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Incentive compatibility in sender-receiver stopping games ☆

Aditya Aradhye [a], János Flesch [b], Mathias Staudigl [c], Dries Vermeulen [b,*]

[a] Czech Technical University, FEE, AI Center, Prague, Czech Republic
[b] Maastricht University, SBE, QE Dept, Maastricht, the Netherlands
[c] Maastricht University, FSE, DKE, Maastricht, the Netherlands

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ABSTRACT

We introduce a model of sender-receiver stopping games, in which the sender observes the current state, and sends a message to the receiver to either stop the game, or to continue. The receiver, only seeing the message, then decides to stop the game, or to continue. The payoff to each player is a function of the state when the receiver quits, with higher states leading to better payoffs.

We prove existence and uniqueness of responsive Perfect Bayesian Equilibrium (PBE) when players are sufficiently patient. The responsive PBE has a simple structure, with a threshold strategy for the sender, and the receiver obediently following the recommendations of the sender. Hence, the sender alone plays the decisive role, and therefore always obtains the best possible payoff for himself.

1. Introduction

Information transmission is a fundamental element of economic models. In various settings, a better informed party (sender) is in the position to transmit information to a lesser informed or even uninformed party (receiver). Typically, the action choices of the receiver have an influence on the payoff of the sender, and hence the information transmission has a strategic aspect. In their seminal paper, Crawford and Sobel (1982), analyze strategic information transmission with a single interaction between the sender and the receiver. Their model and its variations (Crawford and Sobel, 1982; Green and Stokey, 2007) have a wide range of applications, notably in economics, computer science, political science but also in biology and philosophy (Azaria et al., 2012; Skyrms, 2010b; Huttegger et al., 2014). Recently, a few models have been introduced in which the information transmission takes place in a dynamic setting, see for instance Renault et al. (2013) and Golosov et al. (2014). In these models, the sender-receiver game is played repeatedly either on a finite or an infinite horizon and the payoff for the sender and the receiver is the total discounted sum of the stage payoffs. The key focus in these papers is the characterization of the set of equilibrium payoffs where, in the spirit of the folk theorem, the players are sufficiently patient.

This paper sets the stage for a different line of research, which we regard as an important conceptual contribution of our work. Specifically, this paper introduces a model of *sender-receiver stopping games*, which combines features from dynamic sender-receiver games and stopping games (for a survey on the latter, see Solan and Vieille (2005)). In these games the

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* Corresponding author.
E-mail addresses: aradhadi@fel.cvut.cz (A. Aradhye), j.flesch@maastrichtuniversity.nl (J. Flesch), m.staudigl@maastrichtuniversity.nl (M. Staudigl), d.vermeulen@maastrichtuniversity.nl (D. Vermeulen).

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strategic information transmission takes place repeatedly until the receiver decides to stop the interaction. We deal in this paper with sender-receiver stopping games of finite as well as infinite horizon. In the finite horizon, the receiver is forced to stop the game before a pre-defined terminal period has been reached. In the infinite horizon the game may be played for unlimited number of periods. The timing of the game is as follows: In each period nature draws a state of the world, which is only revealed to the sender. After observing the state of the world, the sender sends one out of two messages to the receiver. This message is interpreted as a suggestion either to stop the game or to continue. Now the receiver has to take a decision. After seeing the message, but without knowing the state, the receiver has two options: he can decide to stop the game, or he can decide to continue to the next period. The payoff to each player is a function of the state at which the receiver stops, and these payoffs are either discounted or period-independent. The setting could be thought as an investor (the receiver) who must make an irreversible financial decision, without having exact information about the market situation, but using the advice of an expert (the sender).

We assume that rewards are positively correlated with the state of nature, i.e. higher states lead to better payoffs for both players. Thus, both players have identical ordinal preferences over realizations of the state of nature. Yet, as we impose no further restrictions on the payoffs, the cardinal assignment of values of the two payoff functions can be very different. As a consequence, a certain state may be very appealing to one player, but not so much to the other, creating an interesting strategic tension between the parties. This paper investigates to what extent the cardinal differences may hamper coordination between the players.

1.1. *Our contribution*

The main solution concept that we use to analyze these games is *Perfect Bayesian Equilibrium* (PBE), and we identify a class of PBEs which are appealing to the players in terms of payoffs, and moreover easy to compute and implement. We are interested in PBEs in which the receiver plays a responsive strategy. We call a receiver's strategy responsive if at each time period his mixed action is different for different messages sent by the sender. This is just a condition which excludes those PBEs in which the receiver's strategy is babbling, which are fairly uninteresting from a strategic perspective.[1]

Therefore, the crucial question is how the sender should use his additional information to manipulate the choices of the receiver given the current state, and to what extent the receiver can trust the sender's recommendations and be obedient. In particular, we investigate under what conditions the incentives of the sender and the receiver match.

These concerns are captured by the notion of a *regular strategy profile*. In a regular strategy profile, the receiver simply follows the sender's recommendations, whereas the sender sends a sincere message, given the realization of the state, whether or not he would like the receiver to terminate the game at this period. Since the receiver is obedient, he is not playing an active role in such a strategy profile. The sender's sincere strategy is a threshold strategy that sends the message "continue" if the current state is below the threshold and sends the message "quit" otherwise. In other words, the sender's strategy is the optimal solution of the one player maximization problem in which the decision to continue or to quit is delegated to the sender. This means that in the regular strategy profile the sender obtains the best possible payoff for himself. Indeed, this outcome is Pareto optimal. The game we study admits an (essentially) unique[2] regular strategy profile.

In finite horizon, we show that there is no responsive PBE other than the regular strategy profile. Even the regular strategy profile may fail to be a PBE in certain games if the discount factor is small. More precisely, our findings are as follows:

(i) The regular strategy profile is the unique responsive PBE if the discount factor is sufficiently high or if the payoffs are period-independent.
(ii) There are games that have no responsive PBE if the discount factor is small enough.

For the infinite horizon, we focus on strategy profiles where the expected payoffs after any period do not depend on the history. We show that within this class of strategy profiles, the regular strategy profile is the unique responsive PBE provided the discount factor is sufficiently high. More precisely, our findings are as follows:

(iii) The regular strategy profile is the unique responsive PBE if the discount factor is sufficiently high. In this setting, the regular strategy profile is stationary.
(iv) There are games that have no responsive PBE if the discount factor is small enough.

In the extreme case of infinite horizon where the payoffs are period-independent, there does not exist any responsive PBE. This is due to the fact that the sender does not have a best response when the receiver is obeying, and hence, it should not be interpreted as a breakdown of communication.

[1] Regardless, it is easy to study all the PBEs which are not responsive by simply adding babbling periods in any responsive PBE.
[2] By (essentially) unique, we mean that in any two regular strategy profiles, the actions of the players differ at only measure zero sets and thus induce the same expected payoffs for the players.

1.2. Related literature

Crawford and Sobel (1982) introduced a model of strategic information transmission. The model in which the sender and the receiver interact only once is studied extensively, see Crawford and Sobel (1982); Green and Stokey (2007). Recently a lot of work focused on the dynamic extension, where the strategic interaction takes place repeatedly, see Renault et al. (2013); Golosov et al. (2014); Krishna and Morgan (2004); Aumann and Hart (2003). Renault et al. (2013) assume that the sequence of states follows an irreducible Markov chain. They characterize the limit set of equilibrium payoffs, as players become very patient. Golosov et al. (2014) study finite horizon games, and show that, under certain conditions, full information revelation is possible and conditioning future information release on past actions improves incentives for information revelation.

Our paper relates to the large and growing literature on Bayesian persuasion, see Kamenica and Gentzkow (2011); Ely (2017); Renault et al. (2017); Honryo (2018). In these settings, the informed advisor (i.e. the sender) decides how much information to share with a less informed agent (i.e. the receiver) so as to influence his decision. Renault et al. (2017) show that in many cases, the optimal greedy disclosure policy for the sender exists, which at each stage, minimizes the amount of information being disclosed in that stage under the constraint that it maximizes the current payoff of the sender.

This paper also relates to the classical contributions on communication in games (see e.g. Forges (1986); Myerson (1986)). In Skyrms (2010a); Blume et al. (1998), the evolution of information flow is studied in the setting of strategic communication. Our model is also linked to topics in computer science such as automated advice provision. For instance, Azaria et al. (2012) use sender-receiver games to model interaction between computers and humans.

Most of the previous work has focused on the sender-receiver games with fixed duration of time. Compared to these earlier papers, one of the main novelties of our model is that the game is a stopping game. The receiver has the license to stop the game at any period of time. Some of the techniques we use (for example, backward induction) are also similar to the ones in the literature on stopping games, see Solan and Vieille (2005); Ekström and Villeneuve (2006).

In the responsive PBE, the receiver complies with the sender and the sender tries to maximize his expected payoff without knowing the future states. Hence, our setting in a wider sense, is a variant of the secretary problem, see Ferguson (1989).

The structure of the paper is as follows. In section 2 we introduce the model and in the section 3, we discuss the regular strategy profile. In sections 4 and 5 we state our main results for the finite horizon and the infinite horizon respectively. In section 6, we give illustrative examples and in section 7, we have concluding remarks. In section 8, we provide the proofs of the main theorems.

2. The model

We study sender-receiver stopping games. A sender-receiver stopping game is played at periods $t = 1, 2, \dots, T$, if the game has finite time horizon T and at periods $t = 1, 2, \dots$ if the game has infinite time horizon. At period t, play is as follows. First, a state of the world θ^t is drawn from the uniform distribution on the unit interval $I = [0, 1]$, independently of the earlier realizations $\theta^1, \dots, \theta^{t-1}$. The sender learns θ^t, while the receiver only knows the distribution of θ^t. Next, the sender chooses a message $m^t \in \{m_c, m_q\}$ and sends it to the receiver. The message m_c is a request from the sender to the receiver to continue at this period t and m_q is a proposal to quit. On seeing the message, the receiver chooses an action $a^t \in \{a_c, a_q\}$, where a_c stands for continue and a_q stands for quit. If the receiver continues, then the game proceeds to period $t + 1$. If the receiver quits, then the game ends at period t. The sender receives the payoff $\delta^{t-1} \cdot f(\theta^t)$ and the receiver receives the payoff $\delta^{t-1} \cdot g(\theta^t)$. Here, f and g are continuous and strictly increasing functions from I to $\mathbb{R}_+$ with $f(0) = g(0) = 0$, and $\delta \in (0, 1]$ is a discount factor. The payoff if the receiver continues forever is zero for both players.

If $\delta = 1$, we say that the game has *period-independent* payoffs. If $\delta \in (0, 1)$, we say that the game has *discounted* payoffs with discount factor δ. For brevity, we sometimes write $f^t(\theta) = \delta^{t-1} \cdot f(\theta)$ and $g^t(\theta) = \delta^{t-1} \cdot g(\theta)$.

2.1. Strategies and expected payoffs

Histories. For the sender, a history at period t is a sequence $h_s^t = (\theta^1, m^1, \dots, \theta^{t-1}, m^{t-1})$ of past states and messages sent by the sender. By $H_s^t = (I \times M)^{t-1}$ we denote the set of histories for the sender at period t. Given the usual topology on I, we endow H_s^t with the product Borel sigma-algebra.

Since the receiver does not observe the realization of the states, a history of the receiver at period t is a sequence $h_r^t = (m^1, \dots, m^{t-1})$ of past messages sent by the sender. By $H_r^t = M^{t-1}$ we denote the set of histories for the receiver at period t. Note that H_r^t is a finite set.

Strategies. A strategy $\sigma = (\sigma^t)_{t=1}^T$ for the sender is a sequence of measurable functions $\sigma^t \colon H_s^t \times I \to [0, 1]$. We allow $T = \infty$ when the horizon is infinite. The interpretation is that, at each period t, given the history h_s^t and the state θ^t, the strategy σ^t places probability $\sigma^t(h_s^t, \theta^t)$ on the message m_c.

A strategy $\tau = (\tau^t)_{t=1}^T$ for the receiver is a sequence of functions $\tau^t \colon H_r^t \times M \to [0, 1]$. We do not need any measurability conditions for τ^t as the domain of τ^t is finite. The interpretation is that, at each period t, given the history h_r^t and the message m^t, the strategy τ^t places probability $\tau^t(h_r^t, m^t)$ on the action a_c.

When T is finite, for simplicity we require that at period T, regardless the history, the sender's strategy has to send the message m_q and the receiver's strategy has to play the action a_q.

In this model, we focus on *responsive* strategies of the receiver. A strategy τ of the receiver is called responsive if, for each period $t < T$ and each history h_r^t, we have $\tau^t(h_r^t, m_c) > \tau^t(h_r^t, m_q)$. So, upon receiving the message m_c, the receiver chooses action a_c with higher probability than upon receiving m_q.[3] Note that in case of finite horizon, as the receiver must quit if play reached period T, responsiveness does not put any restriction on actions at period T.

2.2. Perfect Bayesian equilibrium

In this section we introduce the solution concept we use to analyze the sender-receiver stopping games defined above. Consider a strategy profile (σ, τ). The expected payoffs of the sender and the receiver are denoted by $U_s(\sigma, \tau)$ and $U_r(\sigma, \tau)$ respectively. If the receiver did not quit until period t, and the histories are h_s^t and h_r^t respectively, then the continuation expected payoffs from period t onward are denoted by $U_s^t(\sigma, \tau)(h_s^t)$ and $U_r^t(\sigma, \tau)(h_r^t)$ respectively. For details on the explicit construction of these notions, we refer the reader to Appendix B.

Definition 1. A strategy profile (σ, τ) is called a Perfect Bayesian Equilibrium (PBE) if for every period t, and every history h_s^t, we have $U_s^t(\sigma, \tau)(h_s^t) \geq U_s^t(\sigma', \tau)(h_s^t)$ for every strategy σ' of the sender, and for every h_r^t, $U_r^t(\sigma, \tau)(h_r^t) \geq U_r^t(\sigma, \tau')(h_r^t)$ for every strategy τ' of the receiver. A PBE is called responsive if the receiver's strategy is responsive.[4]

Notice that in the definition of PBE we do not explicitly talk about beliefs of the players on the realized history consisting of the past states, messages and actions. Since the sender's history contains all this information, he is fully informed and he knows the history of the receiver. On the other hand, the receiver is not informed of the past or current states. Based on his own history and the strategy profile (σ, τ), he has a natural belief on the possible histories of the sender, which is compatible with Bayesian updating. For details, we refer to Appendix A.

We will regularly make use of the fact that the well-known one-shot deviation principle holds in our games whenever the game has a finite horizon (regardless whether the payoffs are period-independent or discounted) or the game has infinite horizon and the payoffs are discounted. We will do so in the remainder of the paper without further explicit mention of the one-shot deviation principle.

2.3. Terminology for strategies

At period t, the expected payoff $U_s^t(\sigma, \tau)$ for the sender is *history independent* if for every $h_s^t, \overline{h}_s^t \in H_s^t$ it holds that $U_s^t(\sigma, \tau)(h_s^t) = U_s^t(\sigma, \tau)(\overline{h}_s^t)$. Note that history independence of $U_s^t(\sigma, \tau)$ is equivalent to saying that the function $U_s^t(\sigma, \tau)$ is constant. In that case, with slight abuse of notation, we identify the function with the (constant) value of that function, and act as if $U_s^t(\sigma, \tau)$ is a real number instead of a function. A similar observation holds for the expected payoff $U_r^t(\sigma, \tau)$ for the receiver.

A strategy profile (σ, τ) is said to have *History Independent Expected Payoffs* (HIEP) at period t, if both $U_s^t(\sigma, \tau)$ and $U_r^t(\sigma, \tau)$ are history independent at period t. We say that the strategy profile (σ, τ) has HIEP, if it has HIEP at every period t.

A strategy σ for the sender is said to have a *threshold* $\alpha^t \in [0, 1]$ at period t, if

$$\sigma^t(\theta^t) = \begin{cases} 1 & \text{if } \theta^t \in [0, \alpha^t) \\ 0 & \text{if } \theta^t \in (\alpha^t, 1]. \end{cases}$$

A strategy σ for the sender is called a *threshold strategy* if it has a threshold at each period t. A threshold strategy σ is called *stationary* if $\alpha^s = \alpha^t$ for all periods s and t. For the sake of exposition, we do not specify the choice of σ at the threshold. In any case, this occurs with probability zero only. Note that the threshold requirement is strong. The threshold is *only* allowed to depend on the period, not on the specific history.

3. The regular strategy profile

The regular strategy profile plays a central role in our paper. A strategy profile (σ, τ) is called *regular* if τ is obedient, and σ is sincere against τ. The strategy τ is *obedient* τ if, for each period t and each history h_r^t, $\tau^t(h_r^t, m_c) = 1$ and $\tau^t(h_r^t, m_q) = 0$. The obedient strategy is pure, and responsive.

For a given strategy τ of the receiver, a strategy σ of the sender is called *sincere against* τ *at period* t (with $t < T$ if the game has finite horizon T) if

[1] $U_s^{t+1}(\sigma, \tau)$ is history independent, and

[3] The reason to restrict our attention to responsive strategies is to avoid PBEs in 'babbling' strategies, which are fairly uninteresting from a game theory perspective.

[4] PBE is a refinement of Bayesian Nash Equilibrium (BNE). Intuitively, it requires that the strategy profile induces a BNE after any history.

[2] the strategy σ has a threshold α^t at period t, where α^t is the unique solution to the equation $f^t(\alpha^t) = U_s^{t+1}(\sigma, \tau)$.

Notice that indeed the equation in condition [2] has a unique solution due to strict monotonicity of f^t, and the fact that, since payoffs are either period-independent or discounted, $U_s^{t+1}(\sigma, \tau) \le f^t(1)$.

A strategy σ is called *sincere against* τ if it is sincere against τ at each period t (with $t < T$ if the game has finite horizon T).

If σ is sincere against τ, as long as the receiver adheres to τ, the sender terminates the game precisely when he wants the game to terminate. In order to see how condition [2] achieves this, assume that (σ, τ) is a regular strategy profile such that σ has threshold α^t at period t. If $\theta^t < \alpha^t$, then $f^t(\theta^t) < f^t(\alpha^t) = U_s^{t+1}(\sigma, \tau)$. In this case the sender would like the receiver to continue the game as the expected continuation payoff is higher than the expected payoff if the receiver quits. Indeed, the strategy σ recommends the message m_c as $\theta^t < \alpha^t$ and α^t is the threshold. Similarly, if $\theta^t > \alpha^t$, then $f^t(\theta^t) > f^t(\alpha^t) = U_s^{t+1}(\sigma, \tau)$ and σ recommends the message m_q.

Notice that a regular strategy profile has HIEP. In the remainder of this section, we argue in both the finite and the infinite horizon model that the regular strategy profile is (essentially) unique. To explain this, we define function $H : I \to I$ by

$$H(x) = f^{-1}\Big(\delta \cdot \Big[x \cdot f(x) + \int_x^1 f(\theta)d\theta\Big]\Big).$$

Lemma 1. *For any regular strategy profile it holds that $\alpha^{t-1} = H(\alpha^t)$ for any two consecutive thresholds α^{t_1} and α^t in the strategy of the sender.*

Proof. Suppose that the players use a regular strategy profile (σ, τ). Assume that σ has a threshold α^t at each period t. Then, at each period t, with probability α^t we have $\theta^t < \alpha^t$ and the receiver continues. In this case the sender gets the expected continuation payoff $U_s^{t+1}(\sigma, \tau)$. Similarly, with probability $1 - \alpha^t$ we have $\theta^t > \alpha^t$ and the receiver quits, with payoff $\frac{1}{1-\alpha^t}\int_{\alpha^t}^1 f^t(\theta)d\theta$. So,

$$U_s^t(\sigma, \tau) = \alpha^t \cdot U_s^{t+1}(\sigma, \tau) + \int_{\alpha^t}^1 f^t(\theta)d\theta$$

As σ is sincere against τ, we have $f^{t-1}(\alpha^{t-1}) = U_s^t(\sigma, \tau)$. This yields

$$f^{t-1}(\alpha^{t-1}) = U_s^t(\sigma, \tau) = \alpha^t \cdot U_s^{t+1}(\sigma, \tau) + \int_{\alpha^t}^1 f^t(\theta)d\theta = \alpha^t \cdot f^t(\alpha^t) + \int_{\alpha^t}^1 f^t(\theta)d\theta.$$

Using the fact that $f^t = \delta^{t-1} \cdot f$ in every period t, we find that

$$\alpha^{t-1} = f^{-1}\Big(\delta \cdot \Big[\alpha^t \cdot f(\alpha^t) + \int_{\alpha^t}^1 f(\theta)d\theta\Big]\Big) = H(\alpha^t).$$

This concludes the proof. □

3.1. Finite horizon

Assume that the game has finite horizon T. Define the numbers $\beta^1, \dots, \beta^T$ as follows. First, $\beta^T = 0$. Then, using a backwards iteration, $\beta^t = H(\beta^{t+1})$ for all $t = T-1, \dots, 1$. By Lemma 3 we have

$$1 > \beta^1 > \beta^2 > \cdots > \beta^T = 0.$$

Note that the function H is defined differently for different values of discount factors δ. Hence, the numbers β^t also depend on the discount factor δ. Moreover, β^t depends on the horizon T. For brevity, we shall still use the notation β^t when the horizon and discount factor are fixed.

Proposition 1. *Assume that the game has finite horizon T. Then the thresholds for the sender's strategy in the regular strategy profile are given by $\beta^1, \dots, \beta^T$ respectively at periods $1, \dots, T$.*

Proof. Let (σ, τ) be a regular strategy profile. Assume that σ has thresholds α^t and α^{t-1} for fixed $t \leq T$. Then $\alpha^{t-1} = H(\alpha^t)$ by Lemma 1. As the sender sends message m_q at period T irrespective of the state, we have $\alpha^T = 0 = \beta^T$. Hence, $\alpha^{t-1} = H(\alpha^t) = H(\beta^t) = \beta^{t-1}$ for $t = T, \dots, 2$. □

The numbers $\beta^1, \dots, \beta^T$ can be computed recursively. Hence the regular strategy profile is in principle entirely computable. It is worth noting that the computation of the regular strategy profile only considers the sender's payoff function; the receiver's payoff function does not play any role at all.

3.2. Infinite horizon

By Lemma 2, the function H has a unique fixed point, which we denote by β. Although β depends on the discount factor δ, for brevity we keep δ fixed, and only use the notation β.

Proposition 2. *Assume that the game has infinite horizon. Then there is a unique regular strategy profile. Moreover, this profile is stationary with threshold β.*

Proof. Let (σ, τ) be a regular strategy profile, with thresholds α^t for every t. We show that $\alpha^t = \beta$ for all t. It suffices to show that $\alpha^1 = \beta$. Assume by way of contradiction that $\alpha^1 > \beta$. (A similar argument holds for $\alpha^1 < \beta$.) Since $\alpha^1 = H(\alpha^2)$ by Lemma 1, Lemma 2 in Appendix D implies that $H\big(\alpha^1\big) < \alpha^1 = H(\alpha^2)$. This implies $\alpha^1 < \alpha^2$. Iterating this argument, we can conclude that $(\alpha^t)_{t=1}^{\infty}$ is strictly increasing. Then, since $\alpha^t \leq 1$ for each t, the sequence $(\alpha^t)_{t=1}^{\infty}$ is convergent, say with limit z. Then, by continuity of H,

$$z = \lim_{t \to \infty} \alpha^t = \lim_{t \to \infty} H(\alpha^t) = H(z).$$

So, $z = \beta$ by Lemma 2. However, $z \geq \alpha^1 > \beta$. Contradiction. □

Thus, the regular strategy profile is determined by the unique solution β of the equation $H(x) = x$. As in the finite horizon, the regular strategy profile only depends on the sender's payoff function.

By Lemma 3 in the Appendix, we have $\beta^t(T) \to \beta$ as $T \to \infty$ for each t. So the following corollary is an immediate consequence of Propositions 1 and 2.

Corollary 1. *Let (σ_T, τ_T) be the regular strategy profile in the game with finite horizon T and (σ, τ) be the regular strategy profile in the game with infinite horizon. Then the sequence $(\sigma_T, \tau_T)_{T=1}^{\infty}$ converges to (σ, τ) as $T \to \infty$, when the payoffs are discounted or period-independent.*

4. Existence and uniqueness of PBE, finite horizon

In this section we consider the case where the game has finite horizon T. We show that, if the payoffs are period-independent or they are discounted with a sufficiently large discount factor, then the regular strategy profile is the unique responsive PBE. We also show that existence of a responsive PBE may fail for small discount factors. Define the function $V : [0, 1] \to \mathbb{R}$ by $V(0) = 0$ and, for $x \in (0, 1]$,

$$V(x) = \frac{1}{x} \cdot \int_0^x g(\theta) d\theta.$$

The amount $V(x)$ is the expected payoff for the receiver if he quits, conditional on the state being in $[0, x]$. For $T \in \mathbb{N}$, let D^T denote the smallest number E in $[0, 1]$ for which $\delta \cdot V(1) \geq V(\beta^1(T))$ for every positive $\delta \in [E, 1]$. It is straightforward to check that $D^T < 1$. Note that $\beta^1(T)$ depends on δ, so that also D^T depends on δ.

Theorem 1. *Consider a sender-receiver stopping game with finite horizon T. Let the payoffs be either period-independent or discounted with discount factor $\delta \geq D^T$. Then, the regular strategy profile is the unique responsive PBE.*

The next Theorem is a partial counterpart to this result, stating that, for low values of the discount factor δ, there does not exist a responsive PBE. The proof is deferred to Section 8.2.

Theorem 2. *Suppose that $0 < \delta < D^2$. Then, for any $T \geq 2$, the sender-receiver game with finite horizon T does not admit a responsive PBE.*

Example 1. Consider the game with finite horizon T in which the payoff functions are δ discounted with $f(x) = x^2$ and $g(x) = x$. Then

$$H(x) = \sqrt{\frac{\delta \cdot (1 + 2x^3)}{3}}.$$

If the message sent by the sender at period $T - 1$ is m_c, then the state is in the interval $[0, \beta^{T-1}]$. So, the payoff for the receiver on quitting is $\delta^{T-2} \cdot V(\beta^{T-1})$ and the payoff on continuing is $\delta^{T-1} \cdot V(1)$. Hence, the receiver prefers to continue if $\delta \cdot V(1) \geq V(\beta^{T-1})$. We have $\beta^{T-1} = \sqrt{\frac{\delta}{3}}$. Further, $V(x) = \frac{x}{2}$. Thus, the inequality $\delta \cdot V(1) \geq V(\beta^{T-1})$ is valid if and only if $\delta \geq \frac{1}{3}$. So, we get $D^2 = \frac{1}{3} > 0$. Thus, by Theorem 2, for $\delta < \frac{1}{3}$ the game does not admit a responsive PBE. □

Note that $D^T \geq D^2$. Thus, the above two Theorems do not necessarily cover all possible values of δ. We briefly discuss this issue. Recall that $\beta^1(T)$ itself depends on the discount factor δ. Hence, in general D^T does not have a closed formula. This makes it difficult to compute precise bounds for the discount factor δ for existence of responsive PBE. We do have the following partial analysis.

Let D be the smallest number E in $[0, 1]$ for which $\delta \cdot V(1) \geq V(\beta)$ for every positive $\delta \in [E, 1]$. Note that if $\delta \geq D$, then for each T, $\delta \geq D^T$ by Lemma 4. So, for $\delta \geq D$, existence and uniqueness of the responsive PBE hold, regardless of the horizon T of the game.

In particular, when $D = 0$, existence and uniqueness of the responsive PBE hold for any horizon T and any discount factor δ. The case $D = 0$ occurs for example when $\gamma \geq \beta$, where γ is the (unique) solution to the equation

$$x = g^{-1}\left(\delta \cdot \left[x \cdot g(x) + \int_x^1 g(\theta)d\theta\right]\right).$$

When $\gamma \geq \beta$, the receiver is at least as patient as the sender. For example, when $f = g$.

5. Existence and uniqueness of PBE, infinite horizon

Now we consider sender-receiver games with infinite horizon. Recall that if the receiver never quits then both players get payoff zero. The payoffs are either period-independent or they are discounted. For the discounted case with sufficiently large discount factors, we prove that the regular strategy profile with stationary threshold strategy for the sender is the unique responsive PBE. We also show that a responsive PBE fails to exist for period-independent payoffs.

Theorem 3. *Consider a sender-receiver stopping game with infinite horizon in which the payoffs are discounted with discount factor $\delta \geq D$. Then the regular strategy profile is the unique responsive PBE among the strategy profiles with HIEP.*

We remark that $D < 1$ for a large class of games. Indeed, $D < 1$ when f is Lipschitz continuous. We conjecture that for $\delta < D$, there does not exist any responsive PBE.

Now we turn to period-independent payoffs. Note that, by Proposition 2, the regular strategy profile is stationary, with threshold β. However, by Lemma 2, $\delta = 1$ implies $\beta = 1$. The outcome of the associated regular strategy profile is to continue forever. This is evidently not an equilibrium outcome. This is also reflected in the following Proposition.

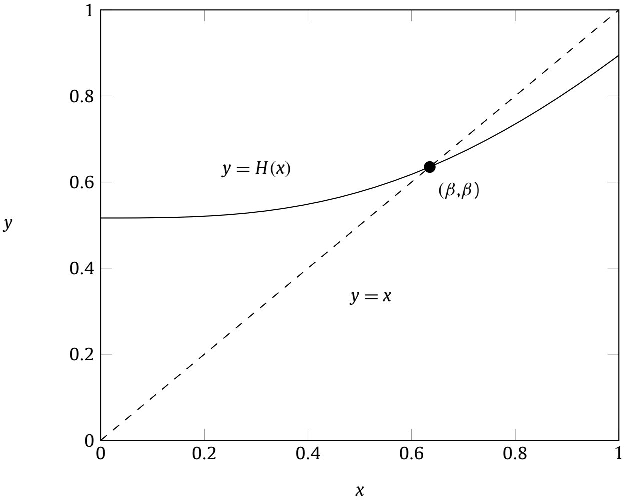
Proposition 3. *In a sender-receiver stopping game with infinite horizon and period-independent payoffs, there does not exist a responsive PBE in strategy profiles with HIEP.*

The result of Proposition 3 is driven by the following observation. At period 1, with probability 1 a state strictly less than 1 is realized. Since the game has an infinite horizon, both players know that if they wait sufficiently long, a strictly better state will be realized later on. Hence, both would like the game to continue. Since this argument holds for each period, the game is never terminated.

Example 2. The example serves several purposes, which we summarize at the end of the example. But its main purpose is to provide an example with a responsive PBE that differs from the regular strategy profile.

This objective forces us to deviate slightly from our standard setting. In this example, players have different discount factors. Let $\delta_S \in (0, 1)$ be the discount factor for the sender, and $\delta_R = 1$ the discount factor for the receiver. So, the receiver has period-independent payoffs. Take $f(x) = g(x) = x$. Take $\alpha \in (0, 1)$.

STRATEGY Y OF RECEIVER. At each period t, the receiver is in one of two possible modes, mode A or mode B. At $t = 1$, the receiver is in mode A. At period t in mode A, if $m^t = m_c$, the receiver chooses action $a^t = a_c$, and switches to mode B at period $t + 1$. If $m^t = m_q$, the receiver chooses action $a^t = a_c$ with probability α, and (if a_c is chosen) at period $t + 1$ continues in mode A. At period t in mode B, the receiver plays obediently, and remains in mode B at period $t + 1$.



T	$\beta^1(T)$
1	0
2	0.51640
3	0.58319
4	0.61029
5	0.62281
10	0.63460
.	
.	
.	
∞	$\beta = 0.63500$

Fig. 1. $f(x) = x^2$, $\delta = 0.8$.

STRATEGY X OF SENDER. The sender is also in one of two possible modes, E or F. He starts in mode E. When at period t the sender has only chosen m_q in previous periods, he is in mode E. Otherwise, he is in mode F. In mode E the sender chooses m_c. In mode F the sender plays the strategy that is sincere against the obedient strategy.

For sufficiently high values of δ_S and α, the strategy pair (X, Y) is a PBE. The following features of this PBE are noteworthy. The strategy Y of the receiver is responsive, but not obedient. The strategy X of the sender uses thresholds after each history, but the thresholds are history dependent. Therefore, X is not a threshold strategy. In particular, X does not satisfy HIEP. This indicates that the HIEP condition in Theorem 3 is relevant. Finally, the example indicates that, in Proposition 3, only the sender needs to have period independent payoffs. □

A strategy profile (σ, τ) is a ϵ-PBE if for every period t, every $\epsilon > 0$, and every history h^t_s, we have $U^t_s(\sigma, \tau)(h^t_s) \geq U^t_s(\sigma', \tau)(h^t_s) - \epsilon$ for every strategy σ' of the sender, and for every h^t_r, $U^t_r(\sigma, \tau)(h^t_r) \geq U^t_r(\sigma, \tau')(h^t_r) - \epsilon$ for every strategy τ' of the receiver. An ϵ-PBE is a responsive ϵ-PBE if the receiver's strategy is responsive. In infinite horizon sender-receiver stopping games with period-independent payoffs, there exists a responsive ϵ-PBE, which is a strategy profile with HIEP. In this PBE, the sender plays a stationary threshold strategy with threshold $\alpha < 1$ such that $f(\alpha) \geq (1 - \epsilon) \cdot f(1)$ and the receiver simply obeys the sender.

Proposition 4. *Let $\epsilon > 0$. For any sender-receiver stopping game with infinite horizon and period-independent payoffs, there exists a stationary responsive ϵ-PBE in strategy profiles with HIEP.*

The following is an immediate consequence of Corollary 1, Theorems 1, 3 and Proposition 3.

Corollary 2. *If the payoffs are discounted ($\delta < 1$), the sequence of PBEs in the finite horizon games converges to a PBE in the infinite horizon game. If the payoffs are period-independent, the sequence of PBE in the finite horizon games converges to the regular strategy profile in the infinite horizon game, which is not a PBE.*

6. Examples

In this section we illustrate our results with the help of the examples.

Example 1 *(revisited).* Consider the game in which the payoff functions are δ discounted with $f(x) = x^2$ and $g(x) = x$. Then $H(x) = \sqrt{\frac{\delta\cdot(1+2x^3)}{3}}$. The graph of H is sketched below in Fig. 1. For $\delta = \frac{4}{5}$, the table below illustrates Lemma 3 [3] in Appendix D.

We first consider the game with finite horizon T. By Proposition 1, the game has a unique regular strategy profile. We determine for which values of discount factor this profile is a PBE. Since $g(x) = x$, we have $V(x) = \frac{x}{2}$. Then the inequality $\delta \cdot V(1) \geq V(\beta^1(T))$ simplifies to $\delta \geq \beta^1(T)$.

For $T = 3$ we compute that $\beta^1(3) = \sqrt{\frac{3\sqrt{3}\cdot\delta+2\cdot\delta^{2.5}}{9\sqrt{3}}}$. The inequality $\delta \geq \beta^1(3)$ then yields $D^3 \approx 0.361$. By Theorem 1 the regular strategy profile is the unique responsive PBE for $\delta \in [D^3, 1]$.

For $T=2$, $D^2=\frac{1}{3}$. So, the regular strategy profile is the unique responsive PBE for $\delta \in [\frac{1}{3},1]$. For $T \geq 2$, if $\delta \in (0, D^2)$, by Theorem 2 the game has no responsive PBE among the profiles with HIEP.

Now we consider the setting in which the game has infinite horizon and $\delta < 1$. By Proposition 2, the game has a unique regular strategy profile. In this profile, the receiver plays the obeying strategy and the sender plays the sincere strategy (against the receiver's strategy) with threshold β at each period t. Here β is the unique solution in $[0,1]$ to the equation $H(x)=x$.

We have $H(\beta)=\beta$, which can be rewritten as $\delta = \frac{3\beta^2}{1+2\beta^3}$. So $\beta < 1$. Further, the inequality $\delta \cdot V(1) \geq V(\beta)$ is equivalent to $\delta \geq \beta$. Then D solves the equation $2\delta x^3 - 3x^2 + \delta = 0$. We find that $D \approx 0.366$. So, by Theorem 3, if $\delta \in [D,1)$, the regular strategy profile is the unique responsive PBE.

If the payoffs are period independent, then by Proposition 3, there is no responsive PBE among the strategy profiles with HIEP. Indeed, in this case, we have $\beta = 1$. So, in the regular strategy profile, the sender sends message m_c whenever the state is less than 1 and the receiver obeys. Hence, the game is played forever with probability 1, and both players receive expected payoff 0. □

Example 3. Here we assume $f(x)=x^2$ and $g(x)=x^3$. As the payoff function of the sender is same as in Example 1, so is the unique regular strategy profile. As $g(x)=x^3$, we have $V(x)=\frac{x^3}{4}$. In this case, we have $D^T=0$ for each $T \geq 1$ and $D=0$. Hence, in the setting with a finite horizon T (for $T \geq 1$), the regular strategy profile is the unique responsive PBE for $\delta \in (0,1]$ and in the setting with the infinite horizon, the regular strategy profile is the unique responsive PBE for $\delta \in (0,1)$. □

Our results are based on the assumption that both payoffs are strictly increasing. For any strictly increasing f and strictly decreasing g, there does not exist any responsive PBE. In the following example as g is neither strictly increasing nor strictly decreasing, the conditions in Theorem 1 are not satisfied, and the game does not admit any responsive PBE.

Example 4. Assume $f(x)=x$, $g(x)=x\cdot(1-x)$ for all x, $\delta=1$, and $T \geq 3$. We argue that there does not exist a responsive PBE. Suppose by way of contradiction that the players play a responsive PBE. At period T, the receiver quits and gets expected payoff $\frac{1}{6}$, while the sender obtains $\frac{1}{2}$. Thus, at period $T-1$, as the receiver is responsive, the sender plays a threshold strategy with threshold $\frac{1}{2}$. Then the receiver is indifferent between quit and continue, so his expected payoff is still $\frac{1}{6}$. Then at period $T-2$, the sender's threshold is $\frac{5}{8}$. When $\theta^{T-2} > \frac{5}{8}$, the sender sends the message m_q. It follows that the receiver gets expected payoff 0.14 on quitting, which is strictly less than his continuation payoff $\frac{1}{6}$. So, the receiver strictly prefers to continue when the suggestion of the sender is to quit. This violates responsiveness. The conclusion also holds for sufficiently large values of $\delta < 1$. □

7. Concluding remarks

This paper shows that the model of sender-receiver stopping games differs from the other models of dynamic sender-receiver games in the literature. The striking feature about this model stated by our main results is that under the responsive PBE the sender plays the optimal threshold strategy and the receiver simply obeys. This is surprising, as the receiver has to comply with the sender regardless his own payoff function. Under the responsive PBE the sender gets the maximum possible payoff for himself. Hence, the delegation of the decision making to the receiver does not hurt the sender.

As the sender can choose only from two messages, full information revelation is not possible. However, limiting the message space of the sender to only two messages does not fundamentally restrict the set of responsive PBEs. In a setting in which the sender can choose from more than two messages, in any responsive PBE, both the sender and the receiver obtain the same expected payoff compared to the unique responsive PBE in the setting in which the sender can only send two messages.

We see many interesting extensions of our model and open question to be addressed in future work. For most of our results, the assumption to have the same discount factor for the sender and the receiver can be easily relaxed.[5] Another immediate extension would be to have arbitrary distributions of the state of the world. The simple case in which nature draws an iid state from a strictly increasing continuous distribution is discussed in Appendix C. A challenging future extension would be the case in which the state of the world follows a Markov chain. We are currently investigating the situation with multiple senders, so that the receiver can make better informed decisions.

8. The proofs

Throughout the remainder of this paper, we fix the strictly increasing continuous functions f and g from I to $\mathbb{R}_+$ with $f(0)=g(0)=0$. Recall that $H: I \to \mathbb{R}$ is defined by

[5] This observation is due to an anonymous referee.

$$H(x) = f^{-1}\left(\delta \cdot \left[x \cdot f(x) + \int_x^1 f(\theta)d\theta\right]\right).$$

We also sometimes use $G: I \to \mathbb{R}$, defined by $G(x) = x \cdot f(x) + \int_x^1 f(\theta)d\theta$. Note that $f(H(x)) = \delta \cdot G(x)$.

8.1. *The proof of Theorem 1*

We will prove Theorem 1 in two parts: Claim 1 and Claim 2. So we fix a sender-receiver stopping game with finite horizon T, with payoffs that are either period-independent ($\delta = 1$) or discounted with discount factor $\delta \in (0, 1)$.

Claim 1. *Let $\delta \in [D^T, 1]$. Suppose that τ is responsive. If (σ, τ) is a PBE, then (σ, τ) is regular.*

Proof. Fix $\delta \in [D^T, 1]$. Suppose that (σ, τ) is a PBE with τ being responsive. We will prove that (σ, τ) is regular. We shall do so by proving that the sender's strategy σ is sincere and the receiver's strategy τ is obedient at each period $t = 1, \dots, T-1$. We apply backward induction by considering the periods in the order $T-1, T-2, \dots, 1$.

We recall from Section 2.1 that at period t, the expected payoffs $U_s^t(\sigma, \tau)$ and $U_r^t(\sigma, \tau)$ are functions of the histories of the sender and the receiver respectively. For $t = T-1, T-2, \dots, 1$, let $Q(t)$ be the following list of statements:

[1] σ is sincere at period t, and the corresponding threshold is equal to β^t.
[2] $U_r^{t+1}(\sigma, \tau) > \frac{1}{\beta^t}\int_0^{\beta^t} g^t(\theta)d\theta$.
[3] $U_r^{t+1}(\sigma, \tau) < \frac{1}{1-\beta^t} \cdot \int_{\beta^t}^1 g^t(\theta)d\theta$.
[4] τ is obedient at period t. That is, for each history h_r^t, $\tau^t(h_r^t, m_c) = 1$ and $\tau^t(h_r^t, m_q) = 0$.
[5] $U_s^t(\sigma, \tau)$ and $U_r^t(\sigma, \tau)$ do not depend on history up to period t.

For ease of notation, and suppressing possible history dependence, for every period $t = 1, \dots, T$ let $p^t = \tau^t(h_r^t, m_c)$ be the probability on the action a_c on seeing the message m_c, and let $q^t = \tau^t(h_r^t, m_q)$ be the probability on the action a_c on seeing the message m_q. As τ is responsive, $p^t > q^t$.

Remark on periods T and $T-1$. At period T, the sender always sends message m_q and the receiver always chooses action a_q. So, $U_s^T(\sigma, \tau)$ and $U_r^T(\sigma, \tau)$ are independent of the history up to period T. Using this, it is straightforward to verify $Q(T-1)$.

The induction step. We assume that $Q(t+1), \dots, Q(T-1)$ are true, for some $t \in \{1, \dots, T-2\}$. We show that $Q(t)$ is also true.

Item [1] of Q(t). Note that by item [5] of $Q(t+1)$ we know that $U_s^{t+1}(\sigma, \tau)$ does not depend on the history up to period $t+1$, and can therefore be treated as a real number. In the same way, $U_s^{t+2}(\sigma, \tau)$ is independent of the history up to period $t+2$. So,

$$\begin{aligned}
U_s^{t+1}(\sigma, \tau) &= \beta^{t+1} \cdot U_s^{t+2}(\sigma, \tau) + \int_{\beta^{t+1}}^1 f^{t+1}(\theta)d\theta \\
&= \beta^{t+1} \cdot f^{t+1}(\beta^{t+1}) + \int_{\beta^{t+1}}^1 f^{t+1}(\theta)d\theta \\
&= \delta^{t-1} \cdot \delta \cdot \left[\beta^{t+1} \cdot f(\beta^{t+1}) + \int_{\beta^{t+1}}^1 f(\theta)d\theta\right] \\
&= \delta^{t-1} \cdot f(H(\beta^{t+1})) = \delta^{t-1} \cdot f(\beta^t) = f^t(\beta^t).
\end{aligned}$$

For the first equality, we use items [1] and [4] of $Q(t+1)$. For the second equality, we use item [1] of $Q(t+1)$ and the definition of sincere strategy. For the fifth equality, we use the definition of β^t.

Thus, when $\theta^t < \beta^t$, the sender expects to get a strictly better payoff if the receiver continues than when the receiver quits. As $p^t > q^t$, the sender strictly prefers to send the message m_c over m_q if $\theta^t < \beta^t$. By the same reasoning, the sender strictly prefers to send the message m_q over m_c if $\theta^t > \beta^t$. Hence, σ is sincere at period t with threshold value β^t.

Item [2] of Q(t). Quitting at period $t+1$ guarantees payoff $\int_0^1 g^{t+1}(\theta)d\theta$ for the receiver. So, using item [2] of Lemma 3, we get $\frac{1}{\beta^t} \cdot \int_0^{\beta^t} g^t(\theta)d\theta < \int_0^1 g^{t+1}(\theta)d\theta \le U_r^{t+1}(\sigma, \tau)$. This proves item [2] of $Q(t)$.

Item [3] of Q(t). We have

$$U_r^{t+1}(\sigma,\tau) = \beta^{t+1}\cdot U_r^{t+2}(\sigma,\tau) + \int_{\beta^{t+1}}^{1} g^{t+1}(\theta)d\theta$$
$$< \frac{1}{1-\beta^{t+1}}\cdot\int_{\beta^{t+1}}^{1} g^{t+1}(\theta)d\theta$$
$$\le \frac{1}{1-\beta^{t+1}}\cdot\int_{\beta^{t+1}}^{1} g^{t}(\theta)d\theta < \frac{1}{1-\beta^{t}}\cdot\int_{\beta^{t}}^{1} g^{t}(\theta)d\theta.$$

In the equality, we use items [1] and [4] of $Q(t+1)$. In the first inequality, we use item [3] of $Q(t+1)$. For the last inequality, we use Lemma 5 and item [1] of Lemma 3 [1].

Item [4] of Q(t). Assume the message is m_c at period t. Then by item [1] of $Q(t)$, we have $\theta^t \in [0,\beta^t]$. We want to show that τ plays action a_c at period t. The expected payoff for the receiver on quitting is $\frac{1}{\beta^t}\int_0^{\beta^t} g^t(\theta)d\theta$. By item [2] of $Q(t)$ we have $U_r^{t+1}(\sigma,\tau) > \frac{1}{\beta^t}\int_0^{\beta^t} g^t(\theta)d\theta$. Since (σ,τ) is a PBE, τ has to play action a_c at period t.

Now, assume the message is m_q at period t. Then by item [1] of $Q(t)$, we have $\theta^t \in [\beta^t, 1]$. We want to show that τ plays action a_q at period t. The expected payoff for the receiver on quitting is $\frac{1}{1-\beta^t}\int_{\beta^t}^{1} g^t(\theta)d\theta$. By [3] of $Q(t)$ we have $U_r^{t+1}(\sigma,\tau) < \frac{1}{1-\beta^t}\int_{\beta^t}^{1} g^t(\theta)d\theta$. Since (σ,τ) is a PBE, τ has to play action a_q at period t.

Item [5] of Q(t). By using items [1] and [4] of $Q(t),\ldots,Q(T-1)$ and the remark on period T, the statement follows immediately. □

Claim 2. *Let $\delta \in [D^T, 1]$. Then the regular strategy profile is a PBE.*

Proof. Let (σ,τ) be the regular strategy profile, and $\delta \ge D^T$. By Proposition 1, the threshold used by σ at period t is β^t. We show that (σ,τ) is a PBE in three subclaims.

Claim 2a. For each period $t = 1,\ldots,T-1$, let $Q(t)$ be the statement that following conditions hold:

[1] If the receiver gets the message m_q at period t, then $U_r^{t+1}(\sigma,\tau) < \frac{1}{1-\beta^t}\cdot\int_{\beta^t}^{1} g^t(\theta)d\theta$.

[2] If the receiver gets the message m_c at period t, then $U_r^{t+1}(\sigma,\tau) > \frac{1}{\beta^t}\int_0^{\beta^t} g^t(\theta)d\theta$.

We prove that $Q(t)$ holds for all $t = 1,\ldots,T-1$ by backwards induction.

Proof of claim 2a. First we prove $Q(T-1)$. By Lemma 5,

$$U_r^{T}(\sigma,\tau) = \int_0^1 g^T(\theta)d\theta \le \int_0^1 g^{T-1}(\theta)d\theta < \frac{1}{1-\beta^{T-1}}\cdot\int_{\beta^{T-1}}^{1} g^{T-1}(\theta)d\theta,$$

which proves item [1] of $Q(T-1)$. Item [2] of $Q(T-1)$ follows from Lemma 3.

Now assume that $Q(t+1)$ is true, where $t \in \{1,\ldots,T-2\}$. We prove that $Q(t)$ also holds. Item [1] of $Q(t)$ can be proved similarly as in item [3] of Claim 1. To prove the item [2] of $Q(t)$, we have

$$
\begin{aligned}
U_r^{t+1}(\sigma,\tau) &= \beta^{t+1}\cdot U_r^{t+2}(\sigma,\tau) + \int_{\beta^{t+1}}^{1} g^{t+1}(\theta)d\theta \\
&\geq \beta^{t+1}\cdot\frac{1}{\beta^{t+1}}\cdot\int_{0}^{\beta^{t+1}} g^{t+1}(\theta)d\theta + \int_{\beta^{t+1}}^{1} g^{t+1}(\theta)d\theta \\
&= \int_0^1 g^{t+1}(\theta)d\theta > \frac{1}{\beta^t}\cdot\int_0^{\beta^t} g^t(\theta)d\theta.
\end{aligned}
$$

In the first inequality, we use item [2] of $Q(t+1)$. In the second inequality, since $\delta \geq D^T$, we can use Lemma 3, item [2]. This completes the proof of item [2] of $Q(t)$.

Claim 2b. For any period $t = 1,\ldots,T-1$, the obedient strategy τ of the receiver is a best response against the sincere strategy σ of the sender.

Proof of claim 2b. Fix period $t \in \{1,\ldots,T-1\}$. Consider the case where the receiver gets message m_q. Because the sender is using threshold β^t, we know that $\theta^t \in [\beta^t, 1]$. So, the expected payoff for the receiver on quitting is $\frac{1}{1-\beta^t}\int_{\beta^t}^{1} g^t(\theta)d\theta$. By claim 2a, we have $U_r^{t+1}(\sigma,\tau) < \frac{1}{1-\beta^t}\cdot\int_{\beta^t}^{1} g^t(\theta)d\theta$. Hence, it is a best response for the receiver to play a_q.

Next, suppose that the receiver gets message m_c. The sender uses threshold β^t, so $\theta^t \in [0,\beta^t]$. So, the payoff for the receiver on quitting is $\frac{1}{\beta^t}\int_0^{\beta^t} g^t(\theta)d\theta$. Then by claim 2a, $U_r^{t+1}(\sigma,\tau) > \frac{1}{\beta^t}\cdot\int_0^{\beta^t} g^t(\theta)d\theta$. Hence, it is a best response for the receiver to play a_c.

Claim 2c. As the final step, we argue that (σ,τ) is a PBE.

Proof of claim 2c. By claim 2b, at any period, τ is best response against σ. Conversely, we show that σ is a best response against τ. Recall that for each period $t = 1,\ldots,T-1$, the expected payoff $U_s^{t+1}(\sigma,\tau)$ is history independent. Since τ is obedient, the sender receives payoff $f^t(\theta^t)$ if σ sends m_q at period t and payoff $U_s^{t+1}(\sigma,\tau)$ if σ sends m_c. So, the sender plays a best response at period t when he sends m_c if $f^t(\theta^t) < U_s^{t+1}(\sigma,\tau)$ and m_q if $f^t(\theta^t) > U_s^{t+1}(\sigma,\tau)$. Hence, since threshold β^t is such that $f^t(\beta^t) < U_s^{t+1}(\sigma,\tau)$, the sender plays a best response at period t when he sends m_c if $\theta^t < \beta^t$ and m_q if $\theta^t > \beta^t$. Thus, the strategy σ is a best response against τ at period t. This completes the proof of the theorem. □

8.2. *The proof of Theorem 2*

In this section we prove Theorem 2. So, let f and g be strictly increasing functions from I to $\mathbb{R}_+$ with $f(0) = g(0) = 0$. Consider a finite horizon $T \geq 2$. Let (σ,τ) be any strategy profile in which τ is responsive. We show that (σ,τ) is not a PBE.

As the receiver must quit at period T, we have $U_s^T(\sigma,\tau) = \int_0^1 \delta^{T-1} f(\theta)d\theta$. Let $\alpha^{T-1} = f^{-1}\big(\delta\cdot\int_0^1 f(\theta)d\theta\big)$. At period $T-1$, if the receiver quits, the sender obtains the payoff $f^{T-1}(\theta^{T-1})$ and if the receiver continues, the sender obtains the expected payoff $U_s^T(\sigma,\tau)$. As τ is responsive, the sender strictly prefers to send the message m_c if $\theta^{T-1} < \alpha^{T-1}$ and the message m_q if $\theta^{T-1} > \alpha^{T-1}$. Hence, the strategy σ has a threshold α^{T-1} at period $T-1$.

As $\alpha^{T-1} > 0$, the receiver gets the message m_c at period $T-1$ with positive probability. If the receiver continues, he obtains the expected payoff $U_r^T(\sigma,\tau) = \int_0^1 \delta^{T-1} g(\theta)d\theta = \delta^{T-1}\cdot V(1)$. And if the receiver quits, he obtains the expected payoff $\frac{1}{\alpha^{T-1}}\int_0^{\alpha^{T-1}} \delta^{T-2} g(\theta)d\theta = \delta^{T-2}\cdot V(\alpha^{T-1})$.

As $\delta < D^2$, we have $\delta\cdot V(1) < V(\alpha^{T-1})$. So, the receiver strictly prefers to quit on receiving the message m_c. Thus, τ is not a best response. Hence, the game does not admit a responsive PBE. □

8.3. The proof of Theorem 3

We will prove Theorem 3 in two parts: Claim 3 and Claim 4. So we fix a sender-receiver stopping game with infinite horizon, and discounted payoffs with discount factor $\delta \in (0,1)$.

Claim 3. *Suppose that $\delta \in (D,1)$. Let (σ,τ) be a responsive PBE that satisfies HIEP. Then, (σ,τ) is the regular strategy profile.*

Proof. Suppose that $\delta \in (D,1)$. Let (σ,τ) be a responsive PBE and a strategy profile with HIEP.

Again, let $p^t = \tau^t(h_r^t, m_c)$ be the probability on the action a_c on seeing the message m_c, and let $q^t = \tau^t(h_r^t, m_q)$ be the probability on the action a_c on seeing the message m_q. Since τ is responsive, $p^t > q^t$ for each t. We prove the statement in a series of steps.

Step 1. The strategy σ has threshold α^t at each period $t \in \mathbb{N}$ that satisfies $f^t(\alpha^t) = U_s^{t+1}(\sigma,\tau)$. Hence, σ is sincere.

Proof of step 1. Fix a period $t \in \mathbb{N}$. Since (σ,τ) is a strategy profile with HIEP, $U_s^{t+1}(\sigma,\tau)$ is independent of the history up to period $t+1$. Further, note that $f^t(1) > f^{t+1}(1) \geq U_s^{t+1}(\sigma,\tau) \geq 0 = f^t(0)$. So, the equation $f^t(x) = U_s^{t+1}(\sigma,\tau)$ has the unique solution, say $\alpha^t \in [0,1)$.

Assume first that $\theta^t < \alpha^t$. Then, $f^t(\theta^t) < f^t(\alpha^t) = U_s^{t+1}(\sigma,\tau)$. Thus, the sender expects to get strictly better payoff when receiver continues than when receiver quits. As $p^t > q^t$, the sender strictly prefers message m_c over m_q. Because (σ,τ) is a PBE, σ sends the message m_c at period t.

Assume that $\theta^t > \alpha^t$. Then, $f^t(\theta^t) > f^t(\alpha^t) = U_s^{t+1}(\sigma,\tau)$. By the same reasoning, σ sends message m_q at period t. Thus, σ is sincere at period t with threshold α^t. This completes the proof of step 1.

Step 2. Define for each period $t \in \mathbb{N}$, $K(t) = \frac{1}{\alpha^t} \cdot \int_0^{\alpha^t} g^t(\theta)d\theta$ and $L(t) = \frac{1}{1-\alpha^t} \cdot \int_{\alpha^t}^1 g^t(\theta)d\theta$. Then, for every $t \in \mathbb{N}$ we have $K(t) \leq U_r^{t+1}(\sigma,\tau) \leq L(t)$.

Proof of step 2. Fix $t \in \mathbb{N}$. As τ is responsive, we know that $1 \geq p^t > q^t \geq 0$. We know that at period t, the strategy σ uses threshold α^t. So, $K(t)$ is the expected payoff to the receiver when he quits, upon getting the message m_c. Since $p^t > 0$, it follows that $K(t) \leq U_r^{t+1}(\sigma,\tau)$. Similarly, $L(t)$ is the expected payoff to the receiver when he quits, upon getting the message m_q. Since $q^t < 1$, it follows that $U_r^{t+1}(\sigma,\tau) \leq L(t)$. This completes the proof of step 2.

Step 3. For each period $t \in \mathbb{N}$ we have $\alpha^t \leq \beta$.

Proof of Step 3. Fix $t \in \mathbb{N}$. Suppose by way of contradiction that $\alpha^t > \beta$. Note that

$$f^t(\alpha^t) = U_s^{t+1}(\sigma,\tau) \leq \alpha^{t+1} \cdot U_s^{t+2}(\sigma,\tau) + \int_{\alpha^{t+1}}^1 f^{t+1}(\theta)d\theta$$

$$= \alpha^{t+1} \cdot f^{t+1}(\alpha^{t+1}) + \int_{\alpha^{t+1}}^1 f^{t+1}(\theta)d\theta = \delta^t \cdot G(\alpha^{t+1}).$$

The first and second equalities follow from the definitions of α^t and α^{t+1} and the last equality from the definition of G. It follows that $f(\alpha^t) \leq \delta \cdot G(\alpha^{t+1})$. So, $\alpha^t \leq H(\alpha^{t+1})$. As $\alpha^t > \beta$ by assumption, by item [3] of Lemma 2, $H\big(\alpha^t\big) < \alpha^t \leq H(\alpha^{t+1})$. This implies $\alpha^t < \alpha^{t+1}$. In particular, $\alpha^{t+1} > \beta$. By iterating the argument, we can conclude that the sequence $(\alpha^t)_{t=1}^{\infty}$ is convergent, since the sequence is bounded from above, and a tail of the sequence is strictly increasing.

Say, the sequence $(\alpha^t)_{t=1}^{\infty}$ converges to $z \leq 1$. By continuity of H, we have $z = \beta$. On the other hand, $\beta < \alpha^t \leq z$. This is a contradiction. This completes the proof of step 3.

Step 4. The strategy τ is obedient.

Proof of Step 4. We prove this claim in two parts.

Part A. We show that $\tau^t(h_r^t, m_c) = p^t = 1$ for every period t and history h_r^t of the receiver. Take any $t \in \mathbb{N}$. It is sufficient to show that $K(t) < U_r^{t+1}(\sigma,\tau)$. By step 3, $\alpha^t \leq \beta$. It holds that

$$K(t) = \frac{1}{\alpha^t} \cdot \int_0^{\alpha^t} g^t(\theta)d\theta \leq \frac{1}{\beta} \cdot \int_0^{\beta} g^t(\theta)d\theta < \int_0^1 g^{t+1}(\theta)d\theta \leq U_r^{t+1}(\sigma,\tau).$$

In the first inequality, we use Lemma 5. In the second inequality, we use item [3] of Lemma 4 and the assumption that $\delta \cdot V(1) > V(\beta)$. Since (σ,τ) is a PBE, the last inequality follows because quitting at period $t+1$ cannot be better for the receiver than playing τ against σ. This completes the proof of part A.

Part B. We show that $\tau^t(h_r^t, m_q) = q^t = 0$ for every period t and history h_r^t of the receiver. For every $t \in \mathbb{N}$, let $Q(t)$ be the statement $U_r^{t+1}(\sigma,\tau) < L(t)$. As (σ,τ) is a PBE, $Q(t)$ implies that $q^t = 0$. Thus, it is sufficient to show that $Q(t)$ is true for every $t \in \mathbb{N}$.

Part B1. We show that $Q(t+1)$ implies $Q(t)$. Suppose that $Q(t+1)$ is true, so $q^{t+1}=0$. Together with part A, it follows that τ is obedient at period $t+1$. Then

$$f^t(\alpha^t)=U_s^{t+1}(\sigma,\tau)=\alpha^{t+1}\cdot U_s^{t+2}(\sigma,\tau)+\int_{\alpha^{t+1}}^{1} f^{t+1}(\theta)d\theta$$
$$=\alpha^{t+1}\cdot f^{t+1}(\alpha^{t+1})+\int_{\alpha^{t+1}}^{1} f^{t+1}(\theta)d\theta=\delta^t\cdot G(\alpha^{t+1}).$$

The first and third equalities follow from the definitions of α^t and α^{t+1} and the last equality from the definition of G. The second equality follows from τ being obedient at period $t+1$.

It follows that $\alpha^t=H(\alpha^{t+1})$. By step 3, we have $\alpha^t\le\beta$. Hence by item [3] of Lemma 2, $\alpha^t\ge\alpha^{t+1}$. Then by following the calculations similar to Item [3] of Claim 1, we get $U_r^{t+1}(\sigma,\tau)<\frac{1}{1-\alpha^t}\cdot\int_{\alpha^t}^{1} g^t(\theta)d\theta=L(t)$.

Part B2. We prove that $Q(t)$ is true for every t. Assume by way of contradiction that there is a $t\in\mathbb{N}$ for which $Q(t)$ is not true. By step 4.2.1, $Q(t')$ is not true for all $t'\ge t$. Then, by step 2, $U_r^{t'+1}(\sigma,\tau)=L(t')$ for all $t'\ge t$. Denote by E_r^{t+1} the expected payoff of the receiver conditional on getting the message m_q at period $t+1$. We have

$$E_r^{t+1}=q^{t+1}\cdot U_r^{t+2}(\sigma,\tau)+(1-q^{t+1})\cdot L(t+1)$$
$$=q^{t+1}\cdot U_r^{t+2}(\sigma,\tau)+(1-q^{t+1})\cdot U_r^{t+2}(\sigma,\tau)=U_r^{t+2}(\sigma,\tau).$$

The first equality can be explained as follows. On receiving message m_q at period $t+1$, strategy τ continues with probability q^{t+1} and on continuing the receiver gets payoff $U_r^{t+2}(\sigma,\tau)$, whereas τ quits with probability $1-q^{t+1}$ and on quitting the receiver gets payoff $L(t+1)$. It follows that

$$U_r^{t+1}(\sigma,\tau)=\alpha^{t+1}\cdot U_r^{t+2}(\sigma,\tau)+(1-\alpha^{t+1})\cdot E_r^{t+1}=U_r^{t+2}(\sigma,\tau).$$

In the first equality we use the step 4.1.

So, $U_r^{t+1}(\sigma,\tau)=U_r^{t+2}(\sigma,\tau)$. Iterating this argument implies that $U_r^{t+1}(\sigma,\tau)=U_r^{t+j}(\sigma,\tau)$ for any j. As $U_r^{t+j}\le g^{t+j}(1)=\delta^{t+j-1}g(1)$, it then follows that $U_r^{t+1}(\sigma,\tau)=0$. However, if the receiver quits at period $t+1$ instead, regardless of the message, then he receives the expected payoff $\int_0^1 g^{t+1}(\theta)d\theta>0$. This contradicts the assumption that (σ,τ) is PBE. This completes the proof of step 4.2.2.

Now Claim 3 follows immediately from Proposition 2. □

Claim 4. *Suppose that $\delta\in(D,1)$. Then the regular strategy profile is a PBE.*

Proof. Let $\delta\in(D,1)$ and (σ,τ) be the regular strategy profile. By Proposition 2, σ is stationary with threshold β. Clearly, $U_r^{t+1}(\sigma,\tau)=\delta\cdot U_r^t(\sigma,\tau)$ for all periods t. We show that (σ,τ) is a PBE. Since (σ,τ) is regular, we have $U_r^t(\sigma,\tau)=\beta\cdot U_r^{t+1}(\sigma,\tau)+\int_\beta^1 g^t(\theta)d\theta=\beta\cdot\delta\cdot U_r^t(\sigma,\tau)+\int_\beta^1 g^t(\theta)d\theta.$

$$U_r^t(\sigma,\tau)=\frac{\delta^{t-1}}{1-\delta\cdot\beta}\cdot\int_\beta^1 g(\theta)d\theta. \tag{1}$$

To prove that the receiver prefers to play a_q on seeing m_q at period t, we need to show that $U_r^{t+1}(\sigma,\tau)\le\frac{\delta^{t-1}}{1-\beta}\int_\beta^1 g(\theta)d\theta$. This follows easily from (1). Finally, to prove that the receiver prefers to play a_c on seeing m_c at period t, we need to show that $U_r^{t+1}(\sigma,\tau)\ge\frac{\delta^{t-1}}{\beta}\int_0^\beta g(\theta)d\theta$. Using (1), the above inequality can be rewritten to $\frac{1}{\beta}\cdot\int_0^\beta g(\theta)d\theta<\delta\cdot\int_0^1 g(\theta)d\theta$. This follows from Lemma 4.3 and the condition $V(\beta)<\delta\cdot V(1)$, which is due to $\delta\in(D,1)$. □

Declaration of competing interest

None.

Data availability

No data was used for the research described in the article.

Appendix A. The receiver's belief on the history of the sender

In this appendix, we describe the receiver's conditional probability distribution (or belief) $\mathbb{P}_{\sigma,\tau,h_r^t}$ on the set H_s^t of possible histories for the sender, given the strategy profile (σ,τ) and the receiver's history $h_r^t=(m^1,m^2,\dots,m^{t-1})$.

Let $\sigma^k(m^k|h_s^k,\theta^k)$ denote the probability on the message m^k under the strategy σ, given the history h_s^k and the state θ^k. For numbers $y^1,y^2,\dots,y^{t-1}\in[0,1]$, the expression

$$\int_0^{y^1}\int_0^{y^2}\cdots\int_0^{y^{t-1}}\left[\prod_{k=1}^{t-1}\sigma^k(m^k|\theta^1,m^1,\dots,\theta^{k-1},m^{k-1},\theta^k)\right]d\theta^{t-1}\,d\theta^{t-2}\dots d\theta^1$$

is the probability of the event that $\theta^1\le y^1$, $\theta^2\le y^2$,..., $\theta^{t-1}\le y^{t-1}$ and the messages sent are $m^1,m^2,\dots,m^{t-1}$. We denote this probability by $\chi^t_{(\sigma,\tau)}(h_r^t)(y^1,y^2,\dots,y^{t-1})$.

The quantity $\chi^t_{(\sigma,\tau)}(h_r^t)(1,1,\dots,1)$ is the probability that the history at period t is h_r^t. Thus, the probability of the event that $\theta^1\le y^1$, $\theta^2\le y^2$, ..., $\theta^{t-1}\le y^{t-1}$ conditional on the messages $m^1,m^2,\dots,m^{t-1}$ is

$$\Psi^t_{(\sigma,\tau)}(h_r^t)(y^1,y^2,\dots,y^{t-1})=\frac{\chi^t_{(\sigma,\tau)}(h_r^t)(y^1,y^2,\dots,y^{t-1})}{\chi^t_{(\sigma,\tau)}(h_r^t)(1,1,\dots,1)}.$$

If a certain history h_r^t occurs with probability zero, that is, if $\chi^t_{(\sigma,\tau)}(h_r^t)(1,1,\dots,1)=0$, then we define $\Psi^t_{(\sigma,\tau)}(h_r^t)$ to be any probability distribution. The choice of this probability distribution plays no role in our proofs. The probabilities $\Psi^t_{(\sigma,\tau)}(h_r^t)(y^1,y^2,\dots,y^{t-1})$ induce the desired probability measure $\mathbb{P}_{\sigma,\tau,h_r^t}$ on the possible histories h_s^t for the sender.

Appendix B. Expected payoff

In this appendix, we provide the details of how the expected payoffs $U_s(\sigma,\tau)$, $U_r(\sigma,\tau)$ and the continuation expected payoffs $U_s^t(\sigma,\tau)(h_s^t)$ and $U_r^t(\sigma,\tau)(h_r^t)$ from period t onward can be calculated.

It is both convenient and standard to assume that even if the receiver quits at some period t, play continues indefinitely, but actions in any period beyond t have no influence on the payoffs. With this assumption, a play of the game is a sequence $\omega=(\theta^t,m^t,a^t)_{t=1}^\infty$ where $\theta^t\in I$, $m^t\in M$ and $a^t\in A$. Denote by $\Omega=(I\times M\times A)^{\mathbb{N}}$ the set of all plays. Given the usual Borel sigma-algebra of I, we endow Ω with the product sigma-algebra $\mathcal{B}$.

With abuse of notation, define $\theta^t\colon\Omega\to I$, $m^t\colon\Omega\to M$ and $a^t\colon\Omega\to A$ to be the projection maps from the set of plays, respectively to the state, the message and the action at period t. Let $S\colon\Omega\to\mathbb{N}\cup\{\infty\}$ be the mapping such that, for each $\omega\in\Omega$, $S(\omega)$ is the first period t for which $a^t(\omega)=a_q$. If there is no such t then $S(\omega)=\infty$. It is the stopping time which indicates when the game effectively ends. For a play ω, the payoffs for the players are given as follows

$$\Pi_s(\omega)=f^{S(\omega)}\Big(\theta^{S(\omega)}(\omega)\Big)\cdot\mathbb{1}_{\{S(\omega)<\infty\}},\qquad \Pi_r(\omega)=g^{S(\omega)}\Big(\theta^{S(\omega)}(\omega)\Big)\cdot\mathbb{1}_{\{S(\omega)<\infty\}}.$$

Any fixed strategy profile (σ,τ) induces a probability measure on the measurable space $(\Omega,\mathcal{B})$, denoted by $\mathbb{P}_{\sigma,\tau}$. The expectation with respect to this probability measure is denoted by $\mathbb{E}_{\sigma,\tau}$. The expected payoff for the sender is given by $U_s(\sigma,\tau)=\mathbb{E}_{\sigma,\tau}\big[\Pi_s(\omega)\big]$ and the expected payoff for the receiver is given by $U_r(\sigma,\tau)=\mathbb{E}_{\sigma,\tau}\big[\Pi_r(\omega)\big]$.

Let $\Omega^{\ge t}$ denote the set of all continuation plays $\omega^{\ge t}=(\theta^k,m^k,a^k)_{k=t}^\infty$. Given a history $h_s^t\in H_s^t$ for the sender, the continuation strategy $\sigma[h_s^t]=(\sigma^k[h_s^t])_{k=1}^\infty$ of σ is defined in the usual way: for each period $k\in\mathbb{N}$, history $\overline{h}_s^k\in H_s^k$ and state $\theta^k\in I$ we let

$$\sigma^k[h_s^t](h_s^k,\theta^k)=\sigma^{t+k-1}(h_s^t,h_s^k,\theta^k).$$

Given a history $h_r^t\in H_r^t$ for the receiver, we define in a similar way the continuation strategy $\tau[h_r^t]=(\tau^k[h_r^t])_{k=1}^\infty$ of τ.

For each period t, let $\pi^t\colon H_s^t\to H_r^t$ be the map that projects the sender's history to the receiver's history. For a given history h_s^t of the sender, the continuation strategies $\sigma[h_s^t]$ and $\tau[\pi(h_s^t)]$ induce a probability measure on the space $(\Omega,\mathcal{B})$, denoted by $\mathbb{P}_{\sigma,\tau,h_s^t}$. The expected continuation payoff for the sender is given by $U_s^t(\sigma,\tau)(h_s^t)=\mathbb{E}_{\sigma,\tau,h_s^t}\big[\Pi_s(\omega)\big]$.

As discussed in Appendix A, the receiver has a probability distribution (belief) $\mathbb{P}_{\sigma,\tau,h_r^t}$ on H_s^t, conditional on his history h_r^t. The expected continuation payoff for receiver can be calculated as follows

$$U_r^t(\sigma,\tau)(h_r^t)=\int_{H_s^t}U_r(\sigma[h_s^t],\tau[h_r^t])\,\mathbb{P}_{\sigma,\tau,h_r^t}(dh_s^t).$$

Here $U_r(\sigma[h_s^t], \tau[h_r^t])$ is the receiver's expected payoff given the continuation strategies $\sigma[h_s^t]$ and $\tau[h_r^t]$.

Appendix C. Extension: arbitrary distribution

We consider an extension in which the states at each period are drawn from an arbitrary distribution for the games with finite or infinite horizon and with payoffs that are discounted or period-independent.

Consider a sender-receiver game where the payoffs are either discounted ($\delta < 1$) or period-independent ($\delta = 1$). Let the functions f and g from I to $\mathbb{R}_+$ be strictly increasing with $f(0) = g(0) = 0$. At each period t, the state θ^t is drawn from a fixed cumulative distribution F on $[0, 1]$, independently from realized states of previous periods. We assume that F is strictly increasing and continuous on $[0, 1]$ and $F(0) = 0$. We denote this game by $\mathcal{G}^F$.

Using the game $\mathcal{G}^F$, we define a new game $\mathcal{G}^u$ with the same horizon T in which the states at each period t in the game $\mathcal{G}^u$ are drawn from the uniform distribution independently from states of previous periods. The game $\mathcal{G}^u$ has the same δ as the game $\mathcal{G}^F$ and has the functions $\hat{f}$ and $\hat{g}$ which are defined as follows: $\hat{f}(x) = f(F^{-1}(x))$, $\hat{g}(x) = g(F^{-1}(x))$.

Given a strategy profile (σ, τ) in the game $\mathcal{G}^F$, consider a strategy profile $(\hat{\sigma}, \hat{\tau})$ in the game $\mathcal{G}^u$, defined as follows: $\hat{\sigma}^t(\theta^t) = \sigma^t(F^{-1}(\theta^t))$ and $\hat{\tau}^t(m^t) = \tau^t(m^t)$. It is straightforward, but tedious to show that the payoffs of the players in the game $\mathcal{G}^u$ when the strategy profile is $(\hat{\sigma}, \hat{\tau})$ and in the game $\mathcal{G}^F$ when the strategy profile is (σ, τ) are exactly same.

Under this transformation, the receiver's strategy remains the same. If the sender's strategy σ in $\mathcal{G}^F$ is a threshold strategy with a threshold α^t at period t, then $\hat{\sigma}$ is also a threshold strategy with threshold $F^{-1}(\alpha^t)$ at period t. So, the regular strategy profile in $\mathcal{G}^F$ is transformed into the regular strategy profile in $\mathcal{G}^u$. Hence, the existence and uniqueness results in the game $\mathcal{G}^u$ can be used to derive the existence and uniqueness results in the game $\mathcal{G}^F$.

Appendix D. Auxiliary lemmas

Lemma 2. *The following statements hold:*

[1] *The function H is strictly increasing,*
[2] *The function H has a unique fixed point, denoted by β,*
[3] *$H(y) > y$ for all $y < \beta$ and $H(y) < y$ for all $y > \beta$,*
[4] *$\beta \to 1$ as $\delta \to 1$ and $\beta = 1$ when $\delta = 1$.*

Proof. [1] Take $0 \leq x < y \leq 1$. Because $f(x) < f(y)$, we have

$$G(y) - G(x) = y \cdot f(y) - x \cdot f(x) - \int_x^y f(\theta)d\theta \geq (y - x) \cdot f(y) - \int_x^y f(\theta)d\theta > 0.$$

Hence, G is strictly increasing. The same then holds for H.

[2] We have $H(0) > 0$ and $H(1) \leq 1$. As H is strictly increasing and continuous, there is at least one $x \in I$ with $H(x) = x$. Let $\beta = \inf\{x \in I \mid H(x) = x\}$. By continuity of H, we have $H(\beta) = \beta$. Now we will prove part 3 of the Lemma. This will imply that β is the unique solution of $H(x) = x$.

[3] Take $0 \leq x < y \leq 1$. Using the notation from [1], we have

$$\begin{aligned} G(y) - G(x) &= y \cdot f(y) - x \cdot f(x) - \int_x^y f(\theta)d\theta \\ &< y \cdot f(y) - x \cdot f(x) - (y - x) \cdot f(x) \\ &= y \cdot (f(y) - f(x)) \leq f(y) - f(x). \end{aligned}$$

Then also $\delta \cdot (G(y) - G(x)) < f(y) - f(x)$, so that $\delta \cdot G(y) - f(y) < \delta \cdot G(x) - f(x)$. In particular, if $y > \beta$, then

$$f(H(y)) - f(y) = \delta \cdot G(y) - f(y) < \delta \cdot G(\beta) - f(\beta) = f(H(\beta)) - f(\beta) = f(\beta) - f(\beta) = 0.$$

It follows that $f(H(y)) < f(y)$. Hence, also $H(y) < y$.

[4] As β is the unique fixed point of H, and for $\delta = 1$ we have $H(1) = 1$, it follows that $\beta = 1$ when $\delta = 1$. A continuity argument shows that $\beta \to 1$ as $\delta \to 1$. This completes the proof. □

Lemma 3. *For the model with finite time horizon T, the following statements hold:*

[1] $1 \geq \beta > \beta^1 > \beta^2 > \cdots > \beta^T = 0.$
[2] $\frac{1}{\beta^t} \cdot \int_0^{\beta^t} g^t(\theta)d\theta < \int_0^1 g^{t+1}(\theta)d\theta$ *for $\delta \in [D^T, 1]$ and $t = 1, \ldots, T - 1$.*

[3] $\beta^1(T) \to \beta$ *as* $T \to \infty$. *More generally,* $\beta^t(T) \to \beta$ *as* $T \to \infty$ *for each* t.

Proof. [1] By definition, $\beta^T = 0$. We also have

$$\beta^{T-1} = H(\beta^T) = H(0) = f^{-1}\Big(\delta \cdot \int_0^1 f(\theta)d\theta\Big) > 0.$$

So, $\beta^{T-1} > \beta^T$. Then inductively $\beta^t = H(\beta^{t+1}) > H(\beta^{t+2}) = \beta^{t+1}$ for all $t = T-2, \dots, 1$. Thus, $\beta^1 > \beta^2 > \cdots > \beta^T = 0$.

As $\beta^1 > \beta^2$, we have $H(\beta^1) > H(\beta^2) = \beta^1$. So, by Lemma 2, we have $\beta^1 < \beta$. Finally, $\beta = H(\beta) \le f^{-1}(\delta f(1)) \le f^{-1}(f(1)) = 1$, so $\beta \le 1$. This completes the proof of [1].

[2] It holds that

$$\frac{1}{\beta^t} \cdot \int_0^{\beta^t} g^t(\theta)d\theta < \frac{1}{\beta^1} \cdot \int_0^{\beta^1} g^t(\theta)d\theta \le \delta \cdot \int_0^1 g^t(\theta)d\theta = \int_0^1 g^{t+1}(\theta)d\theta.$$

The first inequality follows from substituting $a = b = 0$, $c = \beta^t$, and $d = \beta^1$ into Lemma 5. The second inequality follows from the fact that $\delta \cdot V(1) \ge V(\beta^1)$, which is true due to $\delta \in [D^T, 1]$.

[3] Let $\beta^t(T)$ denote the threshold at time t when the horizon is T. By definition, we have $\beta^t(T) = \beta^1(T-t+1)$ for $t \le T$. So, it is sufficient to show that $\beta^1(T) \to \beta$ as $T \to \infty$. By part [1], $\beta^1(T) > \beta^2(T) = \beta^1(T-1)$ for any $T > 1$. As H is strictly increasing, we have $H(\beta^1(T)) > H(\beta^1(T-1)) = \beta^1(T)$. Then, by item [3] of Lemma 2, also $\beta^1(T) < \beta$ for all T. Hence, the sequence $(\beta^1(T))_{T\in\mathbb{N}}$ is strictly increasing and bounded above by β. So, the sequence is convergent. Assume it converges to $z \in [0, \beta]$. Then, since $\beta^1(T) = H(\beta^1(T-1))$, we have

$$z = \lim_{T\to\infty} \beta^1(T) = \lim_{T\to\infty} H(\beta^1(T-1)) = H(z)$$

by continuity of H. From Lemma 3, item [2], it follows that $z = \beta$. This completes the proof. □

Recall that D is the smallest number in $[0, 1]$ such that $\delta \cdot V(1) \ge V(\beta)$ for every positive $\delta \in [D, 1]$.

Lemma 4. *The following statements hold:*

[1] $\frac{1}{\beta} \cdot \int_0^{\beta} g^t(\theta)d\theta < \int_0^1 g^{t+1}(\theta)d\theta$ *for* $\delta \in (D, 1)$.

[2] $D \ge D^T$ *for each* $T \ge 1$.

Proof. [1] It holds that

$$\frac{1}{\beta} \cdot \int_0^{\beta} g^t(\theta)d\theta < \delta \cdot \int_0^1 g^t(\theta)d\theta = \int_0^1 g^{t+1}(\theta)d\theta.$$

The first inequality follows from $\delta \cdot V(1) \ge V(\beta)$, which is true due to the assumption $\delta \in (D, 1)$.

[2] By definition, $\delta \cdot V(1) \ge V(\beta)$ for each $\delta \in [D, 1]$. Due to Lemma 3, we have $\beta \ge \beta^1(T)$ for each T. So, $\delta \cdot V(1) \ge V(\beta^1(T))$ for each $\delta \in [D, 1]$. So, by definition of D^T, we have $D^T \le D$. □

The next Lemma easily follows from the basic observation that, when one function is pointwise larger or equal to another function, then also its average value is higher than the average value of the latter function.

Lemma 5. *Let* $g: \mathbb{R} \to \mathbb{R}$ *be a strictly increasing function. Then, for any* $a \le b < c \le d$,

$$\frac{1}{c-a}\int_a^c g(x)dx < \frac{1}{d-b}\int_b^d g(x)dx.$$

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