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ORIGINAL PAPER

Optimal testing and social distancing of individuals with private health signals

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Abstract
In an epidemic, the regulation of social distancing and testing is critical for the large group of individuals who are possibly infected, but have not developed clear, distinct symptoms. Each individual's reaction to a regulation scheme depends on its private probability assessment of being infected. Assuming no monetary transfers, we identify a simple class of schemes for welfare maximization: all individuals who ask for a test are tested with the same probability, independently of their infection probabilities, and the social distancing regulation depends on who asks for a test. Social distancing has a double role: to provide incentives so that the right people get tested, and to curb the spread of the disease. If testing capacities are scarce it can be optimal to test only a randomly selected fraction of those who ask for a test, and require maximal social distancing precisely for those individuals who ask unsuccessfully. If public costs and benefits are small, laissez faire is optimal.

Keywords Infectuous disease · Optimal test allocation · Transfer-free mechanisms · Private health signals

JEL Classification I12 · I18 · D62 · D82

1 Introduction

Social distancing and testing for an infection are the two main tools for curbing the spread of a virus when no vaccination is available. A ready example is the early phase of the Covid-19 pandemic: social-distancing regulation, ranging from the obligation to a wear a mask in public to temporary apartment confinement or a prohibition to

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work, had a strong impact on the lifes of most people in most countries; in addition, a negative infection test result became a precondition for most social activities.

Optimal regulation is critical for the large group of those individuals who, at a given point in time, are possibly infected, but have not (yet) developed clear, distinct symptoms. Each such individual is *privately* informed about their current likelihood of being infected, due to private knowledge of recent social contacts, and possibly due to early symptoms. We investigate how a regulator can and should utilize the private probability assessment of each individual, which we call the person's health signal, type, or belief.[1]

Social distancing and testing are intertwined. The regulator's goal is to foster the treatment of infected individuals and reduce the spread of the disease while minimizing the disruptions that are caused by social distancing. Testing affects the extent to which fast treatment and quarantining of infected people is possible. How can the regulation scheme assure that the right people get tested, and how should the social distancing of untested people be regulated? We investigate these questions.

In line with survey evidence,[2] we assume that an individual's personal value for getting tested is increasing in her type (with small types having a negative value). Our analysis begins with the simple insight that, given that testing is costly and is a hassle, and only infected people can spread the disease, only those individuals who are sufficiently likely to be infected should be tested.[3] However, the marginally tested type in the social optimum will generally have a non-zero value for getting tested. Thus, the regulation scheme uses social distancing to provide incentives for individuals to reveal their health types.

Social distancing is unpleasant for any individual, but putting an infected person in quarantine (i.e., maximal social distancing) has the positive effect of preventing this person from spreading the disease. Thus, any person should be put into quarantine if and only if it is sufficiently likely to be infected. We assume that a tested individual is put in quarantine if the test is positive, and no social distancing is imposed if the test is negative.

If the government can use money to incentivize people, then—as we show—the social optimum can be implemented in a straightforward way. There is a price for a test. The price may be positive or negative (i.e., a subsidy), depending on the circumstances. If the price is set optimally, it induces exactly the types above the socially optimal threshold to get tested. In addition, agents may be offered a monetary compensation if they choose to go into quarantine without being tested first. If set right,

[1] We are not assuming that individuals consciously do calculations with probabilities. Nevertheless, thinking in terms of degrees of likelihood is common sense and is relevant to many aspects of life beginning with the weather forecast. While psychological research has identified many biases in decision-making under uncertainty (e.g., Gigerenzer (2008), Kahneman (2011)), it is fair to say that the individuals in our model face quite simple decision problems, given the testing-and-social-distancing schemes that we propose.

[2] Fallucchi et al. (2021) observe that individuals' willingness to get tested is positively correlated with concerns about contracting Covid-19.

[3] If done so, the rate of infections among the tested individuals will be higher than the base rate in the population. We assume the regulator knows the base rate (and, in fact, the distribution of beliefs). See Stock et al. (2020) on how to identify the base rate from a biased testing sample in the Covid-19 pandemic, using data from Iceland.

this compensation induces the socially optimally range of types to go into quarantine right away without being tested.

Monetary incentives are problematic for two reasons. First, it is well known that using government funds to pay subsidies or collect fees can have shadow costs. Second, our simple model misses the fact that, in reality, the incentive effects of money correlate with the individual's wealth. This correlation can garble the regulation scheme: two individuals of the same health type, but with different wealth levels, will generally react differently to the same monetary incentives. Both reasons probably contributed to the weak role of regulated monetary incentives for testing and social distancing in the Covid-19 pandemic.

We investigate the optimal incentive-compatible regulation of testing and social distancing *without monetary incentives* in a one-period model. We find that the threat of social distancing can replace incentives via monetary transfers to some extent. Thus, we identify a double role of social distancing: to provide incentives to get tested, and to curb the spread of the disease.

In an optimal regulation scheme without monetary incentives there exists a threshold type $\check{p}$ such that an individual will not be tested if its health signal is below $\check{p}$, and all individuals with health signals above will be tested with the same probability. Given that any *tested* individual is quarantined if and only if the test is positive, the social distancing of *untested* individuals is regulated such that an intermediate level of distancing between complete freedom and quarantine is stipulated.

Note the qualitatively new features that are absent from the optimal regulation scheme with monetary incentives. First, the government optimally collects only *binary* information from the individuals, essentially asking only who would like to get tested, whereas with monetary transfers it can be optimal to further divide the group of untested individuals into quarantined and non-quarantined individuals. Second, randomized testing can be optimal if capacities are scarce, whereas with monetary transfers the government would simply increase the price of a test. Third, intermediate levels of required social distancing may be used as a quasi-money that induces the right individuals to get tested, whereas with monetary transfers the government optimally requires either maximal or minimal social distancing for each individual. In addition to these qualitative differences, the range of optimally tested types will generally not be the same as with monetary incentives.

To outline the structure of the optimal regulation scheme, we distinguish four cases. The first case is the simplest: in extreme circumstances it can be optimal to quarantine everybody and test nobody. The other three cases can be described in terms of the threshold type $\check{p}$ that the government would like to implement.

One possibility for the structure of the optimal regulation scheme is that $\check{p}$ equals the type of an individual with the value zero for being tested. We call this the laissez-faire solution because no social distancing is required for those individuals who decide not to get tested, and those who want to get tested do get tested. This scheme is optimal if the benefits of quarantining are small, while testing and quarantining are rather costly for the government. The possible optimality of laissez faire constitutes another qualitative feature that is absent from the optimal regulation with monetary transfers (where such optimality is a non-generic case).

The next possibility is that test capacities are relatively abundant, so that the government would like to test many individuals. In this case, $\check{p}$ will be so low that an individual of type $\check{p}$ will have a strictly negative valuation for a test, and all individuals with types above $\check{p}$ will be tested for sure. The trick to make individuals with all types down to $\check{p}$ reveal themselves so they can be tested, is to require some intermediate level of social distancing for any individual who decides to remain untested. This lowers the payoff from not getting tested so that, if the level of social distancing is chosen right, individuals of type $\check{p}$ become indifferent between being tested and not being tested. Incentive compatibility is then satisfied. This form of the testing-and-social-distancing schedule explains why it can be optimal to require some social distancing even for those individuals who are quite sure to be uninfected. A ready example is the common regulation during the Covid crisis that requires a negative test for certain social activities such as visiting a restaurant.

The remaining, last, possibility is that test capacities are scarce. In this case, the government must carefully select who to test. Here, the marginally tested type $\check{p}$ is so high that an individual of type $\check{p}$ has a strictly positive valuation for a test. How are individuals with a strictly positive valuation and a type below $\check{p}$ prevented from snatching a test by claiming a type above? The solution is to introduce probabilistic testing. Only a randomly selected fraction of the individuals who claim to have types above $\check{p}$ are tested. For any individual who claims a type above $\check{p}$, if the randomization implies that this individual does not belong to the tested fraction, maximal social distancing is required. Each individual now faces a gamble if she claims a type above $\check{p}$: on the one hand, this allows her to grab a test with some probability, but, on the other hand, it sends her in quarantine for sure if (through the randomization) she ends up not getting tested. Higher types are more willing than lower types to take such a gamble because for them the test is more valuable, while the hassle of being put in quarantine for those who do not get a test is type-independent.[4]

1.1 Literature

There is a huge literature on the dynamics of infectious diseases.[5] Our model has only one period. This should be interpreted as a snapshot view at a particular point in an epidemic, and can be a building block for a dynamic model in which the government would continuously adapt its regulation scheme as costs and benefits (which are exogeneous in our model) evolve dynamically and endogeneously.

Behavioral aspects have been taken into account only recently in the literature on epidemology (see the surveys by Klein et al. (2007), Avery et al. (2020), and McAdams (2021)). The behavioral aspect that is modelled most frequently is that each individual chooses their degree of social distancing. Many papers focus on calibration results. See, e.g., Kremer (1996) concerning the HIV/AIDS epidemic, and

[4] As for a concrete application example, imagine this schedule to be used for the group of individuals who arrive at an airport on a given day if tests are too scarce to test everybody who wants a test. A random selection of those who state a willingness to be tested get a test, while the others are put in quarantine.

[5] See von Thadden (2020) for an adaptation to the epidemiological specifics of the Covid-19 pandemic. See Ellison (2024) for variants that allow for heterogeneity in contact rates.

Fenichel et al. (2011), Eichenbaum et al. (2021), Droste and Stock (2021) concerning Covid-19. Given the negative externalities of social contacts in an epidemic, a social planner's role to enforce social distancing arises. Farboodi et al. (2021) have estimated welfare gains from optimal social-distancing restrictions in the Covid-19 pandemic. Jones et al. (2020) introduce congestion in the health-care system; they show that in a laissez faire equilibrium agents have rather weak social distancing incentives because they expect to get infected eventually no matter what, implying that a social planner who can enforce social distancing saves many lifes. Makris (2021) shows the empirical importance of modeling social-distancing preferences that are heterogenous across the population if the regulator imposes a minimum social-distancing restriction. Relatedly, Brotherhood et al. (2020) assume that social-distancing preferences are age-dependent and so should be their regulation.

Complementing these calibration results, there are some analytical contributions. Kruse and Strack (2020) present results on the optimal timing of social-distancing restrictions; McAdams et al. (2023) emphasize the importance of modelling social contacts as strategic complements; Carnehl et al. (2023) distinguish between the affects of changes in the transmission rate and changes in the contact rate.

The impact of testing has also been studied. Droste et al. (2024) show that widespread screening tests have huge economic benefits, especially if supported by confirmatory tests. Eichenbaum et al. (2022) compare societies with different levels of aversion to social distancing in a world where testing capacities arise gradually. Berger et al. (2022) emphasize the positive role of targeted quarantining through frequent testing of asymptomatic individuals.

Virtually all these models augment the basic distinction between susceptible, infected, and recovered individuals,[6] but do not analyze the role of private information. We consider the sub-population of susceptible and possibly infected individuals and consider incentive constraints with respect to the private probability assessment of carrying an infection.[7]

An individual's believed probability of being infected is behaviorally relevant. Gong (2015) present evidence for the HIV/AIDS epidemic, showing that individuals who are surprised by a positive (negative) HIV test tend to increase (decrease) their sexual activity. This suggests that, in the context of HIV, an individual's social-distancing preferences are *decreasing* in the belief of being infected. Paula et al. (2014), however, in their study of HIV in Malawi, present evidence that points in the opposite relation. Similarly, Brotherhood et al. (2020) for their calibration to the Covid pandemic, rely on social-distancing preferences that are *increasing* in the belief of being infected: time at home is valued more by an infected than a healthy individual, and in between by an uncertain ("fevery") individual.

In contrast to the above literature, our analysis focusses on a group of individuals who, independently of their private health signals, have the same social-distancing

[6] A mathematically different approach is Acemoglu et al. (2020) who study the spread of an infection on a network of agents who choose their social activities depending on their preferences. More testing can lead to more social activity and more infections.

[7] Chari et al. (2021) model heterogeneity that arises from *public* information that is obtained via a contact-tracing technology. This allows the government to condition their social-distancing regulation on these beliefs without having to satisfy incentive constraints.

preferences. In essence, we are assuming that a high-risk type's smaller expected cost of becoming infected through social contacts is balanced by her altruistic concern of possibly infecting others. The assumption simplifies our analysis because it means that the regulation of social contacts serves as a quasi-money that has the same value for everybody. With heterogenous social-contact preferences, the value of this quasi-money would be type-dependent, and the valuation for a test would also change because quarantine applies to those who are tested positive.[8]

Piguillem and Shi (2020) analyze whether social distancing and testing are complements or substitutes, and they argue for the latter. This result, however, relies on the assumption that the government forces a random (and, in their calibration to the Italian Covid-19 outbreak, sometimes large) fraction of the population to get tested frequently. No private health signals are considered. That is, Piguillem and Shi (2020) propose to randomly test everybody and to forego any social-distancing restrictions. In our model, higher welfare is obtained when only the individuals with high private signals of an infection get tested, although this needs to be incentivized via social-distancing regulations. In this sense, in our model social distancing and testing are complements.

Deb et al. (2022) consider, like we, a model in which agents make choices based on their private health signals. However, they assume that social distancing and testing can only be regulated indirectly, through monetary incentives. No welfare improvement over the laissez faire is possible without monetary incentives. Our model is complementary to their's because we show how by regulating social distancing directly the government can achieve welfare gains even without monetary transfers. Moreover, we identify major qualitative features of an optimal regulation scheme that are different from their world. First, our regulator may require an *intermediate* level of social distancing whereas in Deb et al. (2022) agents exercise either minimal or maximal distancing. Second, in our world it is always optimal to test only agents with sufficiently high infection risks, whereas in their's at-work testing can be optimal, which implies that only agents with sufficiently *low* infection risks are tested.

Ely et al. (2021) study the problem of optimally allocating scarce imperfect tests of different sensitivities to individuals with heterogenous infection risks. They assume that the regulator observes each individual's infection risk, but remark that "incentive compatibility may be a substantial part of the practical test-allocation problem", a concern that our model takes up.

Chen (2006) considers the welfare effects of vaccination, a tool which we assume is not available. Each individual chooses whether or not to get vaccinated, which incurs a personal cost. Due to the incentive effects of a vaccination, its overall welfare effect can be ambiguous.

Caplin and Eliaz (2003), in a static model, combine individual choices of being tested and contact choices that are conditional on a certificate of the test result. Fear of a positive test result is introduced as a psychological bias, and the optimal certification policy of the government is determined.

[8] If the type-dependence is moderate, then it will still be possible to incentivize high types to get tested by assigning enough social distancing to untested types, and the regulation schemes that we find should come close to optimality.

On a technical level, finding the optimal regulation scheme in our model is a mechanism-design problem in which an individual's required degree of social distancing acts as a quasi-money that steers every individual to reveal her type. From the individual's point of view, getting tested is like receiving a good that may have a positive or negative value for the individual. The technical challenge of solving the government's problem mainly arises from the fact that the probability of becoming quarantined is restricted between 0 and 1, thus restricting the amount of quasi-money that can be paid by any individual.[9]

The rest of the paper is structured as follows. In Sect. 2 we define the government's regulation problem. Section 3 presents all of our results. Section 4 is devoted to the detailed arguments behind our solution to the government's problem. After concluding in Sect. 5 we present the remaining proofs in Sect. 6.

2 Model

We consider an individual who is uncertain about whether or not she is infected by a given disease. At time 0, the individual possesses a private signal, her type $p \in [\underline{p}, \overline{p}]$, that describes the individual's personal probability assessment that she is infected, given her current symptoms and recent contacts with other people, where $0 \leq \underline{p} \leq \overline{p} < 1$.[10] (At a point in time after time 0, the individual may develop clear, distinct symptoms, but our analysis focusses on time 0.) Although our model considers a single individual, it is instructive to imagine a population of individuals with private health signals out of which the considered individual is a representative member.

We assume that, across the population, individuals' probability assessements are not systematically wrong, that is, for all p, among all individuals who think that they are infected with probability p, the expected fraction p is in fact infected. Let F denote the c.d.f. for the distribution of types p in the population of individuals. We assume that F has a density f that is strictly positive on the open interval $(\underline{p}, \overline{p})$.

At time 0, the individual may be tested for the illness. For an individual of type p, the *expected value of being tested* is

$$v(p) = pb - c^t, \tag{1}$$

where $b > 0$ and $c^t > 0$ are given parameters. The parameters are easiest to interpret if the test is perfect. Then c^t denotes the individual's (hassle) cost of having the test

[9] Again on a technical level, our setup may be seen as a case of mechanism design with costly state-verification (see Ben-Porath et al. (2014)). In this literature, a designer commits to verifying states and implementing outcomes conditional on agents' reports when agents have private information related to these states. In our setting, the government is able to verify an individual's health state by testing for the infection, but she is not able to verify the agent's type. The verification of the health state carries a cost not only to the government, but is also costly to the individual.

[10] The assumption $\overline{p} < 1$ is not used in our formal analysis, but it is needed for the consistent interpretation of our model. We do not dispute the existence of individuals who believe to be infected with a probability above $\overline{p}$, but they are left out of the model. The implicit assumption is that such people have clear, distinct symptoms so that their health signals are not private, but public and verifiable. Thus, they can be regulated separately from those with private signals. Also, their preferences may be different because they may prefer to stay in bed or go to a hospital.

done, and b is the benefit of knowing about one's infection: the individual can adapt its plans such as cancelling a trip, and it can unlock a fast treatment if the medical provider has a rule to only accept patients with confirmed infections. If the test is not perfect, the interpretations of c^t and b change because these numbers then include any expected hassle costs from false negatives or false positives such that the value $v(p)$ is reduced, but the qualitative features of the function v are unchanged if the test imperfections are small.

We assume that there is enough heterogeneity in the population so that a conflict exists between those who, in the absence of any other incentives, would like to get tested and those who do not want to get tested, that is, $v(\underline{p}) < 0 < v(\overline{p})$. Let $p^* \in (\underline{p}, \overline{p})$ denote the indifferent type, that is, $v(p^*) = 0$, or

$$p^* = \frac{c^t}{b}. \tag{2}$$

The individual's payoff also depends on its social distancing (starting at time 0). We consider degrees of social distancing from 0 and 1. We use the term quarantine to indicate the maximal social-distancing level, 1. The social-distancing level of 0 will be identified with the individual's voluntary level of social contacts, which we assume is *independent* of her type. Considering any social contact, an altruistic person would be concerned about the probability of infecting others, which is *increasing* in her type, while any individual will be concerned about the probability of getting infected, which is *decreasing* in her type. For the sake of analytical tractability, we assume that these two motives cancel out across the range of types $[\underline{p}, \overline{p}]$.

For the purposes of the model, any social-distancing level may be thought of as a probability of being put into quarantine. We assume that any positively tested individual is put into quarantine while any negatively tested individual is set free, and this is taken into account in the definition of the testing value $v(p)$.[11] For any individual that is not positively tested, the cost of being in quarantine is denoted $c^q > 0$. We assume that being quarantined is more unpleasant than being tested,

$$c^q > c^t. \tag{3}$$

The government's goal is to set up a rule for determining who gets tested and what level of social distancing will be required for untested individuals.

Even before introducing the government's welfare function, it is pretty clear that the optimal testing-and-social-distancing rule will, in general, be type-dependent. For all p, let

$$0 \leq m(p) \leq 1 \tag{4}$$

denote the probability that an individual of type p is tested and let

$$0 \leq q(p) \leq 1 \tag{5}$$

[11] If we allowed some social distancing for some types of negatively tested individuals, or a reduced social-distancing level for some types of positively tested individuals, the testing-value function would turn into a function that depends both on the actual type and the announced type, thus changing the incentive constraints for all types of individuals. We leave this problem to future research.

denote the required degree of social distancing for the individual conditional on the event that the individual does not get tested. The pair of functions (m, q) defines the government's rule or (direct) mechanism.

The main difficulty for the government is that, given any individual's personal value of getting tested and cost of getting quarantined, individuals may lie about their personal health signal. Due to the revelation principle, there is no loss of generality in restricting attention to mechanisms (m, q) that are *incentive compatible*, that is, direct mechanisms in which no individual can gain from making a false claim about her type. In order to spell out this condition, let

$$U(\hat{p}, p) = v(p)m(\hat{p}) - c^q(1 - m(\hat{p}))q(\hat{p}) \tag{6}$$

denote the expected utility of an individual of any type p who pretends to be of some type $\hat{p}$. The incentive-compability condition requires that

$$U(p, p) \geq U(\hat{p}, p) \quad \text{for all} \quad \hat{p} \text{ and} p. \tag{7}$$

Our model does not attempt to capture the dynamic aspects of the spread of the disease. Rather, we are interested in the problem of which mechanism is optimal at the given current point in time (implicitly assuming that the government can adapt its rule over time). Thus, we take it as given that the goverment is concerned about two things: first, the current expected utility of an (average) individual, which should be kept high; second, the probability that any given individual spreads the disease, which should be kept low.

An individual can spread the disease if and only if it is infected and is not quarantined. Let $b^q > 0$ denote the social benefit of quarantining an infected individual. Let y denote the probability that the test result is positive if the individual is infected. We assume that the probability of a false negative is small in the sense that

$$1 - y < \frac{b}{b^q}. \tag{8}$$

The *expected quarantining benefit* that is achieved by the goverment's rule with respect to type p is equal to

$$(b^q y)m(p)p + b^q(1 - m(p))pq(p).$$

This is because, in case the individual is tested (probability $m(p)$), the benefit b^q occurs if and only if the individual is infected and tested positive (probability yp); if, however, the individual is not tested (probability $1 - m(p)$), then being infected (probability p) and getting quarantined (probability $q(p)$) are stochastically independent events, so that the benefit b^q only occurs with probability $pq(p)$.

The government's welfare objective is given by

$$W = E_{p\sim F}\Big[U(p, p) + (b^q y)m(p)p + b^q(1 - m(p))pq(p)$$
$$w^1 U(p, p) - c^{gt}m(p) - c^{gq}(1 - m(p))q(p)\Big], \tag{9}$$

where $c^{gt} > 0$ denotes the government's cost of performing a test, and $c^{gq} \geq 0$ denotes the government's surveillance cost of enforcing the quarantine of an untested person.[12] Interpreting c^{gt} as an opportunity cost, we can view c^{gt} as a measure of the current scarcity of test medication units or test facilities, that is, the higher c^{gt} the higher is the cost of using a test unit for any particular individual. In this view, c^{gt} is the government's value of saving a test unit for a different point in time or of using it for an individual outside the considered population of individuals. The cost c^{gq} can be interpreted as a measure of the availability of surveillance and enforcement infrastructure.

By scaling the parameters b, c^q, and c^t, the utility U can be scaled arbitrarily. This scale defines the relative weight of the individual's utility in the welfare function. In our static model all parameters are exogeneous.[13]

The government's goal is to solve the following (second-best) welfare-maximization problem:

$$\max_{m(\cdot),q(\cdot)} W \quad \text{s.t. } (4), \quad (5), \quad (7),$$

where the expected utilities that occur in (7) are computed via (6).

2.1 What constitutes a population?

For a regulator, it is important to define what constitutes the population of individuals to which a regulation scheme $(m(\cdot), q(\cdot))$ is applied. Generally, the government may achieve welfare gains by conditioning its regulation scheme on any verifiable trait that allows for statistical discrimination. For example, the quarantining benefit b^q can depend on a person's profession, where, say, a school teacher may have a higher b^q than a lighthouse keeper, suggesting welfare gains if individuals of these two professions are assigned to different populations. Another important discrimination possibility would be based on age. A young person without preexisting illness will typically have a lower testing benefit b than an old person, so optimal regulation will generally depend on age. Similarly, the regulation of teenagers should be based on their—likely large—quarantining cost c^q. Alternatively, a group of people may also be characterized by a particular type distribution F, such as the group of individuals who arrive in an airplane from a particular country with a known high infection rate,

[12] For simplicity, we assume that there is no cost of enforcing the quarantine of a positively tested person. While such a cost could be easily incorporated into our model, it is reasonable to assume that the cost of enforcing the quarantine of a positively tested person is much smaller than of an untested individual that lacks clear, distinct symptoms and can hide its infection risk. A person who is tested positive may prefer (possibly due to moral obligation or legal concerns) to stay at home or go to a medical facility, which will enforce the quarantine without incurring significant extra costs beyond the cost of caring for and treating the individual.

[13] In a dynamic model, because the fraction of infected individuals in the population becomes a variable, the definition of an agent's current expected utility must be altered such that she obtains a benefit from being healthy. The testing-and-social-distancing rule would be adapted dynamically. The relative weight on the individual utility in the welfare function would depend on the impact of the current spread of the disease on the discounted expected utility of forward-looking agents.

or the group of people who live in a particular geographic region (see Allcott et al. (2024) on the empirical importance of geographic variation).

3 Results

We first solve the government's problem without informational frictions (Sect. 3.1), then provide the solution structure to the actual regulation problem (Sect. 3.2) and illustrate the solution via numerical examples (Sect. 3.3). Finally, we fully characterize optimal regulation schemes and provide comparative-statics results (Sect. 3.4).

3.1 The first-best social optimum and monetary transfers

As a benchmark, we first describe the rule the government would implement if it could either use monetary incentives, or could directly observe the individual's type and thus could set up any rule without relying on the individual's type report. Such a social planner can regulate each type separately and solves the following first-best problem:

$$\max_{m(\cdot),q(\cdot)} W \quad \text{s.t. } (4), \quad (5).$$

To state the solution, we define three threshold types. Let $p = \underline{p}^q$ denote the type of an individual for whom the social benefit of quarantining without a test, pb^q, equals the total (i.e., individual and social) cost of quarantining, $c^q + c^{gq}$. That is,

$$\underline{p}^q = \frac{c^q + c^{gq}}{b^q}.$$

Next, let $p = \underline{p}^t$ denote the type such that the total (i.e., individual and social) benefit of testing an otherwise unquarantined individual of type p, $p(b + b^q y)$, equals the total cost of testing, $c^t + c^{gt}$. That is,

$$\underline{p}^t = \frac{c^t + c^{gt}}{b + b^q y}.$$

Lastly, let $p = \underline{p}^{qt}$ denote the type of an individual for whom the individual benefit of testing, pb, equals the total cost differential between testing and quarantining. That is,

$$\underline{p}^{qt} = \frac{c^{gt} + c^t - c^{gq} - c^q}{b}.$$

The optimal scheme partitions the type space in up to three intervals.

Proposition 1 *A first-best testing-and-quarantining schedule is given as follows.*

If $\underline{p}^q \geq \underline{p}^t$, then an individual is tested if and only if her type is at least $\underline{p}^t$; no social distancing is required for untested individuals.

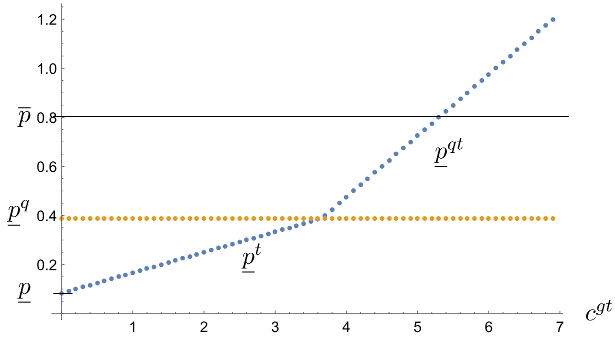

Fig. 1 Example of a government's first-best optimal rule as a function of the cost of a test unit, c^{gt}. The blue curve given by the value of $\underline{p}^t$ or, resp., $\underline{p}^{qt}$, indicates the marginally tested type. The orange line indicates the marginally quarantined type. For this diagram, it is assumed F is the uniform distribution on the interval $[\underline{p}, \overline{p}] = [0.1, 0.8]$, $y = 1$, $b = 4$, $c^q = 3$, $c^t = 1$, $b^q = 8$, $c^{gq} = 0.1$. The computations were performed using Mathematica 12

If $\underline{p}^q < \underline{p}^t$, then an individual is tested if and only if her type is at least $\underline{p}^{qt}$; there is a nonempty interval of types—the types in $[\underline{p}^q, \underline{p}^{qt})$—such that an individual with such a type is quarantined right away, without being tested; no social distancing is required for individuals with types in $[\underline{p}, \underline{p}^q)$.

The proof is straightforward and is relegated to the "Appendix".

We illustrate Proposition 1 via a discussion of the comparative statics with respect to the cost of testing, c^{gt}. If test capacities are abundant (i.e., $c^{gt} \approx 0$), then the first "If" case applies because

$$\underline{p}^t \approx \frac{c^t}{b + b^q} \overset{(3)}{<} \frac{c^q}{b + b^q y} \overset{(8)}{<} \frac{c^q + c^{gq}}{b^q} = \underline{p}^q.$$

All individuals with types above $\underline{p}^t$ are tested, and no social distancing is required for untested types. Note that $\underline{p}^t < \underline{p}$ if $\underline{p} > 0$ and b^q is sufficiently large. That is, unless some individuals are almost certain to be healthy (i.e., $\underline{p} = 0$), all individuals will be tested if the public benefit b^q is sufficiently large.

As c^{gt} increases, the marginally tested type $\underline{p}^t$ increases, so that fewer and fewer individuals are tested. There exists c^{gt} such that $\underline{p}^t = \underline{p}^q$. If test capacities become even scarcer, the second "If" case in Proposition 1 applies. The set of tested types shrinks ever more as c^{gt} increases further, but the quarantining threshold $\underline{p}^q$ remains constant. At some point, test capacity is so scarce that $\underline{p}^{qt} \geq \overline{p}$. Then nobody is tested anymore, but quarantining of individuals with types in the interval $[\underline{p}^q, \overline{p}]$ persists; depending on the parameters, it can be optimal to quarantine nobody (if $\underline{p}^q > \overline{p}$) or everybody (if $\underline{p}^q \leq \underline{p}$).

Figure 1 provides an illustration. It shows an example of the marginally tested type and the marginally untested quarantined type as functions of the testing cost c^{gt}, keeping the other parameters fixed.

A simple, but very important, observation is that, with the exception of extreme cases or non-generic cases, the first-best solution is not incentive compatible. Generically, the marginally tested type, $\underline{p}^t$ or $\underline{p}^{qt}$, will be different from the type p^* who has a test value of 0. Moreover, if the solution requires that some, but not all, untested individuals are quarantined then these would not reveal their types. Thus, in order to achieve its welfare goal, the government must take the individual's incentive compatibility constraints into account.

The first best can be implemented if incentives can be modulated via monetary transfers. To achieve this, the government sets a (positive or negative) price z^t for a test. In addition, a compensation z^q for individuals who decide to go into quarantine without testing may be set. Specifically, if $\underline{p}^q \geq \underline{p}^t$, then by setting $z^t = v(\underline{p}^t)$, an individual of type $\underline{p}^t$ becomes indifferent between testing and not testing. Because in this case no untested individual is to be quarantined, no further compensation is needed, that is $z^q = 0$.

Suppose now $\underline{p}^q < \underline{p}^t$. Then the price $z^t = v(\underline{p}^t) + c^q$ makes an individual of type $\underline{p}^t$ indifferent between testing and not testing. In addition, by offering the compensation $z^q = c^q$ for any individual who decides to go into quarantine, all untested individuals become indifferent, so that the first best is again incentive compatible.

Actual government regulation of epidemics typically does not rely much on explicit monetary incentives. Accordingly, in the rest of the paper, we focus on optimal regulation without monetary incentives. This will lead to rather different insights concerning what the government should do.

3.2 The structure of optimal regulation schemes

Next we present our core result; it resolves the structure of optimal regulation schemes. The result reveals four different categories of solutions: *no-testing-always-quarantining*, *laissez faire*, setting up *testing incentives*, and *testing disincentives*. The latter three categories correspond to ranges of a one-dimensional parameter, the *testing threshold* $\check{p}$.

Proposition 2 *It is either optimal for the government to test nobody and quarantine everybody, or the government's problem has a solution (m^*, q^*) that takes the following form. There exists $\check{p} \in [\underline{p}, \overline{p}]$ such that, for all types p, the optimal testing schedule is*

$$m^*(p) = \begin{cases} 0 & \text{if } p < \check{p}, \\ \check{m} & \text{if } p \geq \check{p}, \end{cases}$$

where

$$\check{m} = \frac{c^q}{c^q + \max\{v(\check{p}), 0\}}. \tag{10}$$

Moreover, we distinguish the following cases.

"Laissez faire" If $\check{p} = p^*$, then $\check{m} = 1$ and the optimal quarantining probability for untested individuals is

$$q^*(p) = 0 \quad \textit{for all } p < \check{p}.$$

"Testing incentives" If $\check{p} < p^*$, then $\check{m} = 1$ and the optimal quarantining probability for untested individuals is

$$q^*(p) = \frac{-v(\check{p})}{c^q} > 0 \quad \textit{for all } p < \check{p}.$$

"Testing disincentives" If $\check{p} > p^*$, then $\check{m} < 1$ and, for all p, the optimal quarantining probability for untested individuals is

$$q^*(p) = \begin{cases} 0 & \textit{if } p < \check{p}, \\ 1 & \textit{if } p \geq \check{p}. \end{cases}$$

No-testing-always-quarantining should be interpreted here as saving any available testing capacity for a different population of individuals or for use at another point in time.

The central insight underlying the other three solution categories is that the optimal scheme only makes use of a binary information. In essence, it asks whether the individual believes its risk is small (type below $\check{p}$) or large (type above $\check{p}$). The government could implement schemes such that the testing probability is strictly increasing across a range of types, but such complicated schemes would not bring any improvements. Rather, all individuals with risk types below the threshold $\check{p}$ are not tested, and all other individuals are tested with a fixed probability $\check{m}$. The binary-information property is analogous to an optimal auction with a single agent (Myerson 1981). However, we can have $\check{m} < 1$ whereas in an auction it is never optimal for the seller to sell a fraction of the good.

Laissez faire means that the individual behavior remains unregulated. All types with positive values for getting tested (i.e., types above $\check{p} = p^*$) are tested for sure (i.e., $\check{m} = 1$), and all other types are not tested. No social distancing of untested individuals is required (i.e., $q^*(p) = 0$ for all $p < \check{p}$).

Another possibility for the optimum is that the government *sets up testing incentives* (i.e., $\check{p} < p^*$, or $v(\check{p}) < 0$). Intuitively, test capacities are relatively abundant so that the government tests more individuals than would like to get tested in the absence of regulation. In order to make individuals with negative values for getting tested reveal themselves, some social distancing (i.e., $q^*(p) > 0$) is enforced for individuals who remain untested. This lowers each individual's payoff from not getting tested. (Note that an extreme possibility is that $\check{p} = \underline{p}$, that is, everybody is tested.)

The remaining last possibility for the optimum is that the government *sets up testing disincentives* (i.e., $\check{p} > p^*$, or $v(\check{p}) > 0$). Intuitively, test capacities are relatively scarce so that the government tests fewer individuals than would like to get tested in the absence of regulation. In order to prevent individuals with positive values for getting tested from seeking a test, randomized testing is introduced. Only a randomly selected fraction $\check{m} < 1$ of the individuals who claim to have types above $\check{p}$ are tested, and each individual of such a type that does not belong to the tested fraction is put in quarantine for sure. Each individual now faces a gamble if she asks to get tested: with some probability she is then *not* tested and is *still* put in quarantine, whereas she would not have been put in quarantine had she not asked for a test. Individuals with higher types are more willing than those with lower types to take such a gamble because for them the test is more valuable, while the hassle of being put in quarantine for those who do not get a test is type-independent.[14]

The proof of Proposition 2 is relegated to Sect. 4.

3.3 Numerical examples

In this section, we illustrate the optimal regulation scheme via some numerical examples (the computations rely on the characterization of the optimal testing threshold in the next section).

3.3.1 Combined importance of test scarcity and average infection risk

Our first numerical example illustrates the combined importance of the scarcity of tests and the average infection risk in the population. Both dimensions affect the optimal regulation scheme.

Specifically, we consider a parameterized class of type distributions

$$F(p) = \left(\frac{p - \underline{p}}{\overline{p} - \underline{p}} \right)^{\beta},$$

where we assume that the lowest type $\underline{p} = 0.1$, the highest type $\overline{p} = 0.8$, and the parameter β ranges from 0.1 to 1.5. The larger β the more probability mass is shifted to higher-risk types, in the sense of first-order stochastic dominance. Thus, a larger β captures a population that is more at risk.

To model different degrees of test scarcity, we consider testing costs c^{gt} that range from 1 to 4.

Figure 2 illustrates the respective optimality of three of the four basic regulation regimes (because no-testing-always-quarantining is never optimal in the example), as a function of the testing cost c^{gt} and the riskiness parameter β. All other parameters are fixed as described in the caption of Fig. 2.

Figure 2 shows that the combination of a low testing cost and a relatively high riskiness (the upper-left area) implies that setting up testing incentives is optimal. The opposite, a combination of a high testing cost and a sufficiently low riskiness (the lower-right area) implies that setting up testing disincentives is optimal. In the remaining middle area, laissez faire is optimal.

[14] Note that one possibility is that $\check{p} = \overline{p}$. Such a solution is essentially equivalent to no-testing-no-quarantining (strictly speaking, the highest type, $\overline{p}$, is tested with a positive probability, but this exact type occurs with probability 0, and the government may as well not test this type).

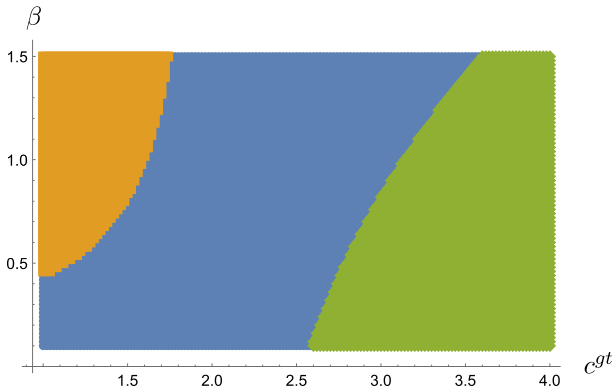

Fig. 2 Parameter constellations such that setting up testing incentives is optimal (upper left region), laissez faire is optimal (middle region), and setting up testing disincentives is optimal (lower right region). For this diagram, it is assumed that $\overline{p} = 0.8$, $\underline{p} = 0.1$, $F(p) = ((p - \underline{p})/(\overline{p} - \underline{p}))^{\beta}$, $y = 1$, $b = 4$, $c^q = 4$, $c^t = 1$, $b^q = 8$, $c^{gq} = 0.1$. Consequently, $p^* = 0.25$. The upper left region represents pairs (c^{gt}, β) such that $\check{p} < p^*$; the middle region represents pairs (c^{gt}, β) such that $\check{p} = p^*$; the lower right region represents pairs (c^{gt}, β) such that $\check{p} > p^*$. (The computations were performed using Mathematica 12.)

The presence of a parameter area where laissez faire is optimal is noteworthy. This is a striking difference to the first-best solution, where laissez faire essentially never occurs. Intuitively speaking, optimality of the laissez faire means that influencing an individuals' testing decisions would be so costly for the regulator (in terms of distorting the quarantining) that she renounces the attempt. The possible optimality of laissez faire is a general feature of optimal regulation, as Corollary 6 below will show.

3.3.2 Dependence of the testing threshold on the testing cost

Our second example illustrates how the testing intensity varies with the scarcity of the government's testing capacity. Formally, we consider the optimal threshold type $\check{p}$ as a function of the government's testing cost c^{gt}. In our example, the testing cost ranges from 0 to 12. All other parameters are defined in the caption of Fig. 3 or Fig. 4, respectively. The only difference between the figures stems from the cost of quarantining c^q.

Consider Fig. 3, where $c^q = 3$. If the testing cost c^{gt} is small, it is optimal to test everybody, that is $\check{p} = \underline{p}$. Next there is a range of testing costs in which it is optimal to set up testing incentives, but not everybody is tested, that is, $\underline{p} < \check{p} < p^* = 0.25$. This is followed by a range of testing costs such that laissez faire is optimal. If the testing cost is even higher, it becomes optimal to provide ever stronger testing disincentives. At the point $\overline{c}^{gt} \approx 6$, testing capacity is so scarce that no-testing-always-quarantine is optimal if the cost is even higher.

The second-best solution is strikingly different from of the first-best solution at the same parameter values that was illustrated in Fig. 1. At low testing costs (c^{gt} below ≈ 3.7), the first best relies on testing some types without extra quarantining of untested types, but the marginally tested type is different in the second best, where type-revelation incentives are provided via social-distancing of untested types or via

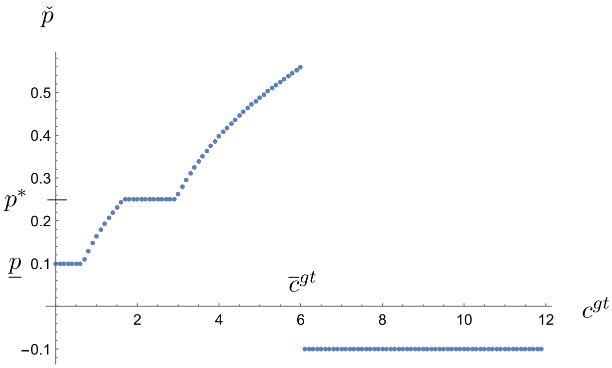

Fig. 3 Example of a government's second-best optimal marginally tested type, $\check{p}$, as a function of the government's cost of a test unit, c^{gt}. The cases in which $c^{gt} > \overline{c}^{gt}$, where no-testing-always-quarantining is optimal, are represented via a negative value of $\check{p}$. For this diagram, it is assumed that F is the uniform distribution on the interval $[\underline{p}, \overline{p}] = [0.1, 0.8]$, $y = 1$, $b = 4$, $c^q = 3$, $c^t = 1$, $b^q = 8$, $c^{gq} = 0.1$. The computations were performed using Mathematica 12

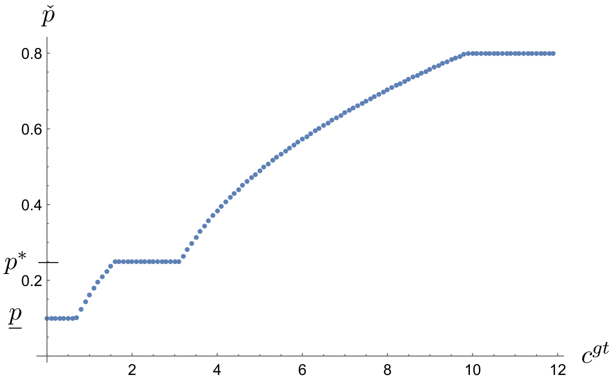

Fig. 4 Example of the government's second-best optimal marginally tested type, $\check{p}$, as a function of the government's cost of a test unit, c^{gt}. If the testing cost is sufficiently high, then $\check{p} = \overline{p}$, that is, no-testing-no-quarantining is optimal. For this diagram, it is assumed that F is the uniform distribution on the interval $[\underline{p}, \overline{p}] = [0.1, 0.8]$, $y = 1$, $b = 4$, $c^q = 4$, $c^t = 1$, $b^q = 8$, $c^{gq} = 0.1$. The computations were performed using Mathematica 12

randomized testing. At higher testing costs (c^{gt} above ≈ 3.7), the first best relies on selectively quarantining a strict subset of the untested types. In the second best, this is not possible: without testing, the regulator can only quarantine everybody or nobody. As a result, the second best relies on some testing longer—up to the cost level $c^{gt} \approx 6$—than the first best, where testing stops already at cost level $c^{gt} \approx 5.3$.

The case of Fig. 4 differs from the case of Fig. 3 because now we assume that quarantine is more unpleasant, $c^q = 4$. As in Fig. 3, there is a range of testing cost levels such that laissez faire is optimal, and if testing cost are even higher, it becomes optimal to provide ever stronger testing disincentives. In contrast to the example of Fig. 3, however, no-testing-*no*-quarantine is optimal if testing capacities are sufficiently scarce. This reflects the high individual cost of quarantine.

Several lessons can be drawn from the example. First, the higher the testing cost the less testing occurs in the optimal regulation scheme; this will be shown as a general result in Corollary 7 below. Second, if the testing cost is very high, then all testing ceases, and two different schemes can be optimal: quarantine everybody or quarantine nobody. Third, as in the example of Fig. 2, laissez faire is optimal for a range of cost levels.

3.4 Optimal regulation: full characterization and comparative statics

In this section, we characterize the cases in which no-testing-always-quarantine is optimal and, concerning the remaining cases, characterize the optimal testing threshold $\check{p}$ (Proposition 4). Using this result, we obtain a simpler characterization of the respective parameter areas where each of the four regimes—no-testing-always-quarantine, laissez faire, testing incentives, and testing disincentives—are optimal (Proposition 5). We use these characterizations to show that the optimality of laissez faire is not non-generic (Corollary 6), and to obtain comparative-statics results with respect to the government's cost of providing tests (Corollary 7).

Some auxiliary functions must be specified. For all types p, define

$$B(p) = \left(-bp - c^{gt} - \frac{c^{gq}}{c^q} v(p)\right) F(p) + \left(b + b^q y + \frac{b^q}{c^q} v(p)\right) E_{p' \sim F}[p' | p' \leq p] F(p). \tag{11}$$

For all $\lambda \geq 0$ and all types p, define

$$A^\lambda(p) = B(p) + \mathbf{1}_{p > p^*} \cdot v(p)(\lambda - \lambda^*), \tag{12}$$

where

$$\lambda^* = -1 + \frac{b^q E_{p' \sim F}[p'] - c^{gq}}{c^q}.$$

For all $\lambda \geq 0$, define

$$\alpha^\lambda = -\min_p A^\lambda(p) + A^0(\overline{p}) - \lambda c^q. \tag{13}$$

The following lemma implies that the function $\lambda \mapsto \alpha^\lambda$ is strictly decreasing on $[0, \infty)$, and, by the Intermediate-value Theorem, intersects the horizontal axis. Thus, there exists a unique $\check{\lambda}$ such that $\alpha^{\check{\lambda}} = 0$. The proof is straightforward and is relegated to the "Appendix".

Lemma 3 *The function $\lambda \mapsto \alpha^\lambda$ is Lipschitz continuous. Its derivative satisfies the inequalities $-v(\overline{p}) - c^q \leq d\alpha^\lambda / d\lambda \leq -c^q$. Moreover, $\alpha^0 \geq 0$, and $\alpha^\lambda < 0$ for all sufficiently large λ.*

Proposition 4 characterizes the optimal value of the threshold type $\check{p}$ that was left as a free parameter in Proposition 2. The result also yields a computational path to solving the government's problem for any parameter constellation.

Proposition 4 *Let $\check{\lambda} \geq 0$ be such that $\alpha^{\check{\lambda}} = 0$. If $\check{\lambda} \leq \lambda^*$, then no-testing-always-quarantine is optimal.*

Alternatively, suppose that $\check{\lambda} \geq \lambda^*$. Let $\check{p}$ be a minimizer of $A^{\check{\lambda}}$. Then $\check{p}$ yields a solution for the government's problem as described in Proposition 2.

The proof of Proposition 4 is relegated to Sect. 4.

Our first application of these results is Proposition 5, where we characterize in a computationally tractable way the cases in which each of our four solution categories applies. The characterization refers to the five numbers λ^*, α^{λ^*}, $B(p^*)$,

$$\underline{B} = \min_{p \leq p^*} B(p),$$

and $\underline{A}^{\overline{\lambda}}$, which relies on the auxiliary definitions

$$\bar{l} = \frac{1}{c^q}(A^0(\overline{p}) - \underline{B}), \quad \overline{\lambda} = \max\{0, \bar{l}\},$$
$$\text{and} \quad \underline{A}^{\lambda} = \min_{p \geq p^*} A^{\lambda}(p) \quad \text{for all } \lambda \geq 0.$$

Computing each of the five numbers is straightforward, by plugging in exogeneous model parameters or solving a one-dimensional minimization problem.

Proposition 5 *If $\lambda^* \geq 0$ and $\alpha^{\lambda^*} \leq 0$, then* no-testing-always-quarantine *is optimal.*

Alternatively, suppose that $\lambda^* < 0$, or $\lambda^* \geq 0$ and $\alpha^{\lambda^*} > 0$.

If $\underline{B} \leq \underline{A}^{\overline{\lambda}}$ and $\underline{B} = B(p^*)$, then laissez faire *is optimal, that is, $\check{p} = p^*$.*

If $\underline{B} \leq \underline{A}^{\overline{\lambda}}$ and $\underline{B} < B(p^*)$, then setting up testing incentives *is optimal, that is, $\check{p} < p^*$.*

If $\underline{B} > \underline{A}^{\overline{\lambda}}$, then setting up testing disincentives *is optimal, that is, $\check{p} > p^*$.*

Here is a sketch of the proof (for details see the "Appendix"). The condition for the optimality of no-testing-always-quarantine means that the strictly decreasing function $\lambda \mapsto \alpha^{\lambda}$ has already dipped below the horizontal axis when it reaches the point $\lambda = \lambda^*$. Thus, it intersects the horizontal axis to the left of the point λ^*, which corresponds to the condition on $\check{\lambda}$ given in Proposition 4. To understand where the other conditions arise, suppose for simplicity that $\bar{l} \geq 0$, that is, $\overline{\lambda} = \bar{l}$. Then

$$0 = -\underline{B} + A^0(\overline{p}) - \overline{\lambda} c^q,$$

that is, $\overline{\lambda}$ is the number at which we would have $\alpha^{\overline{\lambda}} = 0$ if the minimizer $\check{p}$ of $A^{\overline{\lambda}}$ belonged to $[\underline{p}, p^*]$, that is, if $\underline{B} \leq \underline{A}^{\overline{\lambda}}$. In this case, by Proposition 4, the government's problem has a solution with $\check{\lambda} = \overline{\lambda}$ and thus $\check{p} \leq p^*$. Similar arguments imply that, if the opposite inequality $\underline{B} > \underline{A}^{\overline{\lambda}}$ holds, then $\check{\lambda} > \overline{\lambda}$ and the minimizer $\check{p}$ of $A^{\check{\lambda}}$

cannot belong to $[\underline{p}, p^*]$, that is, $\check{p} > p^*$. The details of the proof of Proposition 5 are relegated to the "Appendix".

See Fig. 2 for an illustration of Proposition 5.

3.4.1 Optimality of laissez faire

The middle region in Fig. 2 indicates that it is often optimal for the regulator to do nothing. This is an important lesson from the presence of private information in our model: the optimality of laissez faire is not a non-generic case, in contrast to the situation if information is public (or if monetary transfers are feasible). A general sufficient condition is that the individual's current well-being is sufficiently important relative to the spread of the disease.

Corollary 6 *If, for given values of the other parameters, the public cost and benefit parameters c^{gt}, c^{gq}, and b^q, are sufficiently close to 0, then laissez faire is optimal, that is, $\check{p} = p^*$.*

To obtain a heuristic argument towards the proof, consider the hypothetical limit case $c^{gt} = c^{gq} = b^q = 0$. By definition of λ^*, we have $\lambda^* = -1 < 0$. Hence, by Proposition 5 there exists an optimal regulation scheme with some threshold type $\check{p}$. Also note that (11) implies that

$$B(p) = bE_{p' \sim F}[\mathbf{1}_{p' \leq p} \cdot (-p + p')],$$

which has the slope

$$B'(p) = -bF(p) \ < \ 0,$$

implying that B is strictly decreasing.

On the other hand, (12) together with $\lambda^* = -1$ implies that, for all $p > p^*$ and all $\lambda \geq 0$,

$$A^{\lambda}(p) = B(p) + v(p)(\lambda + 1),$$

which has the slope

$$(A^{\lambda})'(p) = B'(p) + b(\lambda + 1) \ \geq \ b(1 - F(p)) \ > \ 0,$$

implying that A^{λ} is strictly increasing for all $p \geq p^*$.

We conclude that A^{λ}, and in particular $A^{\check{\lambda}}$, is minimized at p^*, showing that $\check{p} = p^*$ by Proposition 4. The arguments above are easily extended to the case in which the parameters c^{gt}, c^{gq}, and b^q are not exactly equal to 0, but are sufficiently close to 0; the details are omitted.

The first-best solution described in Proposition 1 is different. If the public cost and benefit parameters c^{gt}, c^{gq}, and b^q are close to 0, but not exactly equal to 0, then, generically, $\underline{p}^t \neq p^*$, that is, the first-best solution differs from laissez faire. In other words, while the omniscient government's optimal rule (or the optimal scheme with monetary transfers) reacts with some active regulation to even the slighest concern about public costs and benefits, a government that must take the incentive constraints into account optimally sticks to laissez faire if public costs and benefits are small.

3.4.2 Comparative statics with respect to the government's testing cost

To further understand the optimal regulation scheme, we describe the role of the cost parameter c^{gt}. We distinguish the case in which the individuals have a strong dislike of quarantining (i.e., c^q above a threshold) and the opposite case where curbing the spread of the disease is considered relatively more important (i.e., c^q below the threshold). In both cases the range of tested types is decreasing in the testing cost c^{gt},[15] and in the infinite-cost limit nobody is tested anymore. The quarantining in the infinite-cost limit, however, is diametrically different across the two cases, with nobody being quarantined in the first case and everybody in the second.

Corollary 7 *Consider the case $c^q \geq b^q E_{p' \sim F}[p'] - c^{gq}$. Then there exists a marginally tested type $\check{p}$. Moreover, choosing either the minimal or the maximal $\check{p}$ in case of multiplicity, $\check{p}$ is weakly increasing in c^{gt}, and $\check{p} \to \overline{p}$ as $c^{gt} \to \infty$.*

Consider the case $c^q < b^q E_{p' \sim F}[p'] - c^{gq}$. Then there exists a threshold $\overline{c}^{gt}$ such that, for all $c^{gt} < \overline{c}^{gt}$, the marginally tested type $\check{p}$ (choose the minimal or maximal $\check{p}$ in case of multiplicity) is weakly increasing in c^{gt}; no-testing-always-quarantine is optimal for all $c^{gt} \geq \overline{c}^{gt}$.

The proof of Corollary 7—which is rather technical due to the implicit nature of the definition of $\check{p}$—is relegated to the "Appendix". Figure 3 illustrates the case $c^q < b^q E_{p' \sim F}[p'] - c^{gq}$; Fig. 4 illustrates the case $c^q > b^q E_{p' \sim F}[p'] - c^{gq}$.

4 Proof of Proposition 2 and Proposition 4

As a first step, we rewrite the government's problem as a convex maximization problem over testing schedules $m(\cdot)$. As a second step, we show that the solution $m^*(\cdot)$ described in Proposition 2 and Proposition 4 satisfies the (Lagrangian first-order) sufficient conditions for solving the problem as rewritten in the first step. As a third step, we show that the optimal social-distancing schedule $q^*(\cdot)$ described in Proposition 2 and Proposition 4 is a consequence of the optimal testing schedule $m^*(\cdot)$.

4.1 Step 1: rewriting the government's problem

Using standard techniques from mechanism design (see, e.g., Börgers (2015), Chapter 3), we have the following result.

[15] From (10) it follows that the testing probability also decreases.

Lemma 8 *A rule* (m, q) *is incentive compatible if and only if*[16]

$$U(p, p) = b \int_{p^*}^{p} m(p')dp' + U(p^*, p^*) \quad \text{for all } p, \tag{14}$$

and

$$m(p) \leq m(p') \ \text{ for all } p < p'. \tag{15}$$

The first condition (14) is an envelope or integrability condition that yields a "revenue-equivalence" result: the testing schedule $m(\cdot)$ determines the individual's expected utility as a function of the type, up to the constant (setting $\hat{p} = p = p^*$ in (6))

$$U(p^*, p^*) = -c^q(1 - m(p^*))q(p^*).$$

Plugging this into the integrability condition (14) and using again the individual utility expression (6), we get, for each type p, a condition for the quarantine probability $q(p)$ such that the integrability condition is satisfied:

$$v(p)m(p) - c^q(1 - m(p))q(p) = b \int_{p^*}^{p} m(p')dp' - c^q(1 - m(p^*))q(p^*).$$

Rearranging, we can express the cost of the expected quarantine of an untested individual,

$$(1 - m(p))q(p) = \psi^{m,q(p^*)}(p), \tag{16}$$

where we use the shortcut

$$\psi^{m,q(p^*)}(p) = \frac{1}{c^q}\left(v(p)m(p) - b \int_{p^*}^{p} m(p')dp'\right) + (1 - m(p^*))q(p^*). \tag{17}$$

Now consider a schedule $m(\cdot)$ that satisfies the monotonicity condition (15).

We would like to characterize the set of $m(\cdot)$ such that (m, q) is incentive compatible for some quarantining schedule $q(\cdot)$. Given some $m(\cdot)$, the question is then whether or not there exists $q(\cdot)$ that satisfies the probability condition (5) such that the equation (16) holds.

Multiplying (5) with $1 - m(q)$, we obtain the essentially equivalent condition

$$0 \leq (1 - m(p))q(p) \leq 1 - m(p) \quad \text{for all } p, \tag{18}$$

(Note that this condition, in contrast to (5), leaves $q(p)$ undetermined if $m(p) = 1$; this change, however, is inessential because the quarantining probability $q(p)$ is irrelevant for an individual who is tested for sure.)

16 We use the convention $\int_{p^*}^{p} \cdots = -\int_{p}^{p^*} \ldots$ for all $p < p^*$.

Plugging (16) into (18), the condition reads

$$0 \leq \psi^{m,q(p^*)}(p) \leq 1 - m(p) \quad \text{for all } p. \tag{19}$$

Note that this condition implies $0 \leq q(p^*) \leq 1$ if $m(p^*) < 1$. Thus, we can consider $q(p^*)$ as a free variable in the following.

The next step is to express the welfare W as a function of the testing schedule $m(\cdot)$ and $q(p^*)$. This is achieved by plugging into (9) the expressions obtained in (14) and (16), giving

$$\begin{aligned} W = E_{p\sim F}\Bigg[b \int_{p^*}^{p} m(p')dp' - c^q(1-m(p^*))q(p^*) \\ \times \int_{\underline{p}}^{p} + \left(b^q yp - c^{gt}\right) m(p) + \left(b^q p - c^{gq}\right) \psi^{m,q(p^*)}(p) \Bigg]. \end{aligned} \tag{20}$$

In summary, the government's goal is to solve the following problem:

$$\max_{m(\cdot),\ q(p^*)} W \quad \text{s.t. } (4), \quad (15), \quad (19).$$

The left condition in (19) is satisfied for all p if and only if it is satisfied for the p that minimizes the function $\psi^{q(p^*),m}(p)$. The minimizer is $p = p^*$; to see this, consider any $p \neq p^*$ and note that

$$\begin{aligned} &\left(\psi^{q(p^*),m}(p) - \psi^{q(p^*),m}(p^*)\right) c^q \\ &= v(p)m(p) - b\int_{p^*}^{p} m(p')dp' \;=\; b(p-p^*)m(p) - b\int_{p^*}^{p} m(p')dp'. \end{aligned}$$

Due to (15), the last integral is bounded above by $(p-p^*)m(p)$, showing that $\psi(p) \geq \psi(p^*)$.

Thus we can replace the left condition in (19) by the simpler condition $0 \leq \psi^{q(p^*),m}(p^*)$ or, equivalently, using (17), by the condition

$$0 \leq (1 - m(p^*))q(p^*). \tag{21}$$

The right condition in (19) is satisfied for all p if and only if it is satisfied for the p that maximizes the function $\psi^{q(p^*),m}(p) + m(p)$. The maximizer is $p = \overline{p}$; to see this, consider any $p < \overline{p}$ and note that

$$\begin{aligned} &\left(\psi^{q(p^*),m}(\overline{p}) + m(\overline{p}) - \psi^{q(p^*),m}(p) - m(p)\right) c^q \\ &= (v(\overline{p}) + c^q)m(\overline{p}) - (v(p) + c^q)m(p) - b \underbrace{\int_{p}^{\overline{p}} m(p')dp'}_{\leq(\overline{p}-p)m(\overline{p}) \text{ by } (15)}. \end{aligned}$$

Due to (3), $v(p) + c^q > 0$. Thus, again using that $m(p) \le m(\overline{p})$ from (15), we can continue the above equation via the estimation

$$
\begin{aligned}
&\ge (v(\overline{p}) + c^q)m(\overline{p}) - (v(p) + c^q)m(\overline{p}) - (b - c^q)(\overline{p} - p)m(\overline{p}) \\
&= (v(\overline{p}) - v(p))m(\overline{p}) - b(\overline{p} - p)m(\overline{p}) \\
&= 0,
\end{aligned}
$$

where the last equality relies on the definition of v in (1).

Thus, we can replace the right condition in (19) by the simpler condition $\psi^{q(p^*),m}(\overline{p}) \le 1 - m(\overline{p})$ or, equivalently, using (16), by the condition

$$
(1 - m(p^*))q(p^*) \le 1 - m(\overline{p}) - \frac{1}{c^q}\left(v(\overline{p})m(\overline{p}) - b\int_{p^*}^{\overline{p}} m(p')dp'\right). \quad (22)
$$

At this point, it is useful to take stock: we have replaced the condition (19), which is required for all p, by two one-dimensional conditions: (21) provides a lower upper bound for $q(p^*)$ if $m(p^*) < 1$, and (22) provides an upper bound.

Now we can eliminate the variable $q(p^*)$ from the government's problem.

According to (17) and (20), the dependence of W on $q(p^*)$ is described by the additive term

$$
\left(b^q E_{p\sim F}[p] - c^{gq} - c^q\right)(1 - m(p^*))q(p^*).
$$

Thus, W is linear with respect to $q(p^*)$, with slope $(1 - m(p^*))\lambda^* c^q$, where

$$
\lambda^* = \frac{b^q E_{p\sim F}[p] - c^{gq}}{c^q} - 1.
$$

In the following computations, we have to distinguish two cases, depending on the sign of λ^*. Suppose first that

$$
\lambda^* \le 0. \quad (23)
$$

Then W is weakly decreasing in $q(p^*)$. Thus, there exists an optimal $q(p^*)$ that hits the lower bound provided by (21), that is,

$$
q(p^*) = 0. \quad (24)
$$

Plugging (24) into (22), we have

$$
0 \le 1 - m(\overline{p}) - \frac{1}{c^q}\left(v(\overline{p})m(\overline{p}) - b\int_{p^*}^{\overline{p}} m(p')dp'\right).
$$

This (one-dimensional) condition replaces (19) in the government's optimization. Rearranging, we obtain the equivalent condition

$$(v(\overline{p}) + c^q)m(\overline{p}) - b\int_{p^*}^{\overline{p}} m(p')dp' \leq c^q. \tag{25}$$

Next we rewrite the welfare W. Plugging (24) into (17), we get

$$\psi^{m,q(p^*)}(p) = \frac{1}{c^q}\left(v(p)m(p) - b\int_{p^*}^{p} m(p')dp'\right), \tag{26}$$

and plugging (24) into (20) we get

$$\begin{aligned}
W &= E_{p\sim F}\Bigg[b\int_{p^*}^{p} m(p')dp' \\
&\quad \times \int_{\underline{p}}^{p} + \left(b^q yp - c^{gt}\right)m(p) + \left(b^q p - c^{gq}\right)\frac{1}{c^q}\left(v(p)m(p) - b\int_{p^*}^{p} m(p')dp'\right)\Bigg]. \\
&= E_{p\sim F}\left[\left(1 - \frac{b^q p - c^{gq}}{c^q}\right)b\int_{p^*}^{p} m(p')dp'\right] \\
&\quad + E_{p\sim F}\left[\left(b^q yp - c^{gt} + \frac{b^q p - c^{gq}}{c^q}v(p)\right)m(p)\right] \\
&= -\int_{\underline{p}}^{\overline{p}}\int_{p^*}^{p} \kappa(p)\, b\, m(p')dp'dp \;+\; \int_{\underline{p}}^{\overline{p}} L(p)m(p)dp,
\end{aligned} \tag{27}$$

where we have used the auxiliary functions

$$\kappa(p) = \left(-1 + \frac{b^q p - c^{gq}}{c^q}\right) f(p) \tag{28}$$

$$\text{and}\quad L(p) = \left(b^q yp - c^{gt} + \frac{b^q p - c^{gq}}{c^q}v(p)\right) f(p). \tag{29}$$

The first of the two terms in (27) can be rewritten into a more useful form. To do this, we split it into two integrals:

$$\begin{aligned}
-\int_{\underline{p}}^{\overline{p}}\int_{p^*}^{p} m(p')\kappa(p)dp'dp &= \int_{\underline{p}}^{p^*}\int_{p}^{p^*} m(p')\kappa(p)dp'dp \\
&\quad - \int_{p^*}^{\overline{p}}\int_{p^*}^{p} m(p')\kappa(p)dp'dp.
\end{aligned}$$

Each of these double integrals can be simplified via changing the order of integration.

$$\int_{\underline{p}}^{p^*}\int_{p}^{p^*} m(p')\kappa(p)dp'dp = \int_{\underline{p}}^{p^*}\int_{\underline{p}}^{p'} m(p')\kappa(p)dpdp' = \int_{\underline{p}}^{p^*} K(p')m(p')dp',$$

where we have used the auxiliary function

$$K(p) = \int_{\underline{p}}^{p} \kappa(p')dp' \tag{30}$$

Similarly, the second integral can be written as

$$\begin{aligned} -\int_{p^*}^{\overline{p}} \int_{p^*}^{p} m(p')\kappa(p)dp'dp &= -\int_{p^*}^{\overline{p}} \int_{p'}^{\overline{p}} m(p')\kappa(p)dpdp' \\ &= \int_{p^*}^{\overline{p}} (K(p') - K(\overline{p}))m(p')dp'. \end{aligned}$$

Summing up,

$$-\int_{\underline{p}}^{\overline{p}} \int_{p^*}^{p} m(p')\kappa(p)dp'dp = \int_{\underline{p}}^{\overline{p}} \left(K(p') - \mathbf{1}_{p \geq p^*} \cdot K(\overline{p})\right) m(p')dp'.$$

Note that

$$K(\overline{p}) = \lambda^*.$$

Thus, (27) has been simplified as

$$W = \int_{\underline{p}}^{\overline{p}} \left(bK(p) + L(p) - \mathbf{1}_{p \geq p^*} \cdot b\lambda^*\right) m(p)dp. \tag{31}$$

So far we have achieved the following reformulation of the government's problem

$$(\text{case } \lambda^* \leq 0) \qquad \text{s.t. } \max_{m(\cdot)} (31) \quad \text{s.t. } (4), \ (15), \ (25).$$

We can use, e.g., the space $PC[\underline{p}, \overline{p}]$ of right-continuous and piecewise continuous functions for the testing chedules $m(\cdot)$; this is a linear vector space. The constraints (4) and (15) define a convex subset Ω of $PC[\underline{p}, \overline{p}]$. Then the government's problem can be written as

$$(\text{case } \lambda^* \leq 0) \qquad \text{s.t. } \max_{m(\cdot)\in\Omega} (31) \quad \text{s.t. } (25).$$

Now suppose that

$$\lambda^* \geq 0. \tag{32}$$

Then the government's objective W is weakly increasing in $q(p^*)$. Hence, it is optimal to choose $q(p^*)$ such that it hits the upper bound provided by (22). We can proceed analogously to the case (23), obtain the welfare expression

$$W = \int_{\underline{p}}^{\overline{p}} (bK(p) + L(p))\, m(p)dp - K(\overline{p}) \left((v(\overline{p}) + c^q) m(\overline{p}) - c^q \right), \quad (33)$$

and can write the government's problem as

$$(\text{case } \lambda^* \geq 0) \qquad \text{s.t.} \ \max_{m(\cdot)\in\Omega} (33) \quad \text{s.t.} \ (25).$$

4.1.1 Step 2: solving the rewritten problem

We will now show that the solution m^* described in Proposition 2 and Proposition 4 solves the government's problem as reformulated in *Step 1*.

As in Step 1, we distinguish two cases depending on the sign of λ^*. Suppose first that $\lambda^* \leq 0$.

Consider the reformulated problem from *Step 1* (case $\lambda^* \leq 0$). The following two Lagrangian conditions are sufficient for a solution (see, e.g., Luenberger (1968), Chapter 8). First, there exists a number $\lambda \geq 0$ ("Lagrange multiplier") such that $m^*(\cdot)$ solves the problem

$$\max_{m(\cdot)\in\Omega} \int_{\underline{p}}^{\overline{p}} \left(bK(p) + L(p) - \mathbf{1}_{p \geq p^*} \cdot b\lambda^* \right) m(p)dp$$
$$- \lambda \left((v(\overline{p}) + c^q) m(\overline{p}) - b \int_{p^*}^{\overline{p}} m(p')dp' \right). \quad (34)$$

Second, (25) is satisfied with equality at $m = m^*$.

In order to show that m^* satisfies these conditions, we begin by rewriting the objective of the Lagrangian problem (34):

$$W^\lambda = \int_{\underline{p}}^{\overline{p}} \overbrace{\left(bK(p) + L(p) + \mathbf{1}_{p \geq p^*} \cdot b\left(\lambda - \lambda^*\right) \right)}^{\equiv a^\lambda(p)} m(p)dp$$
$$- \lambda (v(\overline{p}) + c^q) m(\overline{p}). \quad (35)$$

In order to further rewrite W^λ, we introduce additional notation. For any type p, define the conditional expectations

$$\eta(p) = E_{p' \sim F}[p' | p' \leq p],$$
$$\eta_2(p) = E_{p' \sim F}[(p')^2 | p' \leq p].$$

Thus, using integration by parts,

$$\int_{\underline{p}}^{p} F(p')dp' = -\int_{\underline{p}}^{p} p' f(p')dp' + pF(p) \ = \ (p - \eta(p))F(p). \quad (36)$$

Similarly,

$$\int_{\underline{p}}^{p}\int_{\underline{p}}^{p'} p'' f(p'')dp''dp' = -\int_{\underline{p}}^{p}(p')^2 f(p')dp' + p\int_{\underline{p}}^{p} f(p'')dp''$$
$$= (p\eta(p) - \eta_2(p))F(p). \tag{37}$$

Using (28) and (30),

$$K(p) = -\left(1 + \frac{c^{gq}}{c^q}\right)F(p) + \frac{b^q}{c^q}\int_{\underline{p}}^{p} p' f(p')dp'.$$

Thus, using (36) and (37),

$$\int_{\underline{p}}^{p} K(p')dp' = -\left(1 + \frac{c^{gq}}{c^q}\right)(p - \eta(p))F(p) + \frac{b^q}{c^q}(p\eta(p) - \eta_2(p))F(p).$$

Using the definition (29),

$$L(p) = \left(-c^{gt} + \frac{c^{gq}}{c^q}c^t\right)f(p) + \left(b^q y - \frac{b^q}{c^q}c^t - \frac{c^{gq}}{c^q}b\right)pf(p) + \frac{b^q}{c^q}bp^2 f(p).$$

Thus,

$$\int_{\underline{p}}^{p} L(p')dp' = \left(-c^{gt} + \frac{c^{gq}}{c^q}c^t\right)F(p) + \left(b^q y - \frac{b^q}{c^q}c^t - \frac{c^{gq}}{c^q}b\right)\eta(p)F(p)$$
$$+\frac{b^q}{c^q}b\eta_2(p)F(p). \tag{38}$$

Combining the derived expressions,

$$\begin{aligned}\int_{\underline{p}}^{p}\left(bK(p') + L(p')\right)dp' &= \left(-b\left(1 + \frac{c^{gq}}{c^q}\right)p - c^{gt} + \frac{c^{gq}}{c^q}c^t\right)F(p)\\ &\quad + \left(b\left(1 + \frac{b^q}{c^q}p\right) + b^q y - \frac{b^q}{c^q}c^t\right)\eta(p)F(p)\\ &= \left(-bp - c^{gt} - \frac{c^{gq}}{c^q}v(p)\right)F(p)\\ &\quad + \left(b + b^q y + \frac{b^q}{c^q}v(p)\right)\eta(p)F(p)\\ &= B(p).\end{aligned}$$

where we have used the definition (11). Thus, by definition of the function a^λ,

$$\int_{\underline{p}}^{p} a^\lambda(p')dp' = \int_{\underline{p}}^{p}\left(bK(p') + L(p') + \mathbf{1}_{p' \geq p^*}\cdot b\left(\lambda - \lambda^*\right)\right)dp'$$

$$= B(p) + \mathbf{1}_{p \geq p^*} \cdot \underbrace{b(p - p^*)}_{=v(p) \text{ by (2)}} \left(\lambda - \lambda^*\right)$$
$$= A^\lambda(p),$$

where the last equality follows from (12).

With this in mind, we apply integration by parts to the right-hand side of (35), yielding

$$W^\lambda = -\int_{\underline{p}}^{\overline{p}} A^\lambda(p) dm(p) + A^\lambda(\overline{p}) m(\overline{p}) - \lambda(v(\overline{p}) + c^q) m(\overline{p}),$$

where m is interpreted as a c.d.f.

Note that, using the definition (12),

$$A^\lambda(\overline{p}) = B(\overline{p}) + v(\overline{p})(\lambda - \lambda^*).$$

Thus, we obtain the simplified formula

$$\begin{aligned} W^\lambda &= -\int_{\underline{p}}^{\overline{p}} A^\lambda(p) dm(p) + \left(B(\overline{p}) - v(\overline{p})\lambda^* - \lambda c^q\right) m(\overline{p}) \\ &= -\int_{\underline{p}}^{\overline{p}} A^\lambda(p) dm(p) + \left(A^0(\overline{p}) - \lambda c^q\right) m(\overline{p}). \end{aligned} \tag{39}$$

Now consider specifically the Lagrange multiplier $\lambda = \check{\lambda}$ from Proposition 4. Fixing any $m(\overline{p})$ $(0 \leq m(\overline{p}) \leq 1)$, $W^{\check{\lambda}}$ is maximized if m puts all of the mass $m(\overline{p})$ on a point $\check{p}$ where $A^{\check{\lambda}}$ is minimized, that is,

$$m(p) = \begin{cases} 0 & \text{if } p < \check{p}, \\ m(\overline{p}) & \text{if } p \geq \check{p}. \end{cases}$$

Given such an m, the value of the Lagrangian can be written as

$$W^{\check{\lambda}} = \left(-\min_p A^{\check{\lambda}}(p) + A^0(\overline{p}) - \check{\lambda} c^q\right) m(\overline{p}) = \alpha^{\check{\lambda}} m(\overline{p}) = 0.$$

In particular, m^* as described in Proposition 2 maximizes $W^{\check{\lambda}}$. Thus, the first of the two Lagrangian conditions is satisfied.

It remains to verify the second condition, that the constraint (25) is satisfied with equality.

Suppose that $\check{p} \leq p^*$. Then $m^*(\overline{p}) = \check{m} = 1$ according to the formula given for $\check{m}$ in Proposition 2. Thus, (25) is satisfied with equality because

$$(v(\overline{p}) + c^q)m^*(\overline{p}) - b\int_{p^*}^{\overline{p}} m^*(p')dp' = (v(\overline{p}) + c^q)\check{m} - b(\overline{p} - p^*)\check{m} = c^q\check{m}$$
$$= c^q,$$

where we have used the definitions of $v(\overline{p})$ and p^*.

Now suppose that $\check{p} > p^*$. Then at $m(\overline{p}) = 1$ the left-hand side of (25) would be strictly larger than c^q. Thus, there exists $\check{m} < 1$ such that, at $m(\overline{p}) = \check{m}$, (25) is satisfied with equality. It is straighforward to check that the formula for $\check{m}$ given in Proposition 2 yields the required value.

Now consider the reformulated problem from *Step 1* (case $\lambda^* \geq 0$). The following three Lagrangian conditions are sufficient for a solution (see, e.g., Luenberger (1968), Chapter 8). First, there exists a number $\lambda_2 \geq 0$ ("Lagrange multiplier") such that $m^*(\cdot)$ solves the problem

$$\max_{m(\cdot)\in\Omega} \int_{\underline{p}}^{\overline{p}} (bK(p) + L(p))\, m(p)dp - \lambda^* \left((v(\overline{p}) + c^q)m(\overline{p}) - c^q\right)$$
$$- \lambda_2 \left((v(\overline{p}) + c^q)m(\overline{p}) - b\int_{p^*}^{\overline{p}} m(p')dp'\right). \tag{40}$$

Second, (25) is satisfied at $m = m^*$. Third, if (25) is satisfied with strict inequality at $m = m^*$, then $\lambda_2 = 0$.

In order to show that m^* satisfies these conditions, we begin by rewriting the objective of the Lagrangian problem (40):

$$= \int_{\underline{p}}^{\overline{p}} \left(bK(p) + L(p) + \lambda_2 b\mathbf{1}_{p\geq p^*}\right) m(p)dp - (\lambda_2 + \lambda^*)(v(\overline{p}) + c^q)m(\overline{p}) + \lambda^* c^q$$
$$= W^{\lambda_2+\lambda^*} + \lambda^* c^q, \tag{41}$$

where the last equality is immediate from a comparison with (35). Note that the term $\lambda^* c^q$ is constant and thus can be dropped from the maximization problem.

First we consider the case $\check{\lambda} \leq \lambda^*$. Fix the Lagrange multiplier $\lambda_2 = 0$. Then,

$$\alpha^{\lambda_2+\lambda^*} \leq 0 \tag{42}$$

because α is a decreasing function.

Fixing any $m(\overline{p})$ $(0 \leq m(\overline{p}) \leq 1)$ and applying (39) with $\lambda = \lambda_2 + \lambda^*$, we see that the objective of the Lagrangian problem is maximized if m puts all of the mass $m(\overline{p})$ on a point $\check{p}$ where $A^{\lambda_2+\lambda^*}$ is minimized, that is,

$$m(p) = \begin{cases} 0 & \text{if } p < \check{p}, \\ m(\overline{p}) & \text{if } p \geq \check{p}. \end{cases}$$

Given such an m, the value of the Lagrangian can be written as

$$W^{\lambda_2+\lambda^*} = \alpha^{\lambda_2+\lambda^*} m(\overline{p}) + \lambda^* c^q,$$

and, due to (42), this expression is maximized by setting $m(\overline{p}) = 0$. That is, no testing is optimal. The constraint (25) is obviously satisfied.

Now suppose that $\check{\lambda} \geq \lambda^*$. Then, we consider the Lagrange multiplier $\lambda_2 = \check{\lambda} - \lambda^*$. Using the fact that the Lagrangian can be written in the form (41), the rest of the proof is as in the case $\lambda^* \leq 0$ that was treated above.

4.1.2 Step 3: optimal quarantining schedule

As in *Step 1* and in *Step 2*, we distinguish two cases depending on the sign of λ^*. Suppose first that $\lambda^* \leq 0$.

Consider first the case $\check{p} \leq p^*$. Then $\check{m} = 1$. Using the definition of m^*, (16), and (26), for all $p < \check{p}$,

$$\begin{aligned} q^*(p) &= (1 - m^*(p))q^*(p) \\ &= \psi^{m^*, q(p^*)}(p) \\ &= \frac{1}{c^q}\left(v(p)m^*(p) - b\int_{p^*}^{p} m^*(p')dp'\right) = -\frac{1}{c^q} b(\check{p} - p^*) = \frac{-v(\check{p})}{c^q}, \end{aligned}$$

as was to be shown.

Now consider the case $\check{p} > p^*$. Then $\check{m} < 1$. Using the definition of m^*, (16), and (26), for all $p < \check{p}$,

$$q^*(p) = (1 - m^*(p))q^*(p) = \psi^{m^*, q(p^*)}(p) = 0,$$

as was to be shown. For all $p \geq \check{p}$, using again the definition of m^*, (16), and (26),

$$\begin{aligned} (1 - \check{m})q^*(p) &= \frac{1}{c^q}\left(v(p)\check{m} - b\int_{\check{p}}^{p} \check{m}dp'\right) = \frac{1}{c^q}\big(v(p) - b(p - \check{p})\big)\,\check{m} \\ &= \frac{1}{c^q} v(\check{p})\check{m}. \end{aligned}$$

Dividing both sides by $1 - \check{m}$ yields the formula

$$q^*(p) = v(\check{p})\frac{\check{m}}{(1 - \check{m})c^q} \quad \text{for all } p \geq \check{p}.$$

Plugging into the right-hand side the formula (10), we obtain the desired conclusion $q^*(p) = 1$.

The arguments in the case $\lambda^* \geq 0$ are similar and are omitted. This completes the proof of Proposition 2 and Proposition 4.

5 Conclusion

We have shown that individuals' private health signals play a crucial role for the optimal regulation of testing and social distancing in a pandemic. Future research may focus on diseases for which a vaccination is available, where an individual's decision to get vaccinated also depends on private information,[17] and on incorporating individuals' type-dependent testing or vaccination decisions into a dynamic regulation model.

Another possible direction may be a model variation in which agents are systematically too pessimistic or optimistic, or have s-shaped probability distortions as in prospect theory (see Kahneman (2011) for an introduction). A psychological bias towards overestimating the probability of being infected may be considered a plausible model variation in a population with a small rate of infections when the illness nevertheless draws a lot of public attention.

6 Appendix

Proof of Proposition 1 We replace the expression (6) for $U(p, p)$ in W and rearrange terms,

$$\begin{aligned} W &= E_{p\sim F}\Big[v(p)m(p) - c^q(1-m(p))q(p) + b^q ypm(p) + b^q p(1-m(p))q(p) \\ &\qquad \times w^1 U(p,p) - c^{gt}m(p) - c^{gq}(1-m(p))q(p)\Big] \\ &= E_{p\sim F}\left[C(p)m(p) + D(p)(1-m(p))q(p)\right], \end{aligned} \tag{43}$$

where we use the shortcuts

$$\begin{aligned} C(p) &= v(p) + b^q yp - c^{gt}, \\ D(p) &= -c^q + b^q p - c^{gq}. \end{aligned}$$

Note that both C and D are linear and strictly increasing functions of p, and C is steeper than D because $v'(p) + b^q y = b + b^q y > b^q$ by (8).

Using (43), the welfare-maximizing value of $m(p)$ and $q(p)$ can be determined separately for each p. Noting that $D(\underline{p}^q) = 0$, constraint (5) together with (43) shows that an optimal quarantining schedule is given by[18]

$$q^{**}(p) = \begin{cases} 1 & \text{if } p \geq \underline{p}^q, \\ 0 & \text{otherwise.} \end{cases} \tag{44}$$

[17] Pancs (2024) considers such a setting. In contrast to us, he assumes that each individual's utility is quasilinear with respect to money and describes a Vickrey–Clarke–Groves auction that allocates vaccine while taking into account vaccination externalities.

[18] It is possible that $\underline{p}^q < \underline{p}$, in which case everybody should be quarantined, or $\underline{p}^q \geq \overline{p}$, in which case nobody should be quarantined.

Using this schedule, the welfare can be expressed as a function of the testing schedule m:

$$W = E_{p\sim F}\left[C(p)m(p) + \max\{0, D(p)\}(1 - m(p))\right].$$

Thus, by constraint (4), an optimal testing schedule is given by

$$m^{**}(p) = \begin{cases} 1 & \text{if } C(p) - \max\{0, D(p)\} \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

In order to achieve a more explicit form for the optimal testing schedule, we distinguish two cases. Suppose first that

$$\underline{p}^q \geq \underline{p}^t. \tag{45}$$

In this case, $D(p) \leq 0$ for all $p < \underline{p}^t$, implying $m^{**}(p) = 0$. For all $p \in [\underline{p}^t, \underline{p}^q]$, we have $C(p) > 0$ and $D(p) \leq 0$, implying $m^{**}(p) = 1$. For all $p > \underline{p}^q$, it is also true that $m^{**}(p) = 1$ because

$$C(p) - D(p) > C(\underline{p}^q) - D(\underline{p}^q) = C(\underline{p}^q) \geq C(\underline{p}^t) = 0,$$

where the first inequality follows from the fact that C is steeper than D.

Summarizing the insights so far, we have seen that, if condition (45) holds, then

$$m^{**}(p) = \begin{cases} 1 & \text{if } p \geq \underline{p}^t, \\ 0 & \text{otherwise.} \end{cases}$$

Note that, under condition (45), the optimal quarantining required by (44) never comes to play: due to (45), all types that are optimally quarantined if they are not tested are tested anyway.

Secondly, consider the case in which (45) does not hold. Here, $C(p) < 0$ for all $p \leq \underline{p}^q$, implying $m^{**}(p) = 0$. For all $p > \underline{p}^q$, we have $D(p) > 0$, implying

$$C(p) - \max\{0, D(p)\} = C(p) - D(p).$$

Note that the definition of $\underline{p}^{qt}$ implies

$$C(\underline{p}^{qt}) - D(\underline{p}^{qt}) = 0.$$

Because $C(\underline{p}^q) - D(\underline{p}^q) = C(\underline{p}^q) < 0$ and C is steeper than D, we have

$$\underline{p}^{qt} > \underline{p}^q,$$

and thus

$$m^{**}(p) = \begin{cases} 1 & \text{if } p \geq \underline{p}^{qt}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, in the case where (45) does not hold, the range of types that are tested is smaller than the range of types that are quarantined. In other words, there is a range of intermediate types that are quarantined right away, without a test being applied; only high types are tested. □

Proof of Lemma 3 Because $A^\lambda(p)$ is strictly increasing in λ if $p > p^*$ and is independent of λ if $p \leq p^*$, the expression $\min_p A^\lambda(p)$ is weakly increasing in λ, showing that α^λ is weakly decreasing in λ.

To show that $\alpha \mapsto \alpha^\lambda$ is Lipschitz continuous, it remains to verify that there exists a number $\overline{L} > 0$ such that, for all $\lambda_2 > \lambda_1$,

$$\alpha^{\lambda_2} - \alpha^{\lambda_1} \geq -\overline{L}(\lambda_2 - \lambda_1). \tag{46}$$

To see this, let p_1 denote a minimizer of A^{λ_1}. Then $\min_p A^{\lambda_2}(p) \leq A^{\lambda_2}(p_1)$, implying

$$\begin{aligned}\alpha^{\lambda_2} - \alpha^{\lambda_1} &\geq -A^{\lambda_2}(p_1) + A^{\lambda_1}(p_1) - (\lambda_2 - \lambda_1)c^q \\ &= -\mathbf{1}_{p_1 > p^*} \cdot v(p_1)(\lambda_2 - \lambda_1) - c^q(\lambda_2 - \lambda_1),\end{aligned}$$

so that Lipschitz continuity is satisfied with $\overline{L} = v(\overline{p}) + c^q$.

By Lipschitz continuity, the derivative $d\alpha^\lambda/d\lambda$ exists almost everywhere. Using the envelope theorem (Milgrom and Segal 2002), and letting p^λ denote a minimizer of A^λ,

$$\frac{d\alpha^\lambda}{d\lambda} = -\frac{d}{d\lambda}\min_p A^\lambda - c^q = -\mathbf{1}_{p^\lambda > p^*} \cdot v(p^\lambda) - c^q,$$

from which the inequalities stated in the lemma are immediate.

Note that $\alpha^0 \geq 0$ from (13).

If we choose λ larger than $\lambda^* - \frac{1}{b}\min_{p>p^*} dB/dp$, then A^λ is strictly increasing on the interval $(p^*, \overline{p}]$, showing that any minimizer of A^λ belongs to the interval $[\underline{p}, p^*]$. For all p in this interval, we have $A^\lambda(p) = B(p)$. Thus, for all sufficiently large λ,

$$\alpha^\lambda = -\min_{p \leq p^*} B(p) + A^0(\overline{p}) - \lambda c^q,$$

showing that $\alpha^\lambda < 0$ if λ is sufficiently large. □

Proof of Proposition 5 Suppose that $\lambda^* \geq 0$ and $\alpha^{\lambda^*} \leq 0$. By Lemma 3, there exists $\check{\lambda} \leq \lambda^*$ such that $\alpha^{\check{\lambda}} = 0$. Thus, Proposition 4 implies that no-testing-always-quarantining solves the government's problem.

Now suppose that $\lambda^* < 0$, or $\lambda^* \geq 0$ and $\alpha^{\lambda^*} > 0$. By Lemma 3, there exists $\check{\lambda} \geq \max\{0, \lambda^*\}$ such that $\alpha^{\check{\lambda}} = 0$. Thus, Proposition 4 implies that the government's problem has a solution with a threshold $\check{p}$. Choose $\check{p}$ minimal if multiple solutions exist.

Note that, for all $\lambda \geq 0$ and all $p \leq p^*$, $A^\lambda(p) = B(p)$. Thus,

$$\min_{p \leq p^*} A^\lambda(p) = \underline{B}.$$

First consider the case $\underline{B} \leq \underline{A}^{\overline{\lambda}}$. It is sufficient to show that $\check{p} \leq p^*$. Note that

$$\underline{B} = \min_p A^{\overline{\lambda}}(p) \leq A^{\overline{\lambda}}(\overline{p}).$$

This implies $\overline{l} \geq 0$ because otherwise we would have $\overline{\lambda} = 0$, implying $\underline{B} \leq A^0(\overline{p})$ by the inequality above, implying $\overline{l} \geq 0$ by the definition of $\overline{l}$.

Thus, $\overline{\lambda} = \overline{l}$.

Suppose that $\check{\lambda} < \overline{\lambda}$. Then $\underline{A}^{\check{\lambda}} \leq \underline{A}^{\overline{\lambda}}$, implying

$$\begin{aligned} \alpha^{\check{\lambda}} &= -\min\{\underline{A}^{\check{\lambda}}, \underline{B}\} + A^0(\overline{p}) - \check{\lambda} c^q > -\min\{\underline{A}^{\overline{\lambda}}, \underline{B}\} + A^0(\overline{p}) - \overline{\lambda} c^q \\ &= -\underline{B} + A^0(\overline{p}) - \overline{l} c^q = 0, \end{aligned}$$

contradicting the definition in Proposition 4.

Thus, $\check{\lambda} \geq \overline{\lambda}$. In the case $\underline{B} < \underline{A}^{\overline{\lambda}}$, we cannot have a solution with $\check{p} > p^*$ because this would imply

$$A^{\check{\lambda}}(\check{p}) \geq A^{\overline{\lambda}}(\check{p}) \geq \underline{A}^{\overline{\lambda}} > \underline{B},$$

contradicting the fact that $\check{p}$ minimizes $A^{\check{\lambda}}$ on the interval $[\underline{p}, \overline{p}]$.

Similarly, in the case $\underline{B} = \underline{A}^{\overline{\lambda}}$ and $\check{\lambda} > \overline{\lambda}$, we cannot have a solution with $\check{p} > p^*$ because this would imply

$$A^{\check{\lambda}}(\check{p}) > A^{\overline{\lambda}}(\check{p}) \geq \underline{A}^{\overline{\lambda}} = \underline{B},$$

again contradicting the fact that $\check{p}$ minimizes $A^{\check{\lambda}}$ on the interval $[\underline{p}, \overline{p}]$.

In the case $\underline{B} = \underline{A}^{\overline{\lambda}}$ and $\check{\lambda} = \overline{\lambda}$, the function $A^{\check{\lambda}}$ has a minimizer that is $\leq p^*$, showing that $\check{p} \leq p^*$, as claimed.

Now consider the case $\underline{B} > \underline{A}^{\overline{\lambda}}$. This implies

$$\min\{\underline{A}^{\overline{\lambda}}, \underline{B}\} < \underline{B}.$$

Suppose first that $\overline{l} \geq 0$. Then $\overline{\lambda} = \overline{l}$, implying

$$\alpha^{\overline{\lambda}} = -\min\{\underline{A}^{\overline{\lambda}}, \underline{B}\} + A^0(\overline{p}) - \overline{\lambda} c^q > -\underline{B} + A^0(\overline{p}) - \overline{l} c^q = 0,$$

Thus, $\check{\lambda} > \overline{\lambda}$ because α^λ is decreasing.

Suppose that $\check{p} \leq p^*$. This would imply $\underline{B} \leq \underline{A}^{\check{\lambda}}$, thus

$$\alpha^{\check{\lambda}} = -\underline{B} + A^0(\overline{p}) - \check{\lambda} c^q < -\underline{B} + A^0(\overline{p}) - \bar{l} c^q = 0,$$

contradicting the definition of $\check{\lambda}$.

Finally, consider the case $\bar{l} < 0$, that is, $A^0(\overline{p}) - \underline{B} < 0$. Suppose that $\check{p} \leq p^*$. This would imply $\underline{B} \leq \underline{A}^{\check{\lambda}}$, thus

$$\alpha^{\check{\lambda}} = -\underline{B} + A^0(\overline{p}) - \check{\lambda} c^q \leq -\underline{B} + A^0(\overline{p}) < 0,$$

contradicting the definition of $\check{\lambda}$. □

Proof of Corollary 7 The condition $c^q \geq b^q E_{p' \sim F}[p'] - c^{gq}$ is equivalent to the condition $\lambda^* \leq 0$.

We indicate the dependence of A^λ on c^{gt} by using the notation $A^\lambda_{c^{gt}}$. Similarly, we will use the notation $\alpha^\lambda_{c^{gt}}$. For any c^{gt}, let $\check{\lambda}(c^{gt})$ denote the unique point λ where $\alpha^\lambda_{c^{gt}} = 0$ (cf. Lemma 3). For any $\lambda \geq 0$ and any $c^{gt} > 0$, let $p^\lambda_{c^{gt}}$ denote the smallest minimizer of $A^\lambda_{c^{gt}}(p)$; the proof will be identical if we select the largest minimizer for all λ and all c^{gt}.

By Proposition 4, if $\check{\lambda}(c^{gt}) < \lambda^*$, then no-testing-always-quarantining is optimal; otherwise, $\check{p} = p^{\check{\lambda}(c^{gt})}_{c^{gt}}$ is the optimal threshold type at the cost c^{gt}. Thus, it is sufficient to prove the following four claims:

$$\text{the function } c^{gt} \mapsto \check{\lambda}(c^{gt}) \text{ is weakly decreasing;} \tag{47}$$

$$\lim_{c^{gt} \to \infty} \check{\lambda}(c^{gt}) = 0. \tag{48}$$

$$\text{the function } c^{gt} \mapsto p^{\check{\lambda}(c^{gt})}_{c^{gt}} \text{ is weakly increasing.} \tag{49}$$

$$\lim_{c^{gt} \to \infty} p^{\check{\lambda}(c^{gt})}_{c^{gt}} = \overline{p}, \tag{50}$$

Recalling the definition (12), the envelope theorem (Milgrom and Segal 2002) yields that the function $c^{gt} \mapsto \min_p A^\lambda_{c^{gt}}(p)$ is Lipschitz continuous and its derivative is, for Lebesgue-almost every c^{gt}, given by

$$\frac{d}{dc^{gt}} \min_p A^\lambda_{c^{gt}}(p) = \frac{\partial A^\lambda_{c^{gt}}}{\partial c^{gt}}(p^\lambda_{c^{gt}}) = -F(p^\lambda_{c^{gt}}).$$

Similarly,

$$\frac{\partial}{\partial c^{gt}} A^0_{c^{gt}}(\overline{p}) = -1.$$

Thus, using (13),

$$\frac{\partial}{\partial c^{gt}} \alpha^\lambda_{c^{gt}} = F(p^\lambda_{c^{gt}}) - 1. \tag{51}$$

By (51), $\partial\alpha^{\lambda}_{c^{gt}}/\partial c^{gt} \leq 0$. Together with the fact that $\alpha^{\lambda}_{c^{gt}}$ is strictly decreasing in λ (cf. Lemma 3), this implies (47). Next we show that the function $c^{gt} \mapsto \check{\lambda}(c^{gt})$ is Lipschitz continuous, implying that its derivative exists almost everywhere.

Consider any two cost levels $c_1^{gt} < c_2^{gt}$. Then

$$\begin{aligned}
0 &= \alpha_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})} - \alpha_{c_1^{gt}}^{\check{\lambda}(c_1^{gt})} \\
&= \alpha_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})} - \alpha_{c_1^{gt}}^{\check{\lambda}(c_2^{gt})} - \left(\alpha_{c_1^{gt}}^{\check{\lambda}(c_1^{gt})} - \alpha_{c_1^{gt}}^{\check{\lambda}(c_2^{gt})}\right) \\
&= \int_{c_1^{gt}}^{c_2^{gt}} \frac{\partial}{\partial c^{gt}} \alpha_{c^{gt}}^{\check{\lambda}(c_2^{gt})} dc^{gt} - \int_{\check{\lambda}(c_2^{gt})}^{\check{\lambda}(c_1^{gt})} \frac{\partial \alpha^{\lambda}_{c_1^{gt}}}{\partial \lambda} d\lambda.
\end{aligned} \tag{52}$$

Thus, using (51) and the estimate $-d\alpha^{\lambda}/d\lambda \geq c^q$ from Lemma 3,

$$0 \geq (-1)(c_2^{gt} - c_1^{gt}) + \left(\check{\lambda}(c_1^{gt}) - \check{\lambda}(c_2^{gt})\right) c^q,$$

implying that

$$\check{\lambda}(c_1^{gt}) - \check{\lambda}(c_2^{gt}) \leq \frac{1}{c^q}(c_2^{gt} - c_1^{gt}).$$

This completes the proof that the function $c^{gt} \mapsto \check{\lambda}(c^{gt})$ is Lipschitz continuous. Because the function is also weakly decreasing,

$$\check{\lambda}'(c^{gt}) \leq 0 \quad \text{for Lebesgue-almost every } c^{gt}.$$

Using (12), for all p,

$$\frac{d}{dc^{gt}} A_{c^{gt}}^{\check{\lambda}(c^{gt})}(p) = -F(p) + \mathbf{1}_{p>p^*} v(p) \check{\lambda}'(c^{gt}).$$

Thus, for all p_1, p_2 with $p_2 > p_1$, and all c_1^{gt}, c_2^{gt} with $c_2^{gt} > c_1^{gt}$,

$$\begin{aligned}
& A_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})}(p_2) - A_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})}(p_1) - \left(A_{c_1^{gt}}^{\check{\lambda}(c_1^{gt})}(p_2) - A_{c_1^{gt}}^{\check{\lambda}(c_1^{gt})}(p_1)\right) \\
&= A_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})}(p_2) - A_{c_1^{gt}}^{\check{\lambda}(c_1^{gt})}(p_2) - \left(A_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})}(p_1) - A_{c_1^{gt}}^{\check{\lambda}(c_1^{gt})}(p_1)\right) \\
&= -\underbrace{(F(p_2) - F(p_1))}_{>0} \underbrace{(c_2^{gt} - c_1^{gt})}_{>0} + \underbrace{\left(\mathbf{1}_{p_2>p^*} v(p_2) - \mathbf{1}_{p_1>p^*} v(p_1)\right)}_{\geq 0} \\
&\quad \underbrace{\left(\check{\lambda}(c_2^{gt}) - \check{\lambda}(c_1^{gt})\right)}_{\leq 0} \\
&< 0.
\end{aligned} \tag{53}$$

Recall that

$$p_{c_1^{gt}}^{\check{\lambda}(c_1^{gt})} \in \arg\min_p A_{c_1^{gt}}^{\check{\lambda}(c_1^{gt})}(p)$$

and

$$p_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})} \in \arg\min_p A_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})}(p).$$

Thus, for all $p < p_{c_1^{gt}}^{\check{\lambda}(c_1^{gt})}$,

$$A_{c_1^{gt}}^{\check{\lambda}(c_1^{gt})}(p_{c_1^{gt}}^{\check{\lambda}(c_1^{gt})}) - A_{c_1^{gt}}^{\check{\lambda}(c_1^{gt})}(p) \leq 0.$$

Applying (53) with $p_2 = p_{c_1^{gt}}^{\check{\lambda}(c_1^{gt})}$ and $p_1 = p$, we conclude that

$$A_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})}(p_{c_1^{gt}}^{\check{\lambda}(c_1^{gt})}) - A_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})}(p) < 0.$$

Thus,

$$p \notin \arg\min_p A_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})}(p),$$

implying that[19]

$$p_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})} \geq p_{c_1^{gt}}^{\check{\lambda}(c_1^{gt})},$$

hence (49) follows.

An analogous argument shows that, for all $\lambda \geq 0$, the marginal-type function $c^{gt} \mapsto p_{c^{gt}}^{\lambda}$ is weakly increasing, implying that $p_{c^{gt}}^{\check{\lambda}(c_2^{gt})} \leq p_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})}$ for all $c^{gt} \leq c_2^{gt}$.

Thus, for any two cost levels $c_1^{gt} < c_2^{gt}$, (51) implies that

$$\begin{aligned}\int_{c_1^{gt}}^{c_2^{gt}} \frac{\partial}{\partial c^{gt}} \alpha_{c^{gt}}^{\check{\lambda}(c_2^{gt})} dc^{gt} &= - \int_{c_1^{gt}}^{c_2^{gt}} \left(1 - F\left(p_{c^{gt}}^{\check{\lambda}(c_2^{gt})}\right)\right) dc^{gt} \\ &\leq -(c_2^{gt} - c_1^{gt}) \left(1 - F\left(p_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})}\right)\right).\end{aligned}$$

[19] For a general background of this type of monotone-comparative-statics argument, see Milgrom and Shannon (1994)).

On the other hand, the estimate $-d\alpha^{\lambda}/d\lambda \leq c^q + v(\overline{p})$ from Lemma 3 implies that

$$-\int_{\check{\lambda}(c_2^{gt})}^{\check{\lambda}(c_1^{gt})} \frac{\partial \alpha_{c_1^{gt}}^{\lambda}}{\partial \lambda} d\lambda \leq \left(\check{\lambda}(c_1^{gt}) - \check{\lambda}(c_2^{gt})\right)\left(c^q + v(\overline{p})\right).$$

In summary, (52) implies that

$$0 \leq -(c_2^{gt} - c_1^{gt})\left(1 - F\left(p_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})}\right)\right) + \left(\check{\lambda}(c_1^{gt}) - \check{\lambda}(c_2^{gt})\right)\left(c^q + v(\overline{p})\right).$$

Rearranging this yields the inequality

$$\frac{\check{\lambda}(c_1^{gt}) - \check{\lambda}(c_2^{gt})}{c_2^{gt} - c_1^{gt}} \geq \frac{1}{c^q + v(\overline{p})}\left(1 - F\left(p_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})}\right)\right).$$

Taking the limit $c_1^{gt} \to c_2^{gt}$ yields that

$$-\check{\lambda}'(c_2^{gt}) \geq \frac{1}{c^q + v(\overline{p})}\left(1 - F\left(p_{c_2^{gt}}^{\check{\lambda}(c_2^{gt})}\right)\right). \tag{54}$$

This implies (50) because otherwise the derivative (54) is bounded away from zero for all c^{gt}, contradicting the fact that $\check{\lambda}(c^{gt}) \geq 0$.

To show (48), suppose otherwise. Then there exists a sequence $(c_n^{gt})_{n=1,2,\ldots}$ with $c_n^{gt} \to \infty$ and a number $\epsilon > 0$ such that $\check{\lambda}(c_n^{gt}) > \epsilon$ for all n. Then (13) implies that

$$\begin{aligned}
&\limsup_n \alpha_{c_n^{gt}}^{\check{\lambda}(c_n^{gt})} \\
&\leq \limsup_n \left(-A_{c_n^{gt}}^{\check{\lambda}(c_n^{gt})}(p_{c_n^{gt}}^{\check{\lambda}(c_n^{gt})}) + A_{c_n^{gt}}^0(\overline{p})\right) - \liminf_n \check{\lambda}(c_n^{gt})c^q \\
&\leq \limsup_n \left(-A_{c_n^{gt}}^{\check{\lambda}(c_n^{gt})}(\overline{p}) + A_{c_n^{gt}}^0(\overline{p})\right) + \limsup_n \left(A_{c_n^{gt}}^{\check{\lambda}(c_n^{gt})}(\overline{p}) - A_{c_n^{gt}}^{\check{\lambda}(c_n^{gt})}(p_{c_n^{gt}}^{\check{\lambda}(c_n^{gt})})\right) \\
&\quad - \liminf_n \check{\lambda}(c_n^{gt})c^q \\
&\leq -v(\overline{p})\epsilon + 0 - \epsilon c^q \overset{(3)}{<} 0,
\end{aligned}$$

contradicting the optimality condition $\alpha_{c_n^{gt}}^{\check{\lambda}(c_n^{gt})} = 0$ from Proposition 4. □

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