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RESEARCH ARTICLE

Optimal allocations in growth models with private information

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Abstract
This paper considers a class of growth models with idiosyncratic human capital risk and private information about individual effort choices (moral hazard). Households are infinitely-lived and have preferences that allow for a time-additive expected utility representation with a one-period utility function that is additive over consumption and effort as well as logarithmic over consumption. Human capital investment is risky due to idiosyncratic shocks that follow a Markov process with transition probabilities that depend on effort choices. The production process is represented by an aggregate production function that uses physical capital and human capital as input factors. We show that constrained optimal allocations are simple in the sense that individual effort levels and individual consumption growth rates are history-independent. Further, constrained optimal allocations are the solutions to a recursive social planner problem that is simple in the sense that exogenous shocks are the only state variables. We also show that constrained optimal allocations can be decentralized as competitive equilibrium allocations of a market economy with a simple tax- and transfer scheme. Finally, it is always optimal to subsidize human capital investment in the market economy.

Keywords Economic growth · Private information · Human capital risk

JEL Classification D51 · D82 · E20

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1 Introduction

Dynamic models with private information about individual effort (moral hazard) have been studied by a large literature in macroeconomics (Ljungqvist and Sargent 2018) and microeconomics (Laffont and Martimort 2002). In these models, constrained optimal allocations often display a dependence on individual histories rendering the analysis of even simple economic problems a challenging task. The literature has tried to circumvent this tractability problem using a recursive approach with additional endogenous state variables (promised utility), but this approach quickly reaches its computational limits when studying economies with multidimensional investment choices or aggregate shocks. Moreover, most applied work has confined attention to steady-state analysis and relied on approximation methods with unknown accuracy.

In this paper, we develop a growth model with private information about individual effort (moral hazard) that is tractable in the sense that optimal allocations do not display a dependence on individual shock histories beyond the current shock realization. Specifically, we consider a dynamic model economy that is populated by a large number of infinitely-lived households who can invest in risk-free physical capital and risky human capital. Human capital investment is risky due to idiosyncratic shocks to the stock of household human capital. Households also make an effort choice that has a utility cost (dis-utility of effort) and affects the probability distribution over idiosyncratic human capital shocks. Specifically, the exogenous shocks follow a Markov process with transition probabilities that depend on effort choices. Households have preferences that allow for a time-additive expected utility representation with a one-period utility function that is additive over consumption and effort as well as logarithmic over consumption. The dis-utility of effort might be subject to idiosyncratic shocks. The production process is represented by a constant-returns-to-scale production function that takes aggregate physical capital and aggregate human capital as input factors.

Constrained optimal allocations are the solution to an infinite-horizon social planner problem with dynamic incentive compatibility constraints. These constraints ensure that households always have an incentive to choose the individual effort level that is part of the allocation (unobserved effort). In other words, we assume that individual effort choices are private information. In contrast, individual shock histories and individual human capital are observed by the social planner. In addition, allocations need to satisfy the standard feasibility constraints that represent the production process, capital accumulation, and the aggregate resource constraint.

We first derive a necessary condition for constrained optimal allocations. Specifically, production efficiency requires that the expected return on risky human capital investment is equal to the risk-free return on physical capital investment for all households. As our proof shows, the result holds for general, separable preferences and general, neoclassical production functions. It does require, however, the assumption that individual human capital is observable and the property that human capital choices are strictly positive, which we show to hold using a mild assumption on the set of admissible allocations.

We use the necessary condition of production efficiency to show that constrained optimal allocations are simple. Specifically, production efficiency in conjunction with the separability of preferences implies that individual effort choices only depend on the current shock realization. Given the simplicity of effort decisions, it is straightforward to show that constrained optimal allocations are the solutions to a recursive social planner problem that is simple in the sense that exogenous shocks are the only state variables. In other words, the model is highly tractable since the computation of constrained optimal allocations does not require the introduction of additional endogenous state variables (promised utility) and their distribution over individual households. Given log-utility preferences, constrained optimal allocations have the further property that individual consumption growth rates only depend on the current exogenous state.

Finally, we show that constrained optimal allocations are equilibrium allocations of a market economy with a simple system of taxes and transfers. Specifically, it is optimal for the government to restrict its fiscal policy to transfer payments and taxes/subsidies that are linear in household wealth/income and only depend on current shock realizations. The corresponding competitive equilibria are simple in the sense that individual household decisions are linear in wealth and prices, respectively rental rates, are independent of the wealth distribution. Thus, the model also constitutes a tractable framework for the analysis of the competitive equilibria of incomplete-market economies. In addition, we show that it is always optimal to subsidize human capital investment in the market economy. In this sense, competitive equilibria of the corresponding incomplete-market model are constrained inefficient.

To sum up, we show that it is not optimal for society to condition allocations on endogenous variables like promised utility once individual households can invest in human capital and the corresponding first-order conditions hold with equality. To streamline the analysis, we develop the main arguments using a basic version of the model with a simple production structure, log-preferences, and without aggregate shocks. However, our analysis suggests that the main arguments and proofs can be extended to a more general version of the model—the specific extensions are discussed in the concluding remarks. We leave a rigourous analysis of possible generalizations for future work.

Literature. Our paper is related to several strands of the literature. First, there is the large literature on (constrained) optimal allocations in moral hazard economies. See, for example, Hopenhayn and Nicolini (1997) and Pavoni and Violante (2007) for well-known applications to unemployment insurance and welfare programs, Ljungqvist and Sargent (2018) for a survey of the macro literature, and Laffont and Martimort (2002) for a survey of the more micro-oriented literature on moral hazard. Going back to the work of Spear and Srivastava (1987), the standard approach in this literature is to render the social-planner problem recursive by introducing an additional (endogenous) state variable—promised utility. Our theoretical tractability result that no such state variable is needed echoes the result derived by Holmstrom and Milgrom (1987) and Fudenberg et al. (1990) for repeated principal-agent problems, but in contrast to these papers we consider a macroeconomic model with an explicit aggregate resources constraint (general equilibrium analysis).

Second, our paper relates to the public finance literature on optimal taxation in dynamic economies with private information about household type (history of shocks)—see Stantcheva (2020) for a survey. Da Costa and Maestri (2007) show in a two-period model with private information about household type that the equality of expected investment returns holds for constrained optimal allocations if human capital is observable. In contrast, in models with unobserved human capital investment, optimality requires a positive human capital premium (Grochulskia and Piskorskib 2010) and the efficiency condition derived in this paper does not hold. Our theoretical tractability result resembles the results of Farhi and Werning (2007) and Phelan (2006), who show that constrained optimal allocations in an OLG-model are the solution to a static social planner problem when the social welfare function puts equal weight on all future generations. In other words, they make an assumption about social preferences. In contrast, in this paper we make assumptions about the production structure and about individual preferences to prove tractability. Finally, Hahn and Yannelis (1997) discuss how constrained optimal allocations depend on the way private information is incorporated into the incentive-compatibility constraint.

Third, our paper is related to the literature on constrained efficient allocations in incomplete-market models (Geanakoplos and Polemarchakis 1986) that assume an exogenous asset payoff structure and therefore take the lack of certain type of insurance as given. Aiyagari (1995) and Davila et al. (2012) analyze constrained optimal allocations in a neoclassical growth model with idiosyncratic productivity risk and incomplete markets. Krebs (2006) and Toda (2015) discuss the efficiency properties of incomplete-market models with human capital and a production structure similar to the one discussed in this paper, and Gottardi et al. (2015) analyze the optimal level of taxation and debt in this class of models. In this paper, we show that competitive equilibrium allocations are constrained inefficient for certain asset payoff structures in the sense that the government can improve social welfare by introducing a subsidy to human capital investment.

2 Model

This section develops the model and defines constrained optimal allocations. Specifically, Sects. 2.1 and 2.2 describe the fundamentals of the economy and Sect. 2.3 defines the social planner problem. The framework combines the production structure of the human capital model developed in Krebs (2003, 2006) with a dynamic model of unobserved effort choices along the lines of Hopenhayn and Nicolini (1997), Pavoni and Violante (2007), and Phelan and Townsend (1991). The basic framework disregards aggregate shocks and confines attention to a simple production structure. In the concluding remarks we discuss possible extensions of the basic framework with aggregate shocks and a more general production structure.

2.1 Preferences and uncertainty

Time is discrete and open ended. The economy is populated by a unit mass of infinitely-lived households. In each period t, the exogenous part of the individual state of a household is represented by s_t, which captures the effect of idiosyncratic shocks on household preferences and human capital accumulation (see below). We denote by $s^t = (s_0, s_1, \ldots, s_t)$ the history of exogenous shocks up to period t. We assume that the probability of history $s^t = (s_0, s_1, \ldots, s_t)$ depends on effort choices, $e^{t-1} = (e_0, \ldots, e_{t-1})$. More precisely, we assume that the probability of s^t given s_0 depends on effort choices as follows: $\pi_t(s^t|s_0, e^{t-1}) = \pi(s_t|s_{t-1}, e_{t-1}) \times \ldots \times \pi(s_1|s_0, e_0)$, where $\pi(s_t|s_{t-1}, e_{t-1})$ is the probability of state s_t in period t given state s_{t-1} and effort choice e_{t-1} in period $t-1$. In other words, for given effort choices, the shock process is a Markov process with transition probabilities given by $\pi(s'|s, e)$.

Each household is assigned an initial stock of human capital, h_0, and there is a given initial distribution (of households) over initial human capital and shocks, $\pi_0(h_0, s_0)$, that is independent of effort choices. We assume that there are a finite number of realizations, $s_t \in \{1, \ldots, S\}$. This assumption implies that the set of possible histories is countable, which sidesteps any measurability problem and simplifies the existence proof. We further assume that the set of possible effort choices, e_t, is a subset of $\mathbb{R}$, which means that possible effort choices can be (completely) ordered.

Households are risk-averse and have identical preferences that allow for a time-additive expected utility representation with one-period utility function that is additive over consumption and effort as well as logarithmic over consumption. Let $\{c_t, e_t|h_0, s_0\}$ stand for the consumption-effort plan of a household of initial type (h_0, s_0). Expected lifetime utility associated with the consumption-effort plan $\{c_t, e_t|h_0, s_0\}$ is then given by

$$U(\{c_t, e_t|h_0, s_0\}, s_0) = \ln c_0(h_0, s_0) - d(e_0(h_0, s_0), s_0) + \sum_{t=1}^{\infty} \sum_{s^t} \beta^t \left[\ln c_t(h_0, s^t) - d(e_t(h_0, s^t), s_t)\right] \pi_t(s^t|s_0, e^{t-1}(h_0, s^{t-1})) \quad (1)$$

where β is the pure discount factor and $d(., s)$ is a—possibly state-dependent—dis-utility function. We assume that for each s, the dis-utility function $d(., s)$ is strictly increasing, which imlies that it is invertible.

2.2 Production, capital accumulation, and resource constraint

There is one consumption good that is produced using the aggregate production function

$$Y_t = F(K_t, H_t) \ , \quad (2)$$

where Y_t is aggregate output in period t, K_t is the aggregate stock of physical capital employed in production, and H_t is the aggregate stock of human capital employed in production. We assume that F is a standard neoclassical production function. In particular, F displays constant returns to scale with respect to the two input factors physical capital, K, and human capital, H.

The consumption good can be transformed into the physical capital good one-for-one. In other words, production of the consumption good and production of physical capital employ the same production function, F. The consumption good is perishable and physical capital depreciates at a constant rate, δ_k. Thus, if X_{kt} denotes aggregate investment in physical capital, then the evolution of aggregate physical capital is given by

$$K_{t+1} = (1 - \delta_k)K_t + X_{kt} \ . \tag{3}$$

Human capital is produced at the household level. An individual household can transform the consumption good into human capital using a quantity x_{ht} of the consumption good to produce ϕx_{ht} units of human capital. Note that $1/\phi$ is the price of human capital in units of the consumption (physical capital) good. Human capital is subject to random shocks, $\eta_t = \eta(s_t)$. The production function and law of motion for household-level human capital, h_t, are described by

$$\begin{aligned} h_{t+1}(h_0, s^t) &= (1 + \eta(s_t))h_t(h_0, s^{t-1}) + \phi x_{ht}(h_0, s^t) \\ h_t(h_0, s^t) &\geq 0 \ , \end{aligned} \tag{4}$$

for all household types and histories (h_0, s^t). Note that h_{t+1} is a linear function of x_{ht} and that, as in Krebs (2003, 2006), we do not impose a non-negativity constraint on human capital investment, x_{ht}.

The η-term in the human capital accumulation equation (4) represents changes in human capital that are affected by effort choices and do not require (substantial) goods investment. For example, positive human capital growth, $\eta(s) > 0$, can represent learning-by-doing, and in this case $\pi(.|s, e)$ summarizes the effect of work effort on the success of on-the-job learning. Unemployment-to-job or job-to-job transition is a second example of a positive human capital shock, and in this case it is (on-the-job) search effort that determines the likelihood that the positive realization occurs (the search is successful). In contrast, job loss and the associated loss of firm- or occupation-specific human capital is a typical example of a negative realization $\eta(s) < 0$. In this case, $\pi(., e)$ may represent both the effect of work effort on the likelihood of job loss and the effect of search effort during unemployment on the size of human capital loss associated with the job loss.[1]

Define for given s the function $\bar{\eta}(s, .) \doteq \sum_{s'} \eta(s')\pi(s'|s, .)$. We assume that the function $\bar{\eta}(s, .)$ is strictly increasing for all s. In other words, more effort leads to better outcome in expected value terms. This assumption imposes a joint restriction on η and π.

We confine attention to plans $\{c_t, h_t|h_0, s_0\}$ that can be represented as $c_t(h_0, s^t) = \tilde{c}_t(h_0, s^t)h_t(h_0, s^{t-1})$, where $\tilde{c}_t$ is bounded. This assumption means that the social planner choice is restricted to allocations for which there is some minimal link between individual consumption and individual human capital. Specifically, it means that $c_t(h_0, s^t) = 0$ if $h_t(h_0, s^{t-1}) = 0$. In conjunction with the assumption that utility over consumption is unbounded from below it implies that it is never optimal for the social planner to choose $h_t = 0$.

Aggregate human capital, H, entering the production function (2) is obtained from individual human capital, h, by taking the expectation over shock histories and initial types:

[1] We use $\eta(s_t)$ instead of $\eta(s_{t+1})$ in (4) in order to simplify the formal proofs, a timing choice also made in Krebs (2003, 2006) and Stantcheva (2017). However, the current analysis and results apply, mutatis mutandis, if the timing is changed and $\eta(s_{t+1})$ is used in (4). See Stokey et al. (1989) for a general discussion of this issue in choice problems under uncertainty.

$$
\begin{aligned}
H_{t+1} &= \mathbf{E}[h_{t+1}] \\
&= \sum_{h_0, s^t} h_{t+1}(h_0, s^t)\pi_t(s^t|s_0, e^{t-1}(h_0, s^{t-1}))\pi_0(h_0, s_0)\ . \qquad (5)
\end{aligned}
$$

Note that for notational ease we assume a finite number of possible levels of initial human capital. In general, we obtain aggregate variables from their individual counterparts as in (5). Taking the expectation over equation (4) yields the aggregate human capital accumulation equation:

$$
H_{t+1} = H_t \ + \mathbf{E}[\eta_t h_t] \ + \phi X_{ht}\ , \qquad (6)
$$

where $X_{ht} = \mathbf{E}[x_{ht}]$ is aggregate investment in human capital. Note that $\mathbf{E}[\eta_t h_t] \neq \mathbf{E}[\eta_t]\mathbf{E}[h_t]$ when e_{t-1} depends on s^{t-1}.

Finally, the aggregate resource constraint in the economy reads:

$$
C_t + X_{kt} + \frac{X_{ht}}{\phi} \ = Y_t \quad . \qquad (7)
$$

The resource constraint (7) says that aggregate output produced is equal to the sum of aggregate consumption, aggregate investment in physical capital, and aggregate goods investment in human capital.

2.3 Constrained optimal allocations

Consider a social planner who directly chooses an allocation, $\{c_t, e_t, h_{t+1}, K_{t+1}\}$ with $H_{t+1} = \mathbf{E}[h_{t+1}]$, subject to the feasibility constraints defined by (2), (3), (4), (7) and additional incentive compatibility constraints. These incentive constraints arise because effort choices are private information (moral hazard). Specifically, an allocation $\{c_t, e_t, h_{t+1}, K_{t+1}\}$ is incentive compatible if $\{c_t, e_t\}$ satisfies:

$$
\begin{aligned}
&\forall\,(h_0, s^t)\,,\ \forall\,\{\hat{e}_{t+n}|h_0, s^t\}\ : \\
&U_t(\{c_{t+n}, e_{t+n}|h_0, s^t\}, s_t) \ \geq \ U_t(\{c_{t+n}, \hat{e}_{t+n}|h_0, s^t\}, s_t)\ , \qquad (8)
\end{aligned}
$$

where $\{c_{t+n}, e_{t+n}|h_0, s^t\}$ is the continuation plan for (h_0, s^t) and $U_t(\{c_{t+n}, e_{t+n}|h_0, s^t\}, s_t)$ is the corresponding continuation lifetime utility.

Equation (8) formalizes the idea that the social planner cannot observe individual effort levels, and that individual households therefore have to have an incentive to adhere to the proposed plan. Private information about individual effort choices (moral hazard) requires that the social planner can only choose consumption-effort allocations, $\{c_t, e_t\}$, that are incentive compatible in the sense that households have an incentive to choose the effort plan for given consumption plan.

We define the constraint set of the social planner problem as the set that satisfies the feasibility constraints and the incentive compatibility constraints:

$$
\mathbf{A} \equiv \{\{c_t, e_t, h_{t+1}, K_{t+1}\}|\{c_t, e_t, h_{t+1}, K_{t+1}\} \text{ satisfies (2), (3), (4), (7), (8)}\}. \qquad (9)
$$

We assume that the social planner's objective function is social welfare defined as the weighted average of the expected lifetime utility of individual households defined in (1), where we use the Pareto weight $\mu(h_0, s_0)$ to weigh the importance of households of type (h_0, s_0). If $\mu(h_0, s_0) = \pi_0(h_0, s_0)$, then each individual household is assigned equal importance by the social planner.

Definition 1 A *contrained optimal allocation* is the solution to the social planner problem

$$\max_{\{c_t, e_t, h_{t+1}, K_{t+1}\}} \sum_{h_0, s_0} U(\{c_t, e_t | h_0, s_0\}, s_0)\mu(h_0, s_0)$$
$$\text{subject } to: \ \{c_t, e_t, h_{t+1}, K_{t+1}\} \in \mathbf{A} \tag{10}$$

where the constraint set $\mathbf{A}$ is defined in (9).

As in Golosov et al. (2003), we assume that physical capital production is not subject to (idiosyncratic) risk and our definition of (optimal) allocations therefore only refers to the aggregate physical capital stock, K. In contrast, human capital is produced at the household level and the allocation of human capital across households is therefore specified as part of an (optimal) allocation.

In (10) we assume that the social planner cannot observe individual effort choices, but can observe individual human capital. In the competitive equilibrium of a market economy, this assumption amounts to the observability of labor income, that is, taxes can depend on individual labor income and therefore indirectly on human capital. Note that in moral-hazard models of the type considered here, these two assumptions are internally consistent in the sense that the observation of the history of human capital stocks, h^t, does not reveal information about the choice of effort, e_t. In contrast, the history of human capital stocks, h^t, does reveal information about the history of shocks, s^{t-1}, since $h_t = h_t(s^{t-1})$. Thus, in economies with private information about histories and types, the assumption of observability of human capital is somewhat questionable.

3 Constrained optimal allocations

This section states and discusses the main properties of constrained optimal allocations. Section 3.1 states a necessary conditions of efficiency: Expected returns are equalized across investment opportunities (Proposition 1). This result immediately implies that effort choices are history independent (Corollary 1). Section 3.2 builds on the efficiency result to provide a full characterization of constrained optimal allocations and shows that they are simple (Proposition 2). Proofs of the propositions are collected in the Appendix.

3.1 Production efficiency

Consider an allocation $\{c_t, e_t, h_{t+1}, K_{t+1}\}$. In economies with complete information, production efficiency requires that expected returns on alternative investment opportunities are equalized if investment levels are positive.[2] In the model considered in this paper, this equalization-of-returns condition reads:

$$\phi F_h(\tilde{K}_{t+1}) + \sum_{s_{t+1}} \eta(s_{t+1})\pi(s_{t+1}|s_t, e_t(h_0, s^t)) = F_k(\tilde{K}_{t+1}) - \delta_k , \qquad (11)$$

where $\tilde{K}$ is the aggregate physical-to-human capital ratio defined as $\tilde{K} = K/H$. Proposition 1 below shows that the optimality condition (11) also characterize optimal allocations in our private information economy for all households. Clearly, the efficiency condition (11) does not have to hold for histories with $h_t(h_0, s^t) = 0$, but such histories cannot be part of an optimal allocation since $h_t(h_0, s^t) = 0$ requires $c_t(h_0, s^t) = 0$ given the assumption made in Sect. 2.2 and this leads to unbounded negative utility. In addition, a standard argument shows that the optimal $\tilde{K}_t$ is independent of t since production displays constant returns to scale with respect to H and K, and these two factors of production can be adjusted at no cost. Thus, we have the following result:

Proposition 1 *Any constrained optimal allocation has the following two properties. First, the efficiency condition* (11) *holds for all household types, h_0, and household histories, s^t. Second, the aggregate capital-to-labor ratio is constant over time: $\tilde{K}_t = \tilde{K}$ for all periods $t = 1, \ldots$.*

Proof See appendix. □

The proof of proposition 1 is quite general and does not hinge on the linearity of individual human capital investment opportunities. The crucial assumptions are that human capital investment is observable and that there is a minimal link between human capital and consumption, but beyond these two assumptions not much is needed for the proof. Indeed, the proof conducted in the Appendix shows that the result holds for any production function (2) and any human capital accumulation equation of the type $h_{t+1} = g(h_t, x_{ht}, l_t, s_t)$ as long as financial investment (borrowing and lending) and human capital investment (labor income) are observable, where l_t is the time spent in human capital production. For the general case the human capital return has to be defined as $r_{h,t+1} = g_{x_{ht}}((1 - l_{t+1})F_{h,t+1} + g_{h,t+1}/g_{x_{h,t+1}}) - 1$.

Proposition 1 states that a standard production efficiency condition has to hold even if there is private information. In this sense, the result resembles the original result by Diamond and Mirrlees (1971). Da Costa and Maestri (2007) show in a one-period model of human capital investment with private information about type that optimality implies that expected investment returns are equalized. The current paper

[2] More precisely, if a capital allocation maximizes aggregate output net of depreciation, then the (expected) returns on physical capital investment and human capital investment are equalized. Further, the capital-to-labor ratio that maximizes the expected total investment return for given effort level is determined by the equality-of-returns condition.

shows that this result holds generally when the social planner observes individual investment decisions. If, however, the social planner cannot observe individual human capital investment choices or individual capital investment (saving) choices, then (11) is in general not a necessary condition for constrained optimality (Grochulskia and Piskorskib 2010).

One direct implication of the efficiency condition (11) is that effort choices are the same for households regardless of their history of shocks or initial human capital. This directly follows from (11) since for given value of the aggregate capital-to-labor ratio, $\tilde{K}$, there is a unique value of $\bar{\eta}(s, e)$ solving (11), where $\bar{\eta}(s, e) = \sum_{s'} \eta(s')\pi(s'|s, e)$. Thus, there is a unique effort choice, e solving (11) for given s and $\tilde{K}$ since we assume that $\bar{\eta}(s, e)$ is strictly increasing in e for all s.

Corollary 1 *Let $\{c_t, e_t, h_{t+1}, K_{t+1}\}$ be a constrained optimal allocation. Then effort choices are stationary and only depend on the current shock realization: $e_t(h_0, s^t) = e^*(s_t)$ for all h_0 and s^t.*

Note that for the argument of corollary 1 to go through, the efficiency condition (11) needs to hold with equality. If (11) only holds as an inequality for some histories, s^t, as is the case with private information about individual human capital (Grochulskia and Piskorskib 2010) or when $h_{t+1}(s^t) = 0$, then corollary 1 would not hold for these histories. Thus, proposition 2 below would not hold and constrained efficient allocations might not be simple. Indeed, if we do not rule out $h_{t+1}(s^t) = 0$, as we do by assuming that there needs to be a minimal connection between individual human capital and individual consumption, then social welfare is in general maximized by an allocation that sets $h_{t+1}(s^t) = 0$ for all histories but one. This happens because for this allocation positive effort only has to be applied for one history/houshold (and everybody else gets to shirk) and there is no diminishing returns to human capital at the household level.

3.2 Full characterization

To characterize constrained optimal allocations fully, it is convenient to represent a consumption plan as

$$c_{t+1} = \beta \left(1 + r(\tilde{K}) + \epsilon_{t+1}\right) c_t \tag{12}$$

where $\epsilon_{t+1} = \epsilon_{t+1}(h_0, s^t)$. In other words, we represent $\{c_t\}$ by c_0 and $\{\epsilon_t\}$. Note that any consumption process can be represented as (12) as long as we do not impose any conditions on $\{\epsilon_t\}$.

Proposition 1, respectively Corollary 1, establishes that in our search for constrained optimal allocations we can confine attention to effort choices that are independent of type and history s^{t-1}: $e_t(h_0, s^t) = e^*(s_t)$. The next proposition shows that optimal allocations have the further property that ϵ has mean zero and is independent of type and history:

$$\epsilon_{t+1}(h_0, s^{t+1}) = \epsilon^*(s_t, s_{t+1})$$

$$\sum_{s_{t+1}} \epsilon^*(s_t, s_{t+1})\pi(s_{t+1}|s_t, e^*(s_t)) = 0 \;. \tag{13}$$

In other words, ϵ is a proper risk measure, and this measure of consumption risk is independent of household type and history s^{t-1}. Further, the optimal $(\tilde{K}^*, e^*, \epsilon^*(.))$ are the solution to the simple social planner problem

$$\max_{e,\epsilon,\tilde{K}} \left\{ \sum_s V(s, e(s), \epsilon(s,.), \tilde{K})\, \mu(s) \right\}$$
$$\text{subject to:}$$
$$\forall s \;:\; r(\tilde{K}) = \phi\, F_h(\tilde{K}) + \sum_{s'} \eta(s')\pi(s'|s, e(s))$$
$$\forall s \;:\; \sum_{s'} \epsilon(s, s')\pi(s'|s, e(s)) = 0$$
$$\forall s,\, \hat{e}(s) \;:\; V(s, e(s), \epsilon(s,.), \tilde{K}) \geq V(s, \hat{e}(s), \epsilon(s,.), \tilde{K}) \tag{14}$$

where $\mu(s) = \sum_h \mu(h, s)$ and the intensive-form value function, V, solves the simple recursive equation:

$$V(s) = -d(e(s), s) + B(\beta) + \frac{\beta}{1-\beta} \sum_{s'} \ln(1 + r(\tilde{K}), \epsilon(s, s'))\pi(s'|s, e(s))$$
$$+\beta \sum_{s'} V(s', e(s'), \epsilon(s, s'), \tilde{K})\pi(s'|s, e(s)) \tag{15}$$

with $B(\beta) = \ln(1-\beta) + \frac{\beta}{1-\beta} \ln \beta$.

The social planner problem (14) is simple because only the exogenous state/shock, s, enters into the equation; no additional endogenous state (promised utility) is needed to obtain the solution. Note that the objective function (social welfare) in the maximization problem (14) allows for a recursive representation because it is defined as the sum of recursively defined functions. In this sense the social planner problem defined by (14) and (15) is a recursive problem even though it slightly deviates from the formulation often used in macroeconomics (Stokey et al. 1989) in the sense that the objective function is a weighted average of value functions. Note further that lifetime utility of a household of initial type (s_0, h_0) is given by:

$$U(\{c_t, e_t|h_0, s_0\}, s_0) = \frac{1}{1-\beta} \ln c_0(h_0, s_0) + V(s_0, e(s_0), \epsilon(s_0,.), \tilde{K}) \;. \tag{16}$$

The following proposition summarizes the preceding discussion:

Proposition 2 *Constrained optimal allocations,* $\{c_t, e_t, h_{t+1}, K_{t+1}\}$*, exist and are simple. Specifically, let the triple* $(e^*, \epsilon^*, \tilde{K}^*)$ *be the solution to the static social planner problem* (14)*, where the intensive-form value function,* V*, is defined by the simple* *recursive equation* (15)*. Then the optimal allocation is given by:*

$$\begin{aligned}
e_t(h_0, s^t) &= e^*(s_t) \\
\epsilon_{t+1}(h_0, s^{t+1}) &= \epsilon^*(s_t, s_{t+1}) \\
\tilde{K}_{t+1} &= \tilde{K}^* \\
c_{t+1}(h_0, s^{t+1}) &= \beta \left(1 + r(\tilde{K}^*) + \epsilon^*(s_t, s_{t+1})\right) c_t(h_0, s^t) \\
c_0(h_0, s_0) &= (1-\beta)\left(1 + r(\tilde{K}_0)\right)(K_0 + H_0/\phi)\, \frac{\mu(h_0, s_0)}{\pi_0(h_0, s_0)} \\
C_{t+1} &= \beta(1 + r(\tilde{K}^*))C_t \\
K_{t+1} &= \beta(1 + r(\tilde{K}^*))K_t \\
H_{t+1} &= \beta(1 + r(\tilde{K}^*))H_t \ .
\end{aligned} \tag{17}$$

In addition, lifetime utility of a household of initial type (s_0, h_0) *is given by* (16)*.*

Proof See appendix. □

Several remarks regarding Proposition 2 are in order.

First, even though the optimal aggregate level of human capital investment, X_{ht}, is uniquely determined for all t, the optimal level of individual human capital investment is indeterminate since the optimal effort choice, $e^*(s)$, is common across households with the same s.

Second, the maximization problem (14) has an intuitive interpretation. The maximization problem is the choice problem of a social planner who chooses effort level, e, consumption risk, ϵ, and a capital-to-labor ratio, $\tilde{K}$, so as to maximize welfare defined by the expected utility of households with log-utility function and consumption given by $\ln(1 + r(\tilde{K}) + \epsilon')$ subject to three constraints. The first constraint states that the return to physical capital investment is equal to the expected return to human capital investment, where the social planner can affect returns through the choice of the capital-to-labor ratio and the mean level of human capital shocks (effort). The second constraint says that ϵ is a variable representing risk and therefore has a fixed mean, which is normalized to zero. The final constraint is the incentive compatibility constraint that ensures that individual households will choose the prescribed effort choice.

Third, proposition 2 implies that the cross-sectional distribution of consumption spreads out over time—the well-known immiseration result of Atkeson and Lucas (1992). If we introduce an OLG-structure with stochastic death of households (Constantinides and Duffie 1996) and a social welfare function that puts weight on future generations (Farhi and Werning 2007; Phelan 2006), we can generate a stationary cross-sectional distribution of consumption while still keeping the tractability of the model. However, the cross-sectional distributions of consumption and wealth still exhibit fat tails and obey the double power law (Toda 2014).

Fourth, proposition 2 rules out that households enter an absorbing state in which consumption is constant and effort is zero—the "retirement" state in the language of Sannikov (2008). In the current model, retirement at low levels of consumption does not occur because utility is not bounded from below. In addition, retirement at high levels of consumption is not optimal because preferences are consistent with balanced growth so that the (relative) cost of providing incentives to induce positive effort choices are independent of the level of consumption, that is, income and substitution effect of increases in income/wealth cancel each other out.

Consider now the special case in which the shock process is i.i.d. for given effort choice and the dis-utility function does not depend on shocks. In this case, Eq. (12) in conjunction with (13) says that expected consumption growth is equal to $\beta(1+r)$ for all (h_0, s^t). In other words, optimal individual consumption has the martingale property—see Ljungqvist and Sargent (2018) for a discussion of the martingale property in economics. The optimal individual consumption process follows a sub-martingale if $\beta(1+r) > 1$, a martingale if $\beta(1+r) = 1$, and a super-martingale if $\beta(1+r) < 1$. In this paper, this martingale property is proved using the property $e_t(h_0, s^t) = e^*(s_t)$ and the assumption of log-utility preferences. Alternatively, the martingale property follows from the inverse Euler equation for log-utility (Rogerson 1985b).

For the case of i.i.d. shocks, Proposition 2 implies that effort choices, e, consumption risk, $\epsilon(.)$, and the intensive-form value function, V, are independent of the current state, s. Further, the social planner problem (14) reduces to the following constrained maximization problem:

$$\max_{e,\epsilon,\tilde{K}} \left[-d(e) + \frac{\beta}{1-\beta} \sum_{s'} ln\left(1 + r(\tilde{K}) + \epsilon(s')\right) \pi(s'|e) \right]$$

subject to:

$$r(\tilde{K}) = \phi\, F_h(\tilde{K}) + \sum_{s'} \eta'(s)\pi(s'|e)$$

$$\sum_{s'} \epsilon(s')\pi(s'|e) = 0$$

$$\forall\, \hat{e}: \quad -d(e) + \frac{\beta}{1-\beta} \sum_{s'} ln\left(1 + r(\tilde{K}) + \epsilon(s')\right) \pi(s'|e)$$

$$\geq -d(\hat{e}) + \frac{\beta}{1-\beta} \sum_{s'} ln\left(1 + r(\tilde{K}) + \epsilon(s')\right) \pi(s'|\hat{e}) \tag{18}$$

We can use well-known results for one-period moral hazard problems (Rogerson 1985b) to ensure that in (18) the first-order condition approach is appropriate. Specifically, we can replace the set of inequalities in (18) by the first-order conditions

$$d'(e) = \frac{\beta}{1-\beta} \sum_{s'} ln\left(1 + r(\tilde{K}) + \epsilon(s')\right) \frac{\partial \pi}{\partial e}(s'|e)\,. \tag{19}$$

In contrast, for general repeated moral hazard economies, the first-order conditions might not be sufficient since the product of two concave (probability) functions is not necessarily concave, and there are no results for general repeated moral hazard problems in the literature. Abraham et al. (2011) provide conditions for a two-period moral hazard problem that ensure necessity and sufficiency of first-order conditions.

4 Competitive market equilibria

In this section, we analyze competitive equilibria of the market economy. Subsection defines competitive market equilibria. Section 4.2 provides a full characterization of competitive market equilibria and shows that they are simple (Proposition 3). In Sect. 4.3, this characterization result is used to show that constrained optimal allocations are also the equilibria of a competitive market economy in which human capital investment is subsidized (Corollary 2 and Corollary 3). Proofs of the propositions are collected in the Appendix.

4.1 Definition of competitive equilibria

In this section, we define competitive equilibria in a market economy. At time $t = 0$, an individual household begins life in initial state s_0 and with initial endowment (a_0, h_0), where a_0 is the amount of financial asset holding of the household in period $t = 0$. To ease the notation, we assume that the initial asset holding of an individual household are proportional to the initial human capital of the household: $a_0 = \frac{K_0}{H_0} h_0$. Thus, the initial state/type of an individual household is given by (h_0, s_0). The initial state of the economy is defined by an initial distribution of individual households over types, $\pi_0(h_0, s_0)$, and an initial aggregate stock of physical capital, K_0. Note that taking the expectations over h_0, respectively a_0, using π_0 yields the initial aggregate stock of human capital, H_0, respectively physical capital, K_0.

A household of initial type (h_0, s_0) chooses a plan consisting of a sequence of functions $\{c_t, e_t, a_{t+1}, h_{t+1}|h_0, s_0\}$, where each $(c_t, e_t, a_{t+1}, h_{t+1})$ stands for a function mapping individual histories s^t into a choice of consumption, $c_t(s^t)$, effort, $e_t(s^t)$, financial asset holding, $a_{t+1}(s^t)$, and human capital, $h_{t+1}(s^t)$. Note that the choice of an action $(c_t, e_t, a_{t+1}, h_{t+1})$ amounts to an effort decision, a consumption-saving decision, and a decision how to allocate the saving between investment in financial assets and investment in human capital.

An individual household with financial asset holding a_t in period t receives financial income $r_f a_t$, where r_f is the risk-free real interest rate (the return to financial investments). A household with human capital h_t earns labor income $r_h h_t$, where r_h is the wage rate (rental rate) per unit of human capital. Note that investment of one unit of the consumption good in financial capital yields the risk-free return r_f and investment of one unit of the consumption good in human capital earns the risky return $\phi r_h + \eta(s_t)$. Note further that we confine attention to wage rates and interest rates that are independent of time.

The government chooses a system of taxes and transfers that provides insurance and affects incentives. This tax-and-transfer system consists of a capital income tax/subsidy, $\tau_a r_f a_t$, a labor income (human capital) tax/subsidy, $\tau_h(s_{t-1}) r_h h_t$, and transfer payments that depend on labor income, $tr(s_{t-1}, s_t) r_{ht} h_t$. Note that taxes/subsidies and transfer payments are linear in the choice variables k and h. Further, we assume that capital and labor income taxes/subsidies are constant over time and independent of individual histories, though the labor income tax in period t may depend on the state in period $t-1$. Further, the transfer payments may depend on the current and last period's state: $tr_t = tr(s_{t-1}, s_t)$. The dependence of tr and τ_h on s_{t-1} is needed to decentralize the constrained optimal allocations as competitive equilibrium allocations (Corollary 2) for the general Markov case, though this dependence can be dropped if human capital shocks, η, are i.i.d. for given effort choices. A *tax-and-transfer policy* is a triple (τ_a, τ_h, tr), where τ_a is a real number and τ_h and tr are functions $\tau_h(s_{t-1})$ and $tr(s_{t-1}, s_t)$.

The sequential budget constraint of individual households say that in each period total spending equals total income. Thus, the household budget constraint requires that

$$
\begin{aligned}
&c_t + a_{t+1} - a_t + x_{ht} = (1 - \tau_h(s_{t-1}) + tr(s_{t-1}, s_t))r_h h_t + (1 - \tau_a)r_f a_t \\
&h_{t+1} \geq 0 \ ; \quad a_{t+1} + \frac{h_{t+1}}{\phi} \geq 0 \ , \qquad (20)
\end{aligned}
$$

where the expression $a_{t+1} + \frac{h_{t+1}}{\phi}$ stands for the value of total individual capital, financial plus human. In (20) the individual variables c_t, x_{ht}, a_{t+1}, and h_{t+1} are functions of (h_0, s^t), and the budget constraint (20) has to hold for all types and histories, (h_0, s^t). For notational ease, we have suppressed this dependence in (20). Note that the budget constraint (20) is linear in the household choice variables a and h.

For given tax-and-transfer policy, (τ_a, τ_h, tr), and given rental rates, r_f and r_h, an individual household of initial type (s_0, h_0) chooses a plan $\{c_t, e_t, a_{t+1}, h_{t+1}|h_0, s_0\}$ that solves the utility maximization problem:

$$
\begin{aligned}
&\max_{\{c_t, e_t, a_t, h_t|h_0, s_0\}} U(\{c_t, e_t|s_0\}) \\
&\quad subject\ to: \ \{c_t, e_t, a_{t+1}, h_{t+1}|h_0, s_0\} \in B(h_0, s_0) \qquad (21)
\end{aligned}
$$

where the budget set, $B(h_0, s_0)$, of an household of type (h_0, s_0) is defined by Eq. (20) and the expected lifetime utility, U, associated with a consumption-effort plan, $\{c_t, e_t|s_0\}$, is defined in (1).

The consumption good is produced by a representative firm that rents physical capital, K_t, and human capital, H_t, in competitive markets at rentals rates r_k and r_h, respectively. In each period t, the representative firm rents physical and human capital up to the point where current profit is maximized:

$$
\max_{K_t, H_t} \{F(K_t, H_t) - r_k K_t - r_h H_t\} \qquad (22)
$$

There is a financial sector that can transform household saving into physical capital at no cost. Thus, the no-arbitrage condition

$$
r_f = r_k - \delta_k \qquad (23)
$$

has to hold. We consider a closed economy so that in equilibrium the demand for capital and labor by the representative firm must be equal to the corresponding aggregate supply by all (domestic) households:

$$\begin{aligned} K_t &= \mathbf{E}[a_t] \\ H_t &= \mathbf{E}[h_t] \,. \end{aligned} \tag{24}$$

Note that we assume that an appropriate law of large numbers applies so that aggregate household variables are obtained by taking the expectations over all individual histories and initial types: $\mathbf{E}[a_t] = \sum_{h_0,s_0,s^{t-1}} a_t(h_0, s_0, s^{t-1})\pi_t(s^{t-1}, e^{t-1}(h_0, s_0, s^{t-1})|h_0, s_0)\pi_0(h_0, s_0)$ and $\mathbf{E}[h_t] = \sum_{h_0,s_0,s^t} h_t(h_0, s_0, s^{t-1})\pi_t(s^t, e^{t-1}(h_0, s_0, s^{t-1})|h_0, s_0)\pi_0(h_0, s_0)$.

We assume that the government runs a balanced budget in each period. We further assume that the social insurance system has its own budget that balances in each period:

$$\begin{aligned} \tau_a r_f \mathbf{E}[a_t] + r_h \mathbf{E}[\tau_h(s_{t-1})h_t] &= 0 \\ \mathbf{E}[tr(s_{t-1}, s_t)] &= 0 \end{aligned} \tag{25}$$

In the current setting, the two government budget constraints (25) are equivalent to one consolidated budget constraint in the sense that the same set of equilibrium allocations can be achieved. However, we prefer to work with the two government budget constraints (25) to separate the tax system, which changes investment incentives, from the social insurance system, which changes the incentive to apply effort.

Recall that an individual household of initial type s_0 chooses a household plan $\{c_t, e_t, a_{t+1}, h_{t+1}|h_0, s_0\}$. We denote the family of household plans, one for each household type (h_0, s_0), by $\{c_t, e_t, a_{t+1}, h_{t+1}\}$. Note that a family of household plans also defines an allocation. Our definition of a market equilibrium is standard:

Definition 2 A *competitive market equilibrium* for given tax-and-transfer policy, (τ_a, τ_h, tr), is a family of household plans, $\{c_t, e_t, a_{t+1}, h_{t+1}\}$, a plan for the representative firm, $\{K_t, H_t\}$, an interest rate, r_f, and a wage rate, r_h, so that i) for each household type (h_0, s_0) the plan $\{c_t, e_t, a_{t+1}, h_{t+1}|h_0, s_0\}$ solves the household's utility maximization problem (21), ii) $\{K_t, H_t\}$ solves the firm's profit maximization problem (22) in each period t, iii) the no-arbitrage condition (23) and the market clearing conditions (24) hold, and iv) the government budget constraint (25) is satisfied.

The no-arbitrage condition (23) and the market clearing conditions (24) together with the government budget constraint (25) and the individual budget constraint (11) imply that the aggregate resource constraint (7) is satisfied. Put differently, Walras' law holds.

4.2 Characterization of competitive equilibria

Under constant-returns-to-scale, profit maximization (22) implies that

$$\begin{aligned} r_{kt} &= F_k(\tilde{K}_t) \\ r_{ht} &= F_h(\tilde{K}_t) \end{aligned} \tag{26}$$

where $\tilde{K}_t = \frac{K_t}{H_t}$ is the ratio of aggregate physical capital to aggregate human capital (capital-to-labor ratio) and $F_k(\tilde{K}_t)$ and $F_h(\tilde{K}_t)$ stand for the marginal product of physical capital and human capital, respectively. Equation (26) summarizes the implications of profit maximization by the representative firm.

To characterize the solution to the household problem in a market economy, it is convenient to introduce the following new household-level variables:

$$\begin{aligned} w_t &= k_t + \frac{h_t}{\phi} \ \ , \ \ \theta_t = \frac{k_t}{w_t} \ \ , \ \ 1 - \theta_t = \frac{h_t}{\phi w_t} \\ r_t &= \theta_t(1-\tau_a)\left(F_k(\tilde{K}_t) - \delta_k\right) \\ &\quad + (1-\theta_t)\left((1-\tau_h(s_{t-1}) + tr(s_{t-1}, s_t))\phi\, F_h(\tilde{K}_t) + \eta(s_t)\right) \end{aligned} \tag{27}$$

Here w_t is the value of total wealth, financial and human, measured in units of the consumption good, θ_t is the share of total wealth invested in financial capital (financial asset holding), and $(1-\theta_t)$ is the share of total wealth invested in human capital. The expression $1+r_t$ is the total return on investing one unit of the consumption good. Note further that w_t is total wealth before assets have paid off and depreciation has taken place and $(1+r_t)w_t$ is total wealth after asset payoff and depreciation has occurred.

Using the change-of-variables (27), we can rewrite the budget constraint (11) as:

$$\begin{aligned} w_{t+1} &= (1 + r_t(\theta_t, \tilde{K}_t, s_{t-1}, s_t))w_t - c_t \\ w_{t+1} &\geq 0 \ \ ; \ \ (1-\theta_{t+1})w_{t+1} \geq (1+\eta(s_t))(1-\theta_t)w_t \end{aligned} \tag{28}$$

Note that the second inequality constraint in (28) is the non-negativity constraint on human capital investment. Clearly, (28) is the budget constraint associated with a consumption-saving problem and a portfolio choice problem when there are two investment opportunities, namely risk-free financial capital and risky human capital. The risk-free return to financial capital investment is given by $(1-\tau_a)(F_k(\tilde{K}_t)-\delta_k)$ and the risky return to human capital investment is $(1-\tau_h(s_{t-1})+tr(s_{t-1}, s_t))\phi F_h(\tilde{K}_t)+\eta(s_t)$. Note that the total investment return, r_t, depends on the individual portfolio share θ_t, the aggregate capital-to-labor ratio $\tilde{K}_t$, which captures any general equilibrium effects, and the individual shock s_t, which represents human capital risk. The investment return also depends on the tax-and-transfer rates, $(\tau_a, \tau_h, tr(.))$, but for notational ease this dependence is suppressed in (28).

A household plan is now given by $\{c_t, e_t, w_{t+1}, \theta_{t+1}|w_0, s_0\}$, where $(c_t, e_t, w_{t+1}, \theta_{t+1})$ is a function that maps histories of shocks, s^t, into choices $(c_t(s^t), e_t(s^t), w_{t+1}(s^t), \theta_{t+1}(s^t))$. The definition of a sequential equilibrium using household plans $\{c_t, e_t, w_{t+1}, \theta_{t+1}|w_0, s_0\}$ instead of $\{c_t, e_t, a_{t+1}, h_{t+1}|h_0, s_0\}$ is, mutatis mutandis, the same as definition 1.

In the following, we focus on competitive equilibria in which the value of the aggregate capital-to-labor ratio, $\tilde{K}$, is constant over time and the efficiency condition (17) holds. In other words, we confine attention to competitive equilibria that might decentralize constrained optimal allocation; a general characterization of competitive equilibria is discussed in the Appendix.

The household decision problem has a simple solution. Specifically, effort and portfolio choice only depends on the current state, $e_t = e(s_t)$ and $\theta_t = \theta(s_{t-1})$, and current consumption, c_t, and next period's wealth, w_{t+1}, are linear functions of current wealth, w_t, given by

$$\begin{aligned} c_t(s^t) &= (1-\beta)(1 + r(\theta(s_{t-1}), \tilde{K}, s_{t-1}, s_t))w_t(s^{t-1}) \\ w_{t+1}(s^t) &= \beta(1 + r(\theta(s_{t-1}), \tilde{K}, s_{t-1}, s_t))w_t(s^{t-1}) \end{aligned} \tag{29}$$

where effort and portfolio choice are the solution to the following Bellman equation:

$$\begin{aligned} V(s) = \max_{\theta,e} \Bigg\{ &-d(e,s) \;+\; B(\beta) \;+\; \frac{\beta}{1-\beta} \sum_{s'} \ln(1 + r(\theta(s), \tilde{K}, s, s'))\pi(s'|s,e) \\ &+\; \beta \sum_{s'} V(s')\pi(s'|s,e)) \Bigg\}. \end{aligned} \tag{30}$$

The linearity of individual consumption and individual wealth choices means that aggregate market clearing reduces to the condition that the (common) portfolio choice of households, θ, has to be consistent with the capital-to-labor ratio chosen by the firm, $\tilde{K}$. More precisely, the two market clearing conditions (24) hold if

$$\tilde{K} = \frac{\sum_s \theta(s, \tilde{K})\Omega_0(s)}{\phi(1 - \sum_s \theta(s, \tilde{K})\Omega_0(s))} \tag{31}$$

where $\theta(s, \tilde{K})$ and $e(\tilde{K})$ are the portfolio demand function and the effort function defined by the solution to (30) and Ω_0 is the distribution of total wealth across household types in period $t = 0$ defined as $\Omega_0(s) \doteq \frac{\sum_{w_0}(1+r(s_0))w_0\pi_0(w_0,s_0)}{\sum_{w_0,s_0}(1+r(s_0))w_0\pi_0(w_0,s)}$. Equation (31) is derived from (15) using $k = \theta w$ and $h = \phi(1-\theta)w$ and the fact that—because of the constant-returns-to-scale assumption—the two equations in (24) can be reduced to one equation.

Proposition 3 *Suppose that* $(\theta, e, V, \tilde{K})$ *solve* (30) *and* (31). *Then the allocation* $\{c_t, e_t, h_{t+1}, K_{t+1}\}$, *induced by* $(\theta, e, V, \tilde{K})$ *together with associated wage rates and financial returns given by* (26) *define a stationary (balanced growth) equilibrium.*

Proof See appendix. □

Proposition 3 characterizes equilibria for given tax-and-transfer policy. The government budget constraint (25) is satisfied if (and only if) the condition

$$\tau_a \tilde{K} r_f(\tilde{K}) + \phi r_h(\tilde{K}) \sum_s \tau_h(s)\Omega_0(s) = 0$$
$$\sum_{s,s'} tr(s,s')\pi(s'|s,e(s))\Omega_0(s) = 0\,. \tag{32}$$

holds. Clearly, Eq. (32) imposes a further condition that determines the set of budget-feasible government policies (τ_a, τ_h, tr).

Proposition 3 shows how the household-level variables evolve in equilibrium. The evolution of aggregate variables is obtained by taking the expectations over individual variables using the government budget constraint (32):

$$C_t = (1-\beta)\left(1 + r_f(\tilde{K})\right) W_t$$
$$W_{t+1} = \beta\left(1 + r_f(\tilde{K})\right) W_t$$
$$K_t = \frac{\phi \tilde{K}^*}{1+\phi\tilde{K}^*} W_t \quad ; \quad H_t = \frac{1}{1+\phi\tilde{K}^*} W_t\,. \tag{33}$$

Several comments regarding he interpretation of Proposition 3 are in order.

First, Proposition 3 is the generalization of the tractability result of Krebs (2003, 2006) to incomplete-market models with an effort choice. Proposition 3 in conjunction with the balanced-budget condition (32) provide a convenient equilibrium characterization that has two useful properties. The consumption-saving choice is linear in wealth and the portfolio and effort choice are constant and independent of wealth (histories). In addition, the equilibrium can be computed without the knowledge of the endogenous, infinite-dimensional wealth distribution. These two properties render the computation of equilibria extremely simple since it suffices to solve (30), (31), and (32).

Second, consider the special case when the shock process is i.i.d. for given effort choice and there are no dis-utility shocks. In this case, Proposition 3 implies that effort and portfolio choices are independent of the current shock, s. Suppose further that e is a continuous variable. We can then use the first-order condition approach and find that the solution to the maximization problem (30) is characterized by the following two equations:

$$0 = \sum_s \frac{(1-\tau_h + tr(s))\phi r_h(\tilde{K}) + \eta(s') - (1-\tau_a) r_f(\tilde{K})}{1 + r(\theta, \tilde{K}, s')} \pi(s'|e)$$
$$d'(e) = \frac{\beta}{1-\beta} \sum_{s'} ln\left(1 + r(\theta, \tilde{K}, s')\right) \frac{\partial \pi}{\partial e}(s'|e) \tag{34}$$

The first equation in (34) expresses the optimal portfolio choice of individual households. It states that the expected marginal utility weighted excess return of human capital investment over physical capital investment must be zero, where the marginal utility is represented by the term $(1+r)^{-1}$. The second equation in (34) is the first-order condition with respect to the effort choice and says that the dis-utility of increasing effort is equal to the expected gains associated with an increase in effort. Note that equilibrium values of $(\theta, e, \tilde{K})$ are now determined by the system of three equations defined by (31), (32), and (34) for a given tax-and-transfer system

Third, we can gain a better understanding of the way the social insurance system, $tr(.)$, affects individual consumption and welfare by noticing that

$$c_{t+1}(s^{t+1}) = \beta\left(1+\theta(1-\tau_a)r_f\right. \tag{35}$$

$$\left.+(1-\theta)\left((1-\tau_h+tr(s_{t+1}))\phi\, r_h+\eta(s_{t+1})\right)\right) c_t(s^t) \tag{36}$$

in the case of i.i.d. shocks. Individual consumption grows at a rate that is equal to $\beta(1+r)$, where the total investment returns, r, depends on portfolio choice, θ, financial returns, $r_f = F_k - \delta_k$, human capital returns $\phi\, F_h$, ex-post shocks, $\eta(s_t)$, the tax rates, τ_a and τ_h, and the transfer payments (insurance), $tr(s_{t+1})$. From (35) we immediately conclude that consumption is independent of human capital shocks if $tr(s_{t+1})\phi\, F_h = -\eta(s_{t+1})$. This is intuitive since in the case of a negative human capital shock, $\eta(s_{t+1}) - \bar{\eta}(e) < 0$, the term $(1-\theta)\eta(s_{t+1})w_{t+1} < 0$ is the total amount of human capital lost in units of the consumption good and the term $(1-\theta)tr(s_{t+1})\phi\, r_h w_{t+1} > 0$ is the corresponding transfer payment in consumption units, where we used the notation $\bar{\eta}(e) \doteq \sum_{s'}\eta(s')\pi(s'|e)$.

4.3 Optimal competitive equilibria

A comparison of the equilibrium allocations of a market economy (Proposition 3) and the constrained optimal allocations (Proposition 2) shows the equivalence between the two—up to distribution of initial consumption levels—when the tax- and transfer system is chosen appropriately. In addition, the initial distribution of (financial) wealth can be chosen to ensure that the initial consumption, $c_0(h_0, s_0)$, chosen by the social planner is also the initial consumption in the equilibrium of the market economy. More precisely, we have the following decentralization result:

Corollary 2 *Let* $\{c_t^*, e_t^*, h_{t+1}^*, K_{t+1}^*\}$ *be a constrained optimal allocation with the associated* $(e^*, \epsilon^*, \tilde{K}^*)$ *solving the simple social planner problem* (20). *Then* $\{c_t^*, e_t^*, h_{t+1}^*, K_{t+1}^*\}$ *is the equilibrium allocation of a competitive market economy with a tax-and-transfer system,* (τ^*, tr^*)*, that is the solution to the following equation system:*

$$\begin{aligned}
& r_f(\tilde{K}^*) + \epsilon^*(s,s') = \theta^*(1-\tau_a^*)r_f(\tilde{K}^*) \\
& +(1-\theta^*)[(1-\tau_h^*(s)+tr(s,s'))\phi r_h(\tilde{K}^*)+\eta(s')] \\
& 0 = \sum_{s'} \frac{(1-\tau_h^*(s)+tr^*(s,s'))\phi r_h(\tilde{K}^*)+\eta(s') - (1-\tau_a^*)r_f(\tilde{K}^*)}{1+r_f(\tilde{K}^*)+\epsilon^*(s,s')}\pi(s'|s,e^*) \\
& 0 = \tau_a^*\tilde{K}^* r_f(\tilde{K}^*) + \phi r_h(\tilde{K}^*)\sum_s \tau_h^*(s)\Omega_0(s)
\end{aligned} \tag{37}$$

with $\theta^* = \frac{\phi \tilde{K}^*}{1+\phi \tilde{K}^*}$.

The first equation in (37) ensures that transfer payments in the market economy are set so that social insurance is optimal. The condition is derived from an equalization of equilibrium consumption and socially optimal consumption (growth rates). The second equation in (37) states that taxes and subsidies have to be chosen so that the socially optimal portfolio allocation is an equilibrium outcome in the market economy. The last equation in (37) is the government budget constraint.

The following corollary is straightforward implications of corollary 2.

Corollary 3 *The optimal tax system requires a subsidy on human capital (risky) investment,* $\tau_h^*(s) < 0$*, and a tax on physical capital (risk-free) investment,* $\tau_a^* > 0$.

The intuition underlying the result is simple. The optimality condition (11) requires that the expected return to human capital investment is equal to the risk-free rate. Since households are risk averse and human capital is risky, they can only be induced to invest in human capital if human capital investment is subsidized relative to investment in the risk-free asset. A version of corollary 3 was first shown in Da Costa and Maestri (2007) using a one-period model with private information about household types.

Corollary 3 also provides a link to the literature on constrained efficient allocations in incomplete-market models (Geanakoplos and Polemarchakis 1986) that assume an exogenous asset payoff structure and therefore take the lack of certain type of insurance as given. Corollary 3 implies that competitive equilibrium allocations are constrained inefficient for certain asset payoff structures in the sense that the government can improve social welfare by introducing a subsidy to human capital investment. Krebs (2006) and Toda (2015) discuss the efficiency properties of this class of incomplete-market models more generally, and Gottardi et al. (2015) analyze the optimal level of taxation and debt in this class of models.

5 Concluding remarks

In this paper, we considered a class of growth models with idiosyncratic human capital risk and private information about individual effort choices (moral hazard). We have shown that constrained optimal allocations are simple and that they can be decentralized as competitive equilibria of a market economy with a simple tax- and transfer scheme.

There are three main extensions of the basic framework that could be incorporated without sacrificing the tractability of the model (Propositions 1–3). We leave the formal treatment of the following three extensions for future research.

First, we can introduce additional sources of idiosyncratic investment and production risk. Specifically, the productivity of human capital investment can be subject to idiosyncratic shocks, $\phi = \phi(s_t)$. Further, the productivity of human capital can be subject to idiosyncratic risk, which amounts to replacing H in the production function (2) by $\mathbf{E}[z(s_t)h_t]$. Moreover, Eq. (4) representing the production of human capital can also be generalized. As in Krebs (2003, 2006) and Stantcheva (2017), Eq. (4) assumes that human capital production only uses goods. In contrast, Heckman et al. (1998)

and Huggett et al. (2011) focus on the time investment in human capital. Clearly, in most cases human capital investment uses both goods and time. The tractability result derived in this paper may also hold for the case in which both goods and time are used to produce human capital as long as there is constant-returns-to-scale. For instance, we can introduce a time cost of human capital production by replacing the term ϕx_{ht} in (4) by $\phi\,(h_t l_t)^{\rho}\, x_{xt}^{1-\rho}$, where l_t denotes the time spend in human capital production.

Second, economic fundamentals may depend on aggregate shock, S_t. Specifically, the stochastic process of exogenous shocks can be a Markov process for given effort choices with transition probabilities $\pi(s_{t+1}, S_{t+1}|s_t, S_t, e_t)$, where certain restrictions should be placed on π to ensure that individual effort choices do not affect the probability of the aggregate shock. We conjecture that the main characterization results for optimal allocations still hold in the sense that effort choices and individual consumption growth rates are independent of individual histories and type, but now they depend on (s_t, S_t).

Third, as in Jones and Manuelli (1990) and Rebelo (1991), the aggregate production function (2) displays constant-returns-to-scale with respect to production factors that can be accumulated without bounds, a property that is well-known to generate endogenous growth. The structure of our arguments suggests that the tractability results still hold if (2) is replaced by a production function with diminishing returns or, equivalently, a production function with constant-returns-to-scale and a third (fixed) factor of production (land). However, in this case we have an explicit time-dependence of individual and aggregate variables, and convergence towards a steady state instead of unbounded growth under certain conditions. Finally, it is open question to what extent diminishing returns at the individual investment level can be introduced while keeping the tractability of the model.

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Appendix

Proof of Proposition 1 Clearly, a straightforward approach to deriving the necessity of condition (11) is to write down the Lagrangian associated with the social planner problem and then to take first-order conditions. However, the existence of a vector of Lagrange multipliers requires additional conditions that might not be satisfied.[3] We therefore use a direct approach that does not require any assumptions on the primitives beyond the once already made in the paper.

[3] See Rustichini (1998) for a general treatment of the question of the existence of a Lagrange vector in infinite-dimensional optimization problems with incentive constraints.

Note first that we can confine attention to plans $\{c_t, h_t|h_0, s_0\}$ with $h_t(s^{t-1}) > 0$ for all t and s^{t-1}. This follows from our assumption that $c_t(h_0, s^t) = \tilde{c}_t(h_0, s^t)h_t(h_0, s^{t-1})$, where $\tilde{c}_t$ is bounded, in conjunction with the assumption that the utility function is unbounded from below because these two assumptions ensure that it is never socially optimal to choose $h_t(s^{t-1}) = 0$ for some s^{t-1}.

To prove the claim, suppose not, that is, for the optimal allocation $\{c_t, e_t, K_{t+1}, h_{t+1}\}$ there exist a $\bar{t}$ and $\bar{s}^{\bar{t}}$ with $h_t(\bar{s}^{\bar{t}}) > 0$ and (11) is not satisfied:

$$\phi\, F_h(\tilde{K}_{\bar{t}+1}) + \sum_{s_{\bar{t}+1}} \eta(s_{\bar{t}+1})\pi(s_{\bar{t}+1}|e_t(\bar{s}^{\bar{t}})) > F_k(\tilde{K}_{\bar{t}+1}) - \delta_k\ . \tag{A1}$$

Inequality (A1) states that the expected value of human capital returns (the left-hand-side of A1) exceeds the risk-free return on physical capital investment (the right-hand-side of A1). The proof by contradiction for the reversed case is, mutatis mutandis, the same.

Consider an alternative allocation $\{\hat{c}_t, e_t, \hat{K}_{t+1}, \hat{h}_{t+1}\}$ with identical $\{e_t\}$ and a $\{\hat{c}_t, \hat{K}_{t+1}, \hat{h}_{t+1}\}$ that only differs from $\{c_t, K_{t+1}, h_{t+1}\}$ at history $\bar{s}^{\bar{t}}$ and for all $s_{\bar{t}+1}$ subsequent to $\bar{s}^{\bar{t}}$. More specifically, we define

$$\begin{aligned}
\hat{h}_{\bar{t}+1}(\bar{s}^{\bar{t}}) &= h_{\bar{t}+1}(\bar{s}^{\bar{t}}) + (1+\eta(s_t))h_t + \phi\,(x_{ht} + \Delta x)\\
\hat{K}_{\bar{t}+1}(\bar{s}^{\bar{t}}) &= K_{\bar{t}+1}(\bar{s}^{\bar{t}}) - \pi_t(\bar{s}^{\bar{t}})\,\Delta x\\
\forall s_{\bar{t}+1}\ :\quad \hat{c}_{\bar{t}+1}(\bar{s}^{\bar{t}}, s_{\bar{t}+1}) &= c_{\bar{t}+1}(\bar{s}^{\bar{t}}, s_{\bar{t}+1}) + \Delta c(s_{\bar{t}+1})\ ,
\end{aligned} \tag{A2}$$

where the changes $\Delta x > 0$ and $\Delta c(s_{\bar{t}+1}) > 0$ are strictly positive real numbers and we have suppressed the dependence of $\pi_t(\bar{s}^{\bar{t}})$ on $e^{\bar{t}-1}(\bar{s}^{\bar{t}-1})$. In words: in period $\bar{t}$, the alternative allocation increases human capital investment by Δx for each household with history $\bar{s}^{\bar{t}}$ and reduces physical capital investment by $\pi_t(\bar{s}^{\bar{t}})\Delta x$, and in period $\bar{t}+1$ it increases consumption for these households in all possible states $s_{\bar{t}+1}$. Clearly, this allocation strictly increases social welfare. We now show that such a strictly positive vector $(\Delta x, \vec{\Delta c})$ exists so that $\{\hat{c}_t, e_t, \hat{K}_t, \hat{h}_t\}$ satisfies the aggregate resource constraint and the incentive constraint, which contradicts the claim that $\{c_t, e_t, K_t, h_t\}$ is an optimal allocation. The idea of the proof is to show that the investment change increases available resources in $\bar{t}+1$ for small enough Δx and that the additional resources can be used to increase consumption in each state $s_{\bar{t}+1}$ without affecting the incentive constraint.

Let $\Delta X = \pi_t(\bar{s}^{\bar{t}})\Delta x$ be the aggregate change in investment. Since F is continuously differentiable, the increase in aggregate human capital investment in period $\bar{t}$ by ΔX increases production in period $\bar{t}+1$ by

$$\phi F_{h,\bar{t}+1}\Delta X + \epsilon_1(\Delta X) \tag{A3}$$

with $\lim_{\Delta X\to 0} \frac{\epsilon_1(\Delta X)}{\Delta X} = 0$. To reverse the increase in human capital investment in period $\bar{t}$, in the alternative allocation investment in human capital in period $\bar{t}+1$ is reduced by $\Delta x'(s_{\bar{t}+1})$. Since we require $\hat{h}_{\bar{t}+2} = h_{\bar{t}+2}$, the two investment changes Δx

and $\Delta x'$ need to satisfy

$$\Delta x'(s_{\bar{t}+1}) \;=\; (1+\eta(s_{\bar{t}+1}))\Delta x \tag{A4}$$

Finally, the reduction in investment in physical capital in period $\bar{t}$ by ΔX reduces output by $\left(F_{k,\bar{t}+1}-\delta_k\right)\Delta X \;+\; \epsilon_2(\Delta X)$ and the increase in physical capital investment in period $\bar{t}+1$ by ΔX necessary to achieve $\hat{K}_{\bar{t}+2}(\bar{s}^{\bar{t}}, s_{\bar{t}+1}) \;=\; K_{\bar{t}+2}(\bar{s}^{\bar{t}}, s_{\bar{t}+1})$ reduces available resources in period $\bar{t}+1$ by $\Delta X + \epsilon_3(\Delta X)$, where $\lim_{\Delta X\to 0}\frac{\epsilon_2(\Delta X)}{\Delta X} \;=\; \lim_{\Delta X\to 0}\frac{\epsilon_3(\Delta X)}{\Delta X}=0$.

In sum, for the alternative allocation $\{\hat{c}_t, e_t, \hat{K}_{t+1}, \hat{h}_{t+1}\}$ the additional resources available for consumption in period $\bar{t}+1$ for households with history $\bar{s}^{\bar{t}}$ are

$$\begin{aligned}\Delta\omega =\; &\phi F_{h,\bar{t}+1}\,\Delta X\\ &+\left(1+\sum_{s_{\bar{t}+1}}\eta(s_{\bar{t}+1})\pi(s_{\bar{t}+1}|e_{\bar{t}}(\bar{s}^{\bar{t}})\right)\Delta X\\ &-\left(1+F_{1,\bar{t}+1}-\delta_k\right)\Delta X \;+\; \epsilon(\Delta X)\end{aligned} \tag{A5}$$

with $\lim_{\Delta X\to 0}\frac{\epsilon(\Delta X)}{\Delta X}=0$. Using the assumption that expected human capital returns exceed the financial returns, we conclude that for small enough ΔX we have $\Delta\omega>0$.

We next show that the additional resources, $\Delta\omega>0$, can be distributed in an incentive-compatible manner that benefits all households. To see this, define consumption in period $\bar{t}$ in the alternative allocation as $\hat{c}_{\bar{t}+1}(\bar{s}^{\bar{t}+1}) \;=\; c_{\bar{t}+1}(\bar{s}^{\bar{t}+1}) + \Delta c(s_{\bar{t}+1})$ with $\Delta c(s_{\bar{t}+1})>0$ for all $s_{\bar{t}+1}$. Further, choose the real number $\Delta c(s_{\bar{t}+1})$ so that

$$\ln\left(c_{\bar{t}+1}(\bar{s}^{\bar{t}})+\Delta c(s_{\bar{t}+1})\right) = \ln\left(c_{\bar{t}+1}(\bar{s}^{\bar{t}})\right) \;+\; \Delta u \tag{A6}$$

for a given, strictly positive real number Δu. This can always be done since the logarithmic function is continuous and strictly increasing. Further, continuous differentiability of the logarithmic function implies for sufficiently small Δu that the solution $\Delta\vec{c}$ to (A6) satisfies $\sum_{s_{\bar{t}+1}}\Delta c(s_{\bar{t}+1})\pi(s_{\bar{t}+1}|e_t(\bar{s}^{\bar{t}}))\,\pi_t(\bar{s}^{\bar{t}}) \;=\; \Delta\omega$. Thus, the alternative allocation $\{\hat{c}_t, e_t, \hat{K}_t, \hat{h}_t\}$ satisfies the aggregate resource constraint. It also satisfies the incentive constraint (8) since

$$\begin{aligned}\sum_{s_{\bar{t}+1}}\ln\left(\hat{c}_{\bar{t}+1}(\bar{s}^{\bar{t}})\right)\pi(s_{\bar{t}+1}|e_t(\bar{s}^{\bar{t}})) &= \sum_{s_{\bar{t}+1}}\ln\left(c_{\bar{t}+1}(\bar{s}^{\bar{t}})+\Delta c(s_{\bar{t}+1})\right)\pi(s_{\bar{t}+1}|e_t(\bar{s}^{\bar{t}}))\\ &= \sum_{s_{\bar{t}+1}}\ln\left(c_{\bar{t}+1}(\bar{s}^{\bar{t}})\pi(s_{\bar{t}+1}|e_t(\bar{s}^{\bar{t}}))\right) \;+\; \Delta u\end{aligned} \tag{A7}$$

for any probability distribution π over states $s_{\bar{t}+1}$. (A7) implies that the incentive constraint also holds for the new consumption allocation, $\{\hat{c}_t\}$, since (i) the incentive constraint is not changed for all $t\geq\bar{t}$ and (ii) a constant independent of effort choice (the discounted value of the discounted value of Δ has been added to both sides to the inequality. This completes the proof of the first part of Proposition 1.

The proof that $\tilde{K}_{t+1} = \tilde{K}$ also proceeds by contradiction. Suppose not, that is, there exists a $\bar{t}$ with $\tilde{K}_{\bar{t}+1} \neq \tilde{K}_{\bar{t}}$. Without loss of generality, suppose that $\tilde{K}_{\bar{t}+1} > \tilde{K}_{\bar{t}}$ and

$$F_k(\tilde{K}_{\bar{t}+1}) - \delta_k \;<\; F_k(\tilde{K}_{\bar{t}}) - \delta_k \; . \tag{A8}$$

In addition, the efficiency condition (17) requires:

$$\begin{aligned} \phi\, F_h(\tilde{K}_{\bar{t}+1}) \;+\; \sum_{s_{\bar{t}+1}} \eta(s_{\bar{t}+1})\pi(s_{\bar{t}+1}|s_{\bar{t}}, e_{\bar{t}}(s_{\bar{t}})) \;=\; F_k(\tilde{K}_{\bar{t}+1}) - \delta_k \\ \phi\, F_h(\tilde{K}_{\bar{t}}) \;+\; \sum_{s_{\bar{t}}} \eta(s_{\bar{t}})\pi(s_{\bar{t}}|s_{\bar{t}-1}, e_{\bar{t}-1}(s_{\bar{t}-1})) \;=\; F_k(\tilde{K}_{\bar{t}}) - \delta_k \end{aligned} \tag{A9}$$

It is straightforward to show that (A8) and (A9) imply that there is an alternative allocation that increases output in period $\bar{t}+1$ by decreasing $\tilde{K}_{\bar{t}+1}$ through an increase in human capital investment and a simultaneous decrease in physical capital investment keeping total investment fixed. Further, this increase in aggregate output comes at no consumption cost, and can be used to increase consumption and utility of a group of households in an incentive-compatible manner leading to a Pareto improvement, which contradicts the claim that the original allocation is constrained optimal. This completes the proof of Proposition 1. □

Proof of Proposition 2 According to the Weierstrass Theorem, it suffices to show that the objective function in the maximization problem (10) is upper semi-continuous and the constraint set is compact. Using a variant of the arguments made in Becker and Boyd (1997), a straightforward argument shows that both properties hold if we choose the product topology to define the underlying metric space.

Note that $\tilde{K}_{t+1} = \tilde{K}$ implies time-invariant effort choices: $e_{t+1}(s) = e(s)$. It remains to be shown that consumption risk is time- and history independent: $\epsilon_{t+1}(s^t, s_{t+1}) = \epsilon(s_{t+1})$. For notational ease, we consider the case in which the shock process is i.i.d. for given effort choice and disregard dis-utility shocks, which means that effort choice is a real number, $e(s_t) = e$. Given the structure of preferences, we can write lifetime utility as:

$$U(\{c_t, h_0, s_0\}, e, \tilde{K}) \;=\; \frac{1}{1-\beta} \ln c_0(h_0, s_0) \;+\; \tilde{U}_0(\{\epsilon_t|s_0\}, e, \tilde{K}) \; , \tag{A10}$$

with $\tilde{U}_0$ given by

$$\begin{aligned} \tilde{U}_0(\{\epsilon_t|s_0\}, e, \tilde{K}) \;=&\; -d(e) \\ &+ \sum_{t=1}^{\infty} \sum_{s^t|s_0} \beta^t \left[\ln(1 + r(\tilde{K}) + \epsilon_t(s^t)) - d(e)\right] \pi_t(s^t|e) \; , \end{aligned}$$

where $\pi_t(s^t|e) = \pi(s_t|e) \times \ldots \times \pi(s_1|e)$. Denote the continuation plan for history s^t by $\{\epsilon_{t+1+n}|s^t\}$ and denote the corresponding continuation value by $\tilde{U}_t$. The function

$\tilde{U}_t$ satisfies the recursive equation

$$\tilde{U}_t(\{\epsilon_{t+n}|s^{t-1}\}, e, \tilde{K}) = -d(e) + B(\beta) + \frac{\beta}{1-\beta}\sum_{s_{t+1}} \ln(1 + r(\tilde{K}) + \epsilon_{t+1}(s^{t+1}))\pi(s_{t+1}|e)$$
$$\beta \sum_{s_{t+1}} \tilde{U}_{t+1}(\{\epsilon_{t+1+n}|s^t\}, e, \tilde{K})\pi(s_{t+1}|e)\ , \tag{A11}$$

with $B(\beta) = \ln(1-\beta) + \frac{\beta}{1-\beta}\ln\beta$.

Clearly, any allocation $\{c_t, e_t, K_{t+1}, h_{t+1}\}$ is equivalent to an allocation $\{c_0, \epsilon_{t+1}, e_t, K_{t+1}, h_{t+1}\}$ so that the social planner problem (10) can be rewritten accordingly. Further, we know from proposition 1 that effort choices are independent of time and individual history/type, and that the aggregate capital-to-labor ratio is time-invariant. Thus, we can confine attention to allocations $\{c_0, \epsilon_{t+1}, e(.), \tilde{K}, h_{t+1}\}$. Rewriting the social planner problem (10) as a maximization problem over choices $\{c_0, \epsilon_{t+1}, e(.), \tilde{K}, h_{t+1}\}$ using (A10), (A11), and the efficiency condition (11) shows that the social planner problem (10) reduces to the maximization problem

$$\max_{e, \tilde{K}, \{\epsilon_t|s_0\}} \left\{ \sum_{s_0} \tilde{U}_0(\{\epsilon_t|s_0\}, e, \tilde{K})\, \mu(s_0) \right\}$$
$$\text{subject to:}$$
$$r(\tilde{K}) = \phi F_h(\tilde{K}) + \sum_{s'} \eta(s')\pi(s'|e))$$
$$\forall t : \sum_{s^{t+1}} \epsilon_{t+1}(s^{t+1})\pi_{t+1}(s^{t+1}|e) = 0$$
$$\forall t, s^t, \hat{e} : \tilde{U}_{t+1}(\{\epsilon_{t+1+n}|s^t\}, e, \tilde{K}) \geq \tilde{U}_{t+1}(\{\epsilon_{t+1+n}|s^t\}, \hat{e}, \tilde{K}), \tag{A12}$$

where $\tilde{U}_t$ are defined by the recursive equation (A11).

We next show that the solution to (A12) has the property that $\epsilon_{t+1}(s^t, .)$ is independent of t and s^t. Suppose not, that is, there exist $\bar{t}$, $\tilde{s}^{\bar{t}}$, and $\hat{s}^{\bar{t}}$ with $\epsilon_{\bar{t}+1}(\hat{s}^{\bar{t}}, .) \neq \epsilon_{\bar{t}+1}(\tilde{s}^{\bar{t}}, .)$. Without loss of generality, assume

$$\sum_{s_{\bar{t}+1}} \ln(1 + r(\tilde{K}) + \epsilon_{\bar{t}+1}(\hat{s}^{\bar{t}}, s_{\bar{t}+1}))\pi(s_{t+1}|e)$$
$$\geq \sum_{s_{\bar{t}+1}} \ln(1 + r(\tilde{K}) + \epsilon_{\bar{t}+1}(\tilde{s}^{\bar{t}}, s_{\bar{t}+1}))\pi(s_{\bar{t}+1}|e) \tag{A13}$$

Define an alternative $\{\epsilon'_t|s_0\}$ that is identical to $\{\epsilon_t|s_0\}$ except at $\tilde{s}^{\bar{t}}$ and $\hat{s}^{\bar{t}}$, where we set

$$\epsilon'_{\bar{t}+1}(\hat{s}^{\bar{t}}, s_{\bar{t}+1}) = \epsilon'_{\bar{t}+1}(\tilde{s}^{\bar{t}}, s_{\bar{t}+1}) = \epsilon_{\bar{t}+1}(\hat{s}^{\bar{t}}, s_{\bar{t}+1})) \tag{A14}$$

for all $s_{\bar{t}+1}$. Simple algebra shows that the allocation $(e, \tilde{K})$ and $\{\epsilon_t'|s_0\}$ lies in the constraint of (A12). The alternative allocation also (weakly increases the value of the objective function in (A12). Thus, there is always an optimal allocation with $\{\epsilon_t|s_0\}$ satisfying $\epsilon_{t+1}(s^{t+1}) = \epsilon(s_{t+1})$ for all t and s^{t+1}. This completes the proof of proposition 2. Note that the last argument is similar to the argument made in Fudenberg et al. (1990) to prove the history-independence of constrained optimal allocations in their simple model without an aggregate resource constraint. In the current human-capital model, the argument works because we can-based on proposition 1—confine attention to effort levels that are type and history-independent. □

Proof of Proposition 3 After using the change-of-variables (27), the Bellman equation associated with the sequential household maximization problem (21) reads

$$v(w, \theta, s) \;=\; \max_{e,c,w',\theta'} \left\{ \ln c - d(e, s) \;+\; \beta \sum_{s'} v(w', \theta', s')\pi(s'|s, e)) \right\}, \quad \text{(A15)}$$

where (w', θ', c) lies in the budget set defined by (28). Guess-and-verify shows that a solution to (A15) is given by (29) and the solution to (30), where the link between value function, v, and intensive-form value function, V, is:

$$v(w, \theta, s) \;=\; \frac{1}{1-\beta} \ln c_0(w, \theta, s) + V(s) \quad \text{(A16)}$$

Using the principle of optimality, we conclude that an individual plan solving (29) and (30) also solves the sequential household maximization problem (21).

There are two technical issues regarding the principle of optimality. First, the Bellman equation (A15) and the associated sequential household maximization problem (21) have the property that probabilities depend on (effort) choices and therefore belong to a class of maximization problems not analyzed in Stokey et al. (1989). However, it is straightforward to show that the standard argument for the principle of optimality still applies in this case.

The second issue is the question of the construction of the appropriate function space since the economic problem is naturally an unbounded problem. To deal with this issue, one can, for example, follow Streufert (1990) and consider the set of continuous functions $\mathbf{B}_W$ that are bounded in the weighted sup-norm $\|V\| \doteq \sup_x \frac{|V(x)|}{W(x)}$, where $x = (w, \theta, s)$ and the weighting function W is given by $W(x) = |L(x)| + |U(x)|$ with U an upper bound and L a lower bound, and endow this function space with the corresponding metric. In other words, $\mathbf{B_W}$ is the set of all functions, V, with $L(x) \leq V(x) \leq U(x)$ for all $x \in \mathbf{X}$. A straightforward but tedious argument shows that confining attention to this function space is without loss of generality. More precisely, one can show that there exist functions L and H so that for all candidate solutions, V, we have $L(x) \leq V(x) \leq H(x)$ for all $x \in \mathbf{X}$.[4]

[4] Alvarez and Stokey (1998) provide a different, but related, argument to prove the existence and uniqueness of a solution to the Bellman equation for a class of unbounded problems similar to the one considered here, though without moral hazard.

It is left to show that the two market clearing conditions (24) hold if (31) holds. To prove this, let

$$\Omega_t(s) \doteq \frac{E_t\left[(1+r_t)w_t|s_t=s\right]}{E_t[(1+r_t)w_t]}$$

be the distribution of total wealth across household types in period t. We first shows that the market clearing condition (15) holds if

$$\tilde{K}_{t+1} = \frac{\sum_s \theta_{t+1}(s)\Omega_t(s)}{\phi(1-\sum_s \theta_{t+1}(s)\Omega_t(s))} \tag{A17}$$

holds. To see this, note that we have

$$\begin{aligned} K_{t+1} &= E\left[\theta_{t+1}w_{t+1}\right] \\ &= \beta E\left[\theta_{t+1}(1+r_t)w_t\right] \\ &= \beta \sum_{s_t} E\left[\theta_{t+1}(1+r_t)w_t|s_t\right]\pi_t(s_t) \\ &= \beta \sum_{s_t} E\left[\theta(s_t)(1+r_t)w_t|s_t\right]\pi_t(s_t) \\ &= \beta E\left[(1+r_t)w_t\right]\sum_{s_t}\theta(s_t)\Omega_t(s_t) \ . \end{aligned} \tag{A18}$$

The second line in (A18) uses the equilibrium law of motion for the individual state variable w, the third line is simply the law of iterated expectations, the fourth line follows from the fact that the portfolio choices only depend on s_t, and the last line is a direct implication of the definition of Ω. A similar expression holds for the aggregate stock of human capital held by households, H_{t+1}. Dividing two expressions proves the equivalence between (A17) and (15).

The law of motion for Ω can be written as:

$$\begin{aligned} \Omega_{t+1}(s_{t+1}) &= \frac{E\left[(1+r_{t+1})w_{t+1}|s_{t+1}\right]\pi(s_{t+1})}{E\left[(1+r_{t+1})w_{t+1}\right]} \\ &= \frac{E\left[(1+r_{t+1})(1+r_t)w_t|s_{t+1}\right]\pi(s_{t+1})}{E\left[(1+r_{t+1})(1+r_t))w_t\right]} \\ &= \frac{\sum_{s_t} E\left[(1+r_{t+1})(1+r_t)w_t|s_t,s_{t+1}\right]\pi(s_t|s_{t+1})\pi(s_{t+1})}{\sum_{s_t,s_{t+1}} E\left[(1+r_{t+1})(1+r_t)w_t|s_t,s_{t+1}\right]\pi(s_t,s_{t+1})} \\ &= \frac{\sum_{s_t} E\left[(1+r_{t+1})(1+r_t)w_t|s_t,s_{t+1}\right]\pi(s_{t+1}|s_t)\pi(s_t)}{\sum_{s_t,s_{t-1}} E\left[(1+r_{t+1})(1+r_t)w_t|s_t,s_{t+1}\right]\pi(s_{t+1}|s_t)\pi(s_t)} \\ &= \frac{\sum_{s_t}(1+r(s_t,s_{t+1}))\pi(s_{t+1}|s_t)E\left[(1+r_t)w_t|s_t\right]\pi(s_t)}{\sum_{s_t,s_{t+1}}(1+r(s_t,s_{t+1}))\pi(s_{t+1}|s_t)E\left[(1+r_t)w_t|s_t\right]\pi(s_t)} \end{aligned}$$

$$= \frac{\sum_{s_t} (1 + r(s_t, s_{t+1}))\pi(s_{t+1}|s_t)\Omega_t(s_t)}{\sum_{s_t, s_{t+1}} (1 + r(s_t, s_{t+1}))\pi(s_{t+1}|s_t)\Omega_t(s_t)} . \tag{A19}$$

where the second line uses the equilibrium law of motion for the individual state variable w, the third line is simply the law of iterated expectations, the fourth line is a rewriting of joint probabilities, the fifth follows from the fact that portfolio choices only depend on s_t in conjunction with the definition of r, and the last line is a direct implication of the definition of Ω. A stationary Ω is the solution to the stationary version of (A19), which reads

$$\Omega(s') = \frac{\sum_s (1 + r(s, s'))\pi(s'|s)\Omega(s)}{\sum_{s,s'} (1 + r(s, s'))\pi(s'|s)\Omega(s)} . \tag{A20}$$

The efficiency condition (11) implies that $\sum_{s'}(1 + r(s, s'))\pi(s'|s)$ is independent of s. In this case, any Ω satisfies the stationarity condition (A20), that is, any Ω—and in particular $\Omega = \Omega_0$—is a stationary distribution. This completes the proof of proposition 3. □

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