Mean Field Limit for Stochastic Particle Systems With and Without Common Noise

Analysis of Singular Interactions

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Abstract

In this thesis, we explore mean-field limit theory, focusing on the derivation of non-linear (stochastic) partial differential equations from large systems of interacting stochastic particles. Our goal is to provide a rigorous description of complex interacting systems through their density functions as the number of particles becomes large. A process also known as "propagation of chaos".

We investigate systems with idiosyncratic noise and those influenced by idiosyncratic and common noise, the latter being particularly relevant in fields like biology and finance. For both types of systems, we demonstrate mean-field limit results, emphasizing the unique challenges and differences in the techniques applicable to each setting. Our primary aim is to handle interaction kernels with low regularity, extending beyond the classical Lipschitz theory..

In the absence of common noise, we focus on moderately interacting particle systems. We illustrate how to combine and extend various techniques such as convergence in probability, the relative entropy method, and the modulated energy method for singular interaction kernels. This includes applications to attractive Keller–Segel models, opinion dynamics, and general sub-Coulomb type kernels. Additionally, we establish well-posedness results for the underlying diffusion-aggregation equations and stochastic differential equations.

When considering systems with common noise, we achieve new well-posedness results for conditional McKean–Vlasov stochastic differential equations by solving the associated stochastic partial differential equations and employing a dual argument. Furthermore, we derive explicit bounds on the relative entropy between the conditional Liouville equation and the stochastic Fokker–Planck equation with a bounded and square-integrable interaction kernel. This extends well-known relative entropy results to the setting of common noise, providing a novel conditional propagation of chaos result.

Our quantitative findings can serve as a foundation for further analysis of the effects of common noise on interacting particle systems and their fluctuations. This work not only advances theoretical understanding but also offers practical insights into the behavior of complex systems under stochastic influence.

Zusammenfassung

In dieser Dissertation erforschen wir den sogennanten “mean-field limit”, wobei der Schwerpunkt auf der Herleitung nichtlinearer (stochastischer) partieller Differentialgleichungen aus stochastischen Interaktionsmodellen liegt. Unser Ziel ist es, dass wenn die Anzahl der Teilchen im Interaktionmodell groß wird, dieses riguros durch eine Dichtefunktion zu beschreiben. Dieser Prozess wird auch “propagation of chaos” genannt.

Wir untersuchen sowohl Systeme mit idiosynkratischem Rauschen als auch solche, die zusätzlich durch ein gemeinsames Rauschen beeinflusst werden, wobei letzteres besonders in Bereichen wie der Biologie und Finanzen relevant ist. Für beide Arten von Systemen untersuchen wir Grenzwertverhalten im Falle steigender Anzahl an Partikel, wobei wir die Herausforderungen und Unterschiede in den Techniken für die unterschiedlichen Fälle hervorheben und vergeleichen. Unser primäres Ziel ist es, mit Interaktionen niedriger Regularität umzugehen, die über die klassische Lipschitz-Theorie hinausgehen.

Dabei konzentrieren wir uns in Abwesenheit vom gemeinsamen Rauchen auf das moderate Regime. Wir veranschaulichen, wie verschiedene Konzepte wie die Konvergenz in Wahrscheinlichkeit, die Methode der relativen Entropie und die modulierte Energiemethode für singuläre Interaktionen kombiniert und erweitert werden können. Dies umfasst Anwendungen auf attraktive Keller-Segel-Modelle, Meinungsdynamiken und allgemeine Sub-Coulomb Interaktionen. Zusätzlich stellen wir Ergebnisse zur Existenz und Eindeutigkeit der zugrunde liegenden Diffusions-Aggregations-Gleichungen und stochastischen Differentialgleichungen auf.

Beim Betrachten von Systemen mit gemeinsamem Rauschen erzielen wir neue Ergebnisse zur Existenz und Eindeutigkeit für bedingte stochastische McKean-Vlasov Differentialgleichungen, indem wir die zugehörigen stochastischen partiellen Differentialgleichungen lösen und ein duales Argument verwenden. Darüber hinaus leiten wir explizite Schranken für die relative Entropie zwischen der bedingten Liouville Gleichung und der stochastischen Fokker–Planck Gleichung mit einem beschränkten und quadratisch integrierbaren Interaktionskern ab. Dieses Abschätzung erweitert bekannte relative Entropie Ergebnisse auf das Setting des gemeinsamen Rauschens und liefert ein neuartiges bedingtes “propagation of chaos” Resultat.

Unsere quantitativen Ergebnisse können als Grundlage für weitere Analysen vom gemeinsamem Rauschen auf interagierende Teilchensysteme und deren Fluktuationen dienen. Diese Dissertation fördert nicht nur das theoretische Verständnis, sondern bietet auch praktische Einblicke in das Verhalten komplexer Systeme unter stochastischem Einfluss.

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Chapter 1

Introduction

In the 19th century, Boltzman [Bol70] proposed the idea of "Stosszahlensatz" or molecular chaos, to elucidate how his famous kinetic equation could be derived from a particle system governed by Newton's laws of dynamics. During this process, he introduced a decreasing quantity, directly linked to thermodynamic entropy in the equilibrium state, suggesting that the solution drives towards Maxwell's equilibrium [Max67]. However, the mathematical tools to rigorously derive Boltzmann equation from interacting particle systems were not yet developed. Initially, significant contributions to this field came from physicists such as Jeans [Jea15] and Vlasov [Vla38]. This underscored the need for an axiomatic mathematical framework, a concept famously advocated by David Hilbert at the International Congress of Mathematics in Paris in 1900. It was not until the 1950s that mathematicians such as Kac and McKean embarked on a concerted effort to formalize the concept molecular chaos. Inspired by Boltzman's work, Kac [Kac56] introduced the idea that for time-evolving systems, chaos should be propagated in time. Following Kac's pioneering work, McKean [McK67] extended the concept of chaos introduced by Kac, broadening its applicability beyond Boltzmann's kinetic theory to include various classical equations in kinetic theory as well as diffusion equations.

One of the most significant contributions of McKean and Kac was the establishment of a comprehensive mathematical framework to analyze the limiting behaviour of interacting particles. Their approach integrated deterministic and probabilistic methods, involving a large system of particles described by (stochastic) ordinary differential equations. In this system each particle experiences the influence of all others through a weighted sum over all interactions mediated by the interaction force k, capturing the microscopic interactions between gas molecules. Intuitively, as the number of particles N becomes large, which is reasonable for many systems such as gas molecules, the microscopic viewpoint transitions to a macroscopic one, described by a density function. This limit procedure, now known as the mean-field limit or the associated concept of propagation of chaos, captures the emergence of collective behaviour from individual interactions.

In this thesis we will further contribute to the above concept of propagation of chaos with respect to a variety of systems. The main goal will be to demonstrate that even if the interaction k between the particles is non-smooth we can still expect that chaos should propagate in time.

Given that our initial motivation is derived from kinetic equations, the density function is expected to satisfy a (stochastic) partial differential equation, allowing us to explore properties of the macroscopic system such as mass conservation, long-term behaviour, or steady states. These are properties we cannot easily observe on the microscopic level. Hence, if we can demonstrate that the microscopic and macroscopic viewpoints are connected, we can make statements about the system without computing all particle interactions. Instead, we can analyze the density, significantly reducing computational costs, making this procedure not only mathematically interesting but also practically valuable in modern numerics. Here, the computational costs of computing interactions grow exponentially, making it easier to simulate the density instead of the interacting particle system by solving the partial differential equations with efficient numerical schemes.

Hence, it is not surprising that the importance of the connection between microscopic and macroscopic systems gained significant recognition, motivating researchers to focus on it. Thus, building upon the groundbreaking works of McKean [McK67] and Kac [Kac56], Sznitman [Szn91] provided an essential comprehensive survey on propagation of chaos, offering important applications and results on random measures. This seminal work serves as an invaluable resource for researchers seeking to explore this topic.

From this point onward, the topic of propagation of chaos grew rapidly, including a wide array of subjects and applications. These include animal herding or flocking [DCBC06, TBL06, BCC11, CCH14, CCHS19], swarming models [CFTV10], various Cucker–Smale models [HL09, CFRT10, CZ21, HZ22], and opinion dynamic models [Lor07, DMPW09, GPY17, Hos20, BPCG24], such as the Hegselmann–Krause model [HK02, RD09, DR10, WLEC17, CSDH19]. Other notable applications include chemotaxis [KS70, Hor04, HP09, NP23], kinetic equations [MM13, CC21], optimal control [ACS23, AMS23], neuroscience [BFFT12, LS14], mean-field games [CDL16, CD18, LZ19, LLF23a], training of neural networks [MS02, MMN18, SS20], and consensus based optimization [GPY17, FKR21, HQ22, Tot22].

One of the main observations leading to the emergence of new results was tackling the problem with a "top-to-bottom" approach. As we have seen, mean-field theory combines stochastic analysis with partial differential equations, famously connected through Itô's and Feynman—Kac's formulas [KS91]. In broad terms, the standard technique nowadays is to guess the limiting partial differential equation and analyze its properties. As it turns out, in most cases, these limiting equations exhibit better regularity than the underlying interacting particle system. Hence, if one can solve the limiting equation with some additional regularity, it can be used to overcome the difficulties in the interacting particle system and, consequently, demonstrate propagation of chaos from a "top-to-bottom" approach. This is also the main approach we are going to follow throughout this thesis.

However, by the superposition principle [BR20], we can identify to each parabolic partial differential equation a stochastic process. Consequently, alongside the limiting density emerges the McKean–Vlasov process or mean-field limit process. Similar to the law of large numbers, we expect that for a large number of particles N, each particle starts behaving like the McKean–Vlasov process. Therefore, the mean-field limit can be viewed as a dynamic version of the law of large numbers.

Moreover, we notice the interplay between partial differential equations and stochastic analysis and how improvements in one area can influence the other. This theme will recur throughout the thesis, where we will demonstrate results in the context of partial differential equations that have implications for the stochastic analysis. Basically, in this context, these two areas can be viewed as sides of the same coin.

Consequently, we will analyze the connection between classic quantities from the theory of partial differential equations such as relative entropy, Fisher information, modulated energy, and modulated free energy within the framework of stochastic analysis. The goal is to demonstrate the connections between these quantities and their implications for mean-field limits, providing new methods for tackling singular interactions. Along the way, we will establish well-posedness results for the equations at hand. These results are significant in their own right and have been extensively studied, such as the well-posedness of McKean–Vlasov equations [HRW21, RZ21, GHM22].

Moreover, we introduce another layer of complexity by adding stochastic perturbations, known as environmental or common noise, to the interacting particle system, affecting all particles. We will highlight the major differences from the classical work by McKean [McK67], Kac [Kac56], and Sznitman [Szn91] in this new setting. Moreover, we will transfer the mentioned quantities from the deterministic setting of partial differential equations into the stochastic setting, providing innovative approaches to tackle mean-field limits with common noise.

In the following we aim to recall fundamental concepts of mean-field limits and propagation of chaos. We start by presenting the historical concepts such as Kac's chaos and defining propagation of chaos. Then, we introduce our primary interacting particle system, which is described by a stochastic differential equation. Next, we will demonstrate some established results in scenarios where the interaction is sufficiently smooth. Additionally, we will discuss the effect of common noise on these systems. This introduction will mirror the main results presented in the upcoming chapters, offering insight into some of the ideas, terminology, and techniques used in this thesis. Furthermore, we will explain the challenges that arise when the interaction is non-smooth, highlighting the necessity of our contributions to the field.

1.1. Notion of chaos

Let us now dive into the various interpretations of chaos. Broadly speaking, propagation of chaos refers to the property that the convergence of measures within a specific topology at initial time $t = 0$ also holds for future times $t > 0$. Throughout this chapter we will consider E as a separable metric space. A measure μ on the product space E^N is said to be symmetric if, for any permutation $\iota : \{1, \dots, N\} \mapsto \{1, \dots, N\}$ and any measurable sets $A_1, \dots, A_N$, the equality

$$\mu(A_1 \times \cdots \times A_N) = \mu(A_{\iota(1)} \times \cdots A_{\iota(A_N)})$$

holds. Furthermore, we write $\mu \ll \nu$, if μ is absolutely continuous with respect to ν [Kle20, Definition 7.30]. We begin by examining interacting particle systems, which are described by an exchangeable collection of $N \in \mathbb{N}$ stochastic processes $\mathbf{X}_t^N = (X_t^1, \dots, X_t^N)$ for $t \geq 0$. Later on, we will find it convenient to represent the dynamics of such a system, $\mathbf{X}_t^N$, through a system of stochastic differential equations (see Section 1.2).

In studying propagation of chaos, our focus shifts to the behaviour of the sequence $\big(\mathbf{X}_t^N, N \in \mathbb{N}\big)$ as $N \to \infty$ for $t > 0$. Later, we will explore various quantities associated with $\mathbf{X}_t^N$ in the limit $N \to \infty$. For example, for a fixed $r \geq 1$ we can examine the law of the first r particles $(X_t^1, \dots, X_t^r)$ or analyze their associated density. Notably, the law may converge in the weak topology or within some Wasserstein distance. Additionally, the density of the r-th marginal can converge almost everywhere or in some L^p-space or in some other topology. Thus, it is not surprising that there exist several notions to describe propagation of chaos.

1.1.1. Kac's chaos. In this section, we provide a brief overview of the foundational work on propagation of chaos by Kac [Kac56] based on the works of Mischler, Mouhot, Hauray [MM13, HM14]. Recall the original objective of Boltzmann, Kac, and McKean, which aimed to derive Boltzmann kinetic equations from the collision of gas molecules. Specifically, they tried to demonstrate that the particle description of a gas, under the assumption of binary collisions, converges to the spatially homogeneous Boltzmann equation

$$\partial_t f(t,v) = Q(f,f)(t,v), \quad v \in \mathbb{R}^d,\ t \geq 0,$$

where v denotes the velocity, Q is the collision operator and f is a density function describing the gas. In his seminal work [Kac56], Kac established the following result [Kac56, Basic Theorem]:

Consider dimension $d = 1$ and let ϕ be a symmetric function defined on the sphere S^N in $\mathbb{R}^N$, satisfying the Master equation

$$\begin{aligned}\partial_t \phi(t, v_1, \dots, v_N) = &\frac{\vartheta}{2N\pi} \sum_{1 \leq i < j \leq N} \int_0^{2\pi} \varphi(v_1, \dots, v_i \cos(\theta) + v_j \sin(\theta), \dots, \\ &- v_i \sin(\theta) + v_j \cos(\theta), \dots, v_N) - \phi(t, v_1, \dots, v_N) \,\mathrm{d}\theta,\end{aligned}$$

where ϑ is a constant. Define the r-th contraction of ϕ as

$$f^{r,N}(t, v_1, \dots, v_r) := \int_{v_{r+1}^2 + \cdots + v_N^2 = N - (v_1^2 + \cdot + v_r^2)} \phi(t, v_1, \dots, v_N) \,\mathrm{d}\mathcal{H}^{N-(r+1)}(v_{r+1}, \dots, v_N),$$

where $(v_1, \dots, v_N) \in S^N$. If, at the initial time $t = 0$ and for $r \in \mathbb{N}$ the function f satisfies the Boltzmann property

$$\lim_{N\to\infty} f^{r,N}(0, v_1, \dots, v_r) = \prod_{i=1}^{r} \lim_{N\to\infty} f^{1,N}(0, v_i),$$

then f satisfies the Boltzmann property for $t > 0$, i.e.

$$\lim_{N\to\infty} f^{r,N}(t, v_1, \dots, v_r) = \prod_{i=1}^{r} \lim_{N\to\infty} f^{1,N}(t, v_i).$$

It is noteworthy that this convergence perspective is rooted in the framework of partial differential equations, without employing probability concepts. This is expected, as stochastic analysis was in its infancy during this period, and probabilistic tools like, for instance, Prokhorov's theorem had just been developed. Moreover, in contemporary research, the Boltzmann property has evolved into more modern forms of convergence suited for various settings. For example, an interacting particle system can be Kac chaotic, Fisher information chaotic, or entropy chaotic [HM14]. These types will be discussed further in subsequent sections.

1.1.2. Propagation of chaos. To analyze the convergence of an interacting particle systems $\mathbf{X}_t^N$, let us introduce the empirical measure. This map is defined from the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ into the space of measures $\mathcal{P}(E)$ on E and is given by

$$\mu_t^{\mathbf{X}^N}(\omega) := \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^i(\omega)}, \tag{1.1}$$

where δ_x is the Dirac measure concentrated at the point $x \in E$. The empirical measure encodes the state of all particles and exhibits the typical law of large numbers scaling of N^{-1}. Understanding the convergence properties of this measure is central to the study of propagation of chaos. Following [Szn91] we also require the concept of f-chaotic.

DEFINITION 1.1. *Let f be a measure on some separable metric space E and $(f^N, N \in \mathbb{N})$ be a sequence of symmetric probability measures on E^N. We say that $(f^N, N \in \mathbb{N})$ is f-chaotic, if for any finite number $r \geq 1$, and collection of bounded continuous functions $\varphi_1, \dots, \varphi_r \in C_b(E)$ we have*

$$\lim_{N\to\infty} \langle f^N, \varphi_1 \otimes \cdots \otimes \varphi_r \otimes 1 \cdots \otimes 1 \rangle = \prod_{i=1}^{r} \langle f, \varphi_i \rangle. \tag{1.2}$$

In its core the property tells us that the r-th marginals of the symmetric probability measures $(f^N, N \in \mathbb{N})$, which we will denote by $f^{r,N}$, converge weakly to the product measure denoted by $f^{\otimes r}$ and hence becomes statistically independent.

As it turns out Sznitman [Szn91, Proposition 2.2] provides a useful characterization of the property in Definition 1.1 in terms of the empirical measure (1.1).

PROPOSITION 1.2. *Let f be a measure on some separable metric space E and f^N be a symmetric probability measure on a E^N. Then, the following statements are equivalent:*

(i) $(f^N, N \in \mathbb{N})$ is f-chaotic.

(ii) *Condition* (1.2) *holds for* $r = 2$. *Thus, it suffices that the second marginal of* f^N *converges weakly towards the product measure* $f \otimes f$.

(iii) *If* $\mathbf{X}^N := (X^1, \dots, X^N)$ *is distributed according to* f^N, *i.e., Law* $(X^1, \dots, X^N) = f^N$ *for all* $N \in \mathbb{N}$, *then the associated empirical measure* $\mu^{\mathbf{X}^N} := \frac{1}{N} \sum_{i=1}^{N} \delta_{X^i}$ *converges in law towards the deterministic measure* f.

We notice that the convergence in law of the measure-valued empirical measure may not be easy to verify at first glance. Thus, we would like to refer to [Kal17, Chapter 4] for a complete analysis on convergence of random measures and in particular [Kal17, Theorem 4.11] for equivalent characterizations. Finally, we are ready to define propagation chaos in terms of measures being chaotic.

Definition 1.3 (Propagation of chaos for a particle system). *Assume that at the time* $t = 0$ *the sequence of joint distributions* $(f_0^N, N \in \mathbb{N})$ *of* $\mathbf{X}_0^N$ *is* f_0*-chaotic. Then, we say propagation of chaos holds, if at any time* $t > 0$ *the sequence of joint distributions* f_t^N *of* $\mathbf{X}_t^N$ *is* f_t*-chaotic.*

We contemplate the precise nature of $(f_t, t \geq 0)$ and its characterization. Conceptually, as the number of particles described by $\mathbf{X}_t^N$ increases, f should represent the density of the entire system, given it has distribution f_0 at time $t = 0$. In the setting of Boltzmann, f solves the Boltzmann equation with initial distribution f_0. In forthcoming discussions, particularly in Section 1.3, we anticipate that for particle systems governed by classical stochastic differential equations driven by independent Brownian motions, f solves a partial differential equation of aggregation-diffusion type. However, this expectation may not universally hold. For instance, substituting the Brownian motions with fractional Brownian motions [GLM24] retains the anticipation of propagation of chaos, with f now defined as the law of the limiting McKean–Vlasov stochastic differential equation (see Section 1.3.2). Additionally, if the system is an abstract Markov process defined by its generator, one needs to pass to the limit inside the associated martingale problem. The underlying theme is that we often "guess" the limiting distribution f.

Subsequently, we have two options at our disposal. We can either pursue a "bottom-to-top" approach, wherein we demonstrate that the system of interacting particles has a solution, while simultaneously proving propagation of chaos and the existence of the limiting object, or we can solve for the limiting object f ex nihilo. The latter approach, known as the "top-to-bottom" method, exploits the fact that the limiting object often has a better structure and becomes more regular through bootstrap arguments. Historically, many proofs have utilized the "bottom-to-top" approach, employing compactness methods or BBGKY hierarchies [Lan75, Gol16, GLM24]. However, it has emerged that the "top-to-bottom" approach is better suited for irregular kernels [LP17, JW18, Ser20, Lac23]. In this thesis, we primarily focus on this approach. For more details on particle systems described by stochastic differential equations, we refer to Section 1.3.

1.1.3. Chaos in Wasserstein distance. Because the notion of Kac's chaos is a non-quantitative property relying on the weak convergence of measure, many mathematician prefer to work with the Wasserstein distance [BGG12, CD18, CC21, PD22]. We start by defining the Wasserstein distance. Let (d) be the metric of the separable metric space E. For $1 \le p < \infty$ let $\mu, \nu \in \mathcal{P}^p(E)$. Then, we define the p-Wasserstein distance as

$$W_p(\mu,\nu) := \inf_{\pi \in \Pi(\nu,\mu)} \left(\int_{E\times E} d(x,y)^p \pi(x,y) \right)^{\frac{1}{p}},$$

where $\Pi(\nu,\mu)$ is the convex non-empty set of all probability measures on $E \times E$ such that π has marginals μ and ν.

For the convenience of the reader we present some simple but crucial facts about the Wasserstein distance applied to the Dirac delta measure and the empirical measure. First, for a point $x \in E$ and a measure μ on E we have the equality

$$W_p(\delta_x,\mu)^p = \int_E d(x,y)^p \,\mathrm{d}\mu(y).$$

Second, for $x_i, y_i \in E$ we have the inequality

$$W_p\left(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}, \frac{1}{N}\sum_{i=1}^N \delta_{y_i} \right)^p \le \frac{1}{N}\sum_{i=1}^N d(x_i,y_i)^p,$$

where we used the natural coupling

$$\pi(x,y) = \frac{1}{N}\sum_{i=1}^N \delta_{(x_i,y_i)}.$$

Hence, if we consider the empirical measure with respect to random variables $X^1,\dots,X^N$ and $Y^1,\dots,Y^N$, we can estimate the Wasserstein distance with respect to the almost everywhere average distance of the variables. Moreover, the concept of propagation of chaos can be naturally extended to incorporate the Wasserstein distance.

Definition 1.4 (Chaos in Wasserstein distance). *Let $p \in \mathbb{N}$, let $(f^N, N \in \mathbb{N})$ be a sequence of symmetric measures on E^N such that $f^N \in \mathcal{P}_p(E^N)$, and let $f \in \mathcal{P}(E)$. The following three notions of chaos in Wasserstein distance were introduced in* [HM14]*:*

(i) Wasserstein Kac's chaos: For all $r \in \mathbb{N}$ and r-the marginal $f^{r,N}$ of f^N it holds that

$$\lim_{N\to\infty} W_p(f^{r,N}, f^{\otimes r}) = 0.$$

(ii) Infinite dimensional Wasserstein chaos:

$$\lim_{N\to\infty} W_p(f^N, f^{\otimes N}) = 0.$$

(iii) Wasserstein empirical chaos: Suppose the E^N-valued random variable $\mathbf{X}^N$ has distribution f^N. Then,

$$\lim_{N\to\infty} W_p(\mathit{Law}\,(\mu^{\mathbf{X}^N}), \delta_f) = 0,$$

where W_p is a Wasserstein distance on $\mathcal{P}(\mathcal{P}(E))$ and $\mu^{\mathbf{X}^N}$ is the empirical measure of $\mathbf{X}^N$.

It is well known that convergence in Wasserstein distance implies the weak convergence of measures [Vil03, Theorem 7.12]. Some advantages of considering the Wasserstein distance is that explicit rates of convergence can be established with respect to a distance. In applications one considers often the expected value of $W_p(\mu_t^{\mathbf{X}^N}, f)$ and tries to prove that it vanishes in the limit [FG15, CD18, Hua24]. However, we need to guarantee that the measures lie in $\mathcal{P}_p(E)$ for $p \geq 1$.

1.1.4. Entropy and Fisher information chaotic. In this section, we present another concept of propagation of chaos by considering the entropy of the system. Let f be a non-negative measurable function on $\mathbb{R}^d$ with mass one. We define the entropy by

$$\mathcal{H}(f) := \int_{\mathbb{R}^d} f(x) \log(f(x)) \, \mathrm{d}x. \tag{1.3}$$

Related to the entropy is the Fisher information, which is given by

$$I(f) := \int_{\mathbb{R}^d} \frac{|\nabla f(x)|^2}{f(x)} \, \mathrm{d}x. \tag{1.4}$$

Obviously, we consider only such f such that the integrals on the right-hand side are well-defined. Notice that for a $f^N \in \mathcal{P}(\mathbb{R}^{dN})$ the Fisher information and entropy scale with the number of particles N. To normalize the effect, we introduce the normalized versions of the above quantities

$$I_N(f^N) := \frac{1}{N} I\big(f^N\big), \quad \mathcal{H}_N\big(f^N\big) := \frac{1}{N}\mathcal{H}\big(f^N\big),$$

where we integrate over $\mathbb{R}^{dN}$ instead of $\mathbb{R}^d$ in (1.3), (1.4).

Definition 1.5. *Consider $f \in \mathcal{P}(\mathbb{R}^d)$ and a sequence $(f^N, N \geq 2)$ of symmetric probability measures on $\mathbb{R}^{dN}$ such that for some $q > 0$, the q-th moment $M_q(f^{1,N}) := \int |z|^q \, df^{1,N}$ of the first marginal $f^{1,N}$ is uniformly bounded in N. We say that*

i) the sequence $(f^N, N \in \mathbb{N})$ is f-Fisher information chaotic if

$$f^{1,N} \rightharpoonup f \text{ weakly in } \mathcal{P}(\mathbb{R}^d), \ I_N(f^N) \to I(f), \ \textit{and } I(f) < \infty;$$

ii) the sequence (f^N) is f-entropy chaotic if

$$f^{1,N} \rightharpoonup f \text{ weakly in } \mathcal{P}(\mathbb{R}^d), \ \mathcal{H}_N\big(f^N\big) \to \mathcal{H}(f), \ \textit{and } H(f) < \infty.$$

A remarkable result by Hauray and Mischler [HM14, Theorem 1.4] shows that under certain assumptions all versions of chaotic are related.

Theorem 1.6. *Consider $(f^N, N \in \mathbb{N})$ a sequence of symmetric probability measures in $P(\mathbb{R}^{dN})$ such that the q-th moment $M_q(f^{1,N})$ is bounded for $q > 2$, and $f \in \mathcal{P}(\mathbb{R}^d)$.*

In the list of assertions below, each one implies the assertion which follows:

(i) $(f^N, N \in \mathbb{N})$ is f-Fisher information chaotic;
(ii) $(f^N, N \in \mathbb{N})$ is f-Kac's chaotic and $I(f^N)$ is bounded;
(iii) $(f^N, N \in \mathbb{N})$ is f-entropy chaotic;
(iv) $(f^N, N \in \mathbb{N})$ is f-Kac's chaotic.

At the end of the chapter we want to introduce the very important concept of relative entropy. In this thesis it will play a crucial role in showing strong convergence of the marginals. Thus, let E as always be a Polish space and μ, ν be two probability measures over E.

The relative entropy $\mathcal{H}(\mu|\nu)$ is defined by

$$\mathcal{H}(\mu|\nu) := \begin{cases} \int_E \frac{\mathrm{d}\mu}{\mathrm{d}\nu} \log\Big(\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\Big) \mathrm{d}\nu, & \mu \ll \nu, \\ \infty, & \text{otherwise}. \end{cases} \tag{1.5}$$

Some properties of the relative entropy include that $\mathcal{H}(\mu \mid \mu) = 0$ and the data processing inequality

$$\mathcal{H}(\mu \circ f^{-1} \mid \nu \circ f^{-1}) \leq \mathcal{H}(\mu \mid \nu)$$

for any measurable function f from E into another measurable space.

Instead of analysing the convergence properties of the entropy, one can compare the interacting particle system to the limit system in the context of the entropy itself. This comparison is achieved through the concept of relative entropy, which will be a significant focus in this thesis. Specifically, we will investigate the relative entropy of the complete joint distribution f^N compared to the distribution of the limiting equation, as well as the relative entropy of the marginals $f^{r,N}$ compared to the marginal distribution of the limiting equation. Understanding, the relationship between the entropy of the complete joint distribution and that of the marginals is clarified by the following lemma [DMM01, Lemma 3.9].

Lemma 1.7. *Let f^N be a symmetric probability measure on E^N and $f \in \mathcal{P}(E)$. Then, the following inequalities hold*

$$\mathcal{H}_r\big(f^{r,N}\big) \leq 2\mathcal{H}_N\big(f^N\big) \quad \text{and} \quad \mathcal{H}_r\big(f^{r,N} \mid f^{\otimes r}\big) \leq 2\mathcal{H}_N\big(f^N \mid f^{\otimes N}\big). \tag{1.6}$$

It is also well-known that the relative entropy is connected with the total variation norm by the Csiszár–Kullback–Pinsker inequality [Vil09, Chapter 22],

$$\|f^r - g^r\|_{L^1(E^r)} \leq \sqrt{2r\mathcal{H}_r(f^r \mid g^r)}. \tag{1.7}$$

By combining this inequality with the previous ones, we observe that the convergence of the normalized relative entropy $\mathcal{H}_N\big(f^N \mid f^{\otimes N}\big)$ of the full marginals is sufficient to demonstrate convergence of the marginals L^1. The convergence in the L^1-norm, in turn, implies the weak convergence of the associated measures and thus Kac's chaos. In the upcoming Chapters 3 and 5, we explore the explicit evolution of relative entropy for measure flows of stochastic differential equations with non-vanishing diffusion coefficients in order to demonstrate the convergence in L^1.

1.2. First order vs. second order systems

In this section, we distinguish between two classical systems commonly discussed in the literature, namely first and second order systems of stochastic differential equations. Firstly, let us delve into first order systems, often interpreted as the small mass limit of Langevin equations in statistical physics [JW17]. We begin with N indistinguishable particles $\mathbf{X}_t^N =$

$(X_t^N, \dots, X_t^N)$ described through the system of stochastic differential equations (SDE's)

$$\mathrm{d}X_t^i = \frac{1}{N}\sum_{j=1}^{N} k(X_t^i - X_t^j)\,\mathrm{d}t + \sigma(t, X_t^i)\,\mathrm{d}B_t^i + \nu(t, X_t^i)\,\mathrm{d}W_t, \quad i = 1, \dots, N, \tag{1.8}$$

where k represents the interaction force between particles, and σ and ν denote the diffusion coefficients. The Brownian motions $(B^i, i \in \mathbb{N})$ are independent, modelling the idiosyncratic noise, while the Brownian motion W is independent of each B^i and captures the environmental/common noise in the system. The initial condition $(X_0^i, i \in \mathbb{N})$ is always chosen to be i.i.d. (independent and identically distributed) with sufficient regularity and independent of all Brownian motions. At this moment we already choose sequences $(B^i, i \in \mathbb{N})$, $(X_0^i, i \in \mathbb{N})$ since our primary focus lies on the limit $N \to \infty$.

If $\sigma = \nu = 0$, then the system reduces to a deterministic dynamic. When we remove the common noise $\nu = 0$, we revert to the classical framework of McKean [McK67] and Sznitman [Szn91]. In this thesis, we explore both variants: one without the common noise $\nu = 0$ and one with common noise $\nu \neq 0$. The inclusion of noise in the model is significant not just from a mathematical perspective of diffusion, but also from a modeling standpoint. This is because we cannot anticipate animals, microorganisms, or opinions to interact with each other or the environment in a deterministic manner.

For fundamental physical principles such as Newton's laws of motion we also need to introduce second-order systems. Thus, let $(\mathbf{X_t^N}, \mathbf{V_t^N}) = (X_t^1, \dots, X_t^N, V_t^1, \dots, V_t^N)$ be the solution of the second order stochastic differential equation

$$\begin{cases} \mathrm{d}X_t^i = c(t, X_t^i, V_t^i)\,\mathrm{d}t, \\ \mathrm{d}V_t^i = \frac{1}{N}\sum\limits_{j=1}^{N} k(X_t^i - X_t^j)\,\mathrm{d}t + \sigma(t, X_t^i, V_t^i)\,\mathrm{d}B_t^i + \nu(t, X_t^i, V_t^i)\,\mathrm{d}W_t, \end{cases} \tag{1.9}$$

for $i = 1, \dots, N$. In contrast to the first order system, only the second equation of this system exhibits the influence of stochastic fluctuations, leading to significant analytical ramifications. In practical scenarios, simplifications often arise, where we set $c(t, X_t^i, V_t^i) = V_t^i$, and the diffusion coefficients σ, ν become constant or even vanish. In the case of vanishing diffusion coefficients the system reduces to the classical deterministic Newton dynamic. Conceptually, the pair (X_t^i, V_t^i) represents the position X_t^i of a particle at time t as well as its velocity V_t^i. It is worth noting that the domain of (X_t^i, V_t^i) remains unbounded, even if the position X_t^i is bounded or situated on a periodic torus, due to the unbounded nature of the velocity V_t^i. Furthermore, the natural connection to kinetic equations like the Vlasov–Poisson equation or Boltzmann's kinetic equation arises as they represent limiting equations of the system (1.9). In this thesis, second-order systems will not be treated in detail. However, we assume that with minor adjustments, similar results to those presented in the upcoming chapters can be obtained for second-order systems.

In both cases the regularity of the interaction force kernel plays a crucial role. Therefore, our primary focus lies on the interaction kernel k and how it affects the system, particularly regarding the concept of propagation of chaos. As an illustration, we will present examples of interaction kernels and their applications, which are relevant to both first and second-order systems.

Example 1.8.

(1) Aggregation models often feature a potential structure $k = -\nabla W$ outside the diagonal, with an additional term V on the diagonal. This results in an stochastic differential equation of the form

$$\mathrm{d}X_t^i = -\frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N} \nabla W(X_t^i - X_t^j)\,\mathrm{d}t - \nabla V(X_t^i)\,\mathrm{d}t + \sigma(t, X_t^i)\,\mathrm{d}B_t^i + \nu(t, X_t^i)\,\mathrm{d}W_t,$$

for $i = 1, \ldots, N$. Observe that we removed the self-interaction term from the sum and instead obtained the additional term ∇V. These systems have been extensively studied, see for instance [Mal01, BGV07, CCH14]*. For second-order models such as the Cucker–Smale model and the related phenomena like swarming of fish or flocking of birds, we refer to* [CFTV10, LLEK10, BCC11, MT11, CCH14, ABCvB14, CCHS19, CZ21]*.*

(2) Alignment models are frequently utilized to simulate phenomena such as the flocking of birds, schools of fish, and swarms of insects. Pioneering works such as [VCBJ+95, CS07] *laid the foundation for this rapidly expanding field, which is evident from the substantial body of literature* [BCC11, CCH14, CCHS19, CZ21]*. One of the classical models in this domain is the Cucker–Smale model* [CS07]*, defined by the system of equations:*

$$\begin{cases} \mathrm{d}X_t^i = V_t^i\,\mathrm{d}t, \\ \mathrm{d}V_t^i = \frac{1}{N}\sum_{j=1}^{N} k(|X_t^i - X_t^j|)(V_t^i - V_t^j)\,\mathrm{d}t + \sigma(t, X_t^i, V_t^i)\,\mathrm{d}B_t^i + \nu(t, X_t^i, V_t^i)\,\mathrm{d}W_t, \end{cases}$$

for each $i = 1, \ldots, N$. Here, the alignment arises in the evolution of the velocity. It is well-documented that models with sufficiently regular interaction kernels k exhibit flocking and alignment behaviour. However, such models lie beyond the scope of this thesis, and interested readers are directed to [HL09, CZ21, BHK22, HZ22] *for more detailed discussions.*

(3) Opinion dynamics models, such as the Hegselmann–Krause model or general bounded confidence models [HK02, Hos20, CNP23, BPCG24]*, explore how opinions spread within a network through interactions among its agents. These networks are characterized by the bounded interaction force kernel, wherein agents disregard ideas that lie beyond a certain distance from their own, thus falling outside their confidence radius.*

(4) Interacting particle systems are widely used for optimization across fields such as economics, physics, finance, and artificial intelligence. Consensus-based optimization methods are commonly employed, featuring a particle system described by

$$\mathrm{d}X_t^i = -\lambda(X_t^i - v_f)H^{\varepsilon}(f(X_t^i) - f(v_f))\,\mathrm{d}t + \sqrt{2\sigma}|X_t^i - v_f|\,\mathrm{d}B_t^i, \quad i = 1, \ldots, N,$$

where λ, σ are constants and v_f is the weighted mean with weight function given by $\exp(-\alpha f(x))$*,*

$$v_f := \left(\sum_{i=1}^{N} \exp(-\alpha f(X_t^i)\right)^{-1} \sum_{i=1}^{N} X_t^i \exp(-\alpha f(X_t^i).$$

The mean-field scaling in this context is embedded within the weight function, and f is the function we wish to minimize. An extensive body of literature exists on this topic. For an overview of recent developments, see [Tot22]*, and for mathematical analysis of the mean-field limit, we refer to* [CCTT18, FKR21, HQ22]*.*

(5) One of the most crucial interaction kernels is the Coulomb kernel given by

$$k(x) = \pm C(d,s)\frac{x}{|x|^s}, \quad x \in \mathbb{R}^d,$$

where $0 < s < d+2$, $C(d,s)$ is some constant depending on the dimension d, the singularity strength s and physical parameters. The sign determines whether the interaction is repulsive (+) [Ser20, NRS22, dCRS23] *or attractive (−)* [BJW23]*. When $s = d$, the Poisson kernel is recovered. Such kernels are used in Vlasov-Poisson systems* [LP17] *to describe electrons or in the field of chemotaxis such as Keller–Segel systems. Obviously, the difficulty lies in the singularity of the kernel, which can lead to collisions of particles and the breakdown of solutions* [FJ17]*.*

(6) From a fluid dynamics perspective, the Biot–Savart kernel

$$k(x) = \frac{(-x_2, x_1)}{|x|^2}, \quad x \in \mathbb{R}^2,$$

is of interest. Consequently, system (1.8) *becomes the vortex model, which is used to approximate the two-dimensional Navier–Stokes equation in vorticity form* [MP82, Osa87, FHM14, JW18]*.*

1.3. Categories of noise and interaction kernels

In the upcoming sections, we will explore various settings for the mean-field limit distinguished by the nature of noise present and irregularity of the interaction kernels. In each of these settings we will provide unique insights into the different frameworks and the challenges they pose, all while considering a simple interaction force except the last subsection, where we talk about singular kernels. This offers a glimpse into the foundational concepts that will be further developed and expanded upon in the main part of the thesis. Given that this thesis is concerned with first-order systems, our analysis in this section will predominantly focus on them.

1.3.1. Deterministic model. Let us provide a brief overview over the deterministic scenario, corresponding to the case $\sigma = \nu = 0$ in equation (1.8). This yields the system

$$\mathrm{d}X_t^i = \frac{1}{N}\sum_{j=1}^N k(X_t^i - X_t^j)\,\mathrm{d}t, \quad i = 1, \dots, N \tag{1.10}$$

with initial condition $X_0^i = x_0^i$ for some points $x_0^i \in \mathbb{R}^d$. In the absence of noise, the effects of regularization by noise are absent, limiting the solution theory to ordinary differential equations, which generally exhibit inferior behaviour compared to stochastic differential equations. Therefore, let us assume $k \in C^1(\mathbb{R}^d)$, which implies that k is Lipschitz continuous, thus resulting in a strong solution for system (1.10). But what is the mean-field limit of this system?

We observe that the empirical measure $\mu^{\mathbf{X}^N}$ satisfies the following Cauchy problem

$$\begin{cases} \partial_t \mu_t^{\mathbf{X}^N} + \nabla \cdot (\mu_t^{\mathbf{X}^N}(k * \mu_t^{\mathbf{X}^N})) = 0, \\ \mu_{t=0} = \mu_0^{\mathbf{X}^N}. \end{cases} \tag{1.11}$$

This equation should be interpreted in a distributional sense and can be derived using the method of characteristics for transport equations. Now, let $\mu_0^{\mathbf{X}^N} \to f_0$ and, for the moment, assume propagation of chaos holds. Taking the limit $N \to \infty$ heuristically in equation (1.11), we observe that the limiting measure f should satisfies the same Cauchy problem but with initial condition f_0, allowing us to describe the dynamics of the limiting measure. Therefore, the strategy for proving propagation of chaos proceeds as follows: we establish the well-posedness of the Cauchy problem and a stability estimate in the Wasserstein distance. Combining both, we obtain the following theorem [Gol16, Theorem 1.4.4].

Theorem 1.9. *Suppose $k \in C^1(\mathbb{R}^d)$, antisymmetric and the initial non-negative density $f_0 \in L^1(\mathbb{R}^d)$ has a first moment bound $M_1(f_0) < \infty$. Then the Cauchy problem* (1.11)

$$\begin{cases} \partial_t f + \nabla \cdot (f_t(k * f_t)) = 0, \\ f_{t=0} = f_0, \end{cases}$$

has a unique weak solution $f_t(\cdot) \in L^1(\mathbb{R}^d) \cap \mathcal{P}(\mathbb{R}^d)$. Additionally, the stability estimate

$$W_1(\mu_t^{\mathbf{X^N}}, f_t) \le C\Big(T, \sup_{x\in\mathbb{R}^d} |\nabla k(x)|\Big) W_1(\mu_0^{\mathbf{X^N}}, f_0)$$

holds, which implies propagation of chaos in the sense of Definition 1.1 and Proposition 1.2.

For a literature review on regular kernel k, we refer to the end of the subsequent section, as most results in the smooth case allow a degeneracy of the diffusion coefficients and, therefore, are also applicable in the deterministic scenario.

1.3.2. Stochastic differential equations without common noise. In the stochastic framework without common noise, the stochastic differential equation (1.8) with i.i.d. initial condition $(X_0^i, i = 1, \dots, N)$ simplifies to

$$\mathrm{d}X_t^i = -\frac{1}{N}\sum_{j=1}^N k(X_t^i - X_t^j)\,\mathrm{d}t + \sigma(t, X_t^i)\,\mathrm{d}B_t^i, \quad i = 1, \dots, N. \tag{1.12}$$

The drift can be rewritten as

$$\frac{1}{N}\sum_{j=1}^N k(X_t^i - X_t^j) = k * \mu_t^{\mathbf{X}^N}(X_t^i). \tag{1.13}$$

Assuming propagation of chaos, where the empirical measure $\mu_t^{\mathbf{X}^N}$ converges to some ρ_t, we can guess the limiting system of processes is given by

$$\begin{cases} \mathrm{d}Y_t^i = -(k * \rho_t)(X_t^i) + \sigma(t, Y_t^i)\,\mathrm{d}B_t^i, \\ \mathrm{Law}(Y_t^i) = \rho_t, \end{cases} \quad i = 1, \dots, N. \tag{1.14}$$

These equations, known as McKean–Vlasov stochastic differential equations, depend on the measure of the solution itself. Moreover, Proposition 1.2 suggests that the first marginal $\rho_t^{1,N}$

of the system (1.12) converges weakly to ρ, establishing a necessary connection between ρ and Y_t^i. Notably, if a solution exists, the processes Y^i and Y^j are independent and identically distributed, reducing the analyze to a single equation. Consequently, the question regarding the well-posedness of McKean–Vlasov SDEs naturally arises in the context of mean-field theory, cf. [Wan18, RZ21, BR20, HRW21, Wan23].

To further understand the relationship between ρ and Y_t^i, it is instructive to examine the associated partial differential equations. As we mentioned, partial differential equations can be linked to stochastic differential equations through Itô's formula [KS91, Theorem 3.6].

THEOREM 1.10 (Itô's formula). *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), \mathbb{P})$ be a complete filtered probability space. Let $((M_t^1, \dots, M_t^d), t \geq 0)$ be a d-dimensional local martingale with respect to $(\mathcal{F}_t, t \geq 0)$ and $((A_t^1, \dots, A_t^d), t \geq 0)$ be a d-dimensional $(\mathcal{F}_t, t \geq 0)$-adapted process of finite variation with $A_0^i = 0$ for all i. Set*

$$X_t^i = X_0^i + A_t^i + M_t^i, \quad 0 \leq t < \infty$$

for all $1 \leq i \leq d$, where $(X_0^1, \dots, X_0^d)$ is an $\mathcal{F}_0$-measurable random vector. Additionally, let $\varphi \colon [0,\infty) \times \mathbb{R}^d \mapsto \mathbb{R}^d$ be continuously differentiable in the first variable and twice continuously differentiable in the second variable, then

$$\begin{aligned} &\varphi(t, X_t) - \varphi(0, X_0) \\ &= \int_0^t \frac{\partial}{\partial t} \varphi(s, X_s) \,\mathrm{d}t + \sum_{i=1}^d \int_0^t \frac{\partial}{\partial x_i} \varphi(s, X_s) \,\mathrm{d}M_s^i + \sum_{i=1}^d \int_0^t \frac{\partial}{\partial x_i} \varphi(s, X_s) \,\mathrm{d}A_s^i \\ &\quad + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} \varphi(s, X_s) \,\mathrm{d}\langle M_s^i, M_s^j \rangle, \quad \mathbb{P}\text{-}a.s., \quad 0 \leq t < \infty. \end{aligned} \tag{1.15}$$

Applying this result to a smooth function $\varphi \colon \mathbb{R}^d \mapsto \mathbb{R}$ and the empirical measure, we obtain

$$\begin{aligned} \langle \varphi, \mu_t^{\mathbf{X}^N} \rangle &= \frac{1}{N} \sum_{i=1}^N \varphi(X_t^i) \\ &= \frac{1}{N} \sum_{i=1}^N \int_0^t -\nabla \varphi(X_s^i) \cdot (k * \mu_s^{\mathbf{X}^N}(X_s^i)) + \frac{1}{2} \mathrm{Tr}\Big(\sigma(s, X_s^i) \sigma(s, X_s^i)^{\mathrm{T}} \nabla^2 \varphi(X_s^i) \Big) \,\mathrm{d}s \\ &\quad + \frac{1}{N} \sum_{i=1}^N \int_0^t \nabla \varphi(X_s^i) \sigma(s, X_s^i) \,\mathrm{d}B_s^i \\ &= \int_0^t -\langle \nabla \varphi \cdot (k * \mu_s^{\mathbf{X}^N}), \mu_s^{\mathbf{X}^N} \rangle + \frac{1}{2} \Big\langle \mathrm{Tr}\Big(\sigma(s, \cdot) \sigma(s, \cdot)^{\mathrm{T}} \nabla^2 \varphi(\cdot) \Big), \mu_s^{\mathbf{X}^N} \Big\rangle \,\mathrm{d}s \\ &\quad + \frac{1}{N} \sum_{i=1}^N \int_0^t \nabla \varphi(X_s^i) \sigma(s, X_s^i) \,\mathrm{d}B_s^i. \end{aligned}$$

Suppose σ is bounded. By applying Itô's isometry A.33, we obtain

$$\mathbb{E}\bigg(\bigg|\frac{1}{N}\sum_{i=1}^{N}\int_0^t \nabla\varphi(X_s^i)\sigma(s,X_s^i)\,\mathrm{d}B_s^i\bigg|\bigg) \leq \frac{C}{N}$$

and consequently by taking the limit, we find

$$\langle\varphi,\rho_t\rangle = \int_0^t -\langle\nabla\varphi\cdot(k*\rho_s),\rho_s\rangle + \Big\langle \mathrm{Tr}\Big(\sigma(s,\cdot)\sigma(s,\cdot)^{\mathrm{T}}\nabla^2\varphi(\cdot)\Big),\rho_s\Big\rangle\,\mathrm{d}s.$$

Hence, we assume that the limiting measure satisfies the following non-local, non-linear partial differential equation

$$\partial_t\rho_t = \nabla\cdot((k*\rho_t)\rho_t) + \sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\Big([\sigma(s,\cdot)\sigma(s,\cdot)^{\mathrm{T}}]_{(i,j)}\rho_t\Big) \tag{1.16}$$

on $[0,T]\times\mathbb{R}^d$, where $[\sigma(s,\cdot)\sigma(s,\cdot)^{\mathrm{T}}]_{(i,j)}$ denotes the (i,j)-entry of the matrix. However, it can be readily verified that the same equation must be satisfied by the law of the process (1.14). Consequently, if uniqueness in the sense of distribution holds, it follows that ρ_t must be the law of Y_t, providing further justification for the inclusion of the second condition in (1.14).

Armed with this insight, we can now demonstrate propagation of chaos in a simple setting where k is Lipschitz continuous and σ is smooth, employing the coupling technique by Sznitman [Szn91]. The approach involves establishing, initially, the existence of a unique strong solution for the McKean–Vlasov stochastic differential equation (1.14) through a fixed-point argument. Subsequently, we can compare the trajectories of both processes in the $L^1(L^\infty)$-norm. Specifically, one can demonstrate the following bound [Szn91, Theorem 1.4]

$$\mathbb{E}\Big(\sup_{0\leq t\leq T}|X_t^i - Y_t^i|\Big) \leq \frac{C}{\sqrt{N}}, \tag{1.17}$$

which is an easy consequence of the Lipschitz continuity of the coefficients k and σ.

We finish this section by providing some literature in the regular case, which in some cases also covers the deterministic case. As mentioned in the begging of this theses, the first mathematical foundation goes back to McKean [Kac56, McK66]. For smooth interaction kernels a variety of results have been achieved such as Gaussian fluctuations [BH77, Dob79, Tan84], well-posedness of the McKean–Vlasov equation [Szn91, CD18] and approximation algorithms [BT97]. Based on monotonicity and Lyapunov-type conditions on the kernel k, Gärtner [Gär88] demonstrated propagation of chaos and existence of martingale solutions to the McKean–Vlasov using a compactness argument.

1.3.3. Moderated interaction. In this section, we explore the scaling of interacting particle systems. We provide a brief overview based on [Oel85] and direct readers to this source for more comprehensive details. Until now, we have primarily focused on weakly interacting systems defined by equations (1.8), (1.9), and (1.12). A common characteristic of these systems is that the interaction $k(X_t^i - X_t^j)$ is of order $\frac{1}{N}$, a natural occurrence in the context of the law of large numbers. Notably, our interaction kernel k remains independent of the number of particles, representing the regime of weakly interacting particles extensively studied in literature [McK67, Szn91, Gär88, JW18, Ser20].

We now extend our analysis to particle systems where the interaction kernel k varies with the number of particles N, denoted as k^N. This extension finds motivation in various partial differential equations arising in biology, physics, and other research areas. These equations are often local and depend on point evaluations, rather than on the whole density. Consequently, we are, for instance, interested in scenarios where k^N converges to the Dirac distribution as $N \to \infty$, yielding

$$\lim_{N\to\infty} k^N * \rho = \rho$$

with the limiting term on the right-hand side being local. In the case of constant diffusion σ, the resulting limiting partial differential equation (1.16) corresponds to the porous medium equation.

Following [Oel85], we introduce the scaling of the interaction force kernel k as

$$k^N(x) = N^{\beta} k(N^{\beta/d} x).$$

This scaling is measure preserving on $\mathbb{R}^d$. Notably, when $\beta = 0$, we recover the weak interaction regime analyzed thus far.

For $\beta = 1$, we enter the strong interaction regime, where different particles only interact, when their distance is of order $N^{-1/d}$, yet the strength of their interaction is of order one. In the mean-field limit, this regime yields the Poisson point process [Szn91].

Finally, for $\beta \in (0, 1)$, we encounter the moderated regime. Here, the strength of the interaction of each each particle is of order $N^{-1+\beta d}$, lying between the weak and strong regimes, hence earning the name moderated regime. In particular, in terms of convergence to the porous medium equation, this regime holds significant importance [Oel90]. Naturally, the question arises how the mean-field limit changes under different regimes. Unfortunately, this question is outside the scope of this thesis and we refer to the works of Oelschläger [Oel85, Oel90]. Nevertheless, we will utilize this concept to derive propagation of chaos for singular interaction kernels, in cases where the interacting particle system may not be well-defined.

1.3.4. Singular interaction kernels. In the case of singular interaction kernels, the standard coupling technique from Section 1.3.2 is no longer viable. Thus, we need to develop a different approach. Recently, prominent ideas have emerged, such as using modulated energy and relative entropy, along with their combination known as the modulated free energy approach. In this section, we introduce these quantities, provide some initial connections between them, and discuss the current state of the art. Consider the interactive particle system with additive noise

$$\mathrm{d}X_t^i = \frac{\kappa}{N} \sum_{j=1}^{N} \nabla g(X_t^i - X_t^j)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t^i, \quad i = 1, \dots, N, \tag{1.18}$$

where we imposed a potential structure $k := -\nabla g$ and introduce an additional parameter $\kappa \in \{-1, +1\}$ to capture either the attractive $\kappa = 1$ or the repulsive $\kappa = -1$ case. We can define the modulated energy as

$$\mathcal{K}_N\big(\mu_t^{\mathbf{X}^N}|\rho_t\big) := \frac{1}{\sigma^2}\mathbb{E}\bigg(\int_{\mathbb{R}^{2d}\setminus\{x=y\}} g(x-y)\ \mathrm{d}(\mu_t^{\mathbf{X}^N}-\rho_t)(x)\ \mathrm{d}(\mu_t^{\mathbf{X}^N}-\rho_t)(y)\bigg), \tag{1.19}$$

where we denote the integrand of the modulated energy by

$$F_N(\mu_t^{\mathbf{X}^N}|\rho_t)(\omega) := \int_{\mathbb{R}^{2d}\setminus\{x=y\}} g(x-y)\ \mathrm{d}(\mu_t^{\mathbf{X}^N}(\omega)-\rho_t)(x)\ \mathrm{d}(\mu_t^{\mathbf{X}^N}(\omega)-\rho_t)(y).$$

Moreover, let us introduce the Liouville equation

$$\begin{cases} \partial_t\rho_t^N(x_1,\dots,x_N) &= \frac{\sigma^2}{2}\sum\limits_{i=1}^N \partial_{x_ix_i}\rho_t^N(x_1,\dots,x_N) \\ &\quad + \kappa\sum\limits_{i=1}^N \partial_{x_i}\Bigg(\rho_t^N(x_1,\dots,x_N)\frac{1}{N}\sum\limits_{j=1}^N k(x_i-x_j)\Bigg), \\ \rho_0^N(x_1,\dots,x_N) &= \prod\limits_{i=1}^N \rho_0(x_i). \end{cases} \tag{1.20}$$

By applying Itô's formula (1.15) one can verify that the law of the interacting particle system (1.18) solves the Liouville equation. Analogously, we also need the N-th tensor product $\rho^{\otimes N}$ of equation (1.16), which satisfies

$$\begin{aligned} &\partial_t\rho_t^{\otimes N}(x_1,\dots,x_N) \\ &= \frac{\sigma^2}{2}\sum_{i=1}^N \partial_{x_ix_i}\rho_t^{\otimes N}(x_1,\dots,x_N) + \sum_{i=1}^N \partial_{x_i}\Big((k*\rho_t)(x_i)\rho_t^{\otimes N}(x_1,\dots,x_N)\Big). \end{aligned}$$

Let us also recall the normalized relative entropy between ρ_t^N and $\rho_t^{\otimes N}$

$$\mathcal{H}_N\big(\rho_t^N|\rho_t^{\otimes N}\big) := \frac{1}{N}\int_{\mathbb{R}^{dN}} \log\left(\frac{\rho_t^N(x_1,\dots,x_N)}{\rho_t^{\otimes N}(x_1,\dots,x_N)}\right)\rho_t^N(x_1,\dots,x_N)\,\mathrm{d}x_1,\dots\,\mathrm{d}x_N.$$

Notice that we divide by N to compensate for the growth of dimension in the limit $N\to\infty$.

To set the stage, let us recall the key inequality and summarize the steps of the relative entropy method introduced by Jabin and Wang [JW18]. The first step is to compute the evolution of the relative entropy (1.21). Skipping the computations from Section 3.4, the evolution of the relative entropy is given by

$$\begin{aligned} &\mathcal{H}_N\big(\rho^N|\rho^{\otimes N}\big) - \mathcal{H}_N\big(\rho_0^N|\rho_0^{\otimes N}\big) \\ &= -\frac{\kappa}{N^2}\sum_{i,j=1}^N \int_0^t\int_{\mathbb{R}^{dN}} \rho_s^N(k(x_i-x_j)-k*\rho_s(x_i))\cdot\nabla_{x_i}\log\left(\frac{\rho_s^N}{\rho_s^N}\right) \\ &\quad - \frac{\sigma^2}{2N}\sum_{i=1}^N\left|\nabla_{x_i}\log\left(\frac{\rho_s^N}{\rho_s^N}\right)\right|^2\rho_s^N\,\mathrm{d}x_1\dots\mathrm{d}x_N\,\mathrm{d}s. \end{aligned} \tag{1.21}$$

However, we can deploy probabilistic arguments [Lac23, Lemma 4.4] to compute the relative entropy. Using Girsanov transformation we find directly

$$\mathcal{H}_N\big(\rho^N|\rho^{\otimes N}\big) = \mathcal{H}_N\big(\rho_0^N|\rho_0^{\otimes N}\big) + \frac{\kappa}{2\sigma^2 N}\sum_{i=1}^N \int_0^t \mathbb{E}\left(\left|k*\big(\mu_s^{\mathbf{X}^N}-\rho_s\big)(X_s^i)\right|^2\right)\mathrm{d}s. \tag{1.22}$$

Let us compare the both expressions. We notice that we can derive the second expression by applying the inequality $ab \leq a^2\sigma^2/2 + (2\sigma^2)^{-1}b^2$ to the first expression. However, it is impossible to reverse this comparison. Therefore, the probabilistic argument hides the dissipation information of the Brownian motion in the Girsanov transformation, which is not recoverable. Although this method yields an equality instead of an inequality, it implies that the dissipation term in equation (1.21) can only be utilized in this manner and does not offer additional benefits for further computation.

This computation also heavily depends on the diffusion coefficient σ, which produces the dissipation term. So, in the deterministic case $\sigma = 0$, we lack the dissipation term, making the above reasoning invalid. Instead, we can suppose more regularity on the solution ρ as well as the concept of entropy solutions [JW18, Definition 2] for ρ^N to handle the case $\sigma = 0$.

The next step in the relative entropy method is to transform the measure of the term

$$\frac{\kappa}{N\sigma^2}\sum_{i=1}^N \int_0^t \int_{\mathbb{R}^{dN}} \left|\frac{1}{N}\sum_{j=1}^N (k(x_i-x_j) - k*\rho_s(x_i))\right|^2 \rho_s^N \,\mathrm{d}x_1\ldots\mathrm{d}x_N\,\mathrm{d}s$$

from the measure of the interacting particle system ρ^N to the mean-field particle system $\rho^{\otimes N}$ by paying the price of an exponential term. This leads to

$$\frac{\kappa}{N^2\sigma^2}\sum_{i=1}^N \log\left(\int_{R^{dN}} \exp\left(\frac{1}{N}\left|\sum_{j=1}^N (k(x_i-x_j) - k*\rho_s(x_i))\right|^2\right)\rho_s^{\otimes N}\,\mathrm{d}x_1\ldots\mathrm{d}x_N\right)\mathrm{d}s$$

with an error term that depends linearly on the relative entropy, which becomes negligible after applying a Gronwall argument. A significant combinatorial result by Jabin and Wang bounds the exponential term using an exponential law of large numbers, resulting in the bound

$$\sup_{0\leq t\leq T} \mathcal{H}_N\big(\rho_t^N|\rho_t^{\otimes N}\big) \leq \frac{C}{N}.$$

Here, the regularity of the kernel is crucial and is hidden in the constant C. In their original work, Jabin and Wang considered kernels $k \in W^{-1,\infty}$ on the torus covering the case of the Biot–Savart kernel. An important extension of this method was introduced by Lacker [Lac23], who further refined the estimate by using a BBGKY hierarchy in the relative entropy estimate, improving the bound from C/N to C/N^2. Lacker also demonstrated the optimality of this bound by presenting a counter example, showing that it cannot be further improved. Previously, Jabin, Bresch, and Soler [BJS22] also used the BBGKY for the Vlasov–Poisson–Fokker–Planck system for plasmas in dimension two.

Now, let us turn our attention to the other important expression: the modulated energy $\mathcal{K}_N$. This quantity is particularly useful because, for the Coulomb potential, the modulated energy implies the weak convergence of the empirical measures [RS23]. We compute its evolution as follows

$$
\begin{aligned}
&F_N\big(\mu_t^{\mathbf{X}^N},\rho_t\big)-F_N\big(\mu_0^{\mathbf{X}^N},\rho_0\big)\\
&\quad=\frac{2\kappa}{N^3}\sum_{\substack{i,j,l=1\\ j,l\neq i}}^{N}\int_0^t \nabla g(X_s^i-X_s^j)\cdot\nabla g(X_s^i-X_s^l)\,\mathrm{d}s\\
&\qquad+\frac{2\kappa}{N}\sum_{i=1}^{N}\int_0^t (g*\operatorname{div}(\nabla g*\rho_s\rho_s))(X_s^i)-2\kappa\int_0^t\langle g*\operatorname{div}(\nabla g*\rho_s\rho_s),\rho_s\rangle\,\mathrm{d}s\,\mathrm{d}s\\
&\qquad-\frac{2\kappa}{N^2}\sum_{i,j=1,i\neq j}^{N}\int_0^t \nabla g*\rho_s(X_s^i)\cdot\nabla g(X_s^i-X_s^j)\,\mathrm{d}s\\
&\qquad+\sigma^2\int_0^t\int_{(\mathbb{R}^d)^2\setminus\{x=y\}}\Delta g(x-y)(\,\mathrm{d}\mu_s^{\mathbf{X}^N}-\,\mathrm{d}\rho_s)^{\otimes 2}(x,y)\,\mathrm{d}s\\
&\qquad+\frac{2\sigma}{N}\sum_{i=1}^{N}\int_0^t\int_{\mathbb{R}^d\setminus\{X_s^i\}}\nabla g(X_s^i-y)(\,\mathrm{d}\mu_s^{\mathbf{X}^N}-\,\mathrm{d}\rho_s)(y)\cdot\,\mathrm{d}B_s^i.
\end{aligned}
$$

Rearranging provides

$$
\begin{aligned}
&F_N\big(\mu_t^{\mathbf{X}^N},\rho_t\big)-F_N\big(\mu_0^{\mathbf{X}^N},\rho_0\big)\\
&\quad=\frac{2\kappa}{N^3}\sum_{\substack{i,j,l=1\\ j,l\neq i}}^{N}\int_0^t (\nabla g(X_s^i-X_s^j)-\nabla g*\rho_s(X_s^i))\cdot(\nabla g(X_s^i-X_s^l)-\nabla g*\rho_s(X_s^i))\,\mathrm{d}s\\
&\qquad+\frac{2\kappa}{N^2}\sum_{\substack{i,j=1\\ j\neq i}}^{N}\int_0^t\langle g*\operatorname{div}(\nabla g*\rho_s\rho_s),\mu_s^{\mathbf{X}^N}-\rho_s\rangle\\
&\qquad+(\nabla g(X_s^i-X_s^j)-\nabla g*\rho_s(X_s^i))\nabla g*\rho_s(X_s^i))\,\mathrm{d}s\\
&\qquad+\sigma^2\int_0^t\int_{(\mathbb{R}^d)^2\setminus\{x=y\}}\Delta g(x-y)(\,\mathrm{d}\mu_s^{\mathbf{X}^N}-\,\mathrm{d}\rho_s)^{\otimes 2}(x,y)\,\mathrm{d}s\\
&\qquad+\frac{2\sigma}{N}\sum_{i=1}^{N}\int_0^t\int_{\mathbb{R}^d\setminus\{X_s^i\}}\nabla g(X_s^i-y)(\,\mathrm{d}\mu_s^{\mathbf{X}^N}-\,\mathrm{d}\rho_s)(y)\cdot\,\mathrm{d}B_s^i.
\end{aligned}
$$

Finally, we obtain

$$
\begin{aligned}
&F_N(\mu_t^N,\rho_t) - F_N(\mu_0^N,\rho_0) \\
&= 2\kappa \int_0^t \int_{\mathbb{R}^d} \nabla g * (\mu_s^{\mathbf{X}^N} - \rho_s)\cdot \nabla g * (\mu_s^{\mathbf{X}^N} - \rho_s)\,\mathrm{d}\mu_s^{\mathbf{X}^N}\,\mathrm{d}s \\
&\quad + 2\kappa \int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla g(x-y)\cdot \nabla g * \rho_s(x)(\,\mathrm{d}\mu_s^{\mathbf{X}^N} - \,\mathrm{d}\rho_s)^{\otimes 2}(x,y)\,\mathrm{d}s \\
&\quad + \sigma^2 \int_0^t \int_{(\mathbb{R}^d)^2\setminus\{x=y\}} \Delta g(x-y)(\,\mathrm{d}\mu_s^{\mathbf{X}^N} - \,\mathrm{d}\rho_s)^{\otimes 2}(x,y)\,\mathrm{d}s \\
&\quad + \frac{2\sigma}{N}\sum_{i=1}^N \int_0^t \int_{\mathbb{R}^d\setminus\{X_s^i\}} \nabla g(X_s^i - y)(\,\mathrm{d}\mu_s^{\mathbf{X}^N} - \,\mathrm{d}\rho_s)(y)\,\mathrm{d}B_s^i.
\end{aligned}
\tag{1.23}
$$

Notice that after the multiplication with σ^{-2} the first term is actually four times the relative entropy if we ignore the scaling factor N, which immediately connects these two quantities. The relationship between the relative entropy expressions (1.21) and (1.22) and their connection to estimates of the modulated free energy will hopefully be explored in future projects. At present, it is unclear how to use this insight, though it has been briefly addressed in recent works.

Continuing with equation (1.23), we observe that the last integral is as a stochastic integral and vanishes in expectation. The second term presents the main challenge in estimation and can be approached in several ways. The first approach in handling this term was proposed by Serfaty and Duernickx [Ser20, Proposition 1.1] in the deterministic setting for the repulsive Coulomb kernel. Their method involves renormalizing the quantity by expanding Dirac measures into spheres with uniform measures and carefully obtaining an estimate on the diagonal terms.

Additionally, it is worth noting that in the case of the repulsive Coulomb kernel ($\kappa = -1$), one can demonstrate that the modulated energy is non-negative. Therefore, we obtain the following bound

$$
\begin{aligned}
&\int_0^t \mathbb{E}\Big(\int_{\mathbb{R}^d} |\nabla g * (\mu_s^{\mathbf{X}^N} - \rho_s)|^2\,\mathrm{d}\mu_s^{\mathbf{X}^N}\Big)\,\mathrm{d}s - F_N(\mu_0^N,\rho_0) \\
&\quad \le 2\int_0^t \Big|\mathbb{E}\Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla g(x-y)\cdot\nabla g*\rho_s(x)(\,\mathrm{d}\mu_s^{\mathbf{X}^N} - \,\mathrm{d}\rho_s)^{\otimes 2}(x,y)\Big)\Big|\,\mathrm{d}s \\
&\qquad + \sigma^2\int_0^t \Big|\mathbb{E}\Big(\int_{(\mathbb{R}^d)^2\setminus\{x=y\}} \Delta W(x-y)(\,\mathrm{d}\mu_s^{\mathbf{X}^N} - \,\mathrm{d}\rho_s)^{\otimes 2}(x,y)\Big)\Big|\,\mathrm{d}s.
\end{aligned}
$$

Hence, comparing it to the relative entropy representation (1.22), we actually found an upper bound on the relative entropy. This provides a further connection between the relative entropy and modulated energy. Unfortunately, a similar bound cannot be obtained in the attractive case, which provides further insight into why the modulated energy method [Ser20] is currently only applied to repulsive kernels.

Another significant extension is the combination of modulated energy and relative entropy, which is referred to as modulated free energy

$$E_N(\rho_t^N|\rho_t^{\otimes N}) := \mathcal{K}_N\big(\mu_t^{\mathbf{X}^N}|\rho_t\big) + \mathcal{H}_N\big(\rho_t^N|\rho_t^{\otimes N}\big). \tag{1.24}$$

This quantity is particularly useful for addressing the attractive Keller–Segel interaction kernel ($g(x) = \lambda \log(|x|)$). Bresch, Jabin, and Wang [BJW23] demonstrated propagation of chaos by deriving crucial upper and lower bounds on the modulated free energy (1.24). The challenge lies in establishing a lower bound on E_N because it is not immediately evident that E_N is non-negative. Interestingly, their approach does not rely on the evolution of the quantity but instead utilizes the specific structure of the kernel and the torus setting. Thus, the evolution is major factor in establishing the upper bound, but plays no role in deriving the lower bound, which is a fascinating observation, because Serfaty and Duernickx [Ser20, Corollary 3.5] use the same strategy in their original work.

In general, we notice that the term

$$\mathbb{E}\bigg(\int_{\mathbb{R}^d} |\nabla g * (\mu_t^{\mathbf{X}^N} - \rho_t)|^2 \,\mathrm{d}\mu_s^{\mathbf{X}^N}\bigg) \tag{1.25}$$

is crucial for deriving mean-field limits. This term plays a significant role in the relative entropy method, the modulated energy method, and the coupling method. From our perspective, it is unavoidable in the estimation process. Regardless of the specific approach chosen, an estimation involving the term (1.25) is essential. Therefore, there should be a fundamental connection between all the quantities discussed here. We will explore this connection further in Chapter 3, aiming to enhance our understanding regarding the relationship between these quantities. While we gained a solid grasp of the individual components during our studies, assembling the full picture remains a tough challenge. This ongoing effort presents an exciting avenue for future research.

Let us also highlight a challenge encountered in the analysis of mean-field limits with singular interaction kernels that at the moment we swept under the rug: the issue of well-posedness of the Liouville equation. This issue becomes particularly relevant for attractive kernels or kernels in the repulsive super-Coulomb case, where the existence of solutions to the associated Liouville equation is unclear. While some authors assert the existence of entropy solutions [JW18, BJW23] in the Coulomb case on the torus, this remains an open problem to the best of our knowledge. For example, in [dCRS23], it is suggested that such solutions may exist for the repulsive case at the level of stochastic differential equations, but as of now, no rigorous proof has been established [RS24].

Another important concept to tackle singular kernels is the convergence in probability method introduced by Pickl and Lazarovici [LP17]. This concept is crucial to our thesis and will be summarized in detail in Chapter 2 for bounded interaction kernel k. One can classify this method between the interface of singular kernels and moderated interactions.

The idea is to regularize the interaction kernel $k \rightsquigarrow k^\varepsilon$ with a parameter ε, which depends on the number of particles $\varepsilon = N^{-\beta}$. This regularization extends to both the interacting particle system $\mathbf{X}^{N,\varepsilon}$ and the mean-field equation $\mathbf{Y}^{N,\varepsilon}$, creating an intermediate framework, avoiding well-posedness issues at the particle level.

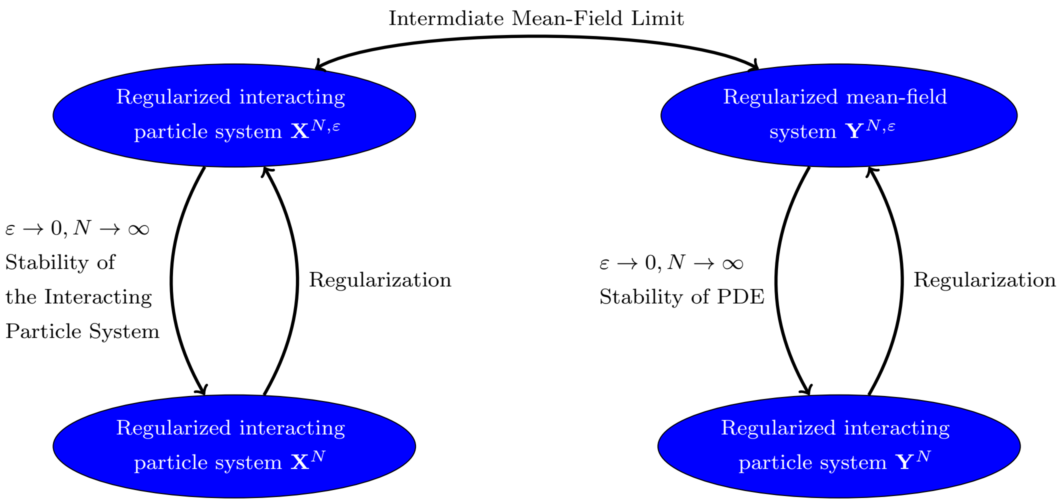

FIGURE 1. Intermediate System

The goal of propagation of chaos in probability is to demonstrate the connection at the top of Figure 1. But, instead of using an estimate in expectation as in the coupling method (1.17), we rely on convergence in probability. This means there exists a $\delta(N) > 0$ with $\delta(N) \to 0$ as $N \to \infty$ such that

$$\mathbb{P}\left(\sup_{t\in[0,T]} |\mathbf{X}_t^{N,\varepsilon} - \mathbf{Y}_t^{N,\varepsilon}|_\infty \geq \delta(N) \right) \to 0, \quad \text{as } N \to \infty.$$

Consequently, the probability that the interacting particles $\mathbf{X}^N$ are further than $\delta(N)$ away from the mean-field particles $\mathbf{Y}^N$ vanishes as $N \to \infty$. We can also demonstrate convergence rates that depend on the number of particles. While the expectation based estimate (1.17) achieves and optimal rate of $1/\sqrt{N}$ [Tan84], convergence in probability can achieve arbitrarily fast convergence rates $N^{-\gamma}$. However, for the "bad set"

$$\sup_{t\in[0,T]} |\mathbf{X}_t^{N,\varepsilon} - \mathbf{Y}_t^{N,\varepsilon}|_\infty \geq \delta(N)$$

the best achievable bound is $\delta(N) = N^{-1/2+\eta}$ for any small $\eta > 0$, which aligns with the law of large numbers [Tan84]. As we will demonstrate in Chapter 3, this concept, under a convolution structure of k^ε, implies convergence in relative entropy and modulated energy in the intermediate regime, underscoring its strength. Further details can be found in Chapter 3. Finally, it is worth noting that this approach is widely applied in singular models such as the Keller–Segel model [HLL19, FHS19].

1.3.5. Stochastic differential equations with common noise. To illustrate the major differences between propagation of chaos with and without common noise, we consider how common noise adds an additional layer of complexity to the analysis of propagation of chaos. This type of noise affects the entire population, and thus, we cannot expect as in the idiosyncratic case, the law of large numbers to negate its effects. As we discussed in Section 1.3.2, there are multiple approaches to derive the mean-field limiting equation. We also showed that the empirical measure satisfies a stochastic differential equation with vanishing quadratic variation as $N \to \infty$. Consequently, we predicted that the limiting measure ρ satisfies a deterministic partial differential equation. Now, let us repeat this calculation with the inclusion of common noise. Applying Itô's formula (1.15) to the empirical measure, we obtain

$$\begin{aligned}\langle \varphi, \mu_t^{\mathbf{X}^N} \rangle = &\int_0^t -\langle \nabla\varphi \cdot (k * \mu_s^{\mathbf{X}^N}), \mu_s^{\mathbf{X}^N} \rangle \,\mathrm{d}s + \frac{1}{N}\sum_{i=1}^N \int_0^t \nabla\varphi(X_s^i)\sigma(s,\cdot)\,\mathrm{d}B_s^i \\ &+ \frac{1}{2}\int_0^t \Big\langle \mathrm{Tr}\Big((\sigma(s,\cdot)\sigma(s,\cdot)^{\mathrm{T}} + \nu(s,\cdot)\nu(s,\cdot)^{\mathrm{T}})\nabla^2\varphi(\cdot) \Big), \mu_s^{\mathbf{X}^N} \Big\rangle \,\mathrm{d}s \\ &+ \langle \nabla\varphi^{\mathrm{T}}(\cdot)\nu(s,\cdot), \mu_s^{\mathbf{X}^N} \rangle \,\mathrm{d}W_t.\end{aligned}$$

We notice that the stochastic term driven by the common noise W does not vanish! Therefore, assuming $\mu_t^{\mathbf{X}^N}$ converges in some topology to ρ such that all limiting procedures are well-defined, we derive the following stochastic partial differential equation

$$\begin{aligned}\mathrm{d}\rho_t = &\nabla \cdot ((k * \rho_t)\rho_t))\,\mathrm{d}t - \nabla \cdot (\nu(t,x)\rho_t \,\mathrm{d}W_t) \\ &+ \frac{1}{2}\sum_{i,j=1}^d \partial_{x_i}\partial_{x_j}\Big(([\sigma(t,x)\sigma(t,x)^{\mathrm{T}}]_{(i,j)} + [\nu(t,x_i)\nu(t,x_j)^{\mathrm{T}}]_{(i,j)})\rho_t \Big)\,\mathrm{d}t,\end{aligned} \tag{1.26}$$

where $[A]_{(i,j)}$ denotes the (i,j)-entry of a matrix A. Thus, the limiting measure ρ solves a stochastic partial differential equation, implying that the measure ρ is no longer deterministic but a truly random measure. This is a significant observation because it indicates that results like Proposition 1.2 are not applicable, making the analysis of such mean-field limits more challenging. Furthermore, the "top-to-bottom" approach, which involves solving the limiting equation first and then using the solution's regularity properties to achieve propagation of chaos, becomes more challenging. The difficulty arises because, in general, we cannot expect to have the same regularity for solutions of the stochastic partial differential equation (1.26) as we do for the deterministic partial differential equation (1.16). In particular, it is challenging to obtain uniform estimates in the probability space, whereas in the deterministic case, one can simply use embedding theorems to guarantee bounded solutions.

Next, let us derive the stochastic partial differential equation (1.26) from the point of view of stochastic analysis. To do this, we introduce the conditional (or stochastic) McKean–Vlasov equation

$$\begin{cases}\mathrm{d}Y_t^i = -(k * \rho_t(Y_t^i))\,\mathrm{d}t + \sigma(t, Y_t^i, \mu_t)\,\mathrm{d}B_t^i + \nu(t, Y_t^i, \mu_t)\,\mathrm{d}W_t, \\ \rho_t = \mathscr{L}_{Y_t^i | \mathcal{F}_t^W}, \quad \forall t \geq 0,\end{cases} \tag{1.27}$$

where $\mathcal{F}_t^W$ is the σ-algebra generated by the Brownian motion W and $\mathscr{L}_{Y_t^i|\mathcal{F}_t^W}$ is the conditional density of Y_t^i given $\mathcal{F}_t^W$. The second equality can be understood in an almost everywhere sense, meaning ρ is a measure on the space of continuous functions $C([0,T];\mathbb{R}^d)$ and a version of the regular conditional law of Y given the filtration $\mathcal{F}^W$ generated by the Brownian motion W. Applying Itô's formula (1.15) and taking the conditional expectation, we obtain

$$\begin{aligned}
&\mathbb{E}\left(\varphi(X_t)\middle|\mathcal{F}_t^W\right)\\
&\quad= \mathbb{E}\left(\varphi(X_0)\middle|\mathcal{F}_t^W\right) - \mathbb{E}\Big((k*\rho_t)\cdot\nabla\varphi\,\mathrm{d}s\Big|\mathcal{F}_t^W\Big)\\
&\qquad+\frac{1}{2}\sum_{i,j=1}^{d}\mathbb{E}\bigg(\int_0^t\bigg(([\sigma(t,x)\sigma(t,X_s^i)^{\mathrm{T}}]_{(i,j)}\\
&\qquad+[\nu(t,x_i)\nu(t,x_j)^{\mathrm{T}}]_{(i,j)})\partial_{x_i}\partial_{x_j}\varphi(s,X_s)\bigg)\,\mathrm{d}s\bigg|\mathcal{F}_t^W\bigg)\\
&\qquad+\mathbb{E}\left(\int_0^t\nabla\varphi(X_s)\sigma(s,X_s)\,dW_s\middle|\mathcal{F}_t^W\right)+\mathbb{E}\left(\int_0^t\partial_x\varphi(X_s)\nu(s,X_s)\,\mathrm{d}B_s\middle|\mathcal{F}_t^W\right)
\end{aligned}$$

for smooth function φ. Fubini's theorem and the stochastic version of the conditional Fubini theorem A.35, which we recalled in the Appendix, imply that the conditional law ρ of Y given $\mathcal{F}^W$ is actually a solution to the equation (1.26). Therefore, as before, there is an interplay between the stochastic partial differential equation and the conditional McKean–Vlasov equations. This relationship is explored in the significant work by Lacker, Shkolnikov, and Zhang [LSZ23], where a superposition principle and mimicking theorems are derived through a complex reduction of the problem to smooth coefficients. For a weak existence result on conditional McKean–Vlasov equations we refer to [HvS21].

To distinguish between the settings with and without common noise, we will refer to the concept of propagation of chaos in the presence of common noise as conditional propagation of chaos. This emphasizes the difference between convergence towards the true distribution and convergence towards the conditional distribution.

In comparison to the non common noise case, this field is not well studied and the literature is sparse. While propagation of chaos for Lipschitz coefficients has been known at least since Sznitman's review [Szn91], the same result for common noise is rather new and can be found, for instance, in Carmona's and Delarue's book [CD18, Chapter 2]. In the context of transport noise was proven by Coghi and Flandoli [CF16] in 2016. Although this regular case can be achieved through standard probabilistic arguments, it still took more than twenty years due to the detailed considerations required. Flandoli and Coghi's approach, inspired by [Gol16], involves constructing a stochastic Liouville equation that maps the initial measure to the solution of the stochastic Fokker–Planck equation. Utilizing sharp estimates in Kolmogorov's continuity theorem and properties of measure-valued solutions of the associated stochastic Fokker–Planck equation, they ultimately demonstrate conditional propagation of chaos.

Additional relevant literature includes the work of Dawson and Vaillancourt [DV95], who formulated a martingale problem and demonstrated the tightness of the empirical measure, as well as the works of Kurtz and Xiong [KX99], and Coghi and Gess [CG19] on the existence of stochastic non-linear Fokker–Planck equations (1.26) under Lipschitz coefficients.

1.3.6. Comments. In the above setting, we use a time-independent interaction $k\colon \mathbb{R}^d \mapsto \mathbb{R}^d$, which expresses the drift term as a convolution with the empirical measure, as shown in equation (1.13). Thus, we can generalize the drift term to $b\big(t, X_t^i, \mu_t^{\mathbf{X}^N}\big)$ instead of a normalized sum. For instance, if b is Lipschitz continuous, the analysis does not change significantly, and we expect all the previous results to hold true [CD18, Theorem 2.12]. The analysis becomes more complex when we involve techniques that use partial differential equations, particularly for singular interaction kernels.

At this point, it is beneficial to have more structure on the measure dependency of b. Typically, this is achieved by expressing the interaction through an integration of the empirical measure with respect to some function

$$b\big(t, X_t^i, \mu_t^{\mathbf{X}^N}\big) = \tilde{b}\bigg(t, X_t^i, \int h(X_t^i, y)\,\mathrm{d}\mu_t^{\mathbf{X}^N}(y)\bigg).$$

By specifying the properties of the new functions $\tilde{b}, h$, we observe that the difference from the weighted sum lies in the transformation applied by $\tilde{b}$ and h. Hence, if we restrict the properties of $\tilde{b}$ and h, we expect all methods to work analogously.

Similar, we can also include measure dependency on the diffusion coefficients, σ, ν by adding a Lipschitz continuity in the Wasserstein distance

$$|\sigma(t,x,\mu) - \sigma(t,y,\tilde{\mu})| + |\nu(t,x,\mu) - \nu(t,y,\tilde{\mu})| \le C\big(|x-y| + W_2(\mu,\tilde{\mu})\big).$$

In this case the coupling technique still works on the space of probability measures with finite second moment. However, even if the coefficients are elliptic it is no clear how to apply the methods from Section 1.3.4 since the associated equations are now also non-linear in the diffusion.

1.4. Setting and notation

Since the complete thesis is based on the analysis of particle systems, we want to fix the probabilistic framework, where particle systems (1.8) are defined. Additionally, we need to fix the space for the solution of our partial differential equations (PDE's) and stochastic partial differential equations (SPDE's). For questions around measurability or completeness and other useful results such as the Aubin–Lions lemma we refer to the Appendix. The setting will be held as general as possible to include models with and without common noise, multiplicative noise, additive noise and arbitrary dimensions. Later, we will reduce the setting in each chapter of this thesis to necessary requirements for this section.

1.4.1. Probabilistic framework. In this thesis we will focus on a finite time interval $[0,T]$ with arbitrary $T > 0$, if not stated otherwise. To that end, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ be a complete probability space with right-continuous and complete filtration $(\mathcal{F}_t)_{t\ge 0}$. Under a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we understand that all sets $A \in \mathcal{N}$ with

$$\mathcal{N} := \{A \subseteq \Omega \mid A \subseteq B \text{ for some } B \in \mathcal{F} \text{ with } P(B) = 0\}$$

are measurable. A filtration $(\mathcal{F}_t)_{t\ge 0}$ is complete if $\mathcal{F}_0$ contains all sets $A \in \mathcal{N}$ and right continuous if

$$\mathcal{F}_t = \mathcal{F}_{t+} :\, = \bigcap_{s>t} \mathcal{F}_s$$

holds for $t \ge 0$. Furthermore, let $(B_t^i = (B_t^{i,1}, \dots, B_t^{i,m}), t \ge 0)$, $i \in \mathbb{N}$ be a sequence independent m-dimensional Brownian motions with respect to the filtration $(\mathcal{F}_t, t \ge 0)$, and $(W_t = (W_t^1, \dots, W_t^{\tilde{m}}), t \ge 0)$ be another $\tilde{m}$-dimensional Brownian motions with respect to the same filtration $(\mathcal{F}_t, t \geq 0)$, which is independent from $(B^i_t, t \geq 0)$ for all $i \in \mathbb{N}$. Moreover, we denote by $\mathcal{F}^W = (\mathcal{F}^W_t, t \geq 0)$ the augmented filtration generated by W and by $\mathcal{P}^W$ the predictable σ-algebra with respect to $\mathcal{F}^W$. More precisely, $\mathcal{P}^W$ is the σ-algebra on $\Omega \times [0,T]$ generated by all $\mathcal{F}^W$-adapted and left-continuous processes. Analogously, we define $\mathcal{F}^B$ and $\mathcal{P}^B$. It remains to construct the initial condition. Our interacting particle systems always have initial distribution ρ_0 and are independent. Therefore, let $(\zeta_i, i \in \mathbb{N})$ be a sequence of i.i.d. random variables with distribution ρ_0, independent of $\mathcal{F}^W$ and $\mathcal{F}^B$. In the stochastic differential equation (1.8), we set $X^i_0 = \zeta^i$ as the initial condition, which accomplishes the desired set-up. This concludes the basic setting in order to define interacting particle systems of the form (1.8), i.e.

$$\mathrm{d}X^i_t = \frac{1}{N}\sum_{j=1}^{N} k(X^i_t - X^j_t)\,\mathrm{d}t + \sigma(t, X^i_t)\,\mathrm{d}B^i_t + \nu(t, X^i_t)\,\mathrm{d}W_t, \quad i = 1, \dots, N$$

with i.i.d. initial condition $(X^i_0, i \in \mathbb{N})$. For a more detailed construction of independent Brownian motions and augmented filtration we refer to [KS91, Chapter 2]. Moreover, for weak and strong uniqueness as well as weak and strong existence of stochastic differential equations and the martingale problem we refer to [KS91, Chapter 5]. Additionally, we adopt the convention that for constant diffusion coefficients σ and/or ν, we omit the associated identity matrix, keeping in mind to ensure that the dimensions in the stochastic integrals are well-defined. Finally, we use the notation $X \sim \rho$ to represent that ρ is the law of random variable X. We write $\mathscr{L}_X$ for the law of X and $\mathscr{L}_{X|\mathcal{F}}$ for the conditional law of X given the filtration $\mathcal{F}$.

1.4.2. Function spaces. For a vector $x \in \mathbb{R}^d$, we write $|x|$ for the standard Euclidean norm and $|x|_\infty$ for the l^∞-Euclidean norm. We denote by $B(a,r)$ a ball with center a and radius r with respect to the Euclidean norm. For an N-particle vector $\mathsf{X}^N = (x_1, \dots, x_N) \in \mathbb{R}^{dN}$ we make the convenient that the l^∞-norm is taken with respect to the dimension d, i.e. $|\mathsf{X}^N|_\infty \colon = \max(|x_1|, \dots, |x_d|)$. We use the generic constant C for inequalities, which may change from line to line.

For $1 \leq p \leq \infty$ we denote by $L^p(\mathbb{R}^d)$ the space of measurable functions whose p-th power is Lebesgue integrable (with the usual modification for $p = \infty$) equipped with the norm $\|\cdot\|_{L^p(\mathbb{R}^d)}$, by $L^1(\mathbb{R}^d, |x|^2\,\mathrm{d}x)$ the space of all measurable functions f such that $\int_{\mathbb{R}^d} |f(x)||x|^2\,\mathrm{d}x < \infty$, by $C^\infty_c(\mathbb{R}^d)$ the space of all infinitely differentiable functions with compact support on $\mathbb{R}^d$, and by $\mathcal{S}(\mathbb{R}^d)$ the space of all Schwartz functions, see [Yos80, Chapter 6] for more details. Even though in most cases functions in $L^p(\mathbb{R}^d)$ will be real-valued, we will use the same notation for $\mathbb{R}^l$-valued functions and replacing the absolute value in the definition by the Euclidean norm. A similar convenient hold for all other normed spaces implicitly by replacing the absolute value with the norm. Additionally, the convolution operator $*$ is understood to act component-wise when one operand is vector-valued. For instance, for $k \colon \mathbb{R}^d \to \mathbb{R}^d$ and $f \colon \mathbb{R}^d \mapsto \mathbb{R}$, the convolution $k * f \colon \mathbb{R}^d \mapsto \mathbb{R}^d$ is defined component-wise.

For a smooth function $f \colon [0,T] \times \mathbb{R}^d \mapsto \mathbb{R}$ and a multi-index κ with length $|\kappa| := \sum_i \kappa_i$, we denote the derivative with respect to $x^\kappa = x_1^{\kappa_1} \cdots x_d^{\kappa_d}$ by $\partial^\kappa f(t,x) := \prod_i \left(\frac{\partial}{\partial x_i}\right)^{\kappa_i} f(t,x)$, where we write $\partial_{x_i} f$ or $f_{x_i}(t,x)$ for $\frac{\partial}{\partial x_i} f(t,x)$. The gradient ∇ and Laplace operator Δ always act on the space variable x. The derivative with respect to time we denote by $\partial_t f(t,x)$ or

$\frac{\mathrm{d}}{\mathrm{d}t}f(t,x)$. For $f \in \mathcal{S}(\mathbb{R}^d)$ we define the Fourier transform $\mathcal{F}[u]$ and inverse Fourier transform $\mathcal{F}^{-1}[u]$ by

$$\mathcal{F}[u](\xi) := \int_{\mathbb{R}^d} e^{-2\pi i \eta \cdot x} f(x)\,\mathrm{d}x \quad \text{and} \quad \mathcal{F}^{-1}[u](\xi) := \int_{\mathbb{R}^d} e^{2\pi i \eta \cdot x} f(x)\,\mathrm{d}x.$$

For $u \in \mathcal{S}'(\mathbb{R}^d)$ we can also define the Fourier transformation by duality

$$\langle \mathcal{F}[u], f\rangle := \langle u, \mathcal{F}[f]\rangle, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

We denote the Bessel potential for each $s \in \mathbb{R}$ and $u \in \mathcal{S}'(\mathbb{R}^d)$ by

$$(1-\Delta)^{s/2}u := \mathcal{F}^{-1}[(1+4\pi^2|\xi|^2)^{s/2}\mathcal{F}[u]]$$

and define the Bessel potential space H_p^s for $p \in (1,\infty)$ and $s \in \mathbb{R}$ by

$$H_p^s := \{u \in \mathcal{S}'(\mathbb{R}^d) \ : \ (1-\Delta)^{s/2}u \in L^p(\mathbb{R}^d)\}$$

with norm

$$\|u\|_{H_p^s(\mathbb{R}^d)} := \left\|(1-\Delta)^{s/2}u\right\|_{L^p(\mathbb{R}^d)}.$$

Applying [Tri83, Theorem 2.5.6] we can characterize the above Bessel potential spaces H_p^m for $1 < p < \infty$ and $m \in \mathbb{N}$ as Sobolev spaces

$$W^{m,p}(\mathbb{R}^d) := \left\{u \in L^p(\mathbb{R}^d) \ : \ \|u\|_{W^{m,p}(\mathbb{R}^d)} := \sum_{\kappa \in \mathcal{A},\ |\kappa| \le m} \|\partial^\kappa f\|_{L^p(\mathbb{R}^d)} < \infty\right\},$$

where $\partial^\kappa u$ is to be understood as weak derivatives [AF03] and $\mathcal{A}$ is the set of all multi-indices. The definition of $W^{m,p}(\mathbb{R}^d)$ extends in the natural way to $p = \infty$. Moreover, we will use the following abbreviation $H^s(\mathbb{R}^d) := H_2^s(\mathbb{R}^d)$.

In general we will denote by $\langle \cdot, \cdot \rangle$ the dual paring and the inner product. If we want to specify the space we write it as a subscript, for instance for the $L^2(\mathbb{R}^d)$ product we write $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^d)}$. But in general it should be clear from context which dual paring we mean. Additionally, weak convergence is denoted by the symbol $\rightharpoonup$. Again, whether it is weak convergence of measures or weak convergence of functions in some normed spaces should be understood from the context.

We also require Banach-valued L^p-spaces as well as the stochastic version of them. Let $(Z, \|\cdot\|_Z)$ be a Banach space. We denote by $L^p([0,T];Z)$ the space of all strongly measurable functions $u\colon [0,T] \to Z$ such that

$$\|u\|_{L^p([0,T];Z)} := \begin{cases} \left(\int_0^T \|u(t)\|_Z^p \,\mathrm{d}t\right)^{\frac{1}{p}} < \infty, & \text{for } 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{t\in[0,T]} \|u(t)\|_Z < \infty, & \text{for } p = \infty. \end{cases}$$

The Banach space $C([0,T];Z)$ consists of all continuous functions $u\colon [0,T] \to Z$, equipped with the norm

$$\max_{t\in[0,T]} \|u(t)\|_Z < \infty.$$

We also introduce another class of L^p-spaces, which should serve as solution spaces for our stochastic partial differential equations. For a filtration $(\mathcal{F}_t)_{t\ge 0}$, $1 \le p \le \infty$ and $0 \le$

$s < t \le T$ we denote by $S^p_{\mathcal{F}}([s,t];Z)$ the set of Z-valued $(\mathcal{F}_t)$-adapted continuous processes $(X_u, u \in [s,t])$ such that

$$\|X\|_{S^p_{\mathcal{F}}([s,t];Z)} := \begin{cases} \left(\mathbb{E}\left(\sup\limits_{u\in[s,t]} \|X_u\|_Z^p\right)\right)^{\frac{1}{p}}, & p \in [1,\infty), \\ \sup\limits_{\omega\in\Omega}\sup\limits_{u\in[s,t]} \|X_u\|_Z\,, & p = \infty, \end{cases}$$

is finite. Similar, $L^p_{\mathcal{F}}([s,t];Z)$ denotes the set of Z-valued predictable processes $(X_u, u \in [s,t])$ such that

$$\|X\|_{L^p_{\mathcal{F}}([s,t];Z)} := \begin{cases} \left(\mathbb{E}\left(\int\limits_s^t \|X_u\|_Z^p \,\mathrm{d}u\right)\right)^{\frac{1}{p}}, & p \in [1,\infty), \\ \sup\limits_{(\omega,u)\in\Omega\times[s,t]} \|X_u\|_Z\,, & p = \infty, \end{cases}$$

is finite. In most case Z will be the Bessel potential space H^s_p, as it is mainly used by Krylov [Kry10] in treating SPDEs. For a more detail introduction to the above function spaces we refer to [Kry99, Section 3].

1.5. Structure of the thesis

We have tried to organize this thesis to follow a logical and natural flow. In Chapter 1, we introduced the general framework of mean-field limits and propagation of chaos, showcasing established results in the Lipschitz case. After the Introduction, the thesis mirrors the content of Chapter 1 but shifts the focus to more complex kernels with lower regularity, specifically those outside the Lipschitz regime.

The thesis is divided into two main parts: the first part deals with systems without common noise ($\nu = 0$ in (1.8)), corresponding to Chapter 2 and Chapter 3. The second part addresses interacting particle systems with common noise ($\nu \neq 0$ in (1.8)), covered in Chapter 4 and Chapter 5.

Each subsequent chapter follows a consistent structure. First, the problem is introduced along with the specific notation for that chapter. For instance, the interacting particle process X_t, the McKean–Vlasov SDE Y_t, and the associated Fokker–Planck equations ρ_t are redefined in each chapter according to its context. This approach is intended to prevent an overload of indices within each chapter.

Thus, any notation, (stochastic) partial differential equations, and stochastic differential equations introduced outside Section 1.4 are only valid within their respective chapters and should not be considered valid outside of them.

Afterwards, we present our contributions to the research area and provide related literature to place our work within it. This supplements the literature already discussed in Chapter 1. Finally, we prove the main results in each chapter, which constitutes the bulk of the content. Therefore, I will only provide a brief overview of the content of each chapter and its main results.

Chapter 2 is based on [CNP24] and includes results on the existence, uniqueness, and regularity for diffusion-aggregation equations with bounded force. Additionally, we demonstrate a convergence of probability result for the moderated interaction particle system towards the

McKean–Vlasov stochastic differential equation with an arbitrary rate, i.e.

$$\mathbb{P}\bigg(\sup_{t\in[0,T]} \big|\mathbf{X}_t^{N,\varepsilon} - \mathbf{Y}_t^{N,\varepsilon}\big|_\infty \geq N^{-\alpha} \bigg) \leq C(\gamma) N^{-\gamma}, \quad \text{for each } N \geq N_0,$$

where $\varepsilon = N^{-\beta}$ is the regularization parameter, $\alpha \in (0, 1/2)$, $\gamma > 0$, and $\mathbf{X}_t^{N,\varepsilon}, \mathbf{Y}_t^{N,\varepsilon}$ are the particles system corresponding to the regularized kernel k^ε.

Chapter 3, based on the work [PN24], serves as a natural continuation of Chapter 2. Here, we utilize the convergence in probability to derive quantitative estimates on the modulated energy and relative entropy. The key insight in this chapter is that, under a convolution structure on the interaction kernel $k^\varepsilon = V^\varepsilon * V^\varepsilon$, we can relate the relative entropy and the modulated energy to a weighted L^2-norm depending on the decomposition V^ε. Consequently, we transform the problem of relative entropy and modulated energy into one concerning an L^2-norm. Our main result is a quantitative inequality regarding the weighted L^2-norm

$$\begin{aligned} &\mathbb{E}\bigg(\sup_{0\leq t\leq T} \Big\| V^\varepsilon * \mu_t^{N,\varepsilon} - V^\varepsilon * \rho_t^\varepsilon \Big\|_{L^2(\mathbb{R}^d)}^2 \bigg) + \frac{\sigma^2}{8} \mathbb{E}\bigg(\int\limits_0^T \big\| \nabla V^\varepsilon * (\mu_s^{N,\varepsilon} - \rho_s^\varepsilon) \big\|_{L^2(\mathbb{R}^d)}^2 \,\mathrm{d}s \bigg) \\ &\leq \frac{C(T, \sigma, \gamma, C_{\mathrm{BDG}}, \|V^\varepsilon\|_{H^2(\mathbb{R}^d)}^2, \|k^\varepsilon\|_{W^{1,\infty}(\mathbb{R}^d)})}{N^{2\alpha}}. \end{aligned}$$

As a corollary, we demonstrate propagation of chaos for the attractive Keller–Segel model and the opinion dynamics model in the moderated regime.

Now, we move into the setting of common noise. Chapter 4 is motivated by the analysis of opinion dynamic models with a bounded confidence radius and is based on the work [CNP23]. In this context, the interaction kernel k has bounded support. Without common noise and with non-degenerate diffusion in the SDE (1.3.2), propagation of chaos is well established using the techniques presented in Section 1.3.4. However, when common noise is present, the well-posedness of the non-linear stochastic Fokker–Planck equation (1.26) and the conditional McKean–Vlasov equation (1.27) is not immediately clear.

Thus, the main result in Chapter 4 is the proof of strong existence and uniqueness for the conditional McKean–Vlasov equation (1.27) in the case $k \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, as well as the existence for the stochastic Fokker–Planck equation (1.26) in the case $k \in L^2(\mathbb{R})$. Additionally, we will establish a conditional propagation of chaos result in the moderated regime with a cut-off ε of order $\log(N)$.

Finally, in Chapter 5, we extend the relative entropy method pioneered by Jabin and Wang [JW18] to the setting of common noise. We will demonstrate that under suitable conditions on the diffusion coefficients σ and ν, we can estimate the relative entropy between the conditional Liouville equation and the stochastic Fokker–Planck equation. More precisely, we obtain

$$\mathbb{E}\bigg(\sup_{0\leq t\leq T} \mathcal{H}\big(\rho_t^N | \rho_t^{\otimes N}\big) \bigg) \leq C\big(\|k\|_{L^\infty(\mathbb{R}^d)} \big)$$

for the non-normalized relative entropy $\mathcal{H}\big(\rho_t^N|\rho_t^{\otimes N}\big)$ between the conditional Liouville equation ρ_t^N and the N-th tensor product $\rho_t^{\otimes N}$ of the the d-dimensional stochastic Fokker–Planck equation (1.26). Additionally, we extend Sznitman's Proposition 1.2 to include random limiting measures, thereby demonstrating conditional propagation of chaos.

In the Appendix, we provide a compilation of relevant facts and inequalities widely acknowledged in the stochastic analysis and partial differential equations community. Although these are well-known, their inclusion here allows for easy reference, gathering essential details into a single location.

Chapter 2

Mean-field limit via convergence in probability

The aim of this chapter is to present the concept of convergence in probability introduced by Peter Pickl and Lazarovici [LP17] in an universal manner without any specific model or kernel. This concept can be regarded as the basic building block for upcoming results. The content of this chapter is taken from [CNP24], which in turn is inspired by the original work [LP17].

2.1. Setting and method

In this section we introduce the problem setting, that is, the necessary notation, the interacting particle systems as well as their associated PDEs, and outline the general method implemented in the present chapter, following [LP17].

On the microscopic level, we model the system, as seen in the Section 1.3.2 by an interacting N-particle system $\mathbf{X}^N = (X^1, \dots, X^N)$, given by stochastic differential equations with additive noise of the form

$$\mathrm{d}X_t^i = -\frac{1}{N}\sum_{j=1}^{N} k(X_t^i - X_t^j)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t^i, \quad i = 1, \dots, N, \;\; \mathbf{X}_0^N \sim \bigotimes_{i=1}^{N} \rho_0 \tag{2.1}$$

for $t \geq 0$, starting from i.i.d. initial condition $(X_0^i, i \in \mathbb{N})$ with $X_0^i \sim \rho_0$, diffusion parameter $\sigma > 0$ and interaction kernel

$$k\colon \mathbb{R}^d \mapsto \mathbb{R}^d.$$

For convenience we also choose B^i to be a d-dimensional Brownian motion, meaning we set $m = d$ as defined in Section 1.4.1, and simplify the notation $\sigma\,\mathrm{d}B_t^i$ by omitting writing the identity matrix. In this chapter our main focus lies on non-smooth interaction kernels k, which satisfy some local Lipschitz assumption.

Since we are interested in non Lipschitz continuous k, the classical coupling method presented in the Section 1.3.2 does not work, that is, comparing the particle $(X_t^i, t \geq 0)$ to the solution $(Y_t^i, t \geq 0)$ of the McKean–Vlasov stochastic differential equations (McKean–Vlasov SDE) $\mathbf{Y}_t^N \colon = (Y_t^1, \dots, Y_t^N)$,

$$\begin{cases} \mathrm{d}Y_t^i = -(k * \mu_t)(Y_t^i)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t^i, \quad i = 1, \dots, N, \;\; \mathbf{Y}_0^N = \mathbf{X}_0^N, \\ \mu_t = \mathrm{Law}(Y_t^i), \end{cases} \tag{2.2}$$

for $t \geq 0$, and, subsequently, showing the weak convergence of the empirical measure $\mu_t^{\mathbf{X^N}} \rightharpoonup \mu_t$ as $N \to \infty$ for all $t \geq 0$. Assuming that the law μ of the solution $(Y_t^i, t \geq 0)$ possesses a probability density ρ that satisfies an associated Fokker–Planck equation, we employ PDE theory to demonstrate propagation of chaos using a "top-to-bottom" approach. However, the result at the end is the probability estimate

$$\mathbb{P}\left(\sup_{t\in[0,T]} |\mathbf{X}_t^{N,\varepsilon} - \mathbf{Y}_t^{N,\varepsilon}|_\infty \geq N^{-\alpha}\right) \leq C(\gamma)N^{-\gamma}, \quad \text{for each } N \geq N_0$$

for $\alpha \in (0, 1/2)$ and $\gamma > 0$.

2.1.1. Approximated interacting particle systems. In order to apply the method of convergence in probability, we need to change our particle systems to the intermediate level (see Figure 1). To introduce the regularized versions of (2.1) and (2.2), we take a smooth approximation $(k^\varepsilon, \varepsilon > 0)$ of k, which will be specified late on. The regularized microscopic N-particle system $\mathbf{X}_t^{N,\varepsilon} := (X_t^{1,\varepsilon}, \ldots, X_t^{N,\varepsilon})$ is given by

$$\mathrm{d}X_t^{i,\varepsilon} = -\frac{1}{N}\sum_{j=1}^{N} k^\varepsilon(X_t^{i,\varepsilon} - X_t^{j,\varepsilon})\,\mathrm{d}t + \sigma\,\mathrm{d}B_t^i, \quad i = 1, \ldots, N, \ \ \mathbf{X}_0^{N,\varepsilon} \sim \bigotimes_{i=1}^{N} \rho_0, \tag{2.3}$$

and the regularized mean-field trajectories $\mathbf{Y}_t^{N,\varepsilon} := (Y_t^{1,\varepsilon}, \ldots, Y_t^{N,\varepsilon})$ by

$$\mathrm{d}Y_t^{i,\varepsilon} = -(k^\varepsilon * \rho_t^\varepsilon)(Y_t^{i,\varepsilon})\,\mathrm{d}t + \sigma\,\mathrm{d}B_t^i, \quad i = 1, \ldots, N, \ \ \mathbf{Y}_0^{N,\varepsilon} = \mathbf{X}_0^{N,\varepsilon}, \tag{2.4}$$

where $\rho_t^\varepsilon := \rho^\varepsilon(t, \cdot)$ denotes the probability density of any of the i.i.d. random variables $Y_t^{i,\varepsilon}$.

Moreover, for $i = 1, \ldots, N$, it is convenient to denote the regularized interaction force $K_i^\varepsilon \colon \mathbb{R}^{dN} \to \mathbb{R}^d$ as

$$K_i^\varepsilon(x_1, \ldots, x_N) := -\frac{1}{N}\sum_{j=1}^{N} k^\varepsilon(x_i - x_j), \quad (x_1, \ldots, x_N) \in \mathbb{R}^{dN}, \tag{2.5}$$

and the mean-field interaction force $\overline{K_{t,i}^\varepsilon} : \mathbb{R}^{dN} \to \mathbb{R}^d$ as

$$\overline{K_{t,i}^\varepsilon}(x_1, \ldots, x_N) := -(k^\varepsilon * \rho_t^\varepsilon)(x_i), \quad (x_1, \ldots, x_N) \in \mathbb{R}^{dN}, \tag{2.6}$$

where ρ_t^ε is the law of $Y_t^{i,\varepsilon}$.

2.1.2. Diffusion-aggregation equations. The associated probability densities of the particle systems, introduced in Subsection 2.1.1, satisfy non-linear, non-local partial differential equations. Indeed, the particle system (2.2) induces the non-linear diffusion-aggregation equation

$$\begin{cases} \partial_t \rho(t,x) = \frac{\sigma^2}{2}\Delta\rho(t,x) + \nabla\cdot((k*\rho)(t,x)\rho(t,x)), & \forall\,(t,x) \in [0,T]\times\mathbb{R}^d, \\ \rho(x,0) = \rho_0, & \forall\, x \in \mathbb{R}^d, \end{cases} \tag{2.7}$$

and the regularized particle system (2.4) the diffusion-aggregation equation

$$\begin{cases} \partial_t \rho^\varepsilon(t,x) = \frac{\sigma^2}{2}\Delta\rho^\varepsilon(t,x) + \nabla\cdot((k^\varepsilon*\rho^\varepsilon)(t,x)\rho^\varepsilon(t,x)), & \forall\,(t,x) \in [0,T]\times\mathbb{R}^d, \\ \rho^\varepsilon(x,0) = \rho_0, & \forall\, x \in \mathbb{R}^d. \end{cases} \tag{2.8}$$

Note that we use ρ_t and ρ_t^ε for the solutions of the PDEs (2.7) and (2.8) as well as for the probability densities of the particle systems (2.2) and (2.4), respectively, since these objects will coincide by the superposition principle, see [BR20]. Thus, if we can solve the

PDE's (2.7), (2.8) the superposition principle provides weak solutions to the McKean–Vlasov SDE's (2.2) and (2.4)

2.1.3. Assumption on initial condition and interaction kernel. Throughout this chapter we make the following assumptions on the interaction kernel k and the initial condition ρ_0 of the interacting particle system.

ASSUMPTION 2.1. *The interaction force kernel* $k\colon \mathbb{R}^d \to \mathbb{R}^d$ *satisfies*

$$k \in L^\infty(\mathbb{R}^d)$$

and the initial condition $\rho_0\colon \mathbb{R}^d \to \mathbb{R}$ *fulfils*

$$\rho_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, |x|^2 \,\mathrm{d}x), \quad \rho_0 \geq 0, \quad \textit{and} \quad \int_{\mathbb{R}^d} \rho_0(x)\,\mathrm{d}x = 1.$$

2.1.4. Outline of the method. The method of the chapter originated from the approach of D. Lazarovici and P. Pickl, developed for the Vlasov–Poisson system in [LP17]. It is based on the coupling method [Szn91] and a regularization of k to k^ε. A key insight of D. Lazarovici and P. Pickl is to prove the convergence in probability with an arbitrary large algebraic rate and algebraic cut-off parameter $\varepsilon \sim N^{-\beta}$, $\beta > 0$, instead of comparing the trajectories $\mathbf{X}^{N,\varepsilon}$ and $\mathbf{Y}^{N,\varepsilon}$ in Wasserstein distance or in L^2-norm, as for instance done in [Szn91, CCH14]. More precisely, for $\alpha \in (0, 1/2)$, $\beta \leq \alpha$ and arbitrary $\gamma > 0$, we shall show that

$$\mathbb{P}\left(\sup_{t\in[0,T]} |\mathbf{X}_t^{N,\varepsilon} - \mathbf{Y}_t^{N,\varepsilon}|_\infty \geq N^{-\alpha}\right) \leq C(\gamma) N^{-\gamma}, \quad \text{for each } N \geq N_0.$$

To implement this strategy and to achieve the aforementioned result, we proceed as follows:

(1) We start with an analysis of the diffusion-aggregation equations (2.7) and (2.8), that is, we prove the well-posedness of the non-local, non-linear PDEs (2.7) and (2.8), together with an $L^\infty([0,T]; L^\infty(\mathbb{R}^d))$-bound on the solution ρ^ε, which is uniform in ε. These results can be obtained via standard PDE techniques such as a compactness method, Aubin–Lions lemma, which provides strong convergence, and a Moser type iteration, see Section 2.2. The uniform bound allows us to have a trade-off between the irregularity of the interaction force kernel and the regularity of the solution ρ^ε.

(2) The main idea of D. Lazarovici and P. Pickl was to recognize that even though the interaction force kernel is not globally Lipschitz continuous, the approximation k^ε satisfies a local Lipschitz bound of order ε^{-d} for $|x - y| \leq 2\varepsilon$, i.e.

$$|k^\varepsilon(x) - k^\varepsilon(y)| \leq l^\varepsilon(y)|x - y|. \tag{2.9}$$

Let us emphasize that the bound depends only on the point y. Hence, the above inequality seems like a Taylor expansion around the point y, where the second order term is missing. Consequently, the bound cannot be achieved by a simple application of the mean-value theorem.

We will assume that the interaction force kernel k satisfies (2.9), see Assumption 2.13 below, and present various examples of such kernels in Section 2.3. We refer to [LP17, CCS19, HLL19] for further models with interaction force kernels satisfying (2.9). In general, whether (2.9) holds true, depends entirely on the interaction force kernel of the considered model, in particular, on the order of discontinuity/singularity of the kernel. Hence, as rule of thumb, if the discontinuity/singularity is of order ε^{-d+1} in a d-dimensional setting, then the local Lipschitz bound assumption can be satisfied.

(3) We need to derive a law of large numbers, see Section 2.4. This allows us to treat every involved object with regard to its expectation on a set with high probability, which enables us to take advantage of the obtained regularity of ρ^ε in Step (1). Unsurprisingly, we need i.i.d. objects to apply the derived law of large numbers. In the present case these objects are going to be the processes $(Y_t^{i,\varepsilon}, t \geq 0)$ for $i \in \mathbb{N}$. Moreover, we would like to emphasize the importance of Step (2) at this moment and the crucial fact that $l^\varepsilon(y)$ only depends on the point y. Replacing in inequality (2.9) the point y with the process $Y_t^{i,\varepsilon}$ and x with the process $X_t^{i,\varepsilon}$, we see that l^ε on the right-hand side of (2.9) is depending on the i.i.d. process $Y_t^{i,\varepsilon}$. Consequently, we can rely on the law of large numbers, Proposition 2.21.

(4) Finally, let us demonstrate how to apply the previous steps to derive propagation of chaos in probability but leaving out the technical difficulties. To that end, for some $\alpha \in (0, 1/2)$ and $\delta > 0$, we define an auxiliary process

$$J_t^N := \min\left(1, N^\alpha |\mathbf{X}_t^{N,\varepsilon} - \mathbf{Y}_t^{N,\varepsilon}|_\infty + N^{-\delta}\right).$$

This process seems to control the difference $|\mathbf{X}_t^{N,\varepsilon} - \mathbf{Y}_t^{N,\varepsilon}|_\infty$ in the limit $N \to \infty$ with weight N^α. Furthermore, the minimum is no restriction, since we only want to show convergence to zero in probability, and we notice that, if $N^\alpha |\mathbf{X}_t^{N,\varepsilon} - \mathbf{Y}_t^{N,\varepsilon}|_\infty$ is too big, the process stays constant one and the time derivative is zero. Therefore, we heuristically obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(N^\alpha |\mathbf{X}_t^{N,\varepsilon} - \mathbf{Y}_t^{N,\varepsilon}|_\infty + N^{-\delta}) \\
&\quad \leq N^\alpha \sup_{i=1,\dots,N} |K_i^\varepsilon(\mathbf{X}_t^{N,\varepsilon}) - \overline{K_{t,i}^\varepsilon}(\mathbf{Y}_t^{N,\varepsilon})| \\
&\quad \leq N^\alpha \sup_{i=1,\dots,N} |K_i^\varepsilon(\mathbf{X}_t^{N,\varepsilon}) - K_i^\varepsilon(\mathbf{Y}_t^{N,\varepsilon}))| + N^\alpha \sup_{i=1,\dots,N} |K_i^\varepsilon(\mathbf{Y}_t^{N,\varepsilon}) - \overline{K_{t,i}^\varepsilon}(\mathbf{Y}_t^{N,\varepsilon})|.
\end{aligned}$$

The last term depends on the i.i.d. particles $(Y_t^i, i = 1, \dots, N)$ and can be estimated via the law of large numbers, Proposition 2.21, with a rate of $N^{-\delta-\alpha}$. For the first term we can use the local Lipschitz bound (having in mind that the particles are close because of the minimum in the process) to complete a Gronwall argument. As mentioned before, the crucial point in this step is the fact that the local Lipschitz bound only depends on the i.i.d. particles $\mathbf{Y}_t^{N,\varepsilon}$ and not on the particles system $\mathbf{X}_t^{N,\varepsilon}$. This allows us to exchange the local Lipschitz bound $\frac{1}{N}\sum_{j=1}^N l^\varepsilon(Y_t^{i,\varepsilon} - Y_t^{j,\varepsilon})$ with its conditional expectation $l^\varepsilon * \rho_t^\varepsilon(Y_t^{i,\varepsilon})$. Using the regularity properties, obtained from the PDE analysis in Step (1), we can bound $\|k^\varepsilon * \rho^\varepsilon\|_{L^\infty([0,T];L^\infty(\mathbb{R}^d))}$. Hence, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}(N^\alpha |\mathbf{X}_t^{N,\varepsilon} - \mathbf{Y}_t^{N,\varepsilon}|_\infty + N^{-\delta}) \leq C(N^\alpha |\mathbf{X}_t^{N,\varepsilon} - \mathbf{Y}_t^{N,\varepsilon}|_\infty + N^{-\delta}).$$

Applying Gronwall's lemma completes the proof. We remark that we implicitly used the fact that the law of large numbers holds for large $N \in \mathbb{N}$ and, consequently, the above Gronwall inequality only holds in the limit $N \to \infty$. In the actual proof we will use a version of the process J_t^N which is multiplied by an exponential, which just leads to a rewriting of the above Gronwall argument.

The remaining of the chapter is devoted to establish Step (1)-(4) with all technical details for bounded interaction force kernels.

Our contribution: In the present chapter we establish the approach of Lazarovici and Pickl [LP17] in a general setting allowing for interacting particle systems and diffusion-aggregation equations with bounded interaction force kernels which can be approximated in a suitable manner by smooth kernels. One main objective is to provide a transparent road map how to utilize this approach. To that end, we give a brief summary of the approach and explain its core concepts.

The first contribution is to demonstrate the well-posedness of the diffusion-aggregation equation (2.7), which is derived from the interacting particle system (2.1), for bounded interaction force kernels k. The main challenge lies in the non-linearity in the transport term, which is treated by a strong-weak convergence argument provided by Aubin's lemma A.31. The presented well-posedness result expands previous existence results regarding similar partial differential equations, for instance, regarding bounded confidence models [Lor07, CJLW17] used in social science.

The second contribution is to provide L^p and L^∞-estimates for the solution ρ through a Moser iteration. Following [LP17], we introduce a uniform local Lipschitz assumption, see Assumption 2.13 below. For instance, we verify that models for the opinion formation of interacting agents, such as the Hegselmann–Krause model [HK02], satisfy this uniform local Lipschitz assumption. As a rule of thumb, Assumption 2.13 is fulfilled by interaction force kernels with jump/singularity that are at most the same order as the space dimension.

As third contribution, we establish propagation of chaos in probability supposing the local Lipschitz assumption for the bounded interaction force kernel k. This is achieved by proving a suitable law of large numbers, demonstrating the convergence of the regularized particle system to the regularized mean-field system in a suitable topology and, subsequently, proving the convergence of the regularized probability density ρ^ε to the probability density ρ as $\varepsilon \to 0$.

Related literature: The influential approach of Lazarovici and Pickl [LP17] deals with the Vlasov–Poisson system, which is a second order system with a singular interaction force kernel k. For the Vlasov–Poisson kernel, the underlying particle system is a priori not well-posed. Therefore, a regularization k^ε of the kernel k is introduced in [LP17], where k^ε is a smooth approximation of the interaction force kernel k depending on the number of particles ($\varepsilon = N^{-\beta}$), such that the interacting particle system is well-posed. Nowadays, the aforementioned approach is widely used, for instance, for the Keller–Segel equation [HLL19, LY19, FHS19], the Cucker–Smale model with singular communication [HKPZ19] and the Vlasov–Poisson–Fokker–Planck equation [CCS19, HLP20, CLPY20]. All of the above system have singular interactions and therefore could collapse, as e.g. the Keller–Segel system [FJ17, Proposition 4]. Hence, an advantage of the method [LP17] is that well-posedness of the underlying particle system is not required since one works directly with the regularized/approximative particle system using the kernel k^ε for which one can apply classical existence and uniqueness theorems for SDE's. Moreover, the approach of Lazarovici and Pickl allows to show propagation of chaos of the regularized particle systems to the regularized mean-field equation. This acts as an intermediate result, as illustrated by the top arrow in Figure 1. However, the remaining limit of the regularized mean-field equation to the mean-field equation (right arrow of Figure 1) is reduced to a convergence analysis on the PDE level. On the other hand, the convergence of the regularized particle system to the non-regularized particles system (left arrow of Figure 1) only requires a stability analysis on the SDE level, which still is, at least in general, a challenging task.

Let us also recall some literature for singular kernels, which do not tackle the problem with the method of convergence in probability. For a more details we refer to the Section 1.3.4 and Chapter 3. Motivated by various models arising especially in physics, which require bounded measurable or even singular interaction force kernels, an enormous amount of work has been dedicated to treat such irregular interaction force kernels. Initially, approaches to treat such irregular kernels were often based on compactness methods in combination with the martingale problems associated to the McKean–Vlasov SDEs, see e.g. [Oel84, Osa87, Gär88, FJ17, GQ15]. More recently, even singular kernels, like the Coulomb potential $x/|x|^s$ for $s > 0$, were investigated in the non-random setting [Ser20, NRS22] ($\sigma = 0$) as well as in a random setting [JW18, BJW19, BJW23, RS23] ($\sigma > 0$). The aforementioned references introduced a novel method called the modulated energy approach, see Section 1.3.4, Chapter 3 or Chapter 5 in the case of common noise. Further results on propagation of chaos were proven for general L^p-interaction force kernels k for first and second order systems on the torus [BJW23] and on the whole space $\mathbb{R}^d$ [HRZ24, Han23, Lac23].

Organization of the chapter: In Section 2.1 we introduce the notation, the interacting particle systems and their associated diffusion-aggregation equations. Moreover, we present a brief outline of the used method, building on the work of D. Lazarovici and P. Pickl [LP17]. The well-posedness and regularity properties of the diffusion-aggregation equations are established in Section 2.2. In Section 2.3 we discuss the local Lipschitz assumption on the approximative interaction force kernels and provide various examples. Section 2.4 contains the law of large numbers and the propagation of chaos in probability is provided in Section 2.5.

2.2. Well-posedness and uniform bounds for the diffusion-aggregation equations

In this section we prove well-posedness of the PDEs (2.7) and (2.8), show the convergence of the solutions $(\rho^\varepsilon, \varepsilon > 0)$ to ρ in the weak topology, and provide regularity results as well as uniform bounds for $(\rho^\varepsilon, \varepsilon > 0)$ and ρ, which are required for propagation of chaos result in probability established later in Section 2.5. We start by introducing an assumption on the approximation sequence $(k^\varepsilon, \varepsilon > 0)$ of interaction force kernels.

Assumption 2.2. *Let $(k^\varepsilon, \varepsilon > 0)$ be a sequence, which satisfies the following:*

(i) For each $\varepsilon > 0$ the interaction force kernel $k^\varepsilon \colon \mathbb{R}^d \mapsto \mathbb{R}^d$ is twice continuously differentiable;
(ii) For each $\varepsilon > 0$ we have $\|k^\varepsilon\|_{L^\infty(\mathbb{R}^d)} \le C \|k\|_{L^\infty(\mathbb{R}^d)} < \infty$;
(iii) We have $\lim_{\varepsilon \to 0} k^\varepsilon = k$ a.e.

Under Assumptions 2.1, 2.2 the strong existence and uniqueness of (2.3) and (2.4) follow directly from the Lipschitz continuity of the approximation k^ε.

For the non-linear, non-local PDE (2.8) we notice that, by Young's inequality, we obtain the following $L^\infty(\mathbb{R}^d)$-bound

$$|(k^\varepsilon * \rho^\varepsilon)(t,x)| \le \|k^\varepsilon\|_{L^\infty(\mathbb{R}^d)} \|\rho_t^\varepsilon\|_{L^1(\mathbb{R}^d)} \le C \|k\|_{L^\infty(\mathbb{R}^d)} . \tag{2.10}$$

Hence, $k^\varepsilon * \rho$ is uniformly bounded in $\varepsilon > 0$ on $[0,T] \times \mathbb{R}^d$. The same statement holds for $k * \rho$. Consequently, the convolution term is bounded and we expect the existence of a weak solution to the PDEs (2.7) and (2.8).

For the partial differential equations (2.7) and (2.8) we rely on the concept of weak solutions, which we recall in the next definition.

Definition 2.3 (Weak solutions). *We say that the function $\rho^\varepsilon \in L^2([0,T]; H^1(\mathbb{R}^d)) \cap L^\infty([0,T]; L^2(\mathbb{R}^d))$ with $\frac{\mathrm{d}}{\mathrm{d}t}\rho^\varepsilon \in L^2([0,T]; H^{-1}(\mathbb{R}^d))$ is a weak solution of* (2.8) *if, for every $\eta \in L^2([0,T]; H^1(\mathbb{R}^d))$,*

$$\begin{aligned} &\int_0^T \langle \partial_t \rho_t^\varepsilon, \eta \rangle_{H^{-1}(\mathbb{R}^d), H^1(\mathbb{R}^d)} \, \mathrm{d}t \\ &= - \int_0^T \int_{\mathbb{R}^d} \left(\frac{\sigma^2}{2} \nabla \rho^\varepsilon(t,x) + (k^\varepsilon * \rho^\varepsilon)(t,x) \rho^\varepsilon(t,x) \right) \cdot \nabla \eta \, \mathrm{d}x \, \mathrm{d}t \end{aligned} \tag{2.11}$$

and $\rho^\varepsilon(0,\cdot) = \rho_0$. Similarly, we define $\rho \in L^2([0,T]; H^1(\mathbb{R}^d)) \cap L^\infty([0,T]; L^2(\mathbb{R}^d))$ with $\partial_t \rho \in L^2([0,T]; H^{-1}(\mathbb{R}^d))$ as a weak solution of (2.7) *if* (2.11) *holds with the interaction force kernel k instead of its approximation k^ε.*

Remark 2.4. *Notice that $\rho^\varepsilon \in L^2([0,T]; H^1(\mathbb{R}^d))$ with $\partial_t \rho^\varepsilon \in L^2([0,T]; H^{-1}(\mathbb{R}^d))$ implies $\rho^\varepsilon \in C([0,T]; L^2(\mathbb{R}^d))$, see* [Eva10, Chapter 5.9]*. The same statement holds for the solution ρ.*

By the regularity of the solution in Definition 2.3 we can actually weaken the assumption on η in equation (2.11) to $\eta \in C([0,T]; C_c^\infty(\mathbb{R}^d))$.

Remark 2.5. *The divergence structure of the PDEs* (2.7) *and* (2.8)*, respectively, implies mass conservation/the normalisation condition*

$$1 = \int_{\mathbb{R}^d} \rho_t(x) \, \mathrm{d}x = \int_{\mathbb{R}^d} \rho_t^\varepsilon(x) \, \mathrm{d}x, \quad t \in [0,T],$$

under Assumption 2.1. This is an immediate consequence by plugging in a cut-off sequence, see [Bre11, Lemma 8.4]*, which converges to the constant function* 1 *as a test function in* (2.11).

Theorem 2.6. *Suppose Assumption 2.1. Then, for each $T > 0$ and $\varepsilon > 0$ there exists a unique non-negative weak solution $\rho^\varepsilon \in L^2([0,T]; H^1(\mathbb{R}^d)) \cap L^\infty([0,T]; L^2(\mathbb{R}^d))$ with $\frac{\mathrm{d}}{\mathrm{d}t}\rho^\varepsilon \in L^2([0,T]; H^{-1}(\mathbb{R}^d))$ to the regularized PDE* (2.8) *in the sense of Definition 2.3. Moreover, the estimate*

$$\begin{aligned} &\|\rho^\varepsilon\|_{L^\infty([0,T];L^2(\mathbb{R}^d))} + \|\rho^\varepsilon\|_{L^2([0,T];H^1(\mathbb{R}^d))} + \|\partial_t \rho^\varepsilon\|_{L^2([0,T];H^{-1}(\mathbb{R}^d))} \\ &\le C\big(T, \|k\|_{L^\infty(\mathbb{R}^d)}\big) \|\rho_0\|_{L^2(\mathbb{R}^d)} \end{aligned} \tag{2.12}$$

holds for all $\varepsilon > 0$.

Proof. Let us explain the main idea of the existence proof. We consider the associated McKean–Vlasov process

$$\begin{cases} \mathrm{d}Y_t^\varepsilon &= -(k^\varepsilon * \rho_t^\varepsilon)(Y_t^\varepsilon)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t^1, \;\; Y_0 \sim \rho_0, \\ \rho_t^\varepsilon &= \mathrm{Law}(Y_t^\varepsilon), \end{cases}$$

for the initial condition ρ_0. Then, by [KS91, Chapter 5. Theorem 2.9] and the Lipschitz continuity of the drift $k^\varepsilon * \rho_t^\varepsilon$, the aforementioned SDE has a unique strong solution and, by [Rom18, Proposition 3.1], it has a density $(\rho_t^\varepsilon, t \geq 0)$. Now, fix ρ_t^ε and consider the solution $\tilde{\rho}^\varepsilon = (\tilde{\rho}_t^\varepsilon, t \geq 0)$ to the linearized parabolic PDE

$$\begin{cases} \partial_t \tilde{\rho}^\varepsilon(t,x) = \frac{\sigma^2}{2}\Delta\tilde{\rho}^\varepsilon(t,x) + \nabla\cdot((k^\varepsilon * \rho^\varepsilon)(t,x)\tilde{\rho}^\varepsilon(t,x)), & \forall\,(t,x) \in [0,T]\times\mathbb{R}^d, \\ \quad \tilde{\rho}^\varepsilon(x,0) = \rho_0, & \forall\, x \in \mathbb{R}^d. \end{cases}$$

By standard second order parabolic PDE theory, we know that the aforementioned PDE is well-posed and

$$\tilde{\rho}^\varepsilon \in L^2([0,T];H^1(\mathbb{R}^d)) \cap L^\infty([0,T];L^2(\mathbb{R}^d)), \quad \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\rho}_t^\varepsilon \in L^2([0,T];H^{-1}(\mathbb{R}^d)),$$

with the estimate (2.12). Applying the superposition principle [BR20, Theorem 4.1], we find a weak solution to

$$\mathrm{d}\tilde{Y}_t^\varepsilon = -(k^\varepsilon * \rho_t^\varepsilon)(\tilde{Y}_t^\varepsilon)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t^1, \;\; \tilde{Y}_0^N \sim \rho_0, \quad t \in [0,T],$$

with $\mathrm{Law}(\tilde{Y}_t^\varepsilon) = \tilde{\rho}_t^\varepsilon\,\mathrm{d}x$. Since strong uniqueness holds [HRZ24, Theorem 4.10] for the above SDE, we have $\tilde{Y}^\varepsilon = Y^\varepsilon$. By the Yamada–Watanabe theorem [KS91, Chapter 5, Proposition 3.20] this implies uniqueness in law and therefore

$$\tilde{\rho}_t^\varepsilon\,\mathrm{d}x = \rho_t^\varepsilon\,\mathrm{d}x, \quad t \in [0,T],$$

in the sense of measures. Hence, $\tilde{\rho}_t^\varepsilon = \rho_t^\varepsilon$ $\mathbb{P}$-a.s. for all $t \in [0,T]$ and ρ^ε has the desired regularity. □

Lemma 2.7. *Suppose Assumption 2.1. Consider a solution ρ^ε of the regularized diffusion-aggregation equation* (2.8) *with initial condition ρ_0, which by Theorem 2.6 exists. Then, we have the following uniform bound*

$$\int_{\mathbb{R}^d} |x|^2\rho^\varepsilon(t,x)\,\mathrm{d}x \leq C\big(T, \|k\|_{L^\infty(\mathbb{R}^d)}\big)\int_{\mathbb{R}^d}|x|^2\rho_0(x)\,\mathrm{d}x$$

for all $t \geq 0$, which depend only upon $\int_{\mathbb{R}^d}(1+|x|^2)\rho_0(x)\,\mathrm{d}x$ and T. Therefore, the function $t \mapsto \int_{\mathbb{R}^d}|x|^2\rho^\varepsilon(t,x)\,\mathrm{d}x$ is bounded in $L^\infty([0,T];\mathbb{R}^d)$.

Proof. The core idea is to use $|x|^2$ as a test function. To that end, we take a sequence of radial antisymmetric functions $(g_n, n \in \mathbb{N})$ with $g_n \in C_c^2(\mathbb{R}^d)$ for all $n \in \mathbb{N}$, such that g_n grows to $|x|^2$ as $n \to \infty$. More precisely, we choose the non-negative function

$$\chi_n(x) := \begin{cases} |x|, & \text{for } |x| \geq \frac{1}{n}, \\ -n^3\frac{|x|^4}{8} + n\frac{3|x|^2}{4} + \frac{3}{8n}, & \text{for } |x| \leq \frac{1}{n}, \end{cases}$$

and let $(\zeta_n, n \in \mathbb{N})$ be a sequence of compactly supported cut-off function defined by $\zeta_n(x) = \zeta(x/n)$, where ζ is a smooth function with support in the ball of radius two and has value one in the unit ball. Notice that the derivatives of ζ_n have support in the annulus $\overline{B(0,2n)}\backslash B(0,n)$ and satisfy the inequality $|\nabla^m \zeta_n(x)| \leq C n^{-m}$. Define $g_n := \zeta_n \chi_n^2$. We notice that the first derivative of g_n is bounded and $|\nabla g_n|^2 \leq C g_n$ for all $n \in \mathbb{N}$. Additionally we have

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}(\chi_n^2(x)) = \frac{\partial}{\partial x_i}\left(2\chi_n(x)\frac{\partial}{\partial x_i}\chi_n(x)\right) = 2\left|\frac{\partial}{\partial x_i}\chi_n(x)\right|^2 + 2\chi_n(x)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}\chi_n(x).$$

Now, for $|x| \geq 1/n$ we have

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}\chi_n(x) = \frac{\partial}{\partial x_i}\frac{x_i}{|x|} = \frac{1}{|x|} - \frac{x_i^2}{|x|^3}.$$

Consequently, we applying Hölder's inequality for sums we obtain

$$|\Delta(\chi_n^2(x))| \leq 2 + 2d^{\frac{1}{2}} + \frac{2}{|x|^2}\sum_{i=1}^{d}|x_i|^2 = 2d^{\frac{1}{2}} + 4.$$

For $|x| \leq 1/n$ we have

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}\chi_n(x) = \frac{\partial}{\partial x_i}\left(-4n^3|x|^2 x_i + \frac{3n}{2}x_i\right) = -8n^3 x_i^2 - 4n^2|x|^2 + \frac{3n}{2},$$

which implies

$$\begin{aligned}|\Delta(\chi_n^2(x))| &= 16n^6|x|^6 + \frac{9n^2}{4}|x|^2 + 2\chi_n(x)\left(-8n^3|x|^2 - 4dn^2|x|^2 + \frac{3dn}{2}\right)\\ &\leq 16 + \frac{9}{4} + \frac{4}{n}\left(8n + 4d + \frac{3dn}{2}\right)\\ &\leq C\end{aligned}$$

for a constant independent of $n \in \mathbb{N}$. Consequently, the Laplace operator of χ^2 is bounded uniformly in $n \in \mathbb{N}$. Then, for $\varphi \in C_c^\infty(0,T)$ we obtain

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^d} g_n(x)\rho^\varepsilon(t,x)\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t)\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^T \int_{\mathbb{R}^d}\left(\frac{\sigma^2}{2}\nabla\rho^\varepsilon(t,x)\cdot\nabla g_n(x) + (k^\varepsilon * \rho^\varepsilon)\nabla g_n(x)\rho^\varepsilon(t,x)\right)\varphi(t)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Furthermore, for $t_1, t_2 \in [0,T]$ we have

$$\left|\int_{\mathbb{R}^d} g_n(x)\rho^\varepsilon(t_1,x)\,\mathrm{d}x - \int_{\mathbb{R}^d} g_n(x)\rho^\varepsilon(t_2,x)\,\mathrm{d}x\right| \leq \|g_n\|_{L^2(\mathbb{R}^d)}\|\rho^\varepsilon(t_1,\cdot) - \rho^\varepsilon(t_2,\cdot)\|_{L^2(\mathbb{R}^d)}.$$

Therefore, $\rho^\varepsilon \in C([0,T]; L^2(\mathbb{R}^d))$ implies that $t \mapsto \int_{\mathbb{R}^d} g_n(x)\rho^\varepsilon(t,x)\,\mathrm{d}x$ is continuous for each $n \in \mathbb{N}$. Then, the fundamental lemma of calculus of variations implies

$$\begin{aligned}&\int_{\mathbb{R}^d} g_n(x)\rho^\varepsilon(t,x)\,\mathrm{d}x - \int_{\mathbb{R}^d} g_n(x)\rho_0(x)\,\mathrm{d}x\\ &= -\int_0^t \int_{\mathbb{R}^d}\frac{\sigma^2}{2}\nabla\rho^\varepsilon(s,x)\cdot\nabla g_n(x) + (k^\varepsilon * \rho^\varepsilon)(s,x)\cdot\nabla g_n(x)\rho^\varepsilon(s,x)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

We can use mass conservation, Sobolev embedding and (2.12) to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} g_n(x)\rho^\varepsilon(t,x)\,\mathrm{d}x \\
&\le \int_{\mathbb{R}^d} |x|^2\rho_0(x)\,\mathrm{d}x + CT\,\|k\|^2_{L^\infty(\mathbb{R}^d)} + \int_0^t \int_{\mathbb{R}^d} \rho^\varepsilon_s(x)|\nabla g_n(x)|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \frac{\sigma^2}{2}\int_0^t \int_{\mathbb{R}^d} \left|\Delta\zeta_n(x)\chi_n^2(x) + 4\nabla\zeta_n(x)\cdot\nabla\chi_n(x)\chi_n(x) + \zeta_n\Delta(\chi_n^2(x))\right| \rho^\varepsilon(s,x)\,\mathrm{d}x\,\mathrm{d}s \\
&\le \int_{\mathbb{R}^d} |x|^2\rho_0(x)\,\mathrm{d}x + CT\,\|k\|^2_{L^\infty(\mathbb{R}^d)} + \int_0^t \int_{\mathbb{R}^d} \rho^\varepsilon_s(x)g_n(x)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \frac{\sigma^2}{2}\int_0^T \int_{B(0,2n)\setminus B(0,n)} \frac{C}{n^2}|\chi_n^2(x)| + \frac{C}{n}|\chi_n(x)|\rho^\varepsilon_s(x)\,\mathrm{d}x\,\mathrm{d}s + \frac{\sigma^2}{2}\int_0^T \int_{\mathbb{R}^d} \rho^\varepsilon_s(x)\,\mathrm{d}x\,\mathrm{d}s \\
&\le \int_{\mathbb{R}} |x|^2\rho_0(x)\,\mathrm{d}x + CT\,\|k\|^2_{L^\infty(\mathbb{R}^d)} + \int_0^t \int_{\mathbb{R}^d} \rho^\varepsilon_s(x)g_n(x)\,\mathrm{d}x + C.
\end{aligned}$$

Applying Gronwall's lemma and Fatou's lemma proves the lemma. □

LEMMA 2.8. *Let $(f_n, n\in\mathbb{N})$ be a sequence in $L^2([0,T];H^1(\mathbb{R}^d))$. If the sequence satisfies*

(i) $\|f_n\|_{L^2([0,T];H^1(\mathbb{R}^d))} \le C$,
(ii) $\|\partial_t f_n\|_{L^2([0,T];H^{-1}(\mathbb{R}^d))} \le C$,
(iii) $\sup\limits_{t\in[0,T]} \int_{\mathbb{R}^d} |x|^2|f_n(t,x)|\,\mathrm{d}x \le C$,

for some constant $C>0$, then $(f_n, n\in\mathbb{N})$ is relative compact in $L^p([0,T];L^p(\mathbb{R}^d))$ for all $p\in[1,2]$.

PROOF. Let us denote by $B(0,R)$ the ball with radius R and center 0. Then, by the Rellich-Kondrachov theorem A.12 the embedding $H^1(B(0,R)) \hookrightarrow L^2(B(0,R))$ is compact. Hence, by Aubin–Lions lemma [Sho97, Chapter 3, Proposition 1.3], $(f_n, n\in\mathbb{N})$ is relative compact in $L^2([0,T];L^2(B(0,R)))$. Since the above spaces is of finite measure, we obtain the relative compactness of $(f_n, n\in\mathbb{N})$ in $L^p([0,T];L^p(B(0,R)))$ for all $p\in[1,2]$. Note that by Cantor's diagonal argument we can extract one subsequence $(f_{n_k}, k\in\mathbb{N})$ such that

$$\lim_{k\to\infty} \|f_{n_k} - f_R\|_{L^p([0,T];L^p(B(0,R)))} = 0.$$

for some limit point $f_R \in L^p([0,T];L^p(B(0,R)))$ and all $p\in[1,2]$, $R\in\mathbb{N}$. Furthermore, using again a Cantor's diagonal argument, we can assume that $(f_{n_k}, k\in\mathbb{N})$ converges almost everywhere to f_R in $B(0,R)$. For $x\in\mathbb{R}^d$ we choose some $R\in\mathbb{N}$ such that $x\in B(0,R)$ and define $f(x) := f_R(x)$. This definition is well-defined, since for $x\in B(0,R)\subset B_{R'}$ we have $f_R = f_{R'}$ on $B(0,R)$ by the local L^p convergence of the sequence $(f_{n_k}, k\in\mathbb{N})$. Consequently, the sequence $(f_{n_k}, k\in\mathbb{N})$ convergence almost everywhere to f on $\mathbb{R}^d$. It remains to prove that $f\in L^p([0,T];L^p(\mathbb{R}^d))$ and $f_{n_k}\to f \in L^p([0,T];L^p(\mathbb{R}^d))$ as $k\to\infty$. First, the uniform second moment estimate and the uniform $L^2(\mathbb{R}^d)$ bound in combination with Fatou's lemma imply $f \in L^p([0,T];L^p(\mathbb{R}^d))$ and

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^d} |x|^2 f(t,x)\,\mathrm{d}x \le \sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \int_{\mathbb{R}^d} |x|^2 f_n(t,x)\,\mathrm{d}x \le C.$$

Second, we find

$$\begin{aligned}
&\|f_{n_k} - f\|_{L^1([0,T];L^1(\mathbb{R}^d))} \\
&\quad = \int_0^T \int_{\mathbb{R}^d} |f_{n_k}(t,x) - f(t,x)|\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^T \int_{B(0,R)} |f_{n_k}(t,x) - f(t,x)|\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{B(0,R)^{\mathrm{c}}} |f_{n_k}(t,x) - f(t,x)|\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le \int_0^T \int_{B(0,R)} |f_{n_k}(t,x) - f(t,x)|\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{R^2} \sup_{k\in\mathbb{N}} \int_0^T \int_{B(0,R)^{\mathrm{c}}} |x|^2 |f_{n_k}(t,x) - f(t,x)|\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Taking $k \to \infty$ and then $R \to \infty$, we find a subsequence $(f_{n_k}, k \in \mathbb{N})$ which converges in $L^1([0,T];L^1(\mathbb{R}^d))$. The uniform $L^p(\mathbb{R}^d)$-bound on $(f_n, n \in \mathbb{N})$ and an interpolation inequality A.3 shows the relative compactness of $(f_n, n \in \mathbb{N})$ in $L^p([0,T];L^p(\mathbb{R}^d))$. □

In the next theorem, we show that the approximation sequence $(\rho^\varepsilon, \varepsilon > 0)$ converges in the weak sense to a weak solution ρ of equation (2.7).

THEOREM 2.9. *Suppose Assumption 2.1. Then, for each $T > 0$ there exists a subsequence $(\rho^{\varepsilon_m}, m \in \mathbb{N})$ such that $\rho^{\varepsilon_m} \rightharpoonup \rho$ as $m \to \infty$ in $L^2([0,T];H^1(\mathbb{R}^d))$. Furthermore, $\rho \in L^2([0,T];H^1(\mathbb{R}^d)) \cap L^\infty([0,T];L^2(\mathbb{R}^d))$ with $\partial_t\rho \in L^2([0,T];H^{-1}(\mathbb{R}^d))$ is the unique non-negative weak solution of equation (2.7), which satisfies*

$$\begin{aligned}
&\|\rho\|_{L^\infty([0,T];L^2(\mathbb{R}^d))} + \|\rho\|_{L^2([0,T];H^1(\mathbb{R}^d))} + \|\partial_t\rho\|_{L^2([0,T];H^{-1}(\mathbb{R}^d))} \\
&\quad \le C\big(T, \|k\|_{L^\infty(\mathbb{R}^d)}\big) \|\rho_0\|_{L^2(\mathbb{R}^d)} .
\end{aligned} \tag{2.13}$$

In addition, there exists a subsequence $(\rho^{\varepsilon_m}, m \in \mathbb{N})$ such that $\rho^{\varepsilon_m} \to \rho$ converges weakly as $m \to \infty$ in $L^1([0,T];L^1(\mathbb{R}^d))$.

PROOF. From (2.12), the Banach–Alaoglu theorem A.28 and the lower semi-continuity we obtain (2.13) and a subsequence $(\rho^{\varepsilon_m}, m \in \mathbb{N})$ such that

$$\begin{aligned}
\rho^{\varepsilon_m} &\rightharpoonup \rho \quad \text{in } L^2([0,T];H^1(\mathbb{R}^d)), \\
\partial_t\rho^{\varepsilon_m} &\rightharpoonup \partial_t\rho \quad \text{in } L^2([0,T];H^{-1}(\mathbb{R}^d)).
\end{aligned}$$

Moreover, we have $\rho \ge 0$ a.e. by Mazur's lemma [Bre11, Corollary 3.8]. Next, we notice that the subsequence $(\rho^{\varepsilon_m}, m \in \mathbb{N})$ fulfills Lemma 2.8. Consequently, without renaming the subsequence we conclude

$$\lim_{m\to\infty} \|\rho^{\varepsilon_m} - \rho\|_{L^p([0,T];L^p(\mathbb{R}^d))} = 0 \tag{2.14}$$

for all $p \in [1,2]$. Hence, it remains to show that we can take the limit in (2.11). From the above weak convergence it immediately follows

$$\int_0^T \langle \partial_t \rho_t^{\varepsilon_m}, \eta \rangle_{H^{-1}(\mathbb{R}^d),H^1(\mathbb{R}^d)} \,\mathrm{d}t \to \int_0^T \langle \partial_t \rho_t, \eta \rangle_{H^{-1}(\mathbb{R}^d),H^1(\mathbb{R}^d)} \,\mathrm{d}t,$$
$$\int_0^T \int_{\mathbb{R}^d} \nabla \rho^{\varepsilon_m}(t,x) \cdot \nabla \eta(t,x) \,\mathrm{d}x \,\mathrm{d}t \to \int_0^T \int_{\mathbb{R}^d} \nabla \rho(t,x) \cdot \nabla \eta(t,x) \,\mathrm{d}x \,\mathrm{d}t$$

for $\eta \in L^2([0,T]; H^1(\mathbb{R}^d))$ as $m \to \infty$. We write the non-linear term as

$$\int_0^T \int_{\mathbb{R}^d} \rho^{\varepsilon_m}(t,x)(k^{\varepsilon_m} * \rho^{\varepsilon_m})(t,x) \cdot \nabla \eta(t,x) \,\mathrm{d}x \,\mathrm{d}t = I_1 + I_2 + I_3 + I_4 \tag{2.15}$$

with

$$I_1 = \int_0^T \int_{\mathbb{R}^d} (\rho^{\varepsilon_m} - \rho)(k^{\varepsilon_m} * \rho^{\varepsilon_m})(t,x) \cdot \nabla \eta(t,x) \,\mathrm{d}x \,\mathrm{d}t,$$
$$I_2 = \int_0^T \int_{\mathbb{R}^d} \rho((k^{\varepsilon_m} - k) * \rho)(t,x) \cdot \nabla \eta(t,x) \,\mathrm{d}x \,\mathrm{d}t,$$
$$I_3 = \int_0^T \int_{\mathbb{R}^d} \rho(k^{\varepsilon_m} * (\rho^{\varepsilon_m} - \rho))(t,x) \cdot \nabla \eta(t,x) \,\mathrm{d}x \,\mathrm{d}t,$$
$$I_4 = \int_0^T \int_{\mathbb{R}^d} \rho(k * \rho)(t,x) \cdot \nabla \eta(t,x) \,\mathrm{d}x \,\mathrm{d}t.$$

For the first term I_1 we notice that it vanishes as $m \to \infty$. Indeed, since $|k^{\varepsilon_m} * \rho^{\varepsilon_m}| \leq C \|k\|_{L^\infty(\mathbb{R}^d)}$ and $\nabla \eta \in L^2([0,T]; L^2(\mathbb{R}^d))$, we have $(k^{\varepsilon_m} * \rho^{\varepsilon_m}) \cdot \nabla n \in L^2([0,T]; L^2(\mathbb{R}^d))$ uniform in ε_m and, thus, $\rho^{\varepsilon_m} \to \rho$ in $L^2([0,T]; L^2(\mathbb{R}^d))$ implies

$$\int_0^T \int_{\mathbb{R}^d} (\rho^{\varepsilon_m} - \rho)(k^{\varepsilon_m} * \rho^{\varepsilon_m})(t,x) \cdot \nabla \eta(t,x) \,\mathrm{d}x \,\mathrm{d}t \to 0, \quad \text{as} \quad m \to \infty$$

by Hölder's inequality. For the second term I_2 we use Assumptions 2.2 to find

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} \rho(t,x)((k^{\varepsilon_m}-k)*\rho)(t,x)\cdot\nabla\eta(t,x)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho(t,x)(k^{\varepsilon_m}-k)(x-y)\rho(t,y)\cdot\nabla\eta(t,x)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le \int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\rho|\nabla\eta|)(t,x)\,\|k^{\varepsilon_m}+k\|_{L^\infty(\mathbb{R}^d)}\,\rho(t,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le (C+1)\int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\rho|\nabla\eta|)(t,x)\,\|k\|_{L^\infty(\mathbb{R}^d)}\,\rho(t,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

The right-hand side is finite and, therefore, by the dominated convergence theorem and the almost everywhere convergence of $k^\varepsilon \to k$, the second term vanishes. For the third term I_3 we apply Young's inequality [LL01, Theorem 4.2] and (2.14) to obtain

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} \rho(t,x)(k^{\varepsilon_m}*(\rho^{\varepsilon_m}-\rho)(t,x))\cdot\nabla\eta(t,x)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le \int_0^T \|\rho^{\varepsilon_m}-\rho(t,\cdot)\|_{L^1(\mathbb{R}^d)}\,\|k\|_{L^\infty(\mathbb{R}^d)}\,\|\rho\nabla\eta(t,\cdot)\|_{L^1(\mathbb{R}^d)}\,\mathrm{d}t \\
&\quad \le \|\eta\|_{L^2([0,T];H^1(\mathbb{R}^d))}\,\|\rho\|_{L^2([0,T];L^2(\mathbb{R}^d))}\,\|k\|_{L^\infty(\mathbb{R}^d)}\int_0^T \|\rho^{\varepsilon_m}-\rho(t,\cdot)\|_{L^1(\mathbb{R}^d)}\,\mathrm{d}t \\
&\quad \to 0, \quad \text{as } m\to\infty.
\end{aligned}$$

Consequently, taking the limit $m\to\infty$ in (2.15), we discover

$$\lim_{m\to\infty}\int_0^T \int_{\mathbb{R}^d} \rho^{\varepsilon_m}(t,x)(k^{\varepsilon_m}*\rho^{\varepsilon_m})(t,x)\cdot\nabla\eta(t,x)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\mathbb{R}^d} \rho(k*\rho)(t,x)\cdot\nabla\eta(t,x)\,\mathrm{d}x\,\mathrm{d}t$$

and therefore ρ is a weak solution. The uniqueness follows by simple L^2-estimates; see for instance [CJLW17, Theorem 3.10] in the case of the Hegselmann–Krause model (notice that the proof of the uniqueness also works for $\mathbb{R}^d$ and $k\in L^\infty(\mathbb{R}^d)$). □

Remark 2.10. *The uniqueness of the solution ρ actually implies that any subsequence convergences to the solution ρ.*

Lemma 2.11. *Suppose Assumption 2.1. Then, for any $T>0$ the weak solutions $(\rho^\varepsilon, \varepsilon>0)$ of (2.8) as well as the weak solution ρ of (2.7) with initial condition ρ_0 are bounded in*

$L^\infty([0,T];L^p(\mathbb{R}^d))$ *for* $p \in [1,\infty)$. *More precisely, we have, for all* $\varepsilon > 0$,

$$\|\rho^\varepsilon\|_{L^\infty([0,T];L^p(\mathbb{R}^d))}, \|\rho\|_{L^\infty([0,T];L^p(\mathbb{R}^d))} \le C\big(p,\sigma,T,\|k\|_{L^\infty(\mathbb{R}^d)}\big)\,\|\rho_0\|_{L^p(\mathbb{R}^d)}.$$

PROOF. Without loss of generality we show the claim only for ρ and we also may assume that ρ is a smooth solution. Otherwise we mollify the initial condition such that there exists a sequence of smooth solutions, which converge weakly in $L^2([0,T];H^1(\mathbb{R}^d))$ to ρ^ε for each fix $\varepsilon > 0$. Applying the lower semi-continuity for each $\varepsilon > 0$ first and then the convergence result in Theorem 2.9 will prove the lemma.

Multiplying (2.7) with $\frac{p}{2(p-1)}\rho^{p-1}$, integrating by parts over $\mathbb{R}^d$ and using inequality (2.10), we obtain

$$\begin{aligned}
&\frac{1}{2(p-1)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\rho^p(t,x)\,\mathrm{d}x\\
&= \frac{p}{2(p-1)}\int_{\mathbb{R}^d}\frac{\mathrm{d}}{\mathrm{d}t}\rho(t,x)\rho^{p-1}(t,x)\,\mathrm{d}x\\
&= \frac{p}{2(p-1)}\int_{\mathbb{R}^d}\Big(\frac{\sigma}{2}\Delta\rho(t,x) + \nabla\cdot((k*\rho)(t,x)\rho(t,x))\Big)\rho^{p-1}(t,x)\,\mathrm{d}x\\
&= \frac{p}{2}\int_{\mathbb{R}^d} -\frac{\sigma}{2}|\nabla\rho(t,x)|^2\rho^{p-2}(t,x) - (k*\rho)(t,x)\cdot\nabla\rho\rho^{p-1}(t,x)\,\mathrm{d}x\\
&= -\frac{\sigma^2}{p}\int_{\mathbb{R}^d}|\nabla(\rho^{p/2})(t,x)|^2\,\mathrm{d}x - \int_{\mathbb{R}^d}\nabla(\rho^{p/2})(t,x)\cdot(k*\rho)(t,x)\rho^{p/2}(t,x)\,\mathrm{d}x\\
&\le -\frac{\sigma^2}{p}\int_{\mathbb{R}^d}|\nabla(\rho^{p/2})(t,x)|^2\,\mathrm{d}x + \|k\|_{L^\infty(\mathbb{R}^d)}\int_{\mathbb{R}^d}|\nabla(\rho^{p/2})(t,x)\rho^{p/2}(t,x)|\,\mathrm{d}x\\
&\le -\frac{\sigma^2}{p}\int_{\mathbb{R}^d}|\nabla(\rho^{p/2})(t,x)|^2\,\mathrm{d}x + \|k\|_{L^\infty(\mathbb{R}^d)}\int_{\mathbb{R}^d}\frac{\sigma^2}{2p\,\|k\|_{L^\infty(\mathbb{R}^d)}}|\nabla(\rho^{p/2})(t,x)|^2\\
&\quad + \frac{p\,\|k\|_{L^\infty(\mathbb{R}^d)}}{2\sigma^2}|\rho^p(t,x)|\,\mathrm{d}x\\
&\le -\frac{\sigma^2}{2p}\int_{\mathbb{R}^d}|\nabla(\rho^{p/2})(t,x)|^2\,\mathrm{d}x + \frac{p\,\|k\|^2_{L^\infty(\mathbb{R}^d)}}{2\sigma^2}\int_{\mathbb{R}^d}|\rho^p(t,x)|\,\mathrm{d}x\\
&\le \frac{p\,\|k\|^2_{L^\infty(\mathbb{R}^d)}}{2\sigma^2}\int_{\mathbb{R}^d}|\rho^p(t,x)|\,\mathrm{d}x,
\end{aligned}$$

where we used Young's inequality with $\varepsilon = \frac{\sigma^2}{\|k\|_{L^\infty(\mathbb{R}^d)}p}$ in the sixth step. An application of Gronwall's inequality leads to

$$\int_{\mathbb{R}^d}\rho^p(t,x)\,\mathrm{d}x \le C\big(p,\sigma,T,\|k\|_{L^\infty(\mathbb{R}^d)}\big)\,\|\rho_0\|_{L^p(\mathbb{R}^d)} \quad \text{for all } t\in[0,T].$$

□

LEMMA 2.12. *Suppose Assumption 2.1. Then, for each* $T > 0$ *there exists a constant* C *such that, for all* $\varepsilon > 0$,

$$\|\rho^\varepsilon\|_{L^\infty([0,T];L^\infty(\mathbb{R}^d))}, \|\rho\|_{L^\infty([0,T];L^\infty(\mathbb{R}^d))} \le C\big(T,\rho_0,\|k\|_{L^\infty(\mathbb{R}^d)}\big)$$

holds for the weak solutions $(\rho^\varepsilon, \varepsilon > 0)$ *of* (2.8) *and for the weak solution* ρ *of* (2.7).

PROOF. As previously, we will only show the claim for ρ and we can assume that ρ is smooth.

Set $\rho_m := \max(\rho - m, 0)$ for some fix strictly positive $m \in \mathbb{R}$ and let $p > 2$. For the sake of notational brevity we drop the depend of the involved on (t, x). Multiplying (2.7) by ρ_m^{p-1} and integrating by parts, we obtain

$$\begin{aligned}
\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\rho_m^p\,\mathrm{d}x &= \int_{\mathbb{R}^d}\left(\frac{\sigma^2}{2}\Delta\rho + \nabla\cdot((k*\rho)\rho)\right)\rho_m^{p-1}\,\mathrm{d}x\\
&= -\int_{\mathbb{R}^d}\frac{\sigma^2(p-1)}{2}\nabla\rho\cdot\nabla(\rho_m)\rho_m^{p-2} - (p-1)\nabla(\rho_m)\rho_m^{p-2}\cdot(k*\rho)\rho\,\mathrm{d}x\\
&= -\int_{\mathbb{R}^d}\frac{\sigma^2(p-1)}{2}|\nabla\rho_m|^2\rho_m^{p-2} - (p-1)\rho_m^{p-1}(\rho_m)_x(k*\rho)\\
&\quad + m(p-1)(\rho_m)_x\rho_m^{p-2}(k*\rho)\,\mathrm{d}x\\
&= -\frac{2\sigma^2(p-1)}{p^2}\int_{\mathbb{R}^d}|\nabla(\rho_m^{p/2})|^2\,\mathrm{d}x - \frac{2(p-1)}{p}\int_{\mathbb{R}^d}\nabla(\rho_m^{p/2})\rho_m^{p/2}\cdot(k*\rho)\,\mathrm{d}x\\
&\quad + \frac{2m(p-1)}{p}\int_{\mathbb{R}^d}\nabla(\rho_m^{p/2})\rho_m^{p/2-1}\cdot(k*\rho)\,\mathrm{d}x.
\end{aligned}$$

Notice, that we can interchange the derivatives of ρ_m and ρ as long as we have big enough p such that at least one ρ_m is present in the integrand. In the next step we estimate the last two terms with Young's inequality. More precisely, we get

$$\begin{aligned}
&2(p-1)\int_{\mathbb{R}^d}\frac{1}{p}\nabla(\rho_m^{p/2})\rho_m^{p/2}\cdot(k*\rho)\,\mathrm{d}x\\
&\le 2(p-1)\,\|k\|_{L^\infty(\mathbb{R}^d)}\int_{\mathbb{R}^d}\frac{1}{p}|\nabla(\rho_m^{p/2})|\,|\rho_m^{p/2}|\,\mathrm{d}x\\
&\le 2(p-1)\int_{\mathbb{R}^d}\frac{\sigma^2}{4p^2}|\nabla(\rho_m^{p/2})|^2 + \frac{\|k\|^2_{L^\infty(\mathbb{R}^d)}}{\sigma^2}|\rho_m^{p/2}|^2\,\mathrm{d}x\\
&= \frac{(p-1)\sigma^2}{2p^2}\int_{\mathbb{R}^d}|\nabla(\rho_m^{p/2})|^2\,\mathrm{d}x + \frac{2(p-1)\,\|k\|^2_{L^\infty(\mathbb{R}^d)}}{\sigma^2}\int_{\mathbb{R}^d}|\rho_m^p|\,\mathrm{d}x
\end{aligned}$$

and

$$\begin{aligned}
&2(p-1)\int_{\mathbb{R}^d}\frac{1}{p}\nabla(\rho_m^{p/2})m\rho_m^{p/2-1}\cdot(k*\rho)\,\mathrm{d}x\\
&\le 2(p-1)\int_{\mathbb{R}^d}\frac{1}{p}|\nabla(\rho_m^{p/2})|\,m\,\|k\|_{L^\infty(\mathbb{R}^d)}\,|\rho_m^{p/2-1}|\,\mathrm{d}x\\
&\le 2(p-1)\int_{\mathbb{R}^d}\frac{\sigma^2}{4p^2}|\nabla(\rho_m^{p/2})|^2 + \frac{m^2\,\|k\|^2_{L^\infty(\mathbb{R}^d)}}{\sigma^2}|\rho_m^{p-2}|\,\mathrm{d}x\\
&\le \frac{(p-1)\sigma^2}{2p^2}\int_{\mathbb{R}^d}|\nabla(\rho_m^{p/2})|^2\,\mathrm{d}x + \frac{2(p-1)\,\|k\|^2_{L^\infty(\mathbb{R}^d)}\,m^2}{\sigma^2}\int_{\mathbb{R}^d}|\rho_m^{p-2}|\,\mathrm{d}x.
\end{aligned}$$

Furthermore, we can estimate

$$\begin{aligned}\int_{\mathbb{R}^d} |\rho_m^{p-2}| \,\mathrm{d}x &= \int_{\mathbb{R}^d} \mathbb{1}_{\{m\le\rho\le m+1\}}|\rho_m^{p-2}| + \mathbb{1}_{\{\rho\ge m+1\}}|\rho_m^{p-2}| \,\mathrm{d}x \\ &\le \int_{\mathbb{R}^d} \mathbb{1}_{\{m\le\rho\le m+1\}} + |\rho_m^{p}| \,\mathrm{d}x \\ &\le \frac{1}{m}\int_{\mathbb{R}^d} \rho \,\mathrm{d}x + \int_{\mathbb{R}^d} |\rho_m^{p}| \,\mathrm{d}x \\ &\le \frac{1}{m} + \int_{\mathbb{R}^d} |\rho_m^{p}| \,\mathrm{d}x.\end{aligned}$$

Hence, we derive for the last term the following inequality

$$\begin{aligned}&2(p-1)\int_{\mathbb{R}^d} \frac{1}{p}\nabla(\rho_m^{p/2}) m\rho_m^{p/2-1}\cdot(k*\rho)\,\mathrm{d}x \\ &\le \frac{(p-1)\sigma^2}{2p^2}\int_{\mathbb{R}^d} |\nabla(\rho_m^{p/2})|^2 \,\mathrm{d}x + \frac{2(p-1)\,\|k\|^2_{L^\infty(\mathbb{R}^d)}\, m}{\sigma^2} \\ &\quad + \frac{2(p-1)\,\|k\|^2_{L^\infty(\mathbb{R}^d)}\, m^2}{\sigma^2}\int_{\mathbb{R}^d} |\rho_m^{p}| \,\mathrm{d}x.\end{aligned}$$

Putting everything together we find

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\rho_m^p\,\mathrm{d}x \le& -\frac{\sigma^2(p-1)}{p^2}\int_{\mathbb{R}^d} |\nabla(\rho_m^{p/2})|^2 \,\mathrm{d}x + \frac{2(p-1)\,\|k\|^2_{L^\infty(\mathbb{R}^d)}\, m}{\sigma^2} \\ &+ \left(\frac{2(p-1)\,\|k\|^2_{L^\infty(\mathbb{R}^d)}}{\sigma^2} + \frac{2(p-1)\,\|k\|^2_{L^\infty(\mathbb{R}^d)}\, m^2}{\sigma^2}\right)\int_{\mathbb{R}^d} |\rho_m^{p}| \,\mathrm{d}x,\end{aligned}$$

from which we can conclude that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\rho_m^p\,\mathrm{d}x \\ &\le -\frac{\sigma^2}{2}\int_{\mathbb{R}^d} |\nabla(\rho_m^{p/2})|^2 \,\mathrm{d}x + \frac{2\,\|k\|^2_{L^\infty(\mathbb{R}^d)}\, p^2(m^2+1)}{\sigma^2}\int_{\mathbb{R}^d} |\rho_m^{p}| \,\mathrm{d}x + \frac{2p^2\,\|k\|^2_{L^\infty(\mathbb{R}^d)}\, m}{\sigma^2}.\end{aligned}$$

By the Gagliardo–Nirenberg–Sobolev inequality [Leo17, Theorem 12.87] and [Nir59] on the whole space as well as Young's inequality with $\tau = \frac{3\sigma^2}{4}$ we have

$$\begin{aligned}\lambda^2\,\|u\|^2_{L^2(\mathbb{R}^d)} &\le C_{\mathrm{GNS}}\lambda^2\,\|u\|^{\frac{4}{2+d}}_{L^1(\mathbb{R}^d)}\,\|\nabla u\|^{\frac{2d}{2+d}}_{L^2(\mathbb{R}^d)} \\ &\le \left(\frac{\sigma^2(2+d)}{4d}\right)^{-\frac{d}{2}}\frac{2}{2+d}(C_{\mathrm{GNS}}\lambda^2)^{\frac{2+d}{2}}\,\|u\|^2_{L^1(\mathbb{R}^d)} + \frac{\sigma^2}{4}\,\|\nabla u\|^2_{L^2(\mathbb{R}^d)},\end{aligned} \tag{2.16}$$

where C_{GNS} is the Gagliardo–Nirenberg–Sobolev constant dimension d. For $u = \rho_m^{p/2}$, $C_1 := \frac{2\|k\|^2_{L^\infty(\mathbb{R}^d)}(m^2+1)}{\sigma^2}$ and $\lambda = \sqrt{C_1}p$ we obtain

$$\begin{aligned}
&C_1 p^2 \int_{\mathbb{R}^d} |\rho_m^p| \,\mathrm{d}x \\
&\quad\le \left(\frac{\sigma^2(2+d)}{4d}\right)^{-\frac{d}{2}} \frac{2}{2+d} C_{\text{GNS}}^{\frac{2+d}{2}} C_1^{2+d} p^{2+d} \left\|\rho_m^{p/2}\right\|^2_{L^1(\mathbb{R}^d)} + \frac{\sigma^2}{4}\left\|(\rho_m^{p/2})_x\right\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \rho_m^p \,\mathrm{d}x \le -\frac{1}{4}\sigma^2 \int_{\mathbb{R}^d} (\rho_m^{p/2})_x^2 \,\mathrm{d}x + p^{2+d} C\big(\sigma, m, \|k\|_{L^\infty(\mathbb{R}^d)}\big)\left(\int_{\mathbb{R}^d} |\rho_m^{p/2}| \,\mathrm{d}x\right)^2 \\
+ \frac{2p^2 \|k\|^2_{L^\infty(\mathbb{R}^d)}\, m}{\sigma^2}.
\end{aligned}$$

Applying the above inequality (2.16) with $u = \rho_m^{p/2}$, $\lambda = p$ and rearranging the terms we discover

$$-\frac{\sigma^2}{4}\left\|(\rho_m^{p/2})_x\right\|^2_{L^2(\mathbb{R}^d)} \le -p^2 \int_{\mathbb{R}^d} |\rho_m^p| \,\mathrm{d}x + \frac{4}{3\sigma} C^{3/2} p^{2+d} \left\|\rho_m^{p/2}\right\|^2_{L^1(\mathbb{R}^d)},$$

which then implies

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \rho_m^p \,\mathrm{d}x \\
&\quad\le -p^2 \int_{\mathbb{R}^d} |\rho_m^p| \,\mathrm{d}x + C\big(\sigma, m, \|k\|_{L^\infty(\mathbb{R}^d)}\big) p^{2+d} \left(\int_{\mathbb{R}^d} \rho_m^{p/2} \,\mathrm{d}x\right)^2 + \frac{2p^2 \|k\|^2_{L^\infty(\mathbb{R}^d)}\, m}{\sigma^2} \\
&\quad= -p^2 \int_{\mathbb{R}^d} |\rho_m^p| \,\mathrm{d}x + C\big(\sigma, m, \|k\|_{L^\infty(\mathbb{R}^d)}\big) p^{2+d} \left(\int_{\mathbb{R}^d} \rho_m^{p/2} \,\mathrm{d}x\right)^2 + C\big(\sigma, m, \|k\|_{L^\infty(\mathbb{R}^d)}\big) p^2
\end{aligned}$$

for some non-negative constant $C(\sigma, m, \|k\|_{L^\infty(\mathbb{R}^d)})$.

Let us define

$$w_j(t) := \int_{\mathbb{R}^d} \rho_m^{2^j}(t,x) \,\mathrm{d}x \quad \text{and} \quad S_j := \sup_{t\in[0,T]} w_j(t),$$

for $j \in \mathbb{N}$. Then, for $p = 2^j$ the above inequality can be written as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} w_j(t) &\le -2^{2j} w_j(t) + 2^{2j}\Big(C\big(\sigma, m, \|k\|_{L^\infty(\mathbb{R}^d)}\big) 2^{dj} w_{j-1}^2(t) + C\big(\sigma, m, \|k\|_{L^\infty(\mathbb{R}^d)}\big)\Big) \\
&\le -2^{2j} w_j(t) + 2^{2j}\Big(C\big(\sigma, m, \|k\|_{L^\infty(\mathbb{R}^d)}\big) 2^{dj} S_{j-1}^2 + C\big(\sigma, m, \|k\|_{L^\infty(\mathbb{R}^d)}\big)\Big).
\end{aligned}$$

Moreover, define $u(x) := -2^{2j}x + 2^{2j}\Big(C\big(\sigma, m, \|k\|_{L^\infty(\mathbb{R}^d)}\big) 2^{dj} S_{j-1}^2 + C\big(\sigma, m, \|k\|_{L^\infty(\mathbb{R}^d)}\big)\Big)$, $\varepsilon := 2^{2j}$ and $A := C\big(\sigma, m, \|k\|_{L^\infty(\mathbb{R}^d)}\big) 2^{2j} S_{j-1}^2 + C\big(\sigma, m, \|k\|_{L^\infty(\mathbb{R}^d)}\big)$. Then, u is globally Lipschitz continuous in x and $v = e^{-\varepsilon t} v_0 + A(1 - e^{-\varepsilon t})$ is a solution of the following ODE

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} v(t) = u(v(t)), \\ v(0) = v_0. \end{cases}$$

Let us choose $v_0 := \int_{\mathbb{R}^d} \rho_0^{2^j}\,\mathrm{d}x \geq w(0)$. Then, we can apply the comparison principle A.23 to obtain

$$\begin{aligned} w_j(t) &\leq v(t) \leq e^{-\varepsilon t} v_0 + A \\ &\leq \|\rho_0\|_{L^\infty(\mathbb{R}^d)}^{2^j-1} + C\big(\sigma, m, \|k\|_{L^\infty(\mathbb{R}^d)}\big) 2^{dj} S_{j-1}^2 + C\big(\sigma, m, \|k\|_{L^\infty(\mathbb{R}^d)}\big). \end{aligned}$$

It follows that

$$S_j = \sup_{t\in[0,T]} w_j(t) \leq C\big(\sigma, m, \|k\|_{L^\infty(\mathbb{R}^d)}\big) \max\big(\|\rho_0\|_{L^\infty(\mathbb{R}^d)}^{2^j-1}, 2^{dj} S_{j-1}^2 + 1\big).$$

To complete the proof, we perform a version of Moser iteration technique to bound the L^∞-norm. Let us assume for the moment that $\|\rho_0\|_{L^\infty(\mathbb{R}^d)} \geq 1$. Let $\tilde{S}_j := \frac{S_j}{\|\rho_0\|_{L^\infty(\mathbb{R}^d)}^{2^j-1}}$. Because $a+b \leq 2\max(a,b)$ the last inequality provides us with

$$\begin{aligned} \tilde{S}_j &\leq C\big(\sigma, m, \|k\|_{L^\infty(\mathbb{R}^d)}\big) \max\big(1, 2^{dj}\tilde{S}_{j-1}^2, \|\rho_0\|_{L^\infty(\mathbb{R}^d)}^{-(2^j-1))}\big) \\ &\leq C\big(\sigma, m, \|k\|_{L^\infty(\mathbb{R}^d)}\big) \max\big(1, 2^{dj}\tilde{S}_{j-1}^2\big). \end{aligned}$$

Adding on both sides $\delta > 0$ and taking the logarithm, we arrive at

$$\begin{aligned} \log(\tilde{S}_j + \delta) &\leq \max(\log\big(C\big(\sigma, m, \|k\|_{L^\infty(\mathbb{R}^d)}\big) + \delta\big), \log\big(C\big(\sigma, m, \|k\|_{L^\infty(\mathbb{R}^d)}\big) 2^{dj}\tilde{S}_{j-1}^2 + \delta)\big) \\ &\leq 2\log\big(\tilde{S}_{j-1} + \delta\big) + dj\log(2) + \log\big(C\big(\sigma, m, \|k\|_{L^\infty(\mathbb{R}^d)}\big)\big) \end{aligned}$$

for some new constant $C\big(\sigma, m, \|k\|_{L^\infty(\mathbb{R}^d)}\big) > 0$. This implies

$$2^{-j}\log\big(\tilde{S}_j + \delta\big) - 2^{1-j}\log\big(\tilde{S}_{j-1} + \delta\big) \leq 2^{-j} dj \log(2) + 2^{-j} C\big(\sigma, m, \|k\|_{L^\infty(\mathbb{R}^d)}\big)$$

for $j \in \mathbb{N}$, where we used Lemma 2.11 to not subtract infinity, i.e. $\log(\tilde{S}_{j-1} + \delta) < \infty$. Adding the above inequality over $j = 1, \dots, J$, we find

$$\begin{aligned} 2^{-J}\log\big(\tilde{S}_J + \delta\big) - \log\big(\tilde{S}_0 + \delta\big) &= \sum_{j=1}^{J} 2^{-j}\log\big(\tilde{S}_j + \delta\big) - 2^{-(j-1)}\log\big(\tilde{S}_{j-1} + \delta\big) \\ &\leq \sum_{j=1}^{\infty} 2^{-j} dj\log(2) + 2^{-j} C\big(\sigma, m, \|k\|_{L^\infty(\mathbb{R}^d)}\big) \\ &\leq C\big(\sigma, m, \|k\|_{L^\infty(\mathbb{R}^d)}\big) \end{aligned}$$

for a constant C independent of J and $\delta > 0$. A straightforward way to see that the series is absolutely convergent is to apply the ratio criterion from elementary analysis.

Now, we have $\tilde{S}_0 = \sup_{t\in[0,T]} \|\rho(\cdot,t)\|_{L^1(\mathbb{R}^d)} = 1$ by mass conservation. Therefore, taking the exponential function on both sides and letting $\delta \to 0$, we discover

$$S_J^{2^{-J}} \leq C\,\|\rho_0\|_{L^\infty(\mathbb{R}^d)}^{(2^J-1)2^{-J}} \leq C\big(\sigma, m, \rho_0, \|k\|_{L^\infty(\mathbb{R}^d)}\big) < \infty.$$

On the other hand, we have

$$S_J^{2^{-J}} = \left(\sup_{t\in[0,T]} \int_{\mathbb{R}^d} \rho_m^{2^J}(t,x)\,\mathrm{d}x\right)^{\frac{1}{2^J}} = \sup_{t\in[0,T]} \left(\int_{\mathbb{R}^d} \rho_m^{2^J}(t,x)\,\mathrm{d}x\right)^{\frac{1}{2^J}}.$$

Finally, we can take the limit $J \to \infty$ to conclude

$$\sup_{t\in[0,T]} \|\rho_m(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} = \sup_{t\in[0,T]} \lim_{J\to\infty} \|\rho_m(t,\cdot)\|_{L^{2^J}(\mathbb{R}^d)} \le \limsup_{J\to\infty} \sup_{t\in[0,T]} \|\rho_m(t,\cdot)\|_{L^{2^J}(\mathbb{R}^d)}$$
$$= \limsup_{J\to\infty} S_J^{2^{-J}} \le C\big(\|k\|_{L^\infty(\mathbb{R}^d)}, \rho_0\big).$$

This finishes the case $\|\rho_0\|_{L^\infty(\mathbb{R}^d)} \ge 1$. In the case $\|\rho\|_{L^\infty(\mathbb{R}^d)} \le 1$ we immediately obtain the inequality

$$S_j \le C\big(\sigma, m, \|k\|_{L^\infty(\mathbb{R}^d)}\big) \max(1, 2^{dj} S_{j-1}^2)).$$

Thus, by the same steps we obtain

$$2^{-J} \log(S_j) \le C\big(\|k\|_{L^\infty(\mathbb{R}^d)}, \rho_0\big).$$

Taking the exponential and the limit we arrive at an uniform bound for the $L^\infty(\mathbb{R}^d)$-norm of ρ. □

2.3. Local Lipschitz bound

In this section we introduce a uniform Lipschitz assumption on the approximation sequence $(k^\varepsilon, \varepsilon > 0)$ and show that most bounded confidence models, as used in the theory of opinion formation [Hos20], satisfy this assumption.

At first glance, we notice that even though the interaction force kernels k^ε is uniformly bounded it is not uniformly Lipschitz continuous in ε. Hence, the classical theory regarding Lipschitz continuous interaction force kernels on mean-field limits cannot be applied directly to the particles systems introduced in Subsection 2.1.1. Instead we need use the properties of the convolution to derive uniform Lipschitz continuity of the mean-field force $k^\varepsilon * \rho^\varepsilon$. Following e.g. [CG17, LP17, FHS19], we derive a Lipschitz bound for certain models in the case where the trajectories X_t^ε and Y_t^ε are close in a suitable sense. This approach requires an approximation with suitable properties and could not be generalized, so far, to arbitrary approximations. The main reason lies in the derivative of the approximation k^ε. If the derivative would be non-negative, then we could use a Taylor approximation, the properties of the solution ρ^ε and the formula $\||\nabla k^\varepsilon| * \rho_t^\varepsilon\|_{L^\infty(\mathbb{R}^d)} = \|k^\varepsilon * \nabla \rho_t^\varepsilon\|_{L^\infty(\mathbb{R}^d)}$ to obtain a local Lipschitz bound for k^ε with estimates on the gradient $\nabla \rho_t^\varepsilon$. Unfortunately, in most cases a simple mollification of k has a derivative becoming non-negative as well as non-positive. Therefore, we have to postulate the following assumptions on the approximation sequence $(k^\varepsilon, \varepsilon > 0)$.

ASSUMPTION 2.13. *The sequence $(k^\varepsilon, \varepsilon > 0)$ satisfies the following:*

(i) There exists a family of functions $(l^\varepsilon, \varepsilon > 0)$ such that

$$|k^\varepsilon(x) - k^\varepsilon(y)| \le l^\varepsilon(y)|x - y|$$

for $x, y \in \mathbb{R}^d$ with $|x - y| \le 2\varepsilon$;

(ii)

$$\sup_{t\in[0,T]} \|l^\varepsilon * \rho_t^\varepsilon\|_{L^\infty(\mathbb{R}^d)} \le C(\|k\|_{L^\infty(\mathbb{R}^d)})(\|\rho_0\|_{L^1(\mathbb{R}^d)} + \|\rho_0\|_{L^\infty(\mathbb{R}^d)}),$$

where $C(\|k\|_{L^\infty(\mathbb{R}^d)})$ is some finite constant depending on the $L^\infty(\mathbb{R}^d)$-norm of k.

REMARK 2.14. *The constant 2 in Assumption 2.13 can be replaced by any positive constant. For simplicity, we choose the most convenient one to avoid cumbersome notation.*

2.3.1. Exemplary interaction kernels. The particle systems, as introduced in Subsection 2.1.1, can be used to model the opinion of interacting individuals, see e.g. [Hos20]. A prominent class are given by so-called bounded confidence models, in which the interaction is described by interaction force kernels of the form

$$k_{BCM}(x) := \mathbb{1}_{[0,R]}(|x|)h(x),$$

where $h\colon \mathbb{R}^d \mapsto \mathbb{R}^d$ is a twice continuously differentiable function. To show that k_{BCM} satisfies Assumption 2.13, we introduce the following approximation sequence $(\psi^\varepsilon_{a,b}, \varepsilon > 0)$ of the indicator function $\mathbb{1}_{[a,b]}(z)$ with $a, b \in \mathbb{R}$, $a < b$, such that the following properties hold for each $\varepsilon > 0$:

- $\psi^\varepsilon_{a,b} \in C_c^\infty(\mathbb{R}^d)$,
- $\psi^\varepsilon_{a,b} \to \mathbb{1}_{[a,b]}$ as $\varepsilon \to 0$ almost everywhere,
- $\operatorname{supp}(\psi^\varepsilon_{a,b}) \subseteq [a-2\varepsilon, b+2\varepsilon]$, $\operatorname{supp}(\frac{\mathrm{d}}{\mathrm{d}z}\psi^\varepsilon_{a,b}) \subset [a-2\varepsilon, a+2\varepsilon] \cup [b-2\varepsilon, b+2\varepsilon]$,
- $0 \le \psi^\varepsilon_{a,b} \le 1$, $\left|\frac{\mathrm{d}}{\mathrm{d}z}\psi^\varepsilon_{a,b}\right| \le \frac{C}{\varepsilon}$ for some constant $C > 0$.

Since we want to take $\varepsilon \to 0$, we consider only the case where ε is small enough. In particular, we can take the mollification of the indicator function of a set. We define the regularized interaction force kernel

$$k^\varepsilon_{BCM}(x) = \psi^\varepsilon_{-R,R}(|x|)h(x)$$

which obviously satisfies Assumptions 2.2. That it also satisfies Assumption 2.13 is verified in the following.

LEMMA 2.15 (Local Lipschitz bound for bounded confidence models). *Consider the regularized interaction force kernel k^ε_{BCM} with cut-off ε. Let $\mathrm{NBR}_\varepsilon := B(-R, 4\varepsilon) \cup B(R, 4\varepsilon)$ ("neighbourhood of R"). Then, we have the following estimates:*

(i) For each $x, y \in \mathbb{R}^d$ with $|x - y| \le 2\varepsilon$ and

$$l^\varepsilon_{BCM}(y) := \begin{cases} C\mathbb{1}_{B(0,R+3)}(y), & y \in \mathrm{NBR}^{\mathrm{c}}_\varepsilon, \\ C\varepsilon^{-1}, & y \in \mathrm{NBR}_\varepsilon, \end{cases}$$

it holds that

$$|k^\varepsilon_{BCM}(x) - k^\varepsilon_{BCM}(y)| \le l^\varepsilon_{BCM}(y)|x - y|;$$

(ii) For each $x, y \in \mathbb{R}^{dN}$ with $|x - y|_\infty \le \varepsilon$ and

$$L^\varepsilon_{i,BCM}(y_1, \dots, y_N) := \frac{1}{N}\sum_{j=1}^N l^\varepsilon_{BCM}(y_i - y_j), \quad (y_1, \dots, y_N) \in \mathbb{R}^{dN},$$

it holds that

$$|K^\varepsilon_{i,BCM}(x) - K^\varepsilon_{i,BCM}(y)| \le 2L^\varepsilon_{i,BCM}(y)|x - y|_\infty,$$

where $K_{i,BCM}$ is defined by (2.5) with k_{BCM}.

PROOF. (i) Let $|x - y| \le 2\varepsilon$. By the mean value theorem, we have the bound

$$|k^\varepsilon_{BCM}(x) - k^\varepsilon_{BCM}(y)| \le \left|\nabla k^\varepsilon(z)\right||x - y|$$

for some z in the line segment between x and y. Let us distinguish between two cases.

Case 1: $y \in \mathrm{NBR}_\varepsilon$. Using the bound

$$\left|\nabla k^\varepsilon_{BCM}(z)\right| \leq \left|\nabla(\psi^\varepsilon(|z|))h(z)\right| + \left|\psi^\varepsilon(z)\nabla h(z)\right| \leq C\varepsilon^{-1}$$

for all $z \in \mathbb{R}^d$ for some constant $C > 0$, which depends on the deterministic function h, it follows

$$|k^\varepsilon_{BCM}(x) - k^\varepsilon_{BCM}(y)| \leq C\varepsilon^{-1}|x-y|.$$

Case 2: $y \in \mathrm{NBR}^{\mathrm{c}}_\varepsilon$. Because z lies on the line segment between x, y, it follows for some $s \in [0,1]$ that

$$|z-y| = |y - s(x-y) - y| \leq |x-y| \leq 2\varepsilon$$

and therefore $|R - z| \geq |R-y| - |z-y| \geq 4\varepsilon - 2\varepsilon = 2\varepsilon$. Analogously, $|-R-z| \geq |-R-y| - |z-y| \geq 2\varepsilon$. Consequently, z is far enough away from the points R and $-R$ such that the derivative of the approximation ψ^ε vanishes. This implies

$$\left|\nabla k^\varepsilon_{BCM}(z)\right| \leq \left|\psi^\varepsilon(|z|)\nabla h(z)\right| \leq \left|\nabla h(z)\right|\mathbb{1}_{[-R-3,R+3]}(|y|) \leq C\mathbb{1}_{[-R-3,R+3]}(|y|),$$

where we used $|y| \leq |y-z| + |z| \leq 2 + |z|$. Together with the mean value theorem this proves the second case.

(ii) We want to apply (i). For $x, y \in \mathbb{R}^{dN}$, $|x-y|_\infty \leq \varepsilon$, it follows

$$\begin{aligned}|K^\varepsilon_{i,BCM}(x) - K^\varepsilon_{i,BCM}(y)| &\leq \frac{1}{N-1}\sum_{\substack{j=1\\ j\neq i}}^N |k^\varepsilon_{BCM}(x_i - x_j) - k^\varepsilon_{BCM}(y_i - y_j)| \\ &\leq \frac{1}{N-1}\sum_{\substack{j=1\\ j\neq i}}^N l^\varepsilon_{BCM}(y_i - y_j)|x_i - x_j - (y_i - y_j)| \\ &\leq 2L^\varepsilon_{i,BCM}(y)|x-y|_\infty.\end{aligned}$$

It is indeed justified to apply (i) since $|x_i - x_j - (y_i - y_j)| \leq 2|x-y|_\infty \leq 2\varepsilon$ for all $i, j = 1, \ldots, N$. □

Remark 2.16. *The second part of Lemma 2.15 is a direct consequence of part one. Hence, if $(k^\varepsilon, \varepsilon > 0)$ satisfies Assumption 2.13, we have*

$$|K^\varepsilon_i(x) - K^\varepsilon_i(y)| \leq 2L^\varepsilon_i(y)|x-y|_\infty$$

for $x, y \in \mathbb{R}^{dN}$ with $|x-y|_\infty \leq \varepsilon$ and

$$L^\varepsilon_i(y_1, \ldots, y_N) := \frac{1}{N}\sum_{j=1}^N l^\varepsilon(y_i - y_j), \quad (y_1, \ldots, y_N) \in \mathbb{R}^{dN}.$$

The convenient properties of the solutions $(\rho^\varepsilon, \varepsilon \geq 0)$ allow us to find a uniform bound of the convolution term $l^\varepsilon * \rho^\varepsilon_t$. This will be the content of the following lemma.

Lemma 2.17. *Suppose Assumption 2.1 and let us define*

$$\bar{L}^\varepsilon_{t,i,BCM}(y_1, \ldots, y_N) := (l^\varepsilon_{BCM} * \rho^\varepsilon_t)(y_i), \quad (y_1, \ldots, y_N) \in \mathbb{R}^{dN},$$

the averaged version of L^ε *for* $i = 1, \dots, N$*. Then, there exists a constant* C*, depending on the deterministic function* h *and* ρ_0*, such that*

$$\sup_{i=1,\dots,N} \sup_{t\in[0,T]} \|\bar{L}^\varepsilon_{t,i,BCM}\|_{L^\infty(\mathbb{R}^{dN})} \le C\big(\|\rho^\varepsilon_0\|_{L^1(\mathbb{R}^d)} + \|\rho^\varepsilon_0\|_{L^\infty(\mathbb{R}^d)} \big),$$

where ρ^ε_t *is the solution of* (2.8) *for the special interaction force kernel* k_{BCM}*.*

Proof. Let $i \in \{1, \dots, N\}$ and $y = (y_1, \dots, y_N) \in \mathbb{R}^{dN}$. Then, by mass conservation and Lemma 2.12, we have

$$\begin{aligned} |\bar{L}^\varepsilon_{t,i,BCM}(y)| &= |(l^\varepsilon_{BCM} * \rho^\varepsilon_t)(y_i)| \\ &\le \int_{\mathbb{R}^d} \mathbb{1}_{\{z\,:\,y_i - z\in \mathrm{NBR}^c_\varepsilon\}} |l^\varepsilon_{BCM}(y_i - z)\rho_t(z)|\,\mathrm{d}z \\ &\quad + \int_{\mathbb{R}^d} \mathbb{1}_{\{z\,:\,y_i - z\in \mathrm{NBR}_\varepsilon\}} |l^\varepsilon_{BCM}(y - z)\rho_t(z)|\,\mathrm{d}z \\ &\le C\int_{\mathbb{R}^d} |\rho^\varepsilon_t(z)|\,\mathrm{d}z + C\varepsilon^d\varepsilon^{-1}\,\|\rho^\varepsilon_t\|_{L^\infty(\mathbb{R}^d)} \\ &\le C\big(\|\rho^\varepsilon_t\|_{L^1(\mathbb{R}^d)} + \|\rho^\varepsilon_t\|_{L^\infty(\mathbb{R}^d)} \big) \\ &\le C\big(\|\rho_0\|_{L^1(\mathbb{R}^d)} + \|\rho_0\|_{L^\infty(\mathbb{R}^d)} \big), \end{aligned}$$

where C again depends on the deterministic function h. □

Consequently, we have shown that $k^\varepsilon = \psi^\varepsilon h$ fulfills Assumption 2.13 and (as previously mentioned) Assumption 2.2, which implies the following corollary.

Corollary 2.18. *The interaction force kernel* k_{BCM} *of the bounded confidence model satisfies Assumptions 2.2 and 2.13. The associated approximation sequence* $\big(k^\varepsilon_{BCM}, \varepsilon > 0\big)$ *is given by* $k^\varepsilon_{BCM}(x) := \psi^\varepsilon_{-R,R}(x)h(x)$*.*

Another example of interest occurs in dimension $d = 1$. The interaction forces with $h(z) := \mathrm{sgn}(z)$, which corresponds to a uniform interaction, i.e., every particle in the interaction radius has the same impact. Unfortunately, $\mathrm{sgn}(z) \notin C^2(\mathbb{R})$ and, hence, we cannot directly apply Lemma 2.15. However, the function $\mathrm{sgn}(z)$ has no effect on the discontinuities $-R$ and R. Therefore, if we can control the function around zero, we can obtain an analogue result to Lemma 2.15. Indeed, we define

$$k_U(z) := -\mathbb{1}_{[-R,0]}(x) + \mathbb{1}_{[0,R]}(z), \quad z \in \mathbb{R},$$

which can be appropriated by $k^\varepsilon_U(z) := \psi^\varepsilon_{-R,0}(z) + \psi^\varepsilon_{0,R}(z)$. Defining

$$\mathrm{NBZR}_\varepsilon := [-R - 4\varepsilon, -R + 4\varepsilon] \cup [-4\varepsilon, 4\varepsilon] \cup [R - 4\varepsilon, R + 4\varepsilon]$$

as the neighbourhood of zero and R, we can perform the same steps as in Lemma 2.15 to prove the following lemma for k_U.

Lemma 2.19 (Local Lipschitz bound for uniform kernel). *Consider the regularized interaction force kernel* k^ε_U *with cut-off* ε*. Then, we have the following estimates:*

(i) For each $x, y \in \mathbb{R}$ *with* $|x-y| \le 2\varepsilon$ *and*

$$l_U^\varepsilon(z) := \begin{cases} 0, & z \in \mathrm{NBZR}_\varepsilon^c, \\ C\varepsilon^{-1}, & z \in \mathrm{NBZR}_\varepsilon, \end{cases}$$

it holds that

$$|k_U^\varepsilon(z) - k_U^\varepsilon(y)| \le l_U^\varepsilon(y)|z-y|;$$

(ii) For each $x, y \in \mathbb{R}^N$ *with* $|x-y|_\infty \le \varepsilon$ *and*

$$L_{i,U}^\varepsilon(y_1,\dots,y_N) := \frac{1}{N}\sum_{j=1}^N l_U^\varepsilon(y_i - y_j), \quad (y_1,\dots,y_N) \in \mathbb{R}^N,$$

it holds that

$$|K_{i,U}^\varepsilon(x) - K_{i,U}^\varepsilon(y)| \le 2L_{i,U}^\varepsilon(y)|x-y|_\infty.$$

Corollary 2.20. *The interaction force kernel* k_U *satisfies Assumptions 2.2 and 2.13 with the associated approximation sequence* $\left(k_U^\varepsilon, \varepsilon > 0\right)$ *given by* $k_U^\varepsilon(z) := \psi_{-R,0}^\varepsilon(z) + \psi_{0,R}^\varepsilon(z)$.

Proof. Apply Lemma 2.19 and similar computations as in proof of Lemma 2.17 to show that Assumption 2.13 is fulfilled. The verification of Assumption 2.2 follows immediately. □

2.4. Law of large numbers

The derivation of propagation of chaos is based on defining several exceptional sets where the desired properties will not hold. Hence, we need to rely on the fact that the probability measure of these sets is extremely small. This fact is the subject of the next proposition.

Proposition 2.21 (Law of large numbers). *Let* $0 < \alpha, \delta$ *such that* $0 < \delta + \alpha < 1/2$ *and* $Z^1, \dots, Z^N$ *be independent random variables in* $\mathbb{R}^d$ *such that* Z^i *has density* u^i *for* $i = 1, \dots, N$. *Let* $h = (h_1, \dots, h_d) : \mathbb{R}^d \to \mathbb{R}^d$ *be a bounded measurable function. Define* $H_i(Z) := \frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^N h(Z^i - Z^j)$ *and*

$$S := \left\{ \sup_{1\le i\le N} |H_i(Z) - \mathbb{E}(H_i(Z))| \ge N^{-(\delta+\alpha)} \right\},$$

$$\widetilde{S} := \left\{ \sup_{1\le i\le N} |H_i(Z) - \mathbb{E}_{(-i)}(H_i(Z))| \ge N^{-(\delta+\alpha)} \right\},$$

where $\mathbb{E}_{(-i)}$ *stands for the expectation with respect to every variable except* Z^i, *i.e.*

$$\mathbb{E}_{(-i)}(H_i(Z)) := \frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^N (h * u^j)(Z_i).$$

Then, for each $\gamma > 0$ *there exists a constant* $C(\gamma) > 0$, *which depends on* γ, C, *such that*

$$\mathbb{P}(S),\, \mathbb{P}(\widetilde{S}) \le C(\gamma)N^{-\gamma}.$$

PROOF. We prove the statement for the set S. The estimate for the set $\widetilde{S}$ can be shown similarly by replacing $\mathbb{E}(H_i(Z))$ with $\mathbb{E}_{(-i)}(H_i(Z))$. First, we notice

$$\mathbb{P}\left(\sup_{1\le i\le N} |H_i(Z) - \mathbb{E}(H_i(Z))| \ge N^{-(\delta+\alpha)}\right) \le \sum_{i=1}^{N}\sum_{l=1}^{d} \mathbb{P}\big(|H_{i,l}(Z) - \mathbb{E}(H_{i,l}(Z))| \ge N^{-(\delta+\alpha)}\big),$$

where $H_{i,l} = \frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N} h_l(Z^i - Z^j)$. Hence, it suffices to prove

$$\mathbb{P}(|H_{i,l}(Z) - \mathbb{E}(H_{i,l}(Z))| \ge N^{-(\delta+\alpha)}) \le C(\gamma)N^{-\gamma}$$

for each $\gamma > 0$, $i = 1, \dots, N$, $l = 1, \dots, d$. Let us assume $i = 1, l = 1$, where we will omit the index l for more convenient notation, and for $j = 2, \dots, N$ let us denote by Θ_j the independent random variables $\Theta_j := h(Z^1 - Z^j)$. Then, applying Chebyshev's inequality to the function $x \mapsto x^{2m}$, we obtain

$$\begin{aligned}\mathbb{P}(|H_1(Z) - \mathbb{E}(H_1(Z))| \ge N^{-(\delta+\alpha)}) &\le N^{2(\delta+\alpha)m}\mathbb{E}(|H_1(Z) - \mathbb{E}(H_1(Z))|^{2m}) \\ &\le N^{2(\delta+\alpha)m}\mathbb{E}\left(\left(\frac{1}{N-1}\sum_{j=2}^{N}(\Theta_j - \mathbb{E}(\Theta_j))\right)^{2m}\right).\end{aligned} \tag{2.17}$$

The expectation on the right-hand side can be rewritten, using the multinomial formula, as

$$(x_2 + x_3 + \dots + x_N)^{2m} = \sum_{a_2+a_3+\dots+a_N=2m} \binom{2m}{a_2, \dots, a_N} \prod_{j=2}^{N} x_j^{a_j},$$

where $a = (a_2, a_3, \dots, a_N) \in \mathbb{N}_0^{N-1}$ is a multi-index of length $|a| = 2m$. Consequently, using the independence of $(\Theta_j, j = 2, \dots, N)$, we get

$$\begin{aligned}&\mathbb{E}\left(\left(\frac{1}{N}\sum_{j=2}^{N}(\Theta_j - \mathbb{E}(\Theta_j))\right)^{2m}\right) \\ &= N^{-2m}\sum_{a_2+a_3+\dots+a_N=2m} \binom{2m}{a_2, \dots, a_N} \prod_{j=2}^{N} \mathbb{E}((\Theta_j - \mathbb{E}(\Theta_j))^{a_j}) \\ &= N^{-2m}\sum_{\substack{a_2+a_3+\dots+a_N=2m\\ |a|_0\le m}} \binom{2m}{a_2, \dots, a_N} \prod_{j=2}^{N} \mathbb{E}((\Theta_j - \mathbb{E}(\Theta_j))^{a_j}),\end{aligned} \tag{2.18}$$

where $|a|_0$ the number of non-zero entries of the multi-index a. Otherwise, if $|a|_0 > m$, then there exists a j such that $a_j = 1$ and the product vanish since $\mathbb{E}(\Theta_j - \mathbb{E}(\Theta_j)) = 0$. From the bound on h we have

$$|\mathbb{E}((\Theta_j - \mathbb{E}(\Theta_j))^{a_j})| = \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} (h(z_1 - z_j) - E(\Theta_j))^{a_j} u^1(z_1) u^j(z_j) \, \mathrm{d}z_1 \, \mathrm{d}z_j \right| \leq C^{a_j}.$$

Using the facts

$$\binom{2m}{a_2, \dots, a_N} \leq (2m)^{2m} \quad \text{and} \quad \sum_{\substack{a_2 + a_3 + \dots + a_N = 2m \\ |a|_0 = k}} 1 \leq N^k (2m)^k$$

for $0 \leq k \leq m$, we can estimate (2.18) to arrive at

$$\begin{aligned} \mathbb{E}\left(\left(\frac{1}{N} \sum_{j=2}^N (\Theta_j - \mathbb{E}(\Theta_j)) \right)^{2m} \right) &\leq N^{-2m} \sum_{\substack{a_2 + a_3 + \dots + a_N = 2m \\ |a|_0 \leq m}} (2m)^{2m} C^{2m} \\ &\leq N^{-2m} \sum_{k=1}^m N^k (2m)^{3m} C^{2m} \leq \frac{C(m) N^m}{N^{2m}} \end{aligned}$$

for some constant $C(m)$. Hence, plugging it into (2.17), we find

$$\mathbb{P}(|H_1(Z) - E(H_1(Z))| \geq N^{-(\delta+\alpha)}) \leq C(m) \frac{N^{2(\delta+\alpha)m+m}}{N^{2m}}.$$

Using the assumption $\delta + \alpha < 1/2$ and choosing m such that $m(-1 + 2(\delta + \alpha)) = \gamma$ proves the proposition. □

The law of large numbers provided in Proposition 2.21 allows to show that the sets, where the desired properties do not hold, are small in probability.

Corollary 2.22. *Let $0 < \alpha, \delta$, $0 < \alpha + \delta < 1/2$, $\varepsilon \sim N^{-\beta}$ with $0 < \beta \leq \alpha$ and define for $0 \leq t \leq T$ the following sets*

$$\begin{aligned} B_t^1 &:= \{ |K^\varepsilon(\mathbf{Y}_t^{N,\varepsilon}) - \overline{K_t^\varepsilon}(\mathbf{Y}_t^{N,\varepsilon})|_\infty \leq N^{-(\delta+\alpha)} \}, \\ B_t^2 &:= \{ |L^\varepsilon(\mathbf{Y}_t^{N,\varepsilon}) - \overline{L_t^\varepsilon}(\mathbf{Y}_t^{N,\varepsilon})|_\infty \leq 1 \}, \end{aligned}$$

where the mean-field particles are close under the kernel K^ε and L^ε, which were defined in Section 2.1.1 and Remark 2.16. Then, for each $\gamma > 0$ there exists a $C(\gamma) > 0$ such that

$$\mathbb{P}\big((B_t^1)^{\mathrm{c}}\big), \mathbb{P}\big((B_t^2)^{\mathrm{c}}\big) \leq C(\gamma) N^{-\gamma}$$

for every $0 \leq t \leq T$, where the constant $C(\gamma)$ is independent of $t \in [0, T]$.

Proof. First, the random variables $(Y_t^{i,\varepsilon}, i = 1, \dots, N)$ are i.i.d. and have a probability density ρ_t^ε given by the solution of the regularized system (2.8). Moreover, we have

$$K_i^\varepsilon(x_1, \dots, x_N) = -\frac{1}{N} \sum_{j=1}^N k^\varepsilon(x_i - x_j), \quad (x_1, \dots, x_N) \in \mathbb{R}^{dN},$$

with k^ε bounded. We recall that we denote by $\mathbb{E}_{(-i)}$ the expectation with respect to every variable but the i-th. Therefore, we get

$$\begin{aligned}\mathbb{E}_{(-i)}(K_i^\varepsilon(\mathbf{Y}_t^{N,\varepsilon})) &= -\frac{1}{N}\sum_{j=1}^N \mathbb{E}(k^\varepsilon(Y_t^{i,\varepsilon} - Y_t^{j,\varepsilon}))\\ &= -\frac{1}{N}\sum_{j=1}^N \int_{\mathbb{R}^d} k^\varepsilon(Y_t^{i,\varepsilon} - z)\rho^\varepsilon(z,t)\,\mathrm{d}z = -(k^\varepsilon * \rho_t)(Y_t^{i,\varepsilon}) = \overline{K_{t,i}^\varepsilon}(\mathbf{Y}_t^{N,\varepsilon})\end{aligned}$$

for all $i = 1, \ldots, N$. As a result, we obtain

$$(B_t^1)^{\mathrm{c}} = \left\{ \sup_{1\le i\le N} |K_i^\varepsilon(\mathbf{Y}_t^{N,\varepsilon}) - \mathbb{E}_{(-i)}(K_i^\varepsilon(\mathbf{Y}_t^{N,\varepsilon}))| > N^{-(\delta+\alpha)} \right\}$$

and therefore, by Proposition 2.21,

$$\mathbb{P}\big((B_t^1)^{\mathrm{c}}\big) \le C(\gamma)N^{-\gamma}.$$

For the set B_t^2 we notice the function $l^\varepsilon N^{-\alpha}$ is bounded since $\varepsilon \sim N^{-\beta}$ and, thus, we can do similar steps as before with the set

$$\begin{aligned}(B_t^2)^{\mathrm{c}} &= \{N^{-\alpha}|L^\varepsilon(\mathbf{Y}_t^{N,\varepsilon}) - \overline{L_t^\varepsilon}(\mathbf{Y}_t^{N,\varepsilon})|_\infty \ge N^{-\alpha}\}\\ &\subseteq \{N^{-\alpha}|L^\varepsilon(\mathbf{Y}_t^{N,\varepsilon}) - \overline{L_t^\varepsilon}(\mathbf{Y}_t^{N,\varepsilon})|_\infty \ge N^{-(\delta+\alpha)}\}.\end{aligned}$$

This proves the corollary. □

2.5. Propagation of chaos in probability

In this section we are going to prove propagation of chaos for the particle system (2.3). We deploy a coupling method with the mean-field SDE (2.2) and show convergence in probability with an arbitrary algebraic rate $N^{-\gamma}$ for $\gamma > 0$. To that end, we present the main result, which states that the trajectory of the N-particle system X^N with $X_0^N \sim \bigotimes_{i=1}^N \rho_0$ typically remains close to the mean-field trajectory Y^N with same starting position $X_0^N = Y_0^N$ during any finite interval $[0, T]$.

Theorem 2.23. *Suppose Assumption 2.1. Let $T > 0$, $\alpha \in \big(0, \frac{1}{2}\big)$ and $(k^\varepsilon, \varepsilon > 0)$ satisfy Assumptions 2.2 and 2.13 with $\varepsilon \sim N^{-\beta}$ for $0 < \beta \le \alpha$. Then, for every $\gamma > 0$, there exists a positive constant $C(\gamma)$ and $N_0 \in \mathbb{N}$ such that*

$$\mathbb{P}\left(\sup_{t\in[0,T]} \big|\mathbf{X}_t^{N,\varepsilon} - \mathbf{Y}_t^{N,\varepsilon}\big|_\infty \ge N^{-\alpha} \right) \le C(\gamma)N^{-\gamma}, \quad \text{for each } N \ge N_0.$$

The constant $C(\gamma)$ depends on the initial density ρ_0, the final time $T > 0$, α and γ. The natural number N_0 also depends on ρ_0, T and α.

To prove Theorem 2.23, we need the following auxiliary lemma.

LEMMA 2.24. [LP17, Lemma 8.1] *For a function $f\colon \mathbb{R}^d \to \mathbb{R}$ we denote the right upper Dini derivative by*

$$\overline{D}_y^+ f(y) := \limsup_{h\to 0^+} \frac{f(y+h)-f(y)}{h}.$$

Let $g \in C^1(\mathbb{R}^d)$ and $h(y) := \sup_{0\le s\le y} g(s)$. Then, one has $\overline{D}_y^+ h(y) \le \max\left(0, \frac{\mathrm{d}}{\mathrm{d}y} g(y)\right)$ for all $y \ge 0$.

PROOF OF THEOREM 2.23. For $T > 0$ and $\alpha \in (0, 1/2)$ and $\delta = \frac{1}{2}(1/2-\alpha) > 0$ let us define the auxiliary process

$$J_t^N := \min\left(1, \sup_{0\le s\le t} e^{\lambda(T-s)}(N^\alpha |\mathbf{X}_s^{N,\varepsilon} - \mathbf{Y}_s^{N,\varepsilon}|_\infty + N^{-\delta})\right),$$

where $\lambda > 0$ is a constant, which will be defined later. In the first step we want to understand how J_t^N helps us to control the maximum distance $|\mathbf{X}_t^{N,\varepsilon} - \mathbf{Y}_t^{N,\varepsilon}|_\infty$. For $0 \le t \le T$ we have

$$\sup_{0\le s\le t} N^\alpha |\mathbf{X}_s^{N,\varepsilon} - \mathbf{Y}_s^{N,\varepsilon}|_\infty \le \sup_{0\le s\le t} e^{\lambda(T-s)}(N^\alpha |\mathbf{X}_s^{N,\varepsilon} - \mathbf{Y}_s^{N,\varepsilon}|_\infty + N^{-\delta}). \tag{2.19}$$

Hence, if $J_t^N < 1$ we obtain $\sup_{0\le s\le t} N^\alpha |\mathbf{X}_s^{N,\varepsilon} - \mathbf{Y}_s^{N,\varepsilon}|_\infty \le J_t^N < 1$. Furthermore, we can assume $N \ge N_0$ such that $J_o^N = e^{\lambda(T-s)} N^{-\delta} < \frac{1}{2}$ with N_0 depending on T, λ. As a result, we find

$$\begin{aligned}\mathbb{P}\left(\sup_{t\in[0,T]} |\mathbf{X}_t^{N,\varepsilon} - \mathbf{Y}_t^{N,\varepsilon}|_\infty \ge N^{-\alpha}\right) &\le \mathbb{P}(J_T^N \ge 1) \le \mathbb{P}\left(J_T^N - J_0^N \ge \frac{1}{2}\right) \\ &\le 2\mathbb{E}(J_T^N - J_0^N) \le 2\mathbb{E}\left(\int_0^T \overline{D}_t^+ J_t^N \,\mathrm{d}t\right) \\ &= 2\int_0^T \mathbb{E}(\overline{D}_t^+ J_t^N)\,\mathrm{d}t,\end{aligned} \tag{2.20}$$

where we used a more general fundamental theorem of calculus, see e.g. [HT06, Theorem 11], in the last inequality. In the next step we want to estimate the Dini derivative $\overline{D}_t^+ J_t^N$. Applying Lemma 2.24, we discover

$$\overline{D}_t^+ J_t^N \le \max\left(0, \frac{\mathrm{d}}{\mathrm{d}t} g(t)\right) \tag{2.21}$$

with $g(t) := e^{\lambda(T-t)}(N^\alpha |\mathbf{X}_t^{N,\varepsilon} - \mathbf{Y}_t^{N,\varepsilon}|_\infty + N^\delta)$. Computing the derivative, we find

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t) \le -\lambda e^{\lambda(T-t)}(N^\alpha |\mathbf{X}_t^{N,\varepsilon} - \mathbf{Y}_t^{N,\varepsilon}|_\infty + N^{-\delta}) + e^{\lambda(T-t)} N^\alpha |K^\varepsilon(\mathbf{X}_t^{N,\varepsilon}) - \overline{K}_t^\varepsilon(\mathbf{Y}_t^{N,\varepsilon})|_\infty \tag{2.22}$$

with K^ε and $\overline{K}_t^\varepsilon$ defined as in (2.5) and (2.6), respectively. Next, let us introduce the set $A_t := \{\overline{D}_t^+ J_t^N > 0\}$ and notice that (2.21) implies $A_t \subseteq \{\overline{D}_t^+ J_t^N \le \frac{\mathrm{d}}{\mathrm{d}t} g(t)\}$. Hence, we discover

$$\mathbb{E}(\overline{D}_t^+ J_t^N) = \mathbb{E}(\overline{D}_t^+ J_t^N \mathbb{1}_{A_t}) + \mathbb{E}(\overline{D}_t^+ J_t^N \mathbb{1}_{A_t^c}) \le \mathbb{E}\left(\frac{\mathrm{d}}{\mathrm{d}t} g(t) \mathbb{1}_{A_t}\right).$$

In combination with (2.20) we see that, in order to prove the theorem, it is enough to show that $\mathbb{E}(\frac{\mathrm{d}}{\mathrm{d}t}g(t)\mathbb{1}_{A_t})$ is bounded by $C(\gamma)N^{-\gamma}$ for some constant $C(\gamma) > 0$ and $t \in [0,T]$.

At this moment let us recall the sets B_t^1, B_t^2 from Section 2.4, where the "good" properties hold to further reduce the problem. We have

$$\begin{aligned}\mathbb{E}\left(\frac{\mathrm{d}}{\mathrm{d}t}g(t)\mathbb{1}_{A_t}\right) &= \mathbb{E}\left(\frac{\mathrm{d}}{\mathrm{d}t}g(t)\mathbb{1}_{A_t\cap B_t^1\cap B_t^2}\right) + \mathbb{E}\left(\frac{\mathrm{d}}{\mathrm{d}t}g(t)\mathbb{1}_{A_t\cap(B_t^1\cap B_t^2)^{\mathrm{c}}}\right)\\ &\leq \mathbb{E}\left(\frac{\mathrm{d}}{\mathrm{d}t}g(t)\mathbb{1}_{A_t\cap B_t^1\cap B_t^2}\right) + CN^{\alpha}\big(\mathbb{P}\big((B_t^1)^{\mathrm{c}}\big) + \mathbb{P}\big((B_t^2)^{\mathrm{c}}\big)\big)\\ &\leq \mathbb{E}\left(\frac{\mathrm{d}}{\mathrm{d}t}g(t)\mathbb{1}_{A_t\cap B_t^1\cap B_t^2}\right) + C(\gamma)N^{-\gamma},\end{aligned}$$

where we used the fact that the interaction force approximation k^ε is uniformly bounded in the first inequality and thus we have $|\frac{\mathrm{d}}{\mathrm{d}t}g(t)| \leq CN^{\alpha}$ with the help of (2.22). The last inequality follows immediately from Corollary 2.22 and relabeling γ. It is therefore enough to prove that $\frac{\mathrm{d}}{\mathrm{d}t}g(t) \leq 0$ holds under the event $A_t \cap B_t^1 \cap B_t^2$. This is equivalent to the inequality

$$e^{\lambda(T-t)}N^{\alpha}|K^\varepsilon(\mathbf{X}_t^{N,\varepsilon}) - \overline{K}_t^\varepsilon(\mathbf{Y}_t^{N,\varepsilon})|_\infty \leq \lambda e^{\lambda(T-t)}(N^\alpha|\mathbf{X}_t^{N,\varepsilon} - \mathbf{Y}_t^{N,\varepsilon}|_\infty + N^{-\delta}). \tag{2.23}$$

We observe that on A_t we have $J_t^N < 1$. In fact, let $J_t^N \geq 1$ and remember that J_t^N is an non-decreasing function bounded by 1. Consequently, the right upper Dini derivative vanishes and we are in the set A_t^{c}. Together with (2.19) this means

$$\sup_{0\leq s\leq t} |\mathbf{X}_s^{N,\varepsilon} - \mathbf{Y}_s^{N,\varepsilon}|_\infty \leq N^{-\alpha} \tag{2.24}$$

holds on A_t. Splitting up the term on the left-hand side of (2.23), we obtain

$$\begin{aligned}|K^\varepsilon(\mathbf{X}_t^{N,\varepsilon}) - \overline{K}_t^\varepsilon(\mathbf{Y}_t^{N,\varepsilon})|_\infty &\leq |K^\varepsilon(\mathbf{X}_t^{N,\varepsilon}) - K^\varepsilon(\mathbf{Y}_t^{N,\varepsilon})|_\infty + |K^\varepsilon(\mathbf{Y}_t^{N,\varepsilon}) - \overline{K}_t^\varepsilon(\mathbf{Y}_t^{N,\varepsilon})|_\infty\\ &\leq |L^\varepsilon(\mathbf{Y}_t^{N,\varepsilon})|_\infty|\mathbf{X}_t^{N,\varepsilon} - \mathbf{Y}_t^{N,\varepsilon}|_\infty + N^{-(\delta+\alpha)}\\ &\leq (C + |\overline{L_t^\varepsilon}(\mathbf{Y}_t^{N,\varepsilon})|_\infty)|\mathbf{X}_t^{N,\varepsilon} - \mathbf{Y}_t^{N,\varepsilon}|_\infty + N^{-(\delta+\alpha)}\\ &\leq C(\rho_0,T)(|\mathbf{X}_t^{N,\varepsilon} - \mathbf{Y}_t^{N,\varepsilon}|_\infty + N^{-(\delta+\alpha)}),\end{aligned}$$

where we used the local Lipschitz bound from Assumption 2.13, inequality (2.24) and the condition of event B_t^1 in the second inequality. Then, we applied the condition of B_t^2 in the third inequality and finally Assumption 2.13 in the last inequality. Inserting this back into the left-hand side of (2.23), we discover

$$\begin{aligned}e^{\lambda(T-t)}N^{\alpha}|K^\varepsilon(\mathbf{X}_t^{N,\varepsilon}) - \overline{K}_t^\varepsilon(\mathbf{Y}_t^{N,\varepsilon})|_\infty &\leq e^{\lambda(T-t)}N^\alpha C(\rho_0,T)(|\mathbf{X}_t^{N,\varepsilon} - \mathbf{Y}_t^{N,\varepsilon}|_\infty + N^{-(\delta+\alpha)})\\ &= C(\rho_0,T)e^{\lambda(T-t)}(N^\alpha|\mathbf{X}_t^{N,\varepsilon} - \mathbf{Y}_t^{N,\varepsilon}|_\infty + N^{-\delta}).\end{aligned}$$

Choosing $\lambda = C(\rho_0,T)$ provides (2.23) and concludes the proof. □

REMARK 2.25. *The cut-off $\alpha \in (0,1/2)$ was only used in Corollary 2.22 to bound the set B_t^2. Hence, one possibility on improving the cut-off is to optimize Proposition 2.21 in order to handle more general cut-off functions.*

From Theorem 2.23 it immediately follows that the marginals of X_t^N and Y_t^N converge in the Wasserstein metric, see e.g. [CG17, Corollary 2.2]. For the sake of completeness we include the statement below.

Corollary 2.26. [CG17, Corollary 2.2.] *Let the assumptions of Theorem 2.23 hold. Consider the probability density $\rho_t^{\otimes N,\varepsilon}$ of $\mathbf{Y}_t^{N,\varepsilon}$ and $\rho_t^{N,\varepsilon}$ the probability density of $\mathbf{X}_t^{N,\varepsilon}$. Then, $\rho_t^{N,k,\varepsilon}$ converges weakly (in the sense of measures) to $\rho_t^{\otimes k,\varepsilon}$ as $N \to \infty$, $\varepsilon(N) \to 0$ for each fixed $k \geq 1$. Furthermore, the probability density $\rho_t^{N,\varepsilon}$ converges weakly (in the sense of measures) to the same measure as $\rho_t^{\otimes N,\varepsilon}$ as $N \to \infty$. More precisely, there exists a positive constant C and $N_0 \in \mathbb{N}$ such that*

$$\sup_{t\in[0,T]} W_1(\rho_t^{N,k,\varepsilon}, \rho_t^{\otimes k,\varepsilon}), \sup_{t\in[0,T]} W_1(\rho_t^{N,\varepsilon}, \rho_t^{\otimes N,\varepsilon}) \leq C(\rho_0, T, \alpha) N^{-\alpha}$$

holds for each $k \geq 1$ and $N \geq N_0$, where W_1 denotes the Wasserstein metric

$$W_1(\mu, \nu) := \inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{1}{k}\sum_{i=1}^{k} |x^i - y^i| \,\mathrm{d}\pi(x, y)$$

and $\Pi(\mu, \nu)$ is the set of all probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with first marginal μ and second marginal ν. The constant $C(\rho_0, T, \alpha)$ depends on the initial condition ρ_0, the final time T and α. Moreover, $N_0 \in \mathbb{N}$ is the same as in Theorem 2.23.

Corollary 2.26 implies the weak convergence in the sense of measures of the k-th marginal $\rho_t^{N,k,\varepsilon}$ to the product measure $\rho_t^{\otimes k}$. Indeed, since $\rho_t^{N,k,\varepsilon}$ converges weakly to $\rho_t^{\otimes k,\varepsilon}$, it is sufficient to show that $\rho_t^{\otimes k,\varepsilon}$ converges weakly to $\rho_t^{\otimes k}$. By the classic result [Szn91, Proposition 2.2] we can consider the special case $k = 2$, i.e $\rho_t^\varepsilon \otimes \rho_t^\varepsilon$ converges weakly to $\rho_t \otimes \rho_t$. We can further reduce it by applying [Pat13, Theorem 2.8], which tells us that it is enough to show ρ_t^ε converges weakly to ρ_t.

Lemma 2.27. *Let $T > 0$ and suppose Assumption 2.1. Moreover, let $(\rho^\varepsilon, \varepsilon > 0)$ and ρ be the weak solutions obtained in theorems 2.6 and 2.9. Then, one has*

$$\sup_{t\in[0,T]} \left| \int_{\mathbb{R}^d} (\rho_t^\varepsilon(x) - \rho_t(x))\phi(x) \,\mathrm{d}x \right| \xrightarrow[\varepsilon\to 0]{} 0 \tag{2.25}$$

for all $\phi \in L^\infty(\mathbb{R}^d)$. In particular, $\rho_t^\varepsilon \otimes \rho_t^\varepsilon$ converges weakly to $\rho_t \otimes \rho_t$ for all $t \geq 0$ in the sense of measures.

Remark 2.28. *Suppose the assumptions of Theorem 2.23 hold. Then, Lemma 2.27 together with the discussion before Lemma 2.27 and Proposition 1.2 imply that, for all $t \in [0, T]$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^{i,\varepsilon}} = \rho_t$$

in law as measure valued random variables if $\varepsilon \sim N^{-\beta}$.

Proof of Lemma 2.27. First, we notice that the convergence is uniform in time. Therefore, the strong convergence result from Lemma 2.8 cannot be applied. We start by showing (2.25) holds for $\phi \in H^1(\mathbb{R}^d)$. To that end, let us assume ϕ is in a dense subset and smooth enough, i.e. $\phi \in C_c^\infty(\mathbb{R}^d)$. Now, let $0 \le t_1 < t_2 \le T$. Then, the uniform bound on $\frac{\mathrm{d}}{\mathrm{d}t}\rho_t^\varepsilon$ (see (2.12)) and integration by parts [Zei90, Theorem 23.23] implies

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} \rho^\varepsilon(t_1, x)\phi(x)\,\mathrm{d}x - \int_{\mathbb{R}^d} \rho^\varepsilon(t_2, x)\phi(x)\,\mathrm{d}x \right| \\
&= \left| \int_{t_1}^{t_2} \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\rho_t^\varepsilon, \phi \right\rangle_{H^{-1}(\mathbb{R}^d), H^1(\mathbb{R}^d)} \mathrm{d}t \right| \\
&\le |t_2 - t_1|^{1/2} \left\| \frac{\mathrm{d}}{\mathrm{d}t}\rho^\varepsilon \right\|_{L^2([0,T];H^{-1}(\mathbb{R}^d))} \|\phi\|_{H^1(\mathbb{R}^d)} \\
&\le C|t_2 - t_1|^{1/2} \|\phi\|_{H^1(\mathbb{R}^d)}.
\end{aligned}$$

Consequently, the sequence of function $t \mapsto \int_{\mathbb{R}^d} \rho_t^\varepsilon(x)\phi(x)\,\mathrm{d}x$ is equicontinuous. Using the $L^\infty([0,T]; L^2(\mathbb{R}^d))$-bound, we also get a uniform bound on the sequence. As a result, we can apply the Arzela–Ascoli theorem to obtain a convergent subsequence, which depends on ϕ and will be denoted by $(\rho^{\varepsilon(\phi)}, \varepsilon(\phi) \in \mathbb{N})$ such that $\int_{\mathbb{R}^d} \rho_t^{\varepsilon(\phi)}\phi\,\mathrm{d}x \to \zeta(\phi)$ in $C([0,T])$. By the fundamental lemma of calculus of variation and the fact that $\rho_t^{\varepsilon(\phi)}$ converges weakly in $L^2([0,T]; L^2(\mathbb{R}^d))$ we can identify the limit $\zeta(\phi) = \int_{\mathbb{R}^d} \rho_t\phi\,\mathrm{d}x$. Since ϕ was taken from a dense subset of $H^1(\mathbb{R}^d)$, we can use a diagonal argument to obtain a subsequence, which will be not renamed, such that, for $\phi \in H^1(\mathbb{R}^d)$,

$$\sup_{t\in[0,T]} \left| \int_{\mathbb{R}^d} (\rho_t^{\varepsilon(\phi)}(x) - \rho_t(x))\phi(x)\,\mathrm{d}x \right| \xrightarrow[\varepsilon(\phi)\to 0]{} 0. \tag{2.26}$$

With another density argument and the uniform bound of $(\rho^\varepsilon, \varepsilon \ge 0)$ in $L^\infty([0,T]; L^2(\mathbb{R}^d))$ we obtain for each $\phi \in L^2(\mathbb{R}^d)$ a subsequence $(\rho_t^{\varepsilon(\phi)}, \varepsilon(\phi) \in \mathbb{N})$ such that (2.26) holds. Again, since $L^2(\mathbb{R}^d)$ is separable we can use another diagonal argument to show that we can obtain a subsequence $(\rho_t^{\varepsilon_k}, k \in \mathbb{N})$ such that (2.26) holds for all $\phi \in L^2(\mathbb{R}^d)$. Notice that this subsequence is independent of the function ϕ. Furthermore, the uniqueness of the limit implies that (2.26) actually holds for any sequence $(\rho_t^{\varepsilon(N)}, \varepsilon(N) > 0)$ itself, where $\varepsilon(N)$ is some sequence depending on N such that $\varepsilon(N) \to 0$ as $N \to \infty$.

Next, for $\phi \in L^\infty(\mathbb{R}^d)$, we apply Lemma 2.7 and the fact that $\phi(x)\mathbb{1}_{\{|x|\le R\}} \in L^2(\mathbb{R}^d)$ to find

$$\begin{aligned}
&\sup_{t\in[0,T]} \left| \int_{\mathbb{R}^d} (\rho_t^\varepsilon(x) - \rho_t(x))\phi(x)\,\mathrm{d}x \right| \\
&\le \sup_{t\in[0,T]} \left| \int_{\mathbb{R}^d} (\rho_t^\varepsilon(x) - \rho_t(x))\phi(x)\mathbb{1}_{\{|x|\le R\}}\,\mathrm{d}x \right| \\
&\quad + \sup_{t\in[0,T]} \left| \int_{\mathbb{R}^d} (\rho_t^\varepsilon(x) - \rho_t(x))\phi(x)\mathbb{1}_{\{|x|\ge R\}}\,\mathrm{d}x \right| \\
&\le \sup_{t\in[0,T]} \left| \int_{\mathbb{R}^d} (\rho_t^\varepsilon(x) - \rho_t(x))\phi(x)\mathbb{1}_{\{|x|\le R\}}\,\mathrm{d}x \right|
\end{aligned}$$

$$+\frac{\|\phi\|_{L^\infty(\mathbb{R}^d)}}{R^2}\sup_{t\in[0,T]}\int_{\mathbb{R}^d}|\rho_t^\varepsilon(x)+\rho_t(x)||x|^2\,\mathrm{d}x$$
$$\le \sup_{t\in[0,T]}\left|\int_{\mathbb{R}^d}(\rho_t^\varepsilon(x)-\rho_t(x))\phi(x)\mathbb{1}_{\{|x|\le R\}}\,\mathrm{d}x\right|+C(\rho_0)\,\|\phi\|_{L^\infty(\mathbb{R}^d)}\,R^{-2}.$$

Letting $\varepsilon \to 0$ and then $R \to \infty$, we obtain (2.25) and the corollary is proven. □

2.6. Comments

The main inequality of Chapter 2 is given in Assumption 2.13, which resembles a Taylor expansion around the point y without an error term. Much of the work by Pickl and Lazarovici is built upon this foundational observation. An alternative, is presented by Prof. Chen's idea to perform an actual Taylor expansion. This approach is more natural and easier to follow. Additionally, it eliminates the need for the Assumption 2.13, allowing the inclusion of more models. However, the drawback is in estimating the Lipschitz bound (Lemma 2.17). In Taylor's approximation, the Lipschitz bound is equivalent to the first derivative. Our proof transfers the local Lipschitz bound on the interacting particle system $\mathbf{X}^N$ to the i.i.d. mean-field particle system $\mathbf{Y}^N$ via the law of large numbers, and then utilizes the properties of the mean-field solution. This corresponds to the "top-to-bottom" approach. Consequently, we need to estimate

$$\sup_{x\in\mathbb{R}^d}|\nabla k^\varepsilon| * \rho_t^\varepsilon(x),$$

if we want to follow Taylor's approximation. Without additional structure on the interaction kernel, we encounter difficulties and cannot proceed further with the known methods. The main problem lies in the fact that, even though we have a convolution structure, we cannot flip the derivative onto the solution, rendering the regularity we obtained from the mean-field problem obsolete. Therefore, similar assumptions to ours are necessary to demonstrate convergence in probability, and there appears to be no mathematical advantage gained.

Chapter 3

Quantitative estimates for the relative entropy

Having established convergence probability for bounded kernels k in Chapter 2, one can ask the question whether the convergence of the same N particle system

$$\mathrm{d}X_t^i = -\frac{1}{N}\sum_{j=1}^{N} k(X_t^i - X_t^j)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t^i, \quad i = 1, \dots, N, \;\; \mathbf{X}_0^N \sim \bigotimes_{i=1}^{N} \rho_0 \tag{3.1}$$

with i.i.d. initial condition $(X_0^i, i = 1, \dots, N)$, d-dimensional Brownian motion B_t^i, $k\colon \mathbb{R}^d \mapsto \mathbb{R}^d$, and $\sigma > 0$, can be demonstrated in some stronger sense?

In this chapter we explore the relative entropy method [JW18] and modulated energy method [Ser20] by using our findings form Chapter 2 about the convergence in probability. The chapter should provide a more detailed view into the modulated energy and relative entropy method, which we have already seen a glimpse of in Section 1.3.4. We know from Section 1.3.4 that all this quantities are in a sense stronger than the notion of weak convergence. We will describe the connection between the various forms of convergences and find a remarkable relation to the earlier work by Oelschläger [Oel87]. This provides new insights into how to tackle singular interaction kernels. In particular, attractive Coulomb kernels.

It also works as a foundation for Chapter 5, where we will investigate the relative entropy approach for interacting particle systems with common noise. Many concepts will reappear, so it's essential to understand them in the non-stochastic setting first, where the limiting equations are deterministic rather than stochastic, as they are in Chapter 5.

This chapter is based on [PN24].

3.1. Problem setting

The setting remains the same as in Chapter 2. For the sake of completeness, we briefly recall it. We will primarily focus on the interacting particle systems and diffusion-aggregation equations on an intermediate level. Therefore, we may temporarily set aside the issue of well-posedness for the non-regularized systems. If we are interested in the non-regularized case, we will always assume $k \in L^\infty(\mathbb{R}^d)$ in this chapter. Additionally, we will need to introduce the Liouville equation, which is satisfied by the density of the interacting particle system itself.

3.1.1. Assumption on initial condition. The assumptions are similar to those in Chapter 2. However, since we are only at an intermediate level, we require only the following assumptions on the initial condition ρ_0 of the interacting particle system throughout this chapter.

ASSUMPTION 3.1. *The initial condition* $\rho_0 \colon \mathbb{R}^d \to \mathbb{R}$ *fulfills*

$$\rho_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, |x|^2 \,\mathrm{d}x), \quad \rho_0 \geq 0, \quad \text{and} \quad \int_{\mathbb{R}^d} \rho_0(x) \,\mathrm{d}x = 1. \tag{3.2}$$

3.1.2. Interacting particle systems. The N-particle system $\mathbf{X}_t^N := (X_t^1, \dots, X_t^N)$ is given by (3.1). The particle system (3.1) induces in the limiting case $N \to \infty$ the following i.i.d. sequence $\mathbf{Y}_t^N := (Y_t^1, \dots, Y_t^N)$ of mean-field particles

$$\mathrm{d}Y_t^i = -(k * \rho_t)(Y_t^i) \,\mathrm{d}t + \sigma \,\mathrm{d}B_t^i, \quad i = 1, \dots, N, \;\; \mathbf{Y}_0^N = \mathbf{X}_0^N, \tag{3.3}$$

where $\rho_t := \rho(t, \cdot)$ denotes the probability density of the i.i.d. random variable Y_t^i.

To introduce the regularized versions of (3.1) and (3.3), we take a smooth approximation $(k^\varepsilon, \varepsilon > 0)$ of k and replace the drift term with its approximation. Hence, the regularized microscopic N-particle system $\mathbf{X}_t^{N,\varepsilon} := (X_t^{1,\varepsilon}, \dots, X_t^{N,\varepsilon})$ is given by

$$\mathrm{d}X_t^{i,\varepsilon} = -\frac{1}{N} \sum_{j=1}^N k^\varepsilon(X_t^{i,\varepsilon} - X_t^{j,\varepsilon}) \,\mathrm{d}t + \sigma \,\mathrm{d}B_t^i, \quad i = 1, \dots, N, \;\; \mathbf{X}_0^{N,\varepsilon} \sim \bigotimes_{i=1}^N \rho_0, \tag{3.4}$$

and the regularized mean-field trajectories $\mathbf{Y}_t^{N,\varepsilon} := (Y_t^{1,\varepsilon}, \dots, Y_t^{N,\varepsilon})$ by

$$\mathrm{d}Y_t^{i,\varepsilon} = -(k^\varepsilon * \rho_t^\varepsilon)(Y_t^{i,\varepsilon}) \,\mathrm{d}t + \sigma \,\mathrm{d}B_t^i, \quad i = 1, \dots, N, \;\; \mathbf{Y}_0^{N,\varepsilon} = \mathbf{X}_0^{N,\varepsilon}, \tag{3.5}$$

where $\rho_t^\varepsilon := \rho^\varepsilon(t, \cdot)$ denotes the probability density of the i.i.d. random variable $Y_t^{i,\varepsilon}$.

In the following computations, we will encounter lengthy expressions, and to maintain clarity in notation, we abbreviate the empirical measure $\mu_t^{\mathbf{X}^{N,\varepsilon}}$ by $\mu_t^{N,\varepsilon}$.

3.1.3. Diffusion-aggregation equations and Liouville equations. The interacting particle system (3.1) induces the following Liouville equation on $\mathbb{R}^{dN}$,

$$\begin{cases} \partial_t \rho_t^N(\mathsf{X}^N) &= \frac{\sigma^2}{2} \sum\limits_{i=1}^N \Delta_{x_i} \rho_t^N(\mathsf{X}^N) + \sum\limits_{i=1}^N \nabla \cdot \left(\rho_t^N(\mathsf{X}^N) \frac{1}{N} \sum\limits_{j=1}^N k(x_i - x_j) \right), \\ \rho_0^N(\mathsf{X}^N) &= \prod\limits_{i=1}^N \rho_0(x_i), \end{cases} \tag{3.6}$$

for $\mathsf{X}^N = (x_1, \dots, x_N) \in \mathbb{R}^{dN}$. The system (3.3) induces the non-linear aggregation-diffusion equation

$$\begin{cases} \partial_t \rho_t = \frac{\sigma^2}{2} \Delta \rho + \nabla \cdot (\rho_t k * \rho_t), & \forall\, (t, x) \in [0, T] \times \mathbb{R}^d, \\ \quad \rho(x, 0) = \rho_0, & \forall\, x \in \mathbb{R}^d. \end{cases}, \tag{3.7}$$

The regularized particle system (3.4) is associated to the Liouville equation

$$\begin{cases} \partial_t \rho_t^{N,\varepsilon}(\mathsf{X}^N) = \frac{\sigma^2}{2} \sum\limits_{i=1}^N \Delta_{x_i} \rho_t^{N,\varepsilon}(\mathsf{X}^N) + \sum\limits_{i=1}^N \nabla_{x_i} \cdot \left(\rho_t^{N,\varepsilon}(\mathsf{X}^N) \frac{1}{N} \sum\limits_{j=1}^N k^\varepsilon(x_i - x_j) \right), \\ \quad \rho_0^{N,\varepsilon}(\mathsf{X}^N) = \prod\limits_{i=1}^N \rho_0(x_i), \end{cases} \tag{3.8}$$

and the regularized system (3.5) to the aggregation-diffusion equation

$$\begin{cases} \partial_t \rho_t^\varepsilon = \frac{\sigma^2}{2}\Delta\rho_t^\varepsilon + \nabla\cdot(\rho_t^\varepsilon k^\varepsilon * \rho_t^\varepsilon), & \forall\,(t,x)\in[0,T]\times\mathbb{R}^d,\\ \rho^\varepsilon(x,0)=\rho_0, & \forall\, x\in\mathbb{R}^d. \end{cases} \tag{3.9}$$

Analogously to Chapter 2 we use ρ_t and ρ_t^ε for the solutions of the PDEs (3.7) and (3.9) as well as for the probability densities of the particle systems (3.3) and (3.5), respectively.

Furthermore, we require the marginal of the system of rank $1\le r\le N$,

$$\rho_t^{N,r} = \int_{\mathbb{R}^{d(N-r)}} \rho_t^N(x_1,\dots,x_N)\,\mathrm{d}x_{r+1}\dots\,\mathrm{d}x_N. \tag{3.10}$$

We remark that the r-th martingale solves the following Liouville equation

$$\begin{aligned} \partial_t\rho_t^{N,r} = \frac{\sigma^2}{2}\sum_{i=1}^r \int_{\mathbb{R}^{d(N-r)}} &\Delta_{x_i}\rho_t^N(x_1,\dots,x_N)\\ &+\nabla_{x_i}\cdot\left(\rho_t^N(x_1,\dots,x_N)\frac{1}{N}\sum_{j=1}^N k(x_i-x_j)\right)\mathrm{d}x_{r+1}\dots\,\mathrm{d}x_N. \end{aligned} \tag{3.11}$$

Similar to (3.10), we denote by $\rho_t^{N,r,\varepsilon}$ the r-th marginal of the approximated Liouville equation, i.e.

$$\rho_t^{N,r,\varepsilon} := \int_{\mathbb{R}^{d(N-m)}} \rho_t^{N,\varepsilon}(x_1,\dots,x_N)\,\mathrm{d}x_{m+1}\dots\,\mathrm{d}x_N,$$

which solves (3.11) with k^ε instead of k. Additionally, we define the chaotic law

$$\rho_t^{\otimes r,\varepsilon}(x_1,\dots,x_m) := \prod_{i=1}^r \rho_t^\varepsilon(x_i),$$

which solves the following equation

$$\begin{aligned} &\partial_t\rho_t^{\otimes r,\varepsilon}(x_1,\dots,x_r)\\ &\quad= \frac{\sigma^2}{2}\sum_{i=1}^r \Delta_{x_i}\rho_t^{\otimes r,\varepsilon}(x_1,\dots,x_r) + \sum_{i=1}^r \nabla_{x_i}\cdot\Big((k*\rho_t^\varepsilon)(x_i)\rho_t^{\otimes r,\varepsilon}(x_1,\dots,x_r)\Big) \end{aligned}$$

with initial condition $\rho_0^{\otimes r,\varepsilon} = \rho_0^{\otimes r}$.

For the partial differential equations (3.6), (3.7), (3.8) and (3.9) we rely on the concept of weak solutions. For the PDE's (3.7) and (3.9) we refer to Definition 2.3. For the Liouville equation (3.8) we define a weak solution in a similar fashion.

Definition 3.2 (Weak solutions). *Let $\varepsilon>0$. A function $\rho^{N,\varepsilon}\in L^2([0,T];H^1(\mathbb{R}^{dN}))\cap L^\infty([0,T];L^2(\mathbb{R}^N))$ with $\partial_t\rho^{N,\varepsilon}\in L^2([0,T];H^{-1}(\mathbb{R}^{dN}))$ is a weak solution of* (3.8) *if for every*

$\eta \in L^2([0,T];H^1(\mathbb{R}^{dN}))$,

$$\int_0^T \langle \partial_t \rho_t^{N,\varepsilon}, \eta_t \rangle_{H^{-1}(\mathbb{R}^{dN}),H^1(\mathbb{R}^{dN})} \,\mathrm{d}t = -\sum_{i=1}^N \int_0^T \int_{\mathbb{R}^{dN}} \Bigg(\frac{\sigma^2}{2} \nabla_{x_i} \rho_t^{N,\varepsilon}(\mathsf{X}^N) \cdot \nabla_{x_i} \eta_t(\mathsf{X}^N)$$

$$+ \frac{1}{N} \sum_{j=1}^N k^\varepsilon(x_i - x_j) \rho_t^{N,\varepsilon}(\mathsf{X}^N) \Bigg) \cdot \nabla_{x_i} \eta_t(\mathsf{X}^N) \,\mathrm{d}\mathsf{X}^N \,\mathrm{d}t \tag{3.12}$$

and $\rho^{N,\varepsilon}(0,\mathsf{X}^N) = \prod_{i=1}^N \rho_0(x_i)$. *Similar,* $\rho^N \in L^2([0,T];H^1(\mathbb{R}^{dN})) \cap L^\infty([0,T];L^2(\mathbb{R}^{dN}))$ *is a solution of* (3.6)*, if* $\partial_t \rho^N \in L^2([0,T];H^{-1}(\mathbb{R}^{dN}))$ *and* (3.12) *holds with the interaction force kernel* k *instead of its approximation* k^ε.

Remark 3.3. *We note that the regularity* $\rho^{N,\varepsilon} \in L^2([0,T];H^1(\mathbb{R}^{dN}))$ *and time regularity* $\partial_t \rho^{N,\varepsilon} \in L^2([0,T];H^{-1}(\mathbb{R}^{dN}))$ *imply* $\rho^{N,\varepsilon} \in C([0,T];L^2(\mathbb{R}^{dN}))$ [Eva10, Chapter 5.9]. *The same statement holds for the solution* ρ^N *of the Liouville equation.*

As in Section 2.2 we can weaken the assumption on the test function η to the regularity $\eta \in C([0,T];C_c^\infty(\mathbb{R}^d))$. Also recall that mass conservation holds by Remark 2.5.

We recall some general facts regarding the well-posedness of the interacting particle systems and Fokker–Planck equations, which will be used throughout the thesis. We may assume that k^ε is bounded. First, we observe that we have a solution $(\rho_t^{N,\varepsilon}, t \geq 0)$ of the regularized PDE (3.8) in the sense of Definition 3.2, which follows from the regularity of k^ε. In the case $k \in L^\infty(\mathbb{R}^d)$, we obtain a solution $(\rho_t^N, t \geq 0)$ by the same methods as well. According to SDE theory [HRZ24, Theorem 3.7 and Theorem 4.10], we also obtain strong solutions $\big(\mathbf{X}_t^{N,\varepsilon}, t \geq 0\big)$, $\big(\mathbf{Y}_t^{N,\varepsilon}, t \geq 0\big)$ to the regularized SDEs (3.4), (3.5). Similar, in the case $k \in L^\infty(\mathbb{R}^d)$ the results imply strong solutions of the particle system (3.1) and McKean–Vlasov SDE (3.3), respectively. Additionally, Section 2.2 guarantees the well-posedness of PDE (3.9) and (3.7) in the case $k \in L^\infty(\mathbb{R}^d)$.

Consequently, our framework is well-defined in the case $k \in L^\infty(\mathbb{R}^d)$ and, in particular, the empirical measure $\mu_t^{\mathbf{X}^{N,\varepsilon}}(\omega)$ given by (1.1) with the associated interacting particle system (3.1) is well-defined. Outside of $k \in L^\infty(\mathbb{R}^d)$, we will only focus on the intermediate regime, where well-posedness is not an issue, since the minimal requirement for k^ε is to be bounded.

Our contribution: We present a novel method to derive propagation of chaos in relative entropy on the whole space for both non-conservative field and potential field possessing a convolution structure. Inspired by Oelschläger [Oel87], the presented method is based on the crucial observation that, under the convolution structure of the interaction kernel k, the expectation of the mollified weighted L^2-norm and the modulated energy (also as a weighted L^2-norm) can be estimated using the dynamics of the underlying systems in conjunction with propagation of chaos in probability, as demonstrated in [LP17, HLP20, FHS19] and Chapter 2. The key contribution of the present work lies in the technique of combining propagation of chaos in probability [LP17, HLL19, FHS19, HKPZ19, CCS19, HLP20, CLPY20, CNP24] with the underlying entropy structure from [JW18, Ser20] and the fluctuation estimates in [Oel87]. Consequently, we prove that convergence in probability, which is obtained by some type of mollification technique, implies convergence in relative entropy for an algebraic cut-off $N^{-\beta}$.

This demonstrates that convergence in probability is actually a quite strong convergence result on the intermediate level.

We emphasize that the main quantitative estimate, Theorem 3.14, is presented in a general manner, allowing its application to a wide range of kernels. We refer to Remark 3.15 for more details and to Section 3.6 for some interesting examples from the fields of chemotaxis and opinion dynamics. In particular, the method can be further applied in handling the attractive and repulsive Coulomb interaction potential in dimension $d \geq 2$, which includes the Keller–Segel model. Finally, we derive an estimate on the supremum norm in time of the relative entropy between the law of the approximated particle system and the chaotic law of the approximated mean-field SDE system of rate greater than $1/2$. Moreover, the approximation is of algebraic order, which is sharper than the logarithmic cut-off derived from the standard coupling method [Szn91, LY19].

Based on the results from Chapter 2 we also provide convergence of the intermediate system to the true mean-field limit for bounded interaction kernels k in the L^1-norm. Consequently, we prove the L^1-convergence of the r-th marginal of the Liouville equation to the r-th chaotic law of the non-linear diffusion-aggregation equation. This final convergence result is only presented for bounded kernels since, in general, the existence for the linear Liouville equation (3.6) on $\mathbb{R}^{dN}$ is not given, see [BJW19, Proposition 4.2] for the torus setting.

Related literature: Motivated by models, particularly from physics, with bounded measurable or even singular interaction force kernels, extensive efforts have been devoted to investigated propagation of chaos for particle systems with such kernels. Initially, approaches to treat such irregular kernels were often based on compactness methods in combination with the martingale problems associated to the McKean–Vlasov SDEs, see e.g. [Oel84, Osa87, Gär88, FHM14, GQ15, FJ17, LLY19]. For general singular L^p-interaction force kernels k, propagation of chaos was demonstrated for first and second order systems on the torus [BJW23] and on the whole space $\mathbb{R}^d$ [Han23, Lac23, HRZ24]. Another approach is the convergence in probability, which we discussed in Chapter 2.

Significant contributions to the understanding of moderately interacting systems were made by Oelschläger several decades ago, particularly in the context of deriving the porous medium equation, as detailed in works such as [Oel84, Oel87]. Especially, for the fluctuation analysis, a weighted L^2-estimate with convergence rate $o(N^{-1/2})$ has been obtained. The convolution structure of the moderate interaction played an important role. In the estimates proposed in [Oel87], the repulsive moderate interaction provides an essential quantity to absorb the rests, which appear from an interacting effect. Related to the work by Oelschläger [Oel87], Olivera, Richard, and Tomašević [ORT23] utilized the semigroup approach and an additional cut-off to demonstrate L^p-estimates for moderate interacting systems.

Our goal is to use the techniques introduced in by Oelschläger's and extend them beyond the porous medium equation. However, the novelty of our work is that we do not follow directly the framework provided by [Oel87], but generate a direct estimation method in a general framework.

Additionally, our work distinguishes itself from the standard modulated energy and relative entropy approach [JW18, BJW19, Ser20, BJW23, RS23]. A drawback of the modulated energy approach in combination with the relative entropy is the torus domain as well as the requirement of entropy solutions on the particle level (microscopic level), see [BJW19, Proposition 4.2]. Such solutions are non-trivial outside the torus setting and in general not unique. Recently, Wang and Feng extended some ideas from the torus setting to the $2D$-viscous point vortex model on the whole space $\mathbb{R}^2$. The idea is to show exponential decay of the solution [FW23, Theorem 4.4] to be able to apply the large deviation result in [JW18]. Again strict restrictions on the initial conditions such as exponential decay are necessary.

In the present chapter we manage to avoid the exponential law of large numbers/large deviation principle [JW18] and the strict conditions on the initial condition by utilizing the convergence in probability. We also treat general forces such as rotational fields or magnetic fields in physics. We manage to derive quantitative bounds on singular forces such as attractive Coulomb interaction kernels on the whole space, which to our knowledge require approximation techniques on the level of the Liouville equation by the nature of their singularities. The price we pay lies in the obtained convergence rate. While [JW18] establish convergence in the sense of the relative entropy of N^{-1}, we achieve a rate of $N^{-1/2-\vartheta}$ for some $\vartheta > 0$. Nevertheless, the convergence is faster than $1/2$ and, therefore, we are optimistic that this result can be used as a stepping stone for Gaussian fluctuation.

Organization of the chapter: In Section 3.1 we recall the interacting particle systems and their associated diffusion-aggregation equations from Chapter 2, and give the necessary assumption on the initial condition. In Section 3.2 we recall the main ingredients from Chapter 2 as an assumption on which we rely for our main estimate in Theorem 3.14. Then, the main results are established in Section 3.3. We provide the idea and the main estimate in Section 3.4. In Section 3.5, we demonstrate propagation of chaos in the case of bounded interaction for the non-regularized systems by establishing the convergence of the approximated PDEs to the non-approximated counterparts. Finally, in Section 3.6, we showcase the applicability of the developed method by discussing, e.g. singular Keller–Segel models and bounded confidence models.

3.2. Assumption of convergence in probability

The analysis of the entropy relies on the convergence of the particle system (3.4) to the particle system (3.5) in probability, which we introduced in Chapter 2. At this point, we want to follow a general approach and not limit ourselves to the results of Chapter 2. Thus, we introduce the following convergence in probability assumption.

Assumption 3.4. *Let $\big(\mathbf{X}_t^{N,\varepsilon}, t \geq 0\big)$, $\big(\mathbf{Y}_t^{N,\varepsilon}, t \geq 0\big)$ be given by (3.4), (3.5). Then for $\alpha \in (0, 1/2)$, $\beta_\alpha \in (0, \alpha)$, $\beta \leq \beta_\alpha$, $\varepsilon \sim N^{-\beta}$ there exists an $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$, $\gamma > 0$ we have*

$$\mathbb{P}\bigg(\sup_{0 \leq t \leq T} \sup_{1 \leq i \leq N} \big| X_t^{i,\varepsilon} - Y_t^{i,\varepsilon} \big| \geq N^{-\alpha} \bigg) \leq C(\gamma) N^{-\gamma}, \tag{3.13}$$

where $C(\gamma)$ depends on the initial density ρ_0, the final time $T > 0$, α and γ.

This assumptions is satisfied by a variety of models [LP17, HLL19, FHS19, HKPZ19, CCS19, HLP20, CLPY20]. In particular for bounded k or even singular kernels this assumption is fulfilled, see Chapter 2.

Furthermore, we need the following law of large numbers result.

ASSUMPTION 3.5. *Let* $(\mathbf{Y}_t^{N,\varepsilon}, t \geq 0)$ *and* ρ_t^ε *be given by* (3.5). *Assume further that* $0 < \alpha, \delta$, $0 < \alpha + \delta < 1/2$, $\varepsilon \sim N^{-\beta}$ *with* $\beta_\alpha \in (0, \alpha)$, $\beta \leq \beta_\alpha$ *and define for* $0 \leq t \leq T$ *the following sets*

$$B_t^\alpha := \left\{ \max_{1 \leq i \leq N} \left| \sum_{j=1}^N k^\varepsilon(Y_t^{i,\varepsilon} - Y_t^{j,\varepsilon}) - (k * \rho_t^\varepsilon)(Y_t^{i,\varepsilon}) \right| \leq N^{-(\delta+\alpha)} \right\}.$$

Then, for each $\gamma > 0$ *there exists a* $C(\gamma) > 0$ *such that*

$$\mathbb{P}\big((B_t^\alpha)^{\mathrm{c}}\big) \leq C(\gamma) N^{-\gamma} \tag{3.14}$$

for every $0 \leq t \leq T$*, where the constant* $C(\gamma)$ *is independent of* $t \in [0, T]$.

We refer again to [LP17, HLL19, FHS19, HKPZ19, CCS19, HLP20, CLPY20]. In particular, the assumption is satisfied for bounded forces k, which satisfy the local Lipschitz bound as we have seen in Proposition 2.21.

3.3. Main results

Let $J^\varepsilon(x) := \frac{1}{\varepsilon^d} J\big(\frac{x}{\varepsilon}\big)$ and $J \colon \mathbb{R}^d \to \mathbb{R}$ be a given mollification kernel, see Section A.2. Let ζ be a cut-off function, which satisfies $|\zeta| \leq 1$, $\zeta = 1$ on $B(0,1)$ and $\zeta = 0$ on $B(0,2)^{\mathrm{c}}$, $\zeta^\varepsilon(x) = \zeta(\varepsilon x)$. We need the following assumptions on the regularized version of interaction force separately to state the main result of this chapter. For concrete examples, such as the Keller–Segel model, we refer to Section 3.6.

DEFINITION 3.6. *We say* $W^\varepsilon = (W_1^\varepsilon, \ldots, W_d^\varepsilon)$, V^ε *are admissible approximations, if* $W_i^\varepsilon \in L^2(\mathbb{R}^d)$ *for all* $i = 1, \ldots, d$ *and* $V^\varepsilon \in H^2(\mathbb{R}^d)$ *with*

$$\|W^\varepsilon\|_{L^2(\mathbb{R}^d)} \leq C\varepsilon^{-a_W}, \quad \|V^\varepsilon\|_{H^2(\mathbb{R}^d)} \leq C\varepsilon^{-a_V} \tag{3.15}$$

for some $C > 0$ *and* $a_W, a_V > 0$. *Additionally, we say* $U^\varepsilon, V^\varepsilon$ *are strongly admissible approximations, if* $U^\varepsilon \in L^2(\mathbb{R}^d)$ *and* $V^\varepsilon \in H^2(\mathbb{R}^d)$ *with*

$$\|U^\varepsilon\|_{H^2(\mathbb{R}^d)} \leq C\varepsilon^{-a_U}, \quad \|V^\varepsilon\|_{H^2(\mathbb{R}^d)} \leq C\varepsilon^{-a_V}. \tag{3.16}$$

Notice that U^ε *is* $\mathbb{R}$*-valued and* W^ε *is* $\mathbb{R}^d$*-valued.*

In general we will consider two type of forces. First, $k^\varepsilon = W^\varepsilon * V^\varepsilon$ and second $k^\varepsilon = \nabla(U^\varepsilon * V^\varepsilon)$. The potential field structure of the latter one will be required for the definition of the modulated energy (see Section 3.4). The assumption on k includes many different forces, where no potential field is needed.

REMARK 3.7. *Some typical examples for the above structure with* $k \colon \mathbb{R}^d \mapsto \mathbb{R}^d$ *are as follows:*

1. *The interaction force kernel* $k \in L^2(\mathbb{R}^d)$. *Then* $W^\varepsilon = k$ *and* $V^\varepsilon = J^\varepsilon$ *is just the standard mollified version of* k.
2. *If* $k \in L^p$ *for* $p < \infty$, *we can choose* $W^\varepsilon = k * J^\varepsilon$ *and* $V^\varepsilon = J^\varepsilon$, *which is also just a mollification of* k.
3. *If* $k \in L^\infty(\mathbb{R}^d)$ *we may choose* $W^\varepsilon = \zeta^\varepsilon(k * J^\varepsilon)$ *and* $V^\varepsilon = J^\varepsilon$, *where* ζ^ε *is defined as a cut-off function to guarantee integrability of the mollification* $k * J^\varepsilon$.

The first main result of this chapter is propagation of chaos on the mollified level with $\varepsilon = N^{-\beta}$:

THEOREM 3.8. *Let $\rho^{N,\varepsilon}$ and ρ^ε be the non-negative solutions of* (3.8) *and of* (3.9) *respectively. Assume that the convergence in probability, Assumption 3.4, and the law of large numbers, Assumption 3.5 hold for $\alpha \in (\frac{1}{4}, \frac{1}{2})$. Let $k^\varepsilon = W^\varepsilon * V^\varepsilon$ and $W^\varepsilon \in L^2(\mathbb{R}^d), V^\varepsilon \in H^2(\mathbb{R}^d)$ be admissible in the sense of Definition 3.6 with rate a_W, a_V. Then there exists a $\beta_1 \in (0, \beta_\alpha)$ depending on a_W, a_V such that $\forall \beta \in (0, \beta_1)$, the following propagation of chaos result holds for $\varepsilon = N^{-\beta}$ between* (3.8) *and of* (3.9).

$$\left\|\rho_t^{N,2,\varepsilon} - \rho_t^\varepsilon \otimes \rho_t^\varepsilon\right\|^2_{L^1(\mathbb{R}^{2d})} \leq 2\mathcal{H}_2\big(\rho_t^{N,2,\varepsilon} | \rho_t^\varepsilon \otimes \rho_t^\varepsilon\big) \leq 4\mathcal{H}_N\big(\rho_t^{N,\varepsilon} | \rho_t^{\otimes N,\varepsilon}\big) = o\bigg(\frac{1}{\sqrt{N}}\bigg). \tag{3.17}$$

where $\rho^{N,2,\varepsilon}$ is the 2-marginal density of $\rho^{N,\varepsilon}$.

Furthermore, if $k^\varepsilon = \nabla(U^\varepsilon * V^\varepsilon)$ with $U^\varepsilon, V^\varepsilon$ being admissible approximations with rate a_U, a_V, then the estimate (3.17) *holds with $\beta \in (0, \tilde{\beta}_1)$ for some $\tilde{\beta}_1 > 0$. If $a_u = a_w$ and a_v is the same as in the previous case, then $\tilde{b}_1 = b_1$ and the estimate* (3.17) *holds with the same constants.*

Moreover, if $U^\varepsilon, V^\varepsilon$ are strongly admissible, then there exists $\beta_2 \in (0, \beta_\alpha)$ such that $\forall \beta \in (0, \beta_2)$, the following estimate for regularized modulated energy holds with $\varepsilon = N^{-\beta}$ between (3.8) *and of* (3.9).

$$\begin{aligned}\mathcal{K}_N\big(\rho_t^{N,\varepsilon} | \rho_t^{\otimes N,\varepsilon}\big) &= \frac{1}{\sigma^2}\mathbb{E}\bigg(\int_{\mathbb{R}^2} (W^\varepsilon * V^\varepsilon)(x-y)\ \mathrm{d}(\mu_t^{N,\varepsilon} - \rho_t^\varepsilon)(x)\ \mathrm{d}(\mu_t^{N,\varepsilon} - \rho_t^\varepsilon)(y)\bigg) \\ &= o\bigg(\frac{1}{\sqrt{N}}\bigg).\end{aligned}$$

REMARK 3.9. *In obtaining the estimate for the modulated energy $\mathcal{K}_N$, the proof has been done with the identity*

$$\mathcal{K}_N\big(\rho_t^{N,\varepsilon} | \rho_t^{\otimes N,\varepsilon}\big) = \frac{1}{\sigma^2}\mathbb{E}\Big(\Big\langle \hat{U}^\varepsilon * (\mu_t^{N,\varepsilon} - \rho_t^\varepsilon), V^\varepsilon * (\mu_t^{N,\varepsilon} - \rho_t^\varepsilon)\Big\rangle\Big),$$

where $\hat{U}(x) = U(-x)$ is the reflection. Again choosing for instance $W^\varepsilon = J^\varepsilon$ we may borrow an additional factor from the mollification kernel J^ε, which will weaken the convergence rate estimate, or in other words, one has to choose even smaller β to achieve the order $o(\frac{1}{\sqrt{N}})$. The restriction $\alpha \in (\frac{1}{4}, \frac{1}{2})$ is in place to guarantee the order $o(\frac{1}{\sqrt{N}})$. The convergence of the relative entropy holds also without this restriction.

Additionally, for bounded force, we know from Chapter 2 that convergence in probability holds for approximations $(k^\varepsilon, \varepsilon > 0)$, which satisfy a local Lipschitz bound. Additionally, we have stability on the PDE level (2.14). Therefore, we can obtain a propagation of chaos result without mollification.

THEOREM 3.10. *Assume that $k \in L^\infty(\mathbb{R}^d)$ and the Assumption 3.1 holds for initial condition ρ_0. Additionally, suppose the Assumptions 3.4, 3.5 hold for the approximation $k^\varepsilon = (\zeta^\varepsilon(k * J^\varepsilon)) * J^\varepsilon$. Then, for any fix $r \in \mathbb{N}$, we have the convergence of the r-th marginal* *of the Liouville equation* (3.6) *to the aggregation-diffusion equation* (3.7) *in the* $L^1(\mathbb{R}^{dr})$*-norm, i.e.*

$$\lim_{N\to\infty} \left\|\rho^{N,r} - \rho^{\otimes r}\right\|_{L^1([0,T];L^1(\mathbb{R}^{dr}))} = 0.$$

REMARK 3.11. *The theorem holds for more general approximation* k^ε *as long as the approximation* $k^\varepsilon \in L^2(\mathbb{R}^d)$ *and the convergence in probability holds.*

Let us finish the section with an overview over the constants:

- $\alpha \in (0, 1/2)$ provides the rate on the distance of the particles in the convergence in probability
$$\sup_{1\le i\le N} |X_t^{i,\varepsilon} - Y_t^{i,\varepsilon}| \ge N^{-\alpha}$$
and in the law of large numbers
$$\left\{\max_{1\le i\le N} \left|\sum_{j=1}^N k^\varepsilon(Y_t^{i,\varepsilon} - Y_t^{j,\varepsilon}) - (k*\rho_t^\varepsilon)(Y_t^{i,\varepsilon})\right| \ge N^{-(\delta+\alpha)}\right\}.$$
- $\beta_\alpha \in (0, \alpha)$ provides the maximum interval $(0, \beta_\alpha)$ for the cut-off parameter β, for which the convergence in probability and law of large numbers hold.
- β is the convergence rate of the approximated particles $X_t^{i,\varepsilon}, Y_t^{i,\varepsilon}$ such that $\varepsilon = N^{-\beta}$.
- β_1, β_2 provide the maximum intervals $(0, \beta_1), (0, \beta_2)$ such that the relative entropy and modulated energy converges with rate greater than 1/2, (see (3.17)).

3.4. Relative entropy method

This section is devoted to present the relative entropy method for the moderate interacting problem and its connection to the L^2-estimate proposed by Oelschläger [Oel87]. We derive the smoothed L^2-estimate for given force k (no requirement as a potential field), and the smoothed modulated energy for potential field with convolution structure. Both lead to the estimate of the relative entropy between $\rho^{N,\varepsilon}$ and $\rho^{\otimes N,\varepsilon}$.

The main idea is to use the assumption of convergence in probability (Assumption 3.4), the structure of the PDEs (3.8), (3.9) and the law of large number (Assumption 3.5). Applying the Csiszár–Kullback–Pinsker inequality 1.6 and the sub-additivity inequality (1.7) we provide an estimate on the $L^1(\mathbb{R}^{dr})$-norm of the marginals $\rho^{N,r,\varepsilon}$ and $\rho^{\otimes r,\varepsilon}$ for fix $r \in \mathbb{N}$.

We emphasize that the method developed in Theorem 3.14 can be applied in different settings. Indeed, since we are working on the approximation level, our assumptions are only needed in the regularized setting. Hence, in general the assumptions on k, V and W itself can be chosen more irregular, extending even to singular models. We refer to Remark 3.15 and the applications Section 3.6 for more details.

3.4.1. Relative entropy and modulated energy. In this section we introduce our main quantities the relative entropy and the modulated free energy. We then show the connection between the L^2-norm

$$\mathbb{E}\left(\left\|V^\varepsilon * (\mu_t^{N,\varepsilon} - \rho_t^\varepsilon)\right\|^2_{L^2(\mathbb{R}^d)}\right), \tag{3.18}$$

the relative entropy $\mathcal{H}_N(\rho_t^{N,\varepsilon}|\rho_t^{\otimes N,\varepsilon})$ as well as the modulated free energy $\mathcal{K}_N(\rho_t^{N,\varepsilon}|\rho_t^{\otimes N,\varepsilon})$. This can be viewed as a combination of Oelschläger's results on moderated interaction and fluctuations [Oel87] and the relative entropy method developed among others by Serfaty, Jabin, Wang, Bresch and Lacker [JW16, JW18, BJW19, BJW23, Ser20, NRS22, RS23, BJW23, Lac23] for the mean-field setting. The aim is to demonstrate how both concepts connect under the convolution assumption. Finally, we derive an estimate on the relative entropy in terms of the above L^2-norm.

Following [BJW19] we recall the relative entropy and modulated free energy from Section 1.3.4. The modulated free energy is given by

$$E_N\Big(\rho^{N,\varepsilon} \mid \rho^{\otimes N,\varepsilon}\Big) := \mathcal{H}_N\big(\rho^{N,\varepsilon}|\rho^{\otimes N,\varepsilon}\big) + \mathcal{K}_N\big(\rho^{N,\varepsilon}|\rho^{\otimes N,\varepsilon}\big),$$

where

$$\mathcal{H}_N\big(\rho_t^{N,\varepsilon}|\rho_t^{\otimes N,\varepsilon}\big) := \frac{1}{N}\int_{\mathbb{R}^{dN}} \rho_t^{N,\varepsilon}(x_1,\dots,x_N)\log\Big(\frac{\rho_t^{N,\varepsilon}(x_1,\dots,x_N)}{\rho_t^{\otimes N,\varepsilon}(x_1,\dots,x_N)}\Big)\ \mathrm{d}x_1,\dots,x_N$$

is the relative entropy introduced in [JW16] and if $k^\varepsilon = \nabla(U^\varepsilon * V^\varepsilon)$ is a potential

$$\mathcal{K}_N\big(\rho_t^{N,\varepsilon}|\rho_t^{\otimes N,\varepsilon}\big) := \frac{1}{\sigma^2}\mathbb{E}\Big(\int_{\mathbb{R}^{2d}} (U^\varepsilon * V^\varepsilon)(x-y)\ \mathrm{d}(\mu_t^{N,\varepsilon} - \rho_t^\varepsilon)(x)\ \mathrm{d}(\mu_t^{N,\varepsilon} - \rho_t^\varepsilon)(y)\Big)$$

is the modulated energy. We refer to [BJW19] and the references therein for more details on the modulated free energy.

Let us now explore some connections between the relative entropy and the structure presented by Oelschläger [Oel87]. We start by rewriting the expectation of the free energy by using our convolution structure. A straightforward calculation shows

$$\mathcal{K}_N\big(\rho_t^{N,\varepsilon}|\rho_t^{\otimes N,\varepsilon}\big) = \frac{1}{\sigma^2}\mathbb{E}\Big(\Big\langle \hat{U}^\varepsilon * (\mu_t^{N,\varepsilon} - \rho_t^\varepsilon), V^\varepsilon * (\mu_t^{N,\varepsilon} - \rho_t^\varepsilon)\Big\rangle\Big), \tag{3.19}$$

where $\hat{U}(x) = U(-x)$ is the reflection. Applying Young's inequality we see that it is enough to control a term of the form

$$\mathbb{E}\Big(\Big\|V^\varepsilon * (\mu_t^{N,\varepsilon} - \rho_t^\varepsilon)\Big\|^2_{L^2(\mathbb{R}^d)}\Big)$$

for some function V^ε, where we just write V^ε for simplicity and understand that we can chose $V^\varepsilon = \hat{U}^\varepsilon$ in all calculations below. Hence, in order to estimate $\mathcal{K}_N\big(\rho_t^{N,\varepsilon}|\rho_t^{\otimes N,\varepsilon}\big)$ we can estimate the L^2-difference between the convoluted empirical measure and the solution the law of the mean-field limit (3.3). This will be accomplished in Theorem 3.14.

But let us recall that our initial goal is to estimate the relative entropy $\mathcal{H}_N\big(\rho_t^{N,\varepsilon}|\rho_t^{\otimes N,\varepsilon}\big)$ and not $\mathcal{K}_N\big(\rho_t^{N,\varepsilon}|\rho_t^{\otimes N,\varepsilon}\big)$. Therefore, let us connect the relative entropy to the L^2-norm of $V^\varepsilon * (\mu_t^{N,\varepsilon} - \rho_t^\varepsilon)$. First, we consider the case $k^\varepsilon = W^\varepsilon * V^\varepsilon$.

LEMMA 3.12. *Let $W^\varepsilon, V^\varepsilon$ be admissible and $k^\varepsilon = W^\varepsilon * V^\varepsilon$. Then for the non-negative solutions $\rho^{N,\varepsilon}$ of* (3.8) *and ρ^ε of* (3.9)*, it holds $\forall t > 0$ that*

$$\mathcal{H}_N\big(\rho_t^{N,\varepsilon}|\rho_t^{\otimes N,\varepsilon}\big) + \frac{\sigma^2}{4N}\int_0^t \sum_{i=1}^N \int_{\mathbb{R}^{dN}} \left|\nabla_{x_i} \log\Big(\frac{\rho_s^{N,\varepsilon}(\mathsf{X}^N)}{\rho_s^{\otimes N,\varepsilon}(\mathsf{X}^N)}\Big)\right|^2 \rho_s^{N,\varepsilon}(\mathsf{X}^N)\,\mathrm{d}\mathsf{X}^N\,\mathrm{d}s$$

$$\leq \frac{\|W^\varepsilon\|_{L^2(\mathbb{R}^d)}^2}{\sigma^2}\mathbb{E}\bigg(\int_0^t \Big(\big\|V^\varepsilon * (\mu_s^{N,\varepsilon} - \rho_s^\varepsilon)\big\|_{L^2(\mathbb{R}^d)}^2\Big)\,\mathrm{d}s\bigg), \tag{3.20}$$

PROOF. Let us compute the time derivative of the relative entropy

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_N\big(\rho_t^{N,\varepsilon}|\rho_t^{\otimes N,\varepsilon}\big)\\
&= \frac{1}{N}\int_{\mathbb{R}^{dN}} \partial_t\rho_t^{N,\varepsilon}(\mathsf{X}^N)\log\Big(\frac{\rho_t^{N,\varepsilon}(\mathsf{X}^N)}{\rho_t^{\otimes N,\varepsilon}(\mathsf{X}^N)}\Big) + \partial_t\rho_t^{N,\varepsilon}(\mathsf{X}^N) - \frac{\rho_t^{N,\varepsilon}(\mathsf{X}^N)}{\rho_t^{\otimes N,\varepsilon}(\mathsf{X}^N)}\partial_t\rho_t^{\otimes N,\varepsilon}(\mathsf{X}^N)\,\mathrm{d}\mathsf{X}^N\\
&= \frac{1}{N}\int_{\mathbb{R}^{dN}}\bigg(\frac{\sigma^2}{2}\sum_{i=1}^N \Delta_{x_i}\rho_t^{N,\varepsilon}(\mathsf{X}^N)\\
&\quad + \sum_{i=1}^N \nabla_{x_i}\cdot\Big(\rho_t^{N,\varepsilon}(\mathsf{X}^N)\frac{1}{N}\sum_{j=1}^N k^\varepsilon(x_i - x_j)\Big)\bigg)\log\Big(\frac{\rho_t^{N,\varepsilon}(\mathsf{X}^N)}{\rho_t^{\otimes N,\varepsilon}(\mathsf{X}^N)}\Big)\\
&\quad - \frac{\rho_t^{N,\varepsilon}(\mathsf{X}^N)}{\rho_t^{\otimes N,\varepsilon}(\mathsf{X}^N)}\sum_{i=1}^N\frac{\sigma^2}{2}\Delta_{x_i}\rho_t^{\otimes N,\varepsilon}(\mathsf{X}^N) - \sum_{i=1}^N \nabla_{x_i}\cdot\big((k^\varepsilon * \rho_t^\varepsilon)(x_i)\rho_t^{\otimes N,\varepsilon}(\mathsf{X}^N)\big)\,\mathrm{d}\mathsf{X}^N\\
&= -\frac{\sigma^2}{2N}\sum_{i=1}^N\int_{\mathbb{R}^{dN}}\left|\nabla_{x_i}\log\Big(\frac{\rho_t^{N,\varepsilon}(\mathsf{X}^N)}{\rho_t^{\otimes N,\varepsilon}(\mathsf{X}^N)}\Big)\right|^2\rho_t^{N,\varepsilon}(\mathsf{X}^N)\,\mathrm{d}\mathsf{X}^N\\
&\quad - \frac{1}{N^2}\sum_{i,j=1}^N\int_{\mathbb{R}^{dN}}\Big(k^\varepsilon(x_i - x_j) - k^\varepsilon * \rho_t^\varepsilon(x_i)\Big)\rho_t^{N,\varepsilon}(\mathsf{X}^N)\cdot\nabla_{x_i}\log\Big(\frac{\rho_t^{N,\varepsilon}(\mathsf{X}^N)}{\rho_t^{\otimes N,\varepsilon}(\mathsf{X}^N)}\Big)\,\mathrm{d}\mathsf{X}^N\\
&\leq -\frac{\sigma^2}{4N}\sum_{i=1}^N\int_{\mathbb{R}^{dN}}\left|\nabla_{x_i}\log\Big(\frac{\rho_t^{N,\varepsilon}(\mathsf{X}^N)}{\rho_t^{\otimes N,\varepsilon}(\mathsf{X}^N)}\Big)\right|^2\rho_t^{N,\varepsilon}(\mathsf{X}^N)\,\mathrm{d}\mathsf{X}^N\\
&\quad + \frac{1}{\sigma^2 N}\sum_{i=1}^N\int_{\mathbb{R}^{dN}}\left|\frac{1}{N}\sum_{j=1}^N k^\varepsilon(x_i - x_j) - k^\varepsilon * \rho_t^\varepsilon(x_i)\right|^2\rho_t^{N,\varepsilon}(\mathsf{X}^N)\,\mathrm{d}\mathsf{X}^N\\
&= -\frac{\sigma^2}{4N}\sum_{i=1}^N\int_{\mathbb{R}^{dN}}\left|\nabla_{x_i}\log\Big(\frac{\rho_t^{N,\varepsilon}(\mathsf{X}^N)}{\rho_t^{\otimes N,\varepsilon}(\mathsf{X}^N)}\Big)\right|^2\rho_t^{N,\varepsilon}(\mathsf{X}^N)\,\mathrm{d}\mathsf{X}^N\\
&\quad + \frac{1}{\sigma^2}\mathbb{E}\Big(\langle\mu_t^{N,\varepsilon}, |k^\varepsilon * (\mu_t^{N,\varepsilon} - \rho_t^\varepsilon)|^2\rangle\Big).
\end{aligned}$$

For $k^\varepsilon = W^\varepsilon * V^\varepsilon$ we have further estimates

$$\frac{1}{\sigma^2}\mathbb{E}\Big(\langle\mu_t^{N,\varepsilon}, |k^\varepsilon * (\mu_t^{N,\varepsilon} - \rho_t^\varepsilon)|^2\rangle\Big) = \frac{1}{\sigma^2}\mathbb{E}\Big(\langle\mu_t^{N,\varepsilon}, |W^\varepsilon * V^\varepsilon * (\mu_t^{N,\varepsilon} - \rho_t^\varepsilon)|^2\rangle\Big)$$

$$\leq \frac{\|W^\varepsilon\|^2_{L^2(\mathbb{R}^d)}}{\sigma^2}\mathbb{E}\bigg(\Big\|V^\varepsilon * (\mu_t^{N,\varepsilon} - \rho_t^\varepsilon)\Big\|^2_{L^2(\mathbb{R}^d)}\bigg)$$

Substituting the estimate into the first inequality, while recalling that $\mathcal{H}_N\big(\rho_0^{N,\varepsilon}|\rho_0^{\otimes N,\varepsilon}\big) = 0$, proves the lemma. □

REMARK 3.13. *Depending on the regularity of* V^ε *and* W^ε *one may choose to interchange the roles in the estimate. Generally, one should choose the more regular function to be* V^ε. *Indeed, in the above estimate we need only the* L^2*-norm of* W^ε*, while later on in Theorem 3.14 we need the* L^∞*-norm as well as the* L^2*-norm of not only the function* V^ε *but also of its derivatives. Moreover, if the force* k^ε *is a potential field, the last term has the following structure*

$$\mathbb{E}\bigg(\Big\|\nabla V^\varepsilon * (\mu_t^{N,\varepsilon} - \rho_t^\varepsilon)\Big\|^2_{L^2(\mathbb{R}^d)}\bigg),$$

which will also be estimated by Theorem 3.14. Hence, we do not lose convergence rates in the case $k^\varepsilon = \nabla(U^\varepsilon * V^\varepsilon)$*, but as already mentioned, we obtained an additional estimate on the modulated energy* $\mathcal{K}_N\big(\rho_t^{N,\varepsilon}|\rho_t^{\otimes N,\varepsilon}\big)$.

Consequently, by the above discussion, in order to control the relative entropy and the modulated energy in the case k^ε is a potential field, we need to find an estimate for the L^2-norm (3.18), which was studied in the moderated regime by Oelschläger [Oel87] nearly forty years ago.

3.4.2. L^2-estimate. In this section we concentrate on estimating the rest term in the entropy estimate (3.20).

We present the main theorem of the chapter, which is formulated for a function V^ε, which depends on ε. This presentation is motivation by our case $k^\varepsilon = W^\varepsilon * V^\varepsilon$. We emphasize that the function in the following theorem can be chosen independent of ε, but than the estimate has no connection to the modulated energy or the relative entropy (see Lemma 3.12).

THEOREM 3.14. *Suppose* (3.2)*, the convergence in probability, Assumption 3.4, and the law of large numbers, Assumption 3.5 hold both with rates* $\beta, \beta_\alpha, \alpha$ *specified therein. Then for any* $V^\varepsilon \in H^2(\mathbb{R}^d)$ *the following* L^2*-estimate holds*

$$\begin{aligned}
&\mathbb{E}\bigg(\sup_{0\leq t\leq T}\Big\|V^\varepsilon * \mu_t^{N,\varepsilon} - V^\varepsilon * \rho_t^\varepsilon\Big\|^2_{L^2(\mathbb{R}^d)}\bigg) + \frac{\sigma^2}{8}\mathbb{E}\bigg(\int_0^T \|\nabla V^\varepsilon * (\mu_s^{N,\varepsilon} - \rho_s^\varepsilon)\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}s\bigg)\\
&\quad\leq \frac{C}{N}(\|V^\varepsilon\|^2_{H^1(\mathbb{R}^d)}\|k^\varepsilon\|^2_{L^\infty(\mathbb{R}^d)} + \big\|\nabla^2 V^\varepsilon\big\|^2_{L^2(\mathbb{R}^d)}) + \frac{C\|V^\varepsilon\|^2_{H^1(\mathbb{R}^d)}(1+\|k^\varepsilon\|^2_{L^\infty(\mathbb{R}^d)})}{N^\gamma}\\
&\qquad+\frac{\|\nabla k^\varepsilon\|_{L^\infty(\mathbb{R}^d)}\|V^\varepsilon\|_{L^2(\mathbb{R}^d)} + \|k^\varepsilon\|^2_{L^\infty(\mathbb{R}^d)}\|\nabla V^\varepsilon\|^2_{L^2(\mathbb{R}^d)} + \|V^\varepsilon\|^2_{L^2(\mathbb{R}^d)}}{N^{2\alpha}}\\
&\qquad+C\frac{\|\nabla V^\varepsilon\|^2_{L^2(\mathbb{R}^d)}(1+\|\nabla k^\varepsilon\|_{L^\infty(\mathbb{R}^d)}) + \|\nabla V^\varepsilon\|_{L^2(\mathbb{R}^d)}\big\|\nabla^2 V^\varepsilon\big\|_{L^2(\mathbb{R}^d)}\|k^\varepsilon\|_{L^\infty(\mathbb{R}^d)}}{N^{\alpha+\frac{1}{2}}},
\end{aligned}$$

where C *depends on* T, σ, γ, C_{BDG}.

Remark 3.15. *The only ingredients we need for completing the proof of theorem 3.14 are the convergence in probability of the particle system* $\mathbf{X}^{N,\varepsilon}$ *to the mean-field limit* $\mathbf{Y}^{N,\varepsilon}$ (3.13) *as well as the law of large numbers* (3.14)*. But the convergence in probability and the law of large numbers are known for a variety of interaction force kernels, see for instance* [LP17, FHS19, HLL19, HLP20]*. Hence, this result can be extended for a variety of interaction force kernels. We refer to Section 3.6 for applicable models such as the case with Coulomb force. In the case of d-dimensional kernels we notice, that our estimates become dimension dependent by the choice of* β *and the approximation* V^ε *and* $\varepsilon \sim N^{-\beta}$*. Consequently, the rate of convergence becomes dependent on the dimension of the problem.*

Remark 3.16. *The results in Theorem 3.14 state that* $\mu_t^{N,\varepsilon}$ *is close to* ρ_t^ε *in the mollified* L^2*-norm. By propagation of chaos we expect that this quantity should be small since* $\mu_t^{N,\varepsilon} - \rho_t^\varepsilon$ *should ideally vanish in the limit. The majority of work, which lies ahead, is to estimate this* L^2*-norm with a good rate. In the process we will also obtain an estimate on the derivative* $V_x^\varepsilon * (\mu_t^{N,\varepsilon} - \rho_t^\varepsilon)$*. This is no surprise, since the estimate follows the structure of the classic a priori* L^2*-estimate for the parabolic equation* [WYW06, Chapter 3]*. As a result, we obtain in the* $L^2(\mathbb{P})$*-norm an* $L^\infty([0,T];L^2(\mathbb{R}^d))$*-bound and as usual an* $L^2([0,T];L^2(\mathbb{R}^d))$*-bound for the derivative. In combination with Lemma 3.12 this will allow us to obtain a bound on the relative entropy* $\mathcal{H}\big(\rho_t^{N,\varepsilon}|\rho_t^{\otimes N,\varepsilon}\big)$*. Additionally, if the interaction force is a potential field we obtain an estimate for* $\mathcal{K}_N\big(\rho_t^{N,\varepsilon}|\rho_t^{\otimes N,\varepsilon}\big)$ *by equality* (3.19)*.*

Let us start by describing the dynamic of the empirical measure $\mu_t^{N,\varepsilon}$. Applying Itô's formula(1.15) to a sufficiently smooth function f, we obtain

$$\begin{aligned}\langle f, \mu_t^{N,\varepsilon}\rangle &= \frac{1}{N}\sum_{i=1}^N f(X_t^{i,\varepsilon})\\ &= \langle f, \mu_0^N\rangle - \frac{1}{N}\sum_{i=1}^N \frac{1}{N}\sum_{j=1}^N \int_0^t \nabla f(X_s^{i,\varepsilon}) \cdot k(X_s^{i,\varepsilon} - X_s^{j,\varepsilon})\,\mathrm{d}s\\ &\quad + \frac{\sigma}{N}\sum_{i=1}^N \int_0^t \nabla^{\mathrm{T}} f(X_s^{i,\varepsilon})\,\mathrm{d}B_s^i + \frac{\sigma^2}{2N}\sum_{i=1}^N \int_0^t \Delta f(X_s^{i,\varepsilon})\,\mathrm{d}s.\end{aligned}$$

Taking the expectation and using the fact that we have a density of $\mathbf{X}_s^{N,\varepsilon}$, provides a weak formulation of the Liouville equation (3.8). If we want to compare it to the mean-field law, we need to make the crucial observation that the stochastic integral in the above equation should vanish after taking the expectation. In other words, we have no term in the regularized PDE (3.9), which corresponds to the stochastic integral. If the integrand is smooth enough then obviously the stochastic integral vanishes. However, we need to compute the following difference

$$\mathbb{E}\bigg(\sup_{0\le t\le T}\Big\|V^\varepsilon * \mu_t^{N,\varepsilon} - V^\varepsilon * \rho_t^\varepsilon\Big\|_{L^2(\mathbb{R}^d)}^2\bigg).$$

Therefore, we need somehow transfer the naive approach to the more complex expected value. Applying the above dynamic we prove the following lemma, which allows us to treat the convolution $V^\varepsilon * \mu_t^{N,\varepsilon}$ as if the stochastic integral vanishes.

Lemma 3.17. *Let $\mu_t^{\mathbf{X}^{N,\varepsilon}}(\omega)$ defined by (1.1) with the interacting particle system (3.4) associated to it. Then, we have the following inequality*

$$\mathbb{E}\bigg(\sup_{0\le t\le T}\Big\|V^\varepsilon * \mu_t^{N,\varepsilon} - V^\varepsilon * \rho_t^\varepsilon\Big\|^2_{L^2(\mathbb{R}^d)}\bigg)$$

$$\le\ 2\mathbb{E}\bigg(\sup_{0\le t\le T}\int_{\mathbb{R}^d}\bigg|\frac{1}{N}\sum_{i=1}^N\bigg(V^\varepsilon(y-X_0^i)+\frac{1}{N}\sum_{j=1}^N\int_0^t\nabla V^\varepsilon(y-X_s^{i,\varepsilon})\cdot k^\varepsilon(X_s^{i,\varepsilon}-X_s^{j,\varepsilon})\,\mathrm{d}s$$

$$+\frac{\sigma^2}{2}\int_0^t\Delta V^\varepsilon(y-X_t^{i,\varepsilon})\,\mathrm{d}s\bigg)-V^\varepsilon*\rho_t^\varepsilon(y)\bigg|^2\mathrm{d}y\bigg)+\frac{2T\sigma^2C_{\mathrm{BDG}}}{N}\|\nabla V^\varepsilon\|^2_{L^2(\mathbb{R}^d)}\,.$$

Proof. We use Itô's formula(1.15), the dynamics (3.4) and the Burkholder–Davis–Gundy inequality (A.1) to find

$$\mathbb{E}\bigg(\sup_{0\le t\le T}\Big\|V^\varepsilon * \mu_t^{N,\varepsilon} - V^\varepsilon * \rho_t^\varepsilon\Big\|^2_{L^2(\mathbb{R}^d)}\bigg)$$

$$=\mathbb{E}\bigg(\sup_{0\le t\le T}\int_{\mathbb{R}^d}\bigg|\frac{1}{N}\sum_{i=1}^N V^\varepsilon(y-X_s^{i,\varepsilon})-V^\varepsilon*\rho_t^\varepsilon(y)\bigg|^2\mathrm{d}y\bigg)$$

$$=\mathbb{E}\bigg(\sup_{0\le t\le T}\int_{\mathbb{R}^d}\bigg|\frac{1}{N}\sum_{i=1}^N\bigg(V^\varepsilon(y-X_0^i)+\frac{1}{N}\sum_{j=1}^N\int_0^t\nabla V^\varepsilon(y-X_s^{i,\varepsilon})\cdot k^\varepsilon(X_s^{i,\varepsilon}-X_s^{j,\varepsilon})\,\mathrm{d}s$$

$$+\frac{\sigma^2}{2}\int_0^t\Delta V^\varepsilon(y-X_t^{i,\varepsilon})\,\mathrm{d}s-\sigma\int_0^t\nabla^{\mathrm{T}}V^\varepsilon(y-X_s^{i,\varepsilon})\,\mathrm{d}B_s^i\bigg)-V^\varepsilon*\rho_t^\varepsilon(y)\bigg|^2\mathrm{d}y\bigg)$$

$$\le 2\mathbb{E}\bigg(\sup_{0\le t\le T}\int_{\mathbb{R}^d}\bigg|\frac{1}{N}\sum_{i=1}^N\bigg(V^\varepsilon(y-X_0^i)+\frac{1}{N}\sum_{j=1}^N\int_0^t\nabla V^\varepsilon(y-X_s^{i,\varepsilon})\cdot k^\varepsilon(X_s^{i,\varepsilon}-X_s^{j,\varepsilon})\,\mathrm{d}s$$

$$+\frac{\sigma^2}{2}\int_0^t V^\varepsilon\Delta(y-X_t^{i,\varepsilon})\,\mathrm{d}s\bigg)-V^\varepsilon*\rho_t^\varepsilon(y)\bigg|^2\mathrm{d}y\bigg)$$

$$+2\sigma^2\mathbb{E}\bigg(\sup_{0\le t\le T}\int_{\mathbb{R}^d}\bigg|\frac{1}{N}\sum_{i=1}^N\int_0^t\nabla^{\mathrm{T}}V_x^\varepsilon(y-X_s^{i,\varepsilon})\,\mathrm{d}B_s^i\bigg|^2\mathrm{d}y\bigg).$$

It remains to estimate the last term by the Burkholder–Davis–Gundy (BDG) inequality (A.1),

$$2\sigma^2\mathbb{E}\bigg(\sup_{0\le t\le T}\int_{\mathbb{R}^d}\bigg|\frac{1}{N}\sum_{i=1}^N\int_0^t\nabla^{\mathrm{T}}V^\varepsilon(y-X_s^{i,\varepsilon})\,\mathrm{d}B_s^i\bigg|^2\mathrm{d}y\bigg)$$

$$\le 2\sigma^2\int_{\mathbb{R}^d}\mathbb{E}\bigg(\sup_{0\le t\le T}\bigg|\frac{1}{N}\sum_{i=1}^N\int_0^t\nabla^{\mathrm{T}}V^\varepsilon(y-X_s^{i,\varepsilon})\,\mathrm{d}B_s^i\bigg|^2\bigg)\mathrm{d}y$$

$$\begin{aligned}
&\leq 2\sigma^2 C_{\mathrm{BDG}} \int_{\mathbb{R}^d} \mathbb{E}\bigg(\bigg\langle \frac{1}{N}\sum_{i=1}^{N}\int_0^{\cdot} \nabla^{\mathrm{T}} V^{\varepsilon}(y - X_s^{i,\varepsilon})\,\mathrm{d}B_s^i \bigg\rangle_T\bigg)\,\mathrm{d}y \\
&\leq \frac{2\sigma^2 C_{\mathrm{BDG}}}{N^2} \int_{\mathbb{R}^d} \mathbb{E}\bigg(\sum_{i=1}^{N}\int_0^T |\nabla V^{\varepsilon}(y - X_s^{i,\varepsilon})|^2\,\mathrm{d}s\bigg)\,\mathrm{d}y \\
&\leq \frac{2T\sigma^2 C_{\mathrm{BDG}}}{N}\, \|\nabla V^{\varepsilon}\|^2_{L^2(\mathbb{R}^d)} \,.
\end{aligned}$$

Inserting this calculation into the previous inequality proves the lemma. □

Proof of Theorem 3.14. By Lemma 3.17 we can ignore the stochastic integral in the processes $(X^i, \varepsilon, t \geq 0)$, which determine the empirical measure $\mu_t^{N,\varepsilon}$. Hence, let us write

$$\begin{aligned}
V^{\varepsilon} \tilde{*} \mu_t^{N,\varepsilon}(y) := \frac{1}{N}\sum_{i=1}^{N}\bigg(V^{\varepsilon}(y - X_0^i) + \frac{1}{N}\sum_{j=1}^{N}\int_0^t \nabla V^{\varepsilon}(y - X_s^{i,\varepsilon}) \cdot k^{\varepsilon}(X_s^{i,\varepsilon} - X_s^{j,\varepsilon})\,\mathrm{d}s \\
+ \frac{\sigma^2}{2}\int_0^t \Delta V^{\varepsilon}(y - X_t^{i,\varepsilon})\,\mathrm{d}s \bigg)
\end{aligned}$$

for the convolution $V^{\varepsilon} * \mu_t^{N,\varepsilon}$ after applying Itô's formula(1.15) but without the stochastic integral. Then, we have

$$\begin{aligned}
&\Big\| V^{\varepsilon} \tilde{*} \mu_t^{N,\varepsilon} - V^{\varepsilon} * \rho_t^{\varepsilon} \Big\|^2_{L^2(\mathbb{R}^d)} - \big\| V^{\varepsilon} * \mu_0^N - V^{\varepsilon} * \rho_0 \big\|^2_{L^2(\mathbb{R}^d)} \\
&= 2\int_0^t \langle \partial_s (V^{\varepsilon} \tilde{*} \mu_s^{N,\varepsilon} - V^{\varepsilon} * \rho_s^{\varepsilon}), V^{\varepsilon} \tilde{*} \mu_s^{N,\varepsilon} - V^{\varepsilon} * \rho_s^{\varepsilon} \rangle_{L^2(\mathbb{R}^d)}\,\mathrm{d}s,
\end{aligned}$$

where we notice that for the initial time $t = 0$, we have $V^{\varepsilon} \tilde{*} \mu_0^N = V^{\varepsilon} * \mu_0^N$ by definition. Let us remark that since all integrands are smooth enough we have $\nabla(V^{\varepsilon} \tilde{*} \mu_t^{N,\varepsilon}) = \nabla V^{\varepsilon} \tilde{*} \mu_t^{N,\varepsilon}$. Next, plugging in $V^{\varepsilon} \tilde{*} \mu_s^{N,\varepsilon}$ and differentiate we obtain

$$\begin{aligned}
&\langle \partial_s V^{\varepsilon} \tilde{*} \mu_s^{N,\varepsilon}, V^{\varepsilon} \tilde{*} \mu_s^{N,\varepsilon} - V^{\varepsilon} * \rho_s^{\varepsilon} \rangle_{L^2(\mathbb{R}^d)} \\
&= \bigg\langle \frac{1}{N}\sum_{i=1}^{N} \frac{1}{N}\sum_{j=1}^{N} \nabla V^{\varepsilon}(\cdot - X_s^{i,\varepsilon}) \cdot k^{\varepsilon}(X_s^{i,\varepsilon} - X_s^{j,\varepsilon}) \\
&\quad + \frac{\sigma^2}{2N}\sum_{i=1}^{N} \Delta V^{\varepsilon}(\cdot - X_s^{i,\varepsilon}), V^{\varepsilon} \tilde{*} \mu_s^{N,\varepsilon} - V^{\varepsilon} * \rho_s^{\varepsilon} \bigg\rangle_{L^2(\mathbb{R}^d)} \\
&= \bigg\langle \frac{1}{N}\sum_{i=1}^{N} \frac{1}{N}\sum_{j=1}^{N} \nabla V^{\varepsilon}(\cdot - X_s^{i,\varepsilon}) \cdot k^{\varepsilon}(X_s^{i,\varepsilon} - X_s^{j,\varepsilon}), V^{\varepsilon} \tilde{*} \mu_s^{N,\varepsilon} - V^{\varepsilon} * \rho_s^{\varepsilon} \bigg\rangle_{L^2(\mathbb{R}^d)} \\
&\quad - \frac{\sigma^2}{2} \langle \nabla V^{\varepsilon} * \mu_s^{N,\varepsilon}, \nabla V^{\varepsilon} \tilde{*} \mu_s^{N,\varepsilon} - \nabla V^{\varepsilon} * \rho_s^{\varepsilon} \rangle_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Similar, ρ_s^ε is a weak solution to our PDE (3.9), which implies

$$\begin{aligned}
&\langle \partial_s V^\varepsilon * \rho_s^\varepsilon), V^\varepsilon \tilde{*} \mu_s^{N,\varepsilon} - V^\varepsilon * \rho_s^\varepsilon \rangle_{L^2(\mathbb{R}^d)} \\
&= \left\langle V^\varepsilon * \left(\frac{\sigma^2}{2} \Delta \rho_s^\varepsilon + \nabla \cdot ((k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon) \right), V^\varepsilon \tilde{*} \mu_s^{N,\varepsilon} - V^\varepsilon * \rho_s^\varepsilon \right\rangle_{L^2(\mathbb{R}^d)} \\
&= -\frac{\sigma^2}{2} \langle \nabla V^\varepsilon * (\rho_s^\varepsilon), \nabla V^\varepsilon \tilde{*} \mu_s^{N,\varepsilon} - \nabla V^\varepsilon * \rho_s^\varepsilon \rangle_{L^2(\mathbb{R}^d)} \\
&\quad + \langle V^\varepsilon * \nabla \cdot ((k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon), V^\varepsilon \tilde{*} \mu_s^{N,\varepsilon} - V^\varepsilon * \rho_s^\varepsilon \rangle_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Combing the last two calculations, we find

$$\begin{aligned}
&\left\| V^\varepsilon \tilde{*} \mu_t^{N,\varepsilon} - V^\varepsilon * \rho_t^\varepsilon \right\|_{L^2(\mathbb{R}^d)}^2 \\
&= \left\| V^\varepsilon * \mu_0^N - V^\varepsilon * \rho_0 \right\|_{L^2(\mathbb{R}^d)}^2 \\
&\quad - 2 \int_0^t \frac{\sigma^2}{2} \langle \nabla (V^\varepsilon * \mu_s^{N,\varepsilon} - V^\varepsilon * \rho_s^\varepsilon), \nabla (V^\varepsilon \tilde{*} \mu_s^{N,\varepsilon} - V^\varepsilon * \rho_s^\varepsilon) \rangle_{L^2(\mathbb{R}^d)} \, \mathrm{d}s \\
&\quad + \int_0^t \left\langle \frac{2}{N^2} \sum_{i,j=1}^N \nabla V^\varepsilon (\cdot - X_s^{i,\varepsilon}) \cdot k^\varepsilon (X_s^{i,\varepsilon} - X_s^{j,\varepsilon}) - \nabla \cdot (V^\varepsilon * (k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon), \right. \\
&\qquad \left. V^\varepsilon \tilde{*} \mu_s^{N,\varepsilon} - V^\varepsilon * \rho_s^\varepsilon \right\rangle_{L^2(\mathbb{R}^d)} \mathrm{d}s.
\end{aligned}$$

The goal is now to insert $V^\varepsilon * \mu_s^{N,\varepsilon}$ back into the equation. Let us define the term

$$M_s^{V^\varepsilon} := \left(\frac{\sigma}{N} \sum_{i=1}^N \int_0^s \nabla^{\mathrm{T}} \partial_{x_1} V^\varepsilon (\cdot - X_u^i) \, \mathrm{d}B_u^i, \dots, \frac{\sigma}{N} \sum_{i=1}^N \int_0^s \nabla^{\mathrm{T}} \partial_{x_d} V^\varepsilon (\cdot - X_u^i) \, \mathrm{d}B_u^i \right)^{\mathrm{T}}.$$

Then, for the absorption term we have

$$\begin{aligned}
&- \int_0^t \frac{\sigma^2}{2} \langle \nabla (V^\varepsilon * \mu_s^{N,\varepsilon} - V^\varepsilon * \rho_s^\varepsilon), \nabla (V^\varepsilon \tilde{*} \mu_s^{N,\varepsilon} - V^\varepsilon * \rho_s^\varepsilon) \rangle_{L^2(\mathbb{R}^d)} \, \mathrm{d}s \\
&= - \int_0^t \frac{\sigma^2}{2} \left\| \nabla V^\varepsilon * \mu_s^{N,\varepsilon} - \nabla V^\varepsilon * \rho_s^\varepsilon \right\|_{L^2(\mathbb{R}^d)}^2 \, \mathrm{d}s \\
&\quad + \int_0^t \frac{\sigma^2}{2} \left\langle \nabla V^\varepsilon * (\mu_s^{N,\varepsilon} - * \rho_s^\varepsilon), M^{V^\varepsilon} \right\rangle_{L^2(\mathbb{R}^d)} \mathrm{d}s \\
&\le - \int_0^t \frac{\sigma^2}{2} \left\| \nabla V^\varepsilon * (\mu_s^{N,\varepsilon} - \rho_s^\varepsilon) \right\|_{L^2(\mathbb{R}^d)}^2 \, \mathrm{d}s + \int_0^t \frac{\sigma^2}{16} \left\| \nabla V^\varepsilon * (\mu_s^{N,\varepsilon} * \rho_s^\varepsilon) \right\|_{L^2(\mathbb{R}^d)}^2 \\
&\quad + 2\sigma^2 \left\| M^{V^\varepsilon} \right\|_{L^2(\mathbb{R}^d)}^2 \, \mathrm{d}s
\end{aligned}$$

$$= -\int_0^t \frac{7\sigma^2}{16} \left\|\nabla V^\varepsilon * (\mu_s^{N,\varepsilon} - \rho_s^\varepsilon)\right\|_{L^2(\mathbb{R}^d)}^2 \,\mathrm{d}s$$
$$+ 2\sigma^2 \int_0^t \left\| \frac{\sigma}{N} \sum_{i=1}^N \int_0^s \Delta V^\varepsilon(\cdot - X_u^i)\,\mathrm{d}B_u \right\|_{L^2(\mathbb{R}^d)}^2 \mathrm{d}s$$

and for the last term

$$\mathbb{E}\Bigg(\sup_{0\le t\le T} \int_0^t \Big\langle \frac{1}{N^2} \sum_{i,j=1}^N \nabla V^\varepsilon(\cdot - X_s^{i,\varepsilon}) \cdot k^\varepsilon(X_s^{i,\varepsilon} - X_s^{j,\varepsilon}) - \nabla(V^\varepsilon * (k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon),$$
$$V^\varepsilon \tilde{*} \mu_s^{N,\varepsilon} - V^\varepsilon * \rho_s^\varepsilon \Big\rangle_{L^2(\mathbb{R}^d)} \mathrm{d}s \Bigg)$$
$$\le \mathbb{E}\Bigg(\sup_{0\le t\le T} \int_0^t \Big| \Big\langle \frac{1}{N^2} \sum_{i,j=1}^N \nabla V^\varepsilon(\cdot - X_s^{i,\varepsilon}) \cdot k^\varepsilon(X_s^{i,\varepsilon} - X_s^{j,\varepsilon}) - \nabla(V^\varepsilon * (k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon),$$
$$V^\varepsilon * \mu_s^{N,\varepsilon} - V^\varepsilon * \rho_s^\varepsilon \Big\rangle_{L^2(\mathbb{R}^d)} \Big| \mathrm{d}s \Bigg)$$
$$+ \mathbb{E}\Bigg(\sup_{0\le t\le T} \int_0^t \Big| \Big\langle \frac{1}{N^2} \sum_{i,j=1}^N \nabla V^\varepsilon(\cdot - X_s^{i,\varepsilon}) \cdot k^\varepsilon(X_s^{i,\varepsilon} - X_s^{j,\varepsilon}) - \nabla(V^\varepsilon * (k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon),$$
$$\frac{\sigma}{N} \sum_{l=1}^N - \int_0^s \nabla^{\mathrm{T}} V^\varepsilon(\cdot - X_u^l)\,\mathrm{d}B_u^l \Big\rangle_{L^2(\mathbb{R}^d)} \Big| \mathrm{d}s \Bigg).$$

Applying Lemma 3.17 and put together the above estimates we have shown

$$\mathbb{E}\Bigg(\sup_{0\le t\le T} \left\| V^\varepsilon * \mu_t^{N,\varepsilon} - V^\varepsilon * \rho_t^\varepsilon \right\|_{L^2(\mathbb{R}^d)}^2 \Bigg)$$
$$\le \mathbb{E}\Bigg(\sup_{0\le t\le T} \left\| V^\varepsilon \tilde{*} \mu_t^{N,\varepsilon} - V^\varepsilon * \rho_t^\varepsilon \right\|_{L^2(\mathbb{R}^d)}^2 \Bigg) + \frac{2T\sigma^2 C_{\mathrm{BDG}}}{N} \left\|\nabla V^\varepsilon\right\|_{L^2(\mathbb{R}^d)}^2$$
$$\le 2\mathbb{E}\Bigg(\sup_{0\le t\le T} \Bigg(\left\| V^\varepsilon * \mu_0^N - V^\varepsilon * \rho_0 \right\|_{L^2(\mathbb{R}^d)}^2 - \int_0^t \frac{7\sigma^2}{16} \left\|\nabla V^\varepsilon * (\mu_s^{N,\varepsilon} - \rho_s^\varepsilon)\right\|_{L^2(\mathbb{R}^d)}^2 \mathrm{d}s$$
$$+ \int_0^t \Big| \Big\langle \frac{1}{N^2} \sum_{i,j=1}^N \nabla V^\varepsilon(\cdot - X_s^{i,\varepsilon}) \cdot k^\varepsilon(X_s^{i,\varepsilon} - X_s^{j,\varepsilon}) - \nabla(V^\varepsilon * (k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon),$$
$$V^\varepsilon * \mu_s^{N,\varepsilon} - V^\varepsilon * \rho_s^\varepsilon \Big\rangle_{L^2(\mathbb{R}^d)} \Big| \mathrm{d}s \Bigg)\Bigg)$$

$$
\begin{aligned}
&+ 2\mathbb{E}\Bigg(\sup_{0\le t\le T} \int_0^t \Bigg| \Bigg\langle \frac{1}{N^2}\sum_{i,j=1}^N \nabla V^\varepsilon(\cdot - X_s^{i,\varepsilon})\cdot k^\varepsilon(X_s^{i,\varepsilon}-X_s^{j,\varepsilon}) - \nabla(V^\varepsilon * (k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon),\\
&\frac{\sigma}{N}\sum_{l=1}^N - \int_0^s \nabla^{\mathrm{T}} V^\varepsilon(\cdot - X_u^l)\,\mathrm{d}B_u^l \Bigg\rangle_{L^2(\mathbb{R}^d)} \Bigg| \,\mathrm{d}s \Bigg)\\
&+ 4\sigma^2\mathbb{E}\Bigg(\sup_{0\le t\le T}\int_0^t \left\| M^{V^\varepsilon}\right\|_{L^2(\mathbb{R}^d)}^2 \,\mathrm{d}s\Bigg) + \frac{2T\sigma^2 C_{\mathrm{BDG}}}{N}\left\|\nabla V^\varepsilon\right\|_{L^2(\mathbb{R}^d)}^2 .
\end{aligned}
\tag{3.21}
$$

Now, we want to estimate each term on its own. We will split the fourth terms into fourth separate lemmata to keep a readable structure. The theorem follows immediately by combining Lemma 3.19 and the inequalities (3.22), (3.30), (3.37) in the lemmata below. We will summarize the estimate after we prove the following lemmata.

LEMMA 3.18 (Initial Value Inequality). *Let the assumptions of Theorem 3.14 hold true. Then*

$$
\mathbb{E}\Bigg(\left\|V^\varepsilon * \mu_0^N - V^\varepsilon * \rho_0\right\|_{L^2(\mathbb{R}^d)}^2 \Bigg) \le \frac{2}{N}\left\|V^\varepsilon\right\|_{L^2(\mathbb{R}^d)}^2 .
\tag{3.22}
$$

PROOF. We compute

$$
\begin{aligned}
&\mathbb{E}\Bigg(\left\|V^\varepsilon * \mu_0^N - V^\varepsilon * \rho_0\right\|_{L^2(\mathbb{R}^d)}^2 \Bigg)\\
&= \int_{\mathbb{R}^d} \mathbb{E}\Bigg(|V^\varepsilon * \mu_0^N(y)|^2 - 2V^\varepsilon * \mu_0^N(y)\cdot V^\varepsilon * \rho_0(y) + |V^\varepsilon * \rho_0(y)|^2\Bigg)\,\mathrm{d}y\\
&= \int_{\mathbb{R}^d} \frac{1}{N^2}\sum_{i,j=1}^N \mathbb{E}\Bigg(V^\varepsilon(y-X_0^i)\cdot V^\varepsilon(y-X_0^j)\Bigg) - \frac{2}{N}\sum_{i=1}^N \mathbb{E}\Bigg(V^\varepsilon(y-X_0^i)\Bigg)\cdot V^\varepsilon * \rho_0(y)\\
&\quad + |V^\varepsilon * \rho_0(y)|^2\,\mathrm{d}y\\
&= \int_{\mathbb{R}^d} \frac{N^2-N}{N^2}|V^\varepsilon * \rho_0(y)|^2 + \frac{1}{N}|V^\varepsilon|^2 * \rho_0(y) - |V^\varepsilon * \rho_0(y)|^2\,\mathrm{d}y\\
&= \frac{1}{N}\int_{\mathbb{R}^d} (V^\varepsilon)^2 * \rho_0(y) - |V^\varepsilon * \rho_0(y)|^2\,\mathrm{d}y\\
&\le \frac{1}{N}\big(\left\||V^\varepsilon|^2 * \rho_0\right\|_{L^1(\mathbb{R}^d)} + \left\|V^\varepsilon * \rho_0\right\|_{L^2(\mathbb{R}^d)}^2\big) \le \frac{2}{N}\left\|V^\varepsilon\right\|_{L^2(\mathbb{R}^d)}^2 \left\|\rho_0\right\|_{L^1(\mathbb{R}^d)},
\end{aligned}
$$

where we used the fact that the initial particles are i.i.d. and Young's inequality for convolutions in the last step. ☐

Lemma 3.19 (Absorption Inequality). *Let the assumptions of Theorem 3.14 hold true. Then*

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le T}\int_0^t \Big|\Big\langle \frac{1}{N^2}\sum_{i,j=1}^N \nabla V^\varepsilon(\cdot - X_s^{i,\varepsilon})\cdot k^\varepsilon(X_s^{i,\varepsilon}-X_s^{j,\varepsilon}) - \nabla(V^\varepsilon * (k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon),\\
&\qquad V^\varepsilon * (\mu_s^{N,\varepsilon}-\rho_s^\varepsilon)\Big\rangle_{L^2}\Big| - \frac{7\sigma^2}{16}\left\|\nabla V^\varepsilon * (\mu_s^{N,\varepsilon}-\rho_s^\varepsilon)\right\|_{L^2(\mathbb{R}^d)}^2 \,\mathrm{d}s\Big)\\
&\le \frac{16T\left\|\nabla k^\varepsilon\right\|_{L^\infty(\mathbb{R}^d)}^2\left\|V^\varepsilon\right\|_{L^2(\mathbb{R}^d)}^2}{\sigma^2 N^{2\alpha}} + \frac{4T\left\|V^\varepsilon\right\|_{L^2(\mathbb{R}^d)}^2}{\sigma^2 N^{2(\alpha+\delta)}}\\
&+\Big(\frac{4\left\|\nabla V^\varepsilon\right\|_{L^2(\mathbb{R}^d)}^2}{N^{2\alpha}\sigma^2} + \frac{16\left\|V^\varepsilon\right\|_{L^2(\mathbb{R}^d)}^2}{N\sigma^2}\Big)\int_0^T \left\|k^\varepsilon * \rho_s^\varepsilon\right\|_{L^\infty(\mathbb{R}^d)}^2 \,\mathrm{d}s\\
&+\frac{C(\gamma)T}{N^\gamma}\Big(\left\|V^\varepsilon\right\|_{L^2(\mathbb{R}^d)}^2\left\|k^\varepsilon\right\|_{L^\infty(\mathbb{R}^d)}^2 + \left\|\nabla V^\varepsilon\right\|_{L^2(\mathbb{R}^d)}^2\Big)\\
&-\frac{\sigma^2}{8}\mathbb{E}\Big(\sup_{0\le t\le T}\int_0^t \left\|\nabla V^\varepsilon * (\mu_s^{N,\varepsilon} - *\rho_s^\varepsilon)\right\|_{L^2(\mathbb{R}^d)}^2 \,\mathrm{d}s\Big).
\end{aligned}$$

Proof. Before we begin the proof of this lemma, we will provide an overview of our approach. Our main strategy is to utilize the convergence in probability of the particle $X_t^{i,\varepsilon}$ to their mean-field limit $Y_t^{i,\varepsilon}$ (Assumption 3.4) in combination with the law of large numbers (Assumption 3.5). This implies that the "bad set", where the particles are apart is small in probability with arbitrary algebraic convergence rate. Therefore, we may assume that $X_t^{i,\varepsilon}$ is close to $Y_t^{i,\varepsilon}$, and we formally replace the empirical measure of $(X_t^{i,\varepsilon}, i=1,\dots,N)$ with the empirical measure associated with $(Y_t^{i,\varepsilon}, i=1,\dots,N)$. However, $(Y_t^{i,\varepsilon}, i=1,\dots,N)$ has more desirable properties. For instance, the particles are independent and have density $\rho_t^\varepsilon \in L^1(\mathbb{R}^d)$ and often even $\rho_t^\varepsilon \in L^\infty(\mathbb{R}^d)$. This allows us to apply the law of large numbers (3.14), which ultimately proves the claim.

Let us start by splitting our probability space Ω into two sets. On one set B_s^α the particles are close to the mean-field particles in probability and "satisfy" the law of large numbers. The other set we take as the complement $(B_s^\alpha)^{\mathrm{c}}$, which has small probability by inequalities (3.13) and (3.14).

More precisely, we have

$$\begin{aligned}
B_s^\alpha =&\Big\{\omega\in\Omega\colon \max_{i=1,\dots,N}\Big|\frac{1}{N}\sum_{j=1}^N k^\varepsilon(Y_s^{i,\varepsilon}(\omega)-Y_s^{j,\varepsilon}(\omega)) - (k^\varepsilon * \rho_s^\varepsilon)(Y_s^{i,\varepsilon}(\omega))\Big| \le N^{-(\alpha+\delta)}\Big\}\\
&\cap\Big\{\omega\in\Omega\colon \max_{i=1,\dots,N}|X_s^{i,\varepsilon}(\omega)-Y_s^{i,\varepsilon}(\omega)|\le N^{-\alpha}\Big\}
\end{aligned}\tag{3.23}$$

for some $\delta>0$ such that $0<\alpha+\delta<1/2$ and we have the estimate $\mathbb{P}((B_s^\alpha)^{\mathrm{c}}) \le C(\gamma)N^{-\gamma}$ for all $\gamma>0$ by (3.13) and (3.14). Let us rewrite the last Lebesgue integral on the left-hand side of our claim as follows

$$\mathbb{E}\Big(\sup_{0\le t\le T}\int_0^t \Big|\Big\langle \frac{1}{N^2}\sum_{i,j=1}^N \nabla V^\varepsilon(\cdot - X_s^{i,\varepsilon})\cdot k^\varepsilon(X_s^{i,\varepsilon}-X_s^{j,\varepsilon}) - \nabla(V^\varepsilon*(k^\varepsilon*\rho_s^\varepsilon)\rho_s^\varepsilon),$$
$$V^\varepsilon*\mu_s^{N,\varepsilon} - V^\varepsilon*\rho_s^\varepsilon\Big\rangle_{L^2(\mathbb{R}^d)}\Big|\,\mathrm{d}s\Big)$$
$$\le \mathbb{E}\Big(\sup_{0\le t\le T}\int_0^t \Big|\Big\langle \frac{1}{N^2}\sum_{i,j=1}^N \nabla V^\varepsilon(\cdot - X_s^{i,\varepsilon})\cdot k^\varepsilon(X_s^{i,\varepsilon}-X_s^{j,\varepsilon}) - \nabla(V^\varepsilon*(k^\varepsilon*\rho_s^\varepsilon)\rho_s^\varepsilon),$$
$$V^\varepsilon*\mu_s^{N,\varepsilon} - V^\varepsilon*\rho_s^\varepsilon\Big\rangle_{L^2(\mathbb{R}^d)}\Big|\Big(\mathbb{1}_{(B_s^\alpha)} + \mathbb{1}_{(B_s^\alpha)^c}\Big)\,\mathrm{d}s\Big).$$

We are going to estimate each term by itself.

On the set B_s^α: In order to estimate the first term above we let $\omega\in B_s^\alpha$ and will not write the indicator function. Then we have

$$\frac{1}{N^2}\sum_{i,j=1}^N \Big\langle \nabla V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega))\cdot k^\varepsilon(X_s^{i,\varepsilon}(\omega)-X_s^{j,\varepsilon}(\omega)) - \nabla(V^\varepsilon*(k^\varepsilon*\rho_s^\varepsilon)\rho_s^\varepsilon)),$$
$$V^\varepsilon*(\mu_s^{N,\varepsilon}(\omega)-\rho_s^\varepsilon)\Big\rangle_{L^2(\mathbb{R}^d)}$$
$$= \frac{1}{N^2}\sum_{i,j=1}^N \Big\langle \nabla V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega))\cdot (k^\varepsilon(X_s^{i,\varepsilon}(\omega)-X_s^{j,\varepsilon}(\omega)) - k^\varepsilon(Y_s^{i,\varepsilon}(\omega)-Y_s^{j,\varepsilon}(\omega))),$$
$$V^\varepsilon*(\mu_s^{N,\varepsilon}(\omega)-\rho_s^\varepsilon)\Big\rangle_{L^2(\mathbb{R}^d)}$$
$$+ \frac{1}{N^2}\sum_{i,j=1}^N \Big\langle \nabla V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega))\cdot k^\varepsilon(Y_s^{i,\varepsilon}(\omega)-Y_s^{j,\varepsilon}(\omega)) - \nabla(V^\varepsilon*(k^\varepsilon*\rho_s^\varepsilon)\rho_s^\varepsilon)),$$
$$V^\varepsilon*(\mu_s^{N,\varepsilon}(\omega)-\rho_s^\varepsilon)\Big\rangle_{L^2(\mathbb{R}^d)}$$
$$= I_s^1(\omega) + I_s^2(\omega).$$

For the first term we obtain

$$|I_s^1(\omega)|$$
$$= \Big|\frac{1}{N^2}\sum_{i,j=1}^N \Big\langle V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega))(k^\varepsilon(X_s^{i,\varepsilon}(\omega)-X_s^{j,\varepsilon}(\omega)) - k^\varepsilon(Y_s^{i,\varepsilon}(\omega)-Y_s^{j,\varepsilon}(\omega))),$$
$$\nabla V^\varepsilon*(\mu_s^{N,\varepsilon}(\omega)-\rho_s^\varepsilon)\Big\rangle_{L^2(\mathbb{R}^d)}\Big|$$
$$\le \frac{1}{N^2}\sum_{i,j=1}^N \Big\langle |(V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega))| \max_{1\le i\le N}|k^\varepsilon(X_s^{i,\varepsilon}(\omega)-X_s^{j,\varepsilon}(\omega)) - k^\varepsilon(Y_s^{i,\varepsilon}(\omega)-Y_s^{j,\varepsilon}(\omega))|,$$

$$
\begin{aligned}
&\quad |\nabla V^\varepsilon * (\mu_s^{N,\varepsilon}(\omega) - \rho_s^\varepsilon| \Big\rangle_{L^2(\mathbb{R}^d)} \\
&\le \frac{2}{N}\sum_{i=1}^N \Big\langle \|\nabla k^\varepsilon\|_{L^\infty(\mathbb{R}^d)} |V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega))| \max_{1\le i\le N} |X_s^{i,N} - Y_s^{i,N}|, \\
&\quad |\nabla V^\varepsilon * (\mu_s^{N,\varepsilon}(\omega) - \rho_s^\varepsilon)| \Big\rangle_{L^2(\mathbb{R}^d)} \\
&\le \frac{2}{N}\sum_{i=1}^N \langle N^{-\alpha} \|\nabla k^\varepsilon\|_{L^\infty(\mathbb{R}^d)} |V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega))|, |\nabla V^\varepsilon * (\mu_s^{N,\varepsilon}(\omega) - \rho_s^\varepsilon|\rangle_{L^2(\mathbb{R}^d)} \\
&\le \frac{2}{N}\sum_{i=1}^N \int_{\mathbb{R}^d} \frac{8\|\nabla k^\varepsilon\|^2_{L^\infty(\mathbb{R}^d)}}{\sigma^2 N^{2\alpha}} |V^\varepsilon(y - X_s^{i,\varepsilon}(\omega))|^2 \,\mathrm{d}y \\
&\quad + \frac{\sigma^2}{32}\int_{\mathbb{R}^d} |\nabla V^\varepsilon * (\mu_s^{N,\varepsilon}(\omega) - \rho_s^\varepsilon)(y)|^2 \,\mathrm{d}y \\
&\le \frac{16}{\sigma^2 N^{2\alpha}} \|\nabla k^\varepsilon\|^2_{L^\infty(\mathbb{R}^d)} \|V^\varepsilon\|^2_{L^2(\mathbb{R}^d)} + \frac{\sigma^2}{16}\left\|\nabla V^\varepsilon * (\mu_s^{N,\varepsilon}(\omega) - \rho_s^\varepsilon)\right\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned} \tag{3.24}
$$

Here we used integration by parts in the first step, the property of the set B_s^α in the fourth step. As always, we neglect the last term by absorbing it into the diffusion in our statement.

We treat the term $I_s^2(\omega)$ using the law of large numbers property of the second term in B_s^α. For $\omega \in B_s^\alpha$ we rewrite

$$
\begin{aligned}
|I_s^2(\omega)| &= \Big| \frac{1}{N^2}\sum_{i,j=1}^N \Big\langle \nabla V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega)) \cdot k^\varepsilon(Y_s^{i,\varepsilon}(\omega) - Y_s^{j,\varepsilon}(\omega)) - \nabla(V^\varepsilon * (k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon)), \\
&\qquad V^\varepsilon * (\mu_s^{N,\varepsilon}(\omega) - \rho_s^\varepsilon) \Big\rangle_{L^2(\mathbb{R}^d)} \Big| \\
&= \Big| \frac{1}{N^2}\sum_{i,j=1}^N \Big\langle V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega)) k^\varepsilon(Y_s^{i,\varepsilon}(\omega) - Y_s^{j,\varepsilon}(\omega)) - V^\varepsilon * ((k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon)), \\
&\qquad \nabla V^\varepsilon * (\mu_s^{N,\varepsilon}(\omega) - \rho_s^\varepsilon) \Big\rangle_{L^2(\mathbb{R}^d)} \Big| \\
&= \Big| \frac{1}{N^2}\sum_{i,j=1}^N \Big\langle V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega))(k^\varepsilon(Y_s^{i,\varepsilon}(\omega) - Y_s^{j,\varepsilon}(\omega)) - (k^\varepsilon * \rho_s^\varepsilon)(Y_s^{i,\varepsilon}(\omega))) \\
&\quad + (V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega)) - V^\varepsilon(\cdot - Y_s^{i,\varepsilon}(\omega)))(k^\varepsilon * \rho_s^\varepsilon)(Y_s^{i,\varepsilon}(\omega)) \\
&\quad + V^\varepsilon(\cdot - Y_s^{i,\varepsilon}(\omega))(k^\varepsilon * \rho_s^\varepsilon)(Y_s^{i,\varepsilon}(\omega)) - V^\varepsilon * ((k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon)), \\
&\qquad \nabla V^\varepsilon * (\mu_s^{N,\varepsilon}(\omega) - \rho_s^\varepsilon) \Big\rangle_{L^2(\mathbb{R}^d)} \Big| \\
&= |I_s^{21}(\omega)| + |I_s^{22}(\omega)| + |I_s^{23}(\omega)|.
\end{aligned} \tag{3.25}
$$

For the first term $I_s^{21}(\omega)$ we obtain

$$
\begin{aligned}
|I_s^{21}(\omega)| &\leq \frac{1}{N}\sum_{i=1}^{N}\left\langle |V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega))| \left|\frac{1}{N}\sum_{j=1}^{N} k^\varepsilon(Y_s^{i,\varepsilon}(\omega) - Y_s^{j,\varepsilon}(\omega)) - (k^\varepsilon * \rho_s^\varepsilon)(Y_s^{i,\varepsilon}(\omega))\right|,\right.\\
&\qquad\qquad \left. |\nabla V^\varepsilon * (\mu_s^{N,\varepsilon}(\omega) - \rho_s^\varepsilon)| \right\rangle_{L^2(\mathbb{R}^d)}\\
&\leq \frac{1}{N}\sum_{i=1}^{N}\langle N^{-(\alpha+\delta)}|V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega))|, |\nabla V^\varepsilon * (\mu_s^{N,\varepsilon}(\omega) - \rho_s^\varepsilon)|\rangle_{L^2(\mathbb{R}^d)}\\
&\leq \frac{4N^{-2(\alpha+\delta)}}{\sigma^2}\,\|V^\varepsilon\|_{L^2(\mathbb{R}^d)}^2 + \frac{\sigma^2}{16}\,\left\|\nabla V^\varepsilon * (\mu_s^{N,\varepsilon}(\omega) - \rho_s^\varepsilon)\right\|_{L^2(\mathbb{R}^d)}^2, \qquad (3.26)
\end{aligned}
$$

where we used the property of the set B_s^α in the second step and Young's inequality.

Using the fact that we are still on the set B_s^α we obtain for the second term $I_s^{22}(\omega)$ the following estimate

$$
\begin{aligned}
&|I_s^{22}(\omega)|\\
&\leq \frac{4\,\|k^\varepsilon * \rho_s^\varepsilon\|_{L^\infty(\mathbb{R}^d)}^2}{N\sigma^2}\sum_{i=1}^{N}\int_{\mathbb{R}^d}|V^\varepsilon(y - X_s^{i,\varepsilon}(\omega)) - V^\varepsilon(y - Y_s^{i,\varepsilon}(\omega)))|^2\,\mathrm{d}y\\
&\quad + \frac{\sigma^2}{16}\int_{\mathbb{R}^d}|\nabla V^\varepsilon * (\mu_s^{N,\varepsilon}(\omega) - \rho_s^\varepsilon)(y)|^2\,\mathrm{d}y\\
&= \frac{4\,\|k^\varepsilon * \rho_s^\varepsilon\|_{L^\infty(\mathbb{R}^d)}^2}{N\sigma^2}\sum_{i=1}^{N}\int_{\mathbb{R}^d}\left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}r}V^\varepsilon(y - Y_s^{i,\varepsilon}(\omega) + r(Y_s^{i,\varepsilon}(\omega) - X_s^{i,\varepsilon}(\omega)))\,\mathrm{d}r\right|^2\mathrm{d}y\\
&\quad + \frac{\sigma^2}{16}\int_{\mathbb{R}^d}|\nabla V^\varepsilon * (\mu_s^{N,\varepsilon}(\omega) - \rho_s^\varepsilon)(y)|^2\,\mathrm{d}y\\
&\leq \frac{4\,\|k^\varepsilon * \rho_s^\varepsilon\|_{L^\infty(\mathbb{R}^d)}^2}{\sigma^2}\max_{1\leq i\leq N}|Y_s^{i,\varepsilon}(\omega) - X_s^{i,\varepsilon}(\omega)|^2\\
&\quad \cdot \frac{1}{N}\sum_{i=1}^{N}\int_0^1\int_{\mathbb{R}^d}\left|\nabla V^\varepsilon(y - Y_s^{i,\varepsilon}(\omega) + r(Y_s^{i,\varepsilon}(\omega) - X_s^{i,\varepsilon}(\omega)))\right|^2\mathrm{d}y\,\mathrm{d}r\\
&\quad + \frac{\sigma^2}{16}\int_{\mathbb{R}^d}|\nabla V^\varepsilon * (\mu_s^{N,\varepsilon}(\omega) - \rho_s^\varepsilon)(y)|^2\,\mathrm{d}y\\
&= \frac{4\,\|k^\varepsilon * \rho_s^\varepsilon\|_{L^\infty(\mathbb{R}^d)}^2}{\sigma^2}\max_{1\leq i\leq N}|Y_s^{i,\varepsilon}(\omega) - X_s^{i,\varepsilon}(\omega)|^2\int_0^1\int_{\mathbb{R}^d}|\nabla V^\varepsilon(z)|^2\,\mathrm{d}z\,\mathrm{d}r\\
&\quad + \frac{\sigma^2}{16}\int_{\mathbb{R}^d}|\nabla V^\varepsilon * (\mu_s^{N,\varepsilon}(\omega) - \rho_s^\varepsilon)(y)|^2\,\mathrm{d}y\\
&\leq \frac{4\,\|k^\varepsilon * \rho_s^\varepsilon\|_{L^\infty(\mathbb{R}^d)}^2}{N^{2\alpha}\sigma^2}\,\|\nabla V^\varepsilon\|_{L^2(\mathbb{R}^d)}^2 + \frac{\sigma^2}{16}\,\left\|\nabla V^\varepsilon * (\mu_s^{N,\varepsilon}(\omega) - \rho_s^\varepsilon)\right\|_{L^2(\mathbb{R}^d)}^2. \qquad (3.27)
\end{aligned}
$$

In the above calculations we used Young's inequality in the first step, Jensen inequality in the second estimate, the property of the set B_s^α in the third estimate.

In order to estimate the last term $I_s^{23}(\omega)$ in (3.25) we use the independence of our mean-field particles $(Y_t^{i,\varepsilon}, i = 1, \dots, N)$. Hence, we can no longer do the estimates pathwise and need to take advantage of the expectation. First, applying Young's inequality we find

$$|I_s^{23}(\omega)| \leq \frac{4}{\sigma^2} \int_{\mathbb{R}^d} \frac{1}{N^2} \left| \sum_{i=1}^N V^\varepsilon(y - Y_s^{i,\varepsilon}(\omega))(k^\varepsilon * \rho_s^\varepsilon)(Y_s^{i,\varepsilon}(\omega)) - V^\varepsilon * ((k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon))(y) \right|^2 \mathrm{d}y$$
$$+ \frac{\sigma^2}{16} \left\| \nabla V^\varepsilon * (\mu_s^{N,\varepsilon}(\omega) - \rho_s^\varepsilon) \right\|_{L^2(\mathbb{R}^d)}^2 .$$

As always, the last term is going to be absorbed. For the first term, we recall that our statement has an supremum over all $0 \leq t \leq T$ and an expectation. Hence, it is enough to estimate

$$\mathbb{E}\Bigg(\sup_{0\leq t\leq T} \int_0^t \frac{4}{\sigma^2} \int_{\mathbb{R}^d} \frac{1}{N^2} \Bigg| \sum_{i=1}^N V^\varepsilon(y - Y_s^{i,\varepsilon}(\omega))(k^\varepsilon * \rho_s^\varepsilon)(Y_s^{i,\varepsilon}(\omega))$$
$$- V^\varepsilon * ((k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon))(y) \Bigg|^2 \mathrm{d}y \Bigg)$$
$$= \int_0^T \frac{4}{N^2\sigma^2} \int_{\mathbb{R}^d} \mathbb{E}\Bigg(\Bigg| \sum_{i=1}^N V^\varepsilon(y - Y_s^{i,\varepsilon}(\omega))(k^\varepsilon * \rho_s^\varepsilon)(Y_s^{i,\varepsilon}(\omega))$$
$$- V^\varepsilon * ((k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon))(y) \Bigg|^2 \Bigg) \mathrm{d}y.$$

Let us denote for fix $y \in \mathbb{R}^d$

$$Z_s^i(y,\omega) := V^\varepsilon(y - Y_s^{i,\varepsilon}(\omega))(k^\varepsilon * \rho_s^\varepsilon)(Y_s^{i,\varepsilon}(\omega)) - V^\varepsilon * ((k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon))(y).$$

Then we notice that

$$\mathbb{E}(Z_s^i) = \mathbb{E}(V^\varepsilon(y - Y_s^{i,\varepsilon})(k^\varepsilon * \rho_s^\varepsilon)(Y_s^{i,\varepsilon})) - V^\varepsilon * ((k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon))(y)$$
$$= \int_{\mathbb{R}^d} V^\varepsilon(y - z)(k^\varepsilon * \rho_s^\varepsilon)(z)\rho_s^\varepsilon(z)\,\mathrm{d}z - V^\varepsilon * ((k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon))(y) = 0.$$

Furthermore, we have the random variables $(Z_s^i, i = 1, \dots, N)$ are pairwise independent. Hence, if $i \neq j$ we find

$$\mathbb{E}(Z_s^i \cdot Z_s^j) = \mathbb{E}(Z_s^i) \cdot \mathbb{E}(Z_s^j) = 0.$$

We notice that we have

$$\mathbb{E}\Bigg(\Bigg| \sum_{i=1}^N V^\varepsilon(y - Y_s^{i,\varepsilon}(\omega))(k^\varepsilon * \rho_s^\varepsilon)(Y_s^{i,\varepsilon}(\omega)) - V^\varepsilon * ((k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon))(y) \Bigg|^2 \Bigg)$$
$$= \mathbb{E}\Bigg(\Bigg| \sum_{i=1}^N Z_s^i \Bigg|^2 \Bigg) = \sum_{i=1}^N \mathbb{E}(|Z_s^i|^2).$$

However, by using the trivial inequality $(a+b)^2 \leq 2(a^2+b^2)$ and Young's inequality for convolution we obtain

$$\begin{aligned}&\int_{\mathbb{R}^d} \mathbb{E}(|Z_s^i(y)|^2)\,\mathrm{d}y \\ &\leq 2\mathbb{E}\bigg(\int_{\mathbb{R}^d} |V^\varepsilon(y - Y_s^{i,\varepsilon}(\omega))(k^\varepsilon * \rho_s^\varepsilon)(Y_s^{i,\varepsilon}(\omega))|^2 + |V^\varepsilon * ((k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon))(y)|^2\,\mathrm{d}y\bigg) \\ &\leq 2\,\|k^\varepsilon * \rho_s^\varepsilon\|^2_{L^\infty(\mathbb{R}^d)}\,\|V^\varepsilon\|^2_{L^2(\mathbb{R}^d)} + 2\,\|V^\varepsilon\|^2_{L^2(\mathbb{R}^d)}\,\|(k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon\|^2_{L^1(\mathbb{R}^d)} \\ &= 4\,\|k^\varepsilon * \rho_s^\varepsilon\|^2_{L^\infty(\mathbb{R}^d)}\,\|V^\varepsilon\|^2_{L^2(\mathbb{R}^d)}\,.\end{aligned}$$

Hence, the estimate for I^{23} follows by the previous law of large numbers argument and is obtained in the following

$$\begin{aligned}&\mathbb{E}\bigg(\sup_{0\leq t\leq T}\int_0^t |I_s^{23}(\omega)|\,\mathrm{d}s\bigg) \\ &\leq \frac{\sigma^2}{16}\int_0^t \big\|\nabla V^\varepsilon * (\mu_s^{N,\varepsilon}(\omega) - \rho_s^\varepsilon)\big\|^2_{L^2(\mathbb{R}^d)} + \frac{16\,\|V^\varepsilon\|^2_{L^2(\mathbb{R}^d)}}{N\sigma^2}\int_0^T \|k^\varepsilon * \rho_s^\varepsilon\|^2_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}s. \end{aligned}\tag{3.28}$$

By combining the estimates (3.26), (3.27), (3.28) with (3.25), and (3.24) we obtain the estimate on the set B_s^α

$$\begin{aligned}&\mathbb{E}\bigg(\sup_{0\leq t\leq T}\int_0^t (|I_s^1(\omega)| + |I_s^2(\omega)|)\mathbb{1}_{(B_s^\alpha)}\,\mathrm{d}s\bigg) \\ &\leq -\frac{3\sigma^2}{16}\int_0^T \big\|\nabla V^\varepsilon * (\mu_s^{N,\varepsilon}(\omega) - \rho_s^\varepsilon)\big\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}s + \frac{16T\,\|\nabla k^\varepsilon\|_{L^\infty(\mathbb{R}^d)}\,\|V^\varepsilon\|^2_{L^2(\mathbb{R}^d)}}{\sigma^2 N^{2\alpha}} \\ &\quad + \frac{4T\,\|V^\varepsilon\|^2_{L^2(\mathbb{R}^d)}}{\sigma^2 N^{2(\alpha+\delta)}} + \Big(\frac{4\,\|\nabla V^\varepsilon\|^2_{L^2(\mathbb{R}^d)}}{N^{2\alpha}\sigma^2} + \frac{16\,\|V^\varepsilon\|^2_{L^2(\mathbb{R}^d)}}{N\sigma^2}\Big)\int_0^T \|k^\varepsilon * \rho_s^\varepsilon\|^2_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}s\end{aligned}\tag{3.29}$$

It remains to obtain an estimate on the complement of B_s^α.

On the set $(B_s^\alpha)^{\mathrm{c}}$*:* Applying Young's inequality, multiple Hölder's inequalities, the fact that $\mathbb{P}((B_s^\alpha)^{\mathrm{c}}) \leq C(\gamma)N^{-\gamma}$, we obtain

$$\begin{aligned}&\mathbb{E}\bigg(\sup_{0\leq t\leq T}\int_0^t \Big|\frac{1}{N^2}\sum_{i,j=1}^N \Big\langle \nabla V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega))\cdot k^\varepsilon(X_s^{i,\varepsilon}(\omega) - X_s^{j,\varepsilon}(\omega)) - \nabla(V^\varepsilon * (k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon)), \\ &\qquad V^\varepsilon * (\mu_s^{N,\varepsilon}(\omega) - \rho_s^\varepsilon)\Big\rangle_{L^2(\mathbb{R}^d)}\Big|\mathbb{1}_{(B_s^\alpha)^{\mathrm{c}}}\,\mathrm{d}s\bigg) \\ &\leq \frac{1}{N^2}\sum_{i,j=1}^N \mathbb{E}\bigg(\int_0^T \mathbb{1}_{(B_s^\alpha)^{\mathrm{c}}}\Big|\Big\langle V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega))k^\varepsilon(X_s^{i,\varepsilon}(\omega) - X_s^{j,\varepsilon}(\omega)) - V^\varepsilon * ((k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon)),\end{aligned}$$

$$\nabla V^\varepsilon * (\mu_s^{N,\varepsilon}(\omega) - \rho_s^\varepsilon)\Big\rangle_{L^2(\mathbb{R}^d)}\Big|\,\mathrm{d}s\Big)$$

$$\le \frac{1}{N^2}\sum_{i,j=1}^N \mathbb{E}\Big(\int_0^T \mathbb{1}_{(B_s^\alpha)^c}\Big(\big\|V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega))k^\varepsilon(X_s^{i,\varepsilon}(\omega) - X_s^{j,\varepsilon}(\omega))\big\|^2_{L^2(\mathbb{R}^d)}$$

$$+ \|V^\varepsilon * ((k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon))\|^2_{L^2(\mathbb{R}^d)}\Big)\,\mathrm{d}s\Big)$$

$$+ \frac{1}{2}\mathbb{E}\Big(\int_0^T \mathbb{1}_{(B_s^\alpha)^c}\,\big\|\nabla V^\varepsilon * (\mu_s^{N,\varepsilon}(\omega) - \rho_s^\varepsilon)\big\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}s\Big)$$

$$\le \frac{1}{N}\sum_{i=1}^N \mathbb{E}\Big(\int_0^T \mathbb{1}_{(B_s^\alpha)^c}\Big(\big\|V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega))\big\|^2_{L^2(\mathbb{R}^d)}\,\|k^\varepsilon\|^2_{L^\infty(\mathbb{R}^d)}$$

$$+ \|V^\varepsilon\|^2_{L^2(\mathbb{R}^d)}\,\|k^\varepsilon * \rho_s^\varepsilon\|^2_{L^\infty(\mathbb{R}^d)}\,\|\rho_s^\varepsilon\|^2_{L^1(\mathbb{R}^d)}\Big)\,\mathrm{d}s\Big) + 2\int_0^T \mathbb{E}\Big(\mathbb{1}_{(B_s^\alpha)^c}\,\|\nabla V^\varepsilon\|^2_{L^2(\mathbb{R}^d)}\Big)\,\mathrm{d}s$$

$$\le 2\int_0^T \mathbb{P}\Big((B_s^\alpha)^c\Big)\Big(\|V^\varepsilon\|^2_{L^2(\mathbb{R}^d)}\,\|k^\varepsilon\|^2_{L^\infty(\mathbb{R}^d)} + \|\nabla V^\varepsilon\|^2_{L^2(\mathbb{R}^d)}\Big)\,\mathrm{d}s$$

$$\le \frac{C(\gamma)T}{N^\gamma}\Big(\|V^\varepsilon\|^2_{L^2(\mathbb{R}^d)}\,\|k^\varepsilon\|^2_{L^\infty(\mathbb{R}^d)} + \|\nabla V^\varepsilon\|^2_{L^2(\mathbb{R}^d)}\Big).$$

Combined with the estimate on the set B_s^α , we obtained the result. □

Lemma 3.20 (Stochastic Remaining Term Inequality). *Let the assumptions of Theorem 3.14 hold true. Then*

$$\mathbb{E}\Big(\sup_{0\le t\le T}\int_0^t \Big|\Big\langle \frac{1}{N^2}\sum_{i,j=1}^N \nabla V^\varepsilon(\cdot - X_s^{i,\varepsilon})\cdot k^\varepsilon(X_s^{i,\varepsilon} - X_s^{j,\varepsilon}) - \nabla(V^\varepsilon * (k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon),$$

$$\frac{\sigma}{N}\sum_{l=1}^N -\int_0^s \nabla^{\mathrm{T}} V^\varepsilon(\cdot - X_u^l)\,\mathrm{d}B_u^l\Big\rangle_{L^2(\mathbb{R}^d)}\Big|\,\mathrm{d}s\Big)$$

$$\le \frac{2\sigma T^{\frac{3}{2}} C_{\mathrm{BDG}}^{\frac{1}{2}}}{N^{\alpha+\frac{1}{2}}}\,\|\nabla V^\varepsilon\|^2_{L^2(\mathbb{R}^d)}\,\|\nabla k^\varepsilon\|_{L^\infty(\mathbb{R}^d)} + \sigma\frac{C_{\mathrm{BDG}}^{\frac{1}{2}} T^{\frac{3}{2}}}{N^{\alpha+\delta+\frac{1}{2}}}\,\|\nabla V^\varepsilon\|^2_{L^2(\mathbb{R}^d)}$$

$$+ \Big(\sigma\frac{C_{\mathrm{BDG}}^{\frac{1}{2}}\,\|\nabla^2 V^\varepsilon\|_{L^2(\mathbb{R}^d)}}{N^{\alpha+\frac{1}{2}}}\,\|\nabla V^\varepsilon\|_{L^2(\mathbb{R}^d)}$$

$$+ \sigma\frac{2C_{\mathrm{BDG}}^{\frac{1}{2}}\,\|\nabla V^\varepsilon\|^2_{L^2(\mathbb{R}^d)}}{N}\Big)\int_0^T \|k^\varepsilon * \rho_s^\varepsilon\|_{L^\infty(\mathbb{R}^d)}\, s^{\frac{1}{2}}\,\mathrm{d}s.$$

$$(3.30)\qquad + \frac{2C(\gamma)C_{\mathrm{BDG}}^{\frac{1}{2}}\sigma}{N^{\frac{1}{2}+\gamma}} \|\nabla V^\varepsilon\|_{L^2(\mathbb{R}^d)}^2 \Big(\|k^\varepsilon\|_{L^\infty(\mathbb{R}^d)} \frac{2}{3} T^{\frac{3}{2}} + \int_0^T \|k^\varepsilon * \rho_s^\varepsilon\|_{L^\infty(\mathbb{R}^d)} s^{\frac{1}{2}} \,\mathrm{d}s \Big).$$

Proof. We carry out a similar strategy as in the previous Lemma 3.19. Again, we want to split Ω into a good and bad set. Remember the definition of set B_s^α in (3.23), we do the estimates on B_s^α and its complement $(B_s^\alpha)^c$ separately.

On the set B_s^α: Let $\omega \in B_s^\alpha$, then we insert the i.i.d. process $\mathbf{Y}^{N,\varepsilon}$ and split the estimate further into two terms

$$\begin{aligned}
&\frac{1}{N^2}\sum_{i,j=1}^N \Big\langle \nabla V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega)) \cdot k^\varepsilon(X_s^{i,\varepsilon}(\omega) - X_s^{j,\varepsilon}(\omega)) - \nabla(V^\varepsilon * (k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon)),\\
&\qquad \frac{\sigma}{N}\sum_{l=1}^N - \int_0^s \nabla^{\mathrm{T}} V^\varepsilon(\cdot - X_u^l)\,\mathrm{d}B_u^l \Big\rangle_{L^2(\mathbb{R}^d)}\\
&= \frac{1}{N^2}\sum_{i,j=1}^N \Big\langle \nabla V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega)) \cdot (k^\varepsilon(X_s^{i,\varepsilon}(\omega) - X_s^{j,\varepsilon}(\omega)) - k^\varepsilon(Y_s^{i,\varepsilon}(\omega) - Y_s^{j,\varepsilon}(\omega))),\\
&\qquad \frac{\sigma}{N}\sum_{l=1}^N - \int_0^s \nabla^{\mathrm{T}} V^\varepsilon(\cdot - X_u^l)\,\mathrm{d}B_u^l \Big\rangle_{L^2(\mathbb{R}^d)}\\
&\quad + \frac{1}{N^2}\sum_{i,j=1}^N \Big\langle \nabla V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega)) \cdot k^\varepsilon(Y_s^{i,\varepsilon}(\omega) - Y_s^{j,\varepsilon}(\omega)) - \nabla(V^\varepsilon * (k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon)),\\
&\qquad \frac{\sigma}{N}\sum_{l=1}^N - \int_0^s \nabla^{\mathrm{T}} V^\varepsilon(\cdot - X_u^l)\,\mathrm{d}B_u^l \Big\rangle_{L^2(\mathbb{R}^d)}\\
&= II_s^1(\omega) + II_s^2(\omega).
\end{aligned}$$

Further, utilizing the property of the set B_s^α and the Burkholder–Davis–Gundy inequality (A.1) we obtain

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le T} \int_0^t |II_s^1(\omega)| \mathbb{1}_{(B_s^\alpha)}\,\mathrm{d}s \Big)\\
&\le \mathbb{E}\Big(\int_0^T \frac{1}{N^2}\sum_{i,j=1}^N \Big\langle \Big| \nabla V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega)) \cdot \Big(k^\varepsilon(X_s^{i,\varepsilon}(\omega) - X_s^{j,\varepsilon}(\omega))\\
&\qquad - k^\varepsilon(Y_s^{i,\varepsilon}(\omega) - Y_s^{j,\varepsilon}(\omega)) \Big) \Big|, \Big| \frac{\sigma}{N}\sum_{l=1}^N \int_0^s \nabla^{\mathrm{T}} V^\varepsilon(\cdot - X_u^l)\,\mathrm{d}B_u^l \Big| \Big\rangle_{L^2(\mathbb{R}^d)} \mathbb{1}_{(B_s^\alpha)}\,\mathrm{d}s \Big)\\
&\le 2\mathbb{E}\Big(\int_0^T \frac{1}{N}\sum_{i=1}^N \Big\langle \|\nabla k^\varepsilon\|_{L^\infty(\mathbb{R}^d)} |\nabla V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega))| \max_{1\le i\le N} |X_s^{i,\varepsilon}(\omega) - Y_s^{i,\varepsilon}(\omega)|,
\end{aligned}$$

$$\left|\frac{\sigma}{N}\sum_{l=1}^{N}\int\limits_0^s \nabla^{\mathrm{T}} V^\varepsilon(\cdot - X_u^l)\,\mathrm{d}B_u^l\right|\Big\rangle_{L^2(\mathbb{R}^d)} \mathbb{1}_{(B_s^\alpha)}\,\mathrm{d}s\Big)$$

$$\leq \frac{2\sigma\,\|k_x^\varepsilon\|_{L^\infty(\mathbb{R}^d)}}{N^{\alpha+1}}\sum_{i=1}^{N}\int\limits_0^T \mathbb{E}\Big(\Big\langle|\nabla V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega))|,$$

$$\left|\frac{1}{N}\sum_{l=1}^{N}\int\limits_0^s \nabla^{\mathrm{T}} V^\varepsilon(\cdot - X_u^l)\,\mathrm{d}B_u^l\right|\Big\rangle_{L^2(\mathbb{R}^d)}\Big)\,\mathrm{d}s$$

$$= \frac{2\sigma\,\|\nabla k^\varepsilon\|_{L^\infty(\mathbb{R}^d)}}{N^\alpha} C_{\mathrm{BDG}}^{\frac{1}{2}}\,\|\nabla V^\varepsilon\|_{L^2(\mathbb{R}^d)}^2\, T^{\frac{3}{2}}\,\frac{1}{N^{\frac{1}{2}}}$$

$$= \frac{2\sigma T^{\frac{3}{2}} C_{\mathrm{BDG}}^{\frac{1}{2}}}{N^{\alpha+\frac{1}{2}}}\,\|\nabla V^\varepsilon\|_{L^2(\mathbb{R}^d)}^2\,\|\nabla k_x^\varepsilon\|_{L^\infty(\mathbb{R}^d)}\,, \tag{3.31}$$

where we have used the estimate

$$\frac{1}{N}\sum_{i=1}^{N}\int\limits_0^T \mathbb{E}\Big(\Big\langle|\nabla V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega))|,\left|\frac{1}{N}\sum_{l=1}^{N}\int\limits_0^s \nabla^{\mathrm{T}} V_x^\varepsilon(\cdot - X_u^l)\,\mathrm{d}B_u^l\right|\Big\rangle_{L^2(\mathbb{R}^d)}\Big)\,\mathrm{d}s$$

$$\leq \frac{1}{N}\sum_{i=1}^{N}\int\limits_0^T\int_{\mathbb{R}^d} \mathbb{E}(|\nabla V^\varepsilon(y - X_s^{i,\varepsilon}(\omega))|^2)^{\frac{1}{2}}\,\mathbb{E}\Big(\left|\frac{1}{N}\sum_{l=1}^{N}\int\limits_0^s \nabla^{\mathrm{T}} V^\varepsilon(y - X_u^l)\,\mathrm{d}B_u^l\right|^2\Big)^{\frac{1}{2}}\,\mathrm{d}y\,\mathrm{d}s$$

$$\leq \frac{1}{N}\sum_{i=1}^{N}\int\limits_0^T\int_{\mathbb{R}^d} \mathbb{E}(|\nabla V^\varepsilon(y - X_s^{i,\varepsilon}(\omega))|^2)^{\frac{1}{2}}\,\frac{C_{\mathrm{BDG}}^{\frac{1}{2}}}{N}\,\mathbb{E}\Big(\sum_{l=1}^{N}\int\limits_0^s |\nabla V^\varepsilon(y - X_u^l)|^2\,\mathrm{d}u\Big)^{\frac{1}{2}}\,\mathrm{d}y\,\mathrm{d}s$$

$$\leq C_{\mathrm{BDG}}^{\frac{1}{2}}\,\frac{1}{N^2}\sum_{i=1}^{N}\int\limits_0^T\Big(\int_{\mathbb{R}^d} \mathbb{E}(|\nabla V^\varepsilon(y - X_s^{i,\varepsilon}(\omega))|^2)\,\mathrm{d}y\Big)^{\frac{1}{2}}$$

$$\cdot\Big(\int_{\mathbb{R}^d}\mathbb{E}\Big(\sum_{l=1}^{N}\int\limits_0^s |\nabla V^\varepsilon(y - X_u^l)|^2\,\mathrm{d}u\Big)\,\mathrm{d}y\Big)^{\frac{1}{2}}\,\mathrm{d}s$$

$$= C_{\mathrm{BDG}}^{\frac{1}{2}}\,\|\nabla V^\varepsilon\|_{L^2(\mathbb{R}^d)}^2\,T^{\frac{3}{2}}\,\frac{1}{N^{\frac{1}{2}}}. \tag{3.32}$$

This completes the estimate of $II_s^1(\omega)$ on the set B_s^α. Next, for $\omega \in B_s^\alpha$ we rewrite $II_s^2(\omega)$ in the following way

$$II_s^2(\omega)$$

$$= \frac{1}{N^2}\sum_{i,j=1}^{N}\Big\langle \nabla V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega))\cdot k^\varepsilon(Y_s^{i,\varepsilon}(\omega) - Y_s^{j,\varepsilon}(\omega)) - \nabla(V^\varepsilon * (k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon)),$$

$$
\begin{aligned}
&\quad\frac{\sigma}{N}\sum_{l=1}^{N} - \int_0^s \nabla^{\mathrm{T}} V^\varepsilon(\cdot - X_u^l)\,\mathrm{d}B_u^l\Big\rangle_{L^2(\mathbb{R}^d)}\\
&= -\frac{\sigma}{N^2}\sum_{i,j=1}^{N}\Big\langle \nabla V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega))\cdot(k^\varepsilon(Y_s^{i,\varepsilon}(\omega) - Y_s^{j,\varepsilon}(\omega)) - (k^\varepsilon * \rho_s^\varepsilon)(Y_s^{i,\varepsilon}(\omega)))\\
&\quad + (\nabla V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega)) - \nabla V^\varepsilon(\cdot - Y_s^{i,\varepsilon}(\omega)))\cdot(k^\varepsilon * \rho_s^\varepsilon)(Y_s^{i,\varepsilon}(\omega))\\
&\quad + \nabla V^\varepsilon(\cdot - Y_s^{i,\varepsilon}(\omega))\cdot(k^\varepsilon * \rho_s^\varepsilon)(Y_s^{i,\varepsilon}(\omega)) - \nabla V^\varepsilon * ((k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon)),\\
&\quad\frac{1}{N}\sum_{l=1}^{N}\int_0^s \nabla^{\mathrm{T}} V^\varepsilon(\cdot - X_u^l)\,\mathrm{d}B_u^l\Big\rangle_{L^2(\mathbb{R}^d)}\\
&= \sigma(II_s^{21}(\omega) + II_s^{22}(\omega) + II_s^{23}(\omega)).
\end{aligned}
$$

For the first term $II_s^{21}(\omega)$, applying the the property of the set B_s^α, we find with the help of the estimate (3.32) for the stochastic term that

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le T}\int_0^t |II_s^{21}(\omega)|\,\mathrm{d}s\Big)\\
&\le \mathbb{E}\Big(\int_0^T \frac{1}{N}\sum_{i=1}^{N}\Big\langle |\nabla V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega))|\Big|\frac{1}{N}\sum_{j=1}^{N} k^\varepsilon(Y_s^{i,\varepsilon}(\omega) - Y_s^{j,\varepsilon}(\omega)) - (k^\varepsilon * \rho_s^\varepsilon)(Y_s^{i,\varepsilon}(\omega))\Big|,\\
&\qquad\Big|\frac{1}{N}\sum_{l=1}^{N}\int_0^s \nabla^{\mathrm{T}} V^\varepsilon(\cdot - X_u^l)\,\mathrm{d}B_u^l\Big|\Big\rangle_{L^2(\mathbb{R}^d)}\mathrm{d}s\Big)\\
&\le N^{-(\alpha+\delta)}\int_0^T \frac{1}{N}\sum_{i=1}^{N}\int_{\mathbb{R}^d}\mathbb{E}\Big(\Big|\nabla V^\varepsilon(y - X_s^{i,\varepsilon}(\omega))\frac{1}{N}\sum_{l=1}^{N}\int_0^s \nabla^{\mathrm{T}} V^\varepsilon(y - X_u^l)\,\mathrm{d}B_u^l\Big|\Big)\,\mathrm{d}y\,\mathrm{d}s
\end{aligned}
$$

(3.33)

$$
\le N^{-(\alpha+\delta)} C_{\mathrm{BDG}}^{\frac{1}{2}}\|\nabla V^\varepsilon\|_{L^2(\mathbb{R}^d)}^2 T^{\frac{3}{2}}\frac{1}{N^{\frac{1}{2}}} = \frac{C_{\mathrm{BDG}}^{\frac{1}{2}} T^{\frac{3}{2}}}{N^{\alpha+\delta+\frac{1}{2}}}\|\nabla V^\varepsilon\|_{L^2(\mathbb{R}^d)}^2,
$$

where we used Fubini's theorem in the second step. For the term $II_s^{22}(\omega)$ we first notice that

$$
\begin{aligned}
&|\partial_{y_j} V^\varepsilon(y - X_s^{i,\varepsilon}(\omega)) - \partial_{y_i} V^\varepsilon(z - Y_s^{i,\varepsilon}(\omega))|\\
&\quad = \Big|\int_0^1 \nabla\partial_{y_j} V^\varepsilon(y - Y_s^{i,\varepsilon}(\omega) - r(Y_s^{i,\varepsilon}(\omega) - X_s^{i\varepsilon}(\omega)))\cdot(Y_s^{i,\varepsilon}(\omega) - X_s^{i\varepsilon}(\omega))\,\mathrm{d}r\Big|
\end{aligned}
$$

and therefore by Minkowski's inequality we obtain

$$
\begin{aligned}
&|\nabla V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega)) - \nabla V^\varepsilon(\cdot - Y_s^{i,\varepsilon}(\omega)))(k^\varepsilon * \rho_s^\varepsilon)(Y_s^{i,\varepsilon}(\omega)| \\
&\quad \leq \left(\sum_{j=1}^d |\partial_{y_j} V^\varepsilon(y - X_s^{i,\varepsilon}(\omega)) - \partial_{y_i} V^\varepsilon(z - Y_s^{i,\varepsilon}(\omega))|^2 \right)^{\frac{1}{2}} \|k^\varepsilon * \rho_s^\varepsilon\|_{L^\infty(\mathbb{R}^d)} \\
&\quad \leq \left(\sum_{j=1}^d \left| \int_0^1 \nabla \partial_{y_j} V^\varepsilon(y - Y_s^{i,\varepsilon}(\omega) - r(Y_s^{i,\varepsilon}(\omega) - X_s^{i\varepsilon}(\omega))) \cdot (Y_s^{i,\varepsilon}(\omega) - X_s^{i\varepsilon}(\omega)) \,\mathrm{d}r \right|^2 \right)^{\frac{1}{2}} \\
&\qquad \cdot \|k^\varepsilon * \rho_s^\varepsilon\|_{L^\infty(\mathbb{R}^d)} \\
&\quad = \int_0^1 |\nabla^2 V^\varepsilon(y - Y_s^{i,\varepsilon}(\omega) - r(Y_s^{i,\varepsilon}(\omega) - X_s^{i\varepsilon}(\omega)))||(Y_s^{i,\varepsilon}(\omega) - X_s^{i\varepsilon}(\omega))| \,\mathrm{d}r \\
&\qquad \cdot \|k^\varepsilon * \rho_s^\varepsilon\|_{L^\infty(\mathbb{R}^d)} \,.
\end{aligned}
$$

Consequently, we arrive at

$$
\begin{aligned}
&|II_s^{22}(\omega)| \\
&\quad \leq \frac{1}{N} \sum_{i=1}^N \Big\langle |\nabla V^\varepsilon(\cdot - X_s^{i,\varepsilon}(\omega)) - \nabla V^\varepsilon(\cdot - Y_s^{i,\varepsilon}(\omega)))(k^\varepsilon * \rho_s^\varepsilon)(Y_s^{i,\varepsilon}(\omega)|, \\
&\qquad\qquad \left| \frac{1}{N} \sum_{l=1}^N \int_0^s \nabla^{\mathrm{T}} V^\varepsilon(\cdot - X_u^l) \,\mathrm{d}B_u^l \right| \Big\rangle_{L^2(\mathbb{R}^d)} \\
&\quad \leq \frac{1}{N} \sum_{i=1}^N \Big\langle \int_0^1 \left| \nabla^2 V^\varepsilon \Big(y - Y_s^{i,\varepsilon}(\omega) - r(X_s^{i,\varepsilon}(\omega) - Y_s^{i,\varepsilon}(\omega)) \Big) \right| |(Y_s^{i,\varepsilon}(\omega) - X_s^{i,\varepsilon}(\omega))| \,\mathrm{d}r, \\
&\qquad\qquad \left| \frac{1}{N} \sum_{l=1}^N \int_0^s \nabla^{\mathrm{T}} V^\varepsilon(\cdot - X_u^l) \,\mathrm{d}B_u^l \right| \Big\rangle_{L^2(\mathbb{R}^d)} \|k^\varepsilon * \rho_s^\varepsilon\|_{L^\infty(\mathbb{R}^d)} \\
&\quad \leq \|k^\varepsilon * \rho_s^\varepsilon\|_{L^\infty(\mathbb{R}^d)} N^{-\alpha} \frac{1}{N} \sum_{i=1}^N \Big\langle \int_0^1 \left| \nabla^2 V^\varepsilon \Big(y - Y_s^{i,\varepsilon}(\omega) - r(X_s^{i,\varepsilon}(\omega) - Y_s^{i,\varepsilon}(\omega)) \Big) \right| \mathrm{d}r, \\
&\qquad\qquad \left| \frac{1}{N} \sum_{l=1}^N \int_0^s \nabla^{\mathrm{T}} V^\varepsilon(\cdot - X_u^l) \,\mathrm{d}B_u^l \right| \Big\rangle_{L^2(\mathbb{R}^d)} \\
&\quad \leq \frac{\|k^\varepsilon * \rho_s^\varepsilon\|_{L^\infty(\mathbb{R}^d)}}{N^{\alpha+1}} \sum_{i=1}^N \left\| \int_0^1 \left| \nabla^2 V_x^\varepsilon \Big(y - Y_s^{i,\varepsilon}(\omega) - r(X_s^{i,\varepsilon}(\omega) - Y_s^{i,\varepsilon}(\omega)) \Big) \right| \mathrm{d}r \right\|_{L^2(\mathbb{R}^d)} \\
&\qquad \cdot \left\| \frac{1}{N} \sum_{l=1}^N \int_0^s \nabla^{\mathrm{T}} V^\varepsilon(\cdot - X_u^l) \,\mathrm{d}B_u^l \right\|_{L^2(\mathbb{R}^d)}
\end{aligned}
$$

$$\leq \|k^\varepsilon * \rho_s^\varepsilon\|_{L^\infty(\mathbb{R}^d)} N^{-\alpha} \left\|\nabla^2 V^\varepsilon\right\|_{L^2(\mathbb{R}^d)} \left\|\frac{1}{N}\sum_{l=1}^N \int_0^s \nabla^{\mathrm{T}} V^\varepsilon(\cdot - X_u^l)\,\mathrm{d}B_u^l\right\|_{L^2(\mathbb{R}^d)},$$

where we utilized the property of B_s^α in the third step, followed by the application of Hölder's inequality and Minkowski's inequality. Consequently, applying the Burkholder–Davis–Gundy inequality (A.1) we obtain

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\leq t\leq T}\int_0^t |II_s^{22}(\omega)|\,\mathrm{d}s\Big)\\
&\leq \frac{\left\|\nabla^2 V^\varepsilon\right\|_{L^2(\mathbb{R}^d)}}{N^\alpha}\mathbb{E}\Big(\int_0^T \|k^\varepsilon * \rho_s^\varepsilon\|_{L^\infty(\mathbb{R}^d)} \left\|\frac{1}{N}\sum_{l=1}^N\int_0^s \nabla^{\mathrm{T}} V^\varepsilon(\cdot - X_u^l)\,\mathrm{d}B_u^l\right\|_{L^2(\mathbb{R}^d)}\mathrm{d}s\Big)\\
&\leq \frac{\left\|\nabla^2 V^\varepsilon\right\|_{L^2(\mathbb{R}^d)}}{N^{\alpha+1}}\int_0^T \|k^\varepsilon * \rho_s^\varepsilon\|_{L^\infty(\mathbb{R}^d)}\\
&\quad\cdot\Big(\int_{\mathbb{R}^d}\mathbb{E}\Big(\Big|\sum_{l=1}^N\int_0^s \nabla^{\mathrm{T}} V^\varepsilon(y - X_u^l)\,\mathrm{d}B_u^l\Big|^2\Big)\,\mathrm{d}y\Big)^{\frac{1}{2}}\mathrm{d}s\\
&\leq \frac{C_{\mathrm{BDG}}^{\frac{1}{2}}\left\|\nabla^2 V^\varepsilon\right\|_{L^2(\mathbb{R}^d)}}{N^{\alpha+1}}\int_0^T \|k^\varepsilon * \rho_s^\varepsilon\|_{L^\infty(\mathbb{R}^d)}\\
&\quad\cdot\Big(\int_{\mathbb{R}^d}\mathbb{E}\Big(\sum_{l=1}^N\int_0^s |\nabla V^\varepsilon(y - X_u^l)|^2\,\mathrm{d}u\Big)\,\mathrm{d}y\Big)^{\frac{1}{2}}\mathrm{d}s\\
&\leq \frac{C_{\mathrm{BDG}}^{\frac{1}{2}}\left\|\nabla^2 V^\varepsilon\right\|_{L^2(\mathbb{R}^d)}}{N^{\alpha+\frac{1}{2}}}\|\nabla V^\varepsilon\|_{L^2(\mathbb{R}^d)}\int_0^T \|k^\varepsilon * \rho_s^\varepsilon\|_{L^\infty(\mathbb{R}^d)}\, s^{\frac{1}{2}}\,\mathrm{d}s.
\end{aligned}\tag{3.34}$$

For $II_s^{23}(\omega)$ we use again the Burkholder–Davis–Gundy (A.1) inequality to estimate the stochastic integral and by the law of large number argument, similar to the term I^{23} in Lemma 3.19, we obtain

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\leq t\leq T}\int_0^t |II_s^{23}(\omega)|\,\mathrm{d}s\Big)\\
&\leq \int_0^T\int_{\mathbb{R}^d}\mathbb{E}\Big(\Big|\frac{1}{N}\sum_{i=1}^N \nabla V^\varepsilon(y - Y_s^{i,\varepsilon}(\omega))\cdot(k^\varepsilon * \rho_s^\varepsilon)(Y_s^{i,\varepsilon}(\omega)) - \nabla(V^\varepsilon * (k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon))(y)\Big|,\\
&\qquad\Big|\frac{1}{N}\sum_{l=1}^N\int_0^s \nabla^{\mathrm{T}} V^\varepsilon(y - X_u^l)\,\mathrm{d}B_u^l\Big|\Big)\,\mathrm{d}y\,\mathrm{d}s
\end{aligned}$$

$$\leq \int_0^T \Bigg(\int_{\mathbb{R}^d} \mathbb{E}\Bigg(\Bigg| \frac{1}{N} \sum_{i=1}^N \nabla V^\varepsilon(y - Y_s^{i,\varepsilon}(\omega)) \cdot (k^\varepsilon * \rho_s^\varepsilon)(Y_s^{i,\varepsilon}(\omega))$$

$$- \nabla(V^\varepsilon * (k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon))(y) \Bigg|^2 \Bigg) \,\mathrm{d}y \Bigg)^{\frac{1}{2}}$$

$$\Bigg(\int_{\mathbb{R}^d} \mathbb{E}\Bigg(\Bigg| \frac{1}{N} \sum_{l=1}^N \int_0^s \nabla^{\mathrm{T}} V^\varepsilon(y - X_u^l) \,\mathrm{d}B_u^l \Bigg|^2 \Bigg) \,\mathrm{d}y \Bigg)^{\frac{1}{2}} \,\mathrm{d}s$$

$$\leq 2C_{\mathrm{BDG}}^{\frac{1}{2}} \int_0^T N^{-1/2} \left\| k^\varepsilon * \rho_s^\varepsilon \right\|_{L^\infty(\mathbb{R}^d)} \left\| \nabla V^\varepsilon \right\|_{L^2(\mathbb{R}^d)} s^{\frac{1}{2}} N^{-1/2} \left\| \nabla V^\varepsilon \right\|_{L^2(\mathbb{R}^d)} \,\mathrm{d}s$$

$$\leq \frac{2C_{\mathrm{BDG}}^{\frac{1}{2}} \left\| \nabla V^\varepsilon \right\|_{L^2(\mathbb{R}^d)}^2}{N} \int_0^T \left\| k^\varepsilon * \rho_s^\varepsilon \right\|_{L^\infty(\mathbb{R}^d)} s^{\frac{1}{2}} \,\mathrm{d}s. \tag{3.35}$$

This completes the estimate on the set B_s^α, namely

$$\mathbb{E}\Bigg(\sup_{0 \leq t \leq T} \int_0^t (|II_s^1(\omega)| + |II_s^2(\omega)|) \mathbb{1}_{(B_s^\alpha)} \,\mathrm{d}s \Bigg)$$

$$\leq \frac{2\sigma T^{\frac{3}{2}} C_{\mathrm{BDG}}^{\frac{1}{2}}}{N^{\alpha + \frac{1}{2}}} \left\| \nabla V^\varepsilon \right\|_{L^2(\mathbb{R}^d)}^2 \left\| \nabla k^\varepsilon \right\|_{L^\infty(\mathbb{R}^d)} + \sigma \frac{C_{\mathrm{BDG}}^{\frac{1}{2}} T^{\frac{3}{2}}}{N^{\alpha + \delta + \frac{1}{2}}} \left\| \nabla V^\varepsilon \right\|_{L^2(\mathbb{R}^d)}^2$$

$$+ \Bigg(\sigma \frac{C_{\mathrm{BDG}}^{\frac{1}{2}} \left\| \nabla^2 V^\varepsilon \right\|_{L^2(\mathbb{R}^d)}}{N^{\alpha + \frac{1}{2}}} \left\| \nabla V^\varepsilon \right\|_{L^2(\mathbb{R}^d)}$$

$$+ \sigma \frac{2C_{\mathrm{BDG}}^{\frac{1}{2}} \left\| \nabla V^\varepsilon \right\|_{L^2(\mathbb{R}^d)}^2}{N} \Bigg) \int_0^T \left\| k^\varepsilon * \rho_s^\varepsilon \right\|_{L^\infty(\mathbb{R}^d)} s^{\frac{1}{2}} \,\mathrm{d}s. \tag{3.36}$$

On the set $(B_s^\alpha)^{\mathrm{c}}$*:* Using $\mathbb{P}((B_s^\alpha)^{\mathrm{c}}) \leq C(\gamma) N^{-\gamma}$ for all $\gamma > 0$ by Assumption 3.5, the Burkholder–Davis–Gundy inequality (A.1), Hölder's inequality, we obtain

$$\mathbb{E}\Bigg(\sup_{0 \leq t \leq T} \int_0^t \Bigg| \Bigg\langle \frac{1}{N^2} \sum_{i,j=1}^N \nabla V^\varepsilon(\cdot - X_s^{i,\varepsilon}) \cdot k^\varepsilon(X_s^{i,\varepsilon} - X_s^{j,\varepsilon}) - (\nabla V^\varepsilon * (k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon),$$

$$\frac{\sigma}{N} \sum_{l=1}^N - \int_0^s \nabla^{\mathrm{T}} V^\varepsilon(\cdot - X_u^l) \,\mathrm{d}B_u^l \Bigg\rangle_{L^2(\mathbb{R}^d)} \Bigg| \mathbb{1}_{(B_s^\alpha)^{\mathrm{c}}} \,\mathrm{d}s \Bigg)$$

$$\leq \frac{\sigma}{N^2} \sum_{i,j=1}^N \int_0^T \int_{\mathbb{R}^d} \mathbb{E}\Bigg(\mathbb{1}_{(B_s^\alpha)^{\mathrm{c}}} \Bigg| (\nabla V^\varepsilon(y - X_s^{i,\varepsilon}) \cdot k^\varepsilon(X_s^{i,\varepsilon} - X_s^{j,\varepsilon}) - \nabla(V^\varepsilon * (k^\varepsilon * \rho_s^\varepsilon)\rho_s^\varepsilon))(y)$$

$$
\begin{aligned}
&\cdot \frac{1}{N}\sum_{l=1}^{N}\int_0^s \nabla^{\mathrm{T}} V^\varepsilon(y-X_u^l)\,\mathrm{d}B_u^l\Big|\Big)\,\mathrm{d}y\,\mathrm{d}s\\
&\le \frac{\sigma}{N^2}\sum_{i,j=1}^{N}\int_0^T\Bigg(\int_{\mathbb{R}^d}\mathbb{E}\Big(\mathbb{1}_{(B_s^\alpha)^{\mathrm{c}}}|(\nabla V^\varepsilon(y-X_s^{i,\varepsilon})\cdot k^\varepsilon(X_s^{i,\varepsilon}-X_s^{j,\varepsilon})\\
&-\nabla(V^\varepsilon*(k^\varepsilon*\rho_s^\varepsilon)(y)\rho_s^\varepsilon)(y)|^2\Big)\,\mathrm{d}y\Bigg)^{\frac12}\\
&\cdot\Bigg(\int_{\mathbb{R}^d}\mathbb{E}\Big(\Big|\frac{1}{N}\sum_{l=1}^{N}\int_0^s\nabla^{\mathrm{T}}V^\varepsilon(y-X_u^l)\,\mathrm{d}B_u^l\Big|^2\Big)\,\mathrm{d}y\Bigg)^{\frac12}\,\mathrm{d}s\\
&\le \frac{2C_{\mathrm{BDG}}^{\frac12}\sigma}{N^2}\sum_{i=1}^{N}\int_0^T\Bigg(\int_{\mathbb{R}^d}\mathbb{E}\Big(\mathbb{1}_{(B_s^\alpha)^{\mathrm{c}}}(|(\nabla V^\varepsilon(y-X_s^{i,\varepsilon})|^2\,\|k^\varepsilon\|^2_{L^\infty(\mathbb{R}^d)}\\
&+|\nabla(V^\varepsilon*(k^\varepsilon*\rho_s^\varepsilon)\rho_s^\varepsilon)(y)|^2)\Big)\,\mathrm{d}y\Bigg)^{\frac12}\Bigg(\int_{\mathbb{R}^d}\mathbb{E}\Big(\sum_{l=1}^{N}\int_0^T|\nabla V^\varepsilon(y-X_u^l)|^2\,\mathrm{d}u\Big)\,\mathrm{d}y\Bigg)^{\frac12}\,\mathrm{d}s\\
&\le \frac{2C_{\mathrm{BDG}}^{\frac12}\sigma}{N^{\frac12}}\,\|V_x^\varepsilon\|_{L^2(\mathbb{R}^d)}\int_0^T\mathbb{E}\Big(\mathbb{1}_{(B_s^\alpha)^{\mathrm{c}}}\big(\,\|\nabla V^\varepsilon\|^2_{L^2(\mathbb{R}^d)}\,\|k^\varepsilon\|^2_{L^\infty(\mathbb{R}^d)}\\
&+\|\nabla(V^\varepsilon*(k^\varepsilon*\rho_s^\varepsilon)\rho_s^\varepsilon)\|^2_{L^2(\mathbb{R}^d)}\big)\Big)^{\frac12}s^{\frac12}\,\mathrm{d}s\\
&\le \frac{2C(\gamma)C_{\mathrm{BDG}}^{\frac12}\sigma}{N^{\frac12+\gamma}}\,\|\nabla V^\varepsilon\|^2_{L^2(\mathbb{R}^d)}\Big(\|k^\varepsilon\|_{L^\infty(\mathbb{R}^d)}\frac{2}{3}T^{\frac32}+\int_0^T\|k^\varepsilon*\rho_s^\varepsilon\|_{L^\infty(\mathbb{R}^d)}\,s^{\frac12}\,\mathrm{d}s\Big).
\end{aligned}
$$

This completes the estimate on the set $(B_s^\alpha)^{\mathrm{c}}$ and we have shown our Lemma. □

LEMMA 3.21 (Stochastic Integral Inequality). *Under the assumptions of Theorem 3.14 we have the following L^2-estimate for the stochastic integral,*

$$
4\sigma^2\mathbb{E}\Big(\sup_{0\le t\le T}\int_0^t\Big\|M^{V^\varepsilon}\Big\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}s\Big)\le\frac{T^{\frac32}}{N}\,\|\nabla^2V^\varepsilon\|\,. \tag{3.37}
$$

with

$$
M^{V^\varepsilon}:=\Big(\frac{\sigma}{N}\sum_{i=1}^{N}\int_0^s\nabla^{\mathrm{T}}\partial_{x_1}V^\varepsilon(\cdot-X_u^i)\,\mathrm{d}B_u^i,\dots,\frac{\sigma}{N}\sum_{i=1}^{N}\int_0^s\nabla^{\mathrm{T}}\partial_{x_d}V^\varepsilon(\cdot-X_u^i)\,\mathrm{d}B_u^i\Big)^{\mathrm{T}}.
$$

Proof. For $j = 1, \dots, d$ an application of the Burkholder–Davis–Gundy inequality (A.1) implies

$$\mathbb{E}\Big(\sup_{0\le t\le T} \int_0^t \Big\| \frac{1}{N} \sum_{i=1}^N \int_0^s \nabla^{\mathrm{T}} \partial_{x_j} V^\varepsilon(\cdot - X_u^i)\,\mathrm{d}B_u \Big\|^2_{L^2(\mathbb{R}^d)} \mathrm{d}s \Big)$$
$$\le \int_0^T \int_{\mathbb{R}^d} \mathbb{E}\Big(\Big| \frac{1}{N} \sum_{i=1}^N \int_0^s \nabla^{\mathrm{T}} \partial_{x_j} V^\varepsilon(y - X_u^i)\,\mathrm{d}B_u \Big|^2 \Big)\,\mathrm{d}y \Big)\,\mathrm{d}s$$
$$\le \frac{1}{N^2} \int_0^T \int_{\mathbb{R}^d} \mathbb{E}\Big(\sum_{i=1}^N \int_0^s |\nabla^{\mathrm{T}} \partial_{x_j} V^\varepsilon(y - X_u^i)|^2\,\mathrm{d}u \Big)\,\mathrm{d}y \Big)\,\mathrm{d}s,$$

which implies

$$4\sigma^2 \mathbb{E}\Big(\sup_{0\le t\le T} \int_0^t \Big\| M^{V^\varepsilon} \Big\|^2_{L^2(\mathbb{R}^d)} \mathrm{d}s \Big) \le \frac{T^{\frac{3}{2}}}{N} \left\|\nabla^2 V^\varepsilon\right\|^2_{L^2(\mathbb{R}^d)}.$$

□

Continuation of the proof of theorem 3.14. We are ready to input the estimates from above lemmata in the the inequality (3.21). We find

$$\mathbb{E}\Big(\sup_{0\le t\le T} \Big\| V^\varepsilon * \mu_t^{N,\varepsilon} - V^\varepsilon * \rho_t^\varepsilon \Big\|^2_{L^2(\mathbb{R}^d)} \Big) + \frac{\sigma^2}{8} \mathbb{E}\Big(\int_0^T \left\| \nabla (V^\varepsilon * \mu_s^{N,\varepsilon} - \rho_s^\varepsilon) \right\|_{L^2(\mathbb{R}^d)} \mathrm{d}s \Big)$$
$$\le \frac{2}{N} \left\|V^\varepsilon\right\|^2_{L^2(\mathbb{R}^d)} + \frac{16T \left\|\nabla k^\varepsilon\right\|^2_{L^\infty(\mathbb{R}^d)} \left\|V^\varepsilon\right\|^2_{L^2(\mathbb{R}^d)}}{\sigma^2 N^{2\alpha}} + \frac{4T \left\|V^\varepsilon\right\|^2_{L^2(\mathbb{R}^d)}}{\sigma^2 N^{2(\alpha+\delta)}}$$
$$+ \Big(\frac{4 \left\|\nabla V^\varepsilon\right\|^2_{L^2(\mathbb{R}^d)}}{N^{2\alpha}\sigma^2} + \frac{16 \left\|\nabla V^\varepsilon\right\|^2_{L^2(\mathbb{R}^d)}}{N\sigma^2} \Big) \int_0^T \left\|k^\varepsilon * \rho_s^\varepsilon\right\|^2_{L^\infty(\mathbb{R}^d)} \mathrm{d}s$$
$$+ \frac{C(\gamma)T}{N^\gamma} \Big(\left\|V^\varepsilon\right\|^2_{L^2(\mathbb{R}^d)} \left\|k^\varepsilon\right\|^2_{L^\infty(\mathbb{R}^d)} + \left\|\nabla V^\varepsilon\right\|^2_{L^2(\mathbb{R}^d)} \Big)$$
$$+ \frac{2\sigma T^{\frac{3}{2}} C_{\mathrm{BDG}}^{\frac{1}{2}}}{N^{\alpha+\frac{1}{2}}} \left\|\nabla V^\varepsilon\right\|^2_{L^2(\mathbb{R}^d)} \left\|\nabla k^\varepsilon\right\|_{L^\infty(\mathbb{R}^d)} + \sigma \frac{C_{\mathrm{BDG}}^{\frac{1}{2}} T^{\frac{3}{2}}}{N^{\alpha+\delta+\frac{1}{2}}} \left\|\nabla V^\varepsilon\right\|^2_{L^2(\mathbb{R}^d)}$$
$$+ \Big(\sigma \frac{C_{\mathrm{BDG}}^{\frac{1}{2}} \left\|\nabla^2 V^\varepsilon\right\|_{L^2(\mathbb{R}^d)}}{N^{\alpha+\frac{1}{2}}} \left\|\nabla V^\varepsilon\right\|_{L^2(\mathbb{R}^d)}$$
$$+ \sigma \frac{2 C_{\mathrm{BDG}}^{\frac{1}{2}} \left\|\nabla V^\varepsilon\right\|^2_{L^2(\mathbb{R}^d)}}{N} \Big) \int_0^T \left\|k^\varepsilon * \rho_s^\varepsilon\right\|_{L^\infty(\mathbb{R}^d)} s^{\frac{1}{2}}\,\mathrm{d}s$$

$$+\frac{2C(\gamma)C_{\mathrm{BDG}}^{\frac{1}{2}}\sigma}{N^{\frac{1}{2}+\gamma}}\left\|\nabla V^{\varepsilon}\right\|_{L^2(\mathbb{R}^d)}^2\Big(\left\|k^{\varepsilon}\right\|_{L^\infty(\mathbb{R}^d)}\frac{2}{3}T^{\frac{3}{2}}+\int\limits_0^T\left\|k^{\varepsilon}*\rho_s^{\varepsilon}\right\|_{L^\infty(\mathbb{R}^d)}s^{\frac{1}{2}}\,\mathrm{d}s\Big).$$
$$+\frac{T^{\frac{3}{2}}}{N}\left\|\nabla^2 V^{\varepsilon}\right\|_{L^2(\mathbb{R}^d)}^2.$$

The above estimate is the most general one we obtain. In the following we simplify it to derive a usable estimates. In the process we may loose some convergence rate, depending on the concrete problem at hand. Noticing that by mass conservation

$$\left\|k^{\varepsilon}*\rho_s^{\varepsilon}\right\|_{L^2(0,T;L^\infty(\mathbb{R}^d))}\leq\left\|k^{\varepsilon}\right\|_{L^\infty(\mathbb{R}^d)}\left\|\rho_s^{\varepsilon}\right\|_{L^2(0,T;L^1(\mathbb{R}^d))}\leq T\left\|k^{\varepsilon}\right\|_{L^\infty(\mathbb{R}^d)},$$

by keeping all the N and ε dependent terms and put all the other constants into a universal constant C, which depends on T, σ, γ, C_{BDG}, we obtain

$$\mathbb{E}\Big(\sup_{0\leq t\leq T}\left\|V^{\varepsilon}*\mu_t^{N,\varepsilon}-V^{\varepsilon}*\rho_t^{\varepsilon}\right\|_{L^2(\mathbb{R}^d)}^2\Big)+\frac{\sigma^2}{8}\mathbb{E}\Big(\int\limits_0^T\left\|\nabla V^{\varepsilon}*(\mu_s^{N,\varepsilon}-\rho_s^{\varepsilon})\right\|_{L^2(\mathbb{R}^d)}\,\mathrm{d}s\Big)$$
$$\leq\frac{C}{N}(\left\|V^{\varepsilon}\right\|_{H^1(\mathbb{R}^d)}^2\left\|k^{\varepsilon}\right\|_{L^\infty(\mathbb{R}^d)}^2+\left\|\nabla^2V^{\varepsilon}\right\|_{L^2(\mathbb{R}^d)}^2)+\frac{C\left\|V^{\varepsilon}\right\|_{H^1(\mathbb{R}^d)}^2(1+\left\|k^{\varepsilon}\right\|_{L^\infty(\mathbb{R}^d)}^2)}{N^{\gamma}}$$
$$+\frac{\left\|\nabla k^{\varepsilon}\right\|_{L^\infty(\mathbb{R}^d)}^2\left\|V^{\varepsilon}\right\|_{L^2(\mathbb{R}^d)}^2+\left\|k^{\varepsilon}\right\|_{L^\infty(\mathbb{R}^d)}^2\left\|\nabla V^{\varepsilon}\right\|_{L^2(\mathbb{R}^d)}^2+\left\|V^{\varepsilon}\right\|_{L^2(\mathbb{R}^d)}^2}{N^{2\alpha}}$$
$$+C\frac{\left\|\nabla V^{\varepsilon}\right\|_{L^2(\mathbb{R}^d)}^2(1+\left\|\nabla k^{\varepsilon}\right\|_{L^\infty(\mathbb{R}^d)})+\left\|V_x^{\varepsilon}\right\|_{L^2(\mathbb{R}^d)}\left\|\nabla^2V^{\varepsilon}\right\|_{L^2(\mathbb{R}^d)}\left\|k^{\varepsilon}\right\|_{L^\infty(\mathbb{R}^d)}}{N^{\alpha+\frac{1}{2}}}.$$

In the above estimates, $\alpha\in(0,\frac{1}{2})$ and $\delta>0$ are also used and the theorem is proven. □

In the our main setting $k^{\varepsilon}=W^{\varepsilon}*V^{\varepsilon}$ we provide the following rough estimate.

Corollary 3.22. *Let $k^{\varepsilon}=W^{\varepsilon}*V^{\varepsilon}$ and $W^{\varepsilon},V^{\varepsilon}$ be admissible with rates a_W,a_V. If Theorem 3.14 holds, then*

$$\mathbb{E}\Big(\sup_{0\leq t\leq T}\left\|V^{\varepsilon}*\mu_t^{N,\varepsilon}-V^{\varepsilon}*\rho_t^{\varepsilon}\right\|_{L^2(\mathbb{R}^d)}^2\Big)+\frac{\sigma^2}{8}\mathbb{E}\Big(\int\limits_0^T\left\|\nabla V^{\varepsilon}*(\mu_s^{N,\varepsilon}-\rho_s^{\varepsilon})\right\|_{L^2(\mathbb{R}^d)}\,\mathrm{d}s\Big)$$
$$\leq\frac{C}{N\varepsilon^{2a_W+4a_V}}+\frac{C}{N^{2\alpha}\varepsilon^{2a_W+4a_V}}+\frac{C}{N^{\alpha+\frac{1}{2}}\varepsilon^{a_W+3a_V}}+\frac{C}{N^{\gamma}\varepsilon^{2a_W+4a_V}}.$$

Proof. Estimating all norms of V^{ε} by $\left\|V^{\varepsilon}\right\|_{H^2(\mathbb{R}^d)}\leq C\varepsilon^{-a_V}$ and using Young's inequality to find

$$\left\|k^{\varepsilon}\right\|_{L^\infty(\mathbb{R}^d)}+\left\|k_x^{\varepsilon}\right\|_{L^\infty(\mathbb{R}^d)}\leq 2\left\|W^{\varepsilon}\right\|_{L^2(\mathbb{R}^d)}\left\|V^{\varepsilon}\right\|_{H^2(\mathbb{R}^d)}\leq C\varepsilon^{-a_W-a_V}.$$

Hence the right hand side of the main inequality in Theorem 3.14 can be estimated by

$$\frac{C}{N\varepsilon^{2a_W+4a_V}}+\frac{C}{N^{2\alpha}\varepsilon^{2a_W+4a_V}}+\frac{C}{N^{\alpha+\frac{1}{2}}\varepsilon^{a_W+3a_V}}+\frac{C}{N^{\gamma}\varepsilon^{2a_W+4a_V}}.$$

□

Now, that we have proven our main estimate, we are ready to demonstrate the relative entropy estimates by combining Theorem 3.14 and Lemma 3.12. We start with the first main result of this chapter

Proof of Theorem 3.8. We combine the assumptions of Theorem 3.14 and the results from Lemma 3.12, Theorem 3.14 and Corollary 3.22, to find a small $\beta_1 \leq \beta_\alpha$ so that for $0 < \beta \leq \beta_1$, $\varepsilon = N^{-\beta}$ and small $0 < \lambda \ll 1$

$$\mathcal{H}_N\big(\rho_t^{N,\varepsilon}|\rho_t^{\otimes N,\varepsilon}\big) \leq \frac{C}{N^{\frac{1}{2}+\lambda}} = o\Big(\frac{1}{\sqrt{N}}\Big).$$

This allows us to demonstrate strong convergence in the $L^\infty([0,T];L^1(\mathbb{R}^d))$-norm. Applying, inequalities (1.6) and (1.7), we obtain

$$\Big\|\rho_t^{N,2,\varepsilon} - \rho_t^\varepsilon \otimes \rho_t^\varepsilon\Big\|_{L^1(\mathbb{R}^{2d})}^2 \leq 2\mathcal{H}_2\big(\rho_t^{N,1,\varepsilon}|\rho_t^\varepsilon\big) \leq 4\mathcal{H}_N\big(\rho_t^{N,\varepsilon}|\rho_t^{\otimes N,\varepsilon}\big) = o\Big(\frac{1}{\sqrt{N}}\Big).$$

In the case $k^\varepsilon = \nabla(U^\varepsilon * V^\varepsilon)$ the estimate (3.17) is derived analogously. The key is to recognize that we actually derived an estimate on the gradient of V^ε, which we have not used so far. In the case of $k^\varepsilon = \nabla(U^\varepsilon * V^\varepsilon)$ we utilize it and as a result we obtain the same convergence rates. The estimate for the modulated energy follows also directly from equality (3.19), Young's inequality and an application of Theorem 3.14 for V^ε and $\hat{U}^\varepsilon$ under the assumption that $U^\varepsilon, V^\varepsilon$ are strongly admissible. □

3.4.3. Special choices of W^ε and V^ε. We present a series of corollaries for Theorem 3.14 for different choices of V^ε. In most applications we want to take a mollified sequence. In the special case $V^\varepsilon = J^\varepsilon$ we obtain the following corollary.

Corollary 3.23. *Suppose Theorem 3.14 holds true. Let $V^\varepsilon = J^\varepsilon$ be a mollification, then for $\varepsilon = N^{-\beta}$ with some $\beta < \beta_\alpha$ and $\|k^\varepsilon\|_{L^\infty(\mathbb{R}^d)} \leq \varepsilon^{-a_k}$, $\|\nabla k^\varepsilon\|_{L^\infty(\mathbb{R}^d)} \leq \varepsilon^{-a_k-1}$ for some $a_k > 0$, then we obtain the following L^2-estimate ,*

$$\begin{aligned}
&\mathbb{E}\bigg(\sup_{0\leq t\leq T}\Big\|J^\varepsilon * \mu_t^{N,\varepsilon} - J^\varepsilon * \rho_t^\varepsilon\Big\|_{L^2(\mathbb{R}^d)}^2\bigg) + \frac{\sigma^2}{2}\mathbb{E}\bigg(\int_0^T \big\|\nabla J^\varepsilon * (\mu_s^{N,\varepsilon} - \rho_s^\varepsilon)\big\|_{L^2(\mathbb{R}^d)}\,\mathrm{d}s\bigg)\\
&\leq \frac{C}{N^{2\alpha-d\beta}} + \frac{C}{N^{\alpha+1/2-(d+2)\beta}} + \frac{C}{N^{2\alpha-d\beta-(2a_k+2)\beta}} + \frac{C}{N^{\alpha+\frac{1}{2}-(d+2)\beta-(a_k+1)\beta}}\\
&\quad + \frac{C}{N^{2\alpha-(d+2)\beta-a_k\beta}} + \frac{C}{N^\gamma N^{(d+2+2a_k)\beta}}\\
&\leq \frac{C}{N^{2\alpha-d\beta-(2a_k+2)\beta}} + \frac{C}{N^{\alpha+\frac{1}{2}-(d+2)\beta-(a_k+1)\beta}} + \frac{C}{N^{2\alpha-(d+2)\beta-a_k\beta}} + \frac{C}{N^\gamma N^{((d+2)+2a_k)\beta}}
\end{aligned}$$

for a constant C, which depends on T, σ, γ, C_{BDG}. In particular if $k \in L^\infty(\mathbb{R}^d)$ and $k^\varepsilon = (\zeta^\varepsilon(J^\varepsilon * k)) * J^\varepsilon$ the above estimate holds with $\varepsilon = N^{-\beta}$ and $a_k = 0$.

Proof. If $V^\varepsilon = J^\varepsilon$, we obtains easily that

$$\begin{aligned}\|J^\varepsilon\|_{H^m(\mathbb{R}^d)} &= \frac{1}{\varepsilon^d}\left\|\mathcal{F}^{-1}\left[\left(1+\left|\frac{\xi}{\varepsilon}\right|^2\right)^{\frac{m}{2}}\mathcal{F}[J](\xi)\right]\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^2(\mathbb{R}^d)}\\ &= \frac{1}{\varepsilon^{\frac{d}{2}+m}}\left\|\mathcal{F}^{-1}[(\varepsilon^2+|\xi|^2)^{\frac{m}{2}}\mathcal{F}[J](\xi)]\right\|_{L^2(\mathbb{R}^d)}\\ &\le \frac{C}{\varepsilon^{\frac{d}{2}+m}}.\end{aligned}$$

Therefore, we obtained with $\varepsilon = N^{-\beta}$,

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\le t\le T}\left\|J^\varepsilon * \mu_t^{N,\varepsilon} - J^\varepsilon * \rho_t^\varepsilon\right\|^2_{L^2(\mathbb{R}^d)}\right) + \frac{\sigma^2}{2}\mathbb{E}\left(\int_0^T \left\|\nabla J^\varepsilon * (\mu_s^{N,\varepsilon} - \rho_s^\varepsilon)\right\|_{L^2(\mathbb{R}^d)}\,\mathrm{d}s\right)\\ &\le \frac{C}{N^{2\alpha-d\beta}} + \frac{C}{N^{\alpha+1/2-(d+2)\beta}} + \frac{C}{N^{2\alpha-d\beta}\varepsilon^{2a_k+2}} + \frac{C}{N^{\alpha+\frac{1}{2}-(d+2)\beta}\varepsilon^{a_k+1}} + \frac{C}{N^{2\alpha-(d+2)\beta}\varepsilon^{a_k}}\\ &\quad+ \frac{C}{N^{\gamma}\varepsilon^{(d+2)+2a_k}}\end{aligned}$$

The second claim follows by Young's inequality and the scaling of the mollifier. More precisely,

$$\|J^\varepsilon\|_{W^{m,1}(\mathbb{R}^d)} = \frac{1}{\varepsilon}\left\|\mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{m}{2}}\mathcal{F}[J](\xi)]\right\|_{L^1(\mathbb{R}^d)} \le \frac{C}{\varepsilon^m}$$

□

Corollary 3.24. *Suppose Assumptions 3.4, 3.5 hold for $\alpha \in (\frac{1}{4},\frac{1}{2})$ and suppose the bounded force k has the approximation $k^\varepsilon = W^\varepsilon * V^\varepsilon$ with $W^\varepsilon = \zeta^\varepsilon(k * J^\varepsilon)$ and $V^\varepsilon = J^\varepsilon$. Then for $\varepsilon = N^{-\beta}$ and $\beta < \min\left(\frac{\alpha}{(d+4)}, \frac{1}{2(d+4)}(4\alpha-1), \beta_\alpha\right)$, there exists an $0<\lambda\ll 1$ such that*

$$\sup_{t\in[0,T]}\mathcal{H}_N\left(\rho_t^{N,\varepsilon}|\rho_t^{\otimes N,\varepsilon}\right) \le \frac{C}{N^{\frac{1}{2}+\lambda}}$$

for a constant C, which depends on T, σ, γ, C_{BDG}.

Proof. Let us start by estimating $\|W^\varepsilon\|^2_{L^2(\mathbb{R}^d)}$ in inequality (3.20),

$$\|W^\varepsilon\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d}|\zeta^\varepsilon(x)|^2\left|\int_{\mathbb{R}^d}k(x-y)J^\varepsilon(y)\,\mathrm{d}y\right|^2\mathrm{d}x \le 4\varepsilon^{-2}\|k\|_{L^\infty(\mathbb{R}^d)} = 4N^{2\beta}\|k\|_{L^\infty(\mathbb{R}^d)}$$

Now, applying Corollary 3.23 to inequality (3.20), keeping track of the powers, we obtain the result. More precisely, notice that we have to first fix $\alpha\in(\frac{1}{4},\frac{1}{2})$, then choose a β_J such that the terms for $0<\beta\le\beta_J$ smaller than $\frac{C}{N^{\frac{1}{2}+\lambda}}$, and then fix γ such that the estimate holds. □

Next, we provide a similar corollary in the case the force k is a potential and has a convolution structure.

Corollary 3.25. *Suppose Assumptions 3.4, 3.5 hold for* $\alpha \in (\frac{1}{4}, \frac{1}{2})$. *Let the force* k *be given by a potential* $k = \nabla(U * V)$ *with* $U, V \in H^{\frac{1}{2}}(\mathbb{R}^d)$ *and its approximation is given by* $k^\varepsilon = \nabla((U * J^\varepsilon) * (V * J^\varepsilon))$. *Then for* $\varepsilon = N^{-\beta}$ *and* $\beta < \min\left(\frac{1}{3}, \alpha - \frac{1}{4}, \beta_\alpha\right)$, *there exists a* $0 < \lambda \ll 1$ *such that*

$$\sup_{t\in[0,T]} \mathcal{H}_N\big(\rho_t^{N,\varepsilon}|\rho_t^{\otimes N,\varepsilon}\big) \le \frac{C}{N^{\frac{1}{2}+\lambda}},$$
$$\sup_{t\in[0,T]} |\mathcal{K}_N\big(\rho_t^{N,\varepsilon}|\rho_t^{\otimes N,\varepsilon}\big)| \le \frac{C}{N^{\frac{1}{2}+\lambda}}.$$

for a constant C, *which depends on* T, σ, γ, C_{BDG}.

Proof. Since $W \in H^{\frac{1}{2}}(\mathbb{R}^d)$ we know that $W * J^\varepsilon \in L^2(\mathbb{R}^d)$ and therefore we only need to estimate the L^2-norm for $\nabla V^\varepsilon = \nabla(V * J^\varepsilon)$ in inequality (3.20) to obtain the convergence rates. We emphasize that in Theorem 3.14 we also obtained an estimate on the gradient V^ε. Consequently, we can use Theorem 3.14 for the function $V^\varepsilon = V * J^\varepsilon$. By the estimate

$$\|J^\varepsilon\|_{W^{m,1}(\mathbb{R}^d)} = \frac{1}{\varepsilon^d}\left\|\mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{m}{2}}\mathcal{F}[J](\xi)]\right\|_{L^1(\mathbb{R}^d)} \le \frac{C}{\varepsilon^m}$$
$$\|\nabla J^\varepsilon\|_{W^{m,1}(\mathbb{R}^d)} = \frac{1}{\varepsilon}\left\|\mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{m}{2}}\mathcal{F}[\nabla J](\xi)]\right\|_{L^1(\mathbb{R}^d)} \le \frac{C}{\varepsilon^{m+1}}$$

for any $m \ge 0$ we know that

$$\begin{aligned}
&\|V^\varepsilon\|_{L^2(\mathbb{R}^d)} \le \|V\|_{L^2(\mathbb{R}^d)} \le C\\
&\|\nabla V^\varepsilon\|_{L^2(\mathbb{R}^d)} \le \|V\|_{H^{\frac{1}{2}}(\mathbb{R}^d)}\|J^\varepsilon\|_{W^{\frac{1}{2},1}(\mathbb{R}^d)} \le \frac{C}{\varepsilon^{\frac{1}{2}}}\\
&\left\|\nabla^2 V^\varepsilon\right\|_{L^2(\mathbb{R}^d)} \le \|V\|_{H^{\frac{1}{2}}(\mathbb{R}^d)}\|\nabla J^\varepsilon\|_{W^{\frac{1}{2},1}(\mathbb{R}^d)} \le \frac{C}{\varepsilon^{\frac{3}{2}}}\\
&\|k^\varepsilon\|_{L^\infty(\mathbb{R}^d)} \le \|W^\varepsilon\|_{H^{\frac{1}{2}}(\mathbb{R}^d)}\|V^\varepsilon\|_{H^{\frac{1}{2}}(\mathbb{R}^d)} \le C\\
&\|\nabla k^\varepsilon\|_{L^\infty(\mathbb{R}^d)} \le \|W^\varepsilon\|_{H^1(\mathbb{R}^d)}\|V^\varepsilon\|_{H^1(\mathbb{R}^d)} \le \frac{C}{\varepsilon}.
\end{aligned} \tag{3.38}$$

Plugging in all estimates with $\varepsilon = N^{-\beta}$ into Theorem 3.14 and having equality (3.19) in mind we obtain the rate of β and the estimate on the modulated energy. □

We have now shown in two cases how to derive explicit estimates on the relative entropy $\mathcal{H}_N\big(\rho_t^{N,\varepsilon}|\rho_t^{\otimes N,\varepsilon}\big)$ with the help of Theorem 3.14. In general, if the function $W^\varepsilon, V^\varepsilon$ have low regularity, we need to mollify them to make them admissible. Hence, we borrow the necessary regularity from J and consequently, get higher rates of N in our estimates. Compare for instance Corollary 3.24 and Corollary 3.25. Therefore the estimate by using the regularity of the J^ε term, will lead to weaker convergence rates. The benefit is of course that one does not require a potential field and the convolution structure of the potential.

3.5. De-regularization of the high dimensional PDE and the limiting PDE

The goal of this section is to prove Theorem 3.10, i.e. the strong form of propagation of chaos on the PDE level in the L^1-norm. For the de-regularization of Liouville equation (3.8) we need $k \in L^\infty(\mathbb{R}^d)$. We take the following approximation $k^\varepsilon = (\zeta^\varepsilon(k * J^\varepsilon)) * J^\varepsilon)$. We need convergence results between $\rho_t^{N,1,\varepsilon}$ and $\rho_t^{N,1}$ as well as ρ_t^ε and ρ_t. The latter convergence was shown as by-product in Theorem 2.9. More precisely,

$$\lim_{\varepsilon \to 0} \|\rho^\varepsilon - \rho\|_{L^1([0,T];L^1(\mathbb{R}^d))} = 0. \tag{3.39}$$

It remains to show that the approximated Liouville equation converges in entropy to the Liouville equation. An application of inequality (1.7) implies also the L^1-convergence.

LEMMA 3.26. *Let $k \in L^\infty(\mathbb{R}^d)$, $\rho^{N,\varepsilon}$ be the solution of the regularized Liouville equation* (3.8) *and ρ^N the solution of the Liouville equation* (3.6)*. Then, we have*

$$\sup_{t\in[0,T]} \mathcal{H}_N\big(\rho_t^{N,\varepsilon} \mid \rho_t^N\big) + \frac{\sigma^2}{4N} \sum_{i=1}^N \int_0^T \int_{\mathbb{R}^{dN}} \rho_s^{N,\varepsilon} \left|\nabla_{x_i} \log\left(\frac{\rho_s^{N,\varepsilon}}{\rho_s^N}\right)\right|^2 \mathrm{d}\mathsf{X}^N \, \mathrm{d}s$$
$$\leq CT\,\|k\|_{L^\infty(\mathbb{R}^d)}^2 \sup_{t\in[0,T]} \sqrt{\mathcal{H}_N\big(\rho_t^{N,\varepsilon} \mid \rho_t^{\otimes N,\varepsilon}\big)} + 2C\,\|k\|_{L^\infty(\mathbb{R}^d)}^2 \|\rho_s^\varepsilon - \rho_s\|_{L^1([0,T];L^1(\mathbb{R}^d))}$$
$$+ \int_0^T \int_{\mathbb{R}^{2d}} |k(x_1 - x_2) - k^\varepsilon(x_1 - x_2)|^2 \rho_s(x_1)\rho_s(x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}s.$$

In particular, the last term vanishes by dominated convergence.

PROOF. We start by computing the time derivative of $\rho_t^{N,\varepsilon} \log\left(\frac{\rho_t^{N,\varepsilon}}{\rho_t^N}\right)$. We have

$$\mathcal{H}_N\big(\rho_t^{N,\varepsilon} \mid \rho_t^N\big) = \mathcal{H}_N\big(\rho^{N,\varepsilon} \mid \rho^N\big)(0) + \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{H}_N\big(\rho_s^{N,\varepsilon} \mid \rho_s^N\big)\,\mathrm{d}s$$
$$= -\frac{\sigma^2}{2N} \sum_{i=1}^N \int_0^t \int_{\mathbb{R}^{dN}} \nabla_{x_i}\rho_s^{N,\varepsilon} \nabla_{x_i} \log\left(\frac{\rho_s^{N,\varepsilon}}{\rho_s^N}\right) \mathrm{d}\mathsf{X}^N\,\mathrm{d}s$$
$$- \frac{1}{N} \sum_{i=1}^N \int_0^t \int_{\mathbb{R}^{dN}} \left(\rho_s^{N,\varepsilon} \frac{1}{N} \sum_{j=1}^N k^\varepsilon(x_i - x_j)\right) \cdot \nabla_{x_i} \log\left(\frac{\rho_s^{N,\varepsilon}}{\rho_s^N}\right) \mathrm{d}\mathsf{X}^N\,\mathrm{d}s$$
$$+ \frac{\sigma^2}{2N} \sum_{i=1}^N \int_0^t \int_{\mathbb{R}^{dN}} \nabla_{x_i}\rho_s^N \nabla_{x_i} \left(\frac{\rho_s^{N,\varepsilon}}{\rho_s^N}\right) \mathrm{d}\mathsf{X}^N\,\mathrm{d}s$$
$$+ \frac{1}{N} \sum_{i=1}^N \int_0^t \int_{\mathbb{R}^{dN}} \left(\rho_s^N \frac{1}{N} \sum_{j=1}^N k(x_i - x_j)\right) \nabla_{x_i} \left(\frac{\rho_s^{N,\varepsilon}}{\rho_s^N}\right) \mathrm{d}\mathsf{X}^N\,\mathrm{d}s$$

$$= -\frac{\sigma^2}{2N}\sum_{i=1}^N \int_0^t \int_{\mathbb{R}^{dN}} \rho_s^{N,\varepsilon}\left|\nabla_{x_i}\log\left(\frac{\rho_s^{N,\varepsilon}}{\rho_s^N}\right)\right|^2 \mathrm{d}\mathsf{X}^N\,\mathrm{d}s$$

$$+ \frac{1}{N^2}\sum_{i,j=1}^N \int_0^t \int_{\mathbb{R}^{dN}} (k(x_i-x_j)-k^\varepsilon(x_i-x_j))\rho_s^{N,\varepsilon}\nabla_{x_i}\log\left(\frac{\rho_s^{N,\varepsilon}}{\rho_s^N}\right)\mathrm{d}\mathsf{X}^N\,\mathrm{d}s$$

$$\le -\frac{\sigma^2}{4N}\sum_{i=1}^N \int_0^t \int_{\mathbb{R}^{dN}} \rho_s^{N,\varepsilon}\left|\nabla_{x_i}\log\left(\frac{\rho_s^{N,\varepsilon}}{\rho_s^N}\right)\right|^2 \mathrm{d}\mathsf{X}^N\,\mathrm{d}s$$

$$+ \frac{1}{\sigma^2 N^2}\sum_{i,j=1}^N \int_0^t \int_{\mathbb{R}^{dN}} |k(x_i-x_j)-k^\varepsilon(x_i-x_j)|^2\rho_s^{N,\varepsilon}\,\mathrm{d}\mathsf{X}^N\,\mathrm{d}s.$$

Now, it is enough to show, that the last term vanishes for $N\to\infty$ and consequently for $\varepsilon\to 0$. We start by using the fact that the particle system (3.4) is exchangeable. We obtain

$$\frac{1}{\sigma^2 N^2}\sum_{i,j=1}^N \int_0^t \int_{\mathbb{R}^{dN}} |k(x_i-x_j)-k^\varepsilon(x_i-x_j)|^2\rho_s^{N,\varepsilon}(\mathsf{X}^N)\,\mathrm{d}\mathsf{X}^N\,\mathrm{d}s$$

$$= \frac{1}{\sigma^2 N^2}\sum_{i,j=1}^N \int_0^t \mathbb{E}\big(|k(X_s^i-X_s^j)-k^\varepsilon(X_s^i-X_s^j)|^2\big)\,\mathrm{d}s$$

$$= \frac{1}{\sigma^2}\int_0^t \mathbb{E}\big(|k(X_s^1-X_s^2)-k^\varepsilon(X_s^1-X_s^2)|^2\big)\,\mathrm{d}s$$

$$= \frac{1}{\sigma^2}\int_0^t \int_{\mathbb{R}^{2d}} |k(x_1-x_2)-k^\varepsilon(x_1-x_2)|^2\rho_s^{N,2,\varepsilon}(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}s.$$

Hence, we obtained an expression in which the dimension does not change in the limit. By applying mass conservation, the Csiszár–Kullback–Pinsker inequality (1.7) and inequality (1.6) we further estimate the term

$$\int_0^t \int_{\mathbb{R}^{2d}} |k(x_1-x_2)-k^\varepsilon(x_1-x_2)|^2\rho_s^{N,2,\varepsilon}(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}s$$

$$= \int_0^t \int_{\mathbb{R}^{2d}} |k(x_1-x_2)-k^\varepsilon(x_1-x_2)|^2(\rho_s^{N,2,\varepsilon}-(\rho_s^\varepsilon\otimes\rho_s^\varepsilon)(x_1,x_2))\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}s$$

$$+ \int_0^t \int_{\mathbb{R}^{2d}} |k(x_1-x_2)-k^\varepsilon(x_1-x_2)|^2(\rho_s^\varepsilon\otimes\rho_s^\varepsilon)(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}s$$

$$
\begin{aligned}
&\le C\,\|k\|^2_{L^\infty(\mathbb{R}^d)} \int_0^t \left\|\rho^{N,2,\varepsilon}_s - \rho^{\otimes 2,\varepsilon}\right\|_{L^1(\mathbb{R}^{2d})} \,\mathrm{d}s + \int_0^t \int_{\mathbb{R}^{2d}} |k(x_1-x_2) - k^\varepsilon(x_1-x_2)|^2 \\
&\quad \cdot \Big((\rho^\varepsilon_s(x_1) - \rho_s(x_1))\rho^\varepsilon_s(x_2) + \rho_s(x_1)(\rho^\varepsilon_s(x_2) - \rho_s(x_2)) + \rho_s(x_1)\rho_s(x_2) \Big) \,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}s \\
&\le CT\,\|k\|^2_{L^\infty(\mathbb{R}^d)} \sup_{t\in[0,T]} \sqrt{\mathcal{H}_N\big(\rho^{N,\varepsilon}_t \mid \rho^{\otimes N,\varepsilon}_t\big)} + 2C\,\|k\|^2_{L^\infty(\mathbb{R}^d)}\,\|\rho^\varepsilon_s - \rho_s\|_{L^1([0,T];L^1(\mathbb{R}^d))} \\
&\quad + \int_0^t \int_{\mathbb{R}^{2d}} |k(x_1-x_2) - k^\varepsilon(x_1-x_2)|^2 \rho_s(x_1)\rho_s(x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}s.
\end{aligned}
$$

Plugging this estimate into our above entropy calculation and taking the supremum in time proves the lemma. □

Combing both implies the strong convergence on the PDE-level of any observable $\rho^{N,r}$ to the law $\rho^{\otimes r}$ in the $L^1(\mathbb{R}^{dr})$-norm.

Proof of theorem 3.10. For $k \in L^\infty(\mathbb{R}^d)$, let $k^\varepsilon = (\zeta^\varepsilon(k * J^\varepsilon)) * J^\varepsilon$, therefore we take $W^\varepsilon = \zeta^\varepsilon(k^\varepsilon * J^\varepsilon)$ and $V^\varepsilon = J^\varepsilon$ with $\varepsilon = \varepsilon(N) = N^{-\beta}$. By assumption of the theorem (see also Theorem 2.23), there exists a $\beta_\alpha \in (0, \frac{1}{2})$ such that for all $\beta \le \beta_\alpha$ the convergence in probability, Assumption 3.4, and the law of large numbers, Assumption 3.5 both hold. Therefore we can apply the result from Corollary 3.24 for $0 < \beta < \min\left(\frac{1}{3}, \alpha - \frac{1}{4}, \beta_\alpha\right)$ and obtain the convergence of the relative entropy $\mathcal{H}\big(\rho^{N,\varepsilon}_t \mid \rho^{\otimes N,\varepsilon}_t\big)$ to zero. We can even get better convergence rate β, since we are not interested in the order of convergence of the relative entropy. Applying (1.7) and (1.6), we obtain

$$
\begin{aligned}
&\left\|\rho^{N,r} - \rho^{\otimes r}\right\|_{L^1([0,T];L^1(\mathbb{R}^{dr}))} \\
&\quad \le \left\|\rho^{N,r} - \rho^{N,r,\varepsilon}\right\|_{L^1([0,T];L^1(\mathbb{R}^{dr}))} + \left\|\rho^{N,r,\varepsilon} - \rho^{\otimes r,\varepsilon}\right\|_{L^1([0,T];L^1(\mathbb{R}^{dr}))} \\
&\qquad + \left\|\rho^{\otimes r,\varepsilon} - \rho^{\otimes r}\right\|_{L^1([0,T];L^1(\mathbb{R}^{dr}))} \\
&\quad \le \int_0^T \sqrt{2r\mathcal{H}_r\big(\rho^{N,r,\varepsilon}_t \mid \rho^{N,r}_t\big)} + \sqrt{2r\mathcal{H}_r\big(\rho^{N,r,\varepsilon}_t \mid \rho^{\otimes r,\varepsilon}_t\big)} + \left\|\rho^{\otimes r,\varepsilon}_t - \rho^{\otimes r}_t\right\|_{L^1(\mathbb{R}^{dr})} \,\mathrm{d}t \\
&\quad \le \int_0^T \sqrt{4r\mathcal{H}_N\big(\rho^{N,\varepsilon}_t \mid \rho^{N}_t\big)} + \sqrt{4r\mathcal{H}_N\big(\rho^{N,\varepsilon}_t \mid \rho^{\otimes N,\varepsilon}_t\big)} + \left\|\rho^{\otimes r,\varepsilon}_t - \rho^{\otimes r}_t\right\|_{L^1(\mathbb{R}^{dr})} \,\mathrm{d}t.
\end{aligned}
$$

As mentioned the second term converges to zero. For the first term we use the inequality in Lemma 3.26 together with the fact that the $\mathcal{H}_N\big(\rho^{N,\varepsilon}_t \mid \rho^{\otimes N,\varepsilon}_t\big)$ converges to zero and the dominated convergence to obtain

$$\begin{aligned}\limsup_{N\to\infty}\int_0^T \sqrt{4r\mathcal{H}_N\left(\rho_t^{N,\varepsilon}\mid\rho_t^N\right)}\,\mathrm{d}t &\le C(\|k\|_{L^\infty(\mathbb{R}^d)},m)\limsup_{N\to\infty}\int_0^T \|\rho_t^\varepsilon-\rho_t\|_{L^1(\mathbb{R}^d)}^{\frac{1}{2}}\,\mathrm{d}t\\ &\le C(\|k\|_{L^\infty(\mathbb{R}^d)},m,T)\limsup_{N\to\infty}\left(\int_0^T \|\rho_t^\varepsilon-\rho_t\|_{L^1(\mathbb{R}^d)}\,\mathrm{d}t\right)^{\frac{1}{2}}\\ &=0.\end{aligned}$$

where the last equality follows by (3.39). Consequently, it remains to show that the third term vanishes, i.e.

$$\limsup_{N\to\infty}\left\|\rho_t^{\otimes r,\varepsilon}-\rho_t^{\otimes r}\right\|_{L^1([0,T];L^1(\mathbb{R}^{dr}))}=0. \tag{3.40}$$

Again this follows by (3.39) and an induction argument. Indeed let us assume $r=2$, then by mass conservation we have

$$\begin{aligned}&\left\|\rho_t^{\otimes 2,\varepsilon}-\rho_t^{\otimes 2}\right\|_{L^1([0,T];L^1(\mathbb{R}^{2d}))}\\ &=\int_0^T\int_{\mathbb{R}^2}\left|(\rho_t^\varepsilon(x_1)-\rho_t(x_1))\rho_t^\varepsilon(x_2)+\rho_t(x_1)(\rho_t^\varepsilon(x_2)-\rho_t(x_2))\right|\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}t\\ &\le 2\,\|\rho_t^\varepsilon-\rho_t\|_{L^1([0,T];L^1(\mathbb{R}))}\xrightarrow{N\to\infty}\quad 0,\end{aligned}$$

which proves the initial case for the induction. Now, by the same argument one can prove the induction step and therefore equation (3.40). □

3.6. Application

We provide some examples for which Theorem 3.8 can be shown with the same techniques developed in Section 3.4. In particular we demonstrate the convergence in relative entropy in the attractive Coulomb case on the whole space. Note that the rate of converges may vary across these examples. As stated in Remark 3.15 we only need the existence of approximated PDE (3.9), the particle system (3.5), the convergence in probability of the particle system $\mathbf{X}^N$ to the mean-field limit $\mathbf{Y}^N$(Assumption (3.4)) and the law of large numbers (Assumption 3.5). Since we are working on the regularized level, we can often assume the existence of the above results.

Although the result in Theorem 3.8 also works with rotational field, it worth to study directly a convolution type of potential field to achieve better cut-off rate, in other words, to allow bigger β. For a given potential field, the challenging part is to find a convolution structure for the potential described in Section 3.4. The first idea to obtain interesting kernels, beside the Delta-Distribution, which was given in [Oel87], is to look at infinite divisible distributions. Assume that k^ε is infinitely divisible. Then, there exists a V^ε such that $k^\varepsilon=V^\varepsilon*V^\varepsilon$ or $k^\varepsilon=\nabla(V^\varepsilon*V^\varepsilon)$. Hence, if we can approximate the antiderivative of our kernel by a infinitely divisible distribution (multiplied by a constant if necessary) we are able to find candidates for interesting kernels.

Another powerful tool is the Fourier analysis. On the Fourier side the equation $k^\varepsilon = V^\varepsilon * W^\varepsilon$ becomes

$$\mathcal{F}(W^\varepsilon) = \mathcal{F}(V^\varepsilon)\mathcal{F}(W^\varepsilon),$$

which can be explored. In particular for singular kernels we have representations of the Fourier transforms, see for instance [Ste70]. Consequently, we can use this approach to obtain a wide range of interesting examples used in biology or physics.

In the rest of the section we provide some fascinating examples for which the case of convolution structure in Theorem 3.14 can be obtained.

3.6.1. Uniform bounded confidence model. Let $V(x) = i\mathbb{1}_{\left[-\frac{R}{2},\frac{R}{2}\right]}(x)$ be a complex-valued function. Then

$$\begin{cases} V * V\colon & \mathbb{R} \to \mathbb{R}, \\ & x \to \begin{cases} 0 & \text{if } x > |R|, \\ -x - R & \text{if } -R \le x \le 0, \\ x - R & \text{if } 0 < x \le R. \end{cases} \end{cases}$$

is a Lipschitz-continuous function with bounded support. Furthermore, we have

$$\nabla(V * V) = -\mathbb{1}_{[-R,0]} + \mathbb{1}_{[0,R]} =: k_U \quad \text{a.e.}$$

Consequently, the uniform bounded confidence model, satisfies the assumption of Section 3.4 with the usual mollification approximation. Also, it is well known that the indicator function $\mathbb{1}_{\left[-\frac{R}{2},\frac{R}{2}\right]} \in H^s(\mathbb{R})$ for all $s < 1/2$. We also have the convergence in probability by Theorem 2.23. Hence, we obtain the following proposition

PROPOSITION 3.27. *Let k_U be given above, then the first marginal $(\rho_t^{N,1}, t \ge 0)$ of the law of the system $\mathbf{X}^N$ converges to the law $(\rho_t, t \ge 0)$ of $\mathbf{Y}^N$ in the $L^1([0,T];L^1(\mathbb{R}))$-norm.*

3.6.2. Parabolic-elliptic Keller–Segel system. In this subsection we provide an approximation for the elliptic-parabolic Keller–Segel model [KS70] in $\mathbb{R}^d$. The underlying PDE is given by

$$\begin{cases} \partial_t \rho_t &= \frac{\sigma^2}{2}\Delta\rho_t - \nabla\cdot(\chi\rho_t\nabla c_t), \\ -\Delta c_t &= \rho_t, \end{cases}$$

for $\chi, \sigma > 0$. Decoupling the above system by setting $c_t = \Phi * \rho_t$ with Φ being the fundamental solution of the Laplace equation we can formally derive the above equation from the particle system (3.4) with the interaction force kernel $k = -\nabla\Phi$. In particular, if $d \ge 2$ we have

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log(|x|), & x \neq 0, \quad \text{if } d \ge 2, \\ \frac{1}{d(d-2)\lambda(B_1(0))}\frac{1}{|x|^{d-2}}, & x \neq 0, \quad \text{if } d \ge 3, \end{cases}$$

is the fundamental solution of the Laplace equation.

In the following we present two approaches to mollify our kernel. For the first approach, let us define a mollification kernel J_{KS}, which satisfies $J_{KS} \ge 0$, $\|J_{KS}\|_{L^1(\mathbb{R}^d)} = 1$ and $\operatorname{supp}(J_{KS}) \subset B(0,1/2)$ and is infinitely differentiable. As always we set $J_{KS}^\varepsilon(x) = \frac{1}{\varepsilon^d}J_{KS}\left(\frac{x}{\varepsilon}\right)$. Then $k^\varepsilon = -\nabla(J_{KS}^\varepsilon * \Phi * J_{KS}^\varepsilon)$ satisfies all properties of [HLL19, Theorem 2.1]. Hence, the convergence in probability Assumption 3.4, the law of large numbers Assumption 3.5 is satisfied.

Hence, under consideration of Remark 3.9 we can obtain a relative entropy convergence results on the approximated d-dimensional attractive Keller–Segel system on the whole space $\mathbb{R}^d$. We formulate the following proposition as combination of Lemma 3.12 and Theorem 3.14.

PROPOSITION 3.28. *Let $k^\varepsilon = -\nabla(W^\varepsilon * V^\varepsilon)$ with $W^\varepsilon = J^\varepsilon_{KS} * \Phi$, $V^\varepsilon = J^\varepsilon_{KS}$. Let $\rho^{N,\varepsilon}$ be the solution of the Liouville equation* (3.8) *and ρ^ε be the solution to regularized Keller–Segel equation, i.e. to the PDE* (3.9)*. Then, there exists a $\beta > 0$ depending on the dimension d such that for $\varepsilon = \varepsilon(N) = N^{-\beta}$ there exists a $\lambda > 0$ such that*

$$\sup_{t\in[0,T]} \mathcal{H}_N\big(\rho_t^{N,\varepsilon(N)} \mid \rho_t^{\otimes N,\varepsilon(N)}\big) \le CN^{-\lambda}.$$

REMARK 3.29. *By going through the proof of Theorem 3.14 one can obtain a convergence rate and precise condition for β. Furthermore, by inequality* (1.7) *we have proven convergence of the $L^1(\mathbb{R}^d)$-norm of the marginals*

$$\lim_{N\to\infty} \left\| \rho_t^{N,2,\varepsilon(N)} - \rho_t^{\varepsilon(N)} \otimes \rho_t^{\varepsilon(N)} \right\|_{L^1(\mathbb{R}^d)} = 0.$$

It is also well-known that under additional assumptions on the initial condition ρ_0 and in the sub-critical regime $\chi < 8\pi$ in the case $d = 2$ the density $\rho_t^{\varepsilon(N)}$ converges in $L^1(\mathbb{R}^d)$. Hence, we have shown that in the sub-critical case the density of the two marginal $\rho_t^{N,2,\varepsilon(N)}$ converges in $L^1(\mathbb{R}^d)$ to the solution of the Keller–Segel equation.

In the case $d \ge 3$ we can obtain an even better approximation, which has a symmetric convolution structure given by $k^\varepsilon = -\nabla(V^\varepsilon * V^\varepsilon)$. Indeed, define the approximation of Φ as $\Phi^\varepsilon := J^\varepsilon_{KS} * \Phi * J^\varepsilon_{KS}$. Then, for $c_\alpha = \pi^{-\alpha/2}\Gamma(\alpha/2)$ and

$$V^\varepsilon = \sqrt{\frac{c_2}{c_{d-2}d(d-2)\lambda(B_1(0))}}\mathcal{F}^{-1}(|\xi|^{-1}\mathcal{F}(J^\varepsilon_{KS})(\xi)) \tag{3.41}$$

we have

$$\Phi^\varepsilon = V^\varepsilon * V^\varepsilon. \tag{3.42}$$

More precisely, for fix $\varepsilon > 0$ we have $J^\varepsilon_{KS} \in L^p(\mathbb{R}^d)$ for all $p \ge 1$. Hence the Fourier transform $\mathcal{F}(J^\varepsilon_{KS})$ is well-defined and by the Hardy–Littlewood–Sobolev inequality [Ste70, Chapter 5, Theorem 1], [LL01, Corollary 5.10] $\Phi^\varepsilon \in L^2(\mathbb{R}^d)$ and the Fourier transform exists. Similar $|\cdot|^{-1}\mathcal{F}(J^\varepsilon_{KS})(\cdot) \in L^2(\mathbb{R}^d)$. A simple calculation shows

$$\begin{aligned}\left\| |\cdot|^{-1}\mathcal{F}^{\frac{1}{2}}(J^\varepsilon_{KS})(\cdot)\right\|^2_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} |\xi|^{-2}|\mathcal{F}(J^\varepsilon_{KS})(\xi)|\,\mathrm{d}\xi \\ &\le \|\mathcal{F}(J^\varepsilon_{KS})\|_{L^\infty(\mathbb{R}^d)} \int_{B_1(0)} |\xi|^{-2}\,\mathrm{d}\xi + \int_{B_1(0)^c} |\mathcal{F}(J^\varepsilon_{KS})(\xi)|\,\mathrm{d}\xi < \infty\end{aligned}$$

since $d > 2$ and $\mathcal{F}(J^\varepsilon_{KS})$ is a Schwartz function. As a result, to verify (3.42) we need to show

$$\mathcal{F}(\Phi^\varepsilon) = \mathcal{F}(V^\varepsilon)^2,$$

where the right-hand is square integrable. Now, by [LL01, Corollary 5.10] we have

$$\mathcal{F}(V^\varepsilon)^2(\xi) = \frac{c_2}{c_{d-2}d(d-2)\lambda(B_1(0))}|\xi|^{-2}\mathcal{F}(J_{KS}^\varepsilon)(\xi) = \mathcal{F}(\Phi^\varepsilon)(\xi),$$

where the left-hand side is in $L^2(\mathbb{R}^d)$ by similar arguments as before. Therefore, (3.42) is proven and we can find an appropriate approximation for the Keller–Segel interaction kernel. In particular, we can derive similar estimates to (3.38) with the help of Fourier analysis and the Hardy–Littlewood–Sobolev inequality [Ste70, Chapter 5. Theorem 1]. Clearly, this estimates will now depend on the dimension d and therefore the convergence rate parameters also depend on the dimension d.

Proposition 3.30. *Let $d \geq 3$ and $k^\varepsilon = -\nabla(W^\varepsilon * V^\varepsilon)$ with $W^\varepsilon, V^\varepsilon$ defined by the same expression* (3.41)*. Then the conclusion of Proposition 3.28 holds. Additionally we have the modulated energy estimate*

$$\sup_{t\in[0,T]} |\mathcal{K}_N\big(\rho_t^{N,\varepsilon}|\rho_t^{\otimes N,\varepsilon}\big)| \leq CN^{-\lambda}$$

for some $C > 0, \lambda > 0$.

Another approach to approximate the Coulomb kernel k is by utilizing the following approximation in dimension $d \geq 3$,

$$\Phi^\varepsilon(x) = \frac{1}{d(d-2)\lambda(B_1(0))}\int_{\mathbb{R}} \frac{h^\varepsilon(y)}{|x-y|^{d-2}}\,\mathrm{d}y, \tag{3.43}$$

where

$$h^\varepsilon(y) = \frac{1}{(4\pi\varepsilon)^{d/2}}\exp\Big(-\frac{|y|^2}{4\varepsilon}\Big) \tag{3.44}$$

is the Weierstrass kernel. Indeed, we note first that the square root $\mathcal{F}^{\frac{1}{2}}(h^\varepsilon)$ is well-defined since the Fourier transform of a Gaussian is still a Gaussian or in other words the normal distribution is infinitely divisible. More precisely, by [LL01, Theorem 5.2] we have

$$\mathcal{F}(h^\varepsilon)(\xi) = \exp\Big(-4\varepsilon\pi^2|\xi|^2\Big).$$

Hence, similar to the first approximation, we obtain Proposition 3.30. By using the Weierstrass kernel over an abstract mollification kernel we obtain explicit sharp convergence rates. For instance, using Plancherel theorem we obtain

$$
\begin{aligned}
\|\nabla V^\varepsilon\|^2_{L^2(\mathbb{R}^d)} &= \frac{c_2}{c_{d-2}d(d-2)\lambda(B_1(0))}\Big\|\nabla\mathcal{F}^{-1}(|\xi|^{-1}\mathcal{F}^{\frac{1}{2}}(h^\varepsilon)(\xi))(\cdot)\Big\|^2_{L^2(\mathbb{R}^d)}\\
&= \frac{c_2}{c_{d-2}d(d-2)\lambda(B_1(0))}\int_{\mathbb{R}^d}\sum_{i=1}^d|\partial_{x_i}\mathcal{F}^{-1}(|\xi|^{-1}\mathcal{F}^{\frac{1}{2}}(h^\varepsilon)(\xi))(x)|^2\,\mathrm{d}x\\
&= \frac{c_2}{c_{d-2}d(d-2)\lambda(B_1(0))}\int_{\mathbb{R}^d}\sum_{i=1}^d|\mathcal{F}^{-1}(2\pi i\xi_i|\xi|^{-1}\mathcal{F}^{\frac{1}{2}}(h^\varepsilon)(\xi))(x)|^2\,\mathrm{d}x\\
&= \frac{2\pi c_2}{c_{d-2}d(d-2)\lambda(B_1(0))}\sum_{i=1}^d\int_{\mathbb{R}^d}|\xi_i\xi^{-1}\mathcal{F}^{\frac{1}{2}}(h^\varepsilon)(\xi)|^2\,\mathrm{d}\xi\\
&= \frac{2\pi c_2}{c_{d-2}d(d-2)\lambda(B_1(0))}\sum_{i=1}^d\int_{\mathbb{R}^d}|\xi_i\xi|^{-1}\exp\Big(-2\varepsilon\pi^2|\xi|^2\Big)|^2\,\mathrm{d}\xi\\
&= \frac{2\pi c_2}{c_{d-2}d(d-2)\lambda(B_1(0))\varepsilon^{d/2}}\sum_{i=1}^d\int_{\mathbb{R}^d}\Big|\xi_i\xi^{-1}\exp\Big(-2\pi^2|\xi|^2\Big)\Big|^2\,\mathrm{d}\xi\\
&\le \frac{2\pi c_2}{c_{d-2}(d-2)\lambda(B_1(0))\varepsilon^{d/2}}\int_{\mathbb{R}^d}\exp\Big(-4\pi^2|\xi|^2\Big)|\,\mathrm{d}\xi\\
&= \frac{c_2}{2^{d-1}\pi^{d/2-1}c_{d-2}(d-2)\lambda(B_1(0))}\varepsilon^{-d/2}.
\end{aligned}
$$

Remark 3.31. *The above potential is attractive and therefore as far as we know regularization/approximation is necessary to obtain a solution of the underlying Liouville equation on* $\mathbb{R}^d$*. Nevertheless, one can obtain tightness of the empirical measure in the super sub-critical regime* [FJ17]*. Our approach provides propagation of chaos of the intermediate system on the level of the relative entropy. Hence, it can be used as a tool to develop further results on propagation of chaos for the Keller–Segel model without regularization.*

3.6.3. Parabolic-elliptic Keller–Segel system with Bessel potential. Let us recall the parabolic-elliptic Keller–Segel model [KS70] in $\mathbb{R}^d$ given by

$$
\begin{cases}
\partial_t\rho_t &= \frac{\sigma^2}{2}\Delta\rho_t - \nabla\cdot(\rho_t\nabla c_t),\\
c_t &= \Delta c_t + \rho_t.
\end{cases}
$$

Again solving the second equation by setting

$$
c_t = (I-\Delta)^{-1}\rho_t = G*\rho_t
$$

with the L^1 function G defined by

$$
G(x) := \mathcal{F}^{-1}[(1+4\pi^2|\xi|^2)^{-1}](x) = \frac{1}{(4\pi)^{d/2}}\int_0^\infty \exp\Big(-t-\frac{|x|^2}{4t}\Big)t^{-\frac{n}{2}}\,\mathrm{d}t, \tag{3.45}
$$

we can decouple the system and obtain an analogous result by using the following approximations of G,

$$G^\varepsilon(x) = G * h^\varepsilon(x), \tag{3.46}$$

or

$$G^\varepsilon(x) = G * J^\varepsilon(x), \tag{3.47}$$

where h^ε is the Weierstrass kernel given by (3.44). Setting

$$V^\varepsilon(x) = \mathcal{F}^{-1}[(1+4\pi^2|\xi|^2)^{-1/2}\mathcal{F}^{\frac{1}{2}}[h^\varepsilon](\xi)](x),$$

it can be shown similar to the elliptic-parabolic Keller–Segel model that

$$G^\varepsilon = V^\varepsilon * V^\varepsilon.$$

Consequently, we obtain the analogous result.

PROPOSITION 3.32. *Let $k^\varepsilon = -\nabla G^\varepsilon$ with G^ε defined by (3.45) (3.46) or (3.47) and suppose the Assumptions 3.4 and 3.5 hold. Moreover, for this k^ε let $\rho^{N,\varepsilon}$ be the solution of the Liouville equation (3.8) and ρ^ε be the solution to regularized Keller–Segel equation, i.e. to the PDE (3.9). Then, there exists a $\beta > 0$ depending on the dimension d such that for $\varepsilon = \varepsilon(N) = N^{-\beta}$ there exists a $\lambda > 0$ such that*

$$\sup_{t\in[0,T]} \mathcal{H}_N\big(\rho_t^{N,\varepsilon(N)} \mid \rho_t^{\otimes N,\varepsilon(N)}\big) + \sup_{t\in[0,T]} |\mathcal{K}_N\big(\rho_t^{N,\varepsilon}|\rho_t^{\otimes N,\varepsilon}\big)| \le CN^{-\lambda}.$$

REMARK 3.33. *By going through the proof of Theorem 3.14 one can obtain a convergence rate and precise condition for β. We also assumed the convergence probability, since we can not reference a concrete result. Nevertheless, we think that this assumption should be true for good enough initial condition.*

3.7. Comments

The results obtained in this chapter are powerful, as they demonstrate the convergence of the marginals in the L^1-norm. In particular, we can utilize PDE methods to eliminate the approximation on the mean-field limit side, as discussed in Remark 3.31. Demonstrating propagation of chaos on the side of interacting particle systems remains an open problem. For instance, for the Keller–Segel model [FJ17], it is possible to show, using a compactness method, that some subsequence converges in the very sub-critical regime, which solves the non-regularized interacting particle system. However, we cannot show uniqueness of the limit, leaving the problem unsolved.

This challenge seems insurmountable with classical PDE arguments since the corresponding Liouville equation (3.8) is linear and lacks the regularization properties of the convolution operator. Known techniques, such as modulated free energy, appear to reach their limits regarding the singularity of the interaction kernel. Therefore, a new method or quantity must be developed to demonstrate not only the mean-field limit at the intermediate level (Figure 1) but the full mean-field limit for the whole space.

Improved regularity of the Fokker–Planck solution could be beneficial [FW23], but this is far from trivial and does not address the solvability of the Liouville equation (3.8) in cases of singular interaction kernels.

Finally, there are still open problems where the techniques developed in this setting could be applied. For example, in fluctuation results, the quantitative nature of these techniques could be very useful. Additionally, in large deviation theory [HHMT24], the use of relative entropy is common, and these techniques could help lower the regularity assumptions on the interaction kernel k. Furthermore, exploring the introduction of control parameters into the equations presents an intriguing and promising direction for future research.

Chapter 4

Well-posedness for conditional McKean–Vlasov equations

In this chapter our aim is to establish the well-posedness of the conditional McKean–Vlasov system with non-Lipschitz kernel k, which where introduced in Section 1.3.5. Our motivation to study such systems is derived from the Hegselmann–Krause model [HK02] (HK model), which belongs to the class of bounded confidence opinion dynamics. More precisely, we are interested in a version of the HK model where the opinions $\mathbf{X}^{N,HK} := (X^{1,HK}, \dots, X^{N,HK})$ of N agents are subject to idiosyncratic noises as well as common noise, i.e., we consider for $i = 1, \dots, N$ the particle system

$$\text{(4.1)} \quad \mathrm{d}X_t^{i,HK} = -\frac{1}{N}\sum_{j=1}^{N} k_{HK}(X_t^{i,HK} - X_t^{j,HK})\,\mathrm{d}t + \sigma(t, X_t^{i,HK})\,\mathrm{d}B_t^i + \nu\,\mathrm{d}W_t, \quad X_0^{i,HK} \sim \rho_0,$$

for $t \geq 0$, where $X_t^{i,HK}$ is the i-th agent's opinion at time t, $k_{HK}(x) := \mathbb{1}_{[0,R]}(|x|)x$ is the one-dimensional (non-Lipschitz) interaction force between the agents, $\sigma\colon [0,T] \times \mathbb{R} \mapsto \mathbb{R}$ is some smooth diffusion coefficients, $\nu > 0$ is a positive constant, and $(X_0^{i,HK}, i \in \mathbb{N})$ is the i.i.d. sequence of initial values independent of all Brownian motions with distribution ρ_0. The local interaction kernel k_{HK} represents the concept of bounded confidence in opinion dynamics, indicating that opinions are influenced only within a certain range. In the above HK model (4.1), the idiosyncratic noises $B = ((B_t^i, t \geq 0), i \in \mathbb{N})$ describe the individual random effects on each agent's opinion and the common noise $(W_t, t \geq 0)$ captures external effects on the agents' opinions. In the mean-field limit we expect as in Section 1.3.5, that the opinion dynamic will follow the conditional McKean–Vlasov equation

$$\text{(4.2)} \qquad \begin{cases} \mathrm{d}Y_t^{i,HK} = -(k_{HK} * \rho_t)(Y_t^{i,HK})\,\mathrm{d}t + \sigma(t, Y_t^{i,HK})\,\mathrm{d}B_t^i + \nu\,\mathrm{d}W_t, \quad Y_0^i = X_0^i, \\ \rho_t^{HK} \text{ is the conditional density of } Y_t^{i,HK} \text{ given } \mathcal{F}_t^W, \end{cases}$$

with the associated stochastic non-linear, non-local Fokker–Planck equation

$$\text{(4.3)} \quad \mathrm{d}\rho_t^{HK} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\Big(\frac{\sigma_t^2 + \nu^2}{2}\rho_t^{HK}\Big)\,\mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}x}((k_{HK} * \rho_t^{HK})\rho_t^{HK})\,\mathrm{d}t - \nu\frac{\mathrm{d}}{\mathrm{d}x}\rho_t^{HK}\,\mathrm{d}W_t, \quad t \geq 0.$$

Let us remark that equation (4.3) is a non-local, non-linear stochastic partial differential equation (SPDE), where the stochastic term is a consequence of the common noise $W = (W_t, t \geq 0)$. Indeed, as we have seen in Section 1.3.5, if the number of agents tends to infinity the effect of the idiosyncratic noises averages out, but the common noise does not.

Thus, our goal is to establish conditional propagation of chaos of the Hegselmann–Krause model with common noise. More precisely, we show that regularized versions of the particle systems (4.1)

$$\mathrm{d}X_t^{i,\varepsilon,HK} = -\frac{1}{N}\sum_{j=1}^{N} k_{HK}(X_t^{i,\varepsilon,HK} - X_t^{j,\varepsilon,HK})\,\mathrm{d}t + \sigma(t, X_t^{i,\varepsilon,HK})\,\mathrm{d}B_t^i + \nu\,\mathrm{d}W_t \tag{4.4}$$

with same initial condition $X_0^{i,\varepsilon HK} = X_0^{i,HK}$ converge in L^2-norm to the conditional McKean–Vlasov equation (4.2) with some cut-off parameter ε depending on the number of particles N. In the proceeding, we require the intermediate problem. Thus, the regularized McKean–Vlasov stochastic differential equation (4.3)

$$\begin{cases} \mathrm{d}Y_t^{i,\varepsilon,HK} = -(k_{HK} * \rho_t)(Y_t^{i,\varepsilon,HK})\,\mathrm{d}t + \sigma(t, Y_t^{i,\varepsilon,HK})\,\mathrm{d}B_t^i + \nu\,\mathrm{d}W_t, \\ \rho_t^{HK} \text{ is the conditional density of } Y_t^{i,\varepsilon,HK} \text{ given } \mathcal{F}_t^W. \end{cases} \tag{4.5}$$

with initial condition $Y_0^{i,HK} = X_0^{i,HK}$. Obviously, we also have the associated regularized conditional McKean–Vlasov equation

$$\mathrm{d}\rho_t^{\varepsilon,HK} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\Big(\frac{\sigma_t^2+\nu^2}{2}\rho_t^{\varepsilon,HK}\Big)\,\mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}x}((k_{HK} * \rho_t^{\varepsilon,HK})\rho_t^{\varepsilon,HK})\,\mathrm{d}t - \nu\frac{\mathrm{d}}{\mathrm{d}x}\rho_t^{\varepsilon,HK}\,\mathrm{d}W_t \tag{4.6}$$

for $t \geq 0$. The regularized systems allows us to estimate, for all i, terms of the form $\mathbb{E}\big(|X_t^{i,\varepsilon,HK} - Y_t^{i,\varepsilon,HK}|\big)$ and $\mathbb{E}\big(|Y_t^{i,\varepsilon,HK} - Y_t^{i,HK}|\big)$ separately. The second term we can treat via SPDE methods by studying the associated non-linear stochastic Fokker–Planck equations. As a result, one can obtain conditional propagation of chaos for the system (4.4). Following the method described above, various versions of propagation of chaos at an intermediate level with a logarithmic cut-off have been demonstrated for a variety of models with general kernels k. This includes the work of [LP17, CG17] on particle systems without common noise, featuring non-Lipschitz, unbounded, and even singular interaction kernels.

Notice that even though the kernel is not Lipschitz continuous, it is bounded and integrable, and therefore $k_{HK} \in L^p(\mathbb{R})$ for all $p \geq 1$. Thus, instead of analysing k_{HK}, we assume that the interaction kernel k satisfies some integrability condition and we focus on the interacting system from Section 1.3.5 with additive common noise. As a corollary, we will then obtain the results for the HK model.

4.1. Problem setting

In this subsection we introduce the interacting particle system with additive common noise, its corresponding mean-field stochastic differential equation with its associated stochastic Fokker–Planck equation. All of the systems are special one-dimensional cases of the general SDE's and SPDE's from Section 1.3.5.

4.1.1. Interacting particle system with common noise. Let us recall the interacting particle system $\mathbf{X}_t^N = (X_t^1, \ldots, X_t^N)$ from Section 1.3.5 with common noise in one dimension with additive common noise, which has the following dynamics

$$\mathrm{d}X_t^i = -\frac{1}{N}\sum_{j=1}^{N} k(X_t^i - X_t^j)\,\mathrm{d}t + \sigma(t, X_t^i)\,\mathrm{d}B_t^i + \nu\,\mathrm{d}W_t, \quad X_0^i \sim \rho_0, , \quad i = 1, \ldots, N \tag{4.7}$$

for $t \in [0,T]$, where $\sigma\colon [0,T] \times \mathbb{R} \mapsto \mathbb{R}$ is the diffusion coefficient, $\nu > 0$ a constant and interaction force $k\colon \mathbb{R} \mapsto \mathbb{R}$. The initial condition $(X_0^i, i \in \mathbb{N})$ is i.i.d. with distribution ρ_0

and independent to all Brownian motions. We point out, that the kernel k denotes a general kernel. However, the kernel k_{HK} always stands for the kernel in the Hegselmann–Krause model.

For establishing conditional propagation of chaos for $k \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ we need to introduce an approximation sequence. Let $(k^\varepsilon, \varepsilon > 0) \subset C_c^\infty(\mathbb{R})$ satisfy

- $\|k^\varepsilon - k\|_{L^2(\mathbb{R})} \to 0$ as $\varepsilon \to \infty$,
- $\operatorname{supp}(k^\varepsilon) \subset F$ for $\varepsilon > 0$ and $\operatorname{supp}\big(\frac{\mathrm{d}}{\mathrm{d}x} k^\varepsilon\big) \subset F$ for some compact set $F \subset \mathbb{R}$,
- $0 \le k^\varepsilon \le C$, $|\frac{\mathrm{d}}{\mathrm{d}x} k^\varepsilon| \le \frac{C}{\varepsilon}$ for some constant $C > 0$.

Notice that we can approximate the indicator function by smooth functions $(\psi^\varepsilon, \varepsilon > 0)$ satisfying

- $\lim_{\varepsilon \to 0} \psi^\varepsilon = \mathbb{1}_{[-R,R]}$ almost everywhere,
- $\operatorname{supp}(\psi^\varepsilon) \subset [-R - 2\varepsilon, R + 2\varepsilon]$, $\operatorname{supp}(\frac{\mathrm{d}}{\mathrm{d}x}\psi^\varepsilon) \subset [-R-2\varepsilon, -R+2\varepsilon] \cup [R-2\varepsilon, R+2\varepsilon]$,
- $0 \le \psi^\varepsilon \le 1$, $|\frac{\mathrm{d}}{\mathrm{d}x}\psi^\varepsilon| \le \frac{C}{\varepsilon}$ for some constant $C > 0$.

Consequently, we can define

$$k^\varepsilon_{HK}(x) = x\psi^\varepsilon(x).$$

Then, $(k^\varepsilon_{HK}, \varepsilon > 0)$ satisfies the above approximation properties. We denote the regularized interacting particle system for the kernel k^ε by $\mathbf{X}_t^{N,\varepsilon} = (X_t^{1,\varepsilon}, \dots, X_t^{N,\varepsilon})$ and its dynamic is given by

$$\mathrm{d}X_t^{i,\varepsilon} = -\frac{1}{N}\sum_{j=1}^N k^\varepsilon(X_t^{i,\varepsilon} - X_t^{j,\varepsilon})\,\mathrm{d}t + \sigma(t, X_t^{i,\varepsilon})\,\mathrm{d}B_t^i + \nu\,\mathrm{d}W_t, \quad X_0^{i,\varepsilon} = X_0^i, \tag{4.8}$$

for $t \in [0,T]$ and $i = 1, \dots, N$. Although the interaction force kernel k is non-Lipschitz continuous, the N-particle systems (4.7) and (4.8) possess unique strong solutions [HRZ24, Theorem 3.7], as k^ε and k are both in $L^2(\mathbb{R})$.

Corresponding to the particle systems (4.7) and (4.8), for $i \in \mathbb{N}$, we obtain the system of mean-field SDEs given by

$$\begin{cases} \mathrm{d}Y_t^i = -(k * \rho_t)(Y_t^i)\,\mathrm{d}t + \sigma(t, Y_t^i)\,\mathrm{d}B_t^i + \nu\,\mathrm{d}W_t, \quad Y_0^i = X_0^i, \\ \rho_t \text{ is the conditional density of } Y_t^i \text{ given } \mathcal{F}_t^W, \end{cases} \tag{4.9}$$

for $t \in [0,T]$, and the system of regularized mean-field SDEs is defined by

$$\begin{cases} \mathrm{d}Y_t^{i,\varepsilon} = -(k^\varepsilon * \rho_t^\varepsilon)(Y_t^{i,\varepsilon})\,\mathrm{d}t + \sigma(t, Y_t^{i,\varepsilon})\,\mathrm{d}B_t^i + \nu\,\mathrm{d}W_t, \quad Y_0^{i,\varepsilon} = X_0^{i,\varepsilon}, \\ \rho_t^\varepsilon \text{ is the conditional density of } Y_t^{i,\varepsilon} \text{ given } \mathcal{F}_t^W, \end{cases} \tag{4.10}$$

for $t \in [0,T]$, where ρ_t denotes the conditional density of Y_t^i given $\mathcal{F}_t^W$, that is, for every bounded continuous function φ, ρ_t satisfies

$$\mathbb{E}(\varphi(Y_t^i) \,|\, \mathcal{F}_t^W) = \int_{\mathbb{R}} \varphi(x)\rho_t(x)\,\mathrm{d}x, \quad \mathbb{P}\text{-a.e.}$$

The same holds for the regularized conditional density ρ_t^ε of $Y_t^{i,\varepsilon}$ given $\mathcal{F}_t^W$. Let us remark that $\rho_t^\varepsilon, \rho_t$ have no superscript i since they are independent of $i \in \mathbb{N}$. Indeed, in our case the (regularized) mean-field particles are conditionally independent given $\mathcal{F}^W$ and identically distributed, thus, the conditional density is the same for each $i \in \mathbb{N}$.

4.1.2. Stochastic Fokker–Planck equations. Associated to the mean-field SDEs (4.9) and (4.10), we have the stochastic Fokker–Planck equation, which reads as

$$\mathrm{d}\rho_t = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\Big(\frac{\sigma_t^2+\nu^2}{2}\rho_t\Big)\,\mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}x}((k*\rho_t)\rho_t)\,\mathrm{d}t - \nu\frac{\mathrm{d}}{\mathrm{d}x}\rho_t\,\mathrm{d}W_t \tag{4.11}$$

and the regularized stochastic Fokker–Planck equation reads as

$$\mathrm{d}\rho_t^\varepsilon = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\Big(\frac{\sigma_t^2+\nu^2}{2}\rho_t^\varepsilon\Big)\,\mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}x}((k^\varepsilon*\rho_t^\varepsilon)\rho_t^\varepsilon)\,\mathrm{d}t - \nu\frac{\mathrm{d}}{\mathrm{d}x}\rho_t^\varepsilon\,\mathrm{d}W_t, \quad t\in[0,T]. \tag{4.12}$$

Let us note that we consistently use the same notations ρ,ρ^ε for the solutions of the stochastic Fokker–Planck equations (4.11) and (4.12), as well as for the conditional densities of the mean-field SDEs (4.9) and (4.10). As discussed in Section 1.3.5, and as we will demonstrate in Theorem 4.15, these notations are justified since they coincide under sufficient regularity assumptions on the initial condition ρ_0. Nevertheless, the meaning of ρ,ρ^ε will always be clear from context. To establish the well-posedness of the stochastic Fokker–Planck equations (4.11) and (4.12), we will employ the function spaces introduced in Section 1.4.2.

4.1.3. Assumptions on initial condition and diffusion coefficients. We make the following assumptions on the diffusion coefficient σ.

ASSUMPTION 4.1. *Let $T>0$ and $\sigma\colon[0,T]\times\mathbb{R}\to\mathbb{R}$ the diffusion coefficient, which satisfies:*

(i) Let $0\le\rho_0\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$ with $\|\rho_0\|_{L^1(\mathbb{R})}=1$

(ii) There exists a constant $\lambda>0$ such that

$$\sigma^2(t,x)\ge\lambda$$

for all $x\in\mathbb{R}$ and $t\in[0,T]$.

(iii) There exists a constant $\Lambda>0$ such that for all $t\in[0,T]$ we have

$$\sigma(t,\cdot)\in C^3(\mathbb{R}) \quad \text{and} \quad \sup_{t\in[0,T]}\sum_{i=1}^{3}\Big\|\frac{\mathrm{d}^i}{\mathrm{d}x^i}\sigma(t,\cdot)\Big\|_{L^\infty(\mathbb{R})}\le\Lambda.$$

Contribution: Our first contribution is to prove the well-posedness of the non-local, non-linear stochastic Fokker–Planck equation (4.11). The main challenge in proving existence and uniqueness results for (4.11) is the non-linear term $(k*\rho_t)\rho_t$ since this prevents us from applying known results in the existing literature on SPDEs, such as those found in textbooks [Kry99, LR15], which consider the well-studied case of linear SPDEs. In the case of non-linear SPDEs, one needs to take advantage of the specific structure of the considered SPDE to employ a fixed point argument, cf. e.g. the recent work [HQ21]. In this line of research, we establish local existence and uniqueness of a weak solution to the non-local, non-linear Fokker–Planck equation (4.11). Additionally, we show global well-posedness of the Fokker–Planck equation (4.11) assuming a sufficiently large diffusion coefficient or a sufficiently small L^2-norm of the initial value. Moreover, in contrast to many recent works like [CF16, CG19, HvS21, CDFM20, BCD21] on interacting particle systems with common noise, we avoid the measure-valued setting and deal with the conditional McKean–Vlasov SDE and the associated Fokker–Planck equations by directly analysing the densities.

Our second contribution is to prove the existence of a unique strong solution to the McKean–Vlasov stochastic differential equation (4.9), which is essential for showing conditional propagation of chaos towards to limiting Fokker–Planck equation (4.11). To obtain the well-posedness of the McKean–Vlasov SDE (4.9), a central insight is to introduce suitable stopping times to ensure sufficient temporary regularity such that a backward stochastic partial differential equation (BSPDE) associated to (4.9) possesses a classical solution, cf. e.g. [DTZ13]. As a result, a duality argument in combination with the Itô–Wentzell formula allows us to deduce the existence of a unique strong solution to the McKean–Vlasov equation (4.9).

Our third contribution is to establish conditional propagation of chaos for regularized particle systems (4.8) with common noise, which then implies the conditional propagation of chaos for the HK model (4.1). Compared to Chapter 2, the cut-off will be logarithmic and not algebraic.

Initially, all results are formulated in dimension one since this is essential to establish the well-posedness of the McKean–Vlasov equation (4.9), see the BSPDE argument in Section 4.4 and in particular [DTZ13, Corollary 2.2]. For all other results, we provide corollaries providing the multi-dimensional versions.

Related literature: For Lipschitz continuous interaction forces, conditional propagation of chaos with transport type common noise has been showed by Coghi and Flandoli [CF16] by utilizing sharp estimates in Kolmogorov's continuity theorem and properties of measure-valued solutions of the associated stochastic Fokker–Planck equation. In particular the approach is from an SDE level to an SPDE level [CF16, CG19]. In contrast we first solve the SPDE and then prove that the associated SDE's are well-posed. For Lipschitz continuous interaction force with multiplicative noise Carmona and Delarue [CD18, Theorem 2.12] demonstrated conditional propagation of chaos by utilizing a coupling argument. Dawson and Vaillancourt [DV95] also formulated a martingale problem and demonstrated tightness of the empirical measure obtaining a qualitative result with no convergence rates.

For particle systems with common noise and non-Lipschitz interaction force k (as in our case), to the best of our knowledge there exists no general theory on stochastic McKean–Vlasov equations or on conditional propagation of chaos. In order to derive conditional propagation of chaos for the interacting particle system (4.7), we essentially rely on the well-posedness of the McKean–Vlasov SDE (4.9) and follow [Szn91] as well as [LP17] to prove that $(\rho_t, t \geq 0)$ characterizes the measure of the mean-field limit.

We should also mention the work by Lacker, Shkolnikov, and Zhang [LSZ23], which establishes a superposition principle between the stochastic Fokker–Planck equation (4.11) and the conditional McKean–Vlasov equation (4.9) by reducing it to cases with smooth coefficients. However, the uniqueness of the solution remains an open problem. As a result, only weak existence in the context of stochastic differential equations has been demonstrated.

Organization of the chapter: In Section 4.2 we provide some background information for the HK model, which was our motivation for studying conditional propagation of chaos for non-smooth interaction kernels. In Section 4.3 the well-posedness of the stochastic Fokker–Planck equation (4.11) is established and in Section 4.4 of the associated McKean–Vlasov equation (4.9). The mean-field limit and conditional propagation of chaos of the HK model with common noise are investigated in Section 4.5.

4.2. Hegselmann–Krause model

In general the theory of opinion dynamics, which has been well-studied since the 1950s, has become a rapidly growing area of research over the last few decades. With the rapid development of the internet and social networks, we have observed significant changes in how opinion dynamics evolve and by what sources they are effected. For example, previous generations were heavily influenced by their geographically nearest social group, but nowadays social networks play a dominant role for expressing and sharing opinions, enabling more people than ever to do so from anywhere in the world. Consequently, the significance of the geographical distance as a factor in shaping public opinion diminishes. Instead, each citizen has a personal filter bubble [Spo17], which affects and in the same time shifts with the opinion. This phenomena is described by so called bounded confidence opinion dynamics. For an overview of opinion dynamics we refer to the surveys [Lor07, Hos20].

The original discrete-time Hegselsmann–Krause model [HK02] is given by

$$x_i(t+1) = \frac{1}{|\mathscr{N}_i(t)|} \sum_{j \in \mathscr{N}_i(t)} x_j(t), \quad t \geq 0, \quad i = 1, \dots, n, \tag{4.13}$$

where $x_i(t)$ is the opinion of agent i at time t, $\mathscr{N}_i(t) := \{1 \leq j \leq n : |x_i(t) - x_j(t)| \leq r_i\}$ denotes the neighbour set of agent i at time t and $|\mathscr{N}_i(t)|$ is the cardinality of the set. The convergence and consensus properties of the discrete-time HK model were extensively studied in the past years, see for instance [HK02, Lor06, BBCN13, KZPS12, NT12]. The main characteristic feature of bounded confidence opinion models, like the HK model (4.13), is that the agents interact only locally, which is modelled by the compactly supported interaction force in the discrete-time HK model (4.13), i.e. $j \in \mathscr{N}_i(t)$, and, thus, opinions outside an agent's moral beliefs get ignored through this local interaction kernel. This phenomena is, e.g., observed in case of liberal and conservative view points and their respective social media bubbles in the USA [ENG$^+$19, GKM17, Spo17]. The discrete-time HK model (4.13) is a fairly simple model to describe opinion dynamics and by now there are numerous generalizations and variants of the original HK model, for instance, the HK model with media literacy [XCW$^+$20] or the HK model with an opinion leader [WCB15]. For further extensions we refer to [DR10, RD09].

An important class of extensions of the original HK model captures external random effect in opinion dynamics, see e.g. [PTHG13, CSDH19], leading naturally to a system of N stochastic processes representing the opinion evolution. In this case, following e.g. [GPY17], the opinion dynamics $\hat{\mathbf{X}}^N := (\hat{X}^i, i = 1, \dots, N)$ of N agents are modelled by a system of stochastic differential equations

$$\mathrm{d}\hat{X}_t^i = -\frac{1}{N} \sum_{j=1}^{N} k_{HK}(\hat{X}_t^i - \hat{X}_t^j)\,\mathrm{d}t + \sigma(t, \hat{X}_t^i)\,\mathrm{d}B_t^i, \quad i = 1, \dots, N, \quad \hat{\mathbf{X}}_0^N \sim \bigotimes_{i=1}^{N} \rho_0, \tag{4.14}$$

for $t \geq 0$, where $\hat{X}_t^i$ is the i-th agent's opinion at time t, k_{HK} is the interaction force between the agents, σ is a smooth diffusion coefficient, $((B_t^i, t \geq 0), i \in \mathbb{N})$ is a sequence of one-dimensional independent Brownian motion and, as previously, ρ_0 is the initial distribution. We note that the interaction force k_{HK} has compact support turning the continuous-time HK model (4.14) into a bounded confidence model. The continuous-time HK model (4.14) has been a topic of active research in the past years, e.g., the convergence to a consensus is studied in [GPY17] and the phase transition was investigated in [WLEC17]. For a more detailed discussion on different types of noises in HK models we refer to [CSDH19] and the references therein.

Our goal is to take it a step further by incorporating common noise into the HK model, which represents the influence of environmental shifts on opinions, resulting in the SDE formulation (4.1). By incorporating common noise, we enhance the HK model to more accurately reflect the stochastic nature of real-world scenarios.

4.3. Well-posedness of the stochastic Fokker–Planck equations

This section is dedicated to establishing the global existence and uniqueness of weak solutions of the stochastic Fokker–Planck equations (4.11) and (4.12) under suitable conditions on the initial condition and coefficients. As a special case we obtain existence and uniqueness of weak solutions for the stochastic Fokker–Planck equations (4.3) and (4.6) for the HK model. Before we start our analysis, we introduce the concept of weak solutions.

DEFINITION 4.2. *For a general interaction force $k \in L^2(\mathbb{R})$, a non-negative stochastic process $(\rho_t, t \geq 0)$ is called a (weak) solution of the SPDE* (4.11) *if*

$$(\rho_t, t \in [0,T]) \in L^2_{\mathcal{F}W}([0,T]; W^{1,2}(\mathbb{R})) \cap S^\infty_{\mathcal{F}W}([0,T]; L^1(\mathbb{R}) \cap L^2(\mathbb{R}))$$

and, for any $\varphi \in C_c^\infty(\mathbb{R})$, ρ satisfies almost surely the equation, for all $t \in [0,T]$,

$$\begin{aligned}\langle \rho_t, \varphi \rangle_{L^2(\mathbb{R})} &= \langle \rho_0, \varphi \rangle_{L^2(\mathbb{R})} + \int_0^t \Big\langle \frac{\sigma_s^2 + \nu^2}{2} \rho_s, \frac{\mathrm{d}^2}{\mathrm{d}x^2} \varphi \Big\rangle_{L^2(\mathbb{R})} \mathrm{d}s \\ &\quad - \int_0^t \Big\langle (k * \rho_s)\rho_s, \frac{\mathrm{d}}{\mathrm{d}x} \varphi \Big\rangle_{L^2(\mathbb{R})} \mathrm{d}s + \int_0^t \nu \Big\langle \rho_s, \frac{\mathrm{d}}{\mathrm{d}x} \varphi \Big\rangle_{L^2(\mathbb{R})} \mathrm{d}W_s \end{aligned} \tag{4.15}$$

REMARK 4.3. *A solution to the stochastic partial differential equation* (4.12) *is defined analogously by replacing k with k^ε.*

REMARK 4.4. *There are multiple solution concepts for SPDEs, see for example* [LR15] *for strong solutions in general separable Hilbert spaces or* [DPZ14] *for mild solutions with respect to a infinitesimal generator. In the present work, we use the concept presented in* [Kry99]. *This has the advantage that we can use Itô's formula for L^p-norms* [Kry10] *as well as the linear SPDE theory in* [Kry99].

Remark 4.5. *Under the assumption that for all $t \in [0,T]$, $\sigma_t \in C^2(\mathbb{R})$ we can rewrite formally equation* (4.15) *such that the leading coefficient is in non-divergence form, i.e.*

$$\begin{aligned}
&\langle \rho_t, \varphi\rangle_{L^2(\mathbb{R})} \\
&\quad = \langle \rho_0, \varphi\rangle_{L^2(\mathbb{R})} + \frac{1}{2}\int_0^t \Big\langle (\sigma_s^2+\nu^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\rho_s, \varphi\Big\rangle_{L^2(\mathbb{R})} + 2\Big\langle \frac{\mathrm{d}}{\mathrm{d}x}(\sigma_s^2+\nu^2)\frac{\mathrm{d}}{\mathrm{d}x}\rho_s, \varphi\Big\rangle_{L^2(\mathbb{R})} \\
&\qquad + \Big\langle \frac{\mathrm{d}^2}{\mathrm{d}x^2}(\sigma_s^2+\nu^2)\rho_s, \varphi\Big\rangle_{L^2(\mathbb{R})} \mathrm{d}s - \int_0^t \Big\langle (k*\rho_s)\rho_s, \frac{\mathrm{d}}{\mathrm{d}x}\varphi\Big\rangle_{L^2(\mathbb{R})} \mathrm{d}s \\
&\qquad - \int_0^t \nu \Big\langle \frac{\mathrm{d}}{\mathrm{d}x}\rho_s, \varphi\Big\rangle_{L^2(\mathbb{R})} \mathrm{d}W_s.
\end{aligned}$$

Hence, $(\rho_t, t\geq 0)$ solves the following SPDE

$$\begin{aligned}
\mathrm{d}\rho_t = {}& \frac{\sigma_t^2+\nu^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\rho_t\,\mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}x}(\sigma_t^2+\nu^2)\frac{\mathrm{d}}{\mathrm{d}x}\rho_t\,\mathrm{d}t \\
&+ \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\sigma_t^2+\nu^2)\rho_t\,\mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}x}((k*\rho_t)\rho_t)\,\mathrm{d}t - \nu\frac{\mathrm{d}}{\mathrm{d}x}\rho_t\,\mathrm{d}W_t, \quad t\in[0,T].
\end{aligned}$$

In the next theorem we establish uniqueness and local existence of weak solutions to the non-local stochastic Fokker–Planck equation (4.11). Furthermore, we are going to see in Corollary 4.9 that the existence will not depend on the L^2-norm of the initial condition ρ_0, allowing us to extend the local solution obtained in Theorem 4.6 to a global solution on an arbitrary interval $[0,T]$.

Theorem 4.6. *Let Assumption 4.1 hold and $k \in L^2(\mathbb{R})$. Then, there exists a $T^* > 0$ and a unique non-negative solution of the SPDE* (4.11) *in the space*

$$\mathbb{B} := L^2_{\mathcal{F}W}([0,T^*]; W^{1,2}(\mathbb{R})) \cap S^\infty_{\mathcal{F}W}([0,T^*]; L^1(\mathbb{R})\cap L^2(\mathbb{R})).$$

Moreover, the solution ρ has the property of mass conservation

$$\|\rho_t\|_{L^1(\mathbb{R})} = 1, \quad \mathbb{P}\text{-a.s.},$$

for all $t\in[0,T]$.

Proof. Let us define the metric space

$$F^{T,M} := \big\{X \in S^\infty_{\mathcal{F}W}([0,T]; L^2(\mathbb{R})) \;:\; \|X\|_{S^\infty_{\mathcal{F}W}([0,T];L^2(\mathbb{R}))} \leq M\big\}$$

for some constant $M > \|\rho_0\|_{L^2(\mathbb{R})}$, for instance $M = 2\|\rho_0\|_{L^2(\mathbb{R})}$. The metric on $F^{T,M}$ is induced by the norm on $S^\infty_{\mathcal{F}W}([0,T]; L^2(\mathbb{R}))$. The solution map $\mathcal{T}\colon F^{T,M}\to F^{T,M}$ is defined as follows. For each $\zeta \in F^{T,M}$ we define $\mathcal{T}(\zeta)$ as the solution of the following linear SPDE

$$\begin{aligned}
\mathrm{d}\rho_t = {}& \frac{\sigma_t^2+\nu^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\rho_t\,\mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}x}(\sigma_t^2+\nu^2)\frac{\mathrm{d}}{\mathrm{d}x}\rho_t\,\mathrm{d}t \\
&+ \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\sigma_t^2+\nu^2)\rho_t\,\mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}x}((k*\zeta_t)\rho_t)\,\mathrm{d}t - \nu\frac{\mathrm{d}}{\mathrm{d}x}\rho_t\,\mathrm{d}W_t, \quad t\in[0,T].
\end{aligned} \tag{4.16}$$

The L^2-bound on k and Hölder's inequality imply

$$|k * \zeta_t(x)| \leq \|k\|_{L^2(\mathbb{R})} \|\zeta_t\|_{L^2(\mathbb{R})} \leq \|k\|_{L^2(\mathbb{R})} M \tag{4.17}$$

for all $x \in \mathbb{R}$, which allows us to check the conditions of the L^p-theory of SPDEs [Kry99, Theorem 5.1 and Theorem 7.1] for the case $n = -1$ therein. For instance, if we define for $q \in W^{1,2}(\mathbb{R})$ the function

$$f(q, t, x) = \frac{\mathrm{d}}{\mathrm{d}x}((k * \zeta_t) q_t),$$

then obviously $f(0, \cdot, \cdot) \in L^2_{\mathcal{F}^W}([0,T]; H^{-1,2}(\mathbb{R}))$ and, since $x/(1+|x|^2)^{1/2}$ is bounded, we can apply the lifting property [Tri78, Theorem 2.3.8] to obtain

$$\|f(q,t,\cdot)\|_{H^{-1,2}(\mathbb{R})} \leq C \|(k * \zeta_t) q_t\|_{L^2(\mathbb{R})} \leq \|k\|_{L^2(\mathbb{R})} M \|q_t\|_{L^2(\mathbb{R})}.$$

By [Kry99, Remark 5.5] this is sufficient to verify [Kry99, Assumption 5.6]. The other assumptions are proven similarly.

Hence, we can deduce that the linear SPDE (4.16) admits a unique solution

$$\rho^\zeta \in L^2_{\mathcal{F}^W}([0,T]; W^{1,2}(\mathbb{R})) \cap S^2_{\mathcal{F}^W}([0,T]; L^2(\mathbb{R})).$$

In the next step we want to demonstrate the non-negativity of the solution ρ^ζ with the regularity of the solution ρ^ζ. Let us denote by k_m the mollification of k and let

$$\rho^{\zeta,m} \in L^2_{\mathcal{F}^W}([0,T]; W^{1,2}(\mathbb{R})) \cap S^2_{\mathcal{F}^W}([0,T]; L^2(\mathbb{R}))$$

be the solution of the SPDE (4.16) with $(k_m * \zeta)\rho^\zeta$ instead of $(k * \zeta)\rho^\zeta$. Then we can write the SPDE in the form

$$\mathrm{d}\rho_t^{\zeta,m} = a(t,x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\rho_t^{\zeta,m}\,\mathrm{d}t + b^m(t,x)\frac{\mathrm{d}}{\mathrm{d}x}\rho_t^{\zeta,m}\,\mathrm{d}t + c^m(t,x)\rho_t^{\zeta,m}\,\mathrm{d}t - \nu\frac{\mathrm{d}}{\mathrm{d}x}\rho_t^{\zeta,m}\,\mathrm{d}W_t,$$

for $t \in [0,T]$, with

$$a(t,x) := \frac{\sigma_t^2 + \nu^2}{2},\ b^m(t,x) := \frac{\mathrm{d}}{\mathrm{d}x}(\sigma_t^2 + \nu^2) + k_m * \zeta_t,\ c^m(t,x) := \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\sigma_t^2 + \nu^2) + \frac{\mathrm{d}}{\mathrm{d}x}k_m * \zeta_t.$$

Now, by Assumption 4.1 the coefficients a^m, b^m, c^m and the coefficient in the stochastic part is bounded. Hence, by the maximum principle [Kry99, Theorem 5.12] the solution $\rho^{\zeta,m}$ is non-negative. On the other hand, we have

$$\begin{aligned}
&\left\| \frac{\mathrm{d}}{\mathrm{d}x}\big((k_m * \zeta_t)\rho_t^\zeta - (k * \zeta_t)\rho_t^\zeta\big) \right\|^2_{L^2_{\mathcal{F}W}([0,T];H^{-1,2}(\mathbb{R}))} \\
&\quad \leq C \left\| ((k_m - k) * \zeta_t)\rho_t^\zeta \right\|^2_{L^2_{\mathcal{F}W}([0,T];L^2(\mathbb{R}))} \\
&\quad \leq C\mathbb{E}\bigg(\int_0^T \|((k_m - k) * \zeta_t)\|^2_{L^\infty(\mathbb{R})} \left\|\rho_t^\zeta\right\|^2_{L^2(\mathbb{R})} \bigg) \\
&\quad \leq C\mathbb{E}\bigg(\int_0^T \|((k_m - k)\|^2_{L^2(\mathbb{R})} \|\zeta_t\|^2_{L^2(\mathbb{R})} \left\|\rho_t^\zeta\right\|^2_{L^2(\mathbb{R})} \bigg) \\
&\quad \leq C \|((k_m - k)\|^2_{L^2(\mathbb{R})} M^2 \left\|\rho_t^\zeta\right\|^2_{L^2_{\mathcal{F}W}([0,T];L^2(\mathbb{R}))} \xrightarrow{m\to\infty} 0.
\end{aligned}$$

Consequently, by [Kry99, Theorem 5.7] we have

$$\lim_{m\to\infty} \left\| \rho^{\zeta,m} - \rho^\zeta \right\|_{L^2_{\mathcal{F}W}([0,T];W^{1,2}(\mathbb{R}))} = 0$$

and therefore $\rho_t^\zeta(\cdot) \geq 0$ for all $t \in [0,T]$ almost surely (by intersecting all sets of measure one, where $\rho^{\zeta,m}$ is non-negative).

The non-negativity of the solution ρ^ζ and the divergence structure of the equation provides us with the normalization condition/mass conservation, that is

$$\left\|\rho_t^\zeta\right\|_{L^1(\mathbb{R})} = \|\rho_0\|_{L^1(\mathbb{R})} = 1, \quad \mathbb{P}\text{-a.s.},$$

for $t \in [0,T]$. This follows immediately by plugging in a cut-off sequence $(\xi_n, n \in \mathbb{N})$ for our test function φ and taking the limit $n \to \infty$ (see [Bre11, p. 212] for properties of the cut-off sequence). Therefore, the map $\mathcal{T}(\zeta) = \rho^\zeta$ will be well-defined if we can obtain a bound on the $S^\infty_{\mathcal{F}W}([0,T];L^2(\mathbb{R}))$-norm. For readability we will from now on drop the superscript ζ in the following. Applying Itô's formula [Kry10], we obtain

$$\begin{aligned}
&\|\rho_t\|^2_{L^2(\mathbb{R})} - \|\rho_0\|^2_{L^2(\mathbb{R})} \\
&\quad = \int_0^t \left\langle \rho_s, \frac{\mathrm{d}}{\mathrm{d}x}(\sigma_s^2 + \nu^2)\frac{\mathrm{d}}{\mathrm{d}x}\rho_s + \rho_s \frac{\mathrm{d}^2}{\mathrm{d}x^2}(\sigma_s^2 + \nu^2) \right\rangle_{L^2(\mathbb{R})} \mathrm{d}s \\
&\qquad - \int_0^t \left\langle (\sigma_s^2 + \nu^2)\frac{\mathrm{d}}{\mathrm{d}x}\rho_s, \frac{\mathrm{d}}{\mathrm{d}x}\rho_s \right\rangle_{L^2(\mathbb{R})} \mathrm{d}s - 2\int_0^t \left\langle (k * \zeta_s)\rho_s, \frac{\mathrm{d}}{\mathrm{d}x}\rho_s \right\rangle_{L^2(\mathbb{R})} \mathrm{d}s \\
&\qquad + \nu^2 \int_0^t \left\| \frac{\mathrm{d}}{\mathrm{d}x}\rho_s \right\|^2_{L^2(\mathbb{R})} \mathrm{d}s + 2\nu \int_0^t \left\langle \rho_s, \frac{\mathrm{d}}{\mathrm{d}x}\rho_s \right\rangle_{L^2(\mathbb{R})} \mathrm{d}W_s
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \left\langle \rho_s, \frac{\mathrm{d}}{\mathrm{d}x}(\sigma_s^2+\nu^2)\frac{\mathrm{d}}{\mathrm{d}x}\rho_s + \rho_s \frac{\mathrm{d}^2}{\mathrm{d}x^2}(\sigma_s^2+\nu^2) \right\rangle_{L^2(\mathbb{R})} \mathrm{d}s \\
&\quad - 2\int_0^t \left\langle (k*\zeta_s)\rho_s, \frac{\mathrm{d}}{\mathrm{d}x}\rho_s \right\rangle_{L^2(\mathbb{R})} \mathrm{d}s - \int_0^t \left\langle \sigma_s^2 \frac{\mathrm{d}}{\mathrm{d}x}\rho_s, \frac{\mathrm{d}}{\mathrm{d}x}\rho_s \right\rangle_{L^2(\mathbb{R})} \mathrm{d}s \\
&\le \int_0^t \left\langle \rho_s, \frac{\mathrm{d}}{\mathrm{d}x}(\sigma_s^2+\nu^2)\frac{\mathrm{d}}{\mathrm{d}x}\rho_s + \rho_s \frac{\mathrm{d}^2}{\mathrm{d}x^2}(\sigma_s^2+\nu^2) \right\rangle_{L^2(\mathbb{R})} \mathrm{d}s \\
&\quad - 2\int_0^t \left\langle (k*\zeta_s)\rho_s, \frac{\mathrm{d}}{\mathrm{d}x}\rho_s \right\rangle_{L^2(\mathbb{R})} \mathrm{d}s - \lambda \int_0^t \left\| \frac{\mathrm{d}}{\mathrm{d}x}\rho_s \right\|_{L^2(\mathbb{R})}^2 \mathrm{d}s,
\end{aligned}
$$

for $0 \le t \le T$, where we used the fact that $\rho_s \frac{\mathrm{d}}{\mathrm{d}x}\rho_s = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(\rho_s^2)$ to get rid of the stochastic integral. At this step, it is crucial that we only have additive common noise. Otherwise the stochastic integral will not vanish and the above estimate will not achieve the L^∞-bound in ω. For the first term we can use Assumption 4.1 and Young's inequality to find

$$
\begin{aligned}
&\left| \int_0^t \left\langle \rho_s, \frac{\mathrm{d}}{\mathrm{d}x}(\sigma_s^2+\nu^2)\frac{\mathrm{d}}{\mathrm{d}x}\rho_s + \rho_s \frac{\mathrm{d}^2}{\mathrm{d}x^2}(\sigma_s^2+\nu^2) \right\rangle_{L^2(\mathbb{R})} \mathrm{d}s \right| \\
&\quad \le \Lambda \int_0^t \left| \left\langle \rho_s, \frac{\mathrm{d}}{\mathrm{d}x}\rho_s + \rho_s \right\rangle_{L^2(\mathbb{R})} \right| \mathrm{d}s \\
&\quad \le \frac{\lambda}{4}\int_0^t \left\| \frac{\mathrm{d}}{\mathrm{d}x}\rho_s \right\|_{L^2(\mathbb{R})}^2 \mathrm{d}s + \left(\frac{\Lambda^2}{\lambda} + \Lambda \right) \int_0^t \|\rho_s\|_{L^2(\mathbb{R})}^2 \mathrm{d}s.
\end{aligned} \tag{4.18}
$$

On the other hand, using (4.17) and Young's inequality, we obtain

$$
\begin{aligned}
\left| \left\langle (k*\zeta_s)\rho_s, \frac{\mathrm{d}}{\mathrm{d}x}\rho_s \right\rangle_{L^2(\mathbb{R})} \right| &\le \|k*\zeta_s\|_{L^\infty(\mathbb{R})} \left\langle |\rho_s|, \left| \frac{\mathrm{d}}{\mathrm{d}x}\rho_s \right| \right\rangle_{L^2(\mathbb{R})} \\
&\le \|k\|_{L^2(\mathbb{R})} M \|\rho_s\|_{L^2(\mathbb{R})} \left\| \frac{\mathrm{d}}{\mathrm{d}x}\rho_s \right\|_{L^2(\mathbb{R})} \\
&\le \frac{\|k\|_{L^2(\mathbb{R})}^2 M^2}{\lambda} \|\rho_s\|_{L^2(\mathbb{R})}^2 + \frac{\lambda}{4} \left\| \frac{\mathrm{d}}{\mathrm{d}x}\rho_s \right\|_{L^2(\mathbb{R})}^2 .
\end{aligned}
$$

After absorbing the terms, we find

$$
\|\rho_t\|_{L^2(\mathbb{R})}^2 - \|\rho_0\|_{L^2(\mathbb{R})}^2 \le \left(\frac{\|k\|_{L^2(\mathbb{R})}^2 M^2 + \Lambda^2}{\lambda} + \Lambda \right) \int_0^t \|\rho_s\|_{L^2(\mathbb{R})}^2 \mathrm{d}s.
$$

For the rest of the proof we define the constant

$$C(\lambda,\Lambda,k,M) := \frac{\|k\|^2_{L^2(\mathbb{R})} M^2 + \Lambda^2}{\lambda} + \Lambda$$

and conclude

$$\|\rho_t\|^2_{L^2(\mathbb{R})} \le \|\rho_0\|^2_{L^2(\mathbb{R})} \exp\big(C(\lambda,\Lambda,k,M)T\big), \tag{4.19}$$

by Gronwall's inequality. Choosing $\hat{T}^* < \ln(M/\|\rho_0\|^2_{L^2(\mathbb{R})}) C(\lambda,\Lambda,k,M)^{-1}$, we have $\rho \in F^{\hat{T}^*,M}$ and the map

$$\mathcal{T}\colon F^{\hat{T}^*,M} \to F^{\hat{T}^*,M}, \quad \zeta \to \rho^{\zeta},$$

is well-defined up to time $\hat{T}^*$.

The next step is to show that $\mathcal{T}$ is a contraction in a small time span ($T \le \hat{T}^*$) and, therefore, has a fixed point. For $\zeta, \tilde{\zeta} \in F^{T,M}$ let $\rho := \mathcal{T}(\zeta), \tilde{\rho} := \mathcal{T}(\tilde{\zeta})$ be the associated solutions of the linear SPDE (4.16). Then, we have

$$\begin{aligned}
&\mathrm{d}(\rho_t - \tilde{\rho}_t)\\
&\quad= \frac{\sigma_t^2+\nu^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\rho_t-\tilde{\rho}_t)\,\mathrm{d}t + (\sigma_t^2+\nu^2)\frac{\mathrm{d}}{\mathrm{d}x}(\rho_t-\tilde{\rho}_t)\,\mathrm{d}t + \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\sigma_t^2+\nu^2)(\rho_t-\tilde{\rho}_t)\,\mathrm{d}t\\
&\qquad+ \frac{\mathrm{d}}{\mathrm{d}x}((k*\zeta_t)\rho_t)\,\mathrm{d}t - \frac{\mathrm{d}}{\mathrm{d}x}((k*\tilde{\zeta}_t)\tilde{\rho}_t)\,\mathrm{d}t - \nu\frac{\mathrm{d}}{\mathrm{d}x}(\rho_t-\tilde{\rho}_t)\,\mathrm{d}W_t, \quad t\in[0,T].
\end{aligned}$$

Applying Itô's formula [Kry10] and multiple Young's inequality again (see (4.18)), we obtain

$$\begin{aligned}
&\|\rho_t-\tilde{\rho}_t\|^2_{L^2(\mathbb{R})}\\
&\quad= -\int_0^t \Big\langle (\sigma_s^2+\nu^2)\frac{\mathrm{d}}{\mathrm{d}x}\rho_s - \frac{\mathrm{d}}{\mathrm{d}x}\tilde{\rho}_s, \frac{\mathrm{d}}{\mathrm{d}x}\rho_s - \frac{\mathrm{d}}{\mathrm{d}x}\tilde{\rho}_s\Big\rangle_{L^2(\mathbb{R})}\,\mathrm{d}s\\
&\qquad- 2\int_0^t \Big\langle (k*\zeta_s)\rho_s - (k*\tilde{\zeta}_s)\tilde{\rho}_s, \frac{\mathrm{d}}{\mathrm{d}x}\rho_s - \frac{\mathrm{d}}{\mathrm{d}x}\tilde{\rho}_s\Big\rangle_{L^2(\mathbb{R})}\,\mathrm{d}s\\
&\qquad+ \nu^2\int_0^t \Big\|\frac{\mathrm{d}}{\mathrm{d}x}(\rho_s-\tilde{\rho}_s)\Big\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s\\
&\qquad+ \int_0^t \Big\langle \rho_s-\tilde{\rho}_s, \frac{\mathrm{d}}{\mathrm{d}x}(\sigma_s^2+\nu^2)\Big(\frac{\mathrm{d}}{\mathrm{d}x}\rho_s - \frac{\mathrm{d}}{\mathrm{d}x}\tilde{\rho}_s\Big) + (\rho_s-\tilde{\rho}_s)\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\sigma_s^2+\nu^2)\Big\rangle_{L^2(\mathbb{R})}\,\mathrm{d}s\\
&\quad\le -\lambda\int_0^t \Big\|\frac{\mathrm{d}}{\mathrm{d}x}\rho_s - \frac{\mathrm{d}}{\mathrm{d}x}\tilde{\rho}_s\Big\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s\\
&\qquad- 2\int_0^t \Big\langle (k*(\zeta_s-\tilde{\zeta}_s))\rho_s + (k*\tilde{\zeta}_s)(\rho_s-\tilde{\rho}_s), \frac{\mathrm{d}}{\mathrm{d}x}\rho_s - \frac{\mathrm{d}}{\mathrm{d}x}\tilde{\rho}_s\Big\rangle_{L^2(\mathbb{R})}\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\lambda}{4} \int_0^t \left\| \frac{\mathrm{d}}{\mathrm{d}x}\rho_s - \frac{\mathrm{d}}{\mathrm{d}x}\tilde{\rho}_s \right\|_{L^2(\mathbb{R})}^2 \mathrm{d}s + \Big(\frac{\Lambda^2}{\lambda} + \Lambda\Big) \int_0^t \|\rho_s - \tilde{\rho}_s\|_{L^2(\mathbb{R})}^2 \,\mathrm{d}s \\
&\leq -\frac{3\lambda}{4} \int_0^t \left\| \frac{\mathrm{d}}{\mathrm{d}x}\rho_s - \frac{\mathrm{d}}{\mathrm{d}x}\tilde{\rho}_s \right\|_{L^2(\mathbb{R})}^2 \mathrm{d}s + \Big(\frac{\Lambda^2}{\lambda} + \Lambda\Big) \int_0^t \|\rho_s - \tilde{\rho}_s\|_{L^2(\mathbb{R})}^2 \,\mathrm{d}s \\
&\quad + \int_0^t \|k\|_{L^2(\mathbb{R})} \left\|\zeta_s - \tilde{\zeta}_s\right\|_{L^2(\mathbb{R})} \|\rho_s\|_{L^2(\mathbb{R})} \left\| \frac{\mathrm{d}}{\mathrm{d}x}\rho_s - \frac{\mathrm{d}}{\mathrm{d}x}\tilde{\rho}_s \right\|_{L^2(\mathbb{R})} \mathrm{d}s \\
&\quad + \|k\|_{L^2(\mathbb{R})} \int_0^t \|\rho_s - \tilde{\rho}_s\|_{L^2(\mathbb{R})} \left\|\tilde{\zeta}_s\right\|_{L^2(\mathbb{R})} \left\| \frac{\mathrm{d}}{\mathrm{d}x}\rho_s - \frac{\mathrm{d}}{\mathrm{d}x}\tilde{\rho}_s \right\|_{L^2(\mathbb{R})} \mathrm{d}s \\
&\leq -\frac{3\lambda}{4} \int_0^t \left\| \frac{\mathrm{d}}{\mathrm{d}x}\rho_s - \frac{\mathrm{d}}{\mathrm{d}x}\tilde{\rho}_s \right\|_{L^2(\mathbb{R})}^2 \mathrm{d}s + \Big(\frac{\Lambda^2}{\lambda} + \Lambda\Big) \int_0^t \|\rho_s - \tilde{\rho}_s\|_{L^2(\mathbb{R})}^2 \,\mathrm{d}s \\
&\quad + \int_0^t \frac{\|k\|_{L^2(\mathbb{R})}^2 M^2}{\lambda} \left\|\zeta_s - \tilde{\zeta}_s\right\|_{L^2(\mathbb{R})}^2 + \frac{\lambda}{4} \left\| \frac{\mathrm{d}}{\mathrm{d}x}\rho_s - \frac{\mathrm{d}}{\mathrm{d}x}\tilde{\rho}_s \right\|_{L^2(\mathbb{R})}^2 \mathrm{d}s \\
&\quad + \int_0^t \frac{\|k\|_{L^2(\mathbb{R})}^2 M^2}{\lambda} \|\rho_s - \tilde{\rho}_s\|_{L^2(\mathbb{R})}^2 + \frac{\lambda}{4} \left\| \frac{\mathrm{d}}{\mathrm{d}x}\rho_s - \frac{\mathrm{d}}{\mathrm{d}x}\tilde{\rho}_s \right\|_{L^2(\mathbb{R})}^2 \mathrm{d}s \\
&\leq \int_0^t \frac{\|k\|_{L^2(\mathbb{R})}^2 M^2}{\lambda} \left\|\zeta_s - \tilde{\zeta}_s\right\|_{L^2(\mathbb{R})}^2 \mathrm{d}s + \int_0^t C(\lambda, \Lambda, k, M) \|\rho_s - \tilde{\rho}_s\|_{L^2(\mathbb{R})}^2 \,\mathrm{d}s \\
&\leq T \frac{\|k\|_{L^2(\mathbb{R})}^2 M^2}{\lambda} \left\|\zeta - \tilde{\zeta}\right\|_{S^\infty_{\mathcal{F}W}([0,T];L^2(\mathbb{R}))}^2 + C(\lambda, \Lambda, k, M) \int_0^t \|\rho_s - \tilde{\rho}_s\|_{L^2(\mathbb{R})}^2 \,\mathrm{d}s.
\end{aligned}
$$

Gronwall's inequality provide us with the estimate

$$
\|\rho - \tilde{\rho}\|_{S^\infty_{\mathcal{F}W}([0,T];L^2(\mathbb{R}))} \leq \sqrt{\frac{T\,\|k\|_{L^2(\mathbb{R})}^2 M^2}{\lambda}} \exp\left(\frac{T}{2} C(\lambda, \Lambda, k, M)\right) \|\zeta - \tilde{\zeta}\|_{S^\infty_{\mathcal{F}W}([0,T];L^2(\mathbb{R}))}.
$$

Now, choosing T^* such that

$$
\sqrt{\frac{T^*\,\|k\|_{L^2(\mathbb{R})}^2 M^2}{\lambda}} \lambda \exp\left(\frac{T^*}{2} C(\lambda, \Lambda, k, M)\right) < 1
$$

and $T^* \leq \hat{T}^*$, we see that the map $\mathcal{T} \colon F^{T^*,M} \to F^{T^*,M}$ is a contraction and consequently we obtain a fixed point ρ, which is a local weak solution up to the time T^* of the SPDE (4.11). $\square$

We notice that in the proof of Theorem 4.6, we only use Hölder's inequality, Young's convolution and product inequality. Hence, the statement of Theorem 4.6 holds also for arbitrary dimension. We state this observation in the following corollary.

COROLLARY 4.7. *Suppose Assumption 4.1 holds in th d-dimensional setting and* $k \in L^2(\mathbb{R}^d)$. *Then, there exists a small* $T^* > 0$ *and a unique non-negative solution of the SPDE* (4.11) *in the space*

$$\mathbb{B} := L^2_{\mathcal{F}W}([0,T^*];W^{1,2}(\mathbb{R}^d)) \cap S^{\infty}_{\mathcal{F}W}([0,T^*];L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)).$$

The solution should be understood in the sense of Definition 4.2, where Definition 4.2 is modified for arbitrary dimension d in the obvious way, see also [Kry99, Definition 3.5].

REMARK 4.8. *Following the steps of the proof of Theorem 4.6, we see that we can obtain not only a local solution but a (global) solution for any* $T > 0$ *by requiring a small* L^2*-norm on the initial condition* ρ_0. *In particular, we can choose a constant* $M > 0$ *such that*

$$\sqrt{\frac{T\,\|k\|^2_{L^2(\mathbb{R})}\,M^2}{\lambda}\lambda}\exp\Bigg(\frac{T}{2}\Bigg(\frac{\|k\|^2_{L^2(\mathbb{R})}\,M^2+\Lambda^2}{\lambda}+\Lambda\Bigg)\Bigg) < 1$$

and then the condition

$$\|\rho_0\|_{L^2(\mathbb{R})} \le M\exp\Bigg(-T\Bigg(\frac{\|k\|^2_{L^2(\mathbb{R})}\,M^2+\Lambda^2}{\lambda}+\Lambda\Bigg)\Bigg)$$

guarantees a unique non-negative solution of the SPDE (4.11) *on the interval* $[0,T]$.

Next, we establish another global existence and uniqueness result. We emphasize that in the following result we do not need any further assumptions on ρ_0 besides being in $L^1(\mathbb{R})\cap L^2(\mathbb{R})$. Instead, we impose a lower bound on the diffusion coefficient σ. Hence, we require a sufficiently high randomness in the stochastic Fokker–Planck equation. We also assert the fact that the continuation of the solution $(\rho_t, t\ge 0)$ is a direct consequence of the L^2-theory of SPDEs.

COROLLARY 4.9. *Let Assumption 4.1 hold and* $k\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$. *Furthermore, assume that the diffusion coefficient* σ *has a derivative* $\frac{\mathrm{d}}{\mathrm{d}x}\sigma$ *with compact support* $[-L,L]$ *and satisfies*

$$\begin{aligned} &2L^2\sup_{0\le t\le T}\left\|\frac{\mathrm{d}}{\mathrm{d}x}(\sigma_t^2)\right\|_{L^\infty(\mathbb{R})}\\ &\quad+\Bigg(C^4_{GNS}\,\|k\|^2_{L^1(\mathbb{R})}+4L^4\sup_{0\le t\le T}\left\|\frac{\mathrm{d}}{\mathrm{d}x}(\sigma_t^2)\right\|^2_{L^\infty(\mathbb{R})}\Bigg)^{1/2}\le\lambda, \end{aligned} \tag{4.20}$$

where λ *is the ellipticity constant in Assumption 4.1* C_{GNS} *is the constant given by the Gagliardo–Nirenberg–Sobolev interpolation in one dimension* [Leo17, Theorem 12.83], *i.e.* $C_{GNS} = \big(\frac{4\pi^2}{9}\big)^{-1/4}$. *Then, for each* $T>0$ *there exist unique global non-negative solutions of the stochastic Fokker–Planck equations* (4.11) *in the space* $\mathbb{B}$.

Proof. Let ρ be the solution given by Theorem 4.6. Following the proof of Theorem 4.6, we apply Itô's formula [Kry10] and obtain, for $0 \leq t \leq T^*$,

$$
\begin{aligned}
&\|\rho_t\|^2_{L^2(\mathbb{R})} - \|\rho_0\|^2_{L^2(\mathbb{R})} \\
&= -\int_0^t \left\| \sigma_s \frac{\mathrm{d}}{\mathrm{d}x}\rho_s \right\|^2_{L^2(\mathbb{R})} \mathrm{d}s + \int_0^t \left\langle \rho_s, \frac{\mathrm{d}}{\mathrm{d}x}(\sigma_s^2+\nu^2)\frac{\mathrm{d}}{\mathrm{d}x}\rho_s + \rho_s \frac{\mathrm{d}^2}{\mathrm{d}x^2}(\sigma_s^2+\nu^2) \right\rangle \mathrm{d}s \\
&\quad - 2\int_0^t \left\langle (k*\rho_s)\rho_s, \frac{\mathrm{d}}{\mathrm{d}x}\rho_s \right\rangle_{L^2(\mathbb{R})} \mathrm{d}s \\
&= -\int_0^t \left\| \sigma_s \frac{\mathrm{d}}{\mathrm{d}x}\rho_s \right\|^2_{L^2(\mathbb{R})} \mathrm{d}s - \int_0^t \left\langle \rho_s \frac{\mathrm{d}}{\mathrm{d}x}(\sigma_s^2), \frac{\mathrm{d}}{\mathrm{d}x}\rho_s \right\rangle_{L^2(\mathbb{R})} \mathrm{d}s \\
&\quad - 2\int_0^t \left\langle (k*\rho_s)\rho_s, \frac{\mathrm{d}}{\mathrm{d}x}\rho_s \right\rangle_{L^2(\mathbb{R})} \mathrm{d}s \\
&\leq -\frac{\lambda}{2}\int_0^t \left\| \frac{\mathrm{d}}{\mathrm{d}x}\rho_s \right\|^2_{L^2(\mathbb{R})} + \sup_{0\leq t\leq T} \left\| \frac{\mathrm{d}}{\mathrm{d}x}(\sigma_t^2) \right\|_{L^\infty(\mathbb{R})} \|\rho_s\|_{L^2([-L,L])} \left\| \frac{\mathrm{d}}{\mathrm{d}x}\rho_s \right\|_{L^2(\mathbb{R})} \mathrm{d}s \\
&\quad + \frac{1}{\lambda}\int_0^t \|(k*\rho_s)\rho_s\|^2_{L^2(\mathbb{R})} \mathrm{d}s \\
&\leq \Big(-\frac{\lambda}{2} + 2L^2 \sup_{0\leq t\leq T} \left\| \frac{\mathrm{d}}{\mathrm{d}x}(\sigma_t^2) \right\|_{L^\infty(\mathbb{R})} \Big) \int_0^t \left\| \frac{\mathrm{d}}{\mathrm{d}x}\rho_s \right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \\
&\quad + \frac{1}{\lambda}\int_0^t \|k*\rho_s\|^2_{L^4(\mathbb{R})} \|\rho_s\|^2_{L^4(\mathbb{R})} \mathrm{d}s \\
&\leq \left(-\frac{\lambda}{2} + 2L^2 \sup_{0\leq t\leq T} \left\| \frac{\mathrm{d}}{\mathrm{d}x}(\sigma_t^2) \right\|_{L^\infty(\mathbb{R})} \right) \int_0^t \left\| \frac{\mathrm{d}}{\mathrm{d}x}\rho_s \right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \\
&\quad + \frac{1}{\lambda}\int_0^t \|k\|^2_{L^1(\mathbb{R})} \|\rho_s\|^2_{L^4(\mathbb{R})} \|\rho_s\|^2_{L^4(\mathbb{R})} \mathrm{d}s \\
&= \left(-\frac{\lambda}{2} + 2L^2 \sup_{0\leq t\leq T} \left\| \frac{\mathrm{d}}{\mathrm{d}x}(\sigma_t^2) \right\|_{L^\infty(\mathbb{R})} \right) \\
&\quad \cdot \int_0^t \left\| \frac{\mathrm{d}}{\mathrm{d}x}\rho_s \right\|^2_{L^2(\mathbb{R})} \mathrm{d}s + \frac{\|k\|^2_{L^1(\mathbb{R})}}{\lambda} \int_0^t \|\rho_s\|^4_{L^4(\mathbb{R})} \mathrm{d}s.
\end{aligned}
$$

We remark that we used integration by parts in the first step, Young's and Hölder's inequality in the third step, Hölder's and Poincaré inequality [Leo17, Theorem 13.19] in the forth step and Young's inequality for convolutions in the fifth step. Let us recall the Gagliardo–Nirenberg–Sobolev interpolation [Leo17, Theorem 12.83], which states that for $u \in L^1(\mathbb{R}) \cap W^{1,2}(\mathbb{R})$ we have

$$\|u\|_{L^4(\mathbb{R})} \leq C_{GNS} \|u\|_{L^1(\mathbb{R})}^{1/2} \left\| \frac{\mathrm{d}}{\mathrm{d}x} u \right\|_{L^2(\mathbb{R})}^{1/2}.$$

Consequently, applying this inequality on the last term in our estimate and having mass conservation in mind we find

$$\begin{aligned}
&\|\rho_t\|_{L^2(\mathbb{R})}^2 - \|\rho_0\|_{L^2(\mathbb{R})}^2 \\
&\quad \leq \bigg(-\frac{\lambda}{2} + 2L^2 \sup_{0 \leq t \leq T} \left\| \frac{\mathrm{d}}{\mathrm{d}x}(\sigma_t^2) \right\|_{L^\infty(\mathbb{R})} + \frac{C_{GNS}^4 \|k\|_{L^1(\mathbb{R})}^2}{\lambda} \bigg) \int_0^t \left\| \frac{\mathrm{d}}{\mathrm{d}x} \rho_s \right\|_{L^2(\mathbb{R})}^2 \mathrm{d}s.
\end{aligned}$$

Hence, if

$$2L^2 \sup_{0 \leq t \leq T} \left\| \frac{\mathrm{d}}{\mathrm{d}x}(\sigma_t^2) \right\|_{L^\infty(\mathbb{R})} + \bigg(C_{GNS}^4 \|k\|_{L^1(\mathbb{R})}^2 + 4L^4 \sup_{0 \leq t \leq T} \left\| \frac{\mathrm{d}}{\mathrm{d}x}(\sigma_t^2) \right\|_{L^\infty(\mathbb{R})}^2 \bigg)^{1/2} \leq \lambda,$$

we discover

$$\|\rho\|_{S^\infty_{\mathcal{F}W}([0,T^*];L^2(\mathbb{R}))} \leq \|\rho_0\|_{L^2(\mathbb{R})}^2. \tag{4.21}$$

Since $\rho \in L^2_{\mathcal{F}W}([0,T^*]; W^{1,2}(\mathbb{R}))$, we may apply [Kry99, Theorem 7.1], which tells us that $\rho \in C([0,T^*], L^2(\mathbb{R}))$, $\mathbb{P}$-a.s., and

$$\mathbb{E}(\|\rho_{T^*}\|_{L^2(\mathbb{R})}^2) < \infty.$$

As a result, we can take ρ_{T^*} as the new initial value and apply [Kry99, Theorem 5.1] in combination with our above arguments in proof of Theorem 4.6 to obtain a solution on $[T^*, 2T^*]$, since the estimate (4.21) and the condition (4.20) are independent of T^*. Hence, after finitely many steps we have a global solution ρ on $[0,T]$. The uniqueness and $\rho \in \mathbb{B}$ follows by repeating the inequalities derived in the contraction argument in Theorem 4.6 or using the uniqueness of the SPDE presented in [Kry99, Theorem 5.1 and Corollary 5.11]. □

REMARK 4.10. *In particular, for a constant diffusion $\sigma > 0$ the condition* (4.20) *reads simply as*

$$C_{GNS}^2 \|k\|_{L^1(\mathbb{R})} \leq \sigma,$$

which can be interpreted such that for a given integrable kernel k the system needs a certain amount of idiosyncratic noise to stay alive for arbitrary $T > 0$. In other word, the diffusion needs to be dominant.

Next, we are going to improve the regularity of the solution ρ by a bootstrap argument.

LEMMA 4.11. *Let $\rho_0 \in L^1(\mathbb{R}) \cap W^{2,2}(\mathbb{R})$ with $\|\rho_0\|_{L^1(\mathbb{R})} = 1$. Moreover, let Assumption 4.1 hold and $k \in L^2(\mathbb{R})$. Assume we have a solution*

$$\rho \in L^2_{\mathcal{F}W}([0,T]; W^{1,2}(\mathbb{R})) \cap S^\infty_{\mathcal{F}W}([0,T]; L^1(\mathbb{R}) \cap L^2(\mathbb{R}))$$

of the SPDE (4.11) *on* $[0,T]$*. Then* ρ *has the following regularity*

$$\rho \in L^2_{\mathcal{F}W}([0,T];W^{3,2}(\mathbb{R})) \cap S^2_{\mathcal{F}W}([0,T],W^{2,2}(\mathbb{R})) \cap S^\infty_{\mathcal{F}W}([0,T];L^1(\mathbb{R})\cap L^2(\mathbb{R})).$$

Proof. Let us explore the following bootstrap argument. By assumptions we know $\rho \in L^2_{\mathcal{F}W}([0,T];W^{1,2}(\mathbb{R})) \cap S^\infty_{\mathcal{F}W}([0,T];L^1(\mathbb{R})\cap L^2(\mathbb{R}))$ and solves the SPDE

$$\mathrm{d}\rho_t = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\Big(\frac{\sigma_t^2+\nu^2}{2}\rho_t\Big)\,\mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}x}((k*\rho_t)\rho_t)\,\mathrm{d}t - \nu\frac{\mathrm{d}}{\mathrm{d}x}\rho_t\,\mathrm{d}W_t. \tag{4.22}$$

Furthermore, $\frac{\mathrm{d}}{\mathrm{d}x}(k^\varepsilon*\rho_t) = k^\varepsilon * \frac{\mathrm{d}}{\mathrm{d}x}\rho_t$ for the smooth approximation k^ε of k and consequently the dominated convergence theorem implies $\frac{\mathrm{d}}{\mathrm{d}x}(k*\rho_t) = k*\frac{\mathrm{d}}{\mathrm{d}x}\rho_t$ in the sense of distributions. As a result

$$\frac{\mathrm{d}}{\mathrm{d}x}((k*\rho_t)\rho_t) = \Big(k*\frac{\mathrm{d}}{\mathrm{d}x}\rho_t\Big)\rho_t + (k*\rho_t)\frac{\mathrm{d}}{\mathrm{d}x}\rho_t$$

is well-defined as a function in $L^1(\mathbb{R})$. Moreover, we find

$$\begin{aligned}
\Big\|\frac{\mathrm{d}}{\mathrm{d}x}((k*\rho_t)\rho_t)\Big\|_{L^2(\mathbb{R})} &\le \Big\|\Big(k*\frac{\mathrm{d}}{\mathrm{d}x}\rho_t\Big)\rho_t\Big\|_{L^2(\mathbb{R})} + \Big\|(k*\rho_t)\frac{\mathrm{d}}{\mathrm{d}x}\rho_t\Big\|_{L^2(\mathbb{R})}\\
&\le \Big\|k*\frac{\mathrm{d}}{\mathrm{d}x}\rho_t\Big\|_{L^\infty(\mathbb{R})}\|\rho_t\|_{L^2(\mathbb{R})} + \|k*\rho_t\|_{L^\infty(\mathbb{R})}\Big\|\frac{\mathrm{d}}{\mathrm{d}x}\rho_t\Big\|_{L^2(\mathbb{R})}\\
&\le \|k\|_{L^2(\mathbb{R})}\|\rho_t\|_{W^{1,2}(\mathbb{R})}\|\rho_t\|_{L^2(\mathbb{R})} + \|k\|_{L^2(\mathbb{R})}\|\rho_t\|_{L^2(\mathbb{R})}\|\rho_t\|_{W^{1,2}(\mathbb{R})}\\
&\le 2\|k\|_{L^2(\mathbb{R})}\|\rho_t\|_{W^{1,2}(\mathbb{R})}\|\rho_t\|_{L^2(\mathbb{R})},
\end{aligned}$$

which implies

$$\Big\|\frac{\mathrm{d}}{\mathrm{d}x}((k*\rho)\rho)\Big\|_{L^2_{\mathcal{F}W}([0,T];L^2(\mathbb{R}))} \le 2\|k\|_{L^2(\mathbb{R})}\|\rho\|_{S^\infty_{\mathcal{F}W}([0,T];L^2(\mathbb{R}))}\|\rho\|_{L^2_{\mathcal{F}W}([0,T];W^{1,2}(\mathbb{R}))}.$$

From the uniqueness of the SPDE (4.22), $\rho_0 \in W^{1,2}(\mathbb{R})$ and [Kry99, Theorem 5.1 and Theorem 7.1] we obtain

$$\rho \in L^2_{\mathcal{F}W}([0,T];W^{2,2}(\mathbb{R})) \cap S^2_{\mathcal{F}W}([0,T];W^{1,2}(\mathbb{R})).$$

With the same strategy and the discovered regularity of ρ one obtains

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}((k*\rho_t)\rho_t) = \Big(k*\frac{\mathrm{d}^2}{\mathrm{d}x^2}\rho_t\Big)\rho_t + 2\Big(k*\frac{\mathrm{d}}{\mathrm{d}x}\rho_t\Big)\frac{\mathrm{d}}{\mathrm{d}x}\rho_t + (k*\rho_t)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\rho_t$$

and consequently

$$\begin{aligned}
\Big\|\frac{\mathrm{d}^2}{\mathrm{d}x^2}((k*\rho_t)\rho_t)\Big\|_{L^2(\mathbb{R})} &\le 2\|k\|_{L^2(\mathbb{R})}\|\rho_t\|_{W^{2,2}(\mathbb{R})}\|\rho_t\|_{L^2(\mathbb{R})} + 2\|k\|_{L^2(\mathbb{R})}\Big\|\frac{\mathrm{d}}{\mathrm{d}x}\rho_t\Big\|^2_{L^2(\mathbb{R})}\\
&\le 4\|k\|_{L^2(\mathbb{R})}\|\rho_t\|_{W^{2,2}(\mathbb{R})}\|\rho_t\|_{L^2(\mathbb{R})},
\end{aligned}$$

where we used Gagliardo–Nirenberg–Sobolev inequality [Leo17, Theorem 7.41] in the last step. Therefore, we have

$$\Big\|\frac{\mathrm{d}}{\mathrm{d}x}((k*\rho)\rho)\Big\|_{L^2_{\mathcal{F}W}([0,T];W^{1,2}(\mathbb{R}))} \le 4\|k\|_{L^2(\mathbb{R})}\|\rho\|_{S^\infty_{\mathcal{F}W}([0,T];L^2(\mathbb{R}))}\|\rho\|_{L^2_{\mathcal{F}W}([0,T];W^{2,2}(\mathbb{R}))}.$$

Again, from the uniqueness of the SPDE (4.22), $\rho_0 \in W^{2,2}(\mathbb{R})$ and [Kry99, Theorem 5.1, Corollary 5.11, Theorem 7.1] we obtain

$$\rho \in L^2_{\mathcal{F}^W}([0,T]; W^{3,2}(\mathbb{R})) \cap S^2_{\mathcal{F}^W}([0,T], W^{2,2}(\mathbb{R})).$$

□

As a consequence of Theorem 4.6, Corollary 4.9 and the fact that $k_{HK}, k^{\varepsilon}_{HK} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ for all $\varepsilon > 0$, we obtain the following corollary.

Corollary 4.12. *Let Assumption 4.1 hold. Further, assume* $0 \le \rho_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ *with* $\|\rho_0\|_{L^1(\mathbb{R})} = 1$. *Then, for the HK kernel* (k_{HK} *there exists a* $T^* > 0$ *and a unique non-negative solution* ρ^{HK}, $\rho^{\varepsilon,HK}$ *of the SPDE* (4.3) *and* (4.6) *in the space*

$$\mathbb{B} = L^2_{\mathcal{F}^W}([0,T^*]; W^{1,2}(\mathbb{R})) \cap S^{\infty}_{\mathcal{F}^W}([0,T^*]; L^1(\mathbb{R}) \cap L^2(\mathbb{R}))$$

Furthermore, if $\frac{\mathrm{d}}{\mathrm{d}x}\sigma_t(x)$ *has compact support and inequality* (4.20) *holds, then we can extend* ρ^{HK}, $\rho^{\varepsilon,HK}$ *to a global solution.*

4.4. Well-posedness of the mean-field SDEs

In this section, we establish the existence of unique strong solutions for the mean-field stochastic differential equations (SDEs) given by (4.9) and (4.10). Analogous to the classical theory of ordinary SDEs, the mean-field SDEs (4.9) and (4.10) are related to the stochastic Fokker–Planck equations (4.11) and (4.12) in the same way that ordinary SDEs are connected to deterministic Fokker–Planck equations.

Similar to Section 4.3, we prove the existence of strong solutions for general interaction force k, i.e. for equation (4.9). We notice that in order to show well-posedness for (4.9), it is enough to show the well-posedness for just one of the identically distributed SDE's of the system (4.9). Hence, in the following we will drop the superscript i. In order to guarantee the well-posedness of the SPDE (4.11) we make the assumption.

Assumption 4.13. *Let* $0 \le \rho_0 \in L^1(\mathbb{R}) \cap W^{2,2}(\mathbb{R})$ *with* $\|\rho_0\|_{L^1(\mathbb{R})} = 1$. *For* $T > 0$ *there exists a unique solution* ρ *in* $L^2_{\mathcal{F}^W}([0,T]; W^{1,2}(\mathbb{R}))$ *of the SPDE* (4.11) *on the interval* $[0,T]$ *with*

$$\|\rho\|_{L^2_{\mathcal{F}^W}([0,T];W^{1,2}(\mathbb{R}))} + \|\rho\|_{S^{\infty}_{\mathcal{F}^W}([0,T];L^1(\mathbb{R})\cap L^2(\mathbb{R}))} \le C$$

for some finite constant $C > 0$.

Remark 4.14. *The existence of a unique solution to the SPDE* (4.11) *in the above assumption is satisfied, for instance, if the conditions stated in Remark 4.8, Theorem 4.6 or Corollary 4.9 are satisfied.*

Theorem 4.15. *Let Assumption 4.1 as well as Assumption 4.13 hold and* $k \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. *Then, the mean-field SDE* (4.9) *has a unique strong solution* $(Y_t, t \in [0,T])$ *and* ρ_t *is the conditional density of* Y_t *given* $\mathcal{F}^W_t$ *for every* $t \in [0,T]$.

The idea to prove Theorem 4.15 is to freeze the measure ρ_t in the SDE (4.9) and use a duality argument by introducing a dual backward stochastic partial differential equation (BSPDE) in Lemma 4.17 in order to prove that ρ_t is the conditional density of Y_t for given $\mathcal{F}^W_t$.

Proof. Let ρ be the unique solution of the SPDE (4.11) as in Assumption 4.13. We recall that by the regularity result presented in Lemma 4.11 we have

$$\rho \in L^2_{\mathcal{F}^W}([0,T];W^{3,2}(\mathbb{R})) \cap S^2_{\mathcal{F}^W}([0,T],W^{2,2}(\mathbb{R})).$$

Step 1. Fix ρ in the mean-field SDE (4.9) and notice that we are dealing with a standard SDE with random coefficients. Hence, we can apply classical results if the drift coefficient $k*\rho$ is Lipschitz continuous. The regularity of the solution, Sobolev embedding theorem [Bre11, Theorem 8.8] and Morrey's inequality [Leo17, Theorem 12.66] yields

$$\begin{aligned}\sup_{0\le t\le T}\sup_{\substack{x,y\in\mathbb{R}\\ x\neq y}}\frac{|(k*\rho_t)(\omega,x)-(k*\rho_t)(\omega,y)|}{|x-y|} &\le \sup_{0\le t\le T}\|k*\rho_t(\omega)\|_{W^{2,2}(\mathbb{R})}\\ &\le \|k\|_{L^1(\mathbb{R})}\sup_{0\le t\le T}\|\rho_t(\omega)\|_{W^{2,2}(\mathbb{R})}\end{aligned}$$

and

$$\begin{aligned}\sup_{0\le t\le T}|(k*\rho_t)(\omega,0)| &\le \sup_{0\le t\le T}\sup_{x\in\mathbb{R}}|(k*\rho_t)(\omega,x)|\\ &\le \|k\|_{L^1(\mathbb{R})}\sup_{0\le t\le T}\|\rho_t(\omega)\|_{W^{1,2}(\mathbb{R})}\,.\end{aligned}$$

Furthermore, the maps $\omega\mapsto \sup_{0\le t\le T}\|\rho_t(\omega)\|_{W^{2,2}(\mathbb{R})}$ and $\omega\mapsto \sup_{0\le t\le T}\|\rho_t(\omega)\|_{W^{1,2}(\mathbb{R})}$ are measurable. Therefore, standard results for the existence of SDEs with Lipschitz continuous drift, see e.g. [KRZ99, Theorem 1.1] or [KHLN97, Theorem 2.2], imply that the following SDE has a unique strong solution

$$\begin{cases}\mathrm{d}\overline{Y}_t = -(k*\rho_t)(\overline{Y}_t)\,\mathrm{d}t + \sigma(t,\overline{Y}_t)\,\mathrm{d}B_t + \nu\,\mathrm{d}W_t,\\ \overline{Y}_0\sim\rho_0\,.\end{cases}\tag{4.23}$$

Step 2. We are going to use a dual argument (see Lemma 4.17 below) to show that ρ_t is the conditional density of Y_t with respect to $\mathcal{F}^W_t$. Hence, let $T_1>0$ and $(u_t, t\in[0,T_1])$ be the solution of the BSPDE (4.24) below with terminal condition $G\in L^\infty(\Omega,\mathcal{F}_{T_1},C_c^\infty(\mathbb{R}))$. Utilizing the dual equation from Lemma 4.17, the dual analysis [Zho92, Corollary 3.1] and the fact that u_0 is $\mathcal{F}^W_0$-measurable, we find

$$\langle\rho_0,u_0\rangle = \mathbb{E}(\langle G,\rho_{T_1}\rangle).$$

On the other hand we can use the explicit representation of u_0 given by Lemma 4.17 to obtain

$$\langle\rho_0,u_0\rangle = \int_{\mathbb{R}} u_0(y)\rho_0(y)\,\mathrm{d}y = \mathbb{E}(u_0(\overline{Y}_0)) = \mathbb{E}(\mathbb{E}(G(\overline{Y}_{T_1})\,|\,\sigma(\mathcal{F}^W_0,\sigma(\overline{Y}_0)))) = \mathbb{E}(G(\overline{Y}_{T_1})).$$

Now, let $G=\phi\xi$ with $\phi\in C_c^\infty(\mathbb{R})$ and $\xi\in L^\infty(\Omega,\mathcal{F}_{T_1})$. Consequently, we obtain

$$\mathbb{E}(\xi\langle\phi,\rho_{T_1}\rangle) = \mathbb{E}(\xi\phi(\overline{Y}_{T_1})) = \mathbb{E}(\xi\mathbb{E}(\phi(\overline{Y}_{T_1})\,|\,\mathcal{F}^W_{T_1})),$$

which proves

$$\langle\phi,\rho_{T_1}\rangle = \mathbb{E}(\phi(\overline{Y}_{T_1})\,|\,\mathcal{F}^W_{T_1}),\quad \mathbb{P}\text{-a.e.},$$

and, therefore, ρ_{T_1} is the conditional density of $\overline{Y}_{T_1}$ given $\mathcal{F}_{T_1}$. Since T_1 is arbitrary, we have proven the existence of a strong solution Y of the mean-field SDE (4.9).

□

REMARK 4.16. *If* (4.9) *has a strong solution with conditional density*

$$\rho \in L^2_{\mathcal{F}^W}([0,T^*];W^{1,2}(\mathbb{R})) \cap S^\infty_{\mathcal{F}^W}([0,T^*];L^1(\mathbb{R})\cap L^2(\mathbb{R})),$$

then the conditional density process of Y^1 *is the solution to the SPDE* (4.11)*. Indeed, if we first apply Itô's formula* (1.15) *with a function* $\varphi \in C_c^\infty(\mathbb{R})$*, then take the conditional expectation with respect to the filtration* $\mathcal{F}^W$ *and subsequently applying stochastic Fubini theorem A.35, we conclude that density process of* Y_t *satisfies* (4.15)*. By the uniqueness of the SPDE* (4.11) *we obtain that* ρ_t*, which is the solution constructed in Theorem 4.6, is the conditional density of* Y_t *given* $\mathcal{F}_t^W$ *for all* $t \in [0,T]$*.*

In the following lemma, we close the gap in the above proof by demonstrating the existence of a solution of the BSPDE (4.24) and the explicit representation of u_0.

LEMMA 4.17 (Dual BSPDE). *Let Assumption 4.1 and Assumption 4.13 hold along with* $k \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$*. Then, for every* $T_1 \in (0,T]$ *and* $G \in L^\infty(\Omega, \mathcal{F}_{T_1}, C_c^\infty(\mathbb{R}))$ *the following BSPDE*

$$\begin{aligned} \mathrm{d}u_t &= -\bigg(\frac{\sigma_t^2+\nu^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}u_t - (k*\rho_t)\frac{\mathrm{d}}{\mathrm{d}x}u_t + \nu\frac{\mathrm{d}}{\mathrm{d}x}v_t\bigg)\,\mathrm{d}t + v_t\,\mathrm{d}W_t, \quad t\in[0,T], \\ u_{T_1} &= G, \end{aligned} \tag{4.24}$$

admits a unique solution

$$(u,v) \in (L^2_{\mathcal{F}^W}([0,T];W^{2,2}(\mathbb{R})) \cap S^2_{\mathcal{F}^W}([0,T];W^{1,2}(\mathbb{R}))) \times L^2_{\mathcal{F}^W}([0,T];W^{1,2}(\mathbb{R})),$$

i.e. for any $\varphi \in C_c^\infty(\mathbb{R})$ *the equality*

$$\begin{aligned} \langle u_t,\varphi\rangle_{L^2(\mathbb{R})} = \langle G,\varphi\rangle_{L^2(\mathbb{R})} &+ \int_t^{T_1} \Big\langle \frac{\sigma_s^2+\nu^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}u_s - (k*\rho_s)\frac{\mathrm{d}}{\mathrm{d}x}u_s + \nu\frac{\mathrm{d}}{\mathrm{d}x}v_s, \varphi\Big\rangle_{L^2(\mathbb{R})}\,\mathrm{d}s \\ &- \int_t^{T_1} \langle v_s,\varphi\rangle_{L^2(\mathbb{R})}\,\mathrm{d}W_s \end{aligned}$$

holds for all $t \in [0,T]$ *with probability one. Moreover, we have*

$$u_0(\overline{Y}_0) = \mathbb{E}(G(\overline{Y}_{T_1}) \,|\, \sigma(\sigma(\overline{Y}_0), \mathcal{F}_0^W)), \tag{4.25}$$

where $(\overline{Y}_t, t\in[0,T])$ *is the solution of the linearised SDE* (4.23) *in the proof of Theorem 4.15.*

PROOF. Recall, by Theorem 4.15 we have

$$\rho \in L^2_{\mathcal{F}}([0,T];W^{3,2}(\mathbb{R})) \cap S^2_{\mathcal{F}}([0,T];W^{2,2}(\mathbb{R})).$$

Our approach is to verify the assumptions of the L^2-theory (see for example [DQT12, Theorem 5.5]) for BSPDEs. Let $u_1,u_2 \in W^{2,2}(\mathbb{R})$, then

$$\begin{aligned} \Big\|(k*\rho_t)\frac{\mathrm{d}}{\mathrm{d}x}u_1 - (k*\rho_t)\frac{\mathrm{d}}{\mathrm{d}x}u_2\Big\|_{L^2(\mathbb{R})} &\le \|k*\rho_t\|_{L^\infty(\mathbb{R})}\Big\|\frac{\mathrm{d}}{\mathrm{d}x}(u_1-u_2)\Big\|_{L^2(\mathbb{R})} \\ &\le \|k\|_{L^2(\mathbb{R})}\|\rho_t\|_{L^2(\mathbb{R})}\Big\|\frac{\mathrm{d}}{\mathrm{d}x}(u_1-u_2)\Big\|_{L^2(\mathbb{R})} \\ &\le \|k\|_{L^2(\mathbb{R})}\|\rho_t\|_{L^2(\mathbb{R})}\|u_1-u_2\|_{W^{1,2}(\mathbb{R})}. \end{aligned}$$

Now, by Theorem 4.6, $\|\rho_t\|_{L^2(\mathbb{R})}$ is uniformly bounded in $(\omega, t) \in \Omega \times [0,T]$ and the interpolation theorem [AF03, Theorem 5.2] implies for all $\varepsilon > 0$,

$$\begin{aligned}&\left\|(k * \rho_t)\frac{\mathrm{d}}{\mathrm{d}x}u_1 - (k * \rho_t)\frac{\mathrm{d}}{\mathrm{d}x}u_2\right\|_{L^2(\mathbb{R})} \\ &\quad \le \varepsilon \|u_1 - u_2\|_{W^{2,2}(\mathbb{R})} + C(k, \|\rho\|_{S^\infty_{\mathcal{F}^W}([0,T];L^2(\mathbb{R}))})\kappa(\varepsilon)\|u_1 - u_2\|_{L^2(\mathbb{R})}\end{aligned}$$

for some non-negative decreasing function κ. Hence, Assumption 5.4 in [DQT12, Theorem 5.5] is satisfied. The other assumptions are also easily verified. As a result we obtain a solution

$$(u,v) \in (L^2_{\mathcal{F}^W}([0,T];W^{2,2}(\mathbb{R})) \cap S^2_{\mathcal{F}^W}([0,T];L^2(\mathbb{R}))) \times L^2_{\mathcal{F}^W}([0,T];W^{1,2}(\mathbb{R}))$$

of the BSPDE (4.24). Here, the fact that $u \in S^2_{\mathcal{F}^W}([0,T];L^2(\mathbb{R}))$ is a direct consequence of [DM10, Theorem 2.2].

It remains to show that the equality (4.25) holds. By the bound

$$\mathbb{E}\left(\sup_{t\le T_1}\|\rho_t\|^2_{W^{1,2}(\mathbb{R})}\right) < \infty$$

given by $\rho \in S^2_{\mathcal{F}^W}([0,T];W^{1,2}(\mathbb{R}))$, we observe that there exists a set Ω' with $\mathbb{P}(\Omega') = 1$ and for all $\omega \in \Omega'$ we have

$$\sup_{t\le T_1}\|\rho_t(\omega,\cdot)\|_{W^{1,2}(\mathbb{R})} < \infty. \tag{4.26}$$

Also, the map $(\omega,t) \to \|\rho_t(\omega,\cdot)\|_{W^{1,2}(\mathbb{R})}$ is predictable with respect to $\mathcal{F}^W$ by the L^2-SPDE theory. Consequently, we can define for each $m \in \mathbb{N}$ the stopping time

$$\tau_m(\omega) = \inf\{t \in [0,T_1] : \|\rho_t(\omega,\cdot)\|_{W^{1,2}(\mathbb{R})} \ge m\}$$

and $\tau_m \uparrow T_1$ by (4.26). Furthermore, let us define

$$F(t,x) := (k * \rho_t)(x)\frac{\mathrm{d}}{\mathrm{d}x}u_t(x) \quad \text{and} \quad F_m(t,x) := F(t,x)\mathbb{1}_{(0,\tau_m]}(t),$$

and note that $F_m \in L^2_{\mathcal{F}^W}([0,T];L^2(\mathbb{R}))$ still satisfies all assumptions of the L^2-BSPDE theory ([DQT12, Theorem 5.5]) and therefore there exists a solution

$$(u^m,v^m) \in (L^2_{\mathcal{F}^W}([0,T];W^{2,2}(\mathbb{R})) \cap S^2_{\mathcal{F}^W}([0,T];L^2(\mathbb{R}))) \times L^2_{\mathcal{F}^W}([0,T];W^{1,2}(\mathbb{R}))$$

of the following BSPDE

$$\begin{aligned}\mathrm{d}u^m_t &= -\left(\frac{\sigma_t^2+\nu^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}u^m_t - F_m(t) + \nu\frac{\mathrm{d}}{\mathrm{d}x}v^m_t\right)\mathrm{d}t + v^m_t\,\mathrm{d}W_t, \\ u^m_{T_1} &= G,\end{aligned}$$

for each $m \in \mathbb{N}$.

In the next step we want to obtain a $L^2_{\mathcal{F}W}([0,T];W^{1,2}(\mathbb{R}))$-bound for F_m. The L^2-estimate follows directly from the above computations. For the weak derivative we compute

$$\begin{aligned}
&\left\| \frac{\mathrm{d}}{\mathrm{d}x}\left((k*\rho_t)\frac{\mathrm{d}}{\mathrm{d}x}u_t\right)\mathbb{1}_{(0,\tau_m]}\right\|_{L^2(\mathbb{R})} \\
&\quad\le \left\|\mathbb{1}_{(0,\tau_m]}\left(k*\frac{\mathrm{d}}{\mathrm{d}x}\rho_t\right)\frac{\mathrm{d}}{\mathrm{d}x}u_t\right\|_{L^2(\mathbb{R})} + \left\|(k*\rho_t)\frac{\mathrm{d}^2}{\mathrm{d}x^2}u_t\right\|_{L^2(\mathbb{R})} \\
&\quad\le \mathbb{1}_{(0,\tau_m]}\left\|\left(k*\frac{\mathrm{d}}{\mathrm{d}x}\rho_t\right)\frac{\mathrm{d}}{\mathrm{d}x}u_t\right\|_{L^2(\mathbb{R})} + \|k\|_{L^2(\mathbb{R})}\|\rho_t\|_{S^\infty_{\mathcal{F}}([0,T];L^2(\mathbb{R})))}\|u_t\|_{W^{2,2}(\mathbb{R})}.
\end{aligned}$$

Since $\rho \in S^\infty_{\mathcal{F}W}([0,T];L^2(\mathbb{R})))$, the last term behaves nicely. However, the first term would be problematic because without the stopping time we do not have a similar L^∞-estimate for the derivative, i.e. $\rho \in S^\infty_{\mathcal{F}W}([0,T];W^{1,2}(\mathbb{R})))$. Hence, in order to overcome this problem we introduced the stopping time τ_m and, therefore, we discover

$$\begin{aligned}
&\left\| \frac{\mathrm{d}}{\mathrm{d}x}\left((k*\rho_t)\frac{\mathrm{d}}{\mathrm{d}x}u_t\right)\mathbb{1}_{(0,\tau_m]}\right\|_{L^2_{\mathcal{F}W}([0,T_1];L^2(\mathbb{R}))} \\
&\quad\le \|k\|^2_{L^2(\mathbb{R})}\,\mathbb{E}\left(\int_0^{T_1}\mathbb{1}_{(0,\tau_m]}(t)\left\|\frac{\mathrm{d}}{\mathrm{d}x}\rho_t\right\|^2_{L^2(\mathbb{R})}\left\|\frac{\mathrm{d}}{\mathrm{d}x}u_t\right\|^2_{L^2(\mathbb{R})}\,\mathrm{d}t\right) \\
&\qquad + \|k\|_{L^2(\mathbb{R})}\|\rho_t\|_{S^\infty_{\mathcal{F}}([0,T];L^2(\mathbb{R})))}\|u_t\|_{L^2_{\mathcal{F}W}([0,T];W^{2,2}(\mathbb{R}))} \\
&\quad\le \|k\|_{L^2(\mathbb{R})}\,m^2\,\|u\|_{L^2_{\mathcal{F}W}([0,T];W^{1,2}(\mathbb{R}))} \\
&\qquad + \|k\|_{L^2(\mathbb{R})}\|\rho_t\|_{S^\infty_{\mathcal{F}}([0,T];L^2(\mathbb{R})))}\|u_t\|_{L^2_{\mathcal{F}W}([0,T];W^{2,2}(\mathbb{R}))}.
\end{aligned}$$

As a result, we obtain

$$\|F_m\|_{L^2_{\mathcal{F}W}([0,T];W^{1,2}(\mathbb{R}))} < \infty$$

for each $m \in \mathbb{N}$. Applying [DQT12, Theorem 5.5] again, we find

$$(u^m, v^m) \in (L^2_{\mathcal{F}W}([0,T];W^{3,2}(\mathbb{R})) \cap S^2_{\mathcal{F}W}([0,T];W^{1,2}(\mathbb{R}))) \times L^2_{\mathcal{F}W}([0,T];W^{2,2}(\mathbb{R})).$$

The above regularity ($p(m-2) > 1$ with $p = 2, m = 3$) allows us to apply [DTZ13, Corollary 2.2], which tells us that there exists a set of full measure Ω''_m maybe different from Ω' such that

$$\begin{aligned}
u^m(t,x) = G(x) + &\int_t^{T_1} \frac{\sigma_t^2+\nu^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}u^m(s,x) - \mathbb{1}_{(0,\tau_m]}(s)(k*\rho)(s,x)\frac{\mathrm{d}}{\mathrm{d}x}u(s,x) \\
&+ \nu\frac{\mathrm{d}}{\mathrm{d}x}v^m(s,x)\,\mathrm{d}s - \int_t^{T_1} v^m(s,x)\,\mathrm{d}W_s
\end{aligned}$$

holds for all $(t,x) \in [0,T_1]\times\mathbb{R}$ on Ω''_m. We use the subscript m to indicate that even though the set Ω''_m is independent of (t,x) it still may depend on $m \in \mathbb{N}$.

Besides, to be precise, [Du20, Corollary 2.2] requires $F_m \in L^2_{\mathcal{F}W}([0,T];W^{3,2}(\mathbb{R}))$, which is more regularity than we have. However, one can modify the proof of [Du20, Corollary 2.2] to obtain the same result with $F_m \in L^2_{\mathcal{F}^W}([0,T];W^{1,2}(\mathbb{R}))$. The crucial part is that a mollification of F_m with the standard mollifier converges in the supremum norm to F_m, which follows from Morrey's inequality even in our case $F_m \in L^2_{\mathcal{F}^W}([0,T];W^{1,2}(\mathbb{R}))$. For a similar result in the SPDE setting we refer to [Roz90, Lemma 4.1].

Next, we want to apply an Itô–Wentzell type formula [YT13, Theorem 3.1] (with $V = u$, $X = \overline{Y}$ therein). Hence, we need to verify the required assumption. First, we can view u_m as a jointly continuous Itô process in (t,x) by [DTZ13, Corollary 2.2] on the set Ω''_m. We also recall that $u^m \in L^2_{\mathcal{F}^W}([0,T];W^{2,2}(\mathbb{R}))$, $v^m \in L^2_{\mathcal{F}^W}([0,T];W^{1,2}(\mathbb{R}))$, $F_m \in L^2_{\mathcal{F}^W}([0,T];L^2(\mathbb{R}))$ and $\overline{Y}$ is a strong solution and therefore a continuous semimartingale.

Moreover, we note that $\rho, \frac{\mathrm{d}}{\mathrm{d}x}u$ are $\mathcal{P}^W \times \mathcal{B}(\mathbb{R})$-measurable and τ_m is $\mathcal{P}^W$-measurable. Hence, the same holds true for F_m. Also as previously mentioned $(k*\rho_t)$ is bounded in $x \in \mathbb{R}$ for almost all $(\omega,t) \in \Omega \times [0,T_1]$ and as we have seen in the proof of Theorem 4.15 (Step 1) is Lipschitz continuous for almost all $(\omega,t) \in \Omega \times [0,T_1]$. Thus, all assumptions of [YT13, Theorem 3.1] are fulfilled and we obtain

$$
\begin{aligned}
&u^m_{T_1}(\overline{Y}_{T_1}) \\
&\quad = u^m_0(\overline{Y}_0) + \int_0^{T_1} \Big(\frac{\sigma_t^2+\nu^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}u^m_t - (k*\rho_t)\frac{\mathrm{d}}{\mathrm{d}x}u_t + \nu\frac{\mathrm{d}}{\mathrm{d}x}v^m_t - \frac{\sigma_t^2+\nu^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}u^m_t \\
&\qquad + \mathbb{1}_{(0,\tau_m]}(t)(k*\rho_t)\frac{\mathrm{d}}{\mathrm{d}x}u_t - \nu\frac{\mathrm{d}}{\mathrm{d}x}v^m_t\Big)(\overline{Y}_t)\,\mathrm{d}t \\
&\qquad + \int_0^{T_1}\Big(v^m_t + \nu\frac{\mathrm{d}}{\mathrm{d}x}u^m_t\Big)(\overline{Y}_t)\,\mathrm{d}W_t + \int_0^{T_1}\sigma_t(\overline{Y}_t)\frac{\mathrm{d}}{\mathrm{d}x}u^m_t(\overline{Y}_t)\,\mathrm{d}B_t \\
&\quad = u^m_0(\overline{Y}_0) + \int_0^{T_1} F(t,\overline{Y}_t)(\mathbb{1}_{(0,\tau_m]}(t)-1)\,\mathrm{d}t + \int_0^{T_1}\Big(v^m_t + \nu\frac{\mathrm{d}}{\mathrm{d}x}u^m_t\Big)(\overline{Y}_t)\,\mathrm{d}W_t \\
&\qquad + \int_0^{T_1}\sigma_t(\overline{Y}_t)\frac{\mathrm{d}}{\mathrm{d}x}u^m_t(\overline{Y}_t)\,\mathrm{d}B_t. \qquad (4.27)
\end{aligned}
$$

With this formula at hand, let us introduce the filtration $\mathcal{G}_t = \sigma(\sigma(\overline{Y}_t), \mathcal{F}^W_t)$, $t \in [0,T]$. Our aim is to take the conditional expectation with respect to $\mathcal{G}_0$ on both sides of the equation in order to cancel the stochastic integrals. We observe that $\mathcal{G}_t \subset \mathcal{F}_t$ and the solution $(\overline{Y}_t, t \in [0,T])$ is predictable with respect to the filtration $\mathcal{G}$. Moreover, B^1 and W are still per definition Brownian motions under the filtration $(\mathcal{F}_t, t \in [0,T])$. Hence, both stochastic integrals are martingales with respect to the filtration $(\mathcal{F}_t, t \in [0,T])$, if we can prove an L^2-bound on the integrands.

By Sobolev's embedding or Morrey's inequality and the bound on σ_t we have

$$
\mathbb{E}\Big(\int_0^{T_1}\Big|\sigma_t(\overline{Y}_t)\frac{\mathrm{d}}{\mathrm{d}x}u^m_t(\overline{Y}_t)\Big|^2\,\mathrm{d}t\Big) \le \Lambda\mathbb{E}\Big(\int_0^{T_1}\Big\|\frac{\mathrm{d}}{\mathrm{d}x}u^m_t\Big\|^2_{L^\infty(\mathbb{R})}\,\mathrm{d}t\Big)
$$

$$\leq \Lambda \mathbb{E}\bigg(\int_0^{T_1} \|u_t^m\|_{W^{2,2}(\mathbb{R})}^2 \,\mathrm{d}t \bigg) < \infty,$$

which verifies that the second stochastic integral of (4.27) is a martingale with respect to the filtration $\mathcal{F}$. Hence, we discover

$$\mathbb{E}\bigg(\int_0^{T_1} \sigma_t(\overline{Y}_t) \frac{\mathrm{d}}{\mathrm{d}x} u_t^m(\overline{Y}_t) \,\mathrm{d}B_t \,\big|\, \mathcal{G}_0 \bigg) = \mathbb{E}\bigg(\mathbb{E}\bigg(\int_0^{T_1} \frac{\mathrm{d}}{\mathrm{d}x} u_t^m(\overline{Y}_t) \,\mathrm{d}B_t \,\big|\mathcal{F}_0 \bigg) \,\big|\, \mathcal{G}_0 \bigg) = 0.$$

Furthermore, we have the estimate

$$\begin{aligned} \mathbb{E}\bigg(\int_0^{T_1} \Big| v_t^m(\overline{Y}_t) + \nu \frac{\mathrm{d}}{\mathrm{d}x} u_t^m(\overline{Y}_t) \Big|^2 \,\mathrm{d}t \bigg) &\leq 2\mathbb{E}\bigg(\int_0^{T_1} \|v_t^m\|_{L^\infty(\mathbb{R})}^2 + \nu^2 \Big\| \frac{\mathrm{d}}{\mathrm{d}x} u_t^m \Big\|_{L^\infty(\mathbb{R})}^2 \,\mathrm{d}t \bigg) \\ &\leq C\mathbb{E}\bigg(\int_0^{T_1} \|v_t^m\|_{W^{1,2}(\mathbb{R})}^2 + \|u_t^m\|_{W^{2,2}(\mathbb{R})}^2 \,\mathrm{d}t \bigg) \\ &< \infty, \end{aligned}$$

where we used Morrey's inequality in the second step. Hence, the first stochastic integral appearing in (4.27) is also a martingale with respect to the filtration $\mathcal{G}$ starting at zero. Taking the conditional expectation with respect to $\mathcal{G}_0$ in (4.27) and having in mind that $\overline{Y}_0 = Y_0$, we obtain

$$\mathbb{E}(u_{T_1}^m(\overline{Y}_{T_1}) \,|\, \mathcal{G}_0) = u_0^m(Y_0) + \mathbb{E}\bigg(\int_0^{T_1} F(t, \overline{Y}_t)(\mathbb{1}_{(0,\tau_m]}(t) - 1) \,\mathrm{d}t \,\bigg|\, \mathcal{G}_0 \bigg).$$

It remains to show that

(4.28) $$\lim_{m\to\infty} (\mathbb{E}(u_{T_1}^m(\overline{Y}_{T_1}) \,|\, \mathcal{G}_0) - u_0^m(Y_0)) = \mathbb{E}(u_{T_1}(\overline{Y}_{T_1}) \,|\, \mathcal{G}_0) - u_0(Y_0), \quad \mathbb{P}\text{-a.e.},$$

and

(4.29) $$\lim_{m\to\infty} \mathbb{E}\bigg(\int_0^{T_1} F(t, \overline{Y}_t)(\mathbb{1}_{(0,\tau_m]}(t) - 1) \,\mathrm{d}t \,\bigg|\, \mathcal{G}_0 \bigg) = 0, \quad \mathbb{P}\text{-a.e.}$$

We first show (4.29). However, we prove L^1-convergence, which then implies (4.29) along a subsequence. We compute

$$\begin{aligned}
&\mathbb{E}\bigg(\bigg|\mathbb{E}\bigg(\int_0^{T_1} F(t,\overline{Y}_t)(\mathbb{1}_{(0,\tau_m]}(t)-1)\,\mathrm{d}t\,\bigg|\,\mathcal{G}_0\bigg)\bigg|\bigg)\\
&\le \mathbb{E}\bigg(\mathbb{E}\bigg(\bigg|\int_0^{T_1} F(t,\overline{Y}_t)(\mathbb{1}_{(0,\tau_m]}(t)-1)\,\mathrm{d}t\bigg|\,\bigg|\,\mathcal{G}_0\bigg)\bigg)\\
&\le \mathbb{E}\bigg(\int_0^{T_1} |F(t,\overline{Y}_t)(\mathbb{1}_{(0,\tau_m]}(t)-1)|\,\mathrm{d}t\bigg)\\
&\le \mathbb{E}\bigg(\int_0^{T_1} \|F(t,\cdot)\|_{L^\infty(\mathbb{R})}\,|\mathbb{1}_{(0,\tau_m]}(t)-1|\,\mathrm{d}t\bigg)\\
&\le \mathbb{E}\bigg(\int_0^{T_1} \|F(t,\cdot)\|_{W^{1,2}(\mathbb{R})}\,|\mathbb{1}_{(0,\tau_m]}(t)-1|\,\mathrm{d}t\bigg)\\
&\le C\mathbb{E}\bigg(\int_0^{T_1}\bigg(\bigg\|(k*\rho_t)\frac{\mathrm{d}}{\mathrm{d}x}u_t\bigg\|_{L^2(\mathbb{R})}+\bigg\|(k*\rho_t)\frac{\mathrm{d}^2}{\mathrm{d}x^2}u_t\bigg\|_{L^2(\mathbb{R})}\\
&\quad+\bigg\|(k*\frac{\mathrm{d}}{\mathrm{d}x}\rho_t)\frac{\mathrm{d}}{\mathrm{d}x}u_t\bigg\|_{L^2(\mathbb{R})}\bigg)|\mathbb{1}_{(0,\tau_m]}(t)-1|\,\mathrm{d}t\bigg)\\
&\le C(T)\mathbb{E}\bigg(\int_0^{T_1}\bigg(2\,\|\rho_t\|_{L^2(\mathbb{R})}+\bigg\|\frac{\mathrm{d}}{\mathrm{d}x}\rho_t\bigg\|_{L^2(\mathbb{R})}\bigg)\,\|u_t\|_{W^{2,2}(\mathbb{R})}\,|\mathbb{1}_{(0,\tau_m]}(t)-1|\,\mathrm{d}t\bigg)\\
&\le C(T)\mathbb{E}\bigg(\int_0^{T_1}(\|\rho_t\|^2_{W^{1,2}(\mathbb{R})}+\|u_t\|^2_{W^{2,2}(\mathbb{R})})|\mathbb{1}_{(0,\tau_m]}(t)-1|\,\mathrm{d}t\bigg),
\end{aligned}$$

where we used Morrey's inequality in the fourth step and Hölder's inequality as well as (4.17) in the sixth step. Finally, $\rho \in L^2_{\mathcal{F}W}([0,T];W^{1,2}(\mathbb{R}))$, $u \in L^2_{\mathcal{F}W}([0,T];W^{2,2}(\mathbb{R}))$, dominated convergence theorem and $\tau_m \uparrow T_1$ tell us that the last term vanishes for $m \to \infty$.

Taking the above subsequence, which we do not rename, we demonstrate (4.28) along a further subsequence by proving L^2-convergence of (4.28). Let us define $\widetilde{u}^m = u - u^m$ and $\widetilde{v}^m = v - v^m$, which solve the following BSPDE

$$\mathrm{d}\widetilde{u}^m_t = -\bigg(\frac{\sigma_t^2+\nu^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\widetilde{u}^m_t - \widetilde{F}_m + \nu\frac{\mathrm{d}}{\mathrm{d}x}\widetilde{v}^m_t\bigg)\,\mathrm{d}t + \widetilde{v}^m_t\,\mathrm{d}W_t$$

with terminal condition $\widetilde{G} = 0$, free term

$$\widetilde{F}_m(t,x) = (k*\rho)(t,x)\frac{\mathrm{d}}{\mathrm{d}x}u(t,x)(1-\mathbb{1}_{(0,\tau_m]}(t)) = F(t,x)(1-\mathbb{1}_{(0,\tau_m]}(t))$$

and $\widetilde{F}_m \in L^2_{\mathcal{F}^W}([0,T];L^2(\mathbb{R}))$. Hence, by [DQT12, Theorem 5.5] the solution of the BSPDE is unique and by [DM10, Proposition 3.2 and Proposition 4.1, Step 1] satisfies the estimate

$$\mathbb{E}\bigg(\sup_{t\le T_1}\|\widetilde{u}^m_t\|^2_{W^{1,2}(\mathbb{R})}\bigg) \le C(T)\left\|\widetilde{F}_m\right\|_{L^2_{\mathcal{F}^W}([0,T];L^2(\mathbb{R}))}. \tag{4.30}$$

Consequently, using Jensen inequality, the $\mathcal{G}_0$-measurability of $u_0(Y_0)$, Morrey's inequality and (4.30) we find

$$\begin{aligned}
&\mathbb{E}(|\mathbb{E}(u^m_{T_1}(\overline{Y}_{T_1}) \,|\, \mathcal{G}_0) - u^m_0(Y_0)) - (\mathbb{E}(u_{T_1}(\overline{Y}_{T_1}) \,|\, \mathcal{G}_0) - u_0(Y_0))|^2) \\
&\quad\le 2\mathbb{E}(|\widetilde{u}^m_{T_1}(\overline{Y}_{T_1})|^2 + |\widetilde{u}^m_0(Y_0)|^2) \\
&\quad\le 2\mathbb{E}(\left\|\widetilde{u}^m_{T_1}\right\|^2_{L^\infty(\mathbb{R})} + \|\widetilde{u}^m_0\|^2_{L^\infty(\mathbb{R})}) \\
&\quad\le 2\mathbb{E}(\left\|\widetilde{u}^m_{T_1}\right\|^2_{W^{1,2}(\mathbb{R})} + \|\widetilde{u}^m_0\|^2_{W^{1,2}(\mathbb{R})}) \\
&\quad\le 4\mathbb{E}\bigg(\sup_{t\le T_1}\|\widetilde{u}^m_t\|^2_{W^{1,2}(\mathbb{R})}\bigg) \\
&\quad\le C(T)\mathbb{E}\bigg(\int_0^{T_1}\left\|(\widetilde{F}_m)_t\right\|^2_{L^2(\mathbb{R})}\,\mathrm{d}t\bigg) \\
&\quad\le C(T)\mathbb{E}\bigg(\int_0^{T_1}|1-\mathbb{1}_{(0,\tau_m]}(t)|^2\,\|F(t,x)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}t\bigg).
\end{aligned}$$

But $F \in L^2_{\mathcal{F}^W}([0,T];L^2(\mathbb{R}))$ and, therefore, an application of the dominated convergence theorem proves (4.28) along a subsequence. As a result, the last inequality together with (4.29) implies (4.25) for all $\omega \in \widetilde{\Omega} := \bigcap_{m\in\mathbb{N}}\Omega''_m \cap \Omega'$. Hence, the lemma is proven. □

Utilizing the same calculations, we obtain the same result for the regularized mean-field SDE (4.10).

Corollary 4.18. *Let Assumption 4.1 as well as Assumption 4.13 hold and $k \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then, for any $\varepsilon > 0$ we obtain a unique strong solution $(Y^\varepsilon_t, t \in [0,T])$ of the mean-field SDE* (4.10)*, and ρ^ε_t is the conditional density of Y^ε_t given $\mathcal{F}^W_t$ for every $t \in [0,T]$.*

As an application of Theorem (4.15) to the HK model, we obtain a solution to the non-regularized and the regularized conditional mean-field SDEs (4.2) and (4.5), respectively.

Corollary 4.19. *Let $0 \le \rho_0 \in L^1(\mathbb{R}) \cap W^{2,2}(\mathbb{R})$ with $\|\rho_0\|_{L^1(\mathbb{R})} = 1$. Suppose that for $T > 0$ there exist unique solutions ρ^{HK} and $\rho^{\varepsilon,HK}$ in $L^2_{\mathcal{F}^W}([0,T];W^{1,2}(\mathbb{R}))$ of the SPDEs* (4.3) *and* (4.6)*, respectively, on the interval $[0,T]$ with*

$$\begin{aligned}
&\|\rho^{HK}\|_{L^2_{\mathcal{F}^w}([0,T];W^{1,2}(\mathbb{R}))} + \|\rho^{HK}\|_{S^\infty_{\mathcal{F}^w}([0,T];L^1(\mathbb{R})\cap L^2(\mathbb{R}))} \le C \quad \textit{and} \\
&\|\rho^{\varepsilon,HK}\|_{L^2_{\mathcal{F}^w}([0,T];W^{1,2}(\mathbb{R}))} + \|\rho^{\varepsilon,HK}\|_{S^\infty_{\mathcal{F}^w}([0,T];L^1(\mathbb{R})\cap L^2(\mathbb{R}))} \le C,
\end{aligned}$$

for some finite constant $C > 0$. Moreover, let the diffusion coefficient $\sigma\colon [0,T]\times\mathbb{R}\to\mathbb{R}$ satisfy Assumption 4.1. Then, for $\varepsilon > 0$ there exists unique solution $(Y^{i,HK}, t \in [0,T])$ and

$(Y^{i,HK,\varepsilon}, t \in [0,T])$ *for the mean-field SDEs* (4.2) *and* (4.5)*, respectively. Moreover,* ρ_t *is the conditional density of* $Y_t^{i,HK}$ *given* $\mathcal{F}_t^W$ *and* $\rho_t^{\varepsilon,HK}$ *is the conditional density of* $Y_t^{i,HK,\varepsilon}$ *given* $\mathcal{F}_t^W$*, for every* $t \in [0,T]$ *and for all* $i \in \mathbb{N}$.

REMARK 4.20. *Because* $k_{HK} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *for the HK model, we can use Remark 4.14 to verify the estimates in Corollary 4.19.*

4.5. Mean-field limits of the interacting particle system with common noise

In this section we establish conditional propagation of chaos for the regularized interacting particle system with common noise (4.8) towards the mean-field stochastic differential equations (4.9), respectively, the (non-regularized) stochastic Fokker–Planck equation (4.11). To prove conditional propagation of chaos, we first derive estimates of the difference of the regularized interacting particle system (4.8) and the regularized mean-field SDE (4.10) (see Proposition 4.22) as well as of the difference of the solutions to the regularized stochastic Fokker–Planck equation (4.12) and of the non-regularized stochastic Fokker–Planck equation (4.11) (see Proposition 4.23). In particular the results hold for the HK model, where $k^{\varepsilon}_{HK}(x)$ is the approximation of $k_{HK}(x) = \mathbb{1}_{[-R,R]}(x)x$. As preparation, we need the following auxiliary lemma.

LEMMA 4.21. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space,* $\mathcal{G} \subseteq \mathcal{F}$ *a sub-*σ*-algebra and* X, Y *conditionally independent random variables with values in* $\mathbb{R}$ *given* $\mathcal{G}$*. Moreover, let* X *have a conditional density* $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ *such that* f *is* $\mathcal{G} \otimes \mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$ *measurable and in* $L^1(\Omega \times \mathbb{R})$*. Then, for every bounded measurable function* $h\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$*, we have*

$$\mathbb{E}(h(X,Y) \,|\, \sigma(\mathcal{G}, \sigma(Y)))(\omega) = \int_{\mathbb{R}} h(z, Y(\omega)) f(\omega, z) \,\mathrm{d}z, \quad \omega \in \Omega', \tag{4.31}$$

on a set $\Omega' \subset \Omega$ *of full probability.*

PROOF. First, we notice that by the measurability statement of Fubini's theorem, the right-hand side of (4.31) is $\sigma(\mathcal{G}, \sigma(Y))$-measurable. By the standard Lebesgue integral approximation technique we may assume $h = \mathbb{1}_{B \times B'}(x,y)$ for some measurable sets $B, B' \in \mathcal{B}(\mathbb{R})$ in order to prove (4.31). Hence, we need to show

$$\mathbb{E}(\mathbb{1}_A \mathbb{1}_{B \times B'}(X,Y)) = \mathbb{E}\bigg(\mathbb{1}_A \int_{\mathbb{R}} \mathbb{1}_{B \times B'}(z, Y(\omega)) f(\omega, z) \,\mathrm{d}z\bigg)$$

for all $A \in \sigma(\mathcal{G}, \sigma(Y))$. Now, we reduce the problem again to $A = C \cap C''$ with $C \in \mathcal{G}$ and $C'' = Y^{-1}(B'')$ for some $B'' \in \mathcal{B}(\mathbb{R})$. Consequently, using the conditional independence we find

$$\begin{aligned}
\mathbb{E}(\mathbb{1}_{C\cap C''}\mathbb{1}_{B\times B'}(X,Y)) &= \mathbb{E}(\mathbb{1}_C\mathbb{E}(\mathbb{1}_{C''}\mathbb{1}_{B\times B'}(X,Y)\,|\,\mathcal{G}))\\
&= \mathbb{E}(\mathbb{1}_C\mathbb{E}(\mathbb{1}_{B''\cap B'}(Y)\mathbb{1}_B(X)\,|\,\mathcal{G}))\\
&= \mathbb{E}(\mathbb{1}_C\mathbb{E}(\mathbb{1}_{B''\cap B'}(Y)\,|\,\mathcal{G})\mathbb{E}(\mathbb{1}_B(X)\,|\,\mathcal{G}))\\
&= \mathbb{E}(\mathbb{1}_C\mathbb{1}_{B''\cap B'}(Y)\mathbb{E}(\mathbb{1}_B(X)\,|\,\mathcal{G}))\\
&= \mathbb{E}\bigg(\mathbb{1}_{C\cap C''}(\omega)\mathbb{1}_{B'}(Y(\omega))\int_{\mathbb{R}}\mathbb{1}_B(z)f(\omega,z)\,\mathrm{d}z\bigg)\\
&= \mathbb{E}\bigg(\mathbb{1}_{C\cap C''}(\omega)\int_{\mathbb{R}}\mathbb{1}_{B\times B'}(z,Y(\omega))f(\omega,z)\,\mathrm{d}z\bigg)
\end{aligned}$$

and the lemma is proven. □

The next proposition provides an estimate of the difference of the regularized particle system and the regularized mean-field SDE.

PROPOSITION 4.22. *Suppose Assumption 4.1 and Assumption 4.13 hold. For each $N\in\mathbb{N}$, let $((Y_t^{i,\varepsilon},t\in[0,T]),i=1,\dots,N)$ be the solutions to the regularized mean-field SDEs* (4.10), *as provided by Corollary 4.19, and let $((X_t^{i,\varepsilon},t\in[0,T]),i=1,\dots,N)$ be the solution to regularized interaction particle system* (4.8). *Then, for any $\varepsilon>0$ and $N\in\mathbb{N}$ we have*

$$\sup_{t\in[0,T]}\sup_{i=1,\dots,N}\mathbb{E}(|X_t^{i,\varepsilon}-Y_t^{i,\varepsilon}|^2)\le\frac{2\,\|k\|_{L^2(\mathbb{R})}^2\,T}{(N-1)\varepsilon}\exp\bigg(\frac{(C+\Lambda)T}{\varepsilon}\bigg),$$

where C is some finite generic constant.

PROOF. Applying Itô's formula (1.15), we find

$$\begin{aligned}
&|X_t^{i,\varepsilon}-Y_t^{i,\varepsilon}|^2\\
&= 2\int_0^t(X_s^{i,\varepsilon}-Y_s^{i,\varepsilon})\bigg(\frac{1}{N}\sum_{j=1}^N-k^\varepsilon(X_s^{i,\varepsilon}-X_s^{j,\varepsilon})+(k^\varepsilon*\rho_s^\varepsilon)(Y_s^{i,\varepsilon})\bigg)\,\mathrm{d}s\\
&\quad+2\int_0^t(X_s^{i,\varepsilon}-Y_s^{i,\varepsilon})(\sigma(s,X_s^{i,\varepsilon})-\sigma(s,Y_s^{i,\varepsilon}))\,\mathrm{d}B_s^i+\int_0^t(\sigma(s,X_s^{i,\varepsilon})-\sigma(s,Y_s^{i,\varepsilon}))^2\,\mathrm{d}s.
\end{aligned}\tag{4.32}$$

Splitting the sum we have

$$\begin{aligned}
&\frac{1}{N}\sum_{j=1}^N-k^\varepsilon(X_s^{i,\varepsilon}-X_s^{j,\varepsilon})+(k^\varepsilon*\rho_s^\varepsilon)(Y_s^{i,\varepsilon})\\
&\quad=\frac{1}{N}\sum_{j=1}^N(k^\varepsilon*\rho_s^\varepsilon)(Y_s^{i,\varepsilon})-k^\varepsilon(Y_s^{i,\varepsilon}-Y_s^{j,\varepsilon})\\
&\qquad+\frac{1}{N}\sum_{j=1}^N k^\varepsilon(Y_s^{i,\varepsilon}-Y_s^{j,\varepsilon})-k^\varepsilon(X_s^{i,\varepsilon}-X_s^{j,\varepsilon})
\end{aligned}$$

$$= I_1^s + I_2^s.$$

For I_2^s, we use the property of our approximation sequence to discover

$$|I_2^s| \le \frac{1}{N}\sum_{j=1}^{N} |k^\varepsilon(Y_s^{i,\varepsilon} - Y_s^{j,\varepsilon}) - k^\varepsilon(X_s^{i,\varepsilon} - X_s^{j,\varepsilon})|$$

$$\le \frac{C}{N\varepsilon}\sum_{j=1}^{N} |X_s^{j,\varepsilon} - Y_s^{j,\varepsilon}| + |X_s^{i,\varepsilon} - Y_s^{i,\varepsilon}|$$

and consequently

$$\mathbb{E}\bigg(\bigg|2\int_0^t (X_s^{i,\varepsilon} - Y_s^{i,\varepsilon}) I_2^s \,\mathrm{d}s\bigg|\bigg) \le \frac{C}{N\varepsilon}\int_0^t \sum_{j=1}^{N} \mathbb{E}(|X_s^{j,\varepsilon} - Y_s^{j,\varepsilon}|^2 + |X_s^{i,\varepsilon} - Y_s^{i,\varepsilon}|^2)\,\mathrm{d}s$$

$$\le \frac{C}{\varepsilon}\int_0^t \sup_{i=1,\dots,N} \mathbb{E}(|X_s^{i,\varepsilon} - Y_s^{i,\varepsilon}|^2)\,\mathrm{d}s,$$

where we used Young's inequality. Next, let us rewrite I_1^s such that

$$I_1^s = \frac{1}{N}\sum_{j=1}^{N} (k^\varepsilon * \rho_s^\varepsilon)(Y_s^{i,\varepsilon}) - k^\varepsilon(Y_s^{i,\varepsilon} - Y_s^{j,\varepsilon}) = \frac{1}{N}\sum_{j=1}^{N} Z_{i,j}^s$$

with

$$Z_{i,j}^s = (k^\varepsilon * \rho_s^\varepsilon)(Y_s^{i,\varepsilon}) - k^\varepsilon(Y_s^{i,\varepsilon} - Y_s^{j,\varepsilon})$$

for $i \neq j$. Furthermore,

$$\mathbb{E}(|I_1^s|^2) = \frac{1}{N^2}\mathbb{E}\bigg(\mathbb{E}\bigg(\sum_{j=1}^{N} Z_{i,j}^s \sum_{l=1}^{N} Z_{i,l}^s \,\bigg|\, \sigma(\mathcal{F}_s^W, \sigma(Y_s^{i,\varepsilon}))\bigg)\bigg)$$

$$= \frac{1}{N^2}\sum_{j=1}^{N}\sum_{l=1}^{N} \mathbb{E}(\mathbb{E}(Z_{i,j}^s Z_{i,l}^s \,|\, \sigma(\mathcal{F}_s^W, \sigma(Y_s^{i,\varepsilon}))).$$

It easy to verify that $(Y_s^{i,\varepsilon}, i = 1, \dots, N)$ are conditionally independent given $\mathcal{F}_s^W$ and by Theorem 4.15 have conditional density ρ_s. Hence, we apply Lemma 4.21 to obtain

$$\mathbb{E}(k^\varepsilon(Y_s^{i,\varepsilon} - Y_s^{j,\varepsilon}) \,|\, \sigma(\mathcal{F}_s^W, \sigma(Y_s^i, \varepsilon))) = (k^\varepsilon * \rho_s^\varepsilon)(Y_s^{i,\varepsilon})$$

and therefore $\mathbb{E}(Z_{i,j}^s \,|\, \sigma(\mathcal{F}_s^W, \sigma(Y_s^{i,\varepsilon}))) = 0$, since $(k^\varepsilon * \rho_s^\varepsilon)(Y_s^{i,\varepsilon})$ is $\sigma(\mathcal{F}_s^W, \sigma(Y_s^{i,\varepsilon}))$-measurable. Consequently, for the cross terms $j \neq k$ one can verify that

$$\mathbb{E}(Z_{i,j}^s Z_{i,k}^s \,|\, \sigma(\mathcal{F}_s^W, \sigma(Y_s^{,\varepsilon}))) = \mathbb{E}(Z_{i,j}^s \,|\, \sigma(\mathcal{F}_s^W, \sigma(Y_s^{,\varepsilon})))\mathbb{E}(Z_{i,k}^s \,|\, \sigma(\mathcal{F}_s^W, \sigma(Y_s^{,\varepsilon}))) = 0$$

by the previous findings. Hence, we have

$$\mathbb{E}(|I_1^s|^2) = \frac{1}{N^2}\sum_{j=1}^{N} \mathbb{E}(|Z_{i,j}^s|^2)$$

and using the boundedness of k^ε, the structure of our approximation and mass conservation, we obtain

$$\mathbb{E}(|Z^s_{i,j}|^2) = \mathbb{E}(|(k^\varepsilon * \rho^\varepsilon_s)(Y^{i,\varepsilon}_s) - k^\varepsilon(Y^{i,\varepsilon}_s - Y^{j,\varepsilon}_s)|^2) \leq 2\,\|k^\varepsilon\|^2_{L^\infty(\mathbb{R})} \leq \frac{2}{\varepsilon}\,\|k\|^2_{L^2(\mathbb{R})}\,.$$

Combining all the above facts, we get

$$\mathbb{E}(|I^s_1|^2) \leq \frac{2\,\|k\|^2_{L^2(\mathbb{R})}}{N\varepsilon}$$

and

$$\begin{aligned}\mathbb{E}\bigg(2\int_0^t (X^{i,\varepsilon}_t - Y^{i,\varepsilon}_t) I^s_1 \,\mathrm{d}s\bigg) &\leq \mathbb{E}\bigg(\int_0^t |X^{i,\varepsilon}_t - Y^{i,\varepsilon}_t|^2 \,\mathrm{d}s + \int_0^t |I^s_1|^2 \,\mathrm{d}s\bigg)\\ &\leq \int_0^t \mathbb{E}(|X^{i,\varepsilon}_t - Y^{i,\varepsilon}_t|^2)\,\mathrm{d}s + \frac{2\,\|k\|^2_{L^2(\mathbb{R})}\,T}{N\varepsilon}.\end{aligned}$$

Moreover, using the Lipschitz continuity of our coefficients σ we obtain

$$\mathbb{E}\bigg(\int_0^t (\sigma(s, X^{i,\varepsilon}_s) - \sigma(s, Y^{i,\varepsilon}_s))^2 \,\mathrm{d}s\bigg) \leq \Lambda \int_0^t \mathbb{E}\big(|X^{i,\varepsilon}_s - Y^{i,\varepsilon}_s|^2\big) \leq \Lambda \int_0^t \sup_{i=1,\ldots,N} \mathbb{E}\big(|X^{i,\varepsilon}_s - Y^{i,\varepsilon}_s|^2\big).$$

Now, combining this with the estimate of I^s_2, as well as the fact that the stochastic integral in equation (4.32) is a martingale (Assumption 4.1), we obtain

$$\begin{aligned}\sup_{i=1,\ldots,N} \mathbb{E}(|X^{i,\varepsilon}_t - Y^{i,\varepsilon}_t|^2) &\leq \frac{C}{\varepsilon}\int_0^t \sup_{i=1,\ldots,N} \mathbb{E}(|X^{i,\varepsilon}_t - Y^{i,\varepsilon}_t|^2)\,\mathrm{d}s\\ &\qquad + \Lambda\int_0^t \sup_{i=1,\ldots,N} \mathbb{E}(|X^{i,\varepsilon}_t - Y^{i,\varepsilon}_t|^2)\,\mathrm{d}s + \frac{2\,\|k\|^2_{L^2(\mathbb{R})}\,T}{N\varepsilon}\\ &\leq \frac{C+\Lambda}{\varepsilon}\int_0^t \sup_{i=1,\ldots,N} \mathbb{E}(|X^{i,\varepsilon}_t - Y^{i,\varepsilon}_t|^2)\,\mathrm{d}s + \frac{2\,\|k\|^2_{L^2(\mathbb{R})}\,T}{N\varepsilon}.\end{aligned}$$

Applying Gronwall's inequality yields

$$\sup_{t\in[0,T]}\sup_{i=1,\ldots,N} \mathbb{E}(|X^{i,\varepsilon}_t - Y^{i,\varepsilon}_t|^2) \leq \frac{2\,\|k\|^2_{L^2(\mathbb{R})}\,T}{N\varepsilon}\exp\bigg(\frac{(C+\Lambda)T}{\varepsilon}\bigg),$$

which proves the proposition. □

In the next step we need to estimate the difference of the solutions to the regularized mean-field SDEs and the non-regularized mean-field SDE. Recall that, by the stochastic Fokker–Planck equations, it is sufficient to consider the associated solutions ρ^ε and ρ of the SPDEs (4.12) and (4.11). For more details regarding this observation we refer to the proof of Theorem 4.15.

PROPOSITION 4.23. *Suppose Assumption 4.1 and Assumption 4.13 hold. Let ρ^ε and ρ be the solutions to the regularized stochastic Fokker–Planck equation* (4.12) *and to the non-regularized stochastic Fokker–Planck equation* (4.11)*, respectively, which are provided by Corollary 4.12. Then,*

$$\begin{aligned}&\|\rho^\varepsilon-\rho\|_{S^\infty_{\mathcal{F}W}([0,T];L^2(\mathbb{R}))}\\&\quad\le C(\lambda,\Lambda,T,\|k\|_{L^2(\mathbb{R})},\|\rho\|_{S^\infty_{\mathcal{F}W}([0,T];L^2(\mathbb{R}))}\|\rho^\varepsilon\|_{S^\infty_{\mathcal{F}W}([0,T];L^2(\mathbb{R}))})\,\|k^\varepsilon-k\|_{L^2(\mathbb{R})}\,.\end{aligned}$$

PROOF. To estimate the difference $\rho_t-\rho^\varepsilon_t$, we notice that

$$\begin{aligned}\rho^\varepsilon_t-\rho_t=\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\sigma_t^2+\nu^2}{2}(\rho^\varepsilon_t-\rho_t)\right)\mathrm{d}t+\frac{\mathrm{d}}{\mathrm{d}x}((k^\varepsilon*\rho^\varepsilon_t)\rho^\varepsilon_t)\,\mathrm{d}t\\-\frac{\mathrm{d}}{\mathrm{d}x}((k*\rho_t)\rho_t)\,\mathrm{d}t-\nu\frac{\mathrm{d}}{\mathrm{d}x}(\rho^\varepsilon_t-\rho_t)\,\mathrm{d}W_t.\end{aligned}$$

Performing similar computations as in the proof of Theorem 4.6 by using Young's inequality, we get

$$\begin{aligned}&\|\rho^\varepsilon_t-\rho_t\|^2_{L^2(\mathbb{R})}\\&\le-\lambda\int_0^t\left\|\frac{\mathrm{d}}{\mathrm{d}x}\rho^\varepsilon_s-\frac{\mathrm{d}}{\mathrm{d}x}\rho_s\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s-\int_0^t\left\langle(\rho_s-\rho^\varepsilon_s)\frac{\mathrm{d}}{\mathrm{d}x}(\sigma_s^2),\frac{\mathrm{d}}{\mathrm{d}x}\rho_s-\frac{\mathrm{d}}{\mathrm{d}x}\rho^\varepsilon_s\right\rangle_{L^2(\mathbb{R})}\mathrm{d}s\\&\quad-2\int_0^t\left\langle(k^\varepsilon*\rho^\varepsilon_s)\rho^\varepsilon_s-(k*\rho_s)\rho_s,\frac{\mathrm{d}}{\mathrm{d}x}\rho^\varepsilon_s-\frac{\mathrm{d}}{\mathrm{d}x}\rho_s\right\rangle_{L^2(\mathbb{R})}\mathrm{d}s\\&\le-\frac{3\lambda}{4}\int_0^t\left\|\frac{\mathrm{d}}{\mathrm{d}x}\rho^\varepsilon_s-\frac{\mathrm{d}}{\mathrm{d}x}\rho_s\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s+\frac{\Lambda^2}{\lambda}\int_0^t\|\rho_s-\rho^\varepsilon_s\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s\\&\quad-2\int_0^t\left\langle(k^\varepsilon*\rho^\varepsilon_s)\rho^\varepsilon_s-(k*\rho_s)\rho_s,\frac{\mathrm{d}}{\mathrm{d}x}\rho^\varepsilon_s-\frac{\mathrm{d}}{\mathrm{d}x}\rho_s\right\rangle_{L^2(\mathbb{R})}\mathrm{d}s.\end{aligned}$$

Rewriting the last term gives

$$\begin{aligned}&(k^\varepsilon*\rho^\varepsilon_s)\rho^\varepsilon_s-(k*\rho_s)\rho_s\\&\quad=((k^\varepsilon-k)*\rho^\varepsilon_s)\rho^\varepsilon_s+(k*\rho^\varepsilon_s)\rho^\varepsilon_s-(k*\rho_s)\rho_s\\&\quad=((k^\varepsilon-k)*\rho^\varepsilon_s)\rho^\varepsilon_s+(k*(\rho^\varepsilon_s-\rho_s))\rho^\varepsilon_s+((k*\rho_s)(\rho^\varepsilon_s-\rho_s).\end{aligned}$$

Hence, for the last two terms we can use Young's inequality, Young's inequality for convolution, mass conservation and (4.17) to obtain

$$\begin{aligned}
&\left\langle (k*\rho_s)(\rho_s^\varepsilon-\rho_s)+(k*(\rho_s^\varepsilon-\rho_s))\rho_s^\varepsilon, \frac{\mathrm{d}}{\mathrm{d}x}\rho_s^\varepsilon-\frac{\mathrm{d}}{\mathrm{d}x}\rho_s\right\rangle_{L^2(\mathbb{R})}\\
&\le \|k\|_{L^2(\mathbb{R})}\|\rho_s\|_{L^2(\mathbb{R})}\|\rho_s^\varepsilon-\rho_s\|_{L^2(\mathbb{R})}\left\|\frac{\mathrm{d}}{\mathrm{d}x}\rho_s^\varepsilon-\frac{\mathrm{d}}{\mathrm{d}x}\rho_s\right\|_{L^2(\mathbb{R})}\\
&\quad+\|k\|_{L^2(\mathbb{R})}\|\rho_s^\varepsilon-\rho_s\|_{L^2(\mathbb{R})}\|\rho_s^\varepsilon\|_{L^2(\mathbb{R})}\left\|\frac{\mathrm{d}}{\mathrm{d}x}\rho_s^\varepsilon-\frac{\mathrm{d}}{\mathrm{d}x}\rho_s\right\|_{L^2(\mathbb{R})}\\
&\le \frac{\lambda}{4}\left\|\frac{\mathrm{d}}{\mathrm{d}x}\rho_s^\varepsilon-\frac{\mathrm{d}}{\mathrm{d}x}\rho_s\right\|_{L^2(\mathbb{R})}^2\\
&+\frac{1}{\lambda}(\|k\|_{L^2(\mathbb{R})}\|\rho_s\|_{L^2(\mathbb{R})}\|\rho_s^\varepsilon-\rho_s\|_{L^2(\mathbb{R})}+\|k\|_{L^2(\mathbb{R})}\|\rho_s^\varepsilon-\rho_s\|_{L^2(\mathbb{R})}\|\rho_s^\varepsilon\|_{L^2(\mathbb{R})})^2\\
&\le \frac{\lambda}{2}\left\|\frac{\mathrm{d}}{\mathrm{d}x}\rho_s^\varepsilon-\frac{\mathrm{d}}{\mathrm{d}x}\rho_s\right\|_{L^2(\mathbb{R})}^2+\frac{2}{\lambda}\|k\|_{L^2(\mathbb{R})}^2\|\rho_s^\varepsilon-\rho_s\|_{L^2(\mathbb{R})}^2(\|\rho_s\|_{L^2(\mathbb{R})}^2+\|\rho_s^\varepsilon\|_{L^2(\mathbb{R})}^2).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\left\langle ((k^\varepsilon-k)*\rho_s^\varepsilon)\rho_s^\varepsilon, \frac{\mathrm{d}}{\mathrm{d}x}\rho_s^\varepsilon-\frac{\mathrm{d}}{\mathrm{d}x}\rho_s\right\rangle_{L^2(\mathbb{R})}\\
&\quad\le \|(k^\varepsilon-k)*\rho_s^\varepsilon\|_{L^\infty(\mathbb{R})}\|\rho_s^\varepsilon\|_{L^2(\mathbb{R})}\left\|\frac{\mathrm{d}}{\mathrm{d}x}\rho_s^\varepsilon-\frac{\mathrm{d}}{\mathrm{d}x}\rho_s\right\|_{L^2(\mathbb{R})}\\
&\quad\le \|k^\varepsilon-k\|_{L^2(\mathbb{R})}\|\rho_s^\varepsilon\|_{L^2(\mathbb{R})}\|\rho_s^\varepsilon\|_{L^2(\mathbb{R})}\left\|\frac{\mathrm{d}}{\mathrm{d}x}\rho_s^\varepsilon-\frac{\mathrm{d}}{\mathrm{d}x}\rho_s\right\|_{L^2(\mathbb{R})}\\
&\quad\le \frac{\lambda}{4}\left\|\frac{\mathrm{d}}{\mathrm{d}x}\rho_s^\varepsilon-\frac{\mathrm{d}}{\mathrm{d}x}\rho_s\right\|_{L^2(\mathbb{R})}^2+\frac{1}{\lambda}\|k^\varepsilon-k\|_{L^2(\mathbb{R})}^2\|\rho_s^\varepsilon\|_{L^2(\mathbb{R})}^4,
\end{aligned}$$

where we used Young's inequality for convolutions in the second inequality and Young's inequality for the last step. Consequently, combining the last two estimates with our previous L^2-norm inequality and absorbing the L^2-norm of the derivatives we obtain

$$\begin{aligned}
&\|\rho_t^\varepsilon-\rho_t\|_{L^2(\mathbb{R})}^2\\
&\quad\le \frac{2\|k\|_{L^2(\mathbb{R})}^2+1}{\lambda}\int_0^t(\|\rho_s\|_{L^2(\mathbb{R})}^2+\|\rho_s^\varepsilon\|_{L^2(\mathbb{R})}^2+\Lambda^2)\|\rho_s^\varepsilon-\rho_s\|_{L^2(\mathbb{R})}^2\,\mathrm{d}s\\
&\qquad+\frac{1}{\lambda}\int_0^t\|k^\varepsilon-k\|_{L^2(\mathbb{R})}^2\|\rho_s^\varepsilon\|_{L^2(\mathbb{R})}^4\,\mathrm{d}s\\
&\quad\le \frac{2\|k\|_{L^2(\mathbb{R})}^2+1}{\lambda}\int_0^t(\|\rho_s\|_{L^2(\mathbb{R})}^2+\|\rho_s^\varepsilon\|_{L^2(\mathbb{R})}^2+\Lambda^2)\|\rho_s^\varepsilon-\rho_s\|_{L^2(\mathbb{R})}^2\,\mathrm{d}s
\end{aligned}$$

$$+\frac{T}{\lambda}\|k^\varepsilon-k\|^2_{L^2(\mathbb{R})}\sup_{t\in[0,T]}\|\rho^\varepsilon_t\|^4_{L^2(\mathbb{R})}.$$

Applying Gronwall's inequality and using the uniform bound (4.19), we obtain

$$\begin{aligned}
&\sup_{t\in[0,T]}\|\rho^\varepsilon_t-\rho_t\|^2_{L^2(\mathbb{R})}\\
&\le\frac{T}{\lambda}\|k^\varepsilon-k\|^2_{L^2(\mathbb{R})}\sup_{t\in[0,T]}\|\rho^\varepsilon_t\|^4_{L^2(\mathbb{R})}\\
&\quad\cdot\exp\Bigg(\frac{2\|k\|^2_{L^2(\mathbb{R})}+1}{\lambda}\int_0^T(\|\rho_s\|^2_{L^2(\mathbb{R})}+\|\rho^\varepsilon_s\|^2_{L^2(\mathbb{R})}+\Lambda)\,\mathrm{d}s\Bigg)\\
&\le\frac{T}{\lambda}\|k^\varepsilon-k\|^2_{L^2(\mathbb{R})}\sup_{t\in[0,T]}\|\rho^\varepsilon_t\|^4_{L^2(\mathbb{R})}\\
&\quad\cdot\exp\Bigg(\frac{2T(\|k\|^2_{L^2(\mathbb{R})}+1)}{\lambda}(\|\rho\|^2_{S^\infty_{\mathcal{F}W}([0,T];L^2(\mathbb{R}))}+\|\rho^\varepsilon\|^2_{S^\infty_{\mathcal{F}W}([0,T];L^2(\mathbb{R}))}+\Lambda)\Bigg).
\end{aligned}$$

After taking the supremum over $\omega\in\Omega$, we arrive at

$$\begin{aligned}
&\|\rho^\varepsilon-\rho\|_{S^\infty_{\mathcal{F}W}([0,T];L^2(\mathbb{R}))}\\
&\quad\le C(\lambda,\Lambda,T,\|k\|_{L^2(\mathbb{R})},\|\rho\|_{S^\infty_{\mathcal{F}W}([0,T];L^2(\mathbb{R}))}\|\rho^\varepsilon\|_{S^\infty_{\mathcal{F}W}([0,T];L^2(\mathbb{R}))})\|k^\varepsilon-k\|_{L^2(\mathbb{R})}.
\end{aligned}$$

□

REMARK 4.24. *Due to Proposition 4.23, we know that the solutions ρ^ε of the regularized stochastic Fokker–Planck equations converges to the solution ρ of the non-regularized stochastic Fokker–Planck equation as the interaction force kernels converge in the L^2-norm for $\varepsilon\to 0$.*

Finally, we are in a position to state and prove the main theorem of this section.

THEOREM 4.25 (Conditional propagation of chaos). *Suppose Assumption 4.1 and Assumption 4.13 hold. Let ρ be the solution of the stochastic Fokker–Planck equation* (4.11) *and let us denote by*

$$\mu_t^{\mathbf{X}^N}(\omega):=\frac{1}{N}\sum_{i=1}^N\delta_{X_t^{i,\varepsilon}(\omega)}$$

the empirical measure of the regularized interaction system $((X_t^{i,\varepsilon},\varepsilon>0),i=1,\dots,N)$ given by (4.8)*. Then, we have, for all $t\in[0,T]$,*

$$\begin{aligned}
&\mathbb{E}(|\langle\mu_t^{\mathbf{X}^N},\varphi\rangle-\langle\rho_t,\varphi\rangle|^2)\\
&\le C\Big(\lambda,\Lambda,T,\|k\|_{L^2(\mathbb{R})},\|\varphi\|_{W^{1,\infty}(\mathbb{R})},\|\varphi\|_{L^2(\mathbb{R})},\|\rho\|_{S^\infty_{\mathcal{F}W}([0,T];L^2(\mathbb{R}))},\|\rho^\varepsilon\|_{S^\infty_{\mathcal{F}W}([0,T];L^2(\mathbb{R}))}\Big)\\
&\quad\cdot\left(\frac{1}{N}+\frac{1}{N\varepsilon}\exp\left(\frac{(C+\Lambda)T}{\varepsilon}\right)+\|k^\varepsilon-k\|^2_{L^2(\mathbb{R})}\right)
\end{aligned}$$

for any $\varphi \in C_c^\infty(\mathbb{R})$ *and a finite constant* C*. Consequently, we have for* $\varepsilon = \log(N)$ *the convergence*

$$\lim_{N\to\infty} \mathbb{E}(|\langle \mu_t^{\mathbf{X}^N}, \varphi\rangle - \langle \rho_t, \varphi\rangle|^2) = 0.$$

Proof. We compute

$$\begin{aligned}
&\mathbb{E}(|\langle \mu_t^{\mathbf{X}^N}, \varphi\rangle - \langle \rho_t^\varepsilon, \varphi\rangle|^2)\\
&= \mathbb{E}\bigg(\bigg(\frac{1}{N}\sum_{i=1}^N \varphi(X_t^{i,\varepsilon}) - \int_{\mathbb{R}} \rho_t^\varepsilon(x)\varphi(x)\,\mathrm{d}x\bigg)^2\bigg)\\
&= 2\mathbb{E}\bigg(\bigg(\frac{1}{N}\sum_{i=1}^N \varphi(X_t^{i,\varepsilon}) - \frac{1}{N}\sum_{i=1}^N \varphi(Y_t^{i,\varepsilon})\bigg)^2\bigg)\\
&\quad + 2\mathbb{E}\bigg(\bigg(\frac{1}{N}\sum_{i=1}^N \varphi(Y_t^{i,\varepsilon}) - \int_{\mathbb{R}} \rho_t^\varepsilon(x)\varphi(x)\,\mathrm{d}x\bigg)^2\bigg)\\
&\le \frac{2}{N^2}\bigg(\sum_{i=1}^N \mathbb{E}(|\varphi(X_t^{i,\varepsilon}) - \varphi(Y_t^{i,\varepsilon})|^2)^{1/2}\bigg)^2\\
&\quad + 2\mathbb{E}\bigg(\frac{1}{N}\sum_{i=1}^N \varphi(Y_t^{i,\varepsilon}) - \int_{\mathbb{R}} \rho_t^\varepsilon(x)\varphi(x)\,\mathrm{d}x\bigg)^2\bigg)
\end{aligned}$$

$$\le 2\sup_{i=1,\dots,N} \mathbb{E}(|\varphi(X_t^{i,\varepsilon}) - \varphi(Y_t^{i,\varepsilon})|^2) + 2\mathbb{E}\bigg(\bigg(\frac{1}{N}\sum_{i=1}^N \varphi(Y_t^{i,\varepsilon}) - \int_{\mathbb{R}} \rho_t^\varepsilon(x)\varphi(x)\,\mathrm{d}x\bigg)^2\bigg), \tag{4.33}$$

where we used Minkowski's inequality in the third step. Now, by Proposition 4.22 and

$$|\varphi(X_t^{i,\varepsilon}) - \varphi(Y_t^{i,\varepsilon})|^2 \le \|\varphi\|_{W^{1,\infty}(\mathbb{R})}^2\, |X_t^{i,\varepsilon} - Y_t^{i,\varepsilon}|^2,$$

we can estimate the first term by $\|\varphi\|_{W^{1,\infty}(\mathbb{R})}^2 \frac{2\|k\|_{L^2(\mathbb{R})}^2 T}{N\varepsilon} \exp\Big(\frac{(C+\Lambda)T}{\varepsilon}\Big)$. For the second term we write out the square to obtain

$$\begin{aligned}
&\frac{2}{N^2}\sum_{i,j=1}^N \mathbb{E}\bigg(\bigg(\varphi(Y_t^{i,\varepsilon}) - \int_{\mathbb{R}} \rho_t^\varepsilon(y)\varphi(y)\,\mathrm{d}y\bigg)\bigg(\varphi(Y_t^{j,\varepsilon}) - \int_{\mathbb{R}} \rho_t^\varepsilon(y)\varphi(y)\,\mathrm{d}y\bigg)\bigg)\\
&\quad = \frac{2}{N^2}\sum_{i,j=1}^N \mathbb{E}\bigg(\varphi(Y_t^{i,\varepsilon})\varphi(Y_t^{j,\varepsilon}) - \varphi(Y_t^{i,\varepsilon})\int_{\mathbb{R}} \rho_t^\varepsilon(y)\varphi(y)\,\mathrm{d}y\\
&\qquad - \varphi(Y_t^{j,\varepsilon})\int_{\mathbb{R}} \rho_t^\varepsilon(y)\varphi(y)\,\mathrm{d}y + \bigg(\int_{\mathbb{R}} \rho_t^\varepsilon(y)\varphi(y)\,\mathrm{d}y\bigg)^2\bigg).
\end{aligned}$$

Now, using the fact that ρ_t^ε is the conditional distribution of $Y^{i,\varepsilon}$ with respect to $\mathcal{F}_t^W$, we find

$$\begin{aligned}\mathbb{E}\Big(\varphi(Y_t^{i,\varepsilon})\int_{\mathbb{R}}\rho_t^\varepsilon(y)\varphi(y)\,\mathrm{d}y\Big) &= \mathbb{E}\Big(\mathbb{E}\Big(\varphi(Y_t^{i,\varepsilon})\int_{\mathbb{R}}\rho_t^\varepsilon(y)\varphi(y)\,\mathrm{d}y\Big|\mathcal{F}_t^W\Big)\Big)\\ &= \mathbb{E}\Big(\int_{\mathbb{R}}\rho_t^\varepsilon(y)\varphi(y)\,\mathrm{d}y\,\mathbb{E}(\varphi(Y_t^{i,\varepsilon})|\mathcal{F}_t^W)\Big)\\ &= \mathbb{E}\Big(\Big(\int_{\mathbb{R}}\rho_t^\varepsilon(y)\varphi(y)\,\mathrm{d}y\Big)^2\Big).\end{aligned}$$

Since $(Y_t^{i,\varepsilon}, i=1,\dots,N)$ have identical conditional distributions given $\mathcal{F}_t^W$, the same equality holds for j instead of i. Additionally, using the fact that $Y_t^{i,\varepsilon}, Y_t^{j,\varepsilon}$ are conditionally independent for $i\neq j$, we obtain

$$\begin{aligned}\mathbb{E}\big(\varphi(Y_t^{i,\varepsilon})\varphi(Y_t^{j,\varepsilon})\big) &= \mathbb{E}\big(\mathbb{E}(\varphi(Y_t^{i,\varepsilon})\varphi(Y_t^{j,\varepsilon})|\mathcal{F}_t^W)\big)\\ &= \mathbb{E}\big(\mathbb{E}(\varphi(Y_t^{i,\varepsilon})|\mathcal{F}_t^W)\mathbb{E}(\varphi(Y_t^{j,\varepsilon})|\mathcal{F}_t^W)\big)\\ &= \mathbb{E}\Big(\Big(\int_{\mathbb{R}}\rho_t^\varepsilon(y)\varphi(y)\,\mathrm{d}y\Big)^2\Big)\end{aligned}$$

for the cross terms. Hence, the cross terms vanish and we can estimate the second term in (4.33) by

$$\frac{2}{N^2}\sum_{i=1}^N\mathbb{E}\Big(\Big(\varphi(Y_t^{i,\varepsilon})-\int_{\mathbb{R}}\rho_t^\varepsilon(y)\varphi(y)\,\mathrm{d}y\Big)^2\Big)\le\frac{C(\|\phi\|_{L^\infty(\mathbb{R})})}{N}$$

for some finite constant $C(\|\varphi\|_{L^\infty(\mathbb{R})})$, which depends only on φ. Putting everything together, we find

$$\begin{aligned}\mathbb{E}(|\langle\mu_t^{\mathbf{X}^N},\varphi\rangle-\langle\rho_t^\varepsilon,\varphi\rangle|^2) &\le \|\varphi\|_{W^{1,\infty}}^2\frac{2\,\|k\|_{L^2(\mathbb{R})}^2\,T}{N\varepsilon}\exp\Big(\frac{(C+\Lambda)T}{\varepsilon}\Big)+\frac{C(\|\varphi\|_{L^\infty(\mathbb{R})})}{N}\\ &\le C\Big(\|k\|_{L^2(\mathbb{R})},T,\|\varphi\|_{W^{1,\infty}}\Big)\Big(\frac{1}{N}+\frac{1}{(N-1)\varepsilon}\exp\Big(\frac{CT}{\varepsilon}\Big)\Big).\end{aligned}$$

Next, using Hölder's inequality and Proposition 4.23, we discover

$$\begin{aligned}&\mathbb{E}(|\langle\rho_t^\varepsilon,\varphi\rangle-\langle\rho_t,\varphi\rangle|^2)\\ &\quad\le\mathbb{E}(\|\varphi\|_{L^2(\mathbb{R})}^2\,\|\rho_t^\varepsilon-\rho_t\|_{L^2(\mathbb{R})}^2)\\ &\quad\le\|\varphi\|_{L^2(\mathbb{R})}^2\,\|\rho^\varepsilon-\rho\|_{S_{\mathcal{F}}^\infty([0,T];L^2(\mathbb{R}))}^2\\ &\quad\le\|\varphi\|_{L^2(\mathbb{R})}^2\,C(\lambda,\Lambda,T,\|k\|_{L^2(\mathbb{R})},\|\rho\|_{S_{\mathcal{F}^W}^\infty([0,T];L^2(\mathbb{R}))}\,\|\rho^\varepsilon\|_{S_{\mathcal{F}^W}^\infty([0,T];L^2(\mathbb{R}))})\,\|k^\varepsilon-k\|_{L^2(\mathbb{R})}^2.\end{aligned}$$

Therefore, combining this estimate with the previous one we obtain

$$
\begin{aligned}
&\mathbb{E}(|\langle \mu_t^{\mathbf{X}^N}, \varphi\rangle - \langle \rho_t, \varphi\rangle|^2) \\
&\quad \le 2\mathbb{E}(|\langle \mu_t^{\mathbf{X}^N}, \varphi\rangle - \langle \rho_t^\varepsilon, \varphi\rangle|^2) + 2\mathbb{E}(|\langle \rho_t^\varepsilon, \varphi\rangle - \langle \rho_t, \varphi\rangle|^2) \\
&\quad \le C\Big(\lambda, \Lambda, T, \|k\|_{L^2(\mathbb{R})}, \|\varphi\|_{W^{1,\infty}(\mathbb{R})}, \|\varphi\|_{L^2(\mathbb{R})}, \|\rho\|_{S^\infty_{\mathcal{F}W}([0,T];L^2(\mathbb{R}))}, \|\rho^\varepsilon\|_{S^\infty_{\mathcal{F}W}([0,T];L^2(\mathbb{R}))}\Big) \\
&\qquad \cdot \left(\frac{1}{N} + \frac{1}{N\varepsilon}\exp\left(\frac{(C+\Lambda)T}{\varepsilon}\right) + \|k^\varepsilon - k\|^2_{L^2(\mathbb{R})}\right).
\end{aligned}
$$

□

In conclusion, we can address the question of propagation of chaos for the HK model.

COROLLARY 4.26. *Let the assumption of Theorem 4.25 hold. Then for* $\varepsilon = \log(N)$ *and* $\varphi \in C_c^\infty(\mathbb{R})$ *we have the following conditional propagation of chaos result for the HK model*

$$\lim_{N\to\infty} \mathbb{E}(|\langle \mu_t^{\mathbf{X}^{N,\varepsilon,HK}}, \varphi\rangle - \langle \rho_t^{HK}, \varphi\rangle|^2) = 0,$$

where $\mathbf{X}^{N,\varepsilon,HK}$ *is the interacting particle system associated to the kernel* k^ε_{HK}.

4.6. Comments

In comparison to the previous chapters, the convergence in probability approach is unfortunately not applicable here. One central component of this approach was compensating for the irregularity of the interaction kernel k by leveraging the regularity of the solution ρ of the Fokker–Planck, specifically using the L^∞ bound in (t,x) for the solution ρ. In the SPDE setting, however, we need to replace this bound with an L^∞ bound in (ω,t,x). The probability space Ω, generally lacks a specific structure, making it unrealistic to expect techniques from parabolic differential equations such as bootstrap arguments or embeddings to work. Thus, we cannot produce a uniform bound in $L^\infty(\Omega)$ with the classical methods. Instead, as demonstrated in Section 4.4, we had to employ a stopping time argument to perform a bootstrap argument, which fortunately did not compromise the dual argument. However, this line of reasoning cannot be reliably extended in general. Consequently, in the end, we cannot achieve an algebraic-order cut-off and are instead confined to the less satisfying regime of a logarithmic cut-off.

Moreover, it would be particularly beneficial to gain a deeper understanding of the stability of SPDEs with respect to mollification and the appropriate topologies. This issue is closely related to the challenges represented by the left and right arrows in Figure 1, indicating that we should not anticipate better results than those achieved in the deterministic case. However, the impact of common noise is still not fully understood, especially when considering different types of noise. For instance, it is known that transport-type noise can prevent blow-up [FGL21]. This raises the question: does common noise actually help? This remains an open question, and we anticipate that the coming years will bring significant advances and insights in this area. Regarding the question of stability, we will explore this further in Chapter 5.

Chapter 5

Quantitative estimates for the relative entropy with common noise

We prove a conditional propagation of chaos result for the interacting particle system

$$\mathrm{d}X_t^i = -\frac{1}{N}\sum_{j=1}^{N} k(X_t^i - X_t^j)\,\mathrm{d}t + \sigma(t, X_t^i)\,\mathrm{d}B_t^i + \nu(t, X_t^i)\,\mathrm{d}W_t \tag{5.1}$$

driven by idiosyncratic noise $(B_t^i, t \geq 0)$, $i \in \mathbb{N}$ and common noise $(W_t, t \geq 0)$. The Brownian motions $(B_t^i, t \geq 0)$ are independent of each other, and $(W_t, t \geq 0)$ is independent of $(B_t^i, t \geq 0)$ for all i. Details of the probabilistic setting are provided in Section 1.4. For the interaction kernel, we require $k \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Systems of the form (5.1) are commonly utilized in the field of mean-field games [CD18, Section 2.1] as well as mathematical finance [Ahu16, DLR20, HvS21, LSZ23].

Our aim is to investigate the asymptotic behavior of the system (5.1) and to derive conditional propagation of chaos. We develop the relative entropy method in the common noise setting (5.1), extending the results from Section 1.3.4. While relative entropy theory is well-established in the absence of common noise [JW18, BJW19], there is relatively little literature addressing relative entropy in the case with common noise. The techniques that relative entropy theory relies on—namely, the Liouville equation, the regularity of the mean-field partial differential equation (PDE), and the exponential law of large numbers—need to be adapted for the common noise setting.

The presence of common noise adds an extra layer of complexity to the problem. As we have seen in Section 1.3.5, the empirical distribution of particles at the limit evolves stochastically according to a non-linear stochastic partial differential equation (SPDE). Moreover, the associated Liouville equation must be modified to a conditional Liouville equation, which also solves an SPDE. The work is based on the preprint [Nik24]

5.1. Problem Setting

The setting in this chapter is very general, corresponding to the framework outlined in Section 1.4.1, which involves a sequence of m dimensional Brownian motion $(B^i\, i \in \mathbb{N})$ and a $\tilde{m}$-dimensional common noise W. Recall that we write a vector in $\mathbb{R}^{dN}$ as $\mathsf{X}^N = (x_1, \ldots, x_N) \in \mathbb{R}^{dN}$, where $x_i = (x_{i,1}, \ldots, x_{i,d}) \in \mathbb{R}^d$. For a standard vector in $\mathbb{R}^d$, we use the variable $z \in \mathbb{R}^d$. For a matrix $A \in \mathbb{R}^{d \times d'}$, we denote the (α, β) entry as $[A]_{(\alpha,\beta)}$.

5.1.1. Interacting particle systems with common noise. In this subsection we introduce the probabilistic setting, in particular, the N-particle system and the associated McKean–Vlasov equation.

We require also the following coefficients

$$k\colon \mathbb{R}^d \mapsto \mathbb{R}^d, \quad \sigma\colon [0,T]\times\mathbb{R}^d \mapsto \mathbb{R}^{d\times m}, \quad \nu\colon [0,T]\times\mathbb{R}^d \mapsto \mathbb{R}^{d\times\tilde{m}}.$$

We define the interacting particle system by

$$\mathrm{d}X_t^i = -\frac{1}{N}\sum_{j=1}^N k(X_t^i - X_t^j)\,\mathrm{d}t + \sigma(t,X_t^i)\,\mathrm{d}B_t^i + \nu(t,X_t^i)\,\mathrm{d}W_t \tag{5.2}$$

and the conditional McKean–Vlasov system by

$$\begin{cases} \mathrm{d}Y_t^i = -(k*\rho_t(Y_t^i))\,\mathrm{d}t + \sigma(t,Y_t^i)\,\mathrm{d}B_t^i + \nu(t,Y_t^i)\,\mathrm{d}W_t, \\ \rho_t = \mathscr{L}_{Y_t^i|\mathcal{F}_t^W}, \end{cases} \tag{5.3}$$

where $\mathscr{L}_{Y_t^i|\mathcal{F}_t^W}$ is the conditional density of Y_t^i given $\mathcal{F}_t^W$. It is well-known that pathwise-uniqueness implies that $\mathscr{L}_{Y_t^i|\mathcal{F}_t^W}$ is independent of $i\in\mathbb{N}$.

5.1.2. Stochastic partial differential equations. Let us introduce the SPDE's associated to the SDE's defined in (5.2), (5.3). From the interacting particle system (5.2) we can derive the stochastic Fokker–Planck equation in $\mathbb{R}^{dN}$ solved by the conditional density ρ_t^N given the filtration $\mathcal{F}_t^W$. We call this SPDE the conditional Liouville equation, which is given by

$$\begin{aligned} \mathrm{d}\rho_t^N &= \frac{1}{2}\sum_{i,j=1}^N \sum_{\alpha,\beta=1}^d \partial_{x_{i,\alpha}}\partial_{x_{j,\beta}}\Big(([\sigma(t,x_i)\sigma(t,x_j)^{\mathrm{T}}]_{(\alpha,\beta)}\delta_{i,j} + [\nu(t,x_i)\nu(t,x_j)^{\mathrm{T}}]_{(\alpha,\beta)})\rho_t^N\Big)\,\mathrm{d}t \\ &\quad + \sum_{i=1}^N \nabla_{x_i}\cdot\left(\frac{1}{N}\sum_{j=1}^N k(x_i-x_j)\rho_t^N\right)\mathrm{d}t - \sum_{i=1}^N \nabla_{x_i}\cdot(\nu(t,x_i)\rho_t^N\,\mathrm{d}W_t), \end{aligned} \tag{5.4}$$

where $\delta_{i,j}=1$ if and only if $i=j$ and otherwise zero. Analogously, we define the d-dimensional non-linear SPDE

$$\begin{aligned} \mathrm{d}\rho_t &= \nabla\cdot((k*\rho_t)\rho_t))\,\mathrm{d}t - \nabla\cdot(\nu_t\rho_t\,\mathrm{d}W_t) \\ &\quad + \frac{1}{2}\sum_{\alpha,\beta=1}^d \partial_{z_\alpha}\partial_{z_\beta}\Big(([\sigma_t\sigma_t^{\mathrm{T}}]_{(\alpha,\beta)} + [\nu_t\nu_t^{\mathrm{T}}]_{(\alpha,\beta)})\rho_t\Big)\,\mathrm{d}t \end{aligned} \tag{5.5}$$

associated to (5.3). The provided equation represents the conditional density of a single particle in the McKean–Vlasov SDE (5.3). Extending this concept to cover N conditionally independent particles, we introduce the SPDE

$$\begin{aligned} \mathrm{d}\rho_t^{\otimes N} =& \frac{1}{2}\sum_{i,j=1}^N \sum_{\alpha,\beta=1}^d \partial_{x_{i,\alpha}}\partial_{x_{j,\beta}}\Big((\delta_{i,j}[\sigma(t,x_i)\sigma(t,x_j)^{\mathrm{T}}]_{(\alpha,\beta)} + [\nu(t,x_i)\nu(t,x_j)^{\mathrm{T}}]_{(\alpha,\beta)})\rho_t^{\otimes N}\Big)\,\mathrm{d}t \\ & + \sum_{i=1}^N \nabla_{x_i}\cdot((k*\rho_t)(x_i)\rho_t^{\otimes N}))\,\mathrm{d}t - \sum_{i=1}^N \nabla_{x_i}\cdot(\nu(t,x_i)\rho_t^{\otimes N}\,\mathrm{d}W_t). \end{aligned} \tag{5.6}$$

The notation $\otimes N$ is intentionally used, as it will become evident that if the d-dimensional SPDE (5.5) has a solution, then equation (5.6) also possesses a solution in the form of a tensor product.

From the above equation we can deduce the following r-marginal SPDE of the interacting particle system

$$\begin{aligned}
\mathrm{d}\rho_t^{r,N} &= \sum_{i=1}^{r} \int_{\mathbb{R}^{(N-r)d}} \nabla_{x_i} \cdot \left(\frac{1}{N} \sum_{j=1}^{N} k(x_i - x_j)\rho_t^N \right) \mathrm{d}x_{r+1} \cdot \mathrm{d}x_N \,\mathrm{d}t \\
&\quad - \sum_{i=1}^{N} \nabla_{x_i} \cdot \Big(\int_{\mathbb{R}^{(N-r)d}} \nu(t, x_i)\rho_t^N \,\mathrm{d}x_{r+1} \cdots \mathrm{d}x_N \,\mathrm{d}W_t \Big) \\
&\quad + \frac{1}{2} \sum_{i,j=1}^{N} \sum_{\alpha,\beta=1}^{d} \partial_{x_{i,\alpha}} \partial_{x_{j,\beta}} \Bigg(\int_{\mathbb{R}^{(N-r)d}} ([\sigma(t, x_i)\sigma(t, x_j)^{\mathrm{T}}]_{(\alpha,\beta)} \delta_{i,j} \\
&\quad + [\nu(t, x_i)\nu(t, x_j)^{\mathrm{T}}]_{(\alpha,\beta)})\rho_t^N \Bigg) \mathrm{d}x_{r+1} \cdots x_N \,\mathrm{d}t
\end{aligned}$$

5.1.3. Assumptions on initial condition and diffusion coefficients. Throughout this chapter we make the following assumption on our diffusion coefficients σ, ν and the initial condition ρ_0.

Assumption 5.1.

(1) For each $(i, l) \in \{1, \ldots, d\} \times \{1, \ldots, m\}$ we require $\sigma^{i,l}(t, \cdot) \in C^1(\mathbb{R}^d)$ and for some positive constant $C > 0$ we have the following uniform estimate

$$\sup_{0 \le t \le T} \left\| \sigma^{i,l}(t, \cdot) \right\|_{C^1(\mathbb{R}^d)} \le C.$$

(2) For each $(i, \hat{l}) \in \{1, \ldots, d\} \times \{1, \ldots, \tilde{m}\}$ we require $\nu^{i,\hat{l}}(t, \cdot) \in C^1(\mathbb{R}^d)$ and for some positive constant $C > 0$ we have the following uniform estimate

$$\sup_{0 \le t \le T} \left\| \nu^{i,\hat{l}}(t, \cdot) \right\|_{C^1(\mathbb{R}^d)} \le C.$$

(3) The diffusion ν is divergence free, i.e. for each $\hat{l} = 1, \ldots, \tilde{m}$ we have

$$\sum_{\beta=1}^{d} \partial_{z_\beta} \nu^{\beta,\hat{l}}(t, z) = 0, \quad (t, z) \in [0, T] \times \mathbb{R}^d.$$

(4) For every $\alpha \in \{1, \ldots, d\}$ we have

$$\sum_{\beta=1}^{d} \partial_{y_\beta} \sum_{l=1}^{m} \sigma^{\alpha,l}(t, z)\sigma^{\beta,l}(t, z) = 0, , \quad (t, z) \in [0, T] \times \mathbb{R}^d,$$

$$\sum_{\beta=1}^{d} \partial_{y_\beta} \sum_{\hat{l}=1}^{\tilde{m}} \nu^{\alpha,\hat{l}}(t, z)\nu^{\beta,\hat{l}}(t, z) = 0, \quad (t, z) \in [0, T] \times \mathbb{R}^d.$$

(5) σ satisfies the ellipticity condition. For all $\lambda \in \mathbb{R}^d$ we have

$$\sum_{\alpha,\beta=1}^{d} [\sigma_s(z)\sigma_s(z)^{\mathrm{T}}]_{(\alpha,\beta)} \lambda_\alpha \lambda_\beta \ge \delta|\lambda|^2.$$

(6) The initial condition $\rho_0 \in L^2(\mathbb{R}^d)$ satisfies the second moment estimate

$$\int_{\mathbb{R}^d} |z|^2 \rho_0(z)\,\mathrm{d}z < \infty.$$

Remark 5.2. *Assumption* (3) *is a standard assumption in the field of SPDEs* [CF16, HQ21]*, implying the non-randomness of the L^p-norms. Condition* (4) *is necessary for reducing the conditional Liouville equation* (5.4) *from a Fokker–Planck equation* [BKRS15] *to a SPDE in divergence structure. The same applies to the non-local SPDE* (5.6)*. We require the ellipticity condition* (5) *for the existence of the SPDE's* [Roz90, Kry99]*, analogously to the parabolic PDE theory.*

A direct consequence of Assumption 5.1 (5) is the existence and strong uniqueness of the interacting particle system (5.2).

Our contribution: Our first contribution lies in proving the global well-posedness of the stochastic partial differential equations associated with the particle systems. Since our kernel k lies in $L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ it prevents us from directly applying established results in the literature on SPDEs, such as those found in textbooks by Krylov [Kry99], Rozovsky [Roz90] to our non-linear stochastic Fokker–Planck equation (5.6). Consequently, for our non-linear SPDE (5.6), we resort to employing a Picard iteration method.

Our second contribution focuses on demonstrating the boundedness of the relative entropy in $\mathbb{R}^{dN}$. To achieve this, we compute the evolution of the relative entropy using Itô's formula. A key insight lies in recognizing that, unlike in the classical setting where distributions are directly compared in the relative entropy, we should instead compare the conditional distribution of the interacting particle system (5.1) with the solution of the stochastic Fokker–Planck equation (5.6).

However, defining the relative entropy for general random measures poses a challenge. To the best of our knowledge, we could not find a reliable definition. But if we consider solutions of the conditional Liouville equation (5.4) and the stochastic Fokker–Planck equation (5.6) with sufficient regularity, we can provide a pointwise interpretation of the SPDE's (5.4), (5.6). This enables us to apply Itô's formula to the SPDE's, thereby deriving an expression for the dynamics of the relative entropy. Subsequently, we can further analyze this expression by employing methods known in the non-common case, such as the exponential law of large numbers [JW18]. Notice, that in the absence of transport-type noise, we must address the quadratic variations arising in the calculations. Consequently, we establish the boundedness of the relative entropy for systems with smooth coefficients. By applying the sub-additivity and the Csiszár–Kullback–Pinsker inequality, we obtain an estimate in the L^1-norm between the marginals of the Liouville equation and the stochastic Fokker–Planck equation, showing a decay rate of $N^{-1/2}$, indicating that results from Jabin and Wang [JW18] are applicable even in the presence of common noise. Additionally, we recover the result from Jabin and Wang [JW18] on the whole space in the case of vanishing common noise $\nu = 0$.

Finally, we demonstrate that in the case $k \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ the conditional Liouville equation (5.4) and the stochastic Fokker–Planck equation (5.6) can be approximated by the associated SPDE's with smooth coefficients. This type of stability result leads to the relative entropy estimate, which seems to be novel. Additionally, we demonstrate conditional propagation of chaos. To our best knowledge, this is the first result on conditional propagation of chaos for general bounded interaction kernels in the L^1-norm. However, it is crucial to exercise caution at this stage, as the standard equivalent characterization provided by Sznitman's Proposition 1.2 cannot be directly applied due to the non-deterministic nature of our limiting measure. Therefore, we need to replicate the proof in the setting of random measures. Consequently, we extend the results for conditional propagation of chaos by Carmona and Delarue [CD18, Chapter 2] to the kernel $k \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

Related literatures: In contrast to interacting particle system, which are driven by idiosyncratic noise [Szn91, JW18, RS23, GLM24], literature on common noise remains limited. For systems with uniformly Lipschitz interaction forces, we know from Chapter 4 that Coghi and Flandoli [CF16] established conditional propagation of chaos in the presence of common noise. Dawson and Vaillancourt [DV95] also formulated a martingale problem and demonstrated tightness of the empirical measure obtaining a qualitative result with no convergence rates.

In a parallel effort to ours, Shao and Zhao [SZ24] recently presented a similar relative entropy estimate for the stochastic two-dimensional Navier–Stokes equation driven by transport noise, utilizing methods from [JW18]. Their approach necessitates the initial independence of intensity and position, which introduces additional assumptions on the physical model. While their work shares similarities with ours, such as the relative entropy estimate and utilization of common noise, distinctions arise in the domain (torus vs $\mathbb{R}^d$) and the nature of noise (transport noise vs. Itô noise). Moreover, the authors [SZ24] capitalized on the specific structure of the Biot–Savart kernel, particularly its divergence-free property.

Rosenzweig [Ros20] addressed a related problem by establishing quantitative estimates for the Biot–Savart kernel using the modulated energy method in a pathwise setting, a method previously introduced by Serfaty [Ser20] in the deterministic setting for Coulomb kernels. Challenges in this context stemmed from the commutator estimate for the second-order correction term. Although the modulated energy method naturally extends to systems with transport noise, the pathwise extension in the case of relative entropy requires a pathwise Radon–Nikodym derivative, posing challenges for random measures. Furthermore, Rosenzweig, Nguyen, and Serfaty [NRS22] obtained similar results for the repulsive Coulomb case under transport noise.

In our work, the presence of idiosyncratic noise, dictated by the ellipticity of the diffusion coefficient σ, plays a significant role. This aspect differentiates our approach from the previous literature discussed. In our setting, the work by Huang, Qiu [HQ21] investigates the Keller–Segel model with Bessel potential and the work of Chen, Prömel and the author [CNP23] demonstrates the existence and uniqueness of non-linear SPDEs and conditional McKean–Vlasov equations. Other relevant literature includes the work of Kurtz, Xiong [KX99], Coghi and Gess [CG19] for the existence of stochastic non-linear Fokker–Planck equations under non-linear Lipschitz coefficients.

A broad area where interacting particle systems of the form (5.1) are studied is in mathematical finance and mean-field games. For instance, Carmona, Delarue, and Lacker [CDL16] investigate mean-field games for Lipschitz drifts with common noise of the form (5.1), as do Delarue, Lacker, and Ramanan [DLR20]. We refer to the references therein for more details.

Finally, let us mention the results of Jabin and Wang [JW18], which serve as a foundational reference in our work. We extend their results in the bounded kernel setting to include common noise. Additionally, we require fewer assumptions on the regularity of the mean-field equation, which is a consequence of the stronger assumption on the interaction kernel k and the ellipticity of the diffusion coefficient σ. However, the focus on the entire space $\mathbb{R}^d$ prevents the inclusion of kernels in $W^{-1,\infty}(\mathbb{R}^d)$, as stability of stochastic partial differential equations cannot be guaranteed using our methods.

Organization of the chapter: In Section 5.1 we provide the definitions of the particle systems and the associated SPDE's along with the introduction of the relative entropy. In Section 5.2 the well-posedness of the Liouville equation (5.4) and the stochastic Fokker–Planck equation (5.6) is established. In Section 5.3 we compute the evolution of the relative entropy in the setting of smooth coefficients. Finally, conditional propagation of chaos with common noise for interaction kernel $k \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is investigated in Section 5.4 as well as a stability estimate for the conditional Liouville equation and stochastic Fokker–Planck equation.

5.2. Well-posedness of the stochastic Fokker–Planck and Liouville equations

In this section we demonstrate the existence and uniqueness of solutions to the linear SPDE (5.4) and the non-linear SPDE (5.6) in the sense of Definition 5.3 and Definition 5.5. Additionally, we provide some a priori bounds uniform in the probability space. We use similar techniques to [CNP23] and for readers familiar with SPDE's the results should not come as a surprise. We provide now definitions to the SPDE's (5.4), (5.5) and (5.6).

Definition 5.3. *For a fix $N \in \mathbb{N}$ a non-negative stochastic process $\big(\rho_t^N, t \geq 0\big)$ is called a (weak) solution of the SPDE* (5.6) *with initial condition $\rho_0^{\otimes N}$ if*

$$\rho^N \in L^2_{\mathcal{F}W}([0,T]; H^1(\mathbb{R}^{dN})) \cap S^\infty_{\mathcal{F}W}([0,T]; L^1(\mathbb{R}^{dN}))$$

and, for any $\varphi \in C_c^\infty(\mathbb{R}^{dN})$, ρ^N satisfies almost surely the equation, for all $t \in [0,T]$,

$$\begin{aligned}
&\langle \rho_t^N, \varphi \rangle_{L^2(\mathbb{R}^{dN})} \\
&= \langle \rho_0^{\otimes N}, \varphi \rangle_{L^2(\mathbb{R}^{dN})} - \sum_{i=1}^N \int_0^t \left\langle \frac{1}{N} \sum_{j=1}^N k(x_i - x_j) \rho_s^N, \nabla_{x_i} \varphi \right\rangle_{L^2(\mathbb{R}^{dN})} \mathrm{d}s \\
&\quad + \frac{1}{2} \sum_{i=1}^N \sum_{\alpha,\beta=1}^d \int_0^t \left\langle [\sigma(s,x_i)\sigma(s,x_i)^{\mathrm{T}}]_{(\alpha,\beta)} \rho_s^N, \partial_{x_{i,\beta}} \partial_{x_{i,\alpha}} \varphi \right\rangle_{L^2(\mathbb{R}^{dN})} \mathrm{d}s \\
&\quad + \frac{1}{2} \sum_{i,j=1}^N \sum_{\alpha,\beta=1}^d \int_0^t \left\langle [\nu(s,x_i)\nu(s,x_j)^{\mathrm{T}}]_{(\alpha,\beta)} \rho_s^N, \partial_{x_{j,\beta}} \partial_{x_{i,\alpha}} \varphi \right\rangle_{L^2(\mathbb{R}^{dN})} \mathrm{d}s \\
&\quad + \sum_{i=1}^N \sum_{\alpha=1}^d \sum_{\hat{l}=1}^{\tilde{m}} \int_0^t \left\langle \nu^{\alpha,\hat{l}}(s,x_i) \rho_s^N, \partial_{x_{i,\alpha}} \varphi \right\rangle_{L^2(\mathbb{R}^{dN})} W_s^{\hat{l}}.
\end{aligned}$$

Definition 5.4. *A non-negative stochastic process $(\rho_t, t \geq 0)$ is a (weak) solution to the SPDE* (5.5) *with initial condition ρ_0, if*

$$\rho \in L^2_{\mathcal{F}W}([0,T]; H^1(\mathbb{R}^d)) \cap S^\infty_{\mathcal{F}W}([0,T]; L^1(\mathbb{R}^d)) \tag{5.7}$$

and

$$\langle \rho_t, \varphi \rangle_{L^2(\mathbb{R}^d)} = \langle \rho_0, \varphi \rangle_{L^2(\mathbb{R}^d)} - \int_0^t \Big\langle (k * \rho_s)\rho_s, \nabla_{x_i}\varphi \Big\rangle_{L^2(\mathbb{R}^d)} \mathrm{d}s + \frac{1}{2}\sum_{\alpha,\beta=1}^d \int_0^t \Big\langle [\sigma(s,z)\sigma(s,z)^{\mathrm{T}}]_{(\alpha,\beta)}\rho_s, \partial_{z_\beta}\partial_{z_\alpha}\varphi \Big\rangle_{L^2(\mathbb{R}^d)} \mathrm{d}s + \frac{1}{2}\sum_{\alpha,\beta=1}^d \int_0^t \Big\langle [\nu(s,z)\nu(s,z)^{\mathrm{T}}]_{(\alpha,\beta)}\rho_s, \partial_{z_\beta}\partial_{z_\alpha}\varphi \Big\rangle_{L^2(\mathbb{R}^d)} \mathrm{d}s + \sum_{\alpha=1}^d \sum_{\hat{l}=1}^{\tilde{m}} \int_0^t \Big\langle \nu^{\alpha,\hat{l}}(s,z)\rho_s, \partial_{z_\alpha}\varphi \Big\rangle_{L^2(\mathbb{R}^d)} W_s^{\hat{l}}.$$

holds.

Definition 5.5. *For a fix $N \in \mathbb{N}$ a non-negative stochastic process $\big(\rho_t^{\otimes N}, t \geq 0\big)$ is called a (weak) solution of the SPDE* (5.4) *with initial condition $\rho_0^{\otimes N}$ if*

$$\rho^{\otimes N} \in L^2_{\mathcal{F}W}([0,T]; H^1(\mathbb{R}^{dN})) \cap S^\infty_{\mathcal{F}W}([0,T]; L^1(\mathbb{R}^{dN}))$$

and, for any $\varphi \in C_c^\infty(\mathbb{R}^{dN})$, $\rho^{\otimes N}$ satisfies almost surely the equation, for all $t \in [0,T]$,

$$\begin{aligned} &\langle \rho_t^{\otimes N}, \varphi \rangle_{L^2(\mathbb{R}^{dN})} \\ &= \langle \rho_0^{\otimes N}, \varphi \rangle_{L^2(\mathbb{R}^{dN})} - \sum_{i=1}^N \int_0^t \Big\langle (k * \rho_s)(x_i)\rho_s^{\otimes N}, \nabla_{x_i}\varphi \Big\rangle_{L^2(\mathbb{R}^{dN})} \mathrm{d}s \\ &\quad + \frac{1}{2}\sum_{i=1}^N \sum_{\alpha,\beta=1}^d \int_0^t \Big\langle [\sigma(s,x_i)\sigma(s,x_i)^{\mathrm{T}}]_{(\alpha,\beta)}\rho_s^{\otimes N}, \partial_{x_{i,\beta}}\partial_{x_{i,\alpha}}\varphi \Big\rangle_{L^2(\mathbb{R}^{dN})} \mathrm{d}s \\ &\quad + \frac{1}{2}\sum_{i,j=1}^N \sum_{\alpha,\beta=1}^d \int_0^t \Big\langle [\nu(s,x_i)\nu(s,x_j)^{\mathrm{T}}]_{(\alpha,\beta)}\rho_s^{\otimes N}, \partial_{x_{j,\beta}}\partial_{x_{i,\alpha}}\varphi \Big\rangle_{L^2(\mathbb{R}^{dN})} \mathrm{d}s \\ &\quad + \sum_{i=1}^N \sum_{\alpha=1}^d \sum_{\hat{l}=1}^{\tilde{m}} \int_0^t \Big\langle \nu^{\alpha,\hat{l}}(s,x_i)\rho_s^{\otimes N}, \partial_{x_{i,\alpha}}\varphi \Big\rangle_{L^2(\mathbb{R}^{dN})} W_s^{\hat{l}}. \end{aligned} \tag{5.8}$$

We start by demonstrating the existence of the Liouville equation (5.4).

Proposition 5.6. *Suppose $k \in L^\infty(\mathbb{R}^d)$. Then, there exists a non-negative solution $\rho^N \in L^2_{\mathcal{F}W}([0,T]; H^1(\mathbb{R}^d))$ of the SPDE* (5.4) *in the sense of Definition 5.3.*

PROOF. Define $f(t,u) = \sum_{i=1}^{N} \nabla_{x_i} \cdot \left(\frac{1}{N} \sum_{j=1}^{N} k(x_i - x_j) u \right)$ for $u \in H^1(\mathbb{R}^{dN})$. Then we find

$$\|f(t,u)\|_{H^{-1}(\mathbb{R}^{dN})} \le \sum_{i=1}^{N} \left\| \frac{1}{N} \sum_{j=1}^{N} k(x_i - x_j) u \right\|_{L^2(\mathbb{R}^{dN})} \le N \|k\|_{L^\infty(\mathbb{R}^d)} \|u\|_{L^2(\mathbb{R}^{dN})}$$

and by linearity

$$\|f(t,u) - f(t,v)\|_{H^{-1}(\mathbb{R}^{dN})} \le N \|k\|_{L^\infty(\mathbb{R}^d)} \|u - v\|_{L^2(\mathbb{R}^{dN})}$$

for $u, v \in H^1(\mathbb{R}^{dN})$. Applying [Kry99, Theorem 5.1] proves the existence. It remains to show that ρ^N is non-negative. However, this is a direct consequence of the maximum principle for SPDE's, which was already applied in Theorem 4.6. □

LEMMA 5.7 (A priori L^2-estimate). *Let $k \in L^\infty(\mathbb{R}^d)$ and let the non-negative process $\rho \in L^2_{\mathcal{F}^W}([0,T]; H^1(\mathbb{R}^d))$ satisfy the SPDE*

$$\begin{aligned} \mathrm{d}\rho_t &= \nabla \cdot ((k * u_t)\rho_t))\,\mathrm{d}t - \nabla \cdot (\nu_t \rho_t \,\mathrm{d}W_t) \\ &\quad + \frac{1}{2} \sum_{\alpha,\beta=1}^{d} \partial_{z_\alpha} \partial_{z_\beta} \Big(([\sigma_t \sigma_t^{\mathrm{T}}]_{(\alpha,\beta)} + [\nu_t \nu_t^{\mathrm{T}}]_{(\alpha,\beta)}) \rho_t \Big) \,\mathrm{d}t, \end{aligned}$$

where u_t is a predictable with respect to $\mathcal{F}^W$, L^1-valued process, which satisfies the inequality $\|u_t\|_{L^1(\mathbb{R}^d)} \le 1$, $\mathbb{P}$-a.s. for all $t \ge 0$. Then

$$\|\rho_t\|_{L^2(\mathbb{R}^d))} \le C(T, d, \delta, \|k\|_{L^\infty(\mathbb{R}^d)}) \|\rho_0\|_{L^2(\mathbb{R}^d)}, \quad \mathbb{P}\text{-a.s.} \tag{5.9}$$

for all $t \ge 0$.

PROOF. Let us compute the norm $\|\rho_t\|_{L^2(\mathbb{R}^d)}$,

$$\begin{aligned} &\|\rho_t\|^2_{L^2(\mathbb{R}^d)} - \|\rho_0\|^2_{L^2(\mathbb{R}^d)} \\ &= -2 \int_0^t \int_{\mathbb{R}^d} ((k * u_s)\rho_s) \cdot \nabla \rho_s \,\mathrm{d}z\,\mathrm{d}s \\ &\quad - \sum_{\alpha,\beta=1}^{d} \int_0^t \int_{\mathbb{R}^d} \partial_{z_\beta} \Big(([\sigma_s \sigma_s^{\mathrm{T}}]_{(\alpha,\beta)} + [\nu_s \nu_s^{\mathrm{T}}]_{(\alpha,\beta)}) \rho_s \Big) \partial_{z_\alpha} \rho_s \,\mathrm{d}z\,\mathrm{d}s \\ &\quad + \sum_{\hat{l}=1}^{\tilde{m}} \int_0^t \int_{\mathbb{R}^d} \left| \sum_{\beta=1}^{d} \partial_{z_\beta} (\nu_s^{\beta,l} \rho_s) \right|^2 \mathrm{d}z\,\mathrm{d}s + 2 \sum_{\hat{l}=1}^{\tilde{m}} \sum_{\beta=1}^{d} \int_0^t \int_{\mathbb{R}^d} \rho_s \partial_{z_\beta} (\nu_s^{\beta,\hat{l}} \rho_s) \,\mathrm{d}W_s^{\hat{l}}. \end{aligned}$$

By the divergence-free assumption of ν we have

$$\begin{aligned}\int_{\mathbb{R}^d} \rho_s \sum_{\beta=1}^{d} \partial_{z_\beta}(\nu_s^{\beta,\hat{l}}\rho_s)\,\mathrm{d}z &= \frac{1}{2}\int_{\mathbb{R}^d} \sum_{\beta=1}^{d} \nu_s^{\beta,\hat{l}}\partial_{z_\beta}(\rho_s^2)\,\mathrm{d}z \\ &= -\frac{1}{2}\int_{\mathbb{R}^d} \sum_{\beta=1}^{d} \partial_{z_\beta}\nu_s^{\beta,\hat{l}}\rho_s^2\,\mathrm{d}z \\ &= 0\end{aligned}$$

and therefore the stochastic integral vanishes. Again using the divergence-free assumption 5.1 (3) we find

$$\begin{aligned}\sum_{\hat{l}=1}^{\tilde{m}} \int_0^t \int_{\mathbb{R}^d} \left|\sum_{\beta=1}^{d} \partial_{z_\beta}(\nu_s^{\beta,l}\rho_s)\right|^2 \mathrm{d}z\,\mathrm{d}s &= \sum_{\hat{l}=1}^{\tilde{m}} \sum_{\alpha,\beta=1}^{d} \int_0^t \int_{\mathbb{R}^d} \partial_{z_\alpha}(\nu_s^{\alpha,l}\rho_s)\partial_{z_\beta}(\nu_s^{\beta,l}\rho_s)\,\mathrm{d}z\,\mathrm{d}s \\ &= \sum_{\hat{l}=1}^{\tilde{m}} \sum_{\alpha,\beta=1}^{d} \int_0^t \int_{\mathbb{R}^d} \nu_s^{\alpha,l}\nu_s^{\beta,l}\partial_{z_\alpha}\rho_s\partial_{z_\beta}\rho_s\,\mathrm{d}z\,\mathrm{d}s.\end{aligned} \tag{5.10}$$

However, we also have

$$\begin{aligned}&\sum_{\alpha,\beta=1}^{d} \int_0^t \int_{\mathbb{R}^d} \partial_{z_\beta}\Big(([\sigma_s\sigma_s^{\mathrm{T}}]_{(\alpha,\beta)} + [\nu_s\nu_s^{\mathrm{T}}]_{(\alpha,\beta)})\rho_s\Big)\partial_{z_\alpha}\rho_s\,\mathrm{d}z\,\mathrm{d}s \\ &\quad= \sum_{\alpha,\beta=1}^{d} \int_0^t \int_{\mathbb{R}^d} \partial_{z_\beta}\Big([\sigma_s\sigma_s^{\mathrm{T}}]_{(\alpha,\beta)} + [\nu_s\nu_s^{\mathrm{T}}]_{(\alpha,\beta)}\Big)\rho_s\partial_{z_\alpha}\rho_s\,\mathrm{d}z\,\mathrm{d}s \\ &\qquad+ \sum_{\alpha,\beta=1}^{d} \int_0^t \int_{\mathbb{R}^d} ([\sigma_s\sigma_s^{\mathrm{T}}]_{(\alpha,\beta)} + [\nu_s\nu_s^{\mathrm{T}}]_{(\alpha,\beta)})\partial_{z_\beta}\rho_s\partial_{z_\alpha}\rho_s\,\mathrm{d}z\,\mathrm{d}s.\end{aligned}$$

Consequently, the last term containing ν is exactly the term (5.10) and therefore both terms cancel. For the first term we use the uniform bound on the derivatives and Young's inequality to estimated it by

$$\begin{aligned}&\sum_{\alpha,\beta=1}^{d} \left|\int_0^t \int_{\mathbb{R}^d} \partial_{z_\beta}\Big([\sigma_s\sigma_s^{\mathrm{T}}]_{(\alpha,\beta)} + [\nu_s\nu_s^{\mathrm{T}}]_{(\alpha,\beta)}\Big)\rho_s\partial_{z_\alpha}\rho_s\,\mathrm{d}z\,\mathrm{d}s\right| \\ &\quad\le C(d,\delta)\int_0^t \|\rho_s\|_{L^2(\mathbb{R}^d)}^2 + \frac{\delta}{2}\|\nabla\rho_s\|_{L^2(\mathbb{R}^d)}^2\,\mathrm{d}s.\end{aligned}$$

Combining the last estimates and plugging them in the evolution of the L^2-norm, we arrive at

$$
\begin{aligned}
\|\rho_t\|_{L^2(\mathbb{R}^d)}^2 - \|\rho_0\|_{L^2(\mathbb{R}^d)}^2 = & -2\int_0^t \int_{\mathbb{R}^d} ((k*u_s)\rho_s)\cdot\nabla\rho_s \,\mathrm{d}z\,\mathrm{d}s + C(d,\delta)\,\|\rho_s\|_{L^2(\mathbb{R}^d)}^2 \\
& - \sum_{\alpha,\beta=1}^d \int_0^t \int_{\mathbb{R}^d} [\sigma_s\sigma_s^{\mathrm{T}}]_{(\alpha,\beta)}\partial_{z_\beta}\rho_s\partial_{z_\alpha}\rho_s\,\mathrm{d}z + \frac{\delta}{2}\,\|\nabla\rho_s\|_{L^2(\mathbb{R}^d)}^2\,\mathrm{d}s \\
\leq & \, 2\int_0^t \|k*u_s\|_{L^\infty(\mathbb{R}^d)}^2\,\|\rho_s\|_{L^2(\mathbb{R}^d)}^2\,\mathrm{d}s + C(d,\delta)\,\|\rho_s\|_{L^2(\mathbb{R}^d)}^2\,\mathrm{d}s \\
\leq & \, (2\,\|k\|_{L^\infty(\mathbb{R}^d)}^2 + C(d,\delta))\int_0^t \|\rho_s\|_{L^2(\mathbb{R}^d)}^2\,\mathrm{d}s,
\end{aligned}
$$

where we used the ellipticity condition in the last step. An application of Gronwall's lemma provides

$$
\|\rho_t\|_{L^2(\mathbb{R}^d)}^2 \leq C(T,d,\delta,\|k\|_{L^\infty(\mathbb{R}^d)})\,\|\rho_0\|_{L^2(\mathbb{R}^d)}^2\,, \quad \mathbb{P}\text{-a.s.}
$$

for all $t \geq 0$. □

LEMMA 5.8. *(A priori moment estimate) Suppose we are in the setting of Lemma 5.7. Then the following moment estimate holds*

$$
\mathbb{E}\bigg(\sup_{0\leq t\leq T}\Big(\int_{\mathbb{R}^d}\rho_t(z)|z|^2\,\mathrm{d}z\Big)^2\bigg) \leq C(T,\sigma,\nu,\|k\|_{L^\infty(\mathbb{R}^d)})\Big(\int_{\mathbb{R}^d}\rho_0(x)|z|^2\,\mathrm{d}z\Big)^2.
$$

PROOF. The core idea is to use $|z|^2$ as a test function. To that end, we take a sequence of radial non-negative anti-symmetric smooth functions $(g_n, n \in \mathbb{N})$ with $g_n \in C_c^2(\mathbb{R}^d)$ for all $n \in \mathbb{N}$, such that g_n grows to $|z|^2$ as $n \to \infty$ and $|\nabla g_n|^2 \leq C g_n$ and $|\Delta g_n|$ is uniformly bounded in $n \in \mathbb{N}$. For instance one can choose

$$
\chi_n(z) := \begin{cases} |z|, & \text{for } |z| \geq \frac{1}{n}, \\ -n^3\frac{|z|^4}{8} + n\frac{3|z|^2}{4} + \frac{3}{8n}, & \text{for } |z| \leq \frac{1}{n}, \end{cases}
$$

and let $(\zeta_n, n \in \mathbb{N})$ be a sequence of compactly supported cut-off function defined by $\zeta_n(x) = \zeta(x/n)$, where ζ is a smooth function with support in the ball of radius two and has value one in the unit ball. The reader can verify that $g_n = \chi_n^2\zeta_n$ satisfies the above properties.

Plugging g_n as our test function, we obtain

$$\begin{aligned}
&\langle \rho_t, g_n\rangle_{L^2(\mathbb{R}^d)} \\
&\quad = \langle \rho_0, g_n\rangle_{L^2(\mathbb{R}^d)} - \int_0^t \Big\langle (k * u_s)\rho_s, \nabla g_n \Big\rangle_{L^2(\mathbb{R}^d)} \,\mathrm{d}s + \sum_{\alpha=1}^{d}\sum_{\hat{l}=1}^{\tilde{m}} \int_0^t \Big\langle \nu_s^{\alpha,\hat{l}}\rho_s, \partial_{z_\alpha} g_n \Big\rangle_{L^2(\mathbb{R}^d)} W_s^{\hat{l}} \\
&\qquad - \frac{1}{2}\int_0^t \Big\langle (\sigma_s^2 + \nu_s^2)\rho_s, \Delta g_n \Big\rangle_{L^2(\mathbb{R}^d)} \mathrm{d}s.
\end{aligned}$$

The first term, can be simply estimated by

$$\begin{aligned}
\int_0^t \Big\langle (k * u_s)\rho_s, \nabla g_n \Big\rangle_{L^2(\mathbb{R}^d)} \mathrm{d}s &\le C\,\|k\|^2_{L^\infty(\mathbb{R}^d)} + \int_0^t \langle \rho_s, |\nabla g_n|^2\rangle_{L^2(\mathbb{R}^d)}\,\mathrm{d}s \\
&\le C\,\|k\|^2_{L^\infty(\mathbb{R}^d)} + C\int_0^t \langle \rho_s, g_n\rangle_{L^2(\mathbb{R}^d)}\,\mathrm{d}s.
\end{aligned}$$

For the other term we obtain

$$\int_0^t \Big\langle (\sigma_s^2 + \nu_s^2)\rho_s, \Delta g_n \Big\rangle_{L^2(\mathbb{R}^d)} \mathrm{d}s \le C\int_0^t \int_{\mathbb{R}^d} \rho_s(x)\,\mathrm{d}x\,\mathrm{d}s \le CT,$$

where we used the uniform bound of $|\Delta g_n|$. Now, we take the square and then the expectation to arrive at

$$\begin{aligned}
\mathbb{E}\big(\langle \rho_t, g_n\rangle^2_{L^2(\mathbb{R}^d)}\big) &\le \langle \rho_0, g_n\rangle_{L^2(\mathbb{R}^d)} + CT^2\,\|k\|^4_{L^\infty(\mathbb{R}^d)} + CT^2 + Ct^{\frac{1}{2}}\int_0^t \mathbb{E}\big(\langle \rho_s, g_n\rangle^2_{L^2(\mathbb{R}^d)}\big)\,\mathrm{d}s \\
&\quad + \mathbb{E}\Bigg(\Bigg|\sum_{\alpha=1}^{d}\sum_{\hat{l}=1}^{\tilde{m}} \int_0^t \Big\langle \nu_s^{\alpha,\hat{l}}\rho_s, \partial_{z_\alpha} g_n \Big\rangle_{L^2(\mathbb{R}^d)} W_s^{\hat{l}}\Bigg|^2\Bigg). \qquad (5.11)
\end{aligned}$$

Using the BDG-inequality to estimate the stochastic integral we arrive at

$$\begin{aligned}
&\mathbb{E}\Bigg(\Bigg|\sum_{\alpha=1}^{d}\sum_{\hat{l}=1}^{\tilde{m}} \int_0^t \Big\langle \nu_s^{\alpha,\hat{l}}\rho_s, \partial_{z_\alpha} g_n \Big\rangle_{L^2(\mathbb{R}^d)} W_s^{\hat{l}}\Bigg|^2\Bigg) \\
&\quad \le \sum_{\alpha,\beta=1}^{d}\sum_{\hat{l}=1}^{\tilde{m}} \mathbb{E}\Bigg(\int_0^t \Big\langle \nu_s^{\alpha,\hat{l}}\rho_s, \partial_{z_\alpha} g_n \Big\rangle_{L^2(\mathbb{R}^d)} \Big\langle \nu_s^{\beta,\hat{l}}\rho_s, \partial_{z_\beta} g_n \Big\rangle_{L^2(\mathbb{R}^d)} \mathrm{d}s\Bigg) \\
&\quad \le C(\|\nu\|_{L^\infty(\mathbb{R}^d)}, \tilde{m}, d)\sum_{\alpha=1}^{d} \mathbb{E}\Bigg(\int_0^t \Big|\langle \rho_s, \partial_{z_\alpha} g_n\rangle_{L^2(\mathbb{R}^d)}\Big|^2\,\mathrm{d}s\Bigg)
\end{aligned}$$

$$\leq C(\|\nu\|_{L^\infty(\mathbb{R}^d)}, \tilde{m}, d)\, \mathbb{E}\bigg(\int_0^t \langle \rho_s, |\nabla g_n|^2 \rangle_{L^2(\mathbb{R}^d)} \,\mathrm{d}s \bigg)$$
$$\leq C(\|\nu\|_{L^\infty(\mathbb{R}^d)}, \tilde{m}, d) \int_0^t \mathbb{E}\Big(\langle \rho_s, g_n \rangle_{L^2(\mathbb{R}^d)} \Big) \,\mathrm{d}s.$$

An application of Gronwall's lemma proves

$$\sup_{0\leq t\leq T} \mathbb{E}\bigg(\bigg| \int_{\mathbb{R}^d} \rho_t(z)|z|^2 \,\mathrm{d}z \bigg|^2 \bigg) \leq C(T, \sigma, \nu, \|k\|_{L^\infty(\mathbb{R}^d)}) \bigg(\int_{\mathbb{R}^d} \rho_0(z)|z|^2 \,\mathrm{d}z \bigg)^2.$$

Now, we can use this inequality to estimate the case, where the supremum is inside the expectation. Till inequality 5.11 we follow the same steps but with the difference that the supremum is now inside the expectation. We notice that we can apply Doob's maximal inequality to estimate the stochastic integral with the previous bound. Thus, obtaining the bound

$$\mathbb{E}\bigg(\sup_{0\leq t\leq T} \bigg(\int_{\mathbb{R}^d} \rho_t(z)|z|^2 \,\mathrm{d}z \bigg)^2 \bigg) \leq C(T, \sigma, \nu, \|k\|_{L^\infty(\mathbb{R}^d)}) \bigg(\int_{\mathbb{R}^d} \rho_0(x)|z|^2 \,\mathrm{d}z \bigg)^2.$$

□

We are ready to prove that the non-local d-dimensional SPDE (5.5) has a unique solution.

PROPOSITION 5.9 (Existence of d-dimensional SPDE ρ). *Let $k \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then the SPDE* (5.5) *has a unique solution solution in the space $L^2_{\mathcal{F}W}([0,T]; H^1(\mathbb{R}^d))$ satisfying*

(5.12) $$\|\rho\|_{L^2_{\mathcal{F}W}([0,T];H^1(\mathbb{R}^d))} \leq C(T, \delta, d, \|k\|_{L^2(\mathbb{R}^d)}, \|k\|_{L^\infty(\mathbb{R}^d)}) \|\rho_0\|_{L^2(\mathbb{R}^d)}.$$

PROOF. Let us proof the existence by a Picard-Lindelöf iteration. Let us define for $n \in \mathbb{N}$ the following SPDE

(5.13) $$\begin{aligned} \mathrm{d}\rho_t^n &= \nabla \cdot ((k * \rho_t^{n-1})\rho_t^n))\,\mathrm{d}t - \nabla \cdot (\nu_t \rho^n \,\mathrm{d}W_t) \\ &\quad + \frac{1}{2} \sum_{\alpha,\beta=1}^d \partial_{z_\alpha}\partial_{z_\beta} \Big(([\sigma_t\sigma_t^{\mathrm{T}}]_{(\alpha,\beta)} + [\nu_t\nu_t^{\mathrm{T}}]_{(\alpha,\beta)})\rho_t^n \Big)\,\mathrm{d}t \end{aligned}$$

with initial value ρ_0 and $\rho^0 = \rho_0$. The initial value holds for all SPDE's in this proof. Then the SPDE is linear and by the estimate

(5.14) $$\|\nabla \cdot ((k * \rho^0)u)\|_{H^{-1}(\mathbb{R}^d)} \leq \|k * \rho^0\|_{L^\infty(\mathbb{R}^d)} \|u\|_{L^2(R^d)} \leq \|k\|_{L^\infty(\mathbb{R}^d)} \|u\|_{L^2(R^d)}$$

for $u \in L^2_{\mathcal{F}W}([0,T]; H^1(\mathbb{R}^d))$ we have a solution of the SPDE by [Kry99, Theorem 5.1] in the case $n = 1$, therein. Furthermore, it is easy consequence of the divergence structure and the maximum principle that ρ^1 is non-negative and satisfies mass conservation

$$\|\rho_t^1\|_{L^1(\mathbb{R}^d)} = \|\rho_0\|_{L^1(\mathbb{R}^d)} = 1, \quad \mathbb{P}\text{-a.s.},$$

for $t \in [0,T]$. A more detailed proof is given in [CNP23] in a similar setting. Hence, since ρ^1 has measure one uniform in time and Ω, we can derive, using the same arguments for each

$n \in \mathbb{N}$, a non-negative solution that satisfies mass conservation and the uniform estimate

$$\|\rho^n\|_{L^2_{\mathcal{F}W}([0,T];H^1(\mathbb{R}^d))} \le C(T,\delta,d)\,\|\rho_0\|_{L^2(\mathbb{R}^d)} \tag{5.15}$$

as provided by [Kry99, Theorem 5.1]. Additionally, ρ^n satisfies the L^2-bound (5.9) uniform in $n \in \mathbb{N}$. Let us consider the difference of the SPDE's

$$\begin{aligned}
&\mathrm{d}(\rho^n_t - \rho^{n-1}_t)\\
&\quad= \nabla\cdot\big((k*\rho^{n-1}_t)(\rho^n_t-\rho^{n-1}_t) + (k*(\rho^{n-1}_t-\rho^{n-2}_t))\rho^{n-1}_t\big)\,\mathrm{d}t - \nabla\cdot(\nu_t(\rho^n_t-\rho^{n-1}_t)\,\mathrm{d}W_t)\\
&\qquad+\frac{1}{2}\sum_{\alpha,\beta=1}^{d}\partial_{z_\alpha}\partial_{z_\beta}\Big(([\sigma_t\sigma_t^{\mathrm{T}}]_{(\alpha,\beta)}+[\nu_t\nu_t^{\mathrm{T}}]_{(\alpha,\beta)})(\rho^n_t-\rho^{n-1}_t)\Big)\,\mathrm{d}t.
\end{aligned}$$

Applying Itô's formula [Kry10], we obtain

$$\begin{aligned}
&\big\|\rho^n_t-\rho^{n-1}_t\big\|^2_{L^2(\mathbb{R}^d)}\\
&\quad= -2\int_0^t\int_{\mathbb{R}^d}\big((k*\rho^{n-1}_s)(\rho^n_s-\rho^{n-1}_s)-(k*(\rho^{n-1}_s-\rho^{n-2}_s))\rho^{n-1}_s\big)\cdot\nabla(\rho^n_s-\rho^{n-1}_s)\,\mathrm{d}z\,\mathrm{d}s\\
&\qquad-\sum_{\alpha,\beta=1}^{d}\int_0^t\int_{\mathbb{R}^d}\partial_{z_\beta}\Big(([\sigma_s\sigma_s^{\mathrm{T}}]_{(\alpha,\beta)}+[\nu_s\nu_s^{\mathrm{T}}]_{(\alpha,\beta)})(\rho^n_s-\rho^{n-1}_s)\Big)\partial_{z_\alpha}(\rho^n_s-\rho^{n-1}_s)\,\mathrm{d}z\,\mathrm{d}s\\
&\qquad+\sum_{\hat{l}=1}^{\tilde{m}}\int_0^t\int_{\mathbb{R}^d}\bigg|\sum_{\beta=1}^{d}\partial_{z_\beta}(\nu^{\beta,\hat{l}}_s(\rho^n_s-\rho^{n-1}_s))\bigg|^2\,\mathrm{d}z\,\mathrm{d}s\\
&\qquad+2\sum_{\hat{l}=1}^{\tilde{m}}\sum_{\beta=1}^{d}\int_0^t\int_{\mathbb{R}^d}(\rho^n_s-\rho^{n-1}_s)\partial_{z_\beta}(\nu^{\beta,\hat{l}}_s(\rho^n_s-\rho^{n-1}_s))\,\mathrm{d}x\,\mathrm{d}W^{\hat{l}}_s.
\end{aligned}$$

Similar as before the stochastic integral vanishes by the divergence free assumption on ν. Again using the divergence-free assumption 5.1 (3) we find

$$\begin{aligned}
&\sum_{\hat{l}=1}^{\tilde{m}}\int_0^t\int_{\mathbb{R}^d}\bigg|\sum_{\beta=1}^{d}\partial_{z_\beta}(\nu^{\beta,\hat{l}}_s(\rho^n_s-\rho^{n-1}_s))\bigg|^2\,\mathrm{d}z\,\mathrm{d}s\\
&\quad=\sum_{\hat{l}=1}^{\tilde{m}}\sum_{\alpha,\beta=1}^{d}\int_0^t\int_{\mathbb{R}^d}\partial_{z_\alpha}(\nu^{\alpha,\hat{l}}_s(\rho^n_s-\rho^{n-1}_s))\partial_{z_\beta}(\nu^{\beta,\hat{l}}_s(\rho^n_s-\rho^{n-1}_s))\,\mathrm{d}z\,\mathrm{d}s\\
&\quad=\sum_{\hat{l}=1}^{\tilde{m}}\sum_{\alpha,\beta=1}^{d}\int_0^t\int_{\mathbb{R}^d}\nu^{\alpha,\hat{l}}_s\nu^{\beta,\hat{l}}_s\partial_{z_\alpha}(\rho^n_s-\rho^{n-1}_s)\partial_{z_\beta}(\rho^n_s-\rho^{n-1}_s)\,\mathrm{d}z\,\mathrm{d}s.
\end{aligned}\tag{5.16}$$

However, we also have

$$\sum_{\alpha,\beta=1}^{d}\int_0^t\int_{\mathbb{R}^d}\partial_{z_\beta}\Big(([\sigma_s\sigma_s^{\mathrm{T}}]_{(\alpha,\beta)}+[\nu_s\nu_s^{\mathrm{T}}]_{(\alpha,\beta)})(\rho_s^n-\rho_s^{n-1})\Big)\partial_{z_\alpha}(\rho_s^n-\rho_s^{n-1})\,\mathrm{d}z\,\mathrm{d}s$$

$$=\sum_{\alpha,\beta=1}^{d}\int_0^t\int_{\mathbb{R}^d}\partial_{z_\beta}\Big([\sigma_s\sigma_s^{\mathrm{T}}]_{(\alpha,\beta)}+[\nu_s\nu_s^{\mathrm{T}}]_{(\alpha,\beta)}\Big)(\rho_s^n-\rho_s^{n-1})\partial_{z_\alpha}(\rho_s^n-\rho_s^{n-1})\,\mathrm{d}z\,\mathrm{d}s$$

$$+\sum_{\alpha,\beta=1}^{d}\int_0^t\int_{\mathbb{R}^d}([\sigma_s\sigma_s^{\mathrm{T}}]_{(\alpha,\beta)}+[\nu_s\nu_s^{\mathrm{T}}]_{(\alpha,\beta)})\partial_{z_\beta}(\rho_s^n-\rho_s^{n-1})\partial_{z_\alpha}(\rho_s^n-\rho_s^{n-1})\,\mathrm{d}z\,\mathrm{d}s.$$

The last term with ν is exactly the term (5.16) and therefore both term cancel. For the first term we use the uniform bound on the derivatives and Young's inequality to estimated it by

$$\sum_{\alpha,\beta=1}^{d}\left|\int_0^t\int_{\mathbb{R}^d}\partial_{z_\beta}\Big([\sigma_s\sigma_s^{\mathrm{T}}]_{(\alpha,\beta)}+[\nu_s\nu_s^{\mathrm{T}}]_{(\alpha,\beta)}\Big)(\rho_s^n-\rho_s^{n-1})\partial_{z_\alpha}(\rho_s^n-\rho_s^{n-1})\,\mathrm{d}z\,\mathrm{d}s\right|$$

$$\leq C(d,\delta)\left\|\rho_s^n-\rho_s^{n-1}\right\|_{L^2(\mathbb{R}^d)}^2+\frac{\delta}{2}\left\|\nabla(\rho_s^n-\rho_s^{n-1})\right\|_{L^2(\mathbb{R}^d)}^2.$$

Combining the last estimates we arrive at

$$\begin{aligned}
&\left\|\rho_t^n-\rho_t^{n-1}\right\|_{L^2(\mathbb{R}^d)}^2\\
&=-2\int_0^t\int_{\mathbb{R}^d}((k*\rho_s^{n-1})(\rho_s^n-\rho_s^{n-1})-(k*(\rho_s^{n-1}-\rho_s^{n-2}))\rho_s^{n-1})\cdot\nabla(\rho_s^n-\rho_s^{n-1})\,\mathrm{d}z\,\mathrm{d}s\\
&\quad-\sum_{\alpha,\beta=1}^{d}\int_0^t\int_{\mathbb{R}^d}[\sigma_s\sigma_s^{\mathrm{T}}]_{(\alpha,\beta)}\partial_{z_\beta}(\rho_s^n-\rho_s^{n-1})\partial_{z_\alpha}(\rho_s^n-\rho_s^{n-1})\,\mathrm{d}z\,\mathrm{d}s\\
&\quad+\frac{\delta}{2}\int_0^t\left\|\nabla(\rho_s^n-\rho_s^{n-1})\right\|_{L^2(\mathbb{R}^d)}^2\,\mathrm{d}s+C(d,\delta)\int_0^t\left\|\rho_s^n-\rho_s^{n-1}\right\|_{L^2(\mathbb{R}^d)}^2\,\mathrm{d}s\\
&\leq 4\delta\int_0^t\left\|k*\rho_s^{n-1}\right\|_{L^\infty(\mathbb{R}^d)}^2\left\|\rho_s^n-\rho_s^{n-1}\right\|_{L^2(\mathbb{R}^d)}^2\,\mathrm{d}s+C(d,\delta)\int_0^t\left\|\rho_s^n-\rho_s^{n-1}\right\|_{L^2(\mathbb{R}^d)}^2\,\mathrm{d}s\\
&\quad+4\delta\int_0^t\left\|k(\rho_s^{n-1}-\rho_s^{n-2})\right\|_{L^\infty(\mathbb{R}^d)}^2\left\|\rho_s^{n-1}\right\|_{L^2(\mathbb{R}^d)}^2\,\mathrm{d}s\\
&\leq C(d,\delta,\|k\|_{L^\infty})\int_0^t\left\|\rho_s^n-\rho_s^{n-1}\right\|_{L^2(\mathbb{R}^d)}^2\,\mathrm{d}s+4\delta\left\|k\right\|_{L^2(\mathbb{R}^d)}^2\int_0^t\left\|\rho_s^{n-1}-\rho_s^{n-2}\right\|_{L^2(\mathbb{R}^d)}^2\,\mathrm{d}s,
\end{aligned}$$

where we used the uniform L^2 bound (5.9) and ellipticity condition in the last step. Taking expectation and using Gronwall's lemma, we find

$$\mathbb{E}\big(\left\|\rho_t^n - \rho_t^{n-1}\right\|^2_{L^2(\mathbb{R}^d)} \big) \leq 4\delta \left\|k\right\|^2_{L^2(\mathbb{R}^d)} e^{C(d,\delta,\|k\|_{L^\infty})T} \int_0^t \mathbb{E}\big(\left\|\rho_s^{n-1} - \rho_s^{n-2}\right\|^2_{L^2(\mathbb{R}^d)}\big) \,\mathrm{d}s.$$

The standard Picard–Lindelöf iteration implies that

$$\sum_{n=1}^{\infty} \sup_{0\leq t\leq T} \mathbb{E}\big(\left\|\rho_t^n - \rho_t^{n-1}\right\|^2_{L^2(\mathbb{R}^d)} \big) < \infty$$

and therefore we can find a function $\rho \in L^2_{\mathcal{F}W}([0,T]; H^1(\mathbb{R}^d))$ such that

$$\lim_{n\to\infty} \left\|\rho^n - \rho\right\|_{L^2_{\mathcal{F}W}([0,T];H^1(\mathbb{R}^d))} = 0. \tag{5.17}$$

Furthermore, ρ satisfies mass conservation and therefore by similar arguments as before there exists a solution $\hat{\rho}$ of the linear SPDE

$$\begin{aligned} \mathrm{d}\hat{\rho}_t &= \nabla \cdot ((k * \rho_t)\hat{\rho}_t)) \,\mathrm{d}t - \nabla \cdot (\nu_t \hat{\rho}_t \,\mathrm{d}W_t) \\ &\quad + \frac{1}{2} \sum_{\alpha,\beta=1}^{d} \partial_{z_\alpha}\partial_{z_\beta} \Big(([\sigma_t \sigma_t^{\mathrm{T}}]_{(\alpha,\beta)} + [\nu_t \nu_t^{\mathrm{T}}]_{(\alpha,\beta)})\hat{\rho}_t \Big) \,\mathrm{d}t, \end{aligned}$$

which also satisfies the L^2-bound (5.9) . Applying the inequality in [Kry99, Theorem 5.1] and the L^2-bound (5.9) we find

$$\begin{aligned} \left\|\hat{\rho} - \rho^n\right\|_{L^2_{\mathcal{F}W}([0,T];H^1(\mathbb{R}^d))} &\leq \left\|(k * (\rho - \rho^n))\hat{\rho}\right\|_{L^2_{\mathcal{F}W}([0,T];L^2(\mathbb{R}^d))} \\ &\leq C \left\|\rho - \rho^n\right\|_{L^2_{\mathcal{F}W}([0,T];L^2(\mathbb{R}^d))} \\ &\xrightarrow{n\to\infty} 0. \end{aligned}$$

Hence $\rho = \hat{\rho}$ and we have a solution to the non-linear SPDE (5.5). The properties of ρ are a direct consequence of the properties of ρ^n and the strong convergence in $L^2_{\mathcal{F}W}([0,T]; H^1(\mathbb{R}^d))$. □

REMARK 5.10. *We cannot use the inequality provided by* [Kry99, Theorem 5.1] *directly since our SPDE is non-linear. Therefore, we need to rely on a Gronwall argument and estimate the* $L^2(\mathbb{R}^d)$*-norm directly. We purposely choose this presentation of the existence of the SPDE* (5.5) *because we will need a stability result in Section 5.4, which follows the same idea. Alternatively, we could have verified the monotonicity conditions of Wei and Röckner* [LR15].

LEMMA 5.11. *Let* ρ_t *be a solution of the d-dimensional SPDE* (5.5) *provided by Proposition 5.9. Then the tensor product* $\rho^{\otimes N}(\mathsf{X}^N) = \prod_{i=1}^{N} \rho(x_i)$ *for* $\mathsf{X}^N = (x_1, \ldots, x_N) \in \mathbb{R}^{dN}$ *solves the dN-dimensional SPDE* (5.6) *in the sense of Definition 5.5.*

Proof. By the structure of the solution we immediately obtain the regularity necessary in Definition 5.5. Let $\varphi_i \in C_c^\infty(\mathbb{R}^d)$ for $i = 1, \ldots, N$) be smooth functions. Then for $\varphi(\mathsf{X}^N) = \prod_{i=1}^N \varphi(x_i)$ we obtain

$$\langle \rho_t^{\otimes N}, \varphi \rangle_{L^2(\mathbb{R}^{dN})} = \prod_{i=1}^N \langle \rho_t, \varphi_i \rangle_{L^2(\mathbb{R}^d)}.$$

The right hand side is now a product of Itô processes. Hence applying the Itô formula (1.15) to the function $f(y) = y_1 y_2 \cdots y_N$ for $y \in \mathbb{R}^N$ we obtain

$$\begin{aligned} &\mathrm{d}\Big(\prod_{i=1}^N \langle \rho_t, \varphi_i \rangle_{L^2(\mathbb{R}^d)}\Big) \\ &= \sum_{i=1}^N \prod_{j\neq i}^N \langle \rho_t, \varphi_j \rangle_{L^2(\mathbb{R}^d)} \,\mathrm{d}\langle \rho_t, \varphi_i \rangle_{L^2(\mathbb{R}^d)} \\ &\quad + \frac{1}{2} \sum_{\substack{i,j=1 \\ i\neq j}}^N \prod_{\substack{q\neq i \\ q\neq j}}^N \langle \rho_t, \varphi_q \rangle_{L^2(\mathbb{R}^d)} \,\mathrm{d}\langle \langle \rho_t, \varphi_i \rangle_{L^2(\mathbb{R}^d)}, \langle \rho_t, \varphi_j \rangle_{L^2(\mathbb{R}^d)} \rangle. \end{aligned}$$

Computing the quadratic variation we find

$$\begin{aligned} &\langle \langle \rho_t, \varphi_i \rangle_{L^2(\mathbb{R}^d)}, \langle \rho_t, \varphi_j \rangle_{L^2(\mathbb{R}^d)} \rangle \\ &= \Big\langle \sum_{\alpha=1}^d \sum_{l=1}^{\tilde m} \int_0^t \Big\langle \nu^{\alpha,l}(s,x_i)\rho_s, \partial_{x_{i,\alpha}} \varphi_i \Big\rangle_{L^2(\mathbb{R}^d)} W_s^l, \\ &\qquad \sum_{\beta=1}^d \sum_{\hat l=1}^{\tilde m} \int_0^t \Big\langle \nu^{\beta,\hat l}(s,x_j)\rho_s, \partial_{x_{j,\beta}} \varphi_j \Big\rangle_{L^2(\mathbb{R}^d)} W_s^{\hat l} \Big\rangle \\ &= \sum_{\alpha,\beta=1}^d \sum_{\hat l=1}^{\tilde m} \int_0^t \Big\langle \nu^{\alpha,\hat l}(s,x_i)\rho_s, \partial_{x_{i,\alpha}} \varphi_i \Big\rangle_{L^2(\mathbb{R}^d)} \Big\langle \nu^{\beta,\hat l}(s,x_j)\rho_s, \partial_{x_{j,\beta}} \varphi_j \Big\rangle_{L^2(\mathbb{R}^d)} \,\mathrm{d}s, \end{aligned}$$

which implies the following form for the covariation term

$$\frac{1}{2} \sum_{\substack{i,j=1 \\ i\neq j}}^N \sum_{\alpha,\beta=1}^d \sum_{\hat l=1}^{\tilde m} \int_0^t \langle \nu^{\alpha,\hat l}(s,x_i) \nu^{\beta,\hat l}(s,x_j) \Big(\prod_{q=1}^N \rho_s\Big)(\cdot), \partial_{x_{i,\alpha}} \partial_{x_{j,\beta}} \varphi \rangle_{L^2(\mathbb{R}^d)} \,\mathrm{d}s.$$

Plugging in the dynamic $\langle \rho_t, \varphi \rangle$, we find

$$\begin{aligned}
&\sum_{i=1}^{N} \prod_{j\neq i}^{N} \langle \rho_t, \varphi_j \rangle_{L^2(\mathbb{R}^d)} \, \mathrm{d} \langle \rho_t, \varphi_i \rangle_{L^2(\mathbb{R}^d)} \\
&\quad = \sum_{i=1}^{N} \int_0^t \prod_{j\neq i}^{N} \langle \rho_s, \varphi_j \rangle_{L^2(\mathbb{R}^d)} \bigg(\Big\langle (k * \rho_s)(x_i) \rho_s, \nabla \varphi_i \Big\rangle_{L^2(\mathbb{R}^d)} \\
&\quad + \frac{1}{2} \sum_{\alpha,\beta=1}^{d} \Big\langle [\sigma(s,\cdot)\sigma(s,\cdot)^{\mathrm{T}}]_{(\alpha,\beta)} \rho_s, \partial_{x_{i,\beta}} \partial_{x_{i,\alpha}} \varphi_i \Big\rangle_{L^2(\mathbb{R}^d)} \\
&\quad + \frac{1}{2} \sum_{\alpha,\beta=1}^{d} \Big\langle [\nu(s,x_i)\nu(s,x_i)^{\mathrm{T}}]_{(\alpha,\beta)} \rho_s, \partial_{x_{i,\beta}} \partial_{x_{i,\alpha}} \varphi_i \Big\rangle_{L^2(\mathbb{R}^d)} \bigg) \, \mathrm{d}s \\
&\quad + \sum_{i=1}^{N} \sum_{\alpha=1}^{d} \sum_{\hat{l}=1}^{\tilde{m}} \int_0^t \prod_{j\neq i}^{N} \langle \rho_s, \varphi_j \rangle_{L^2(\mathbb{R}^d)} \Big\langle \nu^{\alpha,\hat{l}}(s,x_i) \rho_s^{\otimes N}, \partial_{x_{i,\alpha}} \varphi_i \Big\rangle_{L^2(\mathbb{R}^d)} W_s^{\hat{l}}.
\end{aligned}$$

Putting the product into one integral over $\mathbb{R}^{dN}$ as before we see that $\prod_{i=1}^{N} \rho_t(x_i)$ satisfies equation (5.8) for $\varphi \in C_c^{\infty}(\mathbb{R}^{dN})$ of the form $\varphi(\mathsf{X}^N) = \prod_{i=1}^{N} \varphi(x_i)$. Because functions of this form are dense in $L^2(\mathbb{R}^{dN})$ the claim follows by the regularity of ρ. □

REMARK 5.12. *At the end of the section let us make some comments on the regularity of the interaction kernel k. First, we are concerned with the global existence of solutions, meaning for arbitrary $T > 0$. Otherwise, the conditional propagation of chaos results seems not powerful, since we assume our initial condition is independent and, therefore, the independence should at least propagate for some small $T > 0$. Hence, to demonstrate global existence, the map $u \mapsto (k * u)\rho$ needs to be a continuous linear map. One way to enforce this is for $k * u$ to be uniformly bounded in the probability space and in the time variable, as discussed in* [HQ21, CNP23]. *Additionally, Itô's formula for the L^p-norms requires the condition $p \geq 2$. Consequently, lacking concrete properties of k, we require k to belong to $L^{p'}(\mathbb{R}^d)$, where p' is the conjugated exponent.*

However, in order to apply the exponential law of large numbers [JW18, Theorem 3] *in Section 5.3, we require a bounded force k. Therefore, the condition $k \in L^2(\mathbb{R}^d) \cap L^{\infty}(\mathbb{R}^d)$ appears to be necessary and minimal.*

5.3. Relative entropy method in the smooth case

In this section we assume that all coefficients σ, ν, k are smooth uniform in time. More precisely, for all indices $\alpha, \beta, \tilde{l}$ we assume

$$\sup_{0\leq t\leq T} \big\| [\sigma(t,\cdot)\sigma^{\mathrm{T}}(t,\cdot)]_{(\alpha,\beta)} \big\|_{C^{\infty}(\mathbb{R}^d)} + \sup_{0\leq t\leq T} \Big\| \nu^{\beta,\tilde{l}}(t,\cdot) \Big\|_{C^{\infty}(\mathbb{R}^d)} + \|k\|_{C^{\infty}(\mathbb{R}^d)} \leq C \tag{5.18}$$

for some constant $C > 0$ and Assumption 5.1 holds. The idea is to demonstrate the relative entropy estimate in cases where $\rho^N, \rho^{\otimes N}$ solve the SPDE's 5.4, 5.6 , not in the weak sense against a test function, but rather pointwise. The key to achieving this lies in the Sobolev embedding, which mandates at least two derivatives to properly interpret the pointwise second derivatives in the SPDE's 5.4, 5.6.

PROPOSITION 5.13. *For each $N \in \mathbb{N}$ the solutions ρ_t^N and $\rho_t^{\otimes N}$ are smooth and there exists versions, which have an Ito process representation.*

Hence, there exists a set $\tilde{\Omega}$ with $\mathbb{P}(\tilde{\Omega}) = 1$ such that for all $(t, \mathsf{X}^N) \in [0,T] \times \mathbb{R}^{dN}$ we have

$$\begin{aligned}
&\rho^N(\omega, t, \mathsf{X}^N) \\
&\quad = \rho_0(x) + \sum_{i=1}^N \int_0^t \nabla_{x_i} \cdot \Big(\frac{1}{N} \sum_{j=1}^N k(x_i - x_j) \rho^N(\omega, s, \mathsf{X}^N) \Big) \, \mathrm{d}s \\
&\quad - \sum_{i=1}^N \sum_{\alpha=1}^d \sum_{\hat{l}=1}^{\tilde{m}} \int_0^t \nu^{\alpha,\hat{l}}(s, x_i) \partial_{x_{i,\alpha}} \rho^N(\omega, s, \mathsf{X}^N) W_s^{\hat{l}} \\
&\quad + \frac{1}{2} \sum_{i=1}^N \sum_{\alpha,\beta=1}^d \int_0^t \partial_{x_{i,\alpha}} \partial_{x_{i,\beta}} \Big([\sigma(s, x_i) \sigma(s, x_i)^{\mathrm{T}}]_{(\alpha,\beta)} \rho^N(\omega, s, \mathsf{X}^N) \Big) \, \mathrm{d}s \\
&\quad + \frac{1}{2} \sum_{i,j=1}^N \sum_{\alpha,\beta=1}^d \int_0^t \partial_{x_{i,\alpha}} \partial_{x_{j,\beta}} \Big(+ [\nu(s, x_i) \nu(s, x_j)^{\mathrm{T}}]_{(\alpha,\beta)} \rho^N(\omega, s, \mathsf{X}^N) \Big) \, \mathrm{d}s
\end{aligned}$$

and

$$\begin{aligned}
&\rho^{\otimes N}(\omega, t, \mathsf{X}^N) \\
&\quad = \rho_0(x) + \sum_{i=1}^N \int_0^t \nabla_{x_i} \cdot \Big((k * \rho)(\omega, t, \mathsf{X}^N) \rho^{\otimes N}(\omega, s, \mathsf{X}^N) \Big) \, \mathrm{d}s \\
&\quad - \sum_{i=1}^N \sum_{\alpha=1}^d \sum_{\hat{l}=1}^{\tilde{m}} \int_0^t \nu^{\alpha,\hat{l}}(s, x_i) \partial_{x_{i,\alpha}} \rho^{\otimes N}(\omega, s, \mathsf{X}^N) W_s^{\hat{l}} \\
&\quad + \frac{1}{2} \sum_{i,j=1}^N \sum_{\alpha,\beta=1}^d \int_0^t \partial_{x_{i,\alpha}} \partial_{x_{i,\beta}} \Big([\sigma(s, x_i) \sigma(s, x_i)^{\mathrm{T}}]_{(\alpha,\beta)} \rho^{\otimes N}(\omega, s, \mathsf{X}^N) \Big) \, \mathrm{d}s \\
&\quad + \frac{1}{2} \sum_{i,j=1}^N \sum_{\alpha,\beta=1}^d \int_0^t \partial_{x_{i,\alpha}} \partial_{x_{j,\beta}} \Big([\nu(s, x_i) \nu(s, x_j)^{\mathrm{T}}]_{(\alpha,\beta)} \rho^{\otimes N}(\omega, s, \mathsf{X}^N) \Big) \, \mathrm{d}s.
\end{aligned}$$

PROOF. Let us fix $N \in \mathbb{N}$ and choose a $n \in \mathbb{N}$ such that $n > 2 + \frac{dN}{2}$. We start with the linear N-particle SPDE. Similar to the non-smooth case, we can obtain a solution

$$\rho^N \in L^2_{\mathcal{F}W}([0,T]; H^1(\mathbb{R}^d)),$$

which satisfies mass conservation. Since all coefficients are smooth we can write the SPDE's in non-divergence form. Therefore, we can apply [Kry99, Theorem 5.1 and Remark 5.6] in the case $p = 2$ and n as above to obtain

$$\rho^N \in L^2_{\mathcal{F}^W}([0,T]; H^n(\mathbb{R}^d)).$$

The Itô representation, then immediately follows from the regularity and [Roz90, Theorem 4.3(e)]. For the non-linear SPDE $\rho^{\otimes N}$ we repeat the arguments with one slight modification. After obtaining the solution by the same steps as in Section 5.2, we linearize the equation by fixing ρ in the convolution $k * \rho$. Now, the coefficients are again smooth and for arbitrary multi-index γ we have

$$\|\partial^\gamma (k * \rho_t)\|_{L^\infty} \le \|k\|_{C^{|\gamma|}(\mathbb{R}^d)}$$

by Young's inequality and mass conservation of ρ. This provides a smooth solution ρ and consequently, by the same argument as in Lemma 5.11, we obtain a smooth solution $\rho^{\otimes N}$, which again has the Itô representation by [Roz90, Theorem 4.3(e)].

□

For each $\mathsf{X}^N \in \mathbb{R}^{dN}$ we obtain two process $\rho_t^N(\mathsf{X}^N)$ and $\rho_t^{\otimes N}(\mathsf{X}^N)$. Consequently, we can utilize Itô's formula to derive an evolution of the relative entropy

$$\mathcal{H}(\rho_t^N|\rho_t^{\otimes N})\colon = \int_{\mathbb{R}^d} \rho_t^N(\mathsf{X}^N) \log\left(\frac{\rho_t^N(\mathsf{X}^N)}{\rho_t^{\otimes N}(\mathsf{X}^N)}\right) \mathrm{d}\rho_t^N(\mathsf{X}^N).$$

To begin, we split the integrand $x\log(x/y)$ of the relative entropy into two terms $x\log(x)$ and $-x\log(y)$. Subsequently, we apply Itô's formula to each function separately and combine them later.

Lemma 5.14 (Itô's formula for $x\log(x)$). *Let ρ_t^N be the smooth solution provided by Proposition 5.13. Then,*

$$\begin{aligned}
&\mathrm{d}\rho_t^N(\mathsf{X}^N)\log(\rho_t^N)(\mathsf{X}^N) \\
&= (\log(\rho_t^N(\mathsf{X}^N)) + 1)\Bigg(\sum_{i=1}^N \nabla_{x_i}\cdot\left(\frac{1}{N}\sum_{j=1}^N k(x_i - x_j)\rho_t^N(\mathsf{X}^N)\right)\mathrm{d}t \\
&\quad - \sum_{i=1}^N\sum_{\alpha=1}^d\sum_{\hat{l}=1}^{\tilde{m}} \nu_t^{\alpha,\hat{l}}(x_i)\partial_{x_{i,\alpha}}\rho_t^N(\mathsf{X}^N)\,\mathrm{d}W_t^{\hat{l}} \\
&\quad + \frac{1}{2}\sum_{i=1}^N\sum_{\alpha,\beta=1}^d \partial_{x_{i,\alpha}}\partial_{x_{i,\beta}}\left([\sigma_t(x_i)\sigma_t(x_i)^{\mathrm{T}}]_{(\alpha,\beta)}\rho_t^N(\mathsf{X}^N)\right)\mathrm{d}t \\
&\quad + \frac{1}{2}\sum_{i,j=1}^N\sum_{\alpha,\beta=1}^d \partial_{x_{i,\alpha}}\partial_{x_{j,\beta}}\left([\nu_t(x_i)\nu_t(x_j)^{\mathrm{T}}]_{(\alpha,\beta)}\rho_t^N(\mathsf{X}^N)\right)\mathrm{d}t\Bigg) \\
&\quad + \frac{\rho_t^N}{2}\sum_{\hat{l}=1}^{\tilde{m}}\left(\sum_{i=1}^N\sum_{\alpha=1}^d \nu_t^{\alpha,\hat{l}}(x_i)\partial_{x_{i,\alpha}}\log(\rho_t^N(\mathsf{X}^N))\right)^2\mathrm{d}t.
\end{aligned} \tag{5.19}$$

In particular, we can choose a set $\tilde{\Omega} \subseteq \Omega$ with $\mathbb{P}(\tilde{\Omega})$ such that equality (5.19) *holds for all* $\big(t, \mathsf{x}^N\big)$ *and the integrands on the right hand side of equality* (5.19) *are product measurable in* (ω, s, x).

Proof. Let us fix $\mathsf{X}^N \in \mathbb{R}^{dN}$. Then for almost all ω and all $t \geq 0$ we have

$$\mathrm{d}\big(\rho_t^N(\mathsf{X}^N)\log(\rho_t^N)(\mathsf{X}^N)\big) = \big(\log(\rho_t^N)(\mathsf{X}^N)+1\big)\,\mathrm{d}\rho_t^N + \frac{1}{2\rho_t^N(\mathsf{X}^N)}\,\mathrm{d}\langle\rho_t^N(\mathsf{X}^N)\rangle.$$

For the quadratic variation we obtain

$$\begin{aligned}\mathrm{d}\big\langle\rho_t^N(\mathsf{X}^N)\big\rangle &= \mathrm{d}\Big\langle\sum_{i=1}^N\sum_{\alpha=1}^d\sum_{\hat{l}=1}^{\tilde{m}}\nu^{\alpha,\hat{l}}(t,x_i)\partial_{x_{i,\alpha}}\rho_t^N(\mathsf{X}^N)W_t^{\hat{l}}\Big\rangle\\ &= \sum_{i,j=1}^N\sum_{\alpha,\beta=1}^d\sum_{\hat{l}=1}^{\tilde{m}}\big(\nu^{\alpha,\hat{l}}(t,x_i)\partial_{x_{i,\alpha}}\rho_t^N(\mathsf{X}^N)\big)\big(\nu^{\alpha,\hat{l}}(t,x_j)\partial_{x_{j,\beta}}\rho_t^N(\mathsf{X}^N)\big)\,\mathrm{d}t.\end{aligned}$$

Consequently, combining the quadratic variation with the dynamic given in Proposition 5.13 we obtain

$$\begin{aligned}&\mathrm{d}\big(\rho_t^N(\mathsf{X}^N)\log(\rho_t^N)(\mathsf{X}^N)\big)\\ &\quad= \big(\log\big(\rho_t^N(\mathsf{X}^N)\big)+1\big)\Bigg(\sum_{i=1}^N\nabla_{x_i}\cdot\Bigg(\frac{1}{N}\sum_{j=1}^N k(x_i-x_j)\rho_t^N(\mathsf{X}^N)\Bigg)\,\mathrm{d}t\\ &\qquad-\sum_{i=1}^N\sum_{\alpha=1}^d\sum_{\hat{l}=1}^{\tilde{m}}\nu_t^{\alpha,\hat{l}}(x_i)\partial_{x_{i,\alpha}}\rho_t^N(\mathsf{X}^N)\,\mathrm{d}W_t^{\hat{l}}\\ &\qquad+\frac{1}{2}\sum_{i=1}^N\sum_{\alpha,\beta=1}^d\partial_{x_{i,\alpha}}\partial_{x_{i,\beta}}\big([\sigma_t(x_i)\sigma_t(x_i)^{\mathrm{T}}]_{(\alpha,\beta)}\rho_t^N(\mathsf{X}^N)\big)\,\mathrm{d}t\Big)\\ &\qquad+\frac{1}{2}\sum_{i,j=1}^N\sum_{\alpha,\beta=1}^d\partial_{x_{i,\alpha}}\partial_{x_{j,\beta}}\big([\nu_t(x_i)\nu_t(x_j)^{\mathrm{T}}]_{(\alpha,\beta)}\rho_t^N(\mathsf{X}^N)\big)\,\mathrm{d}t\Big)\\ &\qquad+\frac{1}{2\rho_t^N(\mathsf{X}^N)}\sum_{i,j=1}^N\sum_{\alpha,\beta=1}^d\sum_{\hat{l}=1}^{\tilde{m}}\big(\nu_t^{\alpha,\hat{l}}(x_i)\partial_{x_{i,\alpha}}\rho_t^N(\mathsf{X}^N)\big)\big(\nu_t^{\alpha,\hat{l}}(x_j)\partial_{x_{j,\beta}}\rho_t^N(\mathsf{X}^N)\big)\,\mathrm{d}t.\end{aligned}$$

Now, we can rewrite the last term into

$$\begin{aligned}&\frac{1}{2\rho_t^N(\mathsf{X}^N)}\sum_{\hat{l}=1}^{\tilde{m}}\Bigg(\sum_{i=1}^N\sum_{\alpha=1}^d\big(\nu^{\alpha,\hat{l}}(t,x_i)\partial_{x_{i,\alpha}}\rho_t^N(\mathsf{X}^N)\big)\Bigg)^2\mathrm{d}t\\ &\quad=\frac{\rho_t^N(\mathsf{X}^N)}{2}\sum_{\hat{l}=1}^{\tilde{m}}\Bigg(\sum_{i=1}^N\sum_{\alpha=1}^d\nu^{\alpha,\hat{l}}(t,x_i)\partial_{x_{i,\alpha}}\log\big(\rho_t^N(\mathsf{X}^N)\big)\Bigg)^2\mathrm{d}t.\end{aligned}$$

Inserting it into the previous equation we obtain equation 5.19. For the measurability statement we just notice that the integrands are continuous in time and space R^d. Therefore we can apply Kolmogorov's continuity criteria in (t,x) to obtain a continuous version of the stochastic integral, which implies the continuity of the right hand side of equation 5.19 and consequently the product measurability. □

Similar, we obtain an expression for $\rho_t^N(\mathsf{X}^N)\log\left(\rho_t^{\otimes N}(\mathsf{X}^N)\right)$.

LEMMA 5.15 (Itô's formula for $x\log(y)$). *Let $\rho_t^{\otimes N}$ be the smooth solution provided by Proposition 5.13. Then,*

$$
\begin{aligned}
&\mathrm{d}\big(\rho_t^N(\mathsf{X}^N)\log\left(\rho_t^{\otimes N}(\mathsf{X}^N)\right)\big)\\
&= \log\left(\rho_t^{\otimes N}(\mathsf{X}^N)\right)\Bigg(\sum_{i=1}^N \nabla_{x_i}\cdot\Bigg(\frac{1}{N}\sum_{j=1}^N k(x_i-x_j)\rho_t^N(\mathsf{X}^N)\Bigg)\,\mathrm{d}t\\
&\quad-\sum_{i=1}^N\sum_{\alpha=1}^d\sum_{\hat{l}=1}^{\tilde{m}}\nu_t^{\alpha,\hat{l}}(x_i)\partial_{x_{i,\alpha}}\rho_t^N(\mathsf{X}^N)\,\mathrm{d}W_s^{\hat{l}}\\
&\quad+\frac{1}{2}\sum_{i=1}^N\sum_{\alpha,\beta=1}^d\partial_{x_{i,\alpha}}\partial_{x_{i,\beta}}\left([\sigma_t(x_i)\sigma_t(x_i)^{\mathrm{T}}]_{(\alpha,\beta)}\rho_t^N(\mathsf{X}^N)\right)\mathrm{d}t\\
&\quad+\frac{1}{2}\sum_{i,j=1}^N\sum_{\alpha,\beta=1}^d\partial_{x_{i,\alpha}}\partial_{x_{j,\beta}}\left([\nu_t(x_i)\nu_t(x_j)^{\mathrm{T}}]_{(\alpha,\beta)})\rho_t^N(\mathsf{X}^N)\right)\mathrm{d}t\Bigg)\\
&\quad+\frac{\rho_t^N(\mathsf{X}^N)}{\rho_t^{\otimes N}(\mathsf{X}^N)}\Bigg(\sum_{i=1}^N\nabla_{x_i}\cdot\Big((k*\rho_t)(x_i)\rho_t^{\otimes N}(\mathsf{X}^N)\Big)\,\mathrm{d}t\\
&\quad-\sum_{i=1}^N\sum_{\alpha=1}^d\sum_{\hat{l}=1}^{\tilde{m}}\nu_t^{\alpha,\hat{l}}(x_i)\partial_{x_{i,\alpha}}\rho_t^{\otimes N}(\mathsf{X}^N)\,\mathrm{d}W_s^{\hat{l}}\\
&\quad+\frac{1}{2}\sum_{i=1}^N\sum_{\alpha,\beta=1}^d\partial_{x_{i,\alpha}}\partial_{x_{i,\beta}}\left([\sigma_t(x_i)\sigma_t(x_i)^{\mathrm{T}}]_{(\alpha,\beta)}\rho_t^{\otimes N}(\mathsf{X}^N)\right)\mathrm{d}t\\
&\quad+\frac{1}{2}\sum_{i,j=1}^N\sum_{\alpha,\beta=1}^d\partial_{x_{i,\alpha}}\partial_{x_{j,\beta}}\left([\nu_t(x_i)\nu_t(x_j)^{\mathrm{T}}]_{(\alpha,\beta)})\rho_t^{\otimes N}(\mathsf{X}^N)\right)\mathrm{d}t\Bigg)\\
&\quad+\rho_t^N(\mathsf{X}^N)\sum_{i,j=1}^N\sum_{\alpha,\beta=1}^d\sum_{\hat{l}=1}^{\tilde{m}}\left(\nu_t^{\alpha,\hat{l}}(x_i)\partial_{x_{i,\alpha}}\log(\rho_t^N)(\mathsf{X}^N)\right)\\
&\quad\cdot\left(\nu_t^{\alpha,\hat{l}}(x_j)\partial_{x_{j,\beta}}\log\left(\rho_t^{\otimes N}(\mathsf{X}^N)\right)\right)\mathrm{d}t\\
&\quad-\frac{\rho_t^N(\mathsf{X}^N)}{2}\sum_{\hat{l}=1}^{\tilde{m}}\Bigg(\sum_{i=1}^N\sum_{\alpha=1}^d\nu_t^{\alpha,\hat{l}}(x_i)\partial_{x_{i,\alpha}}\log\left(\rho_t^{\otimes N}(\mathsf{X}^N)\right)\Bigg)^2\mathrm{d}t.
\end{aligned}
\tag{5.20}
$$

In particular, we can choose a set $\tilde{\Omega}\subseteq\Omega$ with $\mathbb{P}(\tilde{\Omega})$ such that equality (5.19) *holds for all (t,x) and the integrands on the right hand side of equality* (5.19) *are product measurable in (ω,s,x).*

Proof. Applying Itô's formula(1.15) to $x\log(y)$ we obtain

$$
\begin{aligned}
&\mathrm{d}\big(\rho_t^N(\mathsf{X}^N)\log\big(\rho_t^{\otimes N}(\mathsf{X}^N)\big)\big)\\
&\quad=\log\big(\rho_t^{\otimes N}(\mathsf{X}^N)\big)\,\mathrm{d}\rho_t^N(\mathsf{X}^N)+\frac{\rho_t^N(\mathsf{X}^N)}{\rho_t^{\otimes N}(\mathsf{X}^N)}\,\mathrm{d}\rho_t^{\otimes N}(\mathsf{X}^N)\\
&\qquad+\frac{1}{\rho_t^{\otimes N}(\mathsf{X}^N)}\,\mathrm{d}\big\langle\rho_t^N(\mathsf{X}^N),\rho_t^{\otimes N}(\mathsf{X}^N)\big\rangle-\frac{\rho_t^N(\mathsf{X}^N)}{2(\rho_t^{\otimes N}(\mathsf{X}^N))^2}\,\mathrm{d}\big\langle\rho_t^{\otimes N}(\mathsf{X}^N)\big\rangle
\end{aligned}
$$

Computing the quadratic variation we find

$$
\begin{aligned}
\mathrm{d}\big\langle\rho_t^{\otimes N}(\mathsf{X}^N)\big\rangle&=\mathrm{d}\Big\langle\sum_{i=1}^N\sum_{\alpha=1}^d\sum_{\hat{l}=1}^{\tilde{m}}\nu^{\alpha,\hat{l}}(t,x_i)\partial_{x_{i,\alpha}}\rho_t^{\otimes N}(\mathsf{X}^N)W_t^{\hat{l}}\Big\rangle\\
&=\sum_{i,j=1}^N\sum_{\alpha,\beta=1}^d\sum_{\hat{l}=1}^{\tilde{m}}\big(\nu_t^{\alpha,\hat{l}}(x_i)\partial_{x_{i,\alpha}}\rho_t^{\otimes N}(\mathsf{X}^N)\big)\big(\nu_t^{\alpha,\hat{l}}(x_j)\partial_{x_{j,\beta}}\rho_t^{\otimes N}(\mathsf{X}^N)\big)\,\mathrm{d}t\\
&=\sum_{\hat{l}=1}^{\tilde{m}}\Big(\sum_{i=1}^N\sum_{\alpha=1}^d\nu_t^{\alpha,\hat{l}}(x_i)\partial_{x_{i,\alpha}}\rho_t^{\otimes N}(\mathsf{X}^N)\Big)^2\,\mathrm{d}t,
\end{aligned}
$$

where we used the fact that $\mathrm{d}\langle W_t^l,W_t^{\tilde{l}}\rangle=\delta_{l,\tilde{l}}\,\mathrm{d}t$. Now, by similar arguments we obtain

$$
\begin{aligned}
&\mathrm{d}\big\langle\rho_t^N(\mathsf{X}^N),\rho_t^{\otimes N}(\mathsf{X}^N)\big\rangle\\
&=\mathrm{d}\Big\langle\sum_{i=1}^N\sum_{\alpha=1}^d\sum_{l=1}^{\tilde{m}}\nu_t^{\alpha,l}(x_i)\partial_{x_{i,\alpha}}\rho_t^N(\mathsf{X}^N)W_t^l,\sum_{j=1}^N\sum_{\beta=1}^d\sum_{\hat{l}=1}^{\tilde{m}}\nu_t^{\beta,\hat{l}}(x_j)\partial_{x_{j,\beta}}\rho_t^{\otimes N}(\mathsf{X}^N)W_t^{\hat{l}}\Big\rangle\\
&=\sum_{i,j=1}^N\sum_{\alpha,\beta=1}^d\sum_{\hat{l}=1}^{\tilde{m}}\big(\nu_t^{\alpha,\hat{l}}(x_i)\partial_{x_{i,\alpha}}\rho_t^N(\mathsf{X}^N)\big)\big(\nu_t^{\alpha,\hat{l}}(x_j)\partial_{x_{j,\beta}}\rho_t^{\otimes N}(\mathsf{X}^N)\big)\,\mathrm{d}t.
\end{aligned}
$$

Consequently, we arrive at

$$
\begin{aligned}
&\mathrm{d}\big(\rho_t^N(\mathsf{X}^N)\log(\rho_t^{\otimes N}(\mathsf{X}^N))\big)\\
&=\log\big(\rho_t^{\otimes N}(\mathsf{X}^N)\big)\Big(\sum_{i=1}^N\nabla_{x_i}\cdot\Big(\frac{1}{N}\sum_{j=1}^N k(x_i-x_j)\rho_t^N(\mathsf{X}^N)\Big)\,\mathrm{d}t\\
&\quad-\sum_{i=1}^N\sum_{\alpha=1}^d\sum_{\hat{l}=1}^{\tilde{m}}\nu_t^{\alpha,\hat{l}}(x_i)\partial_{x_{i,\alpha}}\rho_t^N(\mathsf{X}^N)\,\mathrm{d}W_t^{\hat{l}}\\
&\quad+\frac{1}{2}\sum_{i=1}^N\sum_{\alpha,\beta=1}^d\partial_{x_{i,\alpha}}\partial_{x_{i,\beta}}\big([\sigma_t(x_i)\sigma_t(x_i)^{\mathrm{T}}]_{(\alpha,\beta)}\rho_t^N(\mathsf{X}^N)\big)\,\mathrm{d}t\\
&\quad+\frac{1}{2}\sum_{i,j=1}^N\sum_{\alpha,\beta=1}^d\partial_{x_{i,\alpha}}\partial_{x_{j,\beta}}\big([\nu_t(x_i)\nu_t(x_j)^{\mathrm{T}}]_{(\alpha,\beta)})\rho_t^N(\mathsf{X}^N)\big)\,\mathrm{d}t\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\rho_t^N(\mathsf{X}^N)}{\rho_t^{\otimes N}(\mathsf{X}^N)} \Bigg(\sum_{i=1}^N \nabla_{x_i} \cdot \Big((k * \rho_t)(x_i) \rho_t^{\otimes N}(\mathsf{X}^N) \Big) \,\mathrm{d}t \\
&- \sum_{i=1}^N \sum_{\alpha=1}^d \sum_{\hat{l}=1}^{\tilde{m}} \nu_t^{\alpha,\hat{l}}(x_i) \partial_{x_{i,\alpha}} \rho_t^{\otimes N}(\mathsf{X}^N) \,\mathrm{d}W_t^{\hat{l}} \\
&+ \frac{1}{2} \sum_{i=1}^N \sum_{\alpha,\beta=1}^d \partial_{x_{i,\alpha}} \partial_{x_{i,\beta}} \big([\sigma_t(x_i)\sigma_t(x_i)^{\mathrm{T}}]_{(\alpha,\beta)} \rho_t^{\otimes N}(\mathsf{X}^N) \big) \,\mathrm{d}t \\
&+ \frac{1}{2} \sum_{i,j=1}^N \sum_{\alpha,\beta=1}^d \partial_{x_{i,\alpha}} \partial_{x_{j,\beta}} \big([\nu_t(x_i)\nu_t(x_j)^{\mathrm{T}}]_{(\alpha,\beta)}) \rho_t^{\otimes N}(\mathsf{X}^N) \big) \,\mathrm{d}t \Bigg) \\
&+ \frac{1}{\rho_t^{\otimes N}(\mathsf{X}^N)} \sum_{i,j=1}^N \sum_{\alpha,\beta=1}^d \sum_{\hat{l}=1}^{\tilde{m}} \big(\nu_t^{\alpha,\hat{l}}(x_i) \partial_{x_{i,\alpha}} \rho_t^N(\mathsf{X}^N) \big) \big(\nu_t^{\alpha,\hat{l}}(x_j) \partial_{x_{j,\beta}} \rho_t^{\otimes N}(\mathsf{X}^N) \big) \,\mathrm{d}t \\
&- \frac{\rho_t^N(\mathsf{X}^N)}{2\big(\rho_t^{\otimes N}(\mathsf{X}^N)\big)^2} \sum_{\hat{l}=1}^{\tilde{m}} \Bigg(\sum_{i=1}^N \sum_{\alpha=1}^d \nu_t^{\alpha,\hat{l}}(x_i) \partial_{x_{i,\alpha}} \rho_t^{\otimes N}(\mathsf{X}^N) \Bigg)^2 \,\mathrm{d}t.
\end{aligned}
$$

For the last two term we find

$$
\begin{aligned}
&\frac{1}{\rho_t^{\otimes N}(\mathsf{X}^N)} \sum_{i,j=1}^N \sum_{\alpha,\beta=1}^d \sum_{\hat{l}=1}^{\tilde{m}} \big(\nu_t^{\alpha,\hat{l}}(x_i) \partial_{x_{i,\alpha}} \rho_t^N(\mathsf{X}^N) \big) \big(\nu_t^{\alpha,\hat{l}}(x_j) \partial_{x_{j,\beta}} \rho_t^{\otimes N}(\mathsf{X}^N) \big) \,\mathrm{d}t \\
&- \frac{\rho_t^N(\mathsf{X}^N)}{2\big(\rho_t^{\otimes N}(\mathsf{X}^N)\big)^2} \sum_{\hat{l}=1}^{\tilde{m}} \Bigg(\sum_{i=1}^N \sum_{\alpha=1}^d \nu_t^{\alpha,\hat{l}}(x_i) \partial_{x_{i,\alpha}} \rho_t^{\otimes N}(\mathsf{X}^N) \Bigg)^2 \,\mathrm{d}t \\
&= \rho_t^N(\mathsf{X}^N) \sum_{i,j=1}^N \sum_{\alpha,\beta=1}^d \sum_{\hat{l}=1}^{\tilde{m}} \big(\nu_t^{\alpha,\hat{l}}(x_i) \partial_{x_{i,\alpha}} \log\big(\rho_t^N(\mathsf{X}^N)\big) \big) \big(\nu_t^{\alpha,\hat{l}}(x_j) \partial_{x_{j,\beta}} \log\big(\rho_t^{\otimes N}(\mathsf{X}^N)\big) \big) \,\mathrm{d}t \\
&\quad - \frac{\rho_t^N(\mathsf{X}^N)}{2} \sum_{\hat{l}=1}^{\tilde{m}} \Bigg(\sum_{i=1}^N \sum_{\alpha=1}^d \nu_t^{\alpha,\hat{l}}(x_i) \partial_{x_{i,\alpha}} \log\big(\rho_t^{\otimes N}(\mathsf{X}^N)\big) \Bigg)^2 \,\mathrm{d}t,
\end{aligned}
$$

which with the same measurability arguments as in Lemma 5.14 implies our claim. □

In the next step we want to combine both Lemma 5.14 and Lemma 5.15. We make a crucial observation that the difference of the last term in equation (5.19) and the last two terms in equation (5.20) creating a square, i.e.

$$
\begin{aligned}
&\frac{\rho_t^N(\mathsf{X}^N)}{2} \sum_{\hat{l}=1}^{\tilde{m}} \Bigg(\sum_{i=1}^N \sum_{\alpha=1}^d \nu_t^{\alpha,\hat{l}}(x_i) \partial_{x_{i,\alpha}} \log\big(\rho_t^N(\mathsf{X}^N)\big) \Bigg)^2 \,\mathrm{d}t \\
&- \rho_t^N(\mathsf{X}^N) \sum_{i,j=1}^N \sum_{\alpha,\beta=1}^d \sum_{\hat{l}=1}^{\tilde{m}} \big(\nu_t^{\alpha,l}(x_i) \partial_{x_{i,\alpha}} \log\big(\rho_t^N(\mathsf{X}^N)\big) \big) \\
&\quad \cdot \big(\nu_t^{\alpha,\hat{l}}(x_j) \partial_{x_{j,\beta}} \log\big(\rho_t^{\otimes N}(\mathsf{X}^N)\big) \big) \,\mathrm{d}t
\end{aligned}
$$

$$+ \frac{\rho_t^N(\mathsf{X}^N)}{2} \sum_{\hat{l}=1}^{\tilde{m}} \Big(\sum_{i=1}^N \sum_{\alpha=1}^d \nu_t^{\alpha,\hat{l}}(x_i) \partial_{x_{i,\alpha}} \log\big(\rho_t^{\otimes N}(\mathsf{X}^N)\big)\Big)^2 \,\mathrm{d}t$$

$$= \frac{\rho_t^N(\mathsf{X}^N)}{2} \sum_{\hat{l}=1}^{\tilde{m}} \Big(\sum_{i=1}^N \sum_{\alpha=1}^d \nu_t^{\alpha,\hat{l}}(x_i) \partial_{x_{i,\alpha}} \log\Big(\frac{\rho_t^N(\mathsf{X}^N)}{\rho_t^{\otimes N}(\mathsf{X}^N)}\Big)\Big)^2 \,\mathrm{d}t. \tag{5.21}$$

As a result we obtain the following dynamic

$$\begin{aligned}
&\mathrm{d}(\rho_t^N(\mathsf{X}^N)\big(\log\big(\rho_t^N(\mathsf{X}^N)\big) - \log\big(\rho_t^{\otimes N}(\mathsf{X}^N)\big)\big)) \\
&= \Big(\log\Big(\frac{\rho_t^N(\mathsf{X}^N)}{\rho_t^{\otimes N}(\mathsf{X}^N)}\Big) + 1\Big)\Big(\sum_{i=1}^N \nabla_{x_i} \cdot \Big(\frac{1}{N}\sum_{j=1}^N k(x_i - x_j)\rho_t^N(\mathsf{X}^N)\Big)\,\mathrm{d}t \\
&\quad - \sum_{i=1}^N \sum_{\alpha=1}^d \sum_{l=1}^{\tilde{m}} \nu_t^{\alpha,l}(x_i)\partial_{x_{i,\alpha}}\rho_t^N(\mathsf{X}^N)\,\mathrm{d}W_s^l \\
&\quad + \frac{1}{2}\sum_{i=1}^N \sum_{\alpha,\beta=1}^d \partial_{x_{i,\alpha}}\partial_{x_{i,\beta}}\big([\sigma_t(x_i)\sigma_t(x_i)^{\mathrm{T}}]_{(\alpha,\beta)}\rho_t^N(\mathsf{X}^N)\big)\,\mathrm{d}t \\
&\quad + \frac{1}{2}\sum_{i,j=1}^N \sum_{\alpha,\beta=1}^d \partial_{x_{i,\alpha}}\partial_{x_{j,\beta}}\big([\nu_t(x_i)\nu_t(x_j)^{\mathrm{T}}]_{(\alpha,\beta)})\rho_t^N(\mathsf{X}^N)\big)\,\mathrm{d}t\Big) \\
&\quad + \frac{\rho_t^N(\mathsf{X}^N)}{\rho_t^{\otimes N}(\mathsf{X}^N)}\Big(\sum_{i=1}^N \nabla_{x_i}\cdot\Big((k*\rho_t)(x_i)\rho_t^{\otimes N}(\mathsf{X}^N)\Big)\,\mathrm{d}t \\
&\quad - \sum_{i=1}^N \sum_{\alpha=1}^d \sum_{\hat{l}=1}^{\tilde{m}} \nu_t^{\alpha,\hat{l}}(x_i)\partial_{x_{i,\alpha}}\rho_t^{\otimes N}(\mathsf{X}^N)\,\mathrm{d}W_s^{\hat{l}} \\
&\quad + \frac{1}{2}\sum_{i=1}^N \sum_{\alpha,\beta=1}^d \partial_{x_{i,\alpha}}\partial_{x_{i,\beta}}\big([\sigma_t(x_i)\sigma_t(x_i)^{\mathrm{T}}]_{(\alpha,\beta)}\rho_t^{\otimes N}(\mathsf{X}^N)\big)\,\mathrm{d}t \\
&\quad + \frac{1}{2}\sum_{i,j=1}^N \sum_{\alpha,\beta=1}^d \partial_{x_{i,\alpha}}\partial_{x_{j,\beta}}\big([\nu_t(x_i)\nu_t(x_j)^{\mathrm{T}}]_{(\alpha,\beta)})\rho_t^{\otimes N}(\mathsf{X}^N)\big)\,\mathrm{d}t\Big) \\
&\quad + \frac{\rho_t^N(\mathsf{X}^N)}{2}\sum_{\hat{l}=1}^{\tilde{m}}\Big(\sum_{i=1}^N\sum_{\alpha=1}^d \nu_t^{\alpha,\hat{l}}(x_i)\partial_{x_{i,\alpha}}\log\Big(\frac{\rho_t^N(\mathsf{X}^N)}{\rho_t^{\otimes N}(\mathsf{X}^N)}\Big)\Big)^2\,\mathrm{d}t.
\end{aligned}$$

After integrating over $\mathbb{R}^{dN}$, the divergence free assumption of ν and the stochastic Fubini Theorem [Ver12] kills the stochastic integrals. Hence, applying Fubini's theorem (everything is product measurable) for the Lebesgue integrals we find

$$\mathcal{H}\big(\rho_t^N|\rho_t^{\otimes N}\big) = \int_0^t \int_{\mathbb{R}^{dN}} \Big(\log\Big(\frac{\rho_s^N(\mathsf{X}^N)}{\rho_s^{\otimes N}(\mathsf{X}^N)}\Big) + 1\Big)\Big(\sum_{i=1}^N \nabla_{x_i}\cdot\Big(\frac{1}{N}\sum_{j=1}^N k(x_i - x_j)\rho_s^N(\mathsf{X}^N)\Big)$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{i=1}^{N}\sum_{\alpha,\beta=1}^{d}\partial_{x_{i,\alpha}}\partial_{x_{i,\beta}}\big([\sigma_s(x_i)\sigma_s(x_i)^{\mathrm{T}}]_{(\alpha,\beta)}\rho_s^N(\mathsf{X}^N)\big)\\
&+\frac{1}{2}\sum_{i,j=1}^{N}\sum_{\alpha,\beta=1}^{d}\partial_{x_{i,\alpha}}\partial_{x_{j,\beta}}\big([\nu_s(x_i)\nu_s(x_j)^{\mathrm{T}}]_{(\alpha,\beta)})\rho_s^N(\mathsf{X}^N)\big)\Bigg)\\
&-\frac{\rho_s^N(\mathsf{X}^N)}{\rho_s^{\otimes N}(\mathsf{X}^N)}\Bigg(\sum_{i=1}^{N}\nabla_{x_i}\cdot\Big((k*\rho_s)(x_i)\rho_s^{\otimes N}(\mathsf{X}^N)\Big)\\
&-\frac{1}{2}\sum_{i=1}^{N}\sum_{\alpha,\beta=1}^{d}\partial_{x_{i,\alpha}}\partial_{x_{i,\beta}}\big([\sigma_s(x_i)\sigma_s(x_i)^{\mathrm{T}}]_{(\alpha,\beta)}\rho_s^{\otimes N}(\mathsf{X}^N)\big)\\
&-\frac{1}{2}\sum_{i,j=1}^{N}\sum_{\alpha,\beta=1}^{d}\partial_{x_{i,\alpha}}\partial_{x_{j,\beta}}\big([\nu_s(x_i)\nu_s(x_j)^{\mathrm{T}}]_{(\alpha,\beta)})\rho_s^{\otimes N}(\mathsf{X}^N)\big)\Bigg)\\
&+\frac{\rho_s^N(\mathsf{X}^N)}{2}\sum_{\hat{l}=1}^{\tilde{m}}\left(\sum_{i=1}^{N}\sum_{\alpha=1}^{d}\nu_s^{\alpha,\hat{l}}(x_i)\partial_{x_{i,\alpha}}\log\left(\frac{\rho_s^N(\mathsf{X}^N)}{\rho_s^{\otimes N}(\mathsf{X}^N)}\right)\right)^2\mathrm{d}\mathsf{X}^N\,\mathrm{d}s.
\end{aligned}
$$

We notice that the constant one in the first term vanishes by the divergence structure of the equation and the integration over the whole domain $\mathbb{R}^{dN}$. In the next step let us use the cancellation property of σ and ν to rewrite the second order differential operator

$$
\begin{aligned}
&\frac{1}{2}\sum_{i,j=1}^{N}\sum_{\alpha,\beta=1}^{d}\partial_{x_{i,\alpha}}\partial_{x_{j,\beta}}\left(\left(\sum_{l=1}^{\tilde{m}}\sigma_s^{\alpha,l}(x_i)\sigma_s^{\beta,l}(x_j)\delta_{i,j}+\sum_{\hat{l}=1}^{\tilde{m}}\nu_s^{\alpha,\hat{l}}(x_i)\nu_s^{\beta,\hat{l}}(x_j)\right)\rho_s^N(\mathsf{X}^N)\right)\\
&\quad=\frac{1}{2}\sum_{i,j=1}^{N}\sum_{\alpha,\beta=1}^{d}\partial_{x_{i,\alpha}}\left(\left(\sum_{l=1}^{\tilde{m}}\sigma_s^{\alpha,l}(x_i)\sigma_s^{\beta,l}(x_j)\delta_{i,j}+\sum_{\hat{l}=1}^{\tilde{m}}\nu_s^{\alpha,\hat{l}}(x_i)\nu_s^{\beta,\hat{l}}(x_j)\right)\partial_{x_{j,\beta}}\rho_s^N(\mathsf{X}^N)\right).
\end{aligned}
$$

The same inequality holds if ρ^N is replaced by $\rho^{\otimes N}$. Hence, if we only look at the terms containing ν we arrive at the following expression for the coefficient ν,

$$
\begin{aligned}
&\frac{1}{2}\sum_{i,j=1}^{N}\sum_{\alpha,\beta=1}^{d}\sum_{\hat{l}=1}^{\tilde{m}}\int_{\mathbb{R}^{dN}}\log\left(\frac{\rho_s^N(\mathsf{X}^N)}{\rho_s^{\otimes N}(\mathsf{X}^N)}\right)\partial_{x_{i,\alpha}}\left(\nu_s^{\alpha,\hat{l}}(x_i)\nu_s^{\beta,\hat{l}}(x_j)\partial_{x_{j,\beta}}\rho_s^N(\mathsf{X}^N)\right)\mathrm{d}\mathsf{X}^N\\
&-\frac{1}{2}\sum_{i,j=1}^{N}\sum_{\alpha,\beta=1}^{d}\sum_{\hat{l}=1}^{\tilde{m}}\int_{\mathbb{R}^{dN}}\frac{\rho_s^N(\mathsf{X}^N)}{\rho_s^{\otimes N}(\mathsf{X}^N)}\partial_{x_{i,\alpha}}\left(\nu_s^{\alpha,\hat{l}}(x_i)\nu_s^{\beta,\hat{l}}(x_j)\partial_{x_{j,\beta}}\rho_s^{\otimes N}(\mathsf{X}^N)\right)\mathrm{d}\mathsf{X}^N\\
&\quad=-\frac{1}{2}\sum_{i,j=1}^{N}\sum_{\alpha,\beta=1}^{d}\sum_{\hat{l}=1}^{\tilde{m}}\int_{\mathbb{R}^{dN}}\partial_{x_{i,\alpha}}\log\left(\frac{\rho_s^N(\mathsf{X}^N)}{\rho_s^{\otimes N}(\mathsf{X}^N)}\right)\nu_s^{\alpha,\hat{l}}(x_i)\nu_s^{\beta,\hat{l}}(x_j)\partial_{x_{j,\beta}}\rho_s^N(\mathsf{X}^N)\,\mathrm{d}\mathsf{X}^N\\
&\qquad+\frac{1}{2}\sum_{i,j=1}^{N}\sum_{\alpha,\beta=1}^{d}\sum_{\hat{l}=1}^{\tilde{m}}\int_{\mathbb{R}^{dN}}\partial_{x_{i,\alpha}}\left(\frac{\rho_s^N(\mathsf{X}^N)}{\rho_s^{\otimes N}(\mathsf{X}^N)}\right)\nu_s^{\alpha,\hat{l}}(x_i)\nu_s^{\beta,\hat{l}}(x_j)\partial_{x_{j,\beta}}\rho_s^{\otimes N}(\mathsf{X}^N)\,\mathrm{d}\mathsf{X}^N
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{2}\sum_{i,j=1}^{N}\sum_{\alpha,\beta=1}^{d}\sum_{\hat{l}=1}^{\tilde{m}}\int_{\mathbb{R}^{dN}} \rho_s^N(\mathsf{X}^N)\partial_{x_{i,\alpha}}\log\left(\frac{\rho_s^N(\mathsf{X}^N)}{\rho_s^{\otimes N}(\mathsf{X}^N)}\right) \\
&\quad \cdot \nu_s^{\alpha,\hat{l}}(x_i)\nu_s^{\beta,\hat{l}}(x_j)\partial_{x_{j,\beta}}\log\left(\rho_s^N(\mathsf{X}^N)\right)\mathrm{d}\mathsf{X}^N \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{N}\sum_{\alpha,\beta=1}^{d}\sum_{\hat{l}=1}^{\tilde{m}}\int_{\mathbb{R}^{dN}} \rho_s^N(\mathsf{X}^N)\partial_{x_{i,\alpha}}\log\left(\frac{\rho_s^N(\mathsf{X}^N)}{\rho_s^{\otimes N}(\mathsf{X}^N)}\right) \\
&\quad \cdot \nu_s^{\alpha,\hat{l}}(x_i)\nu_s^{\beta,\hat{l}}(x_j)\partial_{x_{j,\beta}}\log(\rho_s^{\otimes N})(\mathsf{X}^N)\,\mathrm{d}\mathsf{X}^N \\
&= -\frac{1}{2}\sum_{\hat{l}=1}^{\tilde{m}}\int_{\mathbb{R}^{dN}} \rho_s^N(\mathsf{X}^N)\left(\sum_{i=1}^{N}\sum_{\alpha=1}^{d}\nu_s^{\alpha,\hat{l}}(x_i)\partial_{x_{i,\alpha}}\log\left(\frac{\rho_s^N(\mathsf{X}^N)}{\rho_s^{\otimes N}(\mathsf{X}^N)}\right)\right)^2\mathrm{d}\mathsf{X}^N.
\end{aligned}
$$

Hence, we obtain exactly the same term as in the covariation calculations (5.21) but with a negative sign. Consequently, they vanish and we do not see any contribution of the common noise ν in the relative entropy. Let us summarize our result in the following proposition.

PROPOSITION 5.16. *Let ρ_t^N, $\rho_t^{\otimes N}$ be given by Proposition 5.13. Then we have the following representation of the relative entropy*

$$
\begin{aligned}
\mathcal{H}\big(\rho_t^N|\rho_t^{\otimes N}\big) = &\int_0^t\int_{\mathbb{R}^{dN}}\sum_{i=1}^{N}\log\left(\frac{\rho_s^N(\mathsf{X}^N)}{\rho_s^{\otimes N}(\mathsf{X}^N)}\right)\nabla_{x_i}\cdot\left(\frac{1}{N}\sum_{j=1}^{N}k(x_i-x_j)\rho_s^N(\mathsf{X}^N)\right) \\
&-\sum_{i=1}^{N}\frac{\rho_s^N(\mathsf{X}^N)}{\rho_s^{\otimes N}(\mathsf{X}^N)}\nabla_{x_i}\cdot\Big((k*\rho_s)(x_i)\rho_s^{\otimes N}(\mathsf{X}^N)\Big) \\
&+\frac{1}{2}\sum_{i=1}^{N}\sum_{\alpha,\beta=1}^{d}\log\left(\frac{\rho_s^N(\mathsf{X}^N)}{\rho_s^{\otimes N}(\mathsf{X}^N)}\right)\partial_{x_{i,\alpha}}\Big([\sigma_s(x_i)\sigma_s(x_i)^{\mathrm{T}}]_{(\alpha,\beta)}\partial_{x_{i,\beta}}\rho_s^N(\mathsf{X}^N)\Big) \\
&-\frac{1}{2}\sum_{i=1}^{N}\sum_{\alpha,\beta=1}^{d}\frac{\rho_s^N(\mathsf{X}^N)}{\rho_s^{\otimes N}(\mathsf{X}^N)}\partial_{x_{i,\alpha}}\Big([\sigma_s(x_i)\sigma_s(x_i)^{\mathrm{T}}]_{(\alpha,\beta)}\partial_{x_{i,\beta}}\rho_s^{\otimes N}(\mathsf{X}^N)\Big)\,\mathrm{d}\mathsf{X}^N\,\mathrm{d}s.
\end{aligned}
$$

REMARK 5.17. *This phenomenon seems maybe strange but it somehow shows that the common noise has no effect on the expected relative entropy $\mathbb{E}(\mathcal{H}(\rho_t^N|\rho_t^{\otimes N}))$ as long as both measures are conditioned on the common noise. This is a crucial observation. If both measures are viewed under the information of the common noise W, we expect that the particles behave similar as in the classical mean-field limit setting and this phenomenon is exactly reflected in the relative entropy.*

From the representation we obtain the classical relative entropy bound.

Corollary 5.18. *Let ρ_t^N, $\rho_t^{\otimes N}$ be given by Proposition 5.13. We have the following relative entropy inequality*

$$\mathcal{H}\big(\rho_t^N|\rho_t^{\otimes N}\big) \leq -\frac{\delta}{4}\sum_{i=1}^N \int_0^t \int_{\mathbb{R}^{dN}} \rho_s^N\big(\mathsf{X}^N\big)\left|\nabla_{x_i}\log\left(\frac{\rho_s^N\big(\mathsf{X}^N\big)}{\rho_s^{\otimes N}\big(\mathsf{X}^N\big)}\right)\right|^2 \mathrm{d}\mathsf{X}^N\,\mathrm{d}s$$
$$+\frac{1}{\delta}\sum_{i=1}^N \int_0^t \rho_s^N\big(\mathsf{X}^N\big)\left|\frac{1}{N}\sum_{j=1}^N k(x_i-x_j)-(k*\rho_s)(x_i)\right|^2 \mathrm{d}\mathsf{X}^N\,\mathrm{d}s.$$

Proof. The proof is a standard entropy computation similar to the ones performed in [LLF23b, PN24]. Applying Young's inequality and the exchangeability of the particles we find

$$\int_{\mathbb{R}^{dN}} \sum_{i=1}^N \log\left(\frac{\rho_s^N\big(\mathsf{X}^N\big)}{\rho_s^{\otimes N}\big(\mathsf{X}^N\big)}\right)\nabla_{x_i}\cdot\left(\frac{1}{N}\sum_{j=1}^N k(x_i-x_j)\rho_s^N\big(\mathsf{X}^N\big)\right)$$
$$-\sum_{i=1}^N \frac{\rho_s^N\big(\mathsf{X}^N\big)}{\rho_s^{\otimes N}\big(\mathsf{X}^N\big)}\nabla_{x_i}\cdot\Big((k*\rho_s)(x_i)\rho_s^{\otimes N}\big(\mathsf{X}^N\big)\Big)\,\mathrm{d}\mathsf{X}^N$$
$$=\sum_{i=1}^N \int_{\mathbb{R}^{dN}} \rho_s^N\big(\mathsf{X}^N\big)\nabla_{x_i}\log\left(\frac{\rho_s^N\big(\mathsf{X}^N\big)}{\rho_s^{\otimes N}\big(\mathsf{X}^N\big)}\right)\cdot\left(\frac{1}{N}\sum_{j=1}^N k(x_i-x_j)-(k*\rho_s)(x_i)\right)\mathrm{d}\mathsf{X}^N$$
$$\leq \sum_{i=1}^N \int_{\mathbb{R}^{dN}} \frac{\delta}{4}\rho_s^N\big(\mathsf{X}^N\big)\left|\nabla_{x_i}\log\left(\frac{\rho_s^N\big(\mathsf{X}^N\big)}{\rho_s^{\otimes N}\big(\mathsf{X}^N\big)}\right)\right|^2$$
$$+\frac{\rho_s^N\big(\mathsf{X}^N\big)}{\delta}\left|\frac{1}{N}\sum_{j=1}^N k(x_i-x_j)-(k*\rho_s)(x_i)\right|^2 \mathrm{d}\mathsf{X}^N.$$

Using integration by parts and the ellipticity condition on σ we find

$$\frac{1}{2}\sum_{i=1}^N \sum_{\alpha,\beta=1}^d \int_{\mathbb{R}^{dN}} \log\left(\frac{\rho_s^N\big(\mathsf{X}^N\big)}{\rho_s^{\otimes N}\big(\mathsf{X}^N\big)}\right)\partial_{x_{i,\alpha}}\Big([\sigma_s(x_i)\sigma_s(x_i)^{\mathrm{T}}]_{(\alpha,\beta)}\partial_{x_{i,\beta}}\rho_s^N\big(\mathsf{X}^N\big)\Big)$$
$$-\frac{1}{2}\sum_{i=1}^N \sum_{\alpha,\beta=1}^d \frac{\rho_s^N\big(\mathsf{X}^N\big)}{\rho_s^{\otimes N}\big(\mathsf{X}^N\big)}\partial_{x_{i,\alpha}}\Big([\sigma_s(x_i)\sigma_s(x_i)^{\mathrm{T}}]_{(\alpha,\beta)}\partial_{x_{i,\beta}}\rho_s^{\otimes N}\big(\mathsf{X}^N\big)\Big)\,\mathrm{d}x$$
$$=-\frac{1}{2}\sum_{i=1}^N \sum_{\alpha,\beta=1}^d \int_{\mathbb{R}^{dN}} \rho_s^N\big(\mathsf{X}^N\big)\partial_{x_{i,\alpha}}\log\left(\frac{\rho_s^N\big(\mathsf{X}^N\big)}{\rho_s^{\otimes N}\big(\mathsf{X}^N\big)}\right)$$
$$\cdot[\sigma_s(x_i)\sigma_s(x_i)^{\mathrm{T}}]_{(\alpha,\beta)}\partial_{x_{i,\beta}}\log\big(\rho_s^N\big(\mathsf{X}^N\big)\big)$$
$$+\frac{1}{2}\sum_{i=1}^N \sum_{\alpha,\beta=1}^d \rho_s^N\big(\mathsf{X}^N\big)\partial_{x_{i,\alpha}}\log\left(\frac{\rho_s^N\big(\mathsf{X}^N\big)}{\rho_s^{\otimes N}\big(\mathsf{X}^N\big)}\right)$$
$$\cdot[\sigma_s(x_i)\sigma_s(x_i)^{\mathrm{T}}]_{(\alpha,\beta)}\partial_{x_{i,\beta}}\log\big(\rho_s^{\otimes N}\big(\mathsf{X}^N\big)\big)\,\mathrm{d}\mathsf{X}^N$$

$$= -\frac{1}{2}\sum_{i=1}^{N}\int_{\mathbb{R}^{dN}} \rho_s^N(\mathsf{X}^N) \sum_{\alpha,\beta=1}^{d} \partial_{x_{i,\alpha}} \log\left(\frac{\rho_s^N(\mathsf{X}^N)}{\rho_s^{\otimes N}(\mathsf{X}^N)}\right)$$

$$\cdot [\sigma_s(x_i)\sigma_s(x_i)^{\mathrm{T}}]_{(\alpha,\beta)}\partial_{x_{i,\beta}} \log\left(\frac{\rho_s^N(\mathsf{X}^N)}{\rho_s^{\otimes N}(\mathsf{X}^N)}\right) \mathrm{d}\mathsf{X}^N$$

$$\leq -\frac{\delta}{2}\sum_{i=1}^{N}\int_{\mathbb{R}^{dN}} \rho_s^N(\mathsf{X}^N)\left|\nabla_{x_i} \log\left(\frac{\rho_s^N(\mathsf{X}^N)}{\rho_s^{\otimes N}(\mathsf{X}^N)}\right)\right|^2 \mathrm{d}\mathsf{X}^N.$$

Combining both inequalities proves the corollary. □

THEOREM 5.19. *In the smooth setting we have the following relative entropy bound between the conditional law of the particles system ρ^N and the the solution $\rho^{\otimes N}$ of the SPDE* (5.6),

$$\mathbb{E}\left(\sup_{0\leq t\leq T} \mathcal{H}\big(\rho_t^N|\rho_t^{\otimes N}\big)\right) \leq C\big(T,\delta,\|k\|_{L^\infty(\mathbb{R}^d)}\big), \tag{5.22}$$

where $C\big(T,\delta,\|k\|_{L^\infty(\mathbb{R}^d)}\big)$ depends on the finial time T, the ellipticity constant δ and $\|k\|_{L^\infty}$.

REMARK 5.20. *The results demonstrate that in the smooth case, the relative entropy estimates presented by Jabin and Wang* [JW18] *hold even in the presence of common noise. However, if our coefficients and interaction kernel are not smooth, we need to resort to an approximation argument. As discussed in Section 5.4, this becomes achievable when $k \in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$. The challenging aspect lies in handling the non-linear SPDE 5.5. Consequently, we have reformulated the inquiry about a quantitative relative entropy estimate with common noise into a question concerning the stability of non-linear SPDEs. It is foreseeable that other results derived from the work of Jabin and Wang* [JW18] *can be adapted to the common noise setting by similar techniques.*

PROOF OF THEOREM5.19. The proof is basically an application of [JW18, Theorem 3]. Let us define the following process

$$\psi(s,z,y) = \frac{1}{16e\,\|k\|_{L^\infty(\mathbb{R}^d)}}(k(z-y) - k*\rho_s(z))$$

and notice that $\|\psi\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{2e}$ uniformly in time and the probability space Ω. Then applying Corollary 5.18 and [JW18, Lemma 1] we find

$$\mathbb{E}\left(\sup_{0\leq t\leq T} \mathcal{H}\big(\rho_t^N|\rho_t^{\otimes N}\big)\right)$$

$$\leq \frac{1}{\delta}\sum_{i=1}^{N}\int_0^T \mathbb{E}\left(\rho_s^N(\mathsf{X}^N)\left|\frac{1}{N}\sum_{j=1}^{N} k(x_i-x_j) - (k*\rho_s)(x_i)\right|^2 \mathrm{d}x\right)\mathrm{d}s$$

$$\leq \frac{16e\,\|k\|_{L^\infty(\mathbb{R}^d)}}{\delta}\int_0^T \mathbb{E}\left(\mathcal{H}\big(\rho_s^N|\rho_s^{\otimes N}\big)\right)$$

$$+\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\bigg(\log\Big(\int_{\mathbb{R}^{dN}}\rho_s^{\otimes N}(\mathsf{X}^N)\exp\Big(\frac{1}{N}\sum_{j_1,j_2=1}^{N}\psi(s,x_i,x_{j_1})\psi(s,x_i,x_{j_2})\Big)\,\mathrm{d}x\Big)\bigg)\,\mathrm{d}s.$$

Additionally, we have the following cancellation property for all $s \in [0,T]$,

$$\int_{\mathbb{R}^d}\psi(s,z,y)\rho_s(y)=0, \quad \mathbb{P}\text{-a.e.}.$$

At this moment we can repeat the proof of [JW18, Theorem 3], since it is based on the cancellation property and combinatorial arguments. Hence, it can be performed path-wise and we arrive at

$$\mathbb{E}\bigg(\sup_{0\le t\le T}\mathcal{H}\big(\rho_t^N|\rho_t^{\otimes N}\big)\bigg)\le\frac{16e\,\|k\|_{L^\infty(\mathbb{R}^d)}}{\delta}\int_0^t\mathbb{E}\Big(\mathcal{H}\big(\rho_s^N|\rho_s^{\otimes N}\big)\Big)+C\,\mathrm{d}s. \tag{5.23}$$

Finally, Gronwall's lemma implies

$$\sup_{0\le t\le T}\mathbb{E}\big(\mathcal{H}\big(\rho_t^N|\rho_t^{\otimes N}\big)\big)\le\frac{16e\,\|k\|_{L^\infty(\mathbb{R}^d)}\,Te^{CT}}{\delta}$$

for some positive constant $C>0$. Utilizing this estimate in (5.23) we arrive at

$$\mathbb{E}\bigg(\sup_{0\le t\le T}\mathcal{H}\big(\rho_t^N|\rho_t^{\otimes N}\big)\bigg)\le\frac{16e\,\|k\|_{L^\infty(\mathbb{R}^d)}\,T^2e^{CT}}{\delta}. \tag{5.24}$$

□

An application of the Csiszár–Kullback–Pinsker inequality and the sub-additivity property proves the following L^1 estimate.

COROLLARY 5.21. *In the setting of Theorem 5.19 we obtain*

$$\mathbb{E}\bigg(\sup_{0\le t\le T}\Big\|\rho_t^{r,N}-\rho_t^{\otimes r}\Big\|_{L^1(\mathbb{R}^{dr})}^2\bigg)\le\frac{C}{N}.$$

REMARK 5.22. *We want to end this section with a small comment how to make the calculations completely rigours. Basically we need to avoid the singularity of the* log *function. We accomplish that by replacing the function* $x\log(y)$ *with* $(x+\varepsilon)\log(y+\varepsilon)$*. This guarantees the application of the Itô's formula and the vanishing of the stochastic integrals in all calculations. Obviously we can not longer integrate over the whole space* $\mathbb{R}^{dN}$*. Therefore a multiplication with a suitable cut-off function depending on some parameter* $\tilde{\varepsilon}$ *and than integrating over* $\mathbb{R}^{dN}$ *does the job. Finally we apply all estimates to the approximating system. Since all estimates will be uniform in the parameters* $\varepsilon, \tilde{\varepsilon}$ *everything will be well-defined and we can take the limit by connecting* $\tilde{\varepsilon}$ *with* ε *such that all appearing approximation terms vanish. Normally* ε *needs to vanish much faster than* $\tilde{\varepsilon}$*, i.e.* $\varepsilon=\mathcal{O}(\tilde{\varepsilon}^L)$ *for big enough* $L>0$ *as* $\tilde{\varepsilon}\to 0$*.*

5.4. Stability for the stochastic Fokker–Planck and Liouville equations

The goal of this section is to lower the smoothness assumptions made in Section 5.3 and obtain the estimate in Theorem 5.19 in the case $k \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Our strategy consists of mollifying the coefficients, applying Theorem 5.19 and then use the almost everywhere convergence along a subsequence to conclude the relative entropy estimate. Hence, let us consider a non-negative smooth function $J^1 : \mathbb{R}^d \to \mathbb{R}$ with compact support in the unit ball and mass one. We replace the coefficients σ, ν, k by its mollified versions. More precisely, we define for each index $\alpha, \beta, \tilde{l}$ and $\varepsilon > 0$ the functions

$$\begin{aligned} [\sigma_t(\cdot)\sigma_t^{\mathrm{T}}(\cdot)]_{(\alpha,\beta)} * J^\varepsilon(z) &= \int_{\mathbb{R}^d} [\sigma_t(y)\sigma_t^{\mathrm{T}}(y)]_{(\alpha,\beta)} J^\varepsilon(z-y)\,\mathrm{d}y, \\ \nu_t^{\beta,\tilde{l}} * J^\varepsilon(z) &= \int_{\mathbb{R}^d} \nu_t^{\beta,\tilde{l}}(y) J^\varepsilon(z-y)\,\mathrm{d}y, \\ k * J^\varepsilon(z) &= \int_{\mathbb{R}^d} k(y) J^\varepsilon(z-y)\,\mathrm{d}y, \\ \rho_0 * J^\varepsilon(z) &= \int_{\mathbb{R}^d} \rho_0(y) J^\varepsilon(z-y)\,\mathrm{d}y, \end{aligned}$$

where $J^\varepsilon(z) = \frac{1}{\varepsilon^d} J^1\left(\frac{z}{\varepsilon}\right)$. With abuse of notation, let us denote the mollified coefficients by $k^\varepsilon(z), [\sigma_t(z)\sigma_t^{\mathrm{T}}(z)]^\varepsilon_{(\alpha,\beta)}, [\nu_t(z)\nu_t^{\mathrm{T}}(z)]^\varepsilon_{(\alpha,\beta)}$ and the mollified initial condition by ρ_0^ε. Notice, that the mollified functions also satisfy the regularity estimates in Assumption 5.1 uniformly in ε. Additionally, by the properties of mollifiers we have $[\sigma_t(z)\sigma_t^{\mathrm{T}}(z)]^\varepsilon_{(\alpha,\beta)}$, $[\nu_t(z)\nu_t^{\mathrm{T}}(z)]^\varepsilon_{(\alpha,\beta)}$ converging to $[\sigma_t(z)\sigma_t^{\mathrm{T}}(z)]_{(\alpha,\beta)}$, $[\nu_t(z)\nu_t^{\mathrm{T}}(z)]_{(\alpha,\beta)}$ uniformly on compact sets of $\mathbb{R}^d$ and $k^\varepsilon, \rho^\varepsilon$ converge in the L^2-norm towards k, ρ_0.

Let $\rho^{N,\varepsilon}$, ρ^ε be the solution to (5.4), (5.5) with the mollified coefficients. In the next step we demonstrate that the mollified solutions $\rho^{N,\varepsilon}$, ρ^ε converge to ρ^N, ρ.

Lemma 5.23. *Fix $N \in \mathbb{N}$ and let $k \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then we have the following convergence between $\rho^{N,\varepsilon}$ and ρ^N,*

$$\lim_{\varepsilon \to 0} \left\| \rho^{N,\varepsilon} - \rho^N \right\|_{L^2_{\mathcal{F}^W}([0,T];H^1(\mathbb{R}^{dN}))} = 0.$$

Proof. We need to verify the stability assumptions of [Kry99, Theorem 5.7]. Since the coefficients σ, ν are continuous uniform in time, the mollified versions convergence in the supremums norm. Moreover, by the properties of mollification

$$\lim_{\varepsilon \to 0} \left\| \rho_0^{\otimes N,\varepsilon} - \rho_0^{\otimes N} \right\|_{L^2(\mathbb{R}^{dN})} = 0. \tag{5.25}$$

Indeed, for the case $N = 2$, we obtain

$$\begin{aligned} \left\| \rho_0^{\otimes 2,\varepsilon} - \rho_0^{\otimes 2} \right\|_{L^2(\mathbb{R}^{2d})} &= \int_{\mathbb{R}^{2d}} |(\rho_0^\varepsilon(x_1) - \rho_0(x_1))\rho_0^\varepsilon(x_2) + \rho_0(x_1)(\rho_0^\varepsilon(x_2) - \rho_0^\varepsilon(x_2))|^2 \,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &\le 2\left\|\rho_0^\varepsilon - \rho_0\right\|_{L^2(\mathbb{R}^d)} \left\|\rho_0^\varepsilon\right\|_{L^2(\mathbb{R}^d)} + 2\left\|\rho_0^\varepsilon - \rho_0\right\|_{L^2(\mathbb{R}^d)} \left\|\rho_0\right\|_{L^2(\mathbb{R}^d)} \\ &\le 4\left\|\rho_0^\varepsilon - \rho_0\right\|_{L^2(\mathbb{R}^d)} \left\|\rho_0\right\|_{L^2(\mathbb{R}^d)} \\ &\to 0, \quad \text{as } \varepsilon \to 0. \end{aligned}$$

The dN-dimensional case follows by an induction argument. Hence, we only need to take care of the drift term. Define

$$f(\mathsf{X}^N,u)=\sum_{i=1}^N\nabla_{x_i}\cdot\bigg(\frac{1}{N}\sum_{j=1}^N k(x_i-x_j)u\bigg),\quad f^\varepsilon(x,u)=\sum_{i=1}^N\nabla_{x_i}\cdot\bigg(\frac{1}{N}\sum_{j=1}^N k^\varepsilon(x_i-x_j)u\bigg)$$

for $u\in L^2_{\mathcal{F}W}([0,T];H^1(\mathbb{R}^{dN}))$ and observe that

$$\|f^\varepsilon-f\|_{L^2_{\mathcal{F}W}([0,T];H^{-1}(\mathbb{R}^{dN}))}\le\frac{1}{N}\sum_{i,j=1}^N\|(k^\varepsilon(x_i-x_j)-k(x_i-x_j))u\|_{L^2_{\mathcal{F}W}([0,T];L^2(\mathbb{R}^{dN}))}.$$

By the properties of mollification we can extract a subsequence, which we do not rename, such that $\lim\limits_{\varepsilon\to0}k^\varepsilon=k$ a.e. and

$$|(k^\varepsilon(x_i-x_j)-k(x_i-x_j))u|^2\le2\,\|k\|^2_{L^\infty(\mathbb{R}^d)}\,|u|^2.$$

Hence, by the dominated convergence theorem we obtain

$$\lim_{\varepsilon\to0}\|f^\varepsilon-f\|_{L^2_{\mathcal{F}W}([0,T];H^{-1}(\mathbb{R}^{dN}))}=0$$

and we can apply [Kry99, Theorem 5.7.]. □

LEMMA 5.24. *Let $k\in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$. Then, there exists a sequence of smooth solutions we have the following convergence between ρ^ε and ρ,*

$$\lim_{\varepsilon\to0}\|\rho^\varepsilon-\rho\|_{L^2_{\mathcal{F}W}([0,T];H^1(\mathbb{R}^d))}=0.$$

PROOF. The proof is based on the evolution of the L^2-norm similar to Proposition 5.9. By equation (5.17) we know that the Picard iteration $(\rho^n,n\in\mathbb{N})$ convergence. Hence, it is sufficient to find a smooth approximation for $\rho^{n,\varepsilon}$ for each fixed $n\in\mathbb{N}$. Consequently, let $\rho^{n,\varepsilon}$ be the solution of the Picard iteration with mollified coefficients

$$\begin{aligned}\mathrm{d}\rho_t^{n,\varepsilon}&=\nabla\cdot((k^\varepsilon*\rho_t^{n-1,\varepsilon})\rho_t^{n,\varepsilon}))\,\mathrm{d}t-\nabla\cdot(\nu_t^\varepsilon\rho^{n,\varepsilon}\,\mathrm{d}W_t)\\&\quad+\frac12\sum_{\alpha,\beta=1}^d\partial_{x_{i,\alpha}}\partial_{x_{i,\beta}}\bigg(([\sigma_t\sigma_t^{\mathrm{T}}]^\varepsilon_{(\alpha,\beta)}+[\nu_t^\varepsilon(\nu_t^\varepsilon)^{\mathrm{T}}]_{(\alpha,\beta)})\rho_t^{n,\varepsilon}\bigg)\,\mathrm{d}t.\end{aligned}$$

Then,

$$\begin{aligned}\|\rho^\varepsilon-\rho\|_{L^2_{\mathcal{F}W}([0,T];L^2(\mathbb{R}^d))}&\le\|\rho^n-\rho\|_{L^2_{\mathcal{F}W}([0,T];L^2(\mathbb{R}^d))}+\|\rho^{\varepsilon,n}-\rho^\varepsilon\|_{L^2_{\mathcal{F}W}([0,T];L^2(\mathbb{R}^d))}\\&\quad+\|\rho^{\varepsilon,n}-\rho^n\|_{L^2_{\mathcal{F}W}([0,T];L^2(\mathbb{R}^d))}.\end{aligned}\tag{5.26}$$

The first term vanishes in the limit for $n \to \infty$ by (5.17) and the second by similar arguments. Notice that $\rho^{\varepsilon,n} - \rho^n$ solves the following SPDE

$$
\begin{aligned}
&\mathrm{d}(\rho_t^{n,\varepsilon} - \rho_t^n) \\
&\quad = \nabla \cdot ((k^\varepsilon * \rho_t^{n-1,\varepsilon})\rho_t^{n,\varepsilon} - (k * \rho_t^{n-1})\rho_t^n)\,\mathrm{d}t - \nabla \cdot (((\nu_t^\varepsilon - \nu_t)\rho^n + \nu_t^\varepsilon(\rho^{n,\varepsilon} - \rho_t^n))\,\mathrm{d}W_t) \\
&\qquad + \frac{1}{2}\sum_{\alpha,\beta=1}^d \partial_{x_{i,\alpha}}\partial_{x_{i,\beta}}\Big(([\sigma_t\sigma_t^{\mathrm{T}}]^\varepsilon_{(\alpha,\beta)} + [\nu_t^\varepsilon(\nu_t^\varepsilon)^{\mathrm{T}}]_{(\alpha,\beta)} - [\sigma_t\sigma_t^{\mathrm{T}}]_{(\alpha,\beta)} + [\nu_t\nu_t^{\mathrm{T}}]_{(\alpha,\beta)})\rho_t^n \\
&\qquad + [\sigma_t\sigma_t^{\mathrm{T}}]^\varepsilon_{(\alpha,\beta)} + [\nu_t^\varepsilon(\nu_t^\varepsilon)^{\mathrm{T}}]_{(\alpha,\beta)}(\rho_t^{n,\varepsilon} - \rho_t^n)\Big)\,\mathrm{d}t,
\end{aligned}
$$

where the coefficients $\sigma^\varepsilon, \nu^\varepsilon, \sigma, \nu$ all satisfy Assumptions 5.1. Indeed, the mollification preserves the uniform bounds and the divergence free property. By the non-negativity of the mollifier, the ellipticity

$$
\sum_{\alpha,\beta=1}^d [\sigma_s\sigma_s^{\mathrm{T}}(x)]^\varepsilon_{(\alpha,\beta)}\lambda_\alpha\lambda_\beta = \int_{\mathbb{R}^d} J^\varepsilon(x-y)\sum_{\alpha,\beta=1}^d [\sigma_s\sigma_s^{\mathrm{T}}(x)]_{(\alpha,\beta)}\lambda_\alpha\lambda_\beta\,\mathrm{d}y \geq \delta|\lambda|^2
$$

also holds. Utilizing the linearity we can perform the same steps as in Proposition 5.9 to obtain

$$
\begin{aligned}
&\|\rho_t^{\varepsilon,n} - \rho_t^n\|^2_{L^2(\mathbb{R}^d)} - \|\rho_0^{\varepsilon,n} - \rho_0^n\|^2_{L^2(\mathbb{R}^d)} \\
&\quad \leq -2\int_0^t \int_{\mathbb{R}^d} ((k^\varepsilon * \rho_s^{n-1,\varepsilon})\rho_s^{n,\varepsilon} - (k * \rho_s^{n-1})\rho_s^n) \cdot \nabla(\rho_s^{n,\varepsilon} - \rho_s^n)\,\mathrm{d}z\,\mathrm{d}s \\
&\qquad - \sum_{\alpha,\beta=1}^d \int_0^t \int_{\mathbb{R}^d} [\sigma_s\sigma_s^{\mathrm{T}}]^\varepsilon_{(\alpha,\beta)}\partial_{x_{i,\beta}}(\rho_s^{n,\varepsilon} - \rho_s^n)\partial_{x_{i,\alpha}}(\rho_s^{n,\varepsilon} - \rho_s^n)\,\mathrm{d}z\,\mathrm{d}s \\
&\qquad + \frac{\delta}{2}\int_0^t \|\nabla(\rho_s^{n,\varepsilon} - \rho_s^n)\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}s + C(d,\delta)\int_0^t \|\rho_s^{n,\varepsilon} - \rho_s^n\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}s \\
&\qquad + \sum_{\alpha,\beta=1}^d \int_0^t \int_{\mathbb{R}^d} ([\sigma_s\sigma_s^{\mathrm{T}}]^\varepsilon_{(\alpha,\beta)} + [\nu_s^\varepsilon(\nu_s^\varepsilon)^{\mathrm{T}}]_{(\alpha,\beta)} - [\sigma_s\sigma_s^{\mathrm{T}}]_{(\alpha,\beta)} + [\nu_s\nu_s^{\mathrm{T}}]_{(\alpha,\beta)})\rho_s^n\,\mathrm{d}z\,\mathrm{d}s \\
&\qquad + \sum_{\hat{l}=1}^{\tilde{m}} \int_0^t \int_{\mathbb{R}^d} \left|\sum_{\beta=1}^d \partial_{x_{i,\beta}}(\nu_s^{\beta,\hat{l},\varepsilon} - \nu_s^{\beta,\hat{l}})\rho_s^n\right|^2 \mathrm{d}z\,\mathrm{d}s \\
&\qquad + 2\sum_{\hat{l}=1}^{\tilde{m}}\sum_{\beta=1}^d \int_0^t \int_{\mathbb{R}^d} \rho_s^n(\rho_s^{n,\varepsilon} - \rho_s^n)\partial_{x_{i,\beta}}(\nu_s^{\beta,\hat{l},\varepsilon} - \nu_s^{\beta,\hat{l}})\,\mathrm{d}z\,\mathrm{d}W_s^{\hat{l}}.
\end{aligned}
$$

At the moment we can ignore the last term, since it will vanish after taking the expectation. For the penultimate term we use Lemma 5.8. Let us start with the case $d \geq 3$ and denote by

$2^* = 2d/(d-2)$ the Sobolev exponent. Then for all $R \in \mathbb{N}$ we find

$$
\begin{aligned}
&\mathbb{E}\Big(\sum_{\hat{l}=1}^{\tilde{m}} \int_0^t \int_{\mathbb{R}^d} \Big| \sum_{\beta=1}^{d} \partial_{x_{i,\beta}} (\nu_s^{\beta,\hat{l},\varepsilon} - \nu_s^{\beta,\hat{l}}) \rho_s^n \Big|^2 \, \mathrm{d}z \, \mathrm{d}s \Big) \\
&\quad \le \sum_{\hat{l}=1}^{\tilde{m}} \sum_{\beta=1}^{d} \mathbb{E}\Big(\int_0^t \int_{B(0,R)} |(\nu_s^{\beta,\hat{l},\varepsilon} - \nu_s^{\beta,\hat{l}})\rho_s^n|^2 + \int_{B(0,R)^{\mathrm{c}}} |(\nu_s^{\beta,\hat{l},\varepsilon} - \nu_s^{\beta,\hat{l}})\rho_s^n|^2 \, \mathrm{d}z \, \mathrm{d}s \Big) \\
&\quad \le \sum_{\hat{l}=1}^{\tilde{m}} \sum_{\beta=1}^{d} TC\big(\|\rho_0\|_{L^2(\mathbb{R}^d)} \big) \Big\| \nu_s^{\beta,\hat{l},\varepsilon} - \nu_s^{\beta,\hat{l}} \Big\|_{C^1(B(0,R))} \\
&\qquad + C\mathbb{E}\Big(\int_0^t \|\rho_s^n\|_{L^1(B(0,R)^{\mathrm{c}})}^{\frac{4}{d+2}} \|\rho_s^n\|_{L^{2^*}(\mathbb{R}^d)}^{\frac{2d}{d+2}} \, \mathrm{d}s \Big) \\
&\quad \le \sum_{\hat{l}=1}^{\tilde{m}} \sum_{\beta=1}^{d} TC\big(\|\rho_0\|_{L^2(\mathbb{R}^d)} \big) \Big\| \nu_s^{\beta,\hat{l},\varepsilon} - \nu_s^{\beta,\hat{l}} \Big\|_{C^1(B(0,R))} \\
&\qquad + \frac{C}{R^{\frac{8}{d+2}}} \mathbb{E}\Big(\int_0^t \big\| \rho_s^n |\cdot|^2 \big\|_{L^1(\mathbb{R}^d)}^{\frac{4}{d+2}} \|\rho_s^n\|_{H^1(\mathbb{R}^d)}^{\frac{2d}{d+2}} \, \mathrm{d}s \Big) \\
&\quad \le \sum_{\hat{l}=1}^{\tilde{m}} \sum_{\beta=1}^{d} TC\big(\|\rho_0\|_{L^2(\mathbb{R}^d)} \big) \Big\| \nu_s^{\beta,\hat{l},\varepsilon} - \nu_s^{\beta,\hat{l}} \Big\|_{C^1(B(0,R))} \\
&\qquad + \frac{CT^{\frac{2}{d+2}}}{R^{\frac{8}{d+2}}} \int_0^T \mathbb{E}\Big(\big\| \rho_s^n |\cdot|^2 \big\|_{L^1(\mathbb{R}^d)}^2 \Big)^{\frac{2}{d+2}} \|\rho_s^n\|_{L^2_{\mathcal{F}W}([0,T];H^1(\mathbb{R}^d))}^{\frac{2d}{d+2}} \\
&\quad \le \sum_{\hat{l}=1}^{\tilde{m}} \sum_{\beta=1}^{d} TC\big(\|\rho_0\|_{L^2(\mathbb{R}^d)} \big) \Big\| \nu_s^{\beta,\hat{l},\varepsilon} - \nu_s^{\beta,\hat{l}} \Big\|_{C^1(B(0,R))} + C(\rho_0, T) R^{-\frac{8}{d+2}},
\end{aligned}
$$

where we used the interpolation inequality for L^p spaces and Assumption 5.1 (2) in the third step, the Sobolev embedding in the fourth step and finally inequality (5.15) and Lemma 5.8 in the last step. In the case $d = 2$ we obtain a similar estimate by using the L^q-bound on ρ^n [Leo17, Theorem 12.33] for all $q \in [2,,\infty)$. Utilizing the same split of domains and applying Lemma 5.8 we obtain

$$
\begin{aligned}
&\sum_{\alpha,\beta=1}^{d} \mathbb{E}\Big(\int_0^t \int_{\mathbb{R}^d} ([\sigma_s \sigma_s^{\mathrm{T}}]_{(\alpha,\beta)}^{\varepsilon} + [\nu_s^{\varepsilon} (\nu_s^{\varepsilon})^{\mathrm{T}}]_{(\alpha,\beta)} - [\sigma_s \sigma_s^{\mathrm{T}}]_{(\alpha,\beta)} + [\nu_s \nu_s^{\mathrm{T}}]_{(\alpha,\beta)}) \rho_s^n \, \mathrm{d}z \, \mathrm{d}s \Big) \\
&\quad \le T \sum_{\alpha,\beta=1}^{d} \Big(\sum_{\hat{l}=1}^{\tilde{m}} \Big\| \nu_s^{\beta,\hat{l},\varepsilon} - \nu_s^{\beta,\hat{l}} \Big\|_{C^1(B(0,R))} + \Big\| [\sigma_s \sigma_s^{\mathrm{T}}]_{(\alpha,\beta)}^{\varepsilon} - [\sigma_s \sigma_s^{\mathrm{T}}]_{(\alpha,\beta)} \Big\|_{C^1(B(0,R))} \Big) \\
&\qquad + C(\rho_0) T^{\frac{1}{2}} R^{-2}.
\end{aligned}
$$

Recall that $[\sigma_s\sigma_s^{\mathrm{T}}]^\varepsilon_{(\alpha,\beta)}$ is still elliptic and therefore

$$\sum_{\alpha,\beta=1}^d \int_0^t \int_{\mathbb{R}^d} [\sigma_s\sigma_s^{\mathrm{T}}]^\varepsilon_{(\alpha,\beta)} \partial_{x_{i,\beta}}(\rho_s^{n,\varepsilon}-\rho_s^n)\partial_{x_{i,\alpha}}(\rho_s^{n,\varepsilon}-\rho_s^n)\,\mathrm{d}z\,\mathrm{d}s \geq \delta \int_0^t \int_{\mathbb{R}^d} |\nabla(\rho_s^{n,\varepsilon}-\rho_s^n)|^2\,\mathrm{d}z\,\mathrm{d}s.$$

Combing the last three inequalities we obtain

$$\begin{aligned}
&\mathbb{E}\Big(\|\rho_t^{\varepsilon,n}-\rho_t^n\|^2_{L^2(\mathbb{R}^d)} \Big) \\
&\quad\leq \mathbb{E}\Big(-\frac{\delta}{2} \int_0^t \int_{\mathbb{R}^d} |\nabla(\rho_s^{n,\varepsilon}-\rho_s^n)|^2\,\mathrm{d}z\,\mathrm{d}s + C(\rho_0)T^{\frac{1}{2}}R^{-1} \\
&\qquad - 2\int_0^t \int_{\mathbb{R}^d} ((k^\varepsilon * \rho_s^{n-1,\varepsilon})\rho_s^{n,\varepsilon} - (k*\rho_s^{n-1})\rho_s^n)\cdot\nabla(\rho_s^{n,\varepsilon}-\rho_s^n)\,\mathrm{d}z\,\mathrm{d}s \\
&\qquad + \sum_{\hat{l}=1}^{\tilde{m}} \sum_{\beta=1}^d TC\big(\|\rho_0\|_{L^2(\mathbb{R}^d)}\big) \Big\|\nu_s^{\beta,\hat{l},\varepsilon}-\nu_s^{\beta,\hat{l}}\Big\|_{C^1(B(0,R))} + C(\rho_0,T)R^{-\frac{4}{d+2}} \\
&\qquad + T\sum_{\alpha,\beta=1}^d \Big(\sum_{\hat{l}=1}^{\tilde{m}} \Big\|\nu_s^{\beta,\hat{l},\varepsilon}-\nu_s^{\beta,\hat{l}}\Big\|_{C^1(B(0,R))} + \Big\|[\sigma_s\sigma_s^{\mathrm{T}}]^\varepsilon_{(\alpha,\beta)} - [\sigma_s\sigma_s^{\mathrm{T}}]_{(\alpha,\beta)}\Big\|_{C^1(B(0,R))} \Big)\Big).
\end{aligned}$$

We notice that by the mollification properties the last terms will vanish. The only difficulty remaining is the drift term. Young's inequality implies

$$\begin{aligned}
&- 2\int_0^t \int_{\mathbb{R}^d} ((k^\varepsilon * \rho_s^{n-1,\varepsilon})\rho_s^{n,\varepsilon} - (k*\rho_s^{n-1})\rho_s^n)\cdot\nabla(\rho_s^{n,\varepsilon}-\rho_s^n)\,\mathrm{d}z\,\mathrm{d}s \\
&\quad\leq \int_0^t \int_{\mathbb{R}^d} \frac{1}{2\delta}|(k^\varepsilon * \rho_s^{n-1,\varepsilon})\rho_s^{n,\varepsilon} - (k*(\rho_s^{n-1})\rho_s^n)|^2 + \frac{\delta}{2}|\nabla(\rho_s^{n,\varepsilon}-\rho_s^n)|^2\,\mathrm{d}z\,\mathrm{d}s.
\end{aligned}$$

Notice, that the last term can be absorbed by the diffusion. For the first term we obtain

$$\begin{aligned}
&\frac{1}{2\delta}\int_0^t \int_{\mathbb{R}^d} |(k^\varepsilon * \rho_s^{n-1,\varepsilon})\rho_s^{n,\varepsilon} - (k*(\rho_s^{n-1})\rho_s^n)|^2\,\mathrm{d}z\,\mathrm{d}s \\
&\quad\leq \frac{2}{\delta}\int_0^t \int_{\mathbb{R}^d} |(k^\varepsilon - k) * \rho_s^{n-1,\varepsilon})\rho_s^{n,\varepsilon}|^2 + (k*(\rho_s^{n-1,\varepsilon}-\rho_s^{n-1}))\rho_s^{n,\varepsilon} \\
&\qquad + k*\rho_s^{n-1}(\rho_s^{n,\varepsilon}-\rho_s^n)|^2\,\mathrm{d}z\,\mathrm{d}s \\
&\quad\leq \frac{2C(\rho_0)}{\delta}\Big(T\,\|k^\varepsilon-k\|^2_{L^2(\mathbb{R}^d)} + \int_0^t \big\|\rho_s^{n-1,\varepsilon}-\rho_s^{n-1}\big\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}s + \int_0^t \|\rho_s^{n,\varepsilon}-\rho_s^n\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}s \Big),
\end{aligned}$$

where we used Lemma 5.7 and Young's inequality for convolutions. Substituting this inequality and applying Gronwall's lemma we find

$$\mathbb{E}\Big(\|\rho_t^{\varepsilon,n} - \rho_t^n\|_{L^2(\mathbb{R}^d)}^2 \Big) \le \Big(A(\nu,\sigma,R,\varepsilon) + \int_0^t \mathbb{E}\big(\|\rho_s^{n-1,\varepsilon} - \rho_s^{n-1}\|_{L^2(\mathbb{R}^d)}^2 \big) \,\mathrm{d}s \Big) e^{CT}, \tag{5.27}$$

where

$$\begin{aligned}
&A(\nu,\sigma,R,\varepsilon) \\
&\quad = \sum_{\hat{l}=1}^{\tilde{m}} \sum_{\beta=1}^{d} TC\big(\|\rho_0\|_{L^2(\mathbb{R}^d)} \big) \Big\| \nu_s^{\beta,\hat{l},\varepsilon} - \nu_s^{\beta,\hat{l}} \Big\|_{C^1(B(0,R))} + C(\rho_0,T) R^{-\frac{8}{d+2}} \\
&\qquad + T \sum_{\alpha,\beta=1}^{d} \Big(\sum_{\hat{l}=1}^{\tilde{m}} \Big\| \nu_s^{\beta,\hat{l},\varepsilon} - \nu_s^{\beta,\hat{l}} \Big\|_{C^1(B(0,R))} + \Big\| [\sigma_s\sigma_s^{\mathrm{T}}]^{\varepsilon}_{(\alpha,\beta)} - [\sigma_s\sigma_s^{\mathrm{T}}]_{(\alpha,\beta)} \Big\|_{C^1(B(0,R))} \Big) \\
&\qquad + C(\rho_0) T^{\frac{1}{2}} R^{-2} + \frac{2C(\rho_0)}{\delta} T \|k - k^{\varepsilon}\|_{L^2(\mathbb{R}^d)}^2 + \|\rho_0^{\varepsilon,n} - \rho_0^n\|_{L^2(\mathbb{R}^d)}^2 .
\end{aligned}$$

Applying the inequality n times we arrive at

$$\mathbb{E}\Big(\|\rho_t^{\varepsilon,n} - \rho_t^n\|_{L^2(\mathbb{R}^d)}^2 \Big) \le A(\nu,\sigma,R,\varepsilon) e^{CT} \sum_{j=0}^{n-1} \frac{T^j e^{CTj}}{j!} + \frac{e^{CTn}}{n!} \|\rho_0^{\varepsilon} - \rho_0\|_{L^2(\mathbb{R}^d)}^2 ,$$

which implies

$$\sup_{n\in\mathbb{N}} \mathbb{E}\Big(\|\rho_t^{\varepsilon,n} - \rho_t^n\|_{L^2(\mathbb{R}^d)}^2 \Big) \le A(\nu,\sigma,R,\varepsilon) e^{CT} e^{Te^{CT}} + C \|\rho_0^{\varepsilon} - \rho_0\|_{L^2(\mathbb{R}^d)}^2$$

and finally the convergence

$$\limsup_{\varepsilon\to 0} \sup_{n\in\mathbb{N}} \|\rho^{\varepsilon,n} - \rho^n\|_{L^2_{\mathcal{F}W}([0,T];L^2(\mathbb{R}^d))} = 0,$$

by taking first $\varepsilon \to 0$ in combination with (5.25) and the properties of mollifiers, and then $R \to \infty$. Together with inequality (5.26) and the subsequent comment, this implies

$$\limsup_{\varepsilon\to 0} \|\rho^{\varepsilon} - \rho\|_{L^2_{\mathcal{F}W}([0,T];L^2(\mathbb{R}^d))} = 0.$$

Applying [Kry99, Theorem 5.1], the L^2-bound (5.9) and $k \in L^2(\mathbb{R}^d)$ we arrive at

$$\begin{aligned}
\limsup_{\varepsilon\to 0} \|\rho^{\varepsilon} - \rho\|_{L^2_{\mathcal{F}W}([0,T];H^1(\mathbb{R}^d))} &\le \limsup_{\varepsilon\to 0} \|k * (\rho^{\varepsilon} - \rho)\rho\|_{L^2_{\mathcal{F}W}([0,T];L^2(\mathbb{R}^d))} \\
&\le C \limsup_{\varepsilon\to 0} \|k * (\rho^{\varepsilon} - \rho)\|_{L^2_{\mathcal{F}W}([0,T];L^\infty(\mathbb{R}^d))} \\
&\le C \|k\|_{L^2(\mathbb{R}^d)} \limsup_{\varepsilon\to 0} \|\rho^{\varepsilon} - \rho\|_{L^2_{\mathcal{F}W}([0,T];L^2(\mathbb{R}^d))} \\
&= 0.
\end{aligned}$$

□

Finally, we can present the analogous result to [JW18, Theorem 1] for bounded kernels.

THEOREM 5.25. *Let $k \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, ρ^N be a solution to the Liouville equation* (5.4) *and $\rho^{\otimes N}$ be a solution to the chaotic SPDE* (5.6)*. Then, we have the following relative entropy bound*

$$\mathbb{E}\bigg(\sup_{0\le t\le T} \mathcal{H}\big(\rho_t^N | \rho_t^{\otimes N}\big)\bigg) \le C\big(T, \delta, \|k\|_{L^\infty(\mathbb{R}^d)}\big) \tag{5.28}$$

for some non-negative constant C depending on $\|k\|_{L^\infty(\mathbb{R}^d)}$.

PROOF. By Lemma 5.23 and Lemma 5.24 we know there exists a subsequence, which we do not rename such that $\rho^{N,\varepsilon}$ and $\rho^{\otimes N,\varepsilon}$ converge almost everywhere on $\Omega\times[0,T]\times\mathbb{R}^{dN}$. Hence, by the lower semicontinuity of the relative entropy, the estimate $\|k^\varepsilon\|_{L^\infty(\mathbb{R}^d)} \le C\,\|k\|_{L^\infty(\mathbb{R}^d)}$ and Theorem 5.19 our claim follows. □

As in the smooth case, we can also use the Csiszár–Kullback–Pinsker inequality and sub-additivity property to prove L^1-convergence of every r-marginal.

COROLLARY 5.26. *In the setting of Theorem 5.25, we obtain*

$$\mathbb{E}\bigg(\sup_{0\le t\le T} \Big\|\rho_t^{r,N} - \rho_t^{\otimes r}\Big\|^2_{L^1(\mathbb{R}^{dr})}\bigg) \le \frac{Cr}{N}$$

for every $r \in \mathbb{N}$.

5.5. Conditional propagation of chaos

At the moment it is unclear whether the convergence results in Section 5.4 implies conditional propagation of chaos in the sense of weak convergence of empirical measures. In this section we will demonstrate some useful comparison results between the classical propagation of chaos towards a deterministic measure and the conditional propagation of chaos in the setting of common noise.

We demonstrate the following conditional exchangeability for the interacting particle system (5.2), which is based on [CF16, Lemma 23] adjusted to our setting.

LEMMA 5.27. *Let $t \ge 0$ and $(X_t^i, i = 1, \ldots, N)$ be given by* (5.2)*. Then for any permutation $\vartheta\colon \{1,\ldots,N\} \mapsto \{1,\ldots,N\}$ the vector of random variables $(X_t^i, i = 1,\ldots,N)$ satisfies*

$$\mathbb{E}(h(X_t^1, X_t^2, \ldots, X_t^N)|\mathcal{F}_t^W) = \mathbb{E}(h(X_t^{\vartheta(1)}, X_t^{\vartheta(2)}, \ldots, X_t^{\vartheta(N)})|\mathcal{F}_t^W)$$

for every $h \in C_b(\mathbb{R}^{dN})$.

REMARK 5.28. *In particular the condition* (5.30) *in Lemma 5.29 is fulfilled with $\mathcal{G} = \mathcal{F}_t^W$.*

PROOF. Consider the particle system (5.2) without common noise. Then by [HRZ24] this SDE has a strong solution. Thus, we know that the particle system (5.2) must also have a strong solution. Additionally, by the exchangeability of the initial condition, the Yamada–Watanabe theorem tells us that strong uniqueness implies uniqueness in law and therefore we obtain

$$\begin{aligned} &\mathrm{Law}\big(((X_t^1, X_t^2, \ldots, X_t^N), (W_t^i, i = 1, \ldots, \tilde{m}))\big) \\ &= \mathrm{Law}\big(((X_t^{\vartheta(1)}, X_t^{\vartheta(2)}, \ldots, X_t^{\vartheta(N)}), (W_t^i, i = 1, \ldots, \tilde{m}))\big). \end{aligned} \tag{5.29}$$

Now choose a cylinder set $A \in \mathcal{F}_t^W$, i.e.

$$A = (W^1_{t_{1,1}})^{-1}(A_{1,1}) \cap \cdots \cap (W^1_{t_{1,r_1}})^{-1}(A_{1,r_1}) \cap (W^2_{t_{2,1}})^{-1}(A_{2,1}) \cap \cdots \cap (W^{\tilde m}_{t_{\tilde m, r_{\tilde m}}})^{-1}(A_{\tilde m, r_{\tilde m}})$$

for $r_1, \dots, r_{\tilde m} \in \mathbb{N}$, $t_{i,\gamma} \in [0,t]$ for $i = 1, \dots, \tilde m$, $\gamma = 1, \dots, \max(r_1, \dots, r_{\tilde m})$. These cylinder set are closed under intersections and generate the σ-algebra $\mathcal{F}_t^W$. For $f \in C_b(\mathbb{R}^{dN})$ we find

$$\mathbb{E}(\mathbb{1}_A h(X_t^1, X_t^2, \dots, X_t^N)) = \mathbb{E}(\mathbb{1}_A h(X_t^{\vartheta(1)}, X_t^{\vartheta(2)}, \dots, X_t^{\vartheta(N)}))$$

by the representation of A as the intersection of inverse images of W and the uniqueness of laws (5.29). Consequently, the conditional expectations must coincide. □

Let us partially transfer the convergence results by Sznitman's Proposition 1.2 to the case of random limiting measure.

Lemma 5.29. *Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Let $\mathbf{Z}^N = (Z^{1,N}, \dots, Z^{N,N})$ be a $\mathbb{R}^{dN}$-valued exchangeable vector of measurable variables and $f^{1,N} \in L^1_{\mathcal{G}}(L^1(\mathbb{R}^d))$, $f^{2,N} \in L^1_{\mathcal{G}}(L^1(\mathbb{R}^{2d})$ be the conditional densities given the σ-algebra $\mathcal{G}$ of the random variables $Z^{1,N}, (Z^{1,N}, Z^{2,N})$, respectively. Additionally, suppose there exists a measurable function $g \in L^1_{\mathcal{G}}(L^1(\mathbb{R}^d)) \cap L^\infty_{\mathcal{G}}(L^1(\mathbb{R}^d))$ such that*

$$\lim_{N\to\infty} \left(\left\| f^{1,N} - g \right\|_{L^1_{\mathcal{G}}(L^1(\mathbb{R}^d))} + \left\| f^{2,N} - g \otimes g \right\|_{L^1_{\mathcal{G}}(L^1(\mathbb{R}^{2d}))} \right) = 0$$

and for any bounded continuous function $h \in C_b(\mathbb{R}^d)$ the equality

$$\mathbb{E}(h(Z^{1,N})|\mathcal{G}) = \mathbb{E}(h(Z^{i,N})|\mathcal{G}), \quad \text{for all } i \le N \tag{5.30}$$

holds. Then, the empirical measure converges to g in the sense of random measures equipped with the topology of weak convergence. More precisely, for any bounded continuous function $\varphi \in \mathbb{R}^d$ we have

$$\left\langle \frac{1}{N} \sum_{i=1}^N \delta_{Z^{i,N}}, \varphi \right\rangle \xrightarrow[N\to\infty]{d} \langle g, \varphi \rangle,$$

where the convergence is in the sense of distributions.

Proof. Let $\varphi\colon \mathbb{R}^d \mapsto \mathbb{R}$ be a bounded continuous function. We show L^2-converges, which then implies converges in distribution. Expanding the square we find

$$\begin{aligned}
&\mathbb{E}\left(\left| \left\langle \frac{1}{N} \sum_{i=1}^N \delta_{Z^{i,N}}, \varphi \right\rangle - \langle g, \varphi \rangle \right|^2 \right) \\
&= \frac{1}{N^2} \sum_{i,j=1}^N \mathbb{E}(\varphi(Z^{i,N})\varphi(Z^{j,N})) - 2\frac{1}{N} \sum_{i=1}^N \mathbb{E}(\varphi(Z^{i,N})\langle g, \varphi \rangle) + \mathbb{E}(\langle g, \varphi \rangle^2) \\
&= \frac{1}{N}\mathbb{E}(\varphi(Z^{i,N})^2) + \frac{(N^2 - N)}{N^2}\mathbb{E}(\varphi(Z^{1,N})\varphi(Z^{2,N})) \\
&\quad - 2\mathbb{E}(\mathbb{E}(\varphi(Z^{1,N})|\mathcal{G})\langle g, \varphi \rangle) + \mathbb{E}(\langle g, \varphi \rangle^2).
\end{aligned}$$

Notice that φ is bounded and therefore the first term vanishes. The factor of the second term converges to one and for the expected value we obtain

$$\begin{aligned}|\mathbb{E}(\varphi(Z^{1,N})\varphi(Z^{2,N})) - \mathbb{E}(\langle\varphi, g\rangle^2)| &= |\mathbb{E}(\varphi(Z^{1,N})\varphi(Z^{2,N})|\mathcal{G}) - \mathbb{E}(\langle\varphi, g\rangle^2)| \\ &= |\mathbb{E}(\langle\varphi\otimes\varphi, f^{2,N}\rangle) - \mathbb{E}(\langle\varphi\otimes\varphi, g\otimes g\rangle)| \\ &\leq \|\varphi\|^2_{L^\infty(\mathbb{R}^d} \left\|f^{2,N} - g\otimes g\right\|_{L^1_{\mathcal{G}}(L^1(\mathbb{R}^{2d}))} \\ &\to 0, \text{ as } N\to\infty.\end{aligned}$$

For the third term we find

$$\begin{aligned}|\mathbb{E}(\mathbb{E}(\varphi(Z^{1,N})|\mathcal{G})\langle\varphi, g\rangle) - \mathbb{E}(\langle\varphi, g\rangle^2)| &= |\mathbb{E}((\langle\varphi, f^{1,N}\rangle - \langle\varphi, g\rangle)\langle\varphi, g\rangle)| \\ &\leq \|\varphi\|_{L^\infty(\mathbb{R}^d)} \left\|f^{1,N} - g\right\|_{L^1_{\mathcal{G}}(L^1(\mathbb{R}^d))} \|\langle\varphi, g\rangle\|_{L^\infty_{\mathcal{G}}} \\ &\leq \|\varphi\|^2_{L^\infty(\mathbb{R}^d)} \|g\|_{L^\infty_{\mathcal{G}}(L^1(\mathbb{R}^d))} \left\|f^{1,N} - g\right\|_{L^1_{\mathcal{G}}(L^1(\mathbb{R}^d))} \\ &\to 0, \text{ as } N\to\infty.\end{aligned}$$

Combining the two convergences proves the lemma. □

Combining Lemma 5.29, Lemma 5.27 and Theorem 5.25 we obtain conditional propagation of chaos.

COROLLARY 5.30. *Let $k \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then conditional propagation of chaos holds for the interacting particle system* (5.2) *towards the random measure with density ρ given by the solution of the SPDE* (5.5).

5.6. Comments

At the conclusion of this thesis, it is fitting to reflect on the outcomes and implications of our study.

The results in this chapter, while significant, appear somewhat unsatisfactory in that the inclusion of common noise did not provide a regularizing effect or introduce new terms in the relative entropy that originate from the common noise. Together with Lemma A.36, our findings essentially extend the work of Jabin and Wang [JW18] to a bounded setting. This raises an intriguing question about the utility of adding common noise to the system: Does it influence the mean-field limit? Could transport noise have a different impact? These questions remain open and validates further investigation.

One potential explanation is that the scaling in mean-field limits might be too strong. To achieve term cancellation in the relative entropy, it is necessary to consider the conditional Liouville equation rather than the original system. Remark 5.17 indicates that we essentially freeze the common noise in all calculations. Nonetheless, Lemma 5.29 shows that considering this frozen scenario is sufficient to demonstrate propagation of chaos. At the beginning of this project, we anticipated that including common noise in the relative entropy framework would yield novel insights, freeing us from classical methods. However, we found ourselves returning to these classical methods, perhaps due to our focus on the conditional Liouville equation. Future research could explore not considering the conditional Liouville equation, potentially revealing new and interesting phenomena.

But what about fluctuations or large deviation principles? We know that relative entropy is crucial in large deviation theory [HHMT24]. Understanding how our results contribute to this context and how they can be applied is an interesting challenge. Comparing our findings with large deviation principles, which were established by mean-field games, it still remains unclear how to connect these results. Thus, deeper analysis of regularization by common noise is necessary.

Our study also highlights the importance of SPDE stability. Often, a priori estimates can be achieved, or, as demonstrated in this thesis, results can be proven at an intermediate level, as shown by the top arrow of Figure 1. However, the left arrow in Figure 1 is not well-studied in the mean-field community. For example, Jabin and Wang [JW18] avoid this by restricting the domain to the torus and requiring so-called entropy solutions [JW18, Definition 2], but uniqueness in this class is still lacking. From a probabilistic viewpoint, this suggests that for the sequence $(\mathbf{X}^\varepsilon, \varepsilon > 0)$, we can show tightness and that it solves the limiting problem, but not that the problem itself is uniquely solvable due to the lack of uniqueness in the class of entropy solutions. This challenge restricts us to $k \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ in this chapter, preventing us from achieving the desired regularity $k \in W^{-1,\infty}$ from Jabin and Wang [JW18], without additional assumptions and a torus setting.

Finally, throughout this thesis, we have explored mean-field limits, propagation of chaos, PDEs, and SPDEs. However, when examining the limiting procedure, it appears that the mean-field limit term (1.25) introduced in Chapter 1 is always present, lurking behind various methods. We have approached it from different perspectives: convergence in probability in Chapter 2, and through relative entropy in Chapters 3 and 5. Each method has its advantages depending on the specific kernel k, which we hope have been clearly presented. The ultimate goal from our perspective, is to develop a unifying theory that integrates these approaches and provides a comprehensive understanding of the quantity described in (1.25).

Chapter A

Miscellanea

For completion purposes we present some useful inequalities and embeddings, which are needed throughout the thesis. Most of them are well-known. We refer to [LL01, Fol99, AF03, Eva10, Emm13, Leo17] for the proofs. Additionally, we recall some stochastic inequalities and the stochastic Fubini theorem. Finally, we demonstrate a nice result about relative entropy and conditional densities in Section A.6.

A.1. Inequalities

Lemma A.1 (Young's inequality with epsilon). *Let $a > 0$, $b > 0$, $\varepsilon > 0$, $p > 1$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$ab \leq \frac{\varepsilon}{p} a^p + \frac{\varepsilon^{-\frac{q}{p}} b^q}{q}.$$

Theorem A.2 (Minkowski's inequality). *Let $1 \leq p < \infty$. Suppose that Ω and Ω' are two spaces with σ-finite measures μ and ν, respectively. Let f be a non-negative function on $\Omega \times \Omega'$, which is $\mu \times \nu$ measurable. Then*

$$\left(\int_\Omega \left(\int_{\Omega'} f(x,y) \, \mathrm{d}\nu(y) \right)^p \mathrm{d}\mu(x) \right)^{\frac{1}{p}} \leq \int_{\Omega'} \left(\int_\Omega f(x,y)^p \, \mathrm{d}\mu(x) \right)^{\frac{1}{p}} \mathrm{d}\nu(y).$$

The equation should be understood in the sense that the finiteness of the right-hand side implies the finiteness of the left-hand side.

Lemma A.3 (Interpolation inequality). *Let $U \subseteq \mathbb{R}^d$ be a measurable set. Let $1 \leq p \leq q \leq r \leq \infty$ and*

$$\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r}, \quad \text{that is, } \theta = \frac{1/q - 1/r}{1/p - 1/r}.$$

Moreover, suppose $f \in L^p(U) \cap L^r(U)$. Then $f \in L^q(U)$ and

$$\|f\|_{L^q(U)} \leq \|f\|_{L^p(U)}^\theta \|f\|_{L^r(U)}^{1-\theta}.$$

Theorem A.4 (Young's Inequality for integrals). *Let $p, q, r \geq 1$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$. Let $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ and $h \in L^r(\mathbb{R}^d)$. Then*

$$\begin{aligned} \left| \int_{\mathbb{R}^d} f(x)(g * h)(x) \, \mathrm{d}x \right| &= \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) g(x-y) h(y) \, \mathrm{d}y \, \mathrm{d}x \right| \\ &\leq C(p,q,r,d) \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)} \|h\|_{L^r(\mathbb{R}^d)}, \end{aligned}$$

where $C(p,q,r,d)$ is a constant which depends on the parameters p, q, r, d.

Corollary A.5 (Young's inequality for convolution). *A special case of Theorem A.4 above is the following inequality. For $g \in L^q(\mathbb{R}^d)$ and $h \in L^r(\mathbb{R}^d)$ we have*

$$\|g * h\|_{L^p(\mathbb{R}^d)} \leq C(p,q,r,d)\, \|g\|_{L^q(\mathbb{R}^d)}\, \|h\|_{L^r(\mathbb{R}^d)}$$

with $\frac{1}{q} + \frac{1}{r} = 1 + \frac{1}{p}$.

Theorem A.6 (Hardy–Littlewood–Sobolev inequality). *Let $p, q > 1$ and $0 < \sigma < n$ with $\frac{1}{p} + \frac{\sigma}{n} + \frac{1}{q} = 2$. Let $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$. Then there exists a sharp constant $C(d, \sigma, p)$, independent of f and g such that*

$$\int_{R^n} \int_{R^n} \frac{|f(x)||g(y)|}{|x-y|^\sigma}\, \mathrm{d}x\, \mathrm{d}y \leq C(d,\sigma,p)\, \|f\|_{L^p(\mathbb{R}^d)}\, \|g\|_{L^q(\mathbb{R}^d)}\,.$$

Alternative the conditions on p, q, σ, d can be rephrased as $p, q > 1$, $1 < \frac{1}{p} + \frac{1}{q} < 2$ and $\sigma = d\left(\frac{p-1}{p} + \frac{q-1}{q}\right)$.

Theorem A.7 (Hardy–Littlewood–Sobolev inequality (second version)). *Let $0 < \sigma < n$, $1 < p < q < \infty$ and $\sigma = d(1 - 1/p + 1/q)$. Then we have the estimate*

$$\left(\int_{\mathbb{R}^d} \left|\frac{1}{|x-y|^\sigma} f(y)\right|^q \mathrm{d}y\right)^{\frac{1}{q}} \leq C(p,q)\, \|f\|_{L^p(\mathbb{R}^d)}\,.$$

Theorem A.8 (Sobolev inequality). *Let $1 \leq p < d$ and $p^* = \frac{dp}{d-p}$. Then there exist a constant $C = C(p,d)$ such that for all $u \in W^{1,p}(\mathbb{R}^d)$*

$$\|u\|_{L^{p^*}(\mathbb{R}^d)} \leq C\, \|\nabla u\|_{L^p(\mathbb{R}^d)}$$

holds true.

Theorem A.9. *Let $U \subset \mathbb{R}^d$ be open with compact closure in $\mathbb{R}^d$ and Lipschitz continuous boundary. Let $m, l \in \mathbb{N}_0$ and $1 \leq p, q < \infty$. Then for $m \geq l$ and $m - \frac{d}{p} \geq l - \frac{d}{q}$ the following embeddings are continuous*

$$W^{m,p}(U) \hookrightarrow W^{l,q}(U).$$

Furthermore, if the inequalities on m, l, p, q are strict, then the above embedding is compact.

Theorem A.10 (Sobolev embedding in the case $kp = d$). *Let $k, n \in \mathbb{N}$ be such that $n > m$. Then there exists a positive constant $C = C(d,k)$ such that for every function $u \in W^{k,p}(\mathbb{R}^d)$ with $p = d/k$*

$$\|u\|_{L^q(\mathbb{R}^d)} \leq C q^{1 - \frac{m}{n} + \frac{1}{q}}\, \|u\|_{W^{k,p}(\mathbb{R}^d)}$$

holds true for every $q \leq p < \infty$.

Theorem A.11 (Morrey's inequality in $W^{1,p}(\mathbb{R}^d)$). *Let $d \in \mathbb{N}$ and $d < p < \infty$. Then $W^{1,p}(\mathbb{R}^d) \hookrightarrow C^{0,1-d/p}(\mathbb{R}^d)$ and $W^{1,p}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$. Hence, for a function $u \in W^{1,p}(\mathbb{R}^d)$ we can find a representative $\bar{u}$ such that $\bar{u} = u$ a.e. and $\bar{u} \in C_b^{0,1-d/p}(\mathbb{R}^d)$.*

Theorem A.12 (Rellich–Kondrachov Compactness Theorem). *Let $U \subset \mathbb{R}^d$ be an open set with compact closure in $\mathbb{R}^d$, Lipschitz continuous boundary, and $1 \leq p < \infty$. Then the following embedding*

$$W^{1,p}(U) \hookrightarrow L^p(U)$$

is compact. Furthermore, if $1 \leq p \leq d$ and $p^* = \frac{dp}{d-p}$ is the Sobolev exponent, then the embedding

$$W^{1,p}(U) \hookrightarrow L^q(U)$$

is compact for each $1 \leq q < p^*$.

For the next theorem we need the following notation. Let $r \in [-\infty, \infty] \setminus \{0\}$ and for $r < 0$ we set $l := \lfloor -\frac{d}{r} \rfloor$ and $a := -l - \frac{d}{r} \in [0, 1)$. For a function $u : \mathbb{R}^d \to \mathbb{R}$ we define

$$|u|_r = \begin{cases} \|u\|_{L^r(\mathbb{R}^d)} & \text{if } r > 0, \\ \left\|\nabla^l u\right\|_{L^\infty(\mathbb{R}^d)} & \text{if } r < 0 \text{ and } a = 0, \\ |\nabla^l u|_{C^{0,a}(\mathbb{R}^d)} & \text{if } r < 0 \text{ and } 0 < a < 1, \end{cases}$$

provided that the right-hand side is well-defined.

Theorem A.13 (Gagliardo–Nirenberg interpolation). *Let $1 \leq p, q \leq \infty$, let $0 \leq \theta \leq 1$ and let r be such that*

$$1 - \theta \left(\frac{1}{p} - \frac{1}{d}\right) + \frac{\theta}{q} = \frac{1}{r} \in (-\infty, 1].$$

Then there exists a constant $C = C(n, p, q, \theta)$ such that

$$|u|_r \leq C \left\|\nabla u\right\|_{L^p(\mathbb{R}^d)}^{1-\theta} \left\|u\right\|_{L^q(\mathbb{R}^d)}^{\theta}$$

for every $u \in L^q(\mathbb{R}^d) \cap W^{1,p}(\mathbb{R}^d)$, with the exception that if $p < d$ and $q = \infty$ we assume that u vanishes at infinity, while if $p = d$ we take $0 < \theta \leq 1$.

A.2. Mollification

For $U \subseteq \mathbb{R}^d$ open we introduce the following set

$$U_\varepsilon := \{z \in U \mid \operatorname{dist}(x, \partial U) > \varepsilon\}.$$

And for $U = \mathbb{R}^d$ we define $\mathbb{R}^d_\varepsilon := \mathbb{R}^d$.

Definition A.14. *Define the standard mollifier $J \in C_c^\infty(\mathbb{R}^d)$ by*

$$J(z) := \begin{cases} C \exp\left(\frac{1}{|z|^2 - 1}\right), & \text{if } |z| < 1, \\ 0, & \text{if } |z| \geq 1, \end{cases}$$

the constant C is selected such that $\int_{\mathbb{R}^d} J(z) \, \mathrm{d}z = 1$. For each $\varepsilon > 0$ let

$$J^\varepsilon(z) := \frac{1}{\varepsilon^n} J\left(\frac{z}{\varepsilon}\right).$$

The functions $J_\varepsilon \in C_c^\infty(\mathbb{R}^d)$ satisfy

$$\int_{\mathbb{R}^d} J^\varepsilon(z)\,\mathrm{d}z = 1,\ \operatorname{supp}(J^\varepsilon) \subseteq B(0,\varepsilon).$$

Definition A.15. *If $f : U \to \mathbb{R}$ is locally integrable, we define the mollification of f as*

$$f_\varepsilon(z) := (f * J^\varepsilon)(z) \quad \text{for } z \in U_\varepsilon.$$

In particular

$$f_\varepsilon(z) = \int_U J^\varepsilon(z-y)f(y)\,\mathrm{d}y = \int_{B(0,\varepsilon)} J^\varepsilon(y)f(z-y)\,\mathrm{d}y.$$

Theorem A.16 (Universal mollification). *Let $U \subseteq \mathbb{R}^d$ be open, $f \in L^p(U)$ for some $1 \le p < \infty$ and $h \in L^1(U)$ with $\int_U h\,\mathrm{d}x = 1$. Set $f_\varepsilon = h_\varepsilon * f$ with $h_\varepsilon(x) = \varepsilon^{-n}h(x/\varepsilon)$. Then*

$$f_\varepsilon \in L^p(\mathbb{R}^d) \quad \text{and} \quad \|f_\varepsilon\|_p \le \|h\|_1 \|f\|_p\,,$$
$$f_\varepsilon \to f \quad \text{strongly in } L^p(\mathbb{R}^d) \text{ as } \varepsilon \to 0.$$

If $h \in C_c^\infty(U)$, then $f_\varepsilon \in C^\infty(U_\varepsilon)$ and

$$D^\alpha f_\varepsilon = (D^\alpha h_\varepsilon) * f.$$

Remark A.17. *In particular for $h = J$ we obtain $f_\varepsilon \in C^\infty(U_\varepsilon)$*

$$f_\varepsilon \to f \quad \text{strongly in } L^p(U) \text{ as } \varepsilon \to 0$$
$$f_\varepsilon \to f \quad \mathbb{P}\text{-}a.s. \quad \text{on } U.$$

Definition A.18 (Cut-off function). *Fix a function $\zeta \in C_c^\infty(B(0,2))$ such that $\zeta = 1$ on $B(0,1)$ and $0 \le \zeta \le 1$. Then*

$$\zeta_n(z) := \zeta\left(\frac{z}{n}\right), \qquad n = 1, 2, \ldots$$

is called a sequence of cut-off functions for $\mathbb{R}^d$. In addition, for a multi-index γ we have the following bound

$$|\partial^\gamma \zeta_n| \le \|\partial^\gamma \zeta\|_{L^\infty(B(0,2))}\, n^{-|\gamma|} \mathbb{1}_{B(0,2n)\setminus B(0,n)},$$

where $|\gamma| := \gamma_1 + \ldots + \gamma_n$.

From the previous theorem we can guess that a function $u \in W^{1,\infty}(\mathbb{R}^d)$ is actually Lipschitz continuous. The next theorem shows us that even the converse is true.

Theorem A.19. *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Then u has a representative that is bounded and Lipschitz continuous if and only if $u \in W^{1,\infty}(\mathbb{R}^d)$.*

A.3. Gronwall's inequalities and fundamental lemma of variations

Lemma A.20 (Gronwall's inequality integral version). *Let I denote an interval on the real line of the form $[a,\infty), [a,b], [a,b)$ with $a < b$. Let u be a non-negative, absolutely continuous function on I, which satisfies for a.e. $t \in I$ the differential inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) \le c(t) + \lambda(t)u(t),$$

where c, λ are non-negative, summable functions on I. Then u satisfies

$$u(t) \leq \exp\left(\int_a^t \lambda(s)\,\mathrm{d}s\right)\left(u(0) + \int_a^t c(s)\,\mathrm{d}s\right)$$

for all $t \in I$. In particular, if α, c are non-negative constant we obtain

$$u(t) \leq e^{(t-a)\lambda}(u(a) + (t-a)c).$$

Lemma A.21 (Gronwall's inequality integral version). *Let I denote an interval on the real line of the form $[a, \infty), [a, b], [a, b)$ with $a < b$. Let u be a non-negative, measurable function such that for a.e. $t \in I$*

$$\int_a^t |u(s)|\,\mathrm{d}s < \infty$$

and

$$u(t) \leq c(t) + \int_a^t \lambda(s)u(s)\,\mathrm{d}s$$

for some measurable and bounded functions $c, \lambda : I \to \mathbb{R}$ with $\lambda \geq 0$. Then for a.e. $t \in I$

$$u(t) \leq c(t) + \int_a^t \exp(\Lambda(t) - \Lambda(s))\lambda(s)c(s)\,\mathrm{d}s$$

where $\Lambda(t) := \int_a^t \lambda(z)\,\mathrm{d}z$. In particular, if $c(t) = C$ and $\lambda(t) = A$ for some non-negative constants C, A, then

$$u(t) \leq C \exp(A(t-a)).$$

If in addition the function u is continuous, the statement holds for all $t \in I$.

Remark A.22. *We will mostly use Gronwall's lemma for the interval $[0, T]$ with $T > 0$ to show that u is bounded in some appropriate sense. The special case $C = 0$ and u non-negative provides us with $u = 0$ a.e.*

Lemma A.23 (Comparison principle). *Let $u : J \times \mathbb{R} \to \mathbb{R}$ be a continuous and locally Lipschitz in x, $J = [0, T]$, $\rho : J \to \mathbb{R}$ a continuous and almost everywhere differentiable function, which satisfies the differential equation $\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) \leq u(t, \rho(t)), t \in J$ with $\rho(0) \leq v_0$. Furthermore, let $v \in C^1(J, \mathbb{R})$ be the solution of the following differential equation*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}v(t) = u(t, v(t)), \\ v(0) = v_0, \end{cases}$$

which should exist on J. Then $\rho(t) \leq v(t)$ for almost all $t \in J$.

A.4. Weak convergence and Aubin lemma

We encounter the use of weak convergence techniques to solve non-linear partial differential equations. Therefore, we recall the most crucial results of weak convergence on normed spaces in this section.

DEFINITION A.24. *Let X be a normed space. We say a sequence $(x_n, n \in \mathbb{N})$ converges weakly to x, written*

$$x_n \rightharpoonup x,$$

if

$$x'(x_n) \to x'(x) \quad \forall x' \in X',$$

where X' denotes the dual space of X.

Since X' separates points, the weak limit is unique.

LEMMA A.25 (Mazur's lemma). *Let X be a normed space, $V \subset X$ a closed convex subset and $(x_n, n \in \mathbb{N})$ a weak convergent sequence in V with $x_n \rightharpoonup x$. Then $x \in V$.*

LEMMA A.26 (Mazur's lemma second version). *Assume that X is a normed space and that $(x_n, n \in \mathbb{N})$ converges weakly to x as $n \to \infty$. Then there exists a sequence of convex combinations $(\tilde{x_n}, n \in \mathbb{N})$ defined by*

$$\tilde{x}_n = \sum_{k=n}^{m_n} a_{n,k} x_n$$

with $a_{n,k} \geq 0$ and $\sum_{k=n}^{m_n} a_{n,k} = 1$ such that $\tilde{x}_n \to x$ in the norm of X as $n \to \infty$.

THEOREM A.27 (Weak compactness). *Let X be a reflexive Banach space and suppose the sequence $(x_n, n \in \mathbb{N})$ is bounded in X. Then there exists a subsequence $(x_{n_k}, k \in \mathbb{N})$ and $x \in X$ such that*

$$x_{n_k} \rightharpoonup x.$$

The next corollary is a special case of the above theorem for L^p-spaces. An interesting point is the fact that in contrast to previous theorem, which relies on the axiom of choice, we do not need the axiom of choice for the following corollary.

THEOREM A.28 (Banach-Alaoglu). *Let Ω be measure space, μ a σ-finite measure and $1 < p < \infty$. Then the following statements hold true.*

(i) The space $L^p(\Omega)$ is reflexive.
(ii) Let $(f_n, n \in \mathbb{N})$ be a bounded sequence in $L^p(\Omega)$, then there exists a f in $L^p(\Omega)$ such that

$$f_{n_k} \rightharpoonup f$$

for a subsequence $(f_{n_k}, k \in \mathbb{N})$.

(iii) *Let $(f_n, n \in \mathbb{N})$ be a sequence and $f_n \rightharpoonup f$ in $L^p(\Omega)$, then the L^p-norm is weak lower semicontinuous, i.e.*

$$\|f\|_{L^p(\Omega)} \leq \liminf_{n\to\infty} \|f_n\|_{L^p(\Omega)} .$$

This also holds true for $p = 1, \infty$.

Proof. Follows from Theorem A.27 and the characterization of the dual space of L^p. □

Lemma A.29. *Let $U \subset \mathbb{R}^d$ be open and $1 \leq p < \infty$. Suppose $(f_n, n \in \mathbb{N})$ is a sequence in $W^{1,p}(U)$ which satisfies*

(i) $f_n \rightharpoonup f$ as $n \to \infty$ *in $L^p(U)$,*
(ii) $\frac{\partial f_n}{\partial x_i} \rightharpoonup g_i$ as $n \to \infty$ *in $L^p(U)$ for $i = 1, \ldots, n$.*

Then $f \in W^{1,p}(U)$ and $\partial_i f = g_i$.

Proof. This is an easy consequence of the fact that $C_c^\infty(U) \subset L^p(U)$ for all $1 \leq p \leq \infty$ and the characterization of the dual space. □

Lemma A.30 (Products of weak and strong converging sequences). *Let $U \subset \mathbb{R}^d$ be open and $1 < p < \infty$. Let $(f_n, n \in \mathbb{N})$ be a sequence with $f_n \in L^p(U)$ and $f \in L^p(U)$. Moreover, let $(g_n, n \in \mathbb{N})$ be a sequence with $g_n \in L^q(U)$, $g \in L^q(U)$ and $1/p + 1/q = 1$. Suppose*

$$\begin{aligned} f_n &\rightharpoonup f \quad \text{in } L^p(U) \\ g_n &\to g \quad \text{in } L^q(U). \end{aligned}$$

Then

$$f_n g_n \rightharpoonup fg \quad \text{in } L^1(U).$$

Lemma A.31 (Aubin Lemma). *Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y), (Z, \|\cdot\|_Z)$ be Banach spaces, such that*

$$X \hookrightarrow Y \hookrightarrow Z,$$

where the first inclusion $X \hookrightarrow Y$ is compact. Let $1 \leq p < \infty$ and $(f_n, n \in \mathbb{N})$ a sequence of functions with

(i) $\|f_n\|_{L^p([0,T];X)} \leq C_1$
(ii) $\left\|\frac{\partial f_n}{\partial t}\right\|_{L^p([0,T];Z)} \leq C_2$

for some positive constants C_1, C_2. Then $(f_n, n \in \mathbb{N})$ is relatively compact in $L^p([0,T]; Y)$.

A.5. Stochastic inequalities and Fubini's theorem

Theorem A.32 (Burkholder–Davis–Gundy inequality). *Consider a continuous martingale M which, along with its quadratic variation process $\langle M\rangle$, is bounded. For every stopping time τ, we have*

$$\mathbb{E}\big(|M_\tau|^{2m}\big) \leq C_m \mathbb{E}\big(\langle M\rangle_\tau^m\big); \quad m > 0 \tag{A.1}$$

for a suitable positive constant C_m, which is universal (i.e., depend only on the number m, not on the martingale M nor the stopping time τ).

THEOREM A.33 (Itô isometry). *Let B be a Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(X_t, t \geq 0)$ and adapted to the natural filtration of B. Then*

$$\mathbb{E}\bigg(\Big(\int_0^T X_t \,\mathrm{d}B_t\Big)^2\bigg) = \mathbb{E}\bigg(\int_0^T X_t^2 \,\mathrm{d}t\bigg)$$

for all $T \geq 0$.

In the following we recall the stochastic Fubini theorem [HvS21, Lemma A.5] , which is a central key in connecting the stochastic Fokker–Planck equation with the conditional McKean–Vlasov equation.

DEFINITION A.34 (Immersion and compatibility). . *Let two filtrations $\mathcal{F}$ and $\mathcal{G}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ be such that $\mathcal{F} \subset \mathcal{G}$. Then $\mathcal{F}$ is said to be immersed in $\mathcal{G}$ under $\mathbb{P}$ if every square integrable $\mathcal{F}$ martingale is a $\mathcal{G}$ martingale. For two stochastic processes X and Y defined on this probability space, X is said to be compatible with Y if $\mathcal{F}^Y$ is immersed in $\mathcal{F}^{X,Y} := \mathcal{F}^X \vee \mathcal{F}^Y$ under $\mathbb{P}$, where $\mathcal{F}^X \vee \mathcal{F}^Y$ is the smallest σ-algebra containing both $\mathcal{F}^X, \mathcal{F}^Y$.*

THEOREM A.35 (Fubini-type theorem for conditional expectation and Itô integrals). *Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, three filtrations $\mathcal{F}^j = (\mathcal{F}_t^j)_{t \in I}$, $j = 1, 2, 3$ and three processes B, H, W satisfying the following conditions:*

(1) $\mathcal{F}^1 \subseteq \mathcal{F}^2 \subseteq \mathcal{F}^3$, i.e., $\forall t \in I$, $\mathcal{F}_t^1 \subseteq \mathcal{F}_t^2 \subseteq \mathcal{F}_t^3$.
(2) $\mathcal{F}^1$ is immersed in $\mathcal{F}^2$ under $\mathbb{P}$.
(3) H is a bounded $\mathcal{F}^2$-predictable process.
(4) W and B are $\mathcal{F}^3$ Brownian motions.
(5) W is $\mathcal{F}^1$-adapted.
(6) For any $s, t \in I$, $s \leq t$, $\sigma(B_r - B_s : s \leq r \leq t) \perp\!\!\!\perp (\mathcal{F}_t^1 \vee \mathcal{F}_s^2)$.

Then the following hold $\mathbb{P}$-a.s. for all $t \in I$:

$$E\left[\int_0^t H_s dB_s \middle| \mathcal{F}_t^1\right] = 0, \tag{A.2}$$

$$E\left[\int_0^t H_s dW_s \middle| \mathcal{F}_t^1\right] = \int_0^t E\left[H_s \middle| \mathcal{F}_s^1\right] dW_s. \tag{A.3}$$

A.6. Conditional relative entropy

We demonstrate that the relative entropy on laws can be estimated by the relative entropy on the conditional laws. Consequently, demonstrating convergence in relative entropy on some conditional laws is always a stronger statement, which justifies the use of the conditional Liouville equation (5.4).

LEMMA A.36. *Let $(E, \mathcal{E})$ be a Banach space and X, Y be two $(E, \mathcal{E})$-valued random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{G}^i$ for $i = 1, 2$ be sub σ-algebras of $\mathcal{F}$. Then, we have the following inequality*

$$\mathcal{H}(\mathscr{L}_X | \mathscr{L}_Y) \leq \mathbb{E}(\mathcal{H}(\mathscr{L}_{X|\mathcal{G}^1} | \mathscr{L}_{Y|\mathcal{G}^2})).$$

Choosing $\mathcal{G}^1$ or $\mathcal{G}^2$ as the trivial σ-algebra we obtain

$$\mathcal{H}(\mathscr{L}_X|\mathscr{L}_Y) \leq \min\Big(\mathbb{E}(\mathcal{H}(\mathscr{L}_{X|\mathcal{G}^1}|\mathscr{L}_Y)), \mathbb{E}(\mathcal{H}(\mathscr{L}_X|\mathscr{L}_{Y|\mathcal{G}^2}))\Big).$$

Proof. Let $\mathcal{B}_b(E)$ denote the set of bounded measurable functions. Using the variational formula and Jensen inequality we obtain

$$\begin{aligned}
\mathcal{H}(\mathscr{L}_X|\mathscr{L}_Y) &= \sup_{\psi\in\mathcal{B}_b(E)} \Big(\mathbb{E}(\psi(X)) - \log\big(\mathbb{E}(\exp(\psi(Y)))\big)\Big) \\
&= \sup_{\psi\in\mathcal{B}_b(E)} \Big(\mathbb{E}(\psi(X)) - \log\big(\mathbb{E}(\mathbb{E}(\exp(\psi(Y))|\mathcal{G}^2))\big)\Big) \\
&\leq \sup_{\psi\in\mathcal{B}_b(E)} \Big(\mathbb{E}(\mathbb{E}(\psi(X)|\mathcal{G}^1)) - \mathbb{E}(\log(\mathbb{E}(\exp(\psi(Y))|\mathcal{G}^2)))\Big) \\
&= \sup_{\psi\in\mathcal{B}_b(E)} \Big(\mathbb{E}\Big(\mathbb{E}(\psi(X)|\mathcal{G}^1) - \log\big(\mathbb{E}(\exp(\psi(Y))|\mathcal{G}^2)\big)\Big)\Big) \\
&\leq \mathbb{E}\Big(\sup_{\psi\in\mathcal{B}_b(E)} \Big(\mathbb{E}(\psi(X)|\mathcal{G}^1) - \log\big(\mathbb{E}(\exp(\psi(Y))|\mathcal{G}^2)\big)\Big)\Big) \\
&= \mathbb{E}(\mathcal{H}(\mathscr{L}_{X|\mathcal{G}^1})|\mathscr{L}_{Y|\mathcal{G}^2})).
\end{aligned}$$

□

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