{\widetilde{\gamma}} = b$ and let $\underline{\gamma}$ be the solution of $D\underline{\gamma} = b$. $(O, \widetilde{O})$ is type-2 preserving if $\|\underline{\widetilde{\gamma}} - \underline{\gamma}\|_\infty < \nu$. Let us consider the two linear equation systems $D\underline{\gamma} = b = \widetilde{D}\underline{\widetilde{\gamma}}$. By (a) we get $\|\widetilde{D} - D\| < 2\omega \leq \min\{\frac{1}{2\|D^{-1}\|_\infty}, \frac{\nu}{2\|D^{-1}\|^2\|b\|_\infty}\}$, since $\|b\|_\infty = 1$. Now we can apply Lemma 7.1.5 (c) with $\varepsilon = \nu$, which proves the claim.

Finally we need to show that $(O, \widetilde{O})$ is type-1 preserving. We get two conditions

$$\left\|(Q + P_1)\underline{\widetilde{\gamma}}^2 - Q\underline{\gamma}^2\right\|_\infty < \mu_1 \text{ and} \tag{7.7}$$

$$\left\|(R + P_2)\underline{\widetilde{\gamma}}^1 - R\underline{\gamma}^1\right\|_\infty < \mu_2. \tag{7.8}$$

We want to apply Lemma 7.1.5 (b) on both equations. For the first equation choose $\varepsilon = \frac{\mu_1}{2}$ and for the second equation choose $\varepsilon = \frac{\mu_2}{2}$. This implies for the first equation $\delta = \frac{\mu_1}{6\|Q\|}$ and $\|P_1\| \leq \frac{\mu_1}{6\|\underline{\gamma}^2\|} \leq \frac{\mu}{6\|\underline{\gamma}\|_\infty}$ and for the second equation another $\delta = \frac{\mu_1}{6\|R\|}$ and $\|P_2\| \leq \frac{\mu_1}{6\|\underline{\gamma}^1\|} \leq \frac{\mu}{6\|\underline{\gamma}\|_\infty}$ respectively. Thus the conditions of Lemma 7.1.5 (b) are fulfilled to show (7.7) and (7.8).

(c) Trivial. □

7.3 von Stengel's game for $m = 6$

In [Ste97] von Stengel illustrated his general construction of 2-player games with $|S^1| = |S^2|$ which realize his lower bound for the number of Nash equilibria for generic games (see Section 3.5) in the case $m = 6$ with explicit utility values. The matrix of utility values is copied in the following definition. The game has 75 Nash equilibria, and 923 points in $\mathbb{P}^5\mathbb{R} \times \mathbb{P}^5\mathbb{R}$ which satisfy the homogeneous versions of the equalities which are used for Nash equilibria. With an extensive computer search we analyzed some properties of the 923 points. The results are documented in the Appendix C.

All dots show points in $\mathbb{P}^5\mathbb{R}\times\mathbb{P}^5\mathbb{R}$ which satisfy the homogeneous versions of the equalities which are used for Nash equilibria. Green dots show Nash equilibria, red dots show points in G which do not satisfy the inequalities of Nash equilibria, blue dots show points in A which satisfy all inequalities except those saying that the point is in G, yellow dots shows points in $A\setminus G$ which do not satisfy all inequalities, black dots show points in $\mathbb{P}^5 R\times\mathbb{P}^5\mathbb{R}\setminus A$.

All points are type-0 generic. A star in the center of a dot indicates that this point is not type-1 generic. Surprisingly, there are quite many points which are not type-1 generic, but all of them are type-2 generic. Even 2 of the 75 Nash equilibria are not type-1 generic.

Definition 7.3.1. (Von Stengel's game [Ste97]) (Q, R^T) is the bimatrix game:

$$Q = \begin{pmatrix} 9504 & -660 & 19976 & -20526 & 1776 & -8976 \\ -111771 & 31680 & -130944 & 168124 & -8514 & 52764 \\ 397584 & -113850 & 451176 & -586476 & 29216 & -178761 \\ 171204 & -45936 & 208626 & -263076 & 14124 & -84436 \\ 1303104 & -453420 & 1227336 & -1718376 & 72336 & -461736 \\ 737154 & -227040 & 774576 & -1039236 & 48081 & -300036 \end{pmatrix}$$

$$R = \begin{pmatrix} 72336 & -461736 & 1227336 & -1718376 & 1303104 & -453420 \\ 48081 & -300036 & 774576 & -1039236 & 737154 & -227040 \\ 29216 & -178761 & 451176 & -586476 & 397584 & -113850 \\ 14124 & -84436 & 208626 & -263076 & 171204 & -45936 \\ 1776 & -8976 & 19976 & -20526 & 9504 & -660 \\ -8514 & 52764 & -130944 & 168124 & -111771 & 31680 \end{pmatrix}$$

Remark 7.3.2. (a) Here are some bounds for all generic $\mathcal{T}_1, \mathcal{T}_2$. $\|D^{-1}\|_\infty \leq \frac{18772169}{72765} < 258$, $\|\underline{\gamma}\|_\infty \leq 35$, $\|Q\|_\infty = \|R\|_\infty = 5236308$, $\|\mu\|_\infty \geq \frac{1078}{9297}$ and $\|\nu\| \geq \frac{3}{1453}$.
By Lemma 7.2.7

$$\omega < 7.75 \cdot 10^{-9}$$

Let P_1, P_2 with $\|P_1\|_\infty < \omega$ and $\|P_2\|_\infty < \omega$. For almost all such choices is the perturbed game $\widetilde{O}$ generic according to Definition 7.2.2 (vii).

(b) The choice in (a) is not sharp at all. It can be coupled by calculating ω individually for each $\mathcal{T}_1$ and $\mathcal{T}_2$, but the factor $\|D^{-1}\|_\infty^2$ is likely going to dominate again.

Now we are briefly explaining the two Nash equilibria that are not generic according to Definition 7.2.7(vi).

Lemma 7.3.3. 2 *of the* 75 *Nash equilibria are not generic, but quasi-strong.*

Proof. According to Appendix C we have two Nash equilibria that are not type-1 generic. One for $|\mathcal{T}_1| = |\mathcal{T}_2| = 2$ at position $(10, 10)$. It is straightforward to check that the 10-th element of $\mathcal{O}_2$ is $\{3, 4\}$.Thus $\mathcal{T}_1 = \mathcal{T}_2 = \{3, 4\}$. One observes $\underline{\gamma}^1 = \underline{\gamma}^2 = (0, 0, \frac{4}{7}, \frac{3}{7}, 0, 0)$. For any g^1 with $\text{supp}(\underline{\gamma}^1) = \mathcal{T}_1$ we have

$$\begin{aligned} 6468 = V^1(g^1, g^2) &> V^1(s_1^1, g^2) = 2618 \\ &> V^1(s_2^1, g^2) = V^1(s_6^1, g^2) = -2772 \\ &> V^1(s_5^1, g^2) = -35112 \end{aligned} \tag{7.9}$$

and for any g^2 with $\text{supp}(\underline{\gamma}^2) = \mathcal{T}_2$ we have

$$\begin{aligned} 6468 = V^2(g^1, g^2) &> V^2(g^1, s_5^2) = 2618 \\ &> V^2(g^1, s_2^2) = V^2(g^1, s_6^2) = -2772 \\ &> V^2(g^1, s_1^2) = -35112. \end{aligned} \tag{7.10}$$

Thus the game is quasi-strong, but not type-1 generic because of either equality in (7.9) or (7.10).

We have another Nash equilibrium for $|\mathcal{T}_1| = |\mathcal{T}_2| = 3$ at position $(13, 13)$. The 13-th element of $\mathcal{O}_3$ is $\{2, 3, 6\}$. Thus $\mathcal{T}_1 = \mathcal{T}_2 = \{2, 3, 6\}$. One observes $\underline{\gamma}^1 = (0, \frac{22}{37}, \frac{10}{37}, 0, 0, \frac{5}{37})$ and $\underline{\gamma}^2 = (0, \frac{5}{37}, \frac{10}{37}, 0, 0, \frac{22}{37})$. For any g^1

with $\operatorname{supp}(\underline{\gamma}^1) = T_1$ we have

$$
\begin{aligned}
264 = V^1(g^1, g^2) > V^1(s_1^1, g^2) = V^1(s_4^1, g^2) &= -2772 \\
> V^1(s_5^1), g^2 &= -35112
\end{aligned} \tag{7.11}
$$

□

Remark 7.3.4. As we wrote in Remark 7.2.3 the fact that these 2 of the 75 Nash equilibria are not type-1 generic, is not harmful. They are type-0 and type-2 generic (so quasi-strong). The equality of the utility values at several unused strategies is a bit surprising, but not important.

Though the fact that not all 923 points are type-1 generic and also the fact that some of them are in the union of hypersurfaces at ∞, namely in $\mathbb{P}^5\mathbb{R} \times \mathbb{P}^5\mathbb{R} \setminus A$ show that $U \in \mathcal{D}$ (see Theorem 2.2.1).

Remark 7.3.5. There were some plans to study the bounds of virtual equilibria of this game. However the hypothesis for the study is that the game is generic (as in Definition 7.2.2(vii)), which it regrettably is not. The construction has too many symmetries that make the game both beautiful and elegant, but at the same time nearly intractable for this specific study.

8 A universal family of finite games

Kohlberg and Mertens gave in Section 3.2 in a parametrization [KM86] of the union of the sets of Nash equilibria of mixed extensions of all finite games with a fixed set of players and fixed sets of pure strategies. This parametrization has not been used much. The presentation in [KM86] is very short. Here we recall the parametrization, we extend it by parametrizations of the graphs of best reply maps, and we prepare this by a detailed discussion of a related construction.

This construction is presented in the first of the two subsections. We consider it as a blow up construction in the frame of polytopes. From the game theory point of view, it comes from the case of 1-player games. From the algebraic geometry point of view, it is reminiscent of the notion of (real or complex) blow ups. But it really should play a role in the theory of convex polytopes, and it should be considered from that point of view.

We consider as in Section 2.2 a set $\mathcal{A} = \{1, \dots, m\}$ of players (with $m \in \mathbb{N}_{\geq 2}$), for each player i his set of pure strategies $S^i = \{s^i_0, \dots, s^i_{n_i}\}$ (with $n_i \in \mathbb{N}$), the product $S = S^1 \times \cdots \times S^m$, the set $\mathcal{U}^i = \mathbb{R}^S$ of all possible utility functions U^i of player i, and the product $\mathcal{U} = \prod_{i=1}^m \mathcal{U}^i$.

8.1 A blow up construction in the frame of polytopes

First we consider games with the only player i. As S^i is fixed, such a game is given by a utility function $U^i : S^i \to \mathbb{R}$, which we identify with an element $\underline{u}^i = (u^i_0, \ldots, u^i_{n_i}) \in \mathbb{R}^{S^i}$. Denote $\max(\underline{u}^i) := \max_j u^i_j$ and $\min(\underline{u}^i) := \min_j u^i_j$. The set of best replies and the set of Nash equilibria of the mixed extension of the game with utility function $\underline{u}^i$ coincide and are

$$R^i(\underline{u}^i) := \operatorname{Conv}\Big(\{s^i_j \mid u^i_j = \max(\underline{u}^i)\}\Big) \subset G^i.$$

We denote the union over all games of these sets of mixed Nash equilibria by

$$\mathcal{R}^i := \bigcup_{\underline{u}^i \in \mathbb{R}^{S^i}} R^i(\underline{u}^i) \times \{\underline{u}^i\} \subset G^i \times \mathbb{R}^{S^i},$$

and $e_{\mathcal{R}^i} : \mathcal{R}^i \hookrightarrow G^i \times \mathbb{R}^{S^i} \hookrightarrow \mathbb{R}^{S^i} \times \mathbb{R}^{S^i}$ means the natural embedding. Here we identify W^i with $\mathbb{R}^{S^i}$ by the natural isomorphism of $\mathbb{R}$-vector spaces

$$W^i \to \mathbb{R}^{S^i}, \quad \sum_{j=0}^{n_i} \gamma^i_j s^i_j \mapsto \underline{\gamma}^i = (\gamma^i_0, \ldots, \gamma^i_n) = (s^i_j \mapsto \gamma^i_j). \tag{8.1}$$

Then also $G^i \hookrightarrow \mathbb{R}^{S^i}$. The crucial observation on which [KM86, section 3.2] builds is the following.

Proposition 8.1.1. *The map*

$$\Xi : \begin{cases} \mathcal{R}^i \to \mathbb{R}^{S^i} \\ (\underline{\gamma}^i, \underline{u}^i) \mapsto \underline{\gamma}^i + \underline{u}^i, \end{cases} \tag{8.2}$$

is a bijection. The composition $e_{\mathcal{R}^i} \circ \Xi^{-1} : \mathbb{R}^{S^i} \to \mathbb{R}^{S^i} \times \mathbb{R}^{S^i}$ is continuous and injective and piecewise affine linear.

This says that the map $\Xi^{-1} : \mathbb{R}^{S^i} \to \mathcal{R}^i$ is in the case of one player i a parametrization of the union $\mathcal{R}^i$ of the sets of mixed Nash equilibria (of the mixed extensions) of all games with fixed set S^i of pure strategies.

The proof of Proposition 8.1.1 will be contained in the proof of Lemma 8.1.4. Definition 8.1.3 and Lemma 8.1.4 give more notations and details around the map $\Xi^{-1} : \mathbb{R}^{S^i} \to \mathcal{R}^i$. Remark 8.1.2 offers pictures which shall illustrate the geometry.

Remark 8.1.2. Here we want to show pictures for the cases $n_i = 1$ and $n_i = 2$ which shall illustrate the geometry in Proposition 8.1.1. One parameter in $\underline{u}^i$ is inessential, namely the sum $\sum_{j=0}^{n_i} u_j^i$. We restrict to

$$\mathbb{R}^{S^i,no} := \{\underline{u}^i \in \mathbb{R}^{S^i} \mid \sum_{j=0}^{n_i} u_j^i = 0\}$$

(*no* for *normalized*) and denote

$$\mathcal{R}^{i,no} := \bigcup_{\underline{u}^i \in \mathbb{R}^{S^i,no}} R^i(\underline{u}^i) \times \{\underline{u}^i\} \subset G^i \times \mathbb{R}^{S^i,no},$$
$$\frac{1}{n_i+1}\mathbf{1}_{n_i+1} := \frac{1}{n_1+1}(1,\dots,1) \in \mathbb{R}^{S^i}.$$

The restricted map

$$\Xi^{no} : \mathcal{R}^{i,no} \to \mathbb{R}^{S^i,no} + \frac{1}{n_i+1}\mathbf{1}_{n_i+1}, \quad (\underline{\gamma}^i, \underline{u}^i) \mapsto \underline{\gamma}^i + \underline{u}^i,$$

is also a bijection. The projection $\mathrm{pr}_2 : G^i \times \mathbb{R}^{S^i,no} \to \mathbb{R}^{S^i,no}$ restricts to a map $\mathrm{pr}_2^{no} : \mathcal{R}^{i,no} \to \mathbb{R}^{S^i,no}$ which is surjective, but not bijective: The preimage of a point $\underline{u}^i$ is the simplex $R^i(\underline{u}^i)$, which is only for generic $\underline{u}^i$ a single point. At non-generic $\underline{u}^i$ the projection *blows down* the simplex $R^i(\underline{u}^i)$ to the single point $\underline{u}^i$. Here are pictures of pr_2^{no} and $\mathrm{pr}_2^{no} \circ (\Xi^{no})^{-1}$ for $n_i = 1$ and of $\mathrm{pr}_2^{no} \circ (\Xi^{no})^{-1}$ for $n_i = 2$.

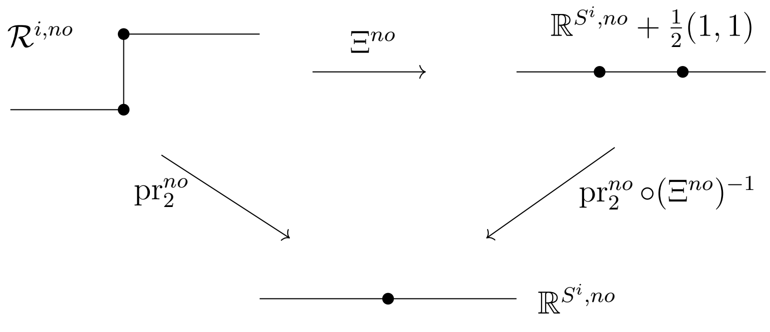


Figure 8.1: $n_i = 1$

In the picture for $n_i = 1$, the two fat points upstairs and the interval between them are mapped to the fat point downstairs. Outside the map is bijective.

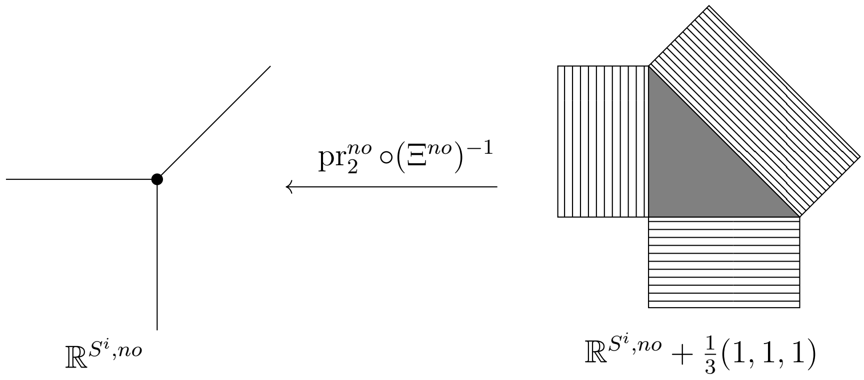


Figure 8.2: $n_i = 2$

In the picture for $n_i = 2$, the grey triangle is mapped to the fat point downstairs, and the intervals in the three 1-parameter families are mapped to the points in the three half lines downstairs. Outside the map is bijective.

These pictures and this construction of blowing down simplices are reminiscent of blow down (and inversely blow up) constructions in (real and complex) algebraic geometry. The construction should play a role in the theory of convex polytopes, but we are not aware of literature on it.

Definition 8.1.3. (a) For $T^i \subset S^i$ with $T^i \neq \emptyset$, we define the affine linear map

$$\Delta^{i,T^i} : \mathbb{R}^{S^i} \to \mathbb{R}, \quad \underline{a}^i = (a_0^i, \dots, a_{n_i}^i) \mapsto |T^i|^{-1}\Big(-1 + \sum_{s_j^i \in T^i} a_j^i\Big)$$

In other words, $\Delta^{i,T^i}(\underline{a}^i) \in \mathbb{R}$ is the unique value with

$$1 = \sum_{s_j^i \in T^i} \Big(a_j^i - \Delta^{i,T^i}(\underline{a}^i)\Big). \tag{8.3}$$

(b) We define the map

$$\Delta^i : \mathbb{R}^{S^i} \to \mathbb{R}, \quad \underline{a}^i \mapsto \max_{T^i \subset S^i, T^i \neq \emptyset} \Delta^{i,T^i}(\underline{a}^i).$$

(c) For $\underline{a}^i \in \mathbb{R}^{S^i}$ define

$$\tau^i(\underline{a}^i) := \{s_j^i \in S^i \mid a_j^i - \Delta^i(\underline{a}^i) > 0\} \subset S^i.$$

(d) For $T^i \subset S^i$ with $T^i \neq \emptyset$ define

$$\mathbb{R}^{S^i,T^i} := \{\underline{a}^i \in \mathbb{R}^{S^i} \,|\, \tau^i(\underline{a}^i) = T^i\}.$$

(e) The map $(\cdot)^+ : \mathbb{R} \to \mathbb{R}$ is the map with

$$(r)^+ := \begin{cases} r & ,\text{if } r \geq 0, \\ 0 & ,\text{if } r < 0. \end{cases}$$

(f) Define the map

$$\delta^i : \mathbb{R}^{S^i} \times \mathbb{R} \to \mathbb{R}, \quad \delta^i(\underline{a}^i, d) := \sum_{j=0}^{n_i} (a^i_j - d)^+.$$

(g) Define the maps

$$\Phi^i_1 : \mathbb{R}^{S^i} \to \mathbb{R}^{S^i}, \quad \underline{a}^i \mapsto \big((a^i_j - \Delta^i(\underline{a}^i))^+\big)_{j=0,\dots,n_i}, \tag{8.4}$$

$$\Phi^i_2 : \mathbb{R}^{S^i} \to \mathbb{R}^{S^i}, \quad \underline{a}^i \mapsto \underline{a}^i - \Phi^i_1(\underline{a}^i) \tag{8.5}$$

$$= \begin{cases} \Delta^i(\underline{a}^i) & \text{if } s^i_j \in \tau^i(\underline{a}^i), \\ a^i_j & \text{if } s^i_j \notin \tau^i(\underline{a}^i). \end{cases} \tag{8.6}$$

(h) For $T^i \subset S^i$ with $T^i \neq \emptyset$, $\mathrm{Conv}(T^i)^{int} \subset \mathbb{R}^{S^i}$ denotes the interior of $\mathrm{Conv}(T^i)$ in its affine hull, i.e. in the smallest affine linear subspace of A^i which contains $\mathrm{Conv}(T^i)$.

Lemma 8.1.4. *(a) For $\underline{a}^i \in \mathbb{R}^{S^i}$, the set $\tau^i(\underline{a}^i)$ is not empty and*

$$\Delta^i(\underline{a}^i) = \Delta^{i,\tau^i(\underline{a}^i)}(\underline{a}^i). \tag{8.7}$$

The set $\mathbb{R}^{S^i}$ is the disjoint union of the sets $\mathbb{R}^{S^i,T^i}$ with $T^i \subset S^i$, $T^i \neq \emptyset$, so they form a stratification of $\mathbb{R}^{S^i}$. For each T^i, the closure $\overline{\mathbb{R}^{S^i,T^i}}$ is a polyhedron in $\mathbb{R}^{S^i}$ (an intersection of half-spaces), of full dimension $|S^i|$, *and*

$$\mathbb{R}^{S^i,T^i} = \bigcup_{d\in\mathbb{R}} \Big((d,\dots,d) + \big\{ \underline{a}^i \in \mathbb{R}^{S^i} \,|\, a^i_j \le 0 \text{ for } s^i_j \notin T^i, \tag{8.8}$$

$$a^i_j = \gamma^i_j \text{ for } j \in T^i \text{ with } \sum_{j\in T^i} \gamma^i_j = 1, \gamma^i_j > 0 \big\} \Big)$$

$$\cong \mathrm{Conv}(T^i)^{int} \times \mathbb{R}^{S^i-T^i}_{\le 0} \times \mathbb{R}. \tag{8.9}$$

(b) Fix a tuple $\underline{a}^i \in \mathbb{R}^{S^i}$*. The map*

$$\delta(\underline{a}^i,\cdot) : \mathbb{R} \to \mathbb{R}, \quad d \mapsto \sum_{j=0}^{n_i} (a^i_j - d)^+,$$

is continuous, it takes value 0 on $[\max(\underline{a}^i), +\infty)$*, it is strictly decreasing on* $(-\infty, \max(\underline{a}^i)]$*, and it takes the value* $\sum_{j=0}^{n_i} a^i_j - (n_i+1)d$ *for* $d \in (-\infty, \min(\underline{a}^i)]$*, especially* $\lim_{d\to-\infty} \delta(\underline{a}^i, d) = +\infty$*.*

(c) Fix a tuple $\underline{a}^i \in \mathbb{R}^{S^i}$*. The value* $\Delta^i(\underline{a}^i)$ *is the unique value* $d \in \mathbb{R}$ *with* $\delta(\underline{a}^i, d) = 1$*.*

(d) The maps $\Delta^i, \Phi^i_1, \Phi^i_2$ *and* (Φ^i_1, Φ^i_2) *are continuous. For any* $T^i \subset S^i$ *with* $T^i \neq \emptyset$*, their restrictions to* $\overline{\mathcal{R}^{S^i,T^i}}$ *are restrictions of affine linear maps.*

(e) The map $\Xi : \mathcal{R}^i \to \mathbb{R}^{S^i}$ *in* (8.2) *is bijective, and*

$$(\Phi^i_1, \Phi^i_2) = e_{\mathcal{R}^i} \circ \Xi^{-1} : \mathbb{R}^{S^i} \to \mathbb{R}^{S^i} \times \mathbb{R}^{S^i},$$

so (Φ^i_1, Φ^i_2) *is the composition of the inverse map* Ξ^{-1} *with the embedding* $e_{\mathcal{R}^i} : \mathcal{R}^i \hookrightarrow \mathbb{R}^{S^i} \times \mathbb{R}^{S^i}$*.*

Proof. (a) By definition of $\Delta^i(\underline{a}^i)$, there is a set $\widetilde{T}^i \subset S^i$ with $\widetilde{T}^i \neq \emptyset$ and $\Delta^i(\underline{a}^i) = \Delta^{i,\widetilde{T}^i}(\underline{a}^i)$. The equality (8.3) for this set $\widetilde{T}^i$ shows that some differences $a^i_j - \Delta^i(\underline{a}^i)$ are positive. Therefore the set $\tau^i(\underline{a}^i)$ is not empty.

Suppose $\Delta^i(\underline{a}^i) > \Delta^{i,\tau^i(\underline{a}^i)}(\underline{a}^i)$. This inequality and (8.3) for $\widetilde{T}^i$ as above and for $\tau^i(\underline{a}^i)$ give

$$\sum_{s^i_j \in \widetilde{T}^i} \Big(a^i_j - \Delta^i(\underline{a}^i) \Big) = 1 = \sum_{s^i_j \in \tau^i(\underline{a}^i)} \Big(a^i_j - \Delta^{i,\tau^i(\underline{a}^i)}(\underline{a}^i) \Big)$$

$$> \sum_{s^i_j \in \tau^i(\underline{a}^i)} \Big(a^i_j - \Delta^i(\underline{a}^i) \Big). \tag{8.10}$$

But for $s^i_j \in S^i \setminus \tau^i(\underline{a}^i)$ we have $a^i_j - \Delta^i(\underline{a}^i) \leq 0$, and for $s^i_j \in \tau^i(\underline{a}^i)$ we have $a^i_j - \Delta^i(\underline{a}^i) > 0$. Therefore the inequality (8.10) is impossible. This shows (8.7).

By definition of the sets $\mathbb{R}^{S^i,T^i}$, the set $\mathbb{R}^{S^i}$ is the disjoint union of these subsets. The equality (8.8) follows from (8.7) and (8.3) ((8.3) for $T^i = \tau^i(\underline{a}^i)$). For an element $(d,\dots,d) + \underline{a}^i$ in the right hand side of (8.8) $\Delta^i((d,\dots,d) + \underline{a}^i) = d$ and $T^i = \tau^i((d,\dots,d) + \underline{a}^i)$.

The isomorphism (8.9) is obvious. This set hat full dimension $|S^i|$.

(b) Trivial.

(c) By part (b) and the intermediate value theorem, there is a unique value d with $\delta(\underline{a}^i, d) = 1$. Define $T^i := \{s^i_j \in S^i \mid a^i_j > d\}$. Then $\sum_{s^i_j \in T^i}(a^i_j - d) = 1$. (8.8) shows $\underline{a}^i \in \mathbb{R}^{S^i,T^i}$ and $d = \Delta^i(\underline{a}^i)$.

(d) The maps are continuous because they are compositions of continuous maps (affine linear maps, max, $(.)^+$, sums or differences). On $\overline{\mathbb{R}^{S^i,T^i}}$, the maps Δ^i and Φ^i_1 coincide with affine linear maps because of (8.7) and because for $\underline{a}^i \in \overline{\mathbb{R}^{S^i,T^i}}$

$$\Phi^i_1(\underline{a}^i) = \begin{cases} a^i_j - \Delta^{i,T^i}(\underline{a}^i) & \text{,if } s^i_j \in T^i, \\ 0 & \text{,if } s^i_j \notin T^i. \end{cases}$$

$\Phi^i_2 = \mathrm{id} - \Phi^i_1$ is then also piecewise affine linear.

(e) Part (c) and the definition of Φ^i_1 in (8.4) imply $\Phi^i_1(\underline{a}^i) \in \mathrm{Conv}\Big(\tau^i(\underline{a}^i)\Big)$. With formula (8.6) for $\Phi^i_2(\underline{a}^i)$ we obtain $(\Phi^i_1, \Phi^i_2)(\underline{a}^i) \in \mathcal{R}^i$. Formula (8.5) gives $\Xi((\Phi^i_1, \Phi^i_2)(\underline{a}^i)) = \underline{a}^i$. If we start with $(\underline{\gamma}^i, \underline{u}^i) \in \mathcal{R}^i$, part (c) shows $\Delta^i(\underline{\gamma}^i + \underline{u}^i) = \max(\underline{u}^i)$: With (8.4) and (8.5) this shows $(\Phi^i_1, \Phi^i_2)(\Xi(\underline{\gamma}^i, \underline{u}^i)) = (\underline{\gamma}^i, \underline{u}^i)$. □

Remark 8.1.5. The characterization of $\Delta^i(\underline{a}^i)$ in part (c) of Lemma 8.1.4 focuses on the central property of $\Delta^i(\underline{a}^i)$. But we started with the characterization in part (b) of Definition 8.1.3 as it is more direct and shows the affine linearity of the map Δ^i.

8.2 The union of sets of Nash equilibria in mixed extensions

Now we come to a parametrization of the union of sets of best replies of one player i in the general case, and after that to the parametrization of the union of sets of Nash equilibria.

Now we consider the general case, the set $\mathcal{A} = \{1, \ldots, m\}$ of players and the set $S = S^1 \times \cdots \times S^m$ of strategy combinations. It will be crucial to split every utility function $U^i \in \mathcal{U}^i = \mathbb{R}^S$ into two pieces, an average part and a normalized part. The notions are introduced in Definition 8.2.1, the splitting property is formulated in the trivial Lemma 8.2.2.

Definition 8.2.1. (a) A utility function $U^i \in \mathcal{U}^i = \mathbb{R}^S$ of player i is an *average utility function* if

$$U^i(s^i, s^{-i}) = U^i(s^i, \widetilde{s}^{-i}) \quad \text{for any } s^i \in S^i \text{ and any } s^{-i}, \widetilde{s}^{-i} \in S^{-i}. \tag{8.11}$$

The set of all average utility functions of player i is denoted by $\mathcal{U}^{i,av} \subset \mathcal{U}^i$. Obviously $\mathcal{U}^{i,av} \cong \mathbb{R}^{S^i}$. We will identify $\mathcal{U}^{i,av}$ with $\mathbb{R}^{S^i}$.

(b) A utility function U^i of player i is a *normalized utility function* if

$$\sum_{s^{-i} \in S^{-i}} U^i(s^i, s^{-i}) = 0 \quad \text{for any } s^i \in S^i. \tag{8.12}$$

The set of all normalized utility functions of player i is denoted by $\mathcal{U}^{i,no} \subset \mathcal{U}^i$.

Lemma 8.2.2. *The map* $\mathrm{pr}^{i,av} : \mathcal{U}^i \to \mathcal{U}^i$ *with*

$$\mathrm{pr}^{i,av}(U^i)(s^i, \widetilde{s}^{-i}) := |S^{-i}|^{-1} \sum_{s^{-i} \in S^{-i}} U^i(s^i, s^{-i})$$

is a projection $\mathrm{pr}^{i,av} : \mathcal{U}^i \to \mathcal{U}^{i,av}$ *(projection:* $(\mathrm{pr}^{i,av})^2 = \mathrm{pr}^{i,av}$*). The map* $\mathrm{pr}^{i,no} := \mathrm{id} - \mathrm{pr}^{i,av}$ *is a projection* $\mathrm{pr}^{i,no} : \mathcal{U}^i \to \mathcal{U}^{i,no}$ *(projection:* $(\mathrm{pr}^{i,no})^2 = \mathrm{pr}^{i,no}$*). This gives a canonical decomposition*

$$U^i = \mathrm{pr}^{i,av}(U^i) + \mathrm{pr}^{i,no}(U^i) \tag{8.13}$$

of an arbitrary utility function $U^i \in \mathcal{U}^i$ *into an average utility function*

$\mathrm{pr}^{i,av}(U^i)$ *and a normalized utility function* $\mathrm{pr}^{i,no}(U^i)$*. It also gives an isomorphism* $\mathcal{U}^{i,av} \times \mathcal{U}^{i,no} \to \mathcal{U}^i, (U^{i,av}, U^{i,no}) \mapsto U^{i,av} + U^{i,no}$*.*

Proof. Trivial. □

Remark 8.2.3. (i) In the case of a utility function $U^i \in \mathcal{U}^i$ we denote (as in Section 2.1) by $V^i : G \to \mathbb{R}$ its mixed extension. In the case of a normalized utility function $U^{i,no} \in \mathcal{U}^{i,no}$ we denote by $V^{i,no} : G \to \mathbb{R}$ its mixed extension.

(iii) Fix a normalized utility function $U^{i,no}$. The set $(\mathrm{pr}^{i,no})^{-1}(U^{i,no})$ can be parametrized by $\mathbb{R}^{S^i}$ in different ways. A canonical way:

$$\begin{aligned} \mathbb{R}^{S^i} &\to (\mathrm{pr}^{i,no})^{-1}(U^{i,no}), \quad U^{i,av} \mapsto U^{i,av} + U^{i,no}, \\ \text{inverse map: } (\mathrm{pr}^{i,no})^{-1}(U^{i,no}) &\to \mathbb{R}^{S^i}, \quad U^i \mapsto \mathrm{pr}^{i,av}(U^i). \end{aligned}$$

A way which depends on the choice of an element $g^{-i} \in G^{-i}$:

$$\begin{aligned} \mathbb{R}^{S^i} &\to (\mathrm{pr}^{i,no})^{-1}(U^{i,no}), \quad \underline{u}^i \mapsto (\underline{u}^i - V^{i,no}(\cdot, g^{-i})) + U^{i,no}, \\ \text{inverse map: } (\mathrm{pr}^{i,no})^{-1}(U^{i,no}) &\to \mathbb{R}^{S^i}, \quad U^i \mapsto V^i(\cdot, g^{-i}). \end{aligned} \tag{8.14}$$

The bijective map in (8.14) is remarkable: A utility function $U^i \in (\mathrm{pr}^{i,no})^{-1}(U^{i,no})$ can be recovered from $U^{i,no}$ and the value $V^i(., g^{-i}) \in \mathbb{R}^{S^i}$ for an arbitrary $g^{-i} \in G^{-i}$, and any value in $\mathbb{R}^{S^i}$ is reached by a suitable $U^i \in (\mathrm{pr}^{i,no})^{-1}(U^{i,no})$. The bijective map in (8.14) allows an easy parametrization of the union of the sets of best replies to g^{-i} for all games in $(\mathrm{pr}^{i,no})^{-1}(U^{i,no})$, see the next Corollary.

Corollary 8.2.4. *Fix a player* i*. The union of the sets of best replies of player* i *for all utility functions* $U^i \in \mathcal{U}^i$ *and all mixed strategy combinations* $g^{-i} \in G^{-i}$ *of the other players can be written in different ways. The way which uses* $(V^i(., g^{-i}), \mathrm{pr}^{i,no}(U^i))$ *in order to characterize* U^i *gives the set*

$$\begin{aligned} &\bigcup_{U^i \in \mathcal{U}^i} \bigcup_{g^{-i} \in G^{-i}} R^i(V^i(., g^{-i})) \times \{(V^i(\cdot, g^{-i}), \mathrm{pr}^{i,no}(U^i), g^{-i})\} \\ =&\mathcal{R}^i \times \mathcal{U}^{i,no} \times G^{-i} \subset G^i \times \mathbb{R}^{S^i} \times \mathcal{U}^{i,no} \times G^{-i}. \end{aligned} \tag{8.15}$$

It is parametrized by the map

$$(\Phi_1^i, \Phi_2^i, \mathrm{pr}_3, \mathrm{pr}_4) : \mathbb{R}^{S^i} \times \mathcal{U}^{i,no} \times G^{-i} \to G^i \times \mathbb{R}^{S^i} \times \mathcal{U}^{i,no} \times G^{-i}, \tag{8.16}$$

where pr_3 *and* pr_4 *are the projections to the third respectively fourth entry. The way which uses* $(\mathrm{pr}^{i,av}(U^i), \mathrm{pr}^{i,no}(U^i))$ *in order to characterize* U^i *gives the set*

$$\begin{aligned}&\bigcup_{U^i \in \mathcal{U}^i} \bigcup_{g^{-i} \in G^{-i}} R^i(V^i(.,g^{-i})) \times \{\mathrm{pr}^{i,av}(U^i), \mathrm{pr}^{i,no}(U^i), g^{-i}\} \\ &\subset G^i \times \mathcal{U}^{i,av} \times \mathcal{U}^{i,no} \times G^{-i}.\end{aligned} \tag{8.17}$$

It is parametrized by the map

$$\begin{aligned}\mathbb{R}^{S^i} \times \mathcal{U}^{i,no} \times G^{-i} &\to G^i \times \mathcal{U}^{i,av} \times \mathcal{U}^{i,no} \times G^{-i}, \\ (\underline{a}^i, U^{i,no}, g^{-i}) &\mapsto \Big(\Phi_1^i(\underline{a}^i), \Phi_2^i(\underline{a}^i) - V^{i,no}(.,g^{-i}), U^{i,no}, g^{-i}\Big).\end{aligned} \tag{8.18}$$

Proof. The fact that the set in (8.15) coincides with the set $\mathcal{R}^i \times \mathcal{U}^{i,no} \times G^{-i}$ and Proposition 8.1.1 respectively Lemma 8.1.4 (e) show that the set in (8.15) can be parametrized by the map in (8.16).

Given $\mathrm{pr}^{i,no}(U^i)$, the values $V^i(\cdot, g^{-i}) \in \mathbb{R}^{S^i}$ and $\mathrm{pr}^{i,av}(U^i) \in \mathbb{R}^{S^i}$ are related by

$$V^i(.,g^{-i}) = \mathrm{pr}^{i,av}(U^i) + \mathrm{pr}^{i,no}(U^i)(.,g^{-i})$$

This shows that the set in (8.17) is parametrized by the map in (8.18). $\square$

Remark 8.2.5. (i) All sets and parametrizations in Corollary 8.2.4 can be restricted to $U^i \in (\mathrm{pr}^{i,no})^{-1}(U^{i,no})$ (instead of $U^i \in \mathcal{U}^i$) for a fixed normalized utility function $U^{i,no}$ and/or to a fixed mixed strategy combination $g^{-i} \in G^{-i}$ of the other players.

(ii) The splitting of a utility function $U^i \in \mathcal{U}^i$ in (8.13) extends to a splitting of a utility map $U \in \mathcal{U} = \prod_{i \in \mathcal{A}}$ as a sum $U = \mathrm{pr}^{av}(U) + \mathrm{pr}^{no}(U)$ with $\mathcal{U}^{av} := \prod_{i \in \mathcal{A}} \mathcal{U}^{i,av}$, $\mathcal{U}^{no} := \prod_{i \in \mathcal{A}} \mathcal{U}^{i,no}$ and the projections $\mathrm{pr}^{av} := (\mathrm{pr}^{i,av})_{i \in \mathcal{A}} : \mathcal{U} \to \mathcal{U}^{av}$, $\mathrm{pr}^{no} := (\mathrm{pr}^{i,no})_{i \in \mathcal{A}} : \mathcal{U} \to \mathcal{U}^{no}$.

(iii) The union of the sets of Nash equilibria for all $U \in \mathcal{U}$ cannot easily be written analogously to (8.15), but it can be written analogously to (8.17). And this set can be parametrized analogously to (8.18). This is the content of Theorem 8.2.6. It is the main result in [KM86, section 3.2].

Theorem 8.2.6. *[KM86, section 3.2] The union of the sets* $\mathcal{N}(U) \subset G$ *of mixed Nash equilibria for all utility maps* $U = (U^1, \dots, U^m) \in \mathcal{U}$ *can be written in the following way. Here* $U \in \mathcal{U}$ *is characterized by the pair*

$(\mathrm{pr}^{av}(U), \mathrm{pr}^{no}(U))$.

$$\bigcup_{U \in \mathcal{U}} \mathcal{N}(U) \times \{\mathrm{pr}^{av}(U), \mathrm{pr}^{no}(U)\} \subset G \times \mathcal{U}^{av} \times \mathcal{U}^{no}. \tag{8.19}$$

This is parametrized by the map

$$\begin{aligned} \Psi : \mathbb{R}^S \times \mathcal{U}^{no} &\to G \times \mathcal{U}^{av} \times \mathcal{U}^{no}, \\ (\underline{a}, U^{no}) &\mapsto \Big(\big(\Phi_1^i(\underline{a}^i) \big)_{i \in \mathcal{A}}, \big(\Phi_2^i(\underline{a}^i) - V^{i,no}(\cdot, (\Phi_1^j(\underline{a}^j)))_{j \in \mathcal{A} \setminus \{i\}} \big)_{i \in \mathcal{A}}, U^{no} \Big). \end{aligned} \tag{8.20}$$

Ψ is continuous and injective. Consider the stratum $\left(\prod_{i \in \mathcal{A}} \mathbb{R}^{S^i, T^i}\right) \times \mathcal{U}^{no}$ in $\mathbb{R}^S \times \mathcal{U}^{no}$ for a tuple $(T^1, \dots, T^m)$ with $T^i \subset S^i$ and $T^i \neq \emptyset$. On such a stratum Ψ is an immersion, and it is multi affine linear with respect to the splitting into linear coordinates on $\mathbb{R}^{S^1}, \dots, \mathbb{R}^{S^m}, \mathcal{U}^{1,no}, \dots, \mathcal{U}^{m,no}$.

Proof. Consider a point $((\underline{\gamma}^i)_{i \in \mathcal{A}}, U^{av}, U^{no})$ in the left hand side of (8.19). Then $\underline{\gamma}^i$ is a best reply to the utility function $U^{i,av} + V^{i,no}(\cdot, (\underline{\gamma}^j)_{j \in \mathcal{A} \setminus \{i\}}) \in \mathbb{R}^{S^i}$. By Lemma 8.1.4 (e), there is a unique value $\underline{a}^i \in \mathbb{R}^{S^i}$ with $(\Phi_1^i(\underline{a}^i), \Phi_2^i(\underline{a}^i)) = (\underline{\gamma}^i, U^{i,av} + V^{i,no}(., (\underline{\gamma}^j)_{j \in \mathcal{A} \setminus \{i\}}))$. The pair $(\underline{a}, U^{no})$ is mapped by Ψ to the point $((\underline{\gamma}^i)_{i \in \mathcal{A}}, U^{av}, U^{no})$.

Vice versa, consider any pair $(\underline{a}, U^{no}) \in \mathbb{R}^S \times \mathcal{U}^{no}$ and denote its image under Ψ by $((\underline{\gamma}^i)_{i \in \mathcal{A}}, U^{av}, U^{no})$, and denote $U^i := U^{i,av} + U^{i,no}$. Then $V^i(\cdot, (\underline{\gamma}^j)_{j \in \mathcal{A} \setminus \{i\}})) = \Phi_2^i(\underline{a}^i)$, so $\underline{\gamma}^i = \Phi_1^i(\underline{a}^i)$ is a best reply to $V^i(\cdot, (\underline{\gamma}^j)_{j \in \mathcal{A} \setminus \{i\}})$. Therefore $((\underline{\gamma}^i)_{i \in \mathcal{A}}, U^{av}, U^{no})$ is in the left hand side of (8.19).

This shows that the left hand side of (8.19) is parametrized by Ψ.

The uniqueness of the value $\underline{a}^i \in \mathbb{R}^{S^i}$ in the first paragraph of this proof shows that Ψ is injective. On a stratum $\left(\prod_{i \in \mathcal{A}} \mathbb{R}^{S^i, T^i}\right) \times \mathcal{U}^{no}$ it is multi affine linear as claimed because Φ_1^i and Φ_2^i are affine linear on $\mathbb{R}^{S^i, T^i}$ by Lemma 8.1.4 (d). It remains to show that on this stratum the map in (8.1) is an immersion. We can consider a fixed U^{no} and the restricted map $\mathbb{R}^S \to G \times \mathcal{U}^{av}$. Its differential at a point $\underline{a}$ is block triangular. Due to the identity $\Phi_1^i(\underline{a}^i) + \Phi_2^i(\underline{a}^i) = \underline{a}^i$, it has maximal rank $|S|$. $\square$

Remark 8.2.7. The set in (8.19) and its parametrization in (8.20) can be restricted to $U \in (\mathrm{pr}^{no})^{-1}(U^{no})$ (instead of $U \in \mathcal{U}$) for a fixed normalized utility map U^{no}. Also this restricted map is continuous, injective, and piecewise multi affine linear and an immersion. Also Lemma 8.2.9 can be restricted in this way and remains correct.

Lemma 8.2.9 gives an amendment to Theorem 8.2.6 from [KM86, section 3.2]. To formulate it, we need Definition 8.2.8.

Definition 8.2.8. (i) Let X be a topological space which is locally compact and Hausdorff. Its one-point compactification X^* is the set $X^* = X \cup \{\infty\}$ whose open subsets are the open subsets of X and the complements of the compact subsets of X. The latter open sets are the open neighborhoods of ∞. It is known that X^* is compact and Hausdorff.

(ii) Denote by $\mathcal{N}_{\mathcal{U}^{av}\times\mathcal{U}^{no}}$ the set on the left hand side of (8.6), by $\mathrm{pr}_{\mathcal{N}} : \mathcal{N}_{\mathcal{U}^{av}\times\mathcal{U}^{no}} \to \mathcal{U}^{av} \times \mathcal{U}^{no}$ the projection, and by $\mathrm{pr}^*_{\mathcal{N}} : \mathcal{N}^*_{\mathcal{U}^{av}\times\mathcal{U}^{no}} \to (\mathcal{U}^{av} \times \mathcal{U}^{no})^*$ its extension to the one-point compactifications with $\mathrm{pr}^*_{\mathcal{N}}(\infty) = \infty$.

Lemma 8.2.9. *[KM86, section 3.2] The map $\mathrm{pr}_{\mathcal{N}} \circ \Psi : \mathbb{R}^S \times \mathcal{U}^{no} \to \mathbb{R}^S \times \mathcal{U}^{no}$ (here $\mathcal{U}^{av}$ is identified with $\mathbb{R}^S$) is homotopic to the identity under a homotopy that extends to $(\mathbb{R}^S \times \mathcal{U}^{no})^*$ and maps ∞ to ∞.*

Proof. The following proof is a copy of the proof in [KM86, section 3.2]. Write $\Psi = (\Phi_1, \Psi_2, \mathrm{id}_{\mathcal{U}^{no}})$ where Ψ_2 has values in $\mathbb{R}^S$. The homotopy is given by

$$H_t := (t\Psi_2 + (1-t)\,\mathrm{id}_{\mathbb{R}^S}, \mathrm{id}_{\mathcal{U}^{no}}) \quad \text{for } t \in [0,1].$$

Obviously $H_0 = (\mathrm{id}_{\mathbb{R}^S}, \mathrm{id}_{\mathcal{U}^{no}})$, $H_1 = (\Psi_2, \mathrm{id}_{\mathcal{U}^{no}}) = \mathrm{pr}_{\mathcal{N}} \circ \Psi$ and $H : [0,1] \times \mathbb{R}^S \times \mathcal{U}^{no} \to \mathbb{R}^S \times \mathcal{U}^{no}$ is continuous.

It remains to see the continuity near ∞. It is sufficient to show for any bound $b > 0$ that $\|(\underline{a}, U^{no})\|_\infty > 2b+1$ implies $\|H_t(\underline{a}, U^{no})\|_\infty > b$ for any $t \in [0,1]$.

Choose a bound $b > 0$ and consider $(\underline{a}, U^{no})$ with $\|(\underline{a}, U^{no})\|_\infty > 2b+1$. If $\|U^{no}\|_\infty > b$, we are ready. Suppose $\|U^{no}\|_\infty \leq b$. Then $\|\underline{a}\|_\infty > 2b+1$. We have

$$\begin{aligned}
&\|(t\Psi_2 + (1-t)\,\mathrm{id}_{\mathbb{R}^S})(\underline{a}, U^{no})\|_\infty \\
\geq &\|\underline{a}\|_\infty - t\|(\Psi_2 - \mathrm{id}_{\mathbb{R}^S})(\underline{a}, U^{no})\|_\infty \\
> &(2b+1) - \|(\Phi_1^i(\underline{a}) - V^{i,no}(\cdot, \Phi_1^j(\underline{a}^j))_{j\in\mathcal{A}\setminus\{i\}})_{i\in\mathcal{A}}\|_\infty \\
> &(2b+1) - (1 + \|U^{no}\|_\infty) \geq (2b+1) - (1+b) = b.
\end{aligned}$$

□

9 Outlook

Consider a pre-tropical game $(\mathcal{A}, G, V)$ with m players. Recall Definition 4.2.1 (c) that C_l^P is the set of mixed strategy combinations $g \in G$ such that the strategies of l players are pure,

$$C_l^P = \left\{ g \in G \;\middle|\; \left| \left\{ i \in \mathcal{A} | \; \gamma^i \in \{0,1\} \right\} \right| = l \right\}$$

In the case of an inner tropical game Theorem 4.2.6 gives for C_l^P-equilibria the upper bounds

$$\begin{aligned} |\mathcal{N}(V) \cap C_l^P| &\leq \; 2^{l-1} \binom{m}{l} \cdot !(m-i) && \text{for } l \geq 1, \\ |\mathcal{N}(V) \cap C_0^P| &\leq \, !m && \text{for } l = 0. \end{aligned}$$

The fundamental theorem for inner tropical games tells that inner tropical games exist where all these inequalities are binding. These are the maximal inner tropical games.

Though for other generic pre-tropical games, these inequalities do not necessarily hold. Our observation from special cases is that when we deform an inner tropical game to a pre-tropical game which is not inner tropical, then the C_l^P-type of an equilibrium can change. We conjecture (optimistically) that the maximal number of all Nash equibria within inner tropical games is also the maximal number of all Nash equilibria within all pre-tropical games. We even conjecture a stronger *semicontinuity* for the possible changes of C_l^P-types: We conjecture that for each generic pre-tropical game and for each $d \in \{0, 1, ..., m\}$, the total number of Nash equilibria of types C_l^P with $l \in \{0, 1, ..., d\}$ cannot be bigger than for the maximal inner tropical games.

Conjecture 9.0.1. *Let $(\mathcal{A}, G, V)$ be a generic pre-tropical game. Then for* *each* $d \in \{1, 2 \ldots, m\}$

$$\sum_{l=0}^{d} |\mathcal{N}(V) \cap C_l^P| \leq !m + \sum_{l=1}^{d} 2^{l-1} \binom{m}{l} \cdot !(m-l).$$

Example 9.0.2. The bound for $l \geq 1$

$$|\mathcal{N}(V) \cap C_l^P| \leq 2^{l-1} \binom{m}{l} \cdot !(m-l)$$

for inner tropical games is wrong in general for other generic pre-tropical games. There exists a pre-tropical 3 player game with

$$|\mathcal{N}(V) \cap C_1^P| = 5 > 2^0 \binom{3}{1} !(3-1) = 1 \cdot 3 \cdot 1 = 3,$$

see [Vuj19, p.36]. Let

$$\begin{aligned}
\lambda^1(\gamma^2, \gamma^3) &= (\gamma^2 - \frac{1}{6}) \cdot (\gamma^3 - \frac{1}{7}) - \frac{1}{10000} \\
\lambda^2(\gamma^1, \gamma^3) &= (-1) \cdot \left((\gamma^1 - \frac{1}{4}) \cdot (\gamma^3 - \frac{1}{5}) - \frac{1}{10000} \right) \\
\lambda^3(\gamma^1, \gamma^2) &= (\gamma^1 - \frac{1}{2}) \cdot (\gamma^2 - \frac{1}{3}) - \frac{1}{11},
\end{aligned}$$

then the set of virtual C_1^P-candidates is given by

$$\begin{aligned}
\mathcal{C} = \Bigg\{ &\left(0, \frac{5}{33}, \frac{499}{2500}\right), \left(1, \frac{17}{33}, \frac{1501}{7500}\right), \\
&\left(\frac{5}{22}, 0, \frac{4979}{35000}\right), \left(\frac{7}{11}, 1, \frac{25021}{175000}\right), \\
&\left(\frac{499}{2000}, \frac{4979}{30000}, 0\right), \left(\frac{2001}{8000}, \frac{10007}{60000}, 1\right) \Bigg\}.
\end{aligned}$$

One checks that all but the last one are Nash equilibria.

Appendices

A C_2^P-density for $m = 6$

This appendix refers to the part of Chapter 6 between Remark

Let us calculate the C_2^P-densities for $m = 6$. The following four tables calculate the partial densities for all player-invariance tuples $(6, q)$ with $0 \leq q \leq 3$. The partial densities are all rational. Let us calculate the smallest positive integer Θ_6 such that $\Theta_6 \cdot \psi_{\Sigma(6,q)}$ only takes integer values. An easy calculation shows that $\Theta_6 = 7776000000$.

$\psi_{\Sigma(n,q)}$ below are the 4 partial densities needed to calculate the full density. The following equation is trivial.

$$\begin{aligned}
\psi_{\mathcal{W}(6)} &\equiv \frac{1}{2^6} \sum_{q=0}^{6} \binom{6}{q} \cdot \psi_{\Sigma(6,q)} \\
&\equiv \frac{1}{64} \left(\left(\sum_{q=0}^{2} 2 \cdot \binom{6}{q} \cdot \psi_{\Sigma(6,q)} \right) + 20 \cdot \psi_{\Sigma(6,3)} \right) \\
&\equiv \frac{1}{32} \Big(\psi_{\Sigma(6,0)} + 6 \cdot \psi_{\Sigma(6,1)} + 15 \cdot \psi_{\Sigma(6,2)} + 10 \cdot \psi_{\Sigma(6,3)} \Big)
\end{aligned}$$

x	$\Theta_6 \cdot \psi_{\Sigma(n,0)}(x)$
1	0
3	0
5	0
7	0
9	0
11	0
13	0
15	0
17	0
19	0
21	480
23	0
25	360
27	5760
29	1560
31	9120
33	49720
35	42360
37	175800
39	456520
41	756720
43	1967010
45	4455420
47	8834490
49	18682980
51	37789140
53	69126120
55	126188220
57	213537560
59	335856120
61	490379100
63	670353440
65	817711020
67	925674660

x	$\Theta_6 \cdot \psi_{\Sigma(n,0)}(x)$
69	938580400
71	868772790
73	723374100
75	560741930
77	391543920
79	251879610
81	153560000
83	82063080
85	42418380
87	21587290
89	10516320
91	4417740
93	2658320
95	983790
97	388080
99	294270
101	95040
103	23430
105	31780
107	7260
109	1800
111	4470
113	0
115	750
117	1740
119	0
121	0
123	0
125	0
127	0
129	0
131	0
133	0
135	30

x	$\Theta_6 \cdot \psi_{\Sigma(n,1)}(x)$
0	0
2	0
4	0
6	0
8	0
10	0
12	0
14	0
16	0
18	1450
20	360
22	0
24	6320
26	2560
28	11120
30	45580
32	28220
34	129260
36	323960
38	469600
40	1180250
42	2540290
44	4483450
46	9713400
48	19428440
50	35119490
52	66887000
54	117408710
56	197465020
58	305509310
60	461587290
62	610185000
64	782586210
66	881792410

x	$\Theta_6 \cdot \psi_{\Sigma(n,1)}(x)$
68	938197210
70	871456180
72	773176390
74	592163550
76	437207540
78	285039680
80	178313000
82	97214940
84	55506750
86	26377460
88	12734770
90	6501030
92	2838270
94	1164800
96	719430
98	279930
100	76300
102	87990
104	19030
106	4940
108	12430
110	1140
112	1000
114	1400
116	0
118	0
120	0
122	0
124	0
126	140
128	0
130	0
132	0
134	0

x	$\Theta_6 \cdot \psi_{\Sigma(n,2)}(x)$	x	$\Theta_6 \cdot \psi_{\Sigma(n,2)}(x)$
1	0	69	897624528
3	0	71	811086246
5	0	73	651000924
7	0	75	495351144
9	416	77	333163728
11	0	79	212077224
13	0	81	125234236
15	0	83	67598706
17	0	85	34234688
19	0	87	17739474
21	4376	89	8355720
23	3208	91	3565964
25	3864	93	2068516
27	31444	95	787740
29	11272	97	292492
31	52136	99	224156
33	170768	101	68900
35	181200	103	15702
37	564840	105	24136
39	1192092	107	4948
41	1881800	109	1232
43	4321216	111	2834
45	8616136	113	0
47	15921164	115	312
49	30505420	117	1032
51	58242388	119	0
53	98399548	121	0
55	170761044	123	0
57	269162860	125	0
59	405110344	127	0
61	555518324	129	0
63	728476726	131	0
65	843221060	133	0
67	923121762	135	10

x	$\Theta_6 \cdot \psi_{\Sigma(n,3)}(x)$	x	$\Theta_6 \cdot \psi_{\Sigma(n,3)}(x)$
0	74	68	924003864
2	0	70	848381403
4	0	72	736878020
6	0	74	558769356
8	0	76	404630838
10	0	78	261902356
12	0	80	161156448
14	0	82	87737784
16	0	84	49499114
18	3753	86	23673438
20	1620	88	11358930
22	0	90	5780133
24	11448	92	2527824
26	3882	94	1026279
28	18300	96	639802
30	77340	98	241404
32	53994	100	64770
34	206976	102	75274
36	479244	104	15924
38	688398	106	3900
40	1653912	108	10726
42	3489998	110	795
44	5955444	112	732
46	12549342	114	972
48	24145470	116	0
50	43069437	118	0
52	79019064	120	0
54	136709231	122	0
56	222116034	124	0
58	338187507	126	108
60	494305576	128	0
62	643480308	130	0
64	801303324	132	0
66	890090130	134	0

B Equivalence classes with non-trivial isotropy groups

This appendix refers to the last part of Section 6.4.

We are interested in calculating the isotropy group up to isomorphism. C_n denotes the cyclic group of order n.

B.1 m = 4

First the case $m = 4$. In this case we have 10 orbits with non-trivial isotropy groups. One is $\simeq C_2 \times C_2 \simeq K_4$, three are $\simeq C_4$ and the other six are $\simeq C_2$.

The following orbit has isotropy groups which are isomorphic to $C_2 \times C_2$.

σ^1	σ^2	σ^3	σ^4
id	(3 4)	(1 2 4)	(1 3)
(3 4)	id	(1 4)	(1 2 3)
(2 3)	(1 3 4)	(2 4)	(1 3 2)
(2 3 4)	(1 3)	(1 4 2)	(2 3)
(2 4 3)	(1 4)	id	(1 2)
(2 4)	(1 4 3)	(1 2)	id

The following three orbits have isotropy groups which are isomorphic to C_4.

σ^1	σ^2	σ^3	σ^4
id	(3 4)	(1 4)	(1 2 3)
(3 4)	id	(1 2 4)	(1 3)
(2 3)	(1 4)	(2 4)	(1 2)
(2 3 4)	(1 4 3)	(1 2)	(2 3)
(2 4 3)	(1 3 4)	id	(1 3 2)
(2 4)	(1 3)	(1 4 2)	id

σ^1	σ^2	σ^3	σ^4
id	(1 3 4)	(2 4)	(1 3)
(3 4)	(1 3)	(1 4)	(2 3)
(2 3)	(3 4)	(1 2 4)	(1 3 2)
(2 3 4)	id	(1 4 2)	(1 2 3)
(2 4 3)	(1 4)	(1 2)	id
(2 4)	(1 4 3)	id	(1 2)

σ^1	σ^2	σ^3	σ^4
id	(1 4 3)	(1 2 4)	id
(3 4)	(1 4)	id	(1 2 3)
(2 3)	(1 3 4)	(1 4 2)	(2 3)
(2 3 4)	(1 3)	(2 4)	(1 3 2)
(2 4 3)	id	(1 4)	(1 2)
(2 4)	(3 4)	(1 2)	(1 3)

The following six orbits have isotropy groups which are isomorphic to C_2.

σ^1	σ^2	σ^3	σ^4
id	(3 4)	(1 2)	id
id	(1 4 3)	(1 2)	(1 3)
(3 4)	id	id	(1 2)
(3 4)	(1 4)	(1 4)	(1 2)
(2 3)	(1 3)	(2 4)	(2 3)
(2 3)	(1 3)	(1 4 2)	(1 3 2)
(2 3 4)	(1 3 4)	(2 4)	(2 3)
(2 3 4)	(1 3 4)	(1 4 2)	(1 3 2)
(2 4 3)	id	id	(1 2 3)
(2 4 3)	(1 4)	(1 4)	(1 2 3)
(2 4)	(3 4)	(1 2 4)	id
(2 4)	(1 4 3)	(1 2 4)	(1 3)

σ^1	σ^2	σ^3	σ^4
id	(3 4)	(1 4 2)	(2 3)
id	(1 4)	(1 2 4)	(1 2)
(3 4)	id	(2 4)	(1 3 2)
(3 4)	(1 4 3)	(1 2)	(1 2 3)
(2 3)	(1 3 4)	(1 4)	(1 2 3)
(2 3)	(1 4 3)	(2 4)	id
(2 3 4)	(1 3)	(1 2 4)	(1 3)
(2 3 4)	(1 4)	id	(2 3)
(2 4 3)	(3 4)	id	(1 3)
(2 4 3)	(1 3)	(1 4 2)	(1 2)
(2 4)	id	(1 4)	id
(2 4)	(1 3 4)	(1 2)	(1 3 2)

σ^1	σ^2	σ^3	σ^4
id	(1 3)	(1 2 4)	(2 3)
id	(1 4)	id	(1 3)
(3 4)	(1 3 4)	(2 4)	(1 2 3)
(3 4)	(1 4 3)	(1 4)	id
(2 3)	id	(1 4)	(1 3 2)
(2 3)	(1 3 4)	(1 2)	id
(2 3 4)	(3 4)	(1 4 2)	(1 3)
(2 3 4)	(1 3)	id	(1 2)
(2 4 3)	(3 4)	(1 2 4)	(1 2)
(2 4 3)	(1 4)	(1 4 2)	(2 3)
(2 4)	id	(1 2)	(1 2 3)
(2 4)	(1 4 3)	(2 4)	(1 3 2)

σ^1	σ^2	σ^3	σ^4
id	id	(1 2 4)	(1 2 3)
id	id	(1 4)	(1 3)
(3 4)	(3 4)	(1 2 4)	(1 2 3)
(3 4)	(3 4)	(1 4)	(1 3)
(2 3)	(1 3 4)	id	(1 2)
(2 3)	(1 4)	id	(1 3 2)
(2 3 4)	(1 3)	(1 2)	id
(2 3 4)	(1 4 3)	(1 4 2)	id
(2 4 3)	(1 3 4)	(2 4)	(1 2)
(2 4 3)	(1 4)	(2 4)	(1 3 2)
(2 4)	(1 3)	(1 2)	(2 3)
(2 4)	(1 4 3)	(1 4 2)	(2 3)

σ^1	σ^2	σ^3	σ^4
id	(3 4)	id	(1 2)
id	(1 3)	(1 4 2)	(1 3)
(3 4)	id	(1 2)	id
(3 4)	(1 3 4)	(1 4)	(1 3 2)
(2 3)	id	(2 4)	(1 2 3)
(2 3)	(1 4 3)	(1 2)	(1 3 2)
(2 3 4)	(3 4)	(1 2 4)	(2 3)
(2 3 4)	(1 4)	(1 4 2)	(1 2)
(2 4 3)	(1 3)	id	(2 3)
(2 4 3)	(1 4)	(1 2 4)	(1 3)
(2 4)	(1 3 4)	(2 4)	id
(2 4)	(1 4 3)	(1 4)	(1 2 3)

σ^1	σ^2	σ^3	σ^4
id	(3 4)	(2 4)	(1 3 2)
id	(1 3 4)	(1 2 4)	(1 3 2)
(3 4)	id	(1 4 2)	(2 3)
(3 4)	(1 3)	(1 4 2)	(1 2 3)
(2 3)	(3 4)	(2 4)	(1 3)
(2 3)	(1 3 4)	(1 2 4)	(1 3)
(2 3 4)	id	(1 4)	(2 3)
(2 3 4)	(1 3)	(1 4)	(1 2 3)
(2 4 3)	(1 4 3)	id	id
(2 4 3)	(1 4 3)	(1 2)	(1 2)
(2 4)	(1 4)	id	id
(2 4)	(1 4)	(1 2)	(1 2)

B.2 m = 5

For $m = 5$ we give all orbits with non-trivial isotropy groups. The isotropy groups here are all isomorphic to C_5.

σ^1	σ^2	σ^3	σ^4	σ^5
id	(1 3 4 5)	(2 5 4)	(1 5 2 3)	(1 3 4)
(4 5)	(1 3 4)	(2 5)	(1 3 5)	(1 4 2 3)
(3 4)	(1 3 5)	(1 5 2 4)	(2 5 3)	(1 3)
(3 4 5)	(1 3 5 4)	(1 5)(2 4)	(1 3)	(2 4 3)
(3 5 4)	(1 3)	(1 4 5)	(2 5)	(1 4)(2 3)
(3 5)	(1 3)(4 5)	(1 4)	(1 5)(2 3)	(2 4)
(2 3)	(3 5 4)	(1 2 4 5)	(1 5 2)	(1 3 2 4)
(2 3)(4 5)	(3 5)	(1 2 4)	(1 3 2 5)	(1 4 2)
(2 3 4)	(4 5)	(1 5 2)	(1 2 3 5)	(1 3)(2 4)
(2 3 4 5)	id	(1 5 4 2)	(1 3)(2 5)	(1 2 3 4)
(2 3 5 4)	(3 4 5)	(1 4 2 5)	(1 2 3)	(1 4 3 2)
(2 3 5)	(3 4)	(1 4)(2 5)	(1 5 3 2)	(1 2 3)
(2 4 3)	(1 5 3 4)	(1 2 5)	(3 5)	(1 3 2)
(2 4 5 3)	(1 5)(3 4)	(1 2 5 4)	(1 3 2)	(3 4)
(2 4)	(1 5 3)	(4 5)	(1 2 5)	(1 3 4 2)
(2 4 5)	(1 5 4 3)	id	(1 3 5 2)	(1 2 4)
(2 4)(3 5)	(1 5)	(1 4 2)	(1 2 5 3)	id
(2 4 3 5)	(1 5 4)	(1 4 5 2)	id	(1 2 4 3)
(2 5 4 3)	(1 4 5)	(1 2)	(2 3 5)	(1 4)
(2 5 3)	(1 4)	(1 2)(4 5)	(1 5)	(2 3 4)
(2 5 4)	(1 4 3 5)	(2 4 5)	(1 2)	(1 4 3)
(2 5)	(1 4)(3 5)	(2 4)	(1 5 3)	(1 2)
(2 5 3 4)	(1 4 3)	(1 5)	(1 2)(3 5)	(2 3)
(2 5)(3 4)	(1 4 5 3)	(1 5 4)	(2 3)	(1 2)(3 4)

σ^1	σ^2	σ^3	σ^4	σ^5
id	(1 3 5 4)	(2 4 5)	(1 5)	(1 3)(2 4)
(4 5)	(1 3 5)	(2 4)	(1 3)(2 5)	(1 4)
(3 4)	(1 3)(4 5)	(1 5)	(2 3 5)	(1 3 2 4)
(3 4 5)	(1 3)	(1 5 4)	(1 3 2 5)	(2 3 4)
(3 5 4)	(1 3 4 5)	(1 4)(2 5)	(2 3)	(1 4 3)
(3 5)	(1 3 4)	(1 4 2 5)	(1 5 3)	(2 3)
(2 3)	(3 4 5)	(1 2 5 4)	(1 5)(2 3)	(1 3 4 2)
(2 3)(4 5)	(3 4)	(1 2 5)	(1 3 5 2)	(1 4)(2 3)
(2 3 4)	(3 5)	(1 5)(2 4)	(1 2 5 3)	(1 3 2)
(2 3 4 5)	(3 5 4)	(1 5 2 4)	(1 3 2)	(1 2 4 3)
(2 3 5 4)	id	(1 4 5 2)	(1 2 5)	(1 4 2 3)
(2 3 5)	(4 5)	(1 4 2)	(1 5 2 3)	(1 2 4)
(2 4 3)	(1 5)	(1 2)(4 5)	(2 5)	(1 3 4)
(2 4 5 3)	(1 5 4)	(1 2)	(1 3 5)	(2 4)
(2 4)	(1 5)(3 4)	(2 5)	(1 2)(3 5)	(1 3)
(2 4 5)	(1 5 3 4)	(2 5 4)	(1 3)	(1 2)(3 4)
(2 4)(3 5)	(1 5 4 3)	(1 4 5)	(1 2)	(2 4 3)
(2 4 3 5)	(1 5 3)	(1 4)	(2 5 3)	(1 2)
(2 5 4 3)	(1 4)(3 5)	(1 2 4 5)	id	(1 4 3 2)
(2 5 3)	(1 4 3 5)	(1 2 4)	(1 5 3 2)	id
(2 5 4)	(1 4 5 3)	id	(1 2 3 5)	(1 4 2)
(2 5)	(1 4 3)	(4 5)	(1 5 2)	(1 2 3 4)
(2 5 3 4)	(1 4 5)	(1 5 4 2)	(1 2 3)	(3 4)
(2 5)(3 4)	(1 4)	(1 5 2)	(3 5)	(1 2 3)

σ^1	σ^2	σ^3	σ^4	σ^5
id	(1 4 5 3)	(1 5)(2 4)	(3 5)	(1 2 4)
(4 5)	(1 4 3)	(1 5 2 4)	(1 2 5)	(3 4)
(3 4)	(1 4)(3 5)	(4 5)	(1 5)(2 3)	(1 2 4 3)
(3 4 5)	(1 4 3 5)	id	(1 2 5 3)	(1 4)(2 3)
(3 5 4)	(1 4)	(1 2 5)	(1 5 2 3)	id
(3 5)	(1 4 5)	(1 2 5 4)	id	(1 4 2 3)
(2 3)	(1 5)(3 4)	(1 4 5 2)	(2 5 3)	(1 2 3 4)
(2 3)(4 5)	(1 5 3 4)	(1 4 2)	(1 2 3 5)	(2 4 3)
(2 3 4)	(1 5)	(2 5 4)	(1 3 5 2)	(1 2 3)
(2 3 4 5)	(1 5 4)	(2 5)	(1 2 3)	(1 3 4 2)
(2 3 5 4)	(1 5 3)	(1 2 4 5)	(1 3 2)	(2 4)
(2 3 5)	(1 5 4 3)	(1 2 4)	(2 5)	(1 3 2)
(2 4 3)	(4 5)	(1 4)(2 5)	(1 5)	(1 2)(3 4)
(2 4 5 3)	id	(1 4 2 5)	(1 2)(3 5)	(1 4)
(2 4)	(3 5 4)	(1 5)	(1 3)(2 5)	(1 2)
(2 4 5)	(3 5)	(1 5 4)	(1 2)	(1 3)(2 4)
(2 4)(3 5)	(3 4)	(1 2)(4 5)	(1 3 2 5)	(1 4 3)
(2 4 3 5)	(3 4 5)	(1 2)	(1 5 3)	(1 3 2 4)
(2 5 4 3)	(1 3 5)	(1 4)	(1 5 2)	(2 3)
(2 5 3)	(1 3 5 4)	(1 4 5)	(2 3)	(1 4 2)
(2 5 4)	(1 3 4 5)	(1 5 2)	(1 3)	(2 3 4)
(2 5)	(1 3 4)	(1 5 4 2)	(2 3 5)	(1 3)
(2 5 3 4)	(1 3)(4 5)	(2 4)	(1 3 5)	(1 4 3 2)
(2 5)(3 4)	(1 3)	(2 4 5)	(1 5 3 2)	(1 3 4)

σ^1	σ^2	σ^3	σ^4	σ^5
id	(1 4 3 5)	(1 5 4)	(2 5)	(1 2 3)
(4 5)	(1 4)(3 5)	(1 5)	(1 2 3)	(2 4)
(3 4)	(1 4 5)	(2 5)	(1 5 3)	(1 2 3 4)
(3 4 5)	(1 4)	(2 5 4)	(1 2 3 5)	(1 4 3)
(3 5 4)	(1 4 5 3)	(1 2 4)	(1 5)	(2 4 3)
(3 5)	(1 4 3)	(1 2 4 5)	(2 5 3)	(1 4)
(2 3)	(1 5 4)	(1 4 2 5)	(2 3 5)	(1 2)
(2 3)(4 5)	(1 5)	(1 4)(2 5)	(1 2)	(2 3 4)
(2 3 4)	(1 5 4 3)	(2 4 5)	(1 3 2 5)	(1 2)(3 4)
(2 3 4 5)	(1 5 3)	(2 4)	(1 2)(3 5)	(1 3 2 4)
(2 3 5 4)	(1 5)(3 4)	(1 2)	(1 3)(2 5)	(2 3)
(2 3 5)	(1 5 3 4)	(1 2)(4 5)	(2 3)	(1 3)(2 4)
(2 4 3)	(3 5)	(1 4 5)	(1 5 3 2)	(1 2 4)
(2 4 5 3)	(3 5 4)	(1 4)	(1 2 5)	(1 4 3 2)
(2 4)	(3 4 5)	(1 5 4 2)	(1 3 5)	(1 2 4 3)
(2 4 5)	(3 4)	(1 5 2)	(1 2 5 3)	(1 3 4)
(2 4)(3 5)	(4 5)	(1 2 5)	(1 3)	(1 4 2)
(2 4 3 5)	id	(1 2 5 4)	(1 5 2)	(1 3)
(2 5 4 3)	(1 3 4)	(1 4 5 2)	(1 5)(2 3)	(3 4)
(2 5 3)	(1 3 4 5)	(1 4 2)	(3 5)	(1 4)(2 3)
(2 5 4)	(1 3)	(1 5)(2 4)	(1 3 5 2)	id
(2 5)	(1 3)(4 5)	(1 5 2 4)	id	(1 3 4 2)
(2 5 3 4)	(1 3 5)	(4 5)	(1 3 2)	(1 4 2 3)
(2 5)(3 4)	(1 3 5 4)	id	(1 5 2 3)	(1 3 2)

σ^1	σ^2	σ^3	σ^4	σ^5
id	(1 5 4 3)	(1 4)(2 5)	(1 2 3 5)	id
(4 5)	(1 5 3)	(1 4 2 5)	id	(1 2 3 4)
(3 4)	(1 5 4)	(1 2 4 5)	(1 3)(2 5)	(3 4)
(3 4 5)	(1 5)	(1 2 4)	(3 5)	(1 3)(2 4)
(3 5 4)	(1 5 3 4)	id	(1 3 2 5)	(1 2 3)
(3 5)	(1 5)(3 4)	(4 5)	(1 2 3)	(1 3 2 4)
(2 3)	(1 4)(3 5)	(1 5 4 2)	(1 2 5)	(2 3)
(2 3)(4 5)	(1 4 3 5)	(1 5 2)	(2 3)	(1 2 4)
(2 3 4)	(1 4 5 3)	(1 2 5)	(1 5 3 2)	(2 3 4)
(2 3 4 5)	(1 4 3)	(1 2 5 4)	(2 3 5)	(1 4 3 2)
(2 3 5 4)	(1 4 5)	(2 4)	(1 5 2)	(1 2 4 3)
(2 3 5)	(1 4)	(2 4 5)	(1 2 5 3)	(1 4 2)
(2 4 3)	(1 3 4 5)	(1 5 4)	(1 3 5 2)	(2 4 3)
(2 4 5 3)	(1 3 4)	(1 5)	(2 5 3)	(1 3 4 2)
(2 4)	(1 3 5)	(1 4 5 2)	(1 5 3)	(2 4)
(2 4 5)	(1 3 5 4)	(1 4 2)	(2 5)	(1 4 3)
(2 4)(3 5)	(1 3)	(2 5 4)	(1 5)	(1 3 2)
(2 4 3 5)	(1 3)(4 5)	(2 5)	(1 3 2)	(1 4)
(2 5 4 3)	id	(1 5 2 4)	(1 3 5)	(1 2)
(2 5 3)	(4 5)	(1 5)(2 4)	(1 2)	(1 3 4)
(2 5 4)	(3 4)	(1 4 5)	(1 5 2 3)	(1 2)(3 4)
(2 5)	(3 4 5)	(1 4)	(1 2)(3 5)	(1 4 2 3)
(2 5 3 4)	(3 5 4)	(1 2)	(1 5)(2 3)	(1 3)
(2 5)(3 4)	(3 5)	(1 2)(4 5)	(1 3)	(1 4)(2 3)

σ^1	σ^2	σ^3	σ^4	σ^5
id	(1 5 3 4)	(1 4 5)	(1 2 5 3)	(2 3 4)
(4 5)	(1 5)(3 4)	(1 4)	(2 3 5)	(1 2 4 3)
(3 4)	(1 5 3)	(1 2 5 4)	(1 3 5)	(2 3)
(3 4 5)	(1 5 4 3)	(1 2 5)	(2 3)	(1 3 4)
(3 5 4)	(1 5)	(2 4 5)	(1 3)	(1 2 4)
(3 5)	(1 5 4)	(2 4)	(1 2 5)	(1 3)
(2 3)	(1 4 5)	(1 5 2 4)	(1 2)(3 5)	(2 4)
(2 3)(4 5)	(1 4)	(1 5)(2 4)	(2 5)	(1 2)(3 4)
(2 3 4)	(1 4 3 5)	(1 2)(4 5)	(1 5 2 3)	(2 4 3)
(2 3 4 5)	(1 4)(3 5)	(1 2)	(2 5 3)	(1 4 2 3)
(2 3 5 4)	(1 4 3)	(2 5)	(1 5)(2 3)	(1 2)
(2 3 5)	(1 4 5 3)	(2 5 4)	(1 2)	(1 4)(2 3)
(2 4 3)	(1 3 5 4)	(1 5 2)	(1 3 2 5)	id
(2 4 5 3)	(1 3 5)	(1 5 4 2)	id	(1 3 2 4)
(2 4)	(1 3)(4 5)	(1 4 2 5)	(1 5 2)	(3 4)
(2 4 5)	(1 3)	(1 4)(2 5)	(3 5)	(1 4 2)
(2 4)(3 5)	(1 3 4 5)	id	(1 5 3 2)	(1 3)(2 4)
(2 4 3 5)	(1 3 4)	(4 5)	(1 3)(2 5)	(1 4 3 2)
(2 5 4 3)	(3 4 5)	(1 5)	(1 3 2)	(1 2 3 4)
(2 5 3)	(3 4)	(1 5 4)	(1 2 3 5)	(1 3 2)
(2 5 4)	(3 5)	(1 4 2)	(1 5)	(1 2 3)
(2 5)	(3 5 4)	(1 4 5 2)	(1 2 3)	(1 4)
(2 5 3 4)	id	(1 2 4 5)	(1 5 3)	(1 3 4 2)
(2 5)(3 4)	(4 5)	(1 2 4)	(1 3 5 2)	(1 4 3)

C Information on von Stengel's game for $m = 6$

This appendix refers to Section 7.3. Explanation of colors:

	Non-virtual equilibrium of a game (Nash equilibrium)
	Non-virtual candidate of a game (strategy combination, not a Nash equilibrium)
	Virtual equilibrium (not a strategy combination, but equilibrium condition holds)
	Virtual candidate (not a strategy combination, equilibrium condition does not hold)
	$\mathcal{C}(\mathcal{T}_1, \mathcal{T}_2) = \emptyset$, the game is not type-0-generic for $(\mathcal{T}_1, \mathcal{T}_2)$
$*$	$\underline{\gamma}$ is not type-1-generic

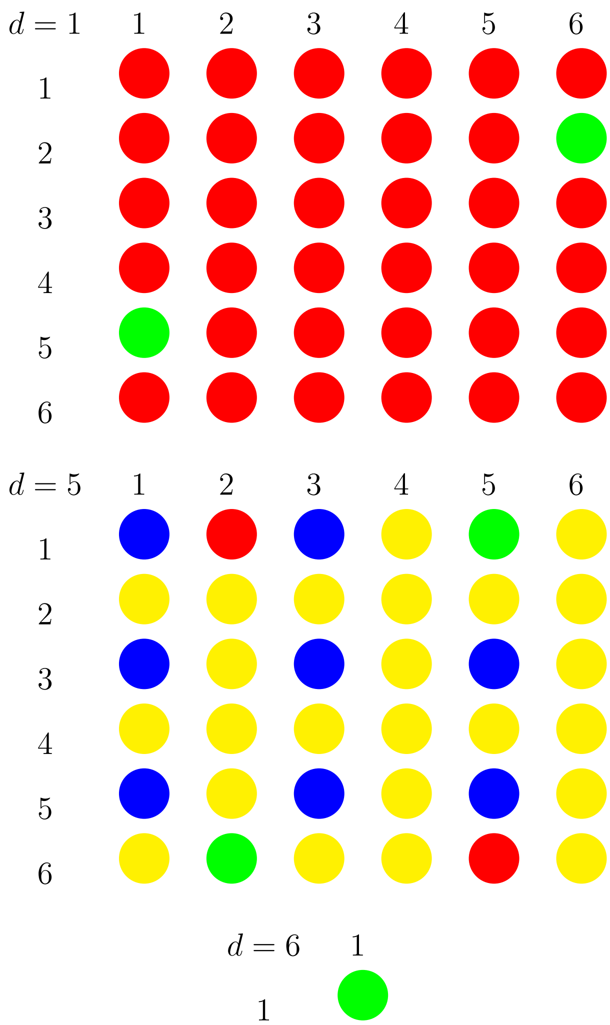
Observe that $N^1 = N^2 = \{1,2,3,4,5,6\} := N$ The following tables list all possible combinations of $\emptyset \neq \mathcal{T}_1 \subset N$ and $\emptyset \neq \mathcal{T}_2 \subset N$ with $|\mathcal{T}_1| = |\mathcal{T}_2| =: d$. Let $\mathcal{O}_d := \{x \in \mathcal{P}(N) \mid d = |x|\}$ The entry of the d-th table with row index i and column index j corresponds to the i-th lexicographically ordered element $\mathcal{T}_1 \in \mathcal{O}_d$ and the j-th lexicographically ordered element $\mathcal{T}_2 \in \mathcal{O}_d$.

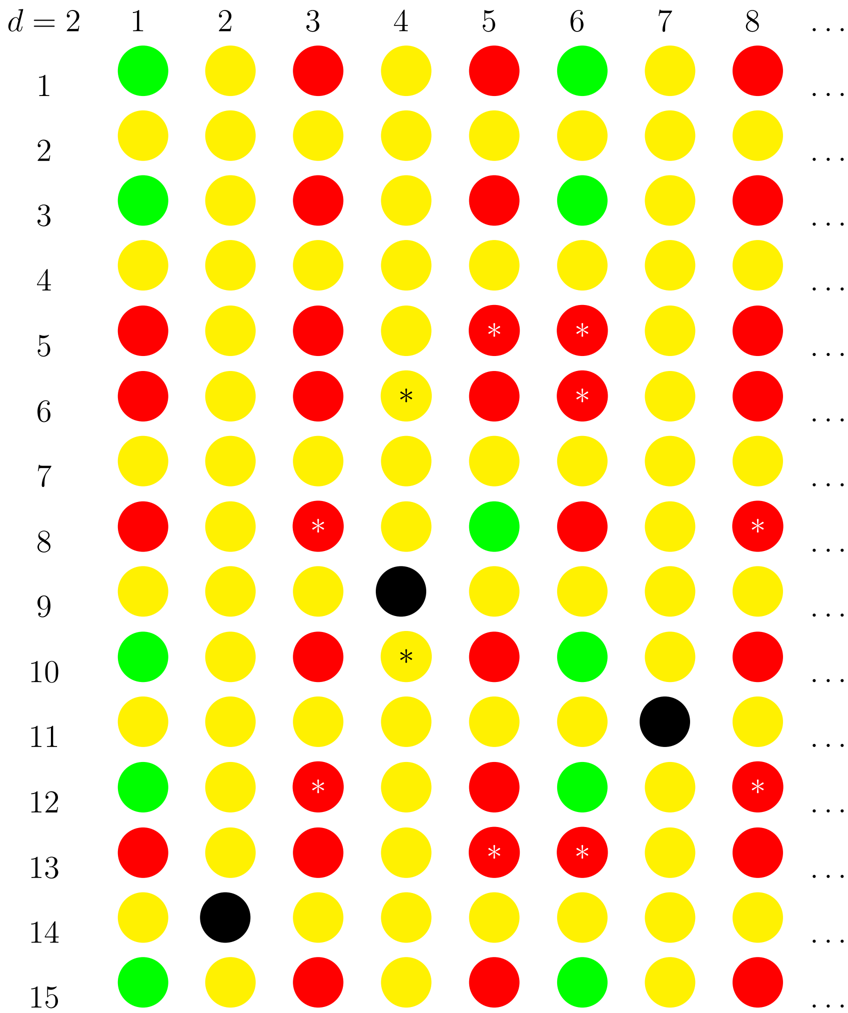
For example $\mathcal{O}_3$ with lexicographic order:

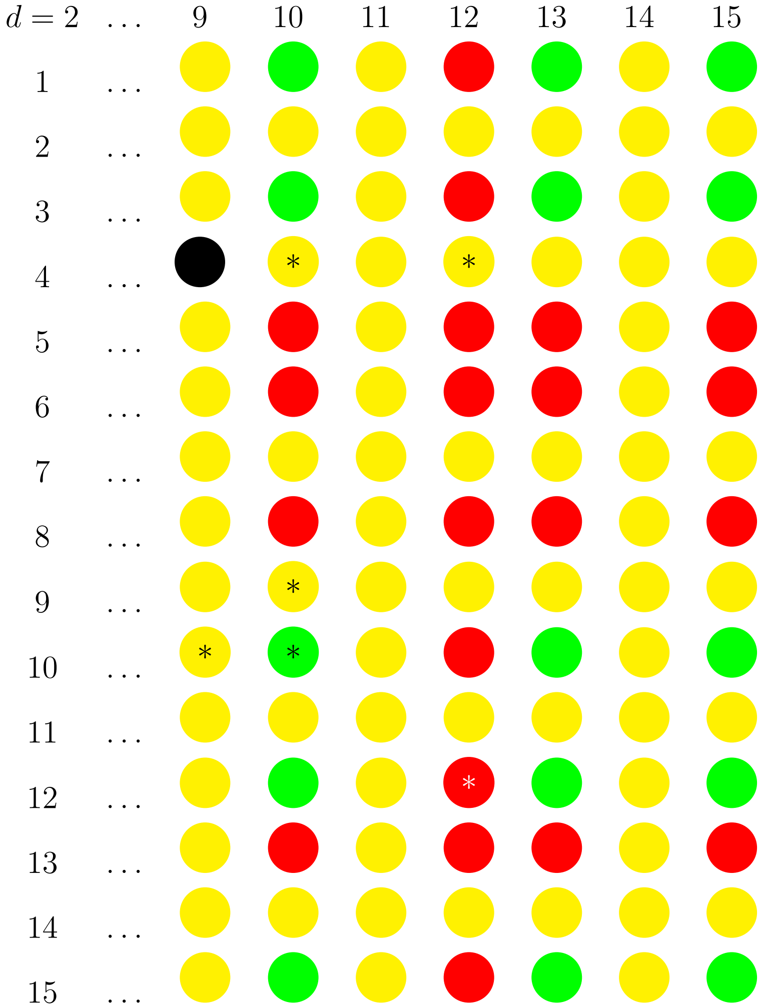
$$\{1,2,3\} < \{1,2,4\} < \{1,2,5\} < \{1,2,6\} < \{1,3,4\} < \{1,3,5\} < \dots$$
$$\dots < \{2,4,6\} < \{2,5,6\} < \{3,4,5\} < \{3,4,6\} < \{3,5,6\} < \{4,5,6\}$$

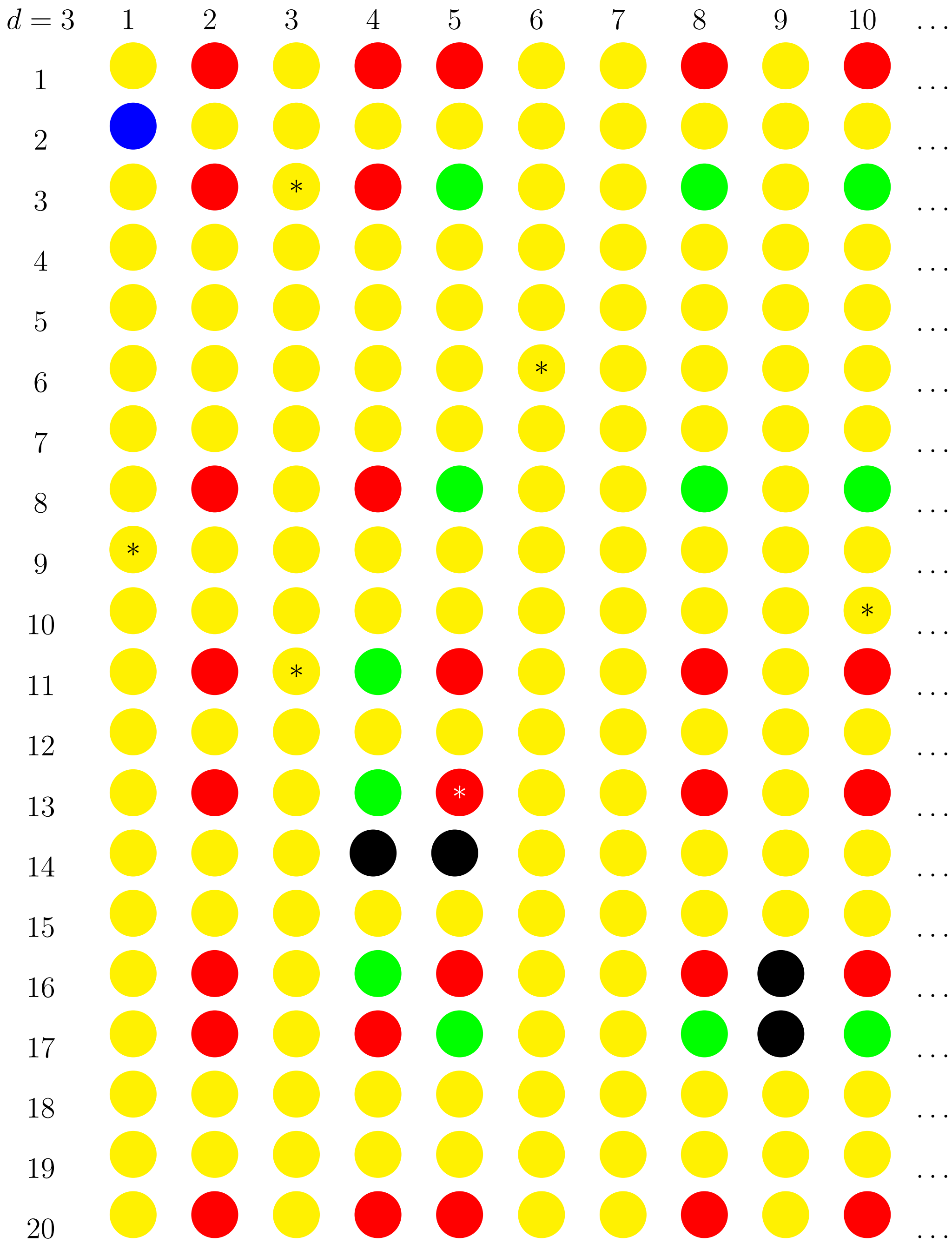
The $d = 3$ table has a Nash equilibrium at position 16 for player 1 and 18 for player 2. Thus $\mathcal{T}_1 = \{2,5,6\}$ and $\mathcal{T}_2 = \{3,4,6\}$. There exists a unique

$\underline{\gamma} \in \mathcal{C}(\mathcal{T}_1, \mathcal{T}_2)$ such that it is a Nash equilibrium. It is type-2 generic, so $\mathrm{supp}(\gamma^1) = \mathcal{T}_1$ and $\mathrm{supp}(\gamma^2) = \mathcal{T}_2$.

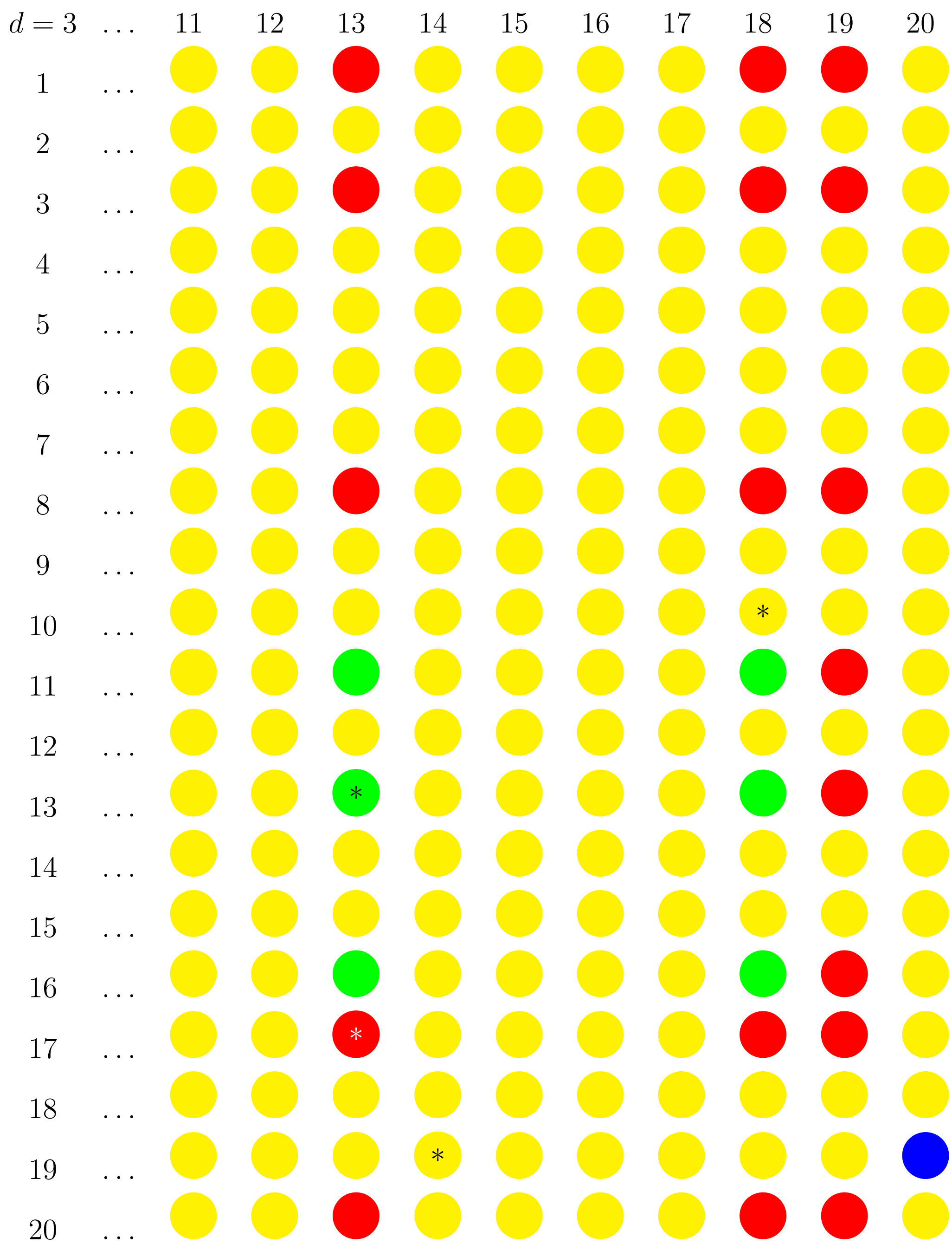


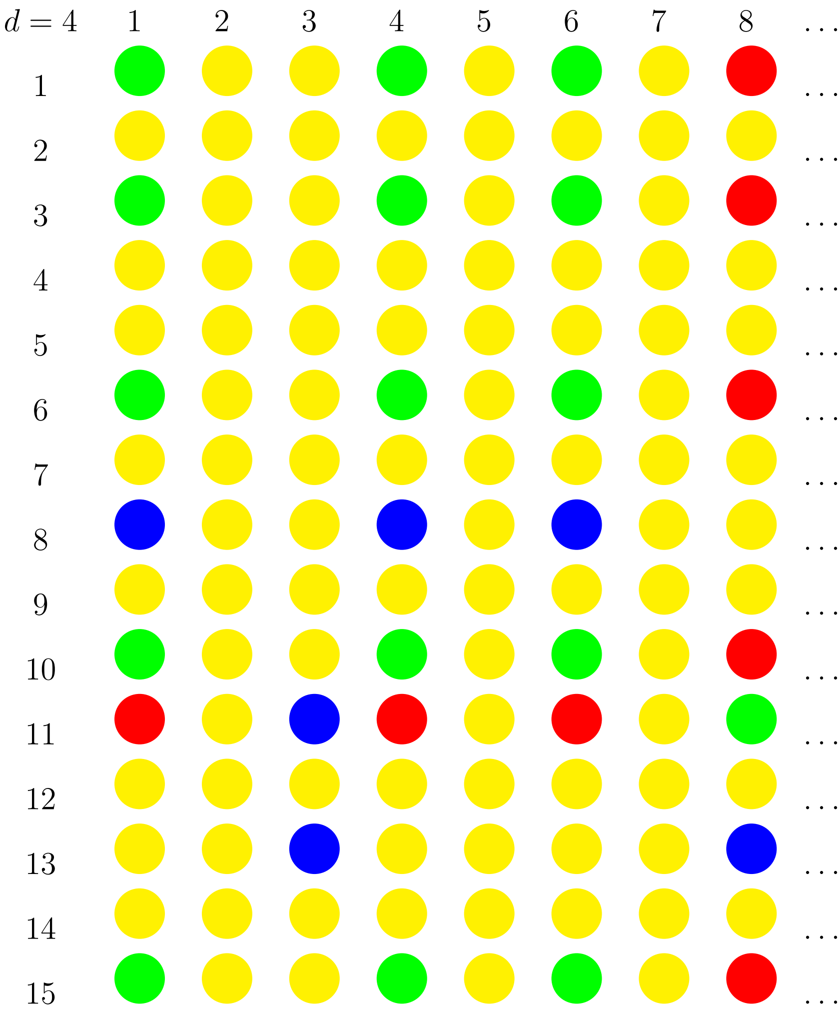

d = 2
1 2 3 4 5 6 7 8 ...
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15

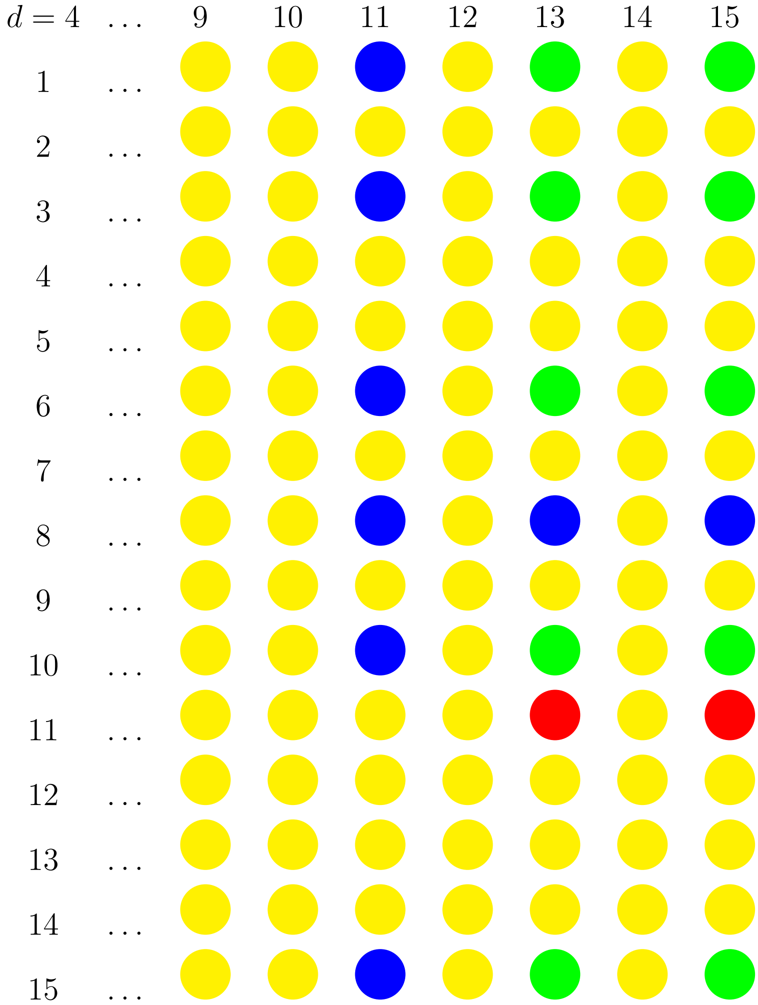

d = 2 . . . 9 10 11 12 13 14 15
1 . . .
2 . . .
3 . . .
4 . . .
5 . . .
6 . . .
7 . . .
8 . . .
9 . . .
10 . . .
11 . . .
12 . . .
13 . . .
14 . . .
15 . . .



$d = 3$
1 2 3 4 5 6 7 8 9 10 ...
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20


d = 3 . . . 11 12 13 14 15 16 17 18 19 20
1 . . .
2 . . .
3 . . .
4 . . .
5 . . .
6 . . .
7 . . .
8 . . .
9 . . .
10 . . .
11 . . .
12 . . .
13 . . .
14 . . .
15 . . .
16 . . .
17 . . .
18 . . .
19 . . .
20 . . .


$d = 4$
1 2 3 4 5 6 7 8 . . .
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15



d = 4 . . . 9 10 11 12 13 14 15
1 . . .
2 . . .
3 . . .
4 . . .
5 . . .
6 . . .
7 . . .
8 . . .
9 . . .
10 . . .
11 . . .
12 . . .
13 . . .
14 . . .
15 . . .

Bibliography

[Bal09] Anne Balthasar. *Geometry and equilibria in bimatrix games.* PhD thesis, London School of Economics and Political Science, 2009.

[BB90] G. Blind and R. Blind. Convex polytopes without triangular faces. *Israel Journal of Mathematics*, 71(2):129–134, 1990.

[BCR98] Jacek Bochnak, Michel Coste, and Marie-Françoise Roy. *Real algebraic geometry.* Springer, 1998.

[Ber75] D.N. Bernstein. The number of roots of a system of equations. *Functional Analysis and its applications*, 9(3):183–185, 1975.

[BL75] R. Bröcker and L. Lander. *Differentiable germs and catastrophes. London Mathematical Society Lecture Note Series 17.* Cambridge University Press, 1975.

[BMMN98] M. Bhattacharjee, D. Macpherson, R.G. Möller, and P.M. Neumann. *Notes on Infinite Permutation Groups.* Number Nr. 1689 in Lecture Notes in Mathematics. Springer, 1998.

[BR90] R. Benedetti and J.J. Risler. *Real algebraic and semi-algebraic sets.* Actualités mathématiques. Hermann, 1990.

[CPR74] H.H. Chin, T. Parthasarathy, and T.E.S. Raghavan. Structure of equilibria in N-person non-cooperative games. *International Journal of Game Theory*, 3(1):1–19, 1974.

[DH08] Peter Deuflhard and Andreas Hohmann. *Numerische Mathematik 1: Eine algorithmisch orientierte Einführung.* Walter de Gruyter GmbH & Co KG, 2008.

[GPS93] Faruk Gül, David Pearce, and Ennio Stacchetti. A bound on the proportion of pure strategy equilibria in generic games. *Mathematics of Operations Research*, 18(3):548–552, 1993.

[GS67] Branko Grünbaum and V.P. Sreedharan. An enumeration of simplicial 4-polytopes with 8 vertices. *Journal of Combinatorial Theory*, 2(4):437–465, 1967.

[Har73] John C. Harsanyi. Oddness of the number of equilibrium points: a new proof. *International Journal of Game Theory*, 2(1):235–250, 1973.

[Kei97] Hans Keiding. On the Maximal Number of Nash Equilibria in an n× n Bimatrix Game. *Games and Economic Behavior*, 21(1-2):148–160, 1997.

[Kho77] A.G. Khovanskii. Newton polyhedra and toroidal varieties. *Functional analysis and its applications*, 11(4):289–296, 1977.

[KM86] Elon Kohlberg and Jean-Francois Mertens. On the strategic stability of equilibria. *Econometrica: Journal of the Econometric Society*, pages 1003–1037, 1986.

[Kre81] V.L. Kreps. Finite N-person non-cooperative games with unique equilibrium points. *International Journal of Game Theory*, 10(3):125–129, 1981.

[Kuh61] Harold W. Kuhn. An algorithm for equilibrium points in bimatrix games. *Proceedings of the National Academy of Sciences*, 47(10):1657–1662, 1961.

[Kus76] A.G. Kushnirenko. Newton polytopes and the Bezout theorem. *Functional analysis and its applications*, 10(3):233–235, 1976.

[LH64] C. E. Lemke and J. T. Howson, Jr. Equilibrium Points of Bimatrix Games. *Journal of the Society for Industrial and Applied Mathematics*, 12(2):413–423, 1964.

[Man64] O.L. Mangasarian. Equilibrium points of bimatrix games. *Journal of the Society for Industrial and Applied Mathematics*, 12(4):778–780, 1964.

[McM70] Peter McMullen. The maximum numbers of faces of a convex polytope. *Mathematika*, 17(2):179–184, 1970.

[MM97] Richard D. McKelvey and Andrew McLennan. The maximal number of regular totally mixed Nash equilibria. *Journal of Economic Theory*, 72(2):411–425, 1997.

[MP99] Andrew McLennan and In-Uck Park. Generic 4× 4 two person games have at most 15 Nash equilibria. *Games and Economic Behavior*, 26(1):111–130, 1999.

[MSW95] Alexander P. Morgan, Andrew J. Sommese, and Charles W. Wampler. A product-decomposition bound for Bezout numbers. *SIAM journal on numerical analysis*, 32(4):1308–1325, 1995.

[Nas51] John Nash. Non-Cooperative Games. *Annals of Mathematics*, 54(2):286–295, 1951.

[QS97] Thomas Quint and Martin Shubik. A theorem on the number of Nash equilibria in a bimatrix game. *International Journal of Game Theory*, 26(3):353–359, 1997.

[Rit94] Klaus Ritzberger. The theory of normal form games from the differentiable viewpoint. *International Journal of Game Theory*, 23(3):207–236, 1994.

[Ros71] J. Rosenmüller. On a generalization of the Lemke–Howson algorithm to noncooperative N-person games. *SIAM Journal on Applied Mathematics*, 21(1):73–79, 1971.

[Rot95] Joseph J. Rotman. *An Introduction to the Theory of Groups.* Springer, 1995.

[Sar42] A. Sard. The measure of the critical values of differentiable maps. *Bulletin of the American Mathematical Society*, 48(12):883–890, 1942.

[Sch05] Arndt von Schemde. Index and Stability in Bimatrix Games. *Lecture Notes in Economics and Mathematical Systems*, 2005.

[Sch10] Günther Schmidt. *Relational Mathematics*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, hardcover edition, 2010.

[Ste97] Bernhard von Stengel. New lower bounds for the number of equilibria in bimatrix games. *Technical report/Departement Informatik, ETH Zürich*, 264, 1997.

[Ste99] Bernhard von Stengel. New maximal numbers of equilibria in bimatrix games. *Discrete & Computational Geometry*, 21(4):557–568, 1999.

[Vid17] Raimundas Vidunas. Counting derangements and Nash equilibria. *Annals of Combinatorics*, 21(1):131–152, 2017.

[Vor58] N.N. Vorob'ev. Equilibrium points in bimatrix games. *Theory of Probability & Its Applications*, 3(3):297–309, 1958.

[Vuj19] Matija Vujić. Geometry of the sets of Nash equilibria in mixed extensions of finite games. Master's thesis, Universität Mannheim, 2019.

[Wil71] Robert Wilson. Computing equilibria of n-person games. *SIAM Journal on Applied Mathematics*, 21(1):80–87, 1971.

[Wil92] Robert Wilson. Computing simply stable equilibria. *Econometrica: Journal of the Econometric Society*, 60:1039–1070, 1992.

[Zie95] Günter M. Ziegler. *Lectures on polytopes*. Springer-Verlag New York, 1995.