Lattice endomorphisms, Seifert forms and upper triangular matrices

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Abstract

This monograph starts with an upper triangular matrix with integer entries and 1's on the diagonal. It develops from this a spectrum of structures, which appear in different contexts, in algebraic geometry, representation theory and the theory of irregular meromorphic connections. It provides general tools to study these structures, and it studies sytematically the cases of rank 2 and 3. The rank 3 cases lead already to a rich variety of phenomena and give an idea of the general landscape. Their study takes up a large part of the monograph.
Special cases are related to Coxeter groups, generalized Cartan lattices and exceptional sequences, or to isolated hypersurface singularities, their Milnor lattices and their distinguished bases. But these make only a small part of all cases. One case in rank 3 which is beyond them, is related to quantum cohomology of $\mathbb{P}^2$ and to Markov triples.
The first structure associated to the matrix is a $\mathbb{Z}$-lattice with a unimodular bilinear form (called Seifert form) and a triangular basis. It leads immediately to an even and an odd intersection form, reflections and transvections, an even and an odd monodromy group, even and odd vanishing cycles. Braid group actions lead to braid group orbits of distinguished bases and of upper triangular matrices.

Zusammenfassung

Diese Monographie beginnt mit einer oberen Dreiecksmatrix mit ganzzahligen Einträgen und Einsen auf der Diagonalen. Ausgehend davon entwickelt sie ein Spektrum von Strukturen, die in verschiedenen mathematischen Kontexten auftreten – insbesondere in der algebraischen Geometrie, der Darstellungstheorie und der Theorie der irregulären meromorphen Zusammenhänge. Sie stellt allgemeine Werkzeuge zur Untersuchung dieser Strukturen bereit und analysiert systematisch die Fälle mit den Rängen 2 und 3. Bereits die Fälle mit Rang 3 führen zu einer Fülle interessanter Phänomene und vermitteln einen Eindruck der allgemeinen Landschaft. Ihre Untersuchung nimmt einen großen Teil der Monographie ein. Einige spezielle Fälle stehen in Verbindung mit Coxeter-Gruppen, verallgemeinerten Cartan-Gittern und exzeptionellen Sequenzen oder mit isolierten Singularitäten von Hyperflächen, ihren Milnor-Gittern und ihren ausgezeichneten Basen.

Diese speziellen Fälle machen jedoch nur einen kleinen Teil der gesamten Theorie aus. Ein bemerkenswertes Beispiel im Rang 3 Fall, das über diese hinausgeht, ist mit der Quantenkohomologie von $\mathbb{P}^2$ sowie mit Markov-Tripeln verbunden. Die erste Struktur, die mit der Matrix assoziiert ist, ist ein $\mathbb{Z}$-Gitter mit einer unimodularen Bilinearform (der sogenannten Seifertform) und einer triangulären Basis. Diese Struktur führt unmittelbar zu einer geraden und einer ungeraden Schnittform, Spiegelungen und Transvektionen, einer geraden und einer ungeraden Monodromiegruppe sowie geraden und ungeraden verschwindenden Zykel. Die Wirkung der Zopfgruppe erzeugt Bahnen von ausgezeichneten Basen sowie von oberen Dreiecksmatrizen.

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Contents

CHAPTER 1

Introduction

This thesis develops many structures, starting from a single upper triangular $n \times n$ matrix S with integer entries and diagonal entries 1. The structures are introduced and are called playing characters in section 1.1.

Section 1.2 tells about the results in this thesis. The thesis provides general tools and facts. It treats the cases $n = 2$ and $n = 3$ systematically.

1.1. Playing characters

$H_{\mathbb{Z}}$ will always be a *$\mathbb{Z}$-lattice*, so a free $\mathbb{Z}$-module of some finite rank $n \in \mathbb{N} = \{1, 2, 3...\}$. Then L will always be a nondegenerate bilinear form $L : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z}$. It is called a *Seifert form*. The pair $(H_{\mathbb{Z}}, L)$ is called a *bilinear lattice*. If for some $\mathbb{Z}$-basis $\underline{e} \in M_{1\times n}(H_{\mathbb{Z}})$ of $H_{\mathbb{Z}}$ the determinant $\det L(\underline{e}^t, \underline{e})$ is 1 the pair $(H_{\mathbb{Z}}, L)$ is called a *unimodular bilinear lattice*. The notion *bilinear lattice* is from [**HK16**]. In chapter 2 we develop the following structures for bilinear lattices, following [**HK16**]. Though in this introduction and in the chapters 3–7 we restrict to unimodular bilinear lattices.

$$T_n^{uni}(\mathbb{Z}) \ := \ \{S \in M_{n\times n}(\mathbb{Z}) \,|\, S_{ij} = 0 \text{ for } i > j,\ S_{ii} = 1\}$$

denotes the set of all upper triangular matrices with integer entries and 1's on the diagonal.

Let $(H_{\mathbb{Z}}, L)$ be a unimodular bilinear lattice of rank n. A basis $\underline{e}$ of $H_{\mathbb{Z}}$ is called *triangular* if $L(\underline{e}^t, \underline{e})^t \in T_n^{uni}(\mathbb{Z})$. The transpose in the matrix is motivated by the case of isolated hypersurface singularities. The set of triangular bases is called $\mathcal{B}^{tri}$. By far not every unimodular bilinear lattice has triangular bases. But here we care only about those which have.

For fixed $n \in \mathbb{N}$ there is an obvious 1-1 correspondence between the set of isomorphism classes of unimodular bilinear lattices $(H_{\mathbb{Z}}, L, \underline{e})$ with triangular bases and the set $T_n^{uni}(\mathbb{Z})$, given by the map $(H_{\mathbb{Z}}, L, \underline{e}) \mapsto S := L(\underline{e}^t, \underline{e})^t$. To a given matrix $S \in T_n^{uni}(\mathbb{Z})$ we always associate the corresponding triple $(H_{\mathbb{Z}}, L, \underline{e})$.

Let a unimodular bilinear lattice $(H_{\mathbb{Z}}, L, \underline{e})$ with a triangular basis $\underline{e}$ be given, with matrix $S = L(\underline{e}^t, \underline{e})^t \in T_n^{uni}(\mathbb{Z})$. The following objects are associated to this triple canonically. The names are motivated by the case of isolated hypersurface singularities.

(i) A symmetric bilinear form $I^{(0)} : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z}$ and a skew-symmetric bilinear form $I^{(1)} : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z}$ with

$$\begin{aligned} I^{(0)} &= L^t + L, \quad \text{so } I^{(0)}(\underline{e}^t, \underline{e}) = S + S^t, \\ I^{(1)} &= L^t - L, \quad \text{so } I^{(1)}(\underline{e}^t, \underline{e}) = S - S^t, \end{aligned}$$

which are called *even* respectively *odd intersection form*.

(ii) An automorphism $M : H_{\mathbb{Z}} \to H_{\mathbb{Z}}$ which is defined by

$$L(Ma, b) = L(b, a), \quad \text{so } M(\underline{e}) = \underline{e} \cdot S^{-1} S^t,$$

and which is called *the monodromy*. It respects L (and $I^{(0)}$ and $I^{(1)}$) because $L(Ma, Mb) = L(Mb, a) = L(a, b)$.

(iii) Six automorphism groups

$$\begin{aligned} O^{(k)} &:= \mathrm{Aut}(H_{\mathbb{Z}}, I^{(k)}) \qquad \text{for } k \in \{0; 1\}, \\ G_{\mathbb{Z}}^M &:= \mathrm{Aut}(H_{\mathbb{Z}}, M) := \{g : H_{\mathbb{Z}} \to H_{\mathbb{Z}} \text{ automorphism } \mid gM = Mg\}, \\ G_{\mathbb{Z}}^{(k)} &:= \mathrm{Aut}(H_{\mathbb{Z}}, I^{(0)}, M) = O^{(k)} \cap G_{\mathbb{Z}}^M \qquad \text{for } k \in \{0; 1\}, \\ G_{\mathbb{Z}} &:= \mathrm{Aut}(H_{\mathbb{Z}}, L) = \mathrm{Aut}(H_{\mathbb{Z}}, L, I^{(0)}, I^{(1)}, M). \end{aligned}$$

(iv) The set of *roots*

$$R^{(0)} := \{a \in H_{\mathbb{Z}} \mid L(a, a) = 1\},$$

and the set

$$R^{(1)} := H_{\mathbb{Z}}.$$

(v) For $k \in \{0; 1\}$ and $a \in R^{(k)}$ the *reflection* (if $k = 0$) respectively *transvection* (if $k = 1$) $s_a^{(k)} \in O^{(k)}$ with

$$s_a^{(k)}(b) := b - I^{(k)}(a, b)a \quad \text{for } b \in H_{\mathbb{Z}}.$$

(vi) For $k \in \{0; 1\}$ the *even* (if $k = 0$) respectively *odd* (if $k = 1$) *monodromy group*

$$\Gamma^{(k)} := \langle s_{e_1}^{(k)}, ..., s_{e_n}^{(k)} \rangle \subset O^{(k)}.$$

(vii) For $k \in \{0; 1\}$ the set of *even* (if $k = 0$) respectively *odd* (if $k = 1$) *vanishing cycles*

$$\Delta^{(k)} := \Gamma^{(k)}\{\pm e_1, ..., \pm e_n\} \subset R^{(k)}.$$

The definitions of all these objects require only $S \in SL_n(\mathbb{Z})$ and $e_1, ..., e_n \in R^{(0)}$, not $S \in T_n^{uni}(\mathbb{Z})$. But the formula (Theorem 2.6)

$$s_{e_1}^{(k)} ... s_{e_n}^{(k)} = (-1)^{k+1} M \quad \text{for } k \in \{0; 1\}$$

depends crucially on $S \in T_n^{uni}(\mathbb{Z})$.

The even data $I^{(0)}, O^{(0)}, \Gamma^{(0)}$ and $\Delta^{(0)}$ are in many areas more important and are usually better understood than the odd data $I^{(1)}, O^{(1)}, \Gamma^{(1)}$ and $\Delta^{(1)}$. But in the area of isolated hypersurface singularities both turn up.

For $k \in \{0; 1\}$ the group $\Gamma^{(k)}$ contains all reflections/transvections $s_a^{(k)}$ with $a \in \Delta^{(k)}$. In the case of a bilinear lattice which is not unimodular this holds for $k = 0$, but not for $k = 1$ (Remark 2.9 (iii)). This is one reason why we restrict in the chapters 3–7 to unimodular bilinear lattices.

Section 3.2 gives an action of a semidirect product $\mathrm{Br}_n \ltimes \{\pm 1\}^n$ of the braid group Br_n of braids with n strings and of a sign group $\{\pm 1\}^n$ on the set $(R^{(k)})^n$ for $k \in \{0; 1\}$. It is compatible with the Hurwitz action of Br_n on $(\Gamma^{(k)})^n$ with connecting map

$$(R^{(k)})^n \to (\Gamma^{(k)})^n, \quad \underline{v} = (v_1, ..., v_n) \mapsto (s_{v_1}^{(k)}, ..., s_{v_n}^{(k)}).$$

Both actions restrict to the same action on $\mathcal{B}^{tri}$. Especially, one obtains the orbit

$$\mathcal{B}^{dist} := \mathrm{Br}_n \ltimes \{\pm 1\}^n (\underline{e}) \subset \mathcal{B}^{tri}$$

of *distinguished bases* of $H_\mathbb{Z}$. The triple $(H_\mathbb{Z}, L, \mathcal{B}^{dist})$ (up to isomorphism) is in many cases a canonical object, whereas the choice of a distinguished basis $\underline{e} \in \mathcal{B}^{dist}$ is a true choice. The question whether $\mathcal{B}^{dist} = \mathcal{B}^{tri}$ or $\mathcal{B}^{dist} \subsetneqq \mathcal{B}^{tri}$ is usually a difficult question. The subgroup

$$G_\mathbb{Z}^\mathcal{B} := \{g \in G_\mathbb{Z} \,|\, g(\mathcal{B}^{dist}) = \mathcal{B}^{dist}\}$$

is in many important cases equal to $G_\mathbb{Z}$. But if $\mathcal{B}^{dist} \subsetneqq \mathcal{B}^{tri}$, then $G_\mathbb{Z}^\mathcal{B} \subsetneqq G_\mathbb{Z}$ is possible.

The action of $\mathrm{Br}_n \ltimes \{\pm 1\}^n$ on $\mathcal{B}^{tri}$ is compatible with an action on $T_n^{uni}(\mathbb{Z})$. The orbit of S is called

$$\mathcal{S}^{dist} := \mathrm{Br}_n \ltimes \{\pm 1\}^n (S) \subset T_n^{uni}(\mathbb{Z}),$$

the matrices in it are called *distinguished matrices*. As $\{\pm 1\}^n$ is the normal subgroup in the semidirect product $\mathrm{Br}_n \ltimes \{\pm 1\}^n$, one can first divide out the action of $\{\pm 1\}^n$. One obtains actions of Br_n on $\mathcal{B}^{tri}/\{\pm 1\}^n$ and on $T_n^{uni}(\mathbb{Z})/\{\pm 1\}^n$. It will be interesting to determine the stabilizers $(\mathrm{Br}_n)_{\underline{e}/\{\pm 1\}^n}$ of $\underline{e}/\{\pm 1\}^n$ and $(\mathrm{Br}_n)_{S/\{\pm 1\}^n}$ of $S/\{\pm 1\}^n$.

1.2. Results

Section 1.1 associated to each matrix $S \in T_n^{uni}(\mathbb{Z})$ an impressive list of algebraic-combinatorial data. For a given matrix S there are many natural questions which all aim at controlling parts of these data, for example:

(i) What can one say about the $\mathbb{Z}$-lattice $(H_\mathbb{Z}, I^{(0)})$ with the even intersection form, e.g. its signature?
(ii) What can one say about the $\mathbb{Z}$-lattice $(H_\mathbb{Z}, I^{(1)})$ with the odd intersection form?
(iii) What are the eigenvalues and the Jordan block structure of the monodromy M?
(iv) How big are the groups $G_\mathbb{Z}$, $G_\mathbb{Z}^{(0)}$, $G_\mathbb{Z}^{(1)}$ and $G_\mathbb{Z}^M$?
(v) How good can one understand the even monodromy group $\Gamma^{(0)}$? Is it determined by the pair $(H_\mathbb{Z}, I^{(0)})$ alone?
(vi) How good can one understand the odd monodromy group $\Gamma^{(1)}$? Is it determined by the pair $(H_\mathbb{Z}, I^{(1)})$ alone?
(vii) Is $\Delta^{(0)} = R^{(0)}$ or $\Delta^{(0)} \subsetneqq R^{(0)}$? How explicitly can one control these two sets? How explicitly can one control $\Delta^{(1)}$?
(viii) Is there an easy description of the set $\mathcal{B}^{dist}$ of distinguished bases? Is $\mathcal{B}^{dist} = \mathcal{B}^{tri}$ or $\mathcal{B}^{dist} \subsetneqq \mathcal{B}^{tri}$?
(ix) Is $G_\mathbb{Z}^\mathcal{B} = G_\mathbb{Z}$ or $G_\mathbb{Z}^\mathcal{B} \subsetneqq G_\mathbb{Z}$?

In this thesis we concentrate on general tools and on the cases of rank 2 and rank 3. The cases of rank 2 are already interesting, but still very special. The cases of rank 3 are still in some sense small, but they show already a big variety of different types and phenomena. We consider them as sufficiently general to give an idea of the landscape for arbitrary rank $n \in \mathbb{N}$. The large number of pages of this thesis is due to the systematic study of all cases of rank 3.

Here the singularity cases form just two cases (A_3, A_2A_1), and also the cases from generalized Cartan lattices form a subset which one can roughly estimate as one third of all cases, not containing some of the most interesting cases ($\mathcal{H}_{1,2}$, $\mathbb{P}^2$).

Examples 1.1. In the following examples, some matrices in $T_2^{uni}(\mathbb{Z})$ and $T_3^{uni}(\mathbb{Z})$ are distinguished. They cover the most important cases in $T_2^{uni}(\mathbb{Z})$ and $T_3^{uni}(\mathbb{Z})$. This will be made precise in Theorem 1.2, which gives results on the braid group action on $T_3^{uni}(\mathbb{Z})$.

$S(A_1^2)$	$S(A_2)$	$S(\mathbb{P}^1)$	$S(x)$ for $x \in \mathbb{Z}$	$S(A_1^3)$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\begin{matrix} S(\mathbb{P}^2) & S(A_2A_1) & S(A_3) & S(\mathbb{P}^1A_1) \\ \begin{pmatrix}1&-3&3\\0&1&-3\\0&0&1\end{pmatrix} & \begin{pmatrix}1&-1&0\\0&1&0\\0&0&1\end{pmatrix} & \begin{pmatrix}1&-1&0\\0&1&-1\\0&0&1\end{pmatrix} & \begin{pmatrix}1&-2&0\\0&1&0\\0&0&1\end{pmatrix}\end{matrix}$$

$$\begin{matrix} & & S(-l,2,-l) & S(x_1,x_2,x_3) \\ S(\widehat{A}_2) & S(\mathcal{H}_{1,2}) & \text{for } l\geq 3 & \text{for } x_1,x_2,x_3\in\mathbb{Z} \\ \begin{pmatrix}1&-1&-1\\0&1&-1\\0&0&1\end{pmatrix} & \begin{pmatrix}1&-2&2\\0&1&-2\\0&0&1\end{pmatrix} & \begin{pmatrix}1&-l&2\\0&1&-l\\0&0&1\end{pmatrix} & \begin{pmatrix}1&x_1&x_2\\0&1&x_3\\0&0&1\end{pmatrix}\end{matrix}$$

The notations A_1^2 , A_2, A_1^3, A_2A_1, A_3 and $\widehat{A}_2$ are due the facts that $(H_\mathbb{Z}, I^{(0)})$ is in these cases the corresponding root lattice respectively in the case $\widehat{A}_2$ the affine root lattice of type $\widehat{A}_2$. The notations $\mathbb{P}^1$ and $\mathbb{P}^2$ come from the quantum cohomology $\mathbb{P}^1$ and $\mathbb{P}^2$, so from algebraic geometry. The notation $\mathcal{H}_{1,2}$ is related to a Hurwitz space, so it also comes from algebraic geometry.

A large part of this thesis is devoted to answering the questions above for the cases of rank 2 and 3. Though the chapters 2, 3 and the sections 5.1, 6.1 and 7.1 offer also a lot of background material and tools. In the following, we present some key results from the chapters 4 to 7.

The action of $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ on $T_3^{uni}(\mathbb{Z})$ boils down to an action of $PSL_2(\mathbb{Z}) \ltimes G^{sign}$ on $T_3^{uni}(\mathbb{Z})$ where $G^{sign} \cong \{\pm 1\}^2$ comes from the action of the sign group $\{\pm 1\}^3$. As the action of $PSL_2(\mathbb{Z})$ is partially nonlinear, it is good to write it as a semidirect product $PSL_2(\mathbb{Z}) \cong G^{phi} \rtimes \langle\gamma\rangle$ where γ acts cyclically and linearly of order 3 and G^{phi} is a free Coxeter group with 3 generators which act nonlinearly.

The sections 4.2–4.4 analyze the action on $T_3^{uni}(\mathbb{Z})$ carefully. The first result Theorem 4.6 builds on coarser classifications of Krüger [**Kr90**, §12] and Cecotti-Vafa [**CV93**, Ch. 6.2]. The following theorem gives a part of Theorem 4.6.

THEOREM 1.2. *(Part of Theorem 4.6)*

(a) The characteristic polynomial of $S^{-1}S^t$ and of the monodromy M of $(H_\mathbb{Z}, L, \underline{e})$ for $S = S(\underline{x}) \in T_3^{uni}(\mathbb{Z})$ with $\underline{x} \in \mathbb{Z}^3$ is

$$\begin{aligned} p_{ch,M} &= (t-1)(t^2-(2-r(\underline{x}))t+1), \\ \text{where} \quad & r:\mathbb{Z}^3\to\mathbb{Z}, \quad \underline{x}\mapsto x_1^2+x_2^2+x_3^2-x_1x_2x_3. \end{aligned}$$

The characteristic polynomial and $r(\underline{x})$ are invariants of the $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ *orbit of $S(\underline{x})$. All eigenvalues of $p_{ch,M}$ are unit roots if and only if $r(\underline{x}) \in \{0,1,2,3,4\}$.*

(b) For $\rho \in \mathbb{Z} - \{4\}$ the fiber $r^{-1}(\rho) \subset \mathbb{Z}^3$ consists only of finitely many $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbits. The following table gives the symbols in Example 1.1 for the fibers over $r \in \{0,1,2,3,4\}$, so there are only seven orbits plus one series of orbits over $r \in \{0,1,2,3,4\}$.

$r(\underline{x})$	0	1	2	3	4
	$A_1^3, \mathbb{P}^2$	A_2A_1	A_3	$-$	$\mathbb{P}^1A_1, \widehat{A}_2, \mathcal{H}_{1,2}, S(-l,2,-l)$ with $l \geq 3$

With the help of certain (beautiful) graphs, in Theorem 4.13 the stabilizers $(\mathrm{Br}_3)_{S/\{\pm 1\}^3}$ are calculated for certain representatives of all $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$-orbits in $T_3^{uni}(\mathbb{Z})$. We work with 14 graphs and 24 sets of representatives.

Lemma 5.8 gives informations on the characteristic polynomial and the signature of $I^{(0)}$ in all rank 3 cases.

LEMMA 1.3. *(Lemma 5.8 (b))*

Consider $\underline{x} \in \mathbb{Z}^3$ with $r = r(\underline{x}) < 0$ or > 4 or with $S(\underline{x})$ one of the cases in the table in Theorem 1.2. Then $p_{ch,M} = (t-\lambda_1)(t-\lambda_2)\Phi_1$ and $\mathrm{sign}\, I^{(0)}$ *are as follows (Φ_m = the cyclotomic polynomial of m-th primitive unit roots).*

$r(\underline{x})$	$p_{ch,M}$	$\mathrm{sign}\, I^{(0)}$	$S(\underline{x})$
$r<0$	$\lambda_1,\lambda_2>0$	$(+--)$	$S(\underline{x})$
$r=0$	Φ_1^3	$(+++)$	$S(A_1^3)$
$r=0$	Φ_1^3	$(+--)$	$S(\mathbb{P}^2)$
$r=1$	$\Phi_6\Phi_1$	$(+++)$	$S(A_2A_1)$
$r=2$	$\Phi_4\Phi_1$	$(+++)$	$S(A_3)$
$r=4$	$\Phi_2^2\Phi_1$	$(++\ 0)$	$S(\mathbb{P}^1A_1)$
$r=4$	$\Phi_2^2\Phi_1$	$(++\ 0)$	$S(\widehat{A}_2)$
$r=4$	$\Phi_2^2\Phi_1$	$(+\ 0\ 0)$	$S(\mathcal{H}_{1,2})$
$r=4$	$\Phi_2^2\Phi_1$	$(+\ 0\ -)$	$S(-l,2,-l)$ with $l \geq 3$
$r>4$	$\lambda_1,\lambda_2<0$	$(++-)$	$S(\underline{x})$

Chapter 5 analyzes the groups $G_\mathbb{Z}, G_\mathbb{Z}^{(0)}, G_\mathbb{Z}^{(1)}$ and $G_\mathbb{Z}^M$ in all rank 3 cases. This leads into an intricate case discussion. The case $\mathcal{H}_{1,2}$ is different from all other cases as it is the only case where $G_\mathbb{Z}$ is not abelian and where the subgroup $\{\pm M^m \mid m \in \mathbb{Z}\}$ does not have finite index in $G_\mathbb{Z}$. The automorphism $Q \in G_\mathbb{Q} := \mathrm{Aut}(H_\mathbb{Q}, L)$ is defined for $r(\underline{x}) \neq 0$. It is id on $\ker(M - \mathrm{id})$ and $-\mathrm{id}$ on $\ker(M^2 - (2-r)M + \mathrm{id})$ (Definition 5.9). It is only in a few cases in $G_\mathbb{Z}$ (Theorem 5.11).

THEOREM 1.4. *(Part of the Theorems 5.11, 5.13, 5.14, 5.16, 5.18, 3.28)*

(a) In the $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbit of $S(\mathcal{H}_{1,2})$

$$G_\mathbb{Z} \cong SL_2(\mathbb{Z}) \times \{\pm 1\}, \quad M = Q,$$

and the subgroup $\{\pm M^m \,|\, m\in\mathbb{Z}\} = \{\pm \mathrm{id}, \pm Q\}$ *has infinite index in* $G_{\mathbb{Z}}$.

(b) In all other rank 3 cases the subgroup $\{\pm M^m \,|\, m\in\mathbb{Z}\}$ *has finite index in* $G_{\mathbb{Z}}$ *and* $G_{\mathbb{Z}}$ *is abelian. Then one of the five possibilities holds,*

$$G_{\mathbb{Z}} = O_3(\mathbb{Z}), \tag{1.1}$$
$$G_{\mathbb{Z}} = \{\mathrm{id}, Q\}\times\{\pm (M^{root})^m \,|\, m\in\mathbb{Z}\}, \tag{1.2}$$
$$G_{\mathbb{Z}} = \{\pm (M^{root})^m \,|\, m\in\mathbb{Z}\}, \tag{1.3}$$
$$G_{\mathbb{Z}} = \{\mathrm{id}, Q\}\times\{\pm M^m \,|\, m\in\mathbb{Z}\}, \tag{1.4}$$
$$G_{\mathbb{Z}} = \{\pm M^m \,|\, m\in\mathbb{Z}\}, \tag{1.5}$$

where M^{root} *is a root of* $\pm M$ *or of* MQ*. The following table gives the index* $[G_{\mathbb{Z}} : \{\pm M^m \,|\, m\in\mathbb{Z}\}]\in\mathbb{N}$ *and informations on* M^{root}.

	matrix	index	M^{root}
(1.1)	$S(A_1^3)$	24	
(1.2)	$S(x,0,0)$ with $x<0$	4	$(M^{root})^2 = MQ$
(1.2)	$S(-l,2,-l)$ with l even	l^2-4	$(M^{root})^{l^2/2-2} = MQ$
(1.2)	$S(4,4,4)$ and $S(5,5,5)$	6	$(M^{root})^3 = -M$
(1.2)	$S(4,4,8)$	4	$(M^{root})^2 = M$
(1.3)	$S(\mathbb{P}^2)$	3	$(M^{root})^3 = M$
(1.3)	$S(\widehat{A}_2)$ and $S(x,x,x)$ with $x\in\mathbb{Z}-\{-1,0,...,5\}$	3	$(M^{root})^3 = -M$
(1.3)	$S(-l,2,-l)$ with l odd	l^2-4	$(M^{root})^{l^2-4} = -M$
(1.3)	$S(2y,2y,2y^2)$ with $x\in\mathbb{Z}_{\geq 3}$	2	$(M^{root})^2 = M$
(1.4)	$S(3,3,4)$ and $S(x,x,0)$ with $x\in\mathbb{Z}_{\geq 2}$	2	
(1.5)	$S(A_3)$ and $S(\underline{x})$ in other $\mathrm{Br}_3\ltimes\{\pm 1\}^3$ orbits	1	

(c) $G_{\mathbb{Z}} = G_{\mathbb{Z}}^{\mathcal{B}}$ *holds for all rank 3 cases except four cases, the* $\mathrm{Br}_3\ltimes\{\pm 1\}^3$ *orbits of* $S(\underline{x})$ *with*

$$\underline{x}\in\{(3,3,4),(4,4,4),(5,5,5),(4,4,8)\}.$$

In these four cases $Q\in G_{\mathbb{Z}} - G_{\mathbb{Z}}^{\mathcal{B}}$.

Though in higher rank it is easier to construct matrices S with $G_{\mathbb{Z}}^{\mathcal{B}} \subsetneqq G_{\mathbb{Z}}$ (Remarks 3.29).

Chapter 6 studies the even and odd monodromy groups and the sets of even and of odd vanishing cycles in the rank 2 and rank 3 cases. The following theorem catches some of the results on the even monodromy group $\Gamma^{(0)}$ and the set $\Delta^{(0)}$ of even vanishing cycles for the rank 3 cases. The group $O^{(0),*}$ (Definition 6.4) is a certain subgroup of $O^{(0)}$ which is determined only by $(H_{\mathbb{Z}}, I^{(0)})$ (so independently of $\mathcal{B}$). Part (b) discusses only the (in general difficult) problem whether $\Delta^{(0)} = R^{(0)}$ or $\Delta^{(0)} \subsetneqq R^{(0)}$. Theorem 6.14 contains many more informations on $\Delta^{(0)}$. Theorem 6.11 contains many more informations on $\Gamma^{(0)}$ than part (a) below. Remarkably, $\Gamma^{(0)} \cong G^{fCox,3}$ (the free Coxeter group with three generators) holds not only for the Coxeter cases $\underline{x} \in \mathbb{Z}^3_{\leq -2}$ (which all satisfy $r(\underline{x}) > 4$), but also in all cases $\underline{x} \in \mathbb{Z}^3$ with $r(\underline{x}) < 0$ and in the case $\mathbb{P}^2$.

THEOREM 1.5. *(a) (Part of Lemma 2.11 and Theorem 6.11)*

(i) (Part of Lemma 2.11) The case A_1^n, $n \in \mathbb{N}$:

$$\Gamma^{(0)} \cong \{\pm 1\}^n, \quad \Gamma^{(1)} = \{\mathrm{id}\}, \quad \Delta^{(0)} = R^{(0)} = \Delta^{(1)} = \{\pm e_1, ..., \pm e_n\}.$$

(ii) The cases with $r(\underline{x}) > 0$ and the cases $A_3, \widehat{A}_2, A_2A_1, \mathbb{P}^1A_1$: They contain all reducible rank 3 cases except A_1^3. Then $\Gamma^{(0)}$ is a Coxeter group. If $\underline{x} \in \mathbb{Z}^3_{\leq -2}$ then $\Gamma^{(0)} \cong G^{fCox,3}$.

(iii) The cases $A_3, \widehat{A}_2, \mathcal{H}_{1,2}$: Then $\Gamma^{(0)} = O^{(0),*}$.

(iv) The cases $S(-l, 2, -l)$ with $l \geq 3$: Then $\Gamma^{(0)} \overset{1:l}{\subset} O^{(0),*}$.

(v) The cases $\mathbb{P}^2$ and $\underline{x} \in \mathbb{Z}^3$ with $r(\underline{x}) < 0$: Then $\Gamma^{(0)} \cong G^{fCox,3}$.

(b) (Part of Theorem 6.14)

(i) $\Delta^{(0)} = R^{(0)}$ holds in the following cases: $A_3, \widehat{A}_2, \mathbb{P}^2$, all $S(\underline{x})$ with $\underline{x} \in \{0, -1, -2\}$, all reducible cases.

(ii) $\Delta^{(0)} \subsetneqq R^{(0)}$ holds in the following cases: $\mathcal{H}_{1,2}$, all $S(-l, 2, -l)$ with $l \geq 3$, $S(3,3,4), S(4,4,4), S(5,5,5), S(4,4,8)$.

(iii) In the cases of the other $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbits in $T_3^{uni}(\mathbb{Z})$, we do not know whether $\Delta^{(0)} = R^{(0)}$ or $\Delta^{(0)} \subsetneqq R^{(0)}$ holds.

Let E_n denote the $n \times n$ unit matrix. Given $S \in T_n^{uni}(\mathbb{Z})$ with associated triple $(H_{\mathbb{Z}}, L, \underline{e})$, consider the matrix $\widetilde{S} := 2E_n - S \in T_n^{uni}(\mathbb{Z})$ with the associated triple $(H_{\mathbb{Z}}, \widetilde{L}, \underline{e})$. Then $\widetilde{L}$, $\widetilde{I}^{(0)}$ and $\widetilde{M}$ are far from L, $I^{(0)}$ and M, but $\widetilde{I}^{(1)} = -I^{(1)}$, $\widetilde{\Gamma}^{(1)} = \Gamma^{(1)}$ and $\widetilde{\Delta}^{(1)} = \Delta^{(1)}$ (Remarks 4.17). For example the cases A_3 and $\widehat{A}_2$ are related in this way, and also the Coxeter case $(-2, -2, -2)$ and the case $\mathcal{H}_{1,2}$ are related in this way (in both cases after an action of $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$).

This motivates in the rank 3 cases to consider the action of the bigger group $(G^{phi} \ltimes \widetilde{G}^{sign}) \rtimes \langle \gamma \rangle$ on $\mathbb{Z}^3$ where $\widetilde{G}^{sign}$ is generated by G^{sign} and the total sign change $\delta^{\mathbb{R}} : \underline{x} \mapsto -\underline{x}$. Lemma 4.18 gives representatives for all orbits of this action on $\mathbb{Z}^3$ (respectively $T_3^{uni}(\mathbb{Z})$). Still it is difficult to see for a given triple $\underline{x} \in \mathbb{Z}^3$ in which orbit it is.

We had for some time the hope that the beautiful facts on the even monodromy group $\Gamma^{(0)}$ for the Coxeter cases $\underline{x} \in \mathbb{Z}^3_{\leq 0}$ would have analoga for the odd monodromy group $\Gamma^{(1)}$, but this does not hold in general. In the case $(-2,-2,-2)$ $\Gamma^{(0)}\cong G^{fCox,3}$, but in the case $\mathcal{H}_{1,2}$ not, and in both cases together $\Gamma^{(1)}\not\cong G^{free,3}$. On the other hand $\Gamma^{(1)}\cong G^{free,3}$ for $\underline{x}\in B_1$, where $B_1\subset\mathbb{Z}^3$ is as follows.

$$\begin{aligned}
B_1 &:= (G^{phi}\ltimes \widetilde{G}^{sign})\rtimes\langle\gamma\rangle(\{\underline{x}\in\mathbb{Z}^3-\{(0,0,0)\}\,|\,r(\underline{x})\leq 0\}),\\
B_2 &:= \{\underline{x}\in\mathbb{Z}^3-\{(0,0,0)\}\,|\,S(\underline{x})\text{ is reducible}\},\\
B_3 &:= \{(0,0,0)\}.
\end{aligned}$$

Though the set B_1 is difficult to understand (see the Examples 4.20). It contains $(3,3,3)$, so the orbit of $\mathbb{P}^2$. B_2 and B_3 consist of the triples $\underline{x}$ with reducible $S(\underline{x})$, so with two or three zero entries.

Consider $\underline{x}\in\mathbb{Z}^3-B_3$. The radical $\operatorname{Rad}I^{(1)}$ has rank 1, so the quotient lattice $\overline{H_\mathbb{Z}}^{(1)}:=H_\mathbb{Z}/\operatorname{Rad}I^{(1)}$ has rank 2. Denote by $\Gamma_s^{(1)}$ the image of $\Gamma^{(1)}$ under the natural homomorphism $\Gamma^{(1)}\to\operatorname{Aut}(\overline{H_\mathbb{Z}}^{(1)})$ and by $\Gamma_u^{(1)}$ the kernel of it. There is an exact sequence

$$\{1\}\to\Gamma_u^{(1)}\to\Gamma^{(1)}\to\Gamma_s^{(1)}\to\{1\}.$$

Denote by $\overline{\Delta^{(1)}}\subset\overline{H_\mathbb{Z}}^{(1)}$ the image of $\Delta^{(1)}$ in $\overline{H_\mathbb{Z}}^{(1)}$. Often $\overline{\Delta^{(1)}}$ is easier to describe than $\Delta^{(1)}$.

The long Theorems 6.18 and 6.21 offer detailed results about $\Gamma^{(1)}$ and $\Delta^{(1)}$ for the representatives in Lemma 4.18 of the $(G^{phi}\ltimes\widetilde{G}^{sign})\rtimes\langle\gamma\rangle$ orbits in $\mathbb{Z}^3$. The next theorem gives only a rough impression.

THEOREM 1.6. *Consider $S=S(\underline{x})\in T_3^{uni}(\mathbb{Z})$ and the associated triple $(H_\mathbb{Z},L,\underline{e})$.*

(a) (Part of Theorem 6.18) Consider $\underline{x}\neq(0,0,0)$.

$$\begin{aligned}
\Gamma^{(1)}\cong G^{free,3} &\iff \underline{x}\in B_1,\\
\Gamma_u^{(1)}=\{\mathrm{id}\} &\iff \underline{x}\in B_1\cup B_2,\\
\Gamma_u^{(1)}\cong\mathbb{Z}^2 &\iff \underline{x}\in\mathbb{Z}^3-(B_1\cup B_2\cup B_3),
\end{aligned}$$

$$\begin{aligned}
\Gamma_s^{(1)} &\cong \text{one of the groups } SL_2(\mathbb{Z}),G^{free,2},G^{free,2}\times\{\pm 1\}\\
&\quad\ \text{for } \underline{x}\in\mathbb{Z}^3-(B_1\cup B_3).
\end{aligned}$$

(b) (Part of Theorem 6.21)

(i) In the cases of A_3 and $\widehat{A}_2$ $\overline{\Delta^{(1)}}=\overline{H_\mathbb{Z}}^{(1),prim}$, so $\overline{\Delta^{(1)}}$ is the set of primitive vectors in $\overline{H_\mathbb{Z}}^{(1)}$, and $\Delta^{(1)}$ is the full preimage in $H_\mathbb{Z}$ of $\overline{\Delta^{(1)}}$.

(ii) Though in many other cases $\overline{\Delta^{(1)}}\not\subset\overline{H_\mathbb{Z}}^{(1),prim}$, and $\Delta^{(1)}$ is not the full preimage in $H_\mathbb{Z}$ of $\overline{\Delta^{(1)}}$, but each fiber has infinitely many elements.

(iii) But for $\underline{x}\in B_1$ the map $\Delta^{(1)}\to\overline{\Delta^{(1)}}$ is a bijection. Especially for $\mathbb{P}^2$ $\overline{\Delta^{(1)}}$ is easy to describe (Theorem 6.21 (h)), but $\Delta^{(1)}$ not.

Chapter 7 studies the set $\mathcal{B}^{dist}=\mathrm{Br}_n\ltimes\{\pm 1\}^n(\underline{e})$ of distinguished bases for a given triple $(H_{\mathbb{Z}},L,\underline{e})$. In general, it is difficult to characterize this orbit in easy terms. We know that the inclusions in (3.3) and (3.4) hold. We are interested when they are equalities.

$$\mathcal{B}^{dist}\subset\{\underline{v}\in(\Delta^{(0)})^n\,|\,s^{(0)}_{v_1}...s^{(0)}_{v_n}=-M\}, \tag{3.3}$$
$$\mathcal{B}^{dist}\subset\{\underline{v}\in(\Delta^{(1)})^n\,|\,s^{(1)}_{v_1}...s^{(1)}_{v_n}=M\}. \tag{3.4}$$

In general, this is a difficult question. In the rank 3 cases Theorem 7.3 and Theorem 7.7 give our results for (3.3) and (3.4).

THEOREM 1.7. *Consider $S(\underline{x})\in T_3^{uni}(\mathbb{Z})$ and the associated triple $(H_{\mathbb{Z}},L,\underline{e})$.*

(a) (Part of Theorem 7.3)

(3.3) *is an equality for all cases except for $\underline{x}$ in the* $\mathrm{Br}_3\ltimes\{\pm 1\}^3$ *orbit of $\mathcal{H}_{1,2}$. There the right hand side of* (3.3) *consists of countably many* $\mathrm{Br}_3\ltimes\{\pm 1\}^3$ *orbits.*

(b) (Part of Theorem 7.7)

(i) The inclusion in (3.4) *is an equality $\iff$ $\underline{x}\in B_1\cup B_2$.*

(ii) The cases A_3, $\widehat{A}_2$, $\mathcal{H}_{1,2}$ and $S(-l,2,l)$ with $l\geq 3$ are not in B_1. But there the inclusion in (3.4) *becomes an equality if one adds on the right hand side of* (3.4) *the condition $\sum_{i=1}^n\mathbb{Z}v_i=H_{\mathbb{Z}}$.*

The last section 7.4 of chapter 7 builds on Theorem 4.16 which determined for a representative $S\in T_3^{uni}(\mathbb{Z})$ of each $\mathrm{Br}_3\ltimes\{\pm 1\}$ orbit in $T_3^{uni}(\mathbb{Z})$ the stabilizer $(Br_3)_{S/\{\pm 1\}^3}$. Theorem 7.11 determines in each of these cases the stabilizer $(\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3}$. The graphs $\mathcal{G}_1,...,\mathcal{G}_{14}$ in section 4.4 used the groups $G^{phi}\rtimes\langle\gamma\rangle$. At the end of section 7.4 different graphs, which use the group Br_3, are introduced for the orbits of matrices as well as the orbits of triangular bases. For the cases of finite orbits and for the case $\widehat{A}_2$ the graphs are given explicitly. In the case of A_3 the orbit $\mathcal{S}^{dist}/\{\pm 1\}^3$ has four elements and the orbit $\mathcal{B}^{dist}/\{\pm 1\}^3$ has 16 elements.

CHAPTER 2

Bilinear lattices and induced structures

This chapter fixes the basic notions, a bilinear lattice and its associated data, namely a Seifert form, an even and an odd intersection form, a monodromy, the roots, the triangular bases, an even and an odd monodromy group, the even and the odd vanishing cycles. The notion of a bilinear lattice and the even part of the associated data are considered in [**HK16**]. The more special case of a unimodular bilinear lattice and even and odd data are considered since long time in singularity theory [**AGV88**][**Eb01**]. In this paper we are mainly interested in unimodular bilinear lattices. Only this chapter 2 treats the general case, partially following [**HK16**].

NOTATIONS 2.1. In these notations, R will be either the ring $\mathbb{Z}$ or one of the fields $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. Later we will work mainly with $\mathbb{Z}$. If $R=\mathbb{Z}$ write $\widetilde{R}:=\mathbb{Q}$, else write $\widetilde{R}:=R$.

In the whole paper, $H_R \supsetneqq \{0\}$ is a finitely generated free R-module, so a $\mathbb{Z}$-lattice if $R=\mathbb{Z}$, and a finite dimensional R-vector space if R is $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. Its rank will usually be called $n\in\mathbb{N}=\{1,2,3,...\}$ (it is its dimension if R is $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$). If R_1 and R_2 are both in the list $\mathbb{Z},\mathbb{Q},\mathbb{R},\mathbb{C}$ and R_1 is left of R_2 and H_{R_1} is given, then $H_{R_2}:=H_{R_1}\otimes_{R_1}R_2$.

In the whole paper, $L:H_R\times H_R\to R$ will be a nondegenerate R-bilinear form. If $U\subset H_R$ is an R-submodule, then $U^\perp:=\{b\in H_R\,|\,L(U,b)=0\}$ and ${}^\perp U:=\{a\in H_R\,|\,L(a,U)=0\}$. In the case $R=\mathbb{Z}$, $U^\perp$ and ${}^\perp U$ are obviously primitive $\mathbb{Z}$-submodules of $H_\mathbb{Z}$.

In Lemma 2.2 we will start with H_R and a symmetric R-bilinear form $I^{[0]}:H_R\times H_R\to R$ or a skew-symmetric R-bilinear form $I^{[1]}:H_R\times H_R\to R$. With the square brackets in the index we distinguish them from the bilinear forms $I^{(0)}$ and $I^{(1)}$, which are induced in Definition 2.3 by a given bilinear form L. Though later they will be identified.

Suppose that $M:H_R\to H_R$ is an automorphism. Then $M_s,M_u,N:H_{\widetilde{R}}\to H_{\widetilde{R}}$ denote the semisimple part, the unipotent part and the nilpotent part of M with $M=M_sM_u=M_uM_s$ and $N=\log M_u, e^N=M_u$. Denote $H_\lambda:=\ker(M_s-\lambda\cdot\textup{id}):H_\mathbb{C}\to H_\mathbb{C}$.

For $m\in\mathbb{N}$ denote by $\Phi_m\in\mathbb{Z}[t]$ the cyclotomic polynomial whose zeros are the primitive m-th unit roots.

The following lemma is elementary and classical. We skip the proof.

LEMMA 2.2. *Let $R\in\{\mathbb{Q},\mathbb{R},\mathbb{C}\}$ and let H_R be an R-vector space of dimension $n\in\mathbb{N}$.*

(a) Let $I^{[0]}:H_R\times H_R\to R$ be a symmetric bilinear form. Consider $a\in H_R$ with $I^{[0]}(a,a)\neq 0$. The map

$$s_a^{[0]}:H_R\to H_R,\quad s_a^{[0]}(b):=b-\frac{2I^{[0]}(a,b)}{I^{[0]}(a,a)}a,$$

is a reflection*, so it is in $\operatorname{Aut}(H_r,I^{[0]})$, it fixes the codimension 1 subspace $\{b\in H_R\,|\,I^{[0]}(a,b)=0\}$ and it maps a to $-a$. Especially $(s_a^{[0]})^2=\mathrm{id}$.*

(b) Let $I^{[1]}:H_R\times H_R\to R$ be a skew-symmetric bilinear form. Consider $a\in H_R$. The map

$$s_a^{[1]}:H_R\to H_R,\quad s_a^{[1]}(b):=b-I^{[1]}(a,b)a,$$

is in $\operatorname{Aut}(H_R,I^{[1]})$ with

$$(s_a^{[1]})^{-1}(b)=b+I^{[1]}(a,b)a.$$

It is id if $a\in\operatorname{Rad}(I^{[1]})$. If $a\notin\operatorname{Rad}(I^{[1]})$ then it fixes the codimension 1 subspace $\{b\in H_R\,|\,I^{[1]}(a,b)=0\}$, and $s_a^{[1]}-\mathrm{id}$ is nilpotent with a single 2×2 Jordan block. Then it is called a transvection.

(c) Fix $k\in\{0;1\}$ and consider $I^{[k]}$ as in (a) or (b). An element $g\in\operatorname{Aut}(H_R,I^{[k]})$ and an element $a\in H_R$ with $I^{[0]}(a,a)\neq 0$ if $k=0$ satisfy

$$g\,s_a^{[k]}\,g^{-1}=s_{g(a)}^{[k]}.$$

DEFINITION 2.3. (a) [**HK16**, ch. 2] A *bilinear lattice* is a pair $(H_\mathbb{Z},L)$ with $H_\mathbb{Z}$ a $\mathbb{Z}$-lattice of some rank $n\in\mathbb{N}$ together with a nondegenerate bilinear form $L:H_\mathbb{Z}\times H_\mathbb{Z}\to\mathbb{Z}$. If $\det L(\underline{e}^t,\underline{e})=1$ for some $\mathbb{Z}$-basis $\underline{e}=(e_1,...,e_n)$ of $H_\mathbb{Z}$ then L and the pair $(H_\mathbb{Z},L)$ are called *unimodular*. The bilinear form is called *Seifert form* in this paper.

(b) A bilinear lattice induces several structures:

(i) [**HK16**, ch. 2] A symmetric bilinear form

$$I^{(0)}=L^t+L:H_\mathbb{Z}\times H_\mathbb{Z}\to\mathbb{Z},\quad\text{so } I^{(0)}(a,b)=L(b,a)+L(a,b),$$

which is called *even intersection form*.

(ii) A skew-symmetric bilinear form

$$I^{(1)} = L^t - L : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z}, \quad \text{so } I^{(1)}(a,b) = L(b,a) - L(a,b),$$

which is called *odd intersection form*.

(iii) [**HK16**, ch. 2] An automorphism $M : H_{\mathbb{Q}} \to H_{\mathbb{Q}}$ which is defined by

$$L(Ma, b) = L(b,a) \quad \text{for } a, b \in H_{\mathbb{Q}},$$

which is called *monodromy*.

(iv) Six automorphism groups

$$\begin{aligned}
O^{(k)} &:= \operatorname{Aut}(H_{\mathbb{Z}}, I^{(k)}) \qquad \text{for } k \in \{0;1\}, \\
G_{\mathbb{Z}}^M &:= \operatorname{Aut}(H_{\mathbb{Z}}, M) := \{g : H_{\mathbb{Z}} \to H_{\mathbb{Z}} \text{ automorphism } \mid gM = Mg\}, \\
G_{\mathbb{Z}}^{(k)} &:= \operatorname{Aut}(H_{\mathbb{Z}}, I^{(0)}, M) = O^{(k)} \cap G_{\mathbb{Z}}^M \qquad \text{for } k \in \{0;1\}, \\
G_{\mathbb{Z}} &:= \operatorname{Aut}(H_{\mathbb{Z}}, L).
\end{aligned}$$

(v) [**HK16**, ch. 2] The set $R^{(0)} \subset H_{\mathbb{Z}}$ of *roots*,

$$R^{(0)} := \{a \in H_{\mathbb{Z}} \mid L(a,a) > 0; \frac{L(a,b)}{L(a,a)}, \frac{L(b,a)}{L(a,a)} \in \mathbb{Z} \text{ for all } b \in H_{\mathbb{Z}}\}.$$

(vi) [**HK16**, ch. 2] The set $\mathcal{B}^{tri}$ of *triangular bases*,

$$\mathcal{B}^{tri} := \{\underline{e} = (e_1, ..., e_n) \in (R^{(0)})^n \mid \bigoplus_{i=1}^n \mathbb{Z}e_i = H_{\mathbb{Z}}, L(e_i, e_j) = 0 \text{ for } i < j\}.$$

(c) Let $n \in \mathbb{N}$ and $R \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}\}$. The sets T_n^{tri} and $T_n^{uni}(R)$ of upper triangular matrices are defined by

$$\begin{aligned}
T_n^{uni}(R) &:= \{S = (s_{ij}) \in M_{n\times n}(R) \quad \mid \quad s_{ii} = 1, s_{ij} = 0 \text{ for } i > j\}, \\
T_n^{tri} &:= \{S = (s_{ij}) \in M_{n\times n}(\mathbb{Z}) \quad \mid \quad s_{ii} \in \mathbb{N}, s_{ij} = 0 \text{ for } i > j, \\
&\qquad \frac{s_{ij}}{s_{ii}}, \frac{s_{ji}}{s_{ii}} \in \mathbb{Z} \text{ for } i \neq j\}.
\end{aligned}$$

Obviously $T_n^{uni}(\mathbb{Z}) \subset T_n^{tri}$.

Remarks 2.4. (i) There are bilinear lattices with $\mathcal{B}^{tri} = \emptyset$. We are interested only in bilinear lattices with $\mathcal{B}^{tri} \neq \emptyset$.

(ii) A triangular basis $\underline{e} \in \mathcal{B}^{tri}$ is called in [**HK16**] a *complete exceptional sequence*.

(iii) In the case $\mathcal{B}^{tri} \neq \emptyset$, [**HK16**] considers the bilinear form L^t (with $L^t(a,b) = L(b,a)$). Our choice L is motivated by singularity theory. Also the names for L, $I^{(0)}$, $I^{(1)}$ and M, namely *Seifert form, even intersection form, odd intersection form* and *monodromy* are motivated by singularity theory. The roots in $R^{(0)}$ are in [**HK16**] also called *pseudo-real roots*.

(iv) In this paper we are mainly interested in the cases of unimodular bilinear lattices with $\mathcal{B}^{tri}\neq\emptyset$. Singularity theory leads to such cases.

(v) [**HK16**] is mainly interested in the cases of *generalized Cartan lattices*. A generalized Cartan lattice is a triple $(H_\mathbb{Z},L,\underline{e})$ with $(H_\mathbb{Z},L)$ a bilinear lattice and $\underline{e}\in\mathcal{B}^{tri}$ with $L(e_i,e_j)\leq 0$ for $i>j$.

REMARKS 2.5. (i) The classification of pairs $(H_\mathbb{R},L)$ and pairs $(H_\mathbb{C},L)$ with L a nondegenerate bilinear form on $H_\mathbb{R}$ respectively $H_\mathbb{C}$ is well understood. Such a pair decomposes into an orthogonal sum of irreducible pairs. This and the classification of the irreducible pairs over $\mathbb{R}$ is carried out in [**Ne98**] and, more explicitly, in [**BH19**].

In both references it is also proved that a pair $(H_\mathbb{R},L)$ of rank $n\in\mathbb{N}$ up to isomorphism is uniquely determined by an unordered tuple of n spectral pairs modulo $2\mathbb{Z}$, i.e. by n pairs $([\alpha_1],l_1),...,([\alpha_n],l_n)\in\mathbb{R}/2\mathbb{Z}\times\mathbb{Z}$. Here $\alpha_1,...,\alpha_n\in\mathbb{R}$. The eigenvalues of the monodromy M are the numbers $e^{-2\pi i\alpha_1},...,e^{-2\pi i\alpha_n}$. The numbers $l_1,...,l_n$ determine the Jordan block structure, see [**BH19**] for details.

The classification over $\mathbb{C}$ follows easily. Though it was carried out before in [**Go94-1**][**Go94-2**], and it is formulated also in [**CDG24**, Theorem 4.22].

(ii) A unimodular bilinear lattice $(H_\mathbb{Z},L)$ is called in [**CDG24**] a *Mukai pair*. In [**CDG24**, 4.1–4.4] basic results of Gorodentsev for $R=\mathbb{Z}$ or $R=\mathbb{C}$ are rewritten. The monodromy is there called *canonical operator*. A triangular basis is there called *exceptional*.

(iii) The classification over $\mathbb{Z}$, so of unimodular bilinear lattices $(H_\mathbb{Z},L)$, is wide open for larger n. The case $n=3$ is treated in great detail in this thesis.

LEMMA 2.6. *(a) Let $(H_\mathbb{Z},L)$ be a bilinear lattice of rank $n\in\mathbb{N}$.*

(i) Let $\underline{e}=(e_1,...,e_n)$ be a $\mathbb{Z}$-basis of $H_\mathbb{Z}$. Define $S:=L^t(\underline{e}^t,\underline{e})=L(\underline{e}^t,\underline{e})^t\in M_{n\times n}(\mathbb{Z})\cap GL_n(\mathbb{Q})$. Then

$$I^{(0)}(\underline{e}^t,\underline{e})=S+S^t,\ \ I^{(1)}(\underline{e}^t,\underline{e})=S-S^t,\ \ M(\underline{e})=\underline{e}S^{-1}S^t.$$

(ii)

$$I^{(0)}(a,b)=L((M+\mathrm{id})a,b),\quad \mathrm{Rad}\,I^{(0)}=\ker((M+\mathrm{id}):H_\mathbb{Z}\to H_\mathbb{Z}),$$
$$I^{(1)}(a,b)=L((M-\mathrm{id})a,b),\quad \mathrm{Rad}\,I^{(1)}=\ker((M-\mathrm{id}):H_\mathbb{Z}\to H_\mathbb{Z}).$$

(iii)

$$G_\mathbb{Z}=\mathrm{Aut}(H_\mathbb{Z},L,I^{(0)},I^{(1)},M)\subset\left\{\begin{array}{c}G_\mathbb{Z}^{(0)}\\ G_\mathbb{Z}^{(1)}\end{array}\right\}\subset G_\mathbb{Z}^M.$$

(iv) $M \in G_{\mathbb{Z}}$ *if* $(H_{\mathbb{Z}}, L)$ *is unimodular or if* $\mathcal{B}^{tri} \neq \emptyset$.

(v) If $a \in R^{(0)}$ *then*

$$s_a^{(0)} := s_a^{[0]} \text{ and } s_a^{(1)} := s^{[1]}_{a/\sqrt{L(a,a)}}, \quad \text{so } s_a^{(1)}(b) = b - \frac{I^{(1)}(a,b)}{L(a,a)}a,$$

are in $O^{(0)}$ *respectively* $O^{(1)}$.

(vi) [**HK16**, Lemma 2.1] *If* $a, b \in R^{(0)}$ *then* $s_a^{(0)}(b) \in R^{(0)}$ *(but not necessarily* $s_a^{(1)}(b) \in R^{(0)}$*).*

(vii) If $a, b \in R^{(0)}$ *with* $L(a,b) = 0$ *then*

$$L(s_a^{(1)}b, s_a^{(1)}b) = L(b,b), \quad s_a^{(1)}(b) \in R^{(0)}, \quad s^{(1)}_{s_a^{(1)}(b)} = s_a^{(1)} s_b^{(1)} (s_a^{(1)})^{-1}.$$

(b) The map

$$\{(H_{\mathbb{Z}}, L, \underline{e}) \,|\, \begin{array}{l} (H_{\mathbb{Z}}, L) \text{ is a bilinear} \\ \text{lattice of rank } n, \ \underline{e} \in \mathcal{B}^{tri} \end{array} \}/\text{isomorphism} \to T_n^{tri}$$

is a bijection and restricts to a bijection

$$\{(H_{\mathbb{Z}}, L, \underline{e}) \,|\, \begin{array}{l} (H_{\mathbb{Z}}, L) \text{ is a unimodular} \\ \text{bilinear lattice of rank } n, \ \underline{e} \in \mathcal{B}^{tri} \end{array} \}/\text{isom.} \to T_n^{uni}(\mathbb{Z}).$$

(c) [**HK16**, Lemma 3.10] *Let* $(H_{\mathbb{Z}}, L)$ *be a unimodular bilinear lattice with* $\mathcal{B}^{tri} \neq \emptyset$. *Then*

$$R^{(0)} = \{a \in H_{\mathbb{Z}} \,|\, L(a,a) = 1\}.$$

(d) Let $(H_{\mathbb{Z}}, L)$ *be a unimodular bilinear lattice with* $\mathcal{B}^{tri} \neq \emptyset$. *Define for* $a \in H_{\mathbb{Z}}$ $s_a^{(1)} := s_a^{[1]} \in O^{(1)}$. *This definition is compatible with the definition of* $s_a^{(1)}$ *for* $a \in R^{(0)}$ *in part (a) (v). Furthermore now for* $a, b \in H_{\mathbb{Z}}$

$$s^{(1)}_{s_a^{(1)}(b)} = s_a^{(1)} s_b^{(1)} (s_a^{(1)})^{-1}.$$

Proof: (a) (i) The defining equation for M can be written as $L((M\underline{e})^t, \underline{e}) = L(\underline{e}, \underline{e})^t$, which implies $M\underline{e} = \underline{e}S^{-1}S^t$. The rest is trivial.

(ii) Trivial.

(iii) $g \in G_{\mathbb{Z}}$ commutes with M because

$$L(gMa, gb) = L(Ma, b) = L(b, a) = L(gb, ga) = L(Mga, gb).$$

Of course it respects $I^{(0)}$ and $I^{(1)}$. Therefore $G_{\mathbb{Z}} = \text{Aut}(H_{\mathbb{Z}}, L, I^{(0)}, I^{(1)}, M)$. The rest is trivial.

(iv) The calculation $L(Ma, Mb) = L(Mb, a) = L(a, b)$ shows that M respects L. It remains to show $M \in \text{Aut}(H_{\mathbb{Z}})$.

This is clear if $(H_\mathbb{Z},L)$ is unimodular. Suppose $\mathcal{B}^{tri}\neq\emptyset$ and $(H_\mathbb{Z},L)$ not unimodular. Consider $\underline{e}\in\mathcal{B}^{tri}$, $S:=L(\underline{e}^t,\underline{e})^t\in T_n^{tri}$ and $D:=\textup{diag}(s_{11},...,s_{nn})\in M_{n\times n}(\mathbb{Z})$. Then $D^{-1}S, SD^{-1}\in T_n^{uni}(\mathbb{Z})$ and

$$S^{-1}S^t=S^{-1}DD^{-1}S^t=(D^{-1}S)^{-1}(SD^{-1})^t\in GL_n(\mathbb{Z}),$$

so $M\in\textup{Aut}(H_\mathbb{Z})$.

(v) If $a\in R^{(0)}$ and $b\in H_\mathbb{Z}$ then $\frac{L(a,b)}{L(a,a)},\frac{L(b,a)}{L(a,a)}\in\mathbb{Z}$, so

$$\frac{2I^{(0)}(a,b)}{I^{(0)}(a,a)},\frac{I^{(1)}(a,b)}{L(a,a)}\in\mathbb{Z}\text{ and }s_a^{(0)}(b),s_a^{(1)}(b),(s_a^{(1)})^{-1}(b)\in H_\mathbb{Z}.$$

(vi) $L(s_a^{(0)}(b),s_a^{(0)}(b))=L(b,b)$ because $s_a^{(0)}\in G_\mathbb{Z}^{(0)}$ and $I^{(0)}=L^t+L$ (in general $L(s_a^{(1)}(b),s_a^{(1)}(b))\neq L(b,b)$). For $c\in H_\mathbb{Z}$

$$\begin{aligned}\frac{L(s_a^{(0)}(b),c)}{L(s_a^{(0)}(b),s_a^{(0)}(b))} &= \frac{L(b-\frac{L(a,b)+L(b,a)}{L(a,a)}a,c)}{L(b,b)}\\ &= \frac{L(b,c)}{L(b,b)}-\frac{L(a,b)+L(b,a)}{L(b,b)}\frac{L(a,c)}{L(a,a)}\in\mathbb{Z},\end{aligned}$$

and analogously $\frac{L(c,s_a^{(0)}(b))}{L(s_a^{(0)}(b),s_a^{(0)}(b))}\in\mathbb{Z}$.

(vii) $I^{(1)}(a,b)=L(b,a)$ because of $L(a,b)=0$.

$$\begin{aligned}L(s_a^{(1)}(b),s_a^{(1)}(b)) &= L(b-\frac{L(b,a)}{L(a,a)}a,b-\frac{L(b,a)}{L(a,a)}a)\\ &= L(b,b)-\frac{L(b,a)}{L(a,a)}L(b,a)-0+(-\frac{L(b,a)}{L(a,a)})^2L(a,a)\\ &= L(b,b).\end{aligned}$$

For $c\in H_\mathbb{Z}$

$$\begin{aligned}\frac{L(s_a^{(1)}(b),c)}{L(s_a^{(1)}(b),s_a^{(1)}(b))} &= \frac{L(b-\frac{L(b,a)}{L(a,a)}a,c)}{L(b,b)}\\ &= \frac{L(b,c)}{L(b,b)}-\frac{L(b,a)}{L(b,b)}\frac{L(a,c)}{L(a,a)}\in\mathbb{Z},\end{aligned}$$

so $s_a^{(1)}(b)\in R^{(0)}$. Finally

$$\begin{aligned}s_a^{(1)}s_b^{(1)}(s_a^{(1)})^{-1} &= s_a^{(1)}s^{[1]}_{b/\sqrt{L(b,b)}}(s_a^{(1)})^{-1}\\ &\overset{\text{Lemma 2.2 (c)}}{=} s^{[1]}_{s_a^{(1)}(b/\sqrt{L(b,b)})}=s^{[1]}_{s_a^{(1)}(b)/\sqrt{L(b,b)}}\\ &= s^{[1]}_{s_a^{(1)}(b)/\sqrt{L(s_a^{(1)}(b),s_a^{(1)}(b))}}=s^{(1)}_{s_a^{(1)}(b)}.\end{aligned}$$

(b) Starting with $S \in T_n^{tri}$, one can define $H_\mathbb{Z} := M_{n\times 1}(\mathbb{Z})$ with standard $\mathbb{Z}$-basis $\underline{e} = (e_1, ..., e_n)$, and one can define $L : H_\mathbb{Z} \times H_\mathbb{Z} \to \mathbb{Z}$ by $L(\underline{e}^t, \underline{e}) = S^t$. Then $\underline{e} \in \mathcal{B}^{tri}$.

If $(H_\mathbb{Z}, L)$ is unimodular then $\pm 1 = \det L(\underline{e}^t, \underline{e}) = L(e_1, e_1)...L(e_n, e_n)$ and $L(e_i, e_i) \in \mathbb{N}$, so $L(e_i, e_i) = 1$ and $L(\underline{e}^t, \underline{e})^t \in T_n^{uni}(\mathbb{Z})$. The rest is trivial.

(c) The inclusion $R^{(0)} \supset \{a \in H_\mathbb{Z} \,|\, L(a, a) = 1\}$ is obvious. Consider $\underline{e} \in \mathcal{B}^{tri}$. By part (b), the matrix $S := L(\underline{e}^t, \underline{e})^t$ is in $T_n^{uni}(\mathbb{Z})$. Consider a root $a = \sum_{i=1}^n \alpha_i e_i \in R^{(0)}$. Then

$$\begin{aligned} (L(a, e_1), ..., L(a, e_n)) &= (\alpha_1, ..., \alpha_n)S, \\ \text{so } \gcd(L(a, e_1), ..., L(a, e_n)) &= \gcd(\alpha_1, ..., \alpha_n). \end{aligned}$$

But $L(a, a)$ divides $\gcd(L(a, e_1), ..., L(a, e_n))$ because a is a root. Therefore $L(a, a)$ divides each α_i. Thus $L(a, a)^2$ divides $\sum_{i=1}^n \sum_{j=1}^n \alpha_i s_{ij} \alpha_j = L(a, a)$, so $L(a, a) = 1$.

(d) By part (c) $L(a, a) = 1$ for $a \in R^{(0)}$. □

Up to now, the triangular shape of the matrix $L(\underline{e}^t, \underline{e})^t \in T_n^{tri}$ has not been used. It leads to the result in Theorem 2.7. In algebraic geometry and the theory of meromorphic differential equations, this result is well known, it is a piece of Picard-Lefschetz theory. In the frame of singularity theory, it is treated in [**AGV88**] and [**Eb01**]. An elementary direct proof for a unimodular lattice is given in [**BH20**]. The case $k = 0$ is proved in [**HK16**].

Theorem 2.7. *Let $(H_\mathbb{Z}, L, \underline{e})$ be a bilinear lattice with a triangular basis $\underline{e}$. Let $k \in \{0, 1\}$. Then*

$$s_{e_1}^{(k)} ... s_{e_n}^{(k)} = (-1)^{k+1} M.$$

Proof: The case $k = 0$ is a special case of Proposition 2.4 in [**HK16**]. The case $k = 1$ can be proved by an easy modification of Lemma 2.3 (5) and Proposition 2.4 in [**HK16**]. Both cases are proved for a unimodular bilinear lattice in [**BH20**, Theorem 4.1]. □

In Picard-Lefschetz theory and singularity theory also the following notions are standard.

Definition 2.8. Let $(H_\mathbb{Z}, L, \underline{e})$ be a bilinear lattice with a triangular basis $\underline{e}$. It induces several structures:

(a) The *even monodromy group* $\Gamma^{(0)} := \langle s_{e_1}^{(0)}, ..., s_{e_n}^{(0)} \rangle \subset O^{(0)}$.
(b) The *odd monodromy group* $\Gamma^{(1)} := \langle s_{e_1}^{(1)}, ..., s_{e_n}^{(1)} \rangle \subset O^{(1)}$.
(c) The set $\Delta^{(0)} := \Gamma^{(0)}\{\pm e_1, ..., \pm e_n\} \subset H_\mathbb{Z}$ of *even vanishing cycles.*

(d) The set $\Delta^{(1)} := \Gamma^{(1)}\{\pm e_1, ..., \pm e_n\} \subset H_{\mathbb{Z}}$ of *odd vanishing cycles.*

Remarks 2.9. (i) The even vanishing cycles are roots, i.e. $\Delta^{(0)} \subset R^{(0)}$, because of Lemma 2.6 (a) (vi). In general $\Delta^{(1)} \not\subset R^{(0)}$. The name *vanishing cycles* for the elements of $\Delta^{(0)}$ and $\Delta^{(1)}$ and the name *monodromy group* stem from singularity theory. In [**HK16**] the elements of $\Delta^{(0)}$ are called *real roots*. $\Gamma^{(1)}$ and $\Delta^{(1)}$ are not considered in [**HK16**].

(ii) A matrix $S \in T_n^{tri}$ or $T_n^{uni}(\mathbb{Z})$ determines by Lemma 2.6 (b) a bilinear lattice $(H_{\mathbb{Z}}, L, \underline{e})$ with a triangular basis (up to isomorphism). This leads to the program to determine for a given matrix S the data $I^{(0)}, I^{(1)}, G_{\mathbb{Z}}, G_{\mathbb{Z}}^{(0)}, G_{\mathbb{Z}}^{(1)}, G_{\mathbb{Z}}^{M}, \Gamma^{(0)}, \Gamma^{(1)}, \Delta^{(0)}$ and $\Delta^{(1)}$. One should start with relevant invariants like $\operatorname{sign} I^{(0)}$, $\operatorname{Rad} I^{(0)}$, $\operatorname{Rad} I^{(1)}$, the characteristic polynomial and the Jordan normal form of M.

(iii) The odd monodromy group $\Gamma^{(1)}$ arises naturally in many cases where $(H_{\mathbb{Z}}, L)$ is a unimodular bilinear lattice, for example in cases from isolated hypersurface singularities. But it is not clear whether it is natural in cases where $(H_{\mathbb{Z}}, L)$ is a bilinear lattice which is not unimodular. Theorem 2.7 is positive evidence. But the following is negative evidence. The monodromy group

$$\Gamma^{(1)} = \langle s^{[1]}_{e_i/\sqrt{L(e_i,e_i)}} \,|\, i \in \{1, ..., n\}\rangle$$

contains because of Lemma 2.2 (c) all transvections $s^{[1]}_{g(e_i)/\sqrt{L(e_i,e_i)}}$ for $g \in \Gamma^{(1)}$. Only in the unimodular cases these coincide with the transvections $s_a^{[1]}$ for $a \in \Delta^{(1)}$. We will only consider the unimodular cases.

(iv) We will work on this program rather systematically in the chapters 5 and 6 for $S \in T_2^{uni}(\mathbb{Z})$ and $S \in T_3^{uni}(\mathbb{Z})$.

Definition 2.10 and Lemma 2.11 discuss the case when a unimodular bilinear lattice $(H_{\mathbb{Z}}, L, \underline{e})$ with triangular basis is *reducible*. Then also the monodromy groups, the set of roots and the sets of vanishing cycles split. But beware that here reducibility involves not only $(H_{\mathbb{Z}}, L)$, but also $\underline{e}$.

Definition 2.10. (a) Let $(H_{\mathbb{Z}}, L, \underline{e})$ be a unimodular bilinear lattice of rank $n \in \mathbb{N}$ with a triangular basis $\underline{e}$. Let $\{1, ..., n\} = I_1 \dot\cup I_2$ be a decomposition into disjoint subsets such that

$$L(e_i, e_j) = L(e_j, e_i) = 0 \quad \text{for } i \in I_1,\ j \in I_2.$$

Then the triple $(H_{\mathbb{Z}}, L, \underline{e})$ is called *reducible*. If such a decomposition does not exist the triple is called *irreducible*.

(b) A matrix $S\in T_n^{uni}(\mathbb{Z})$ is called reducible if the triple $(H_\mathbb{Z},L,\underline{e})$ (which is unique up to isomorphism) is reducible, where $(H_\mathbb{Z},L)$ is a unimodular bilinear lattice and $\underline{e}$ is a triangular basis with $S=L(\underline{e}^t,\underline{e})^t$.

LEMMA 2.11. *Keep the situation of Definition 2.10. For $l\in\{1,2\}$ let $\sigma_l:\{1,2,...,|I_l|\}\to I_l$ be the unique bijection with $\sigma_l(i)<\sigma_l(j)$ for $i<j$. Define*

$$\begin{aligned}\underline{e}_l &:= (e_{1,l},e_{2,l},...,e_{|I_l|,l}):=(e_{\sigma_l(1)},e_{\sigma_l(2)},...,e_{\sigma_l(|I_l|)}),\\ H_{\mathbb{Z},l} &:= \bigoplus_{i=1}^{|I_l|}\mathbb{Z}\cdot e_{i,l},\quad L_l:=L|_{H_{\mathbb{Z},l}}.\end{aligned}$$

Then $(H_{\mathbb{Z},l},L_l,\underline{e}_l)$ is a unimodular bilinear lattice with triangular basis. The decomposition $H_\mathbb{Z}=H_{\mathbb{Z},1}\oplus H_{\mathbb{Z},2}$ is left and right L-orthogonal. Denote by $\Gamma_l^{(0)}$, $\Gamma_l^{(1)}$, $\Delta_l^{(0)}$, $\Delta_l^{(1)}$ and $R_l^{(0)}$ the monodromy groups and sets of vanishing cycles and roots of $(H_{\mathbb{Z},l},L_l,\underline{e}_l)$. Denote by $\widetilde{M_l}$ the automorphism of $H_\mathbb{Z}$ which extends the monodromy M_l on $H_{\mathbb{Z},l}$ by the identity on $H_{\mathbb{Z},m}$, where $\{l,m\}=\{1,2\}$. Then

$$\begin{aligned}\Gamma^{(k)} &= \Gamma_1^{(k)}\times\Gamma_2^{(k)},\\ R^{(0)} &= R_1^{(0)}\ \dot\cup\ R_2^{(0)},\\ \Delta^{(k)} &= \Delta_1^{(k)}\ \dot\cup\ \Delta_2^{(k)},\\ M &= \widetilde{M_1}\widetilde{M_2}=\widetilde{M_2}\widetilde{M_1}.\end{aligned}$$

The proof is trivial. Because of this lemma, we will study the monodromy groups and the sets of vanishing cycles only for irreducible triples. In the Examples 1.1 this excludes the cases $S(A_1^2)$, $S(A_1^3)$, $S(A_2A_1)$, $S(\mathbb{P}^1A_1)$ and all cases $S(x_1,x_2,x_3)$ where two of the three numbers x_1,x_2,x_3 are zero.

The following lemma treats the cases $S(A_1^n):=E_n$ for $n\in\mathbb{N}$. It is a trivial consequence of the special case $S(A_1)=(1)\in M_{1\times 1}(\mathbb{Z})$ and Lemma 2.11, but worth to be stated.

LEMMA 2.12. *The case A_1^n for any $n\in\mathbb{N}$:*

$$H_\mathbb{Z}=\bigoplus_{i=1}^n\mathbb{Z}\cdot e_i,\quad S=S(A_1^n):=E_n,\quad I^{(0)}=2L,\quad I^{(1)}=0,$$

the reflections $s_{e_i}^{(0)}$ with $s_{e_i}^{(0)}(e_j) = \left\{ \begin{array}{ll} e_j & \text{if } j \neq i, \\ -e_i & \text{if } j = i, \end{array} \right\}$ *commute, the transvections* $s_{e_i}^{(1)}$ *are* $s_{e_i}^{(1)} = \mathrm{id}$,

$$\begin{array}{rcl} \Gamma^{(0)} & = & \{\prod_{i=1}^{n}(s_{e_i}^{(0)})^{l_i} \,|\, (l_1, ..., l_n) \in \{0; 1\}^n\} \cong \{\pm 1\}^n, \\ \Gamma^{(1)} & = & \{\mathrm{id}\}, \\ \Delta^{(0)} & = & R^{(0)} = \{\pm e_1, ..., \pm e_n\} = \Delta^{(1)}. \end{array}$$

CHAPTER 3

Braid group actions

In the sections 3.2–3.4 a unimodular bilinear lattice $(H_\mathbb{Z}, L)$ of some rank $n \geq 2$ is considered. The braid group Br_n is introduced in section 3.1. It acts on several sets of n-tuples and of matrices associated to $(H_\mathbb{Z}, L)$.

Section 3.1 starts with the Hurwitz action on G^n where G is a group. Results of Artin, Birman-Hilden and Igusa-Schiffler for G a free group, a free Coxeter group or any Coxeter group are cited and applied. This is relevant as many of the monodromy groups $\Gamma^{(1)}$ and $\Gamma^{(0)}$ of rank 2 or rank 3 unimodular bilinear lattices with triangular bases are such groups.

It turns out that the Hurwitz action of Br_n on $(O^{(k)})^n$ lifts to an action of a semidirect product $\mathrm{Br}_n \ltimes \{\pm 1\}^n$ on sets of certain n-tuples of cycles in $H_\mathbb{Z}$. This is discussed in section 3.2.

Most important is the set $\mathcal{B}^{tri}$ of triangular bases of $(H_\mathbb{Z}, L)$ (if this set is not empty) and the subset $\mathcal{B}^{dist} = \mathrm{Br}_n \ltimes \{\pm 1\}^n(\underline{e})$ of a chosen triangular basis $\underline{e}$. Section 3.3 poses questions on the characterization of such a set $\mathcal{B}^{dist}$ of *distinguished bases* which will guide our work in chapter 7. It also offers several examples with quite different properties.

Section 3.4 connects the group $\mathrm{Br}_n \ltimes \{\pm 1\}^n$ via its action on the orbit of a triangular basis $\underline{e}$ with the group $G_\mathbb{Z}$. There is a group antihomomorphism $Z : (\mathrm{Br}_n \ltimes \{\pm 1\}^n)_S \to G_\mathbb{Z}$, where $(\mathrm{Br}_n \ltimes \{\pm 1\}^n)_S$ denotes the stabilizer of a matrix $S \in T_n^{uni}(\mathbb{Z})$. In this way certain braids induce automorphisms in $G_\mathbb{Z}$, and in many cases these automorphisms generate $G_\mathbb{Z}$, i.e. Z is surjective. Theorem 3.26 (b) states the well known fact $Z((\delta^{1-k}\sigma^{root})^n) = (-1)^{k+1}M$ for $k \in \{0; 1\}$. Theorem 3.26 (c) gives a condition when $Z(\delta^{1-k}\sigma^{root})$ is in $G_\mathbb{Z}$ and thus an n-th root of $(-1)^{k+1}M$. Theorem 3.28 states that for almost all cases with rank ≤ 3 the map Z is surjective. The exceptions are only four cases.

3.1. The braid group and the Hurwitz action, some classical results

Choose $n \in \mathbb{Z}_{\geq 2}$. The braid group Br_n of braids with n strings was introduced by Artin [**Ar25**]. Here we take a purely algebraic point of view. Artin [**Ar25**, Satz 1] showed that Br_n is generated by $n-1$ elementary braids $\sigma_1,...,\sigma_{n-1}$ and that all relations come from the relations

$$\begin{aligned} \sigma_i\sigma_j &= \sigma_j\sigma_i \quad \text{for } i,j\in\{1,...,n-1\} \text{ with } |i-j|\geq 2, \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \text{for } i\in\{1,...,n-2\}. \end{aligned}$$

He also showed [**Ar47**, Theorem 19] that the center of Br_n is

$$\mathrm{Center}(\mathrm{Br}_n) = \langle \sigma^{mon}\rangle,$$

where

$$\sigma^{root} := \sigma_{n-1}\sigma_{n-2}...\sigma_2\sigma_1, \quad \sigma^{mon} := (\sigma^{root})^n.$$

An important action of Br_n is the *Hurwitz action* on the n-th power G^n for any group G. The braid group Br_n acts via

$$\begin{aligned} \sigma_i(g_1,...,g_n) &:= (g_1,...,g_{i-1},g_ig_{i+1}g_i^{-1},g_i,g_{i+2},...,g_n), \\ \sigma_i^{-1}(g_1,...,g_n) &:= (g_1,...,g_{i-1},g_{i+1},g_{i+1}^{-1}g_ig_{i+1},g_{i+2},...,g_n). \end{aligned}$$

The fibers of the map

$$\pi_n: G^n\to G, \quad \underline{g}=(g_1,...,g_n)\mapsto g_1...g_n,$$

are invariant under this action,

$$\pi_n(\underline{g}) = \pi_n(\sigma_i\underline{g}) = \pi_n(\sigma_i^{-1}\underline{g}).$$

We will study this action for $n=3$ in the cases of the monodromy groups for the rank 3 unimodular bilinear lattices. The following results in Theorem 3.2 of Artin [**Ar25**] and Birman-Hilden [**BH73**] will be relevant.

Definition 3.1. (a) Let $G^{free,n}$ be the free group with n generators $x_1,...,x_n$. Let

$$\Delta(G^{free,n}) := \bigcup_{i=1}^n \{wx_iw^{-1}\,|\, w\in G^{free,n}\}$$

be the set of elements conjugate to $x_1,...,x_n$. Obviously $\mathrm{Br}_n((x_1,...,x_n))\subset\Delta(G^{free,n})^n$.

(b) Let $G^{fCox,n}$ be the free Coxeter group with n generators $x_1,...,x_n$, so all relations are generated by the relations $x_1^2=...=x_n^2=e$. Let

$$\Delta(G^{fCox,n}) := \bigcup_{i=1}^n \{wx_iw^{-1}\,|\, w\in G^{fCox,n}\}$$

be the set of elements conjugate to $x_1,...,x_n$. Obviously $\mathrm{Br}_n((x_1,...,x_n))\subset\Delta(G^{fCox,n})^n$.

THEOREM 3.2. *(a)* [**Ar25**, Satz 7 and Satz 9] Br_n *acts simply transitively on the set of tuples*

$$\{(w_1,...,w_n)\in\Delta(G^{free,n})^n\,|\,w_1...w_n=x_1...x_n\}.$$

(b) [**BH73**, Theorem 7] Br_n *acts simply transitively on the set of tuples*

$$\{(w_1,...,w_n)\in\Delta(G^{fCox,n})^n\,|\,w_1...w_n=x_1...x_n\}.$$

REMARKS 3.3. (i) Both results were reproved by Krüger in [**Kr90**, Satz 7.6].

(ii) Theorem 1.31 in [**KT08**] gives a weaker version of Artin's result Theorem 3.2 (a). Theorem 1.31 in [**KT08**] is equivalent to the statement that Br_n acts simply transitively on the set of tuples

$$\begin{aligned}\{(w_1,...,w_n)\in\Delta(G^{free,n})^n\,|\,&w_1...w_n=x_1...x_n,\\ &w_1,...,w_n \text{ generate } G^{free,n},\ \text{ a permutation}\\ &\sigma\in S_n \text{ exists with } w_i \text{ conjugate to } x_{\sigma(i)}\}.\end{aligned}$$

(iii) The formulation of Theorem 1.31 in [**KT08**] is different. There a group automorphism φ of $G^{free,n}$ is called a *braid automorphism* if $\varphi(x_1...x_n)=x_1...x_n$ and if a permutation $\sigma\in S_n$ with $\varphi(x_i)$ conjugate to $x_{\sigma(i)}$ exists. The group of all braid automorphisms is called $\widetilde{\mathrm{Br}}_n$. Theorem 1.31 in [**KT08**] states that the map

$$\begin{aligned}\widetilde{Z}:\{\sigma_1^{\pm 1},...,\sigma_{n-1}^{\pm 1}\}\to\widetilde{Br}_n \quad \text{with}\\ (\widetilde{Z}(\sigma_i)(x_1),...,\widetilde{Z}(\sigma_i)(x_n))=\sigma_i^{-1}(x_1,...,x_n),\\ (\widetilde{Z}(\sigma_i^{-1})(x_1),...,\widetilde{Z}(\sigma_i^{-1})(x_n))=\sigma_i(x_1,...,x_n),\end{aligned}$$

extends to a group isomorphism $\mathrm{Br}_n\to\widetilde{\mathrm{Br}}_n$.

(iv) In order to understand the equivalence of the statements in (ii) and (iii), it is crucial to see that the extension $\widetilde{Z}:\mathrm{Br}_n\to\widetilde{\mathrm{Br}}_n$ which is defined by

$$(\widetilde{Z}(\beta)(x_1),...,\widetilde{Z}(\beta)(x_n))=\beta^{-1}(x_1,...,x_n)\quad\text{for }\beta\in\mathrm{Br}_n$$

is a group homomorphism. This follows from the equations

$$\widetilde{Z}(\beta\sigma_i)=\widetilde{Z}(\beta)\widetilde{Z}(\sigma_i)\quad\text{and}\quad\widetilde{Z}(\beta\sigma_i^{-1})=\widetilde{Z}(\beta)\widetilde{Z}(\sigma_i^{-1})$$

for $\beta \in \mathrm{Br}_n$ and $i \in \{1, ...n-1\}$. The first equation holds because of

$$\begin{aligned}
& (\widetilde{Z}(\beta\sigma_i)(x_1), ..., \widetilde{Z}(\beta\sigma_i)(x_n)) \\
= \ & (\beta\sigma_i)^{-1}(x_1, ..., x_n) \\
= \ & \sigma_i^{-1}(\beta^{-1}(x_1, ..., x_n)) \\
= \ & \sigma_i^{-1}(\widetilde{Z}(\beta)(x_1), ..., \widetilde{Z}(\beta)(x_n)) \\
= \ & (\widetilde{Z}(\beta)(x_1), ..., \widetilde{Z}(\beta)(x_{i-1}), \widetilde{Z}(\beta)(x_{i+1}), \\
& (\widetilde{Z}(\beta)(x_{i+1}))^{-1}\widetilde{Z}(\beta)(x_i)\widetilde{Z}(\beta)(x_{i+1}), \widetilde{Z}(\beta)(x_{i+2}), ..., \widetilde{Z}(\beta)(x_n)) \\
= \ & (\widetilde{Z}(\beta)(x_1), ..., \widetilde{Z}(\beta)(x_{i-1}), \widetilde{Z}(\beta)(x_{i+1}), \\
& \widetilde{Z}(\beta)(x_{i+1}^{-1}x_i x_{i+1}), \widetilde{Z}(\beta)(x_{i+2}), ..., \widetilde{Z}(\beta)(x_n)) \\
= \ & (\widetilde{Z}(\beta)\widetilde{Z}(\sigma_i)(x_1), ..., \widetilde{Z}(\beta)\widetilde{Z}(\sigma_i)(x_n)).
\end{aligned}$$

The second equation is proved by a similar calculation.

EXAMPLE 3.4. Theorem 3.2 will be applied in the following situation, which in fact arises quite often.

Consider a unimodular bilinear lattice $(H_{\mathbb{Z}}, L, \underline{e})$ of rank $n \geq 2$ with triangular basis $\underline{e}$ such that for some $k \in \{0, 1\}$ the following holds:

$$\Gamma^{(k)} = \begin{cases} G^{fCox,n} & \text{with generators } s_{e_1}^{(0)}, ..., s_{e_n}^{(0)} \quad \text{if } k = 0, \\ G^{free,n} & \text{with generators } s_{e_1}^{(1)}, ..., s_{e_n}^{(1)} \quad \text{if } k = 1. \end{cases}$$

Then in the notation of Definition 3.1

$$\begin{aligned}
\Delta(G^{fCox,n}) &= \{s_v^{(0)} \mid v \in \Delta^{(0)}\} \qquad \text{if } k = 0, \\
\Delta(G^{free,n}) &= \{s_v^{(1)} \mid v \in \Delta^{(1)}\} \qquad \text{if } k = 1.
\end{aligned}$$

By Theorem 3.2, two statements hold:

(1) The set

$$\{(s_{v_1}^{(k)}, ..., s_{v_n}^{(k)}) \mid v_1, ..., v_n \in \Delta^{(k)}, s_{v_1}^{(k)}...s_{v_n}^{(k)} = (-1)^{k+1}M\}$$

is a single orbit under the Hurwitz action of Br_n.

(2) The stabilizer of any such tuple $(s_{v_1}^{(k)}, ..., s_{v_n}^{(k)})$ under the Hurwitz action of Br_n is $\{\mathrm{id}\}$.

Theorem 3.2 (b) concerns a free Coxeter group with n generators. The transitivity of the action generalizes to arbitrary Coxeter groups and can be applied to generalize the statement (1) in Example 3.4, as is explained in the following.

DEFINITION 3.5. (Classical, e.g [**Hu90**, 5.1]) A *Coxeter system* (W, S^{gen}) consists of a group W and a finite set $S^{gen} = \{s_1, ..., s_n\} \subset W$ for some $n \in \mathbb{N}$ of generators of the group such that there are generating relations as follows. There is a subset $I \subset \{(i,j) \in \{1,...,n\}^2 \,|\, i<j\}$ and a map $a: I \to \mathbb{Z}_{\geq 2}$ such that the generating relations are

$$s_1^2 = ... = s_n^2 = 1, \quad 1 = (s_i s_j)^{a(i,j)} \text{ for } (i,j) \in I.$$

The group W is then called a *Coxeter group*.

The following theorem was proved by Deligne [**De74**] for the ADE Weyl groups and in general by Igusa and Schiffler [**IS10**]. A short proof was given by Baumeister, Dyer, Stump and Wegener [**BDSW14**].

THEOREM 3.6. [**De74**][**IS10**][**BDSW14**] *Let (W, S^{gen}) with $S^{gen} = \{s_1,...,s_n\}$ be a Coxeter system with $n \geq 2$. Define $\Delta(W, S^{gen}) := \bigcup_{i=1}^n \{w s_i w^{-1} \,|\, w \in W\}$. The set*

$$\{(w_1,...,w_n) \in \Delta(W, S^{gen})^n \,|\, w_1...w_n = s_1...s_n\}$$

is a single orbit under the Hurwitz action of Br_n.

Part (a) of the following theorem is classical if $S_{ij} \in \{0,-1,-2\}$ for $i<j$ and due to Vinberg [**Vi71**] in the general case.

THEOREM 3.7. *Let $(H_\mathbb{Z}, L)$ be a unimodular bilinear lattice of rank $n \geq 2$ and let $\underline{e}$ be a triangular basis such that the matrix $S = L(\underline{e}^t, \underline{e})^t \in T_n^{uni}(\mathbb{Z})$ satisfies $S_{ij} \leq 0$ for $i<j$.*

(a) (Classical for $S_{ij} \in \{0,-1,-2\}$, [**Vi71**, Proposition 6, Theorem 1, Theorem 2, Proposition 17] *for $S_{ij} \leq 0$) The pair $(\Gamma^{(0)}, \{s_{e_1}^{(0)},...,s_{e_n}^{(0)}\})$ is a Coxeter system with*

$$\begin{aligned} I &= \{(i,j) \in \{1,...,n\}^2 \,|\, i<j, S_{ij} \in \{0,-1\}\} \quad \text{and} \\ a(i,j) &= \begin{cases} 2 & \text{if } S_{ij} = 0, \\ 3 & \text{if } S_{ij} = -1. \end{cases} \end{aligned}$$

(b) The set

$$\{(g_1,...,g_n) \in \left(\{s_v^{(0)} \,|\, v \in \Delta^{(0)}\}\right)^n \,|\, g_1...g_n = -M\}$$

is a single orbit under the Hurwitz action of Br_n.

Proof of part (b): Observe

$$\Delta(\Gamma^{(0)}, \{s_{e_1}^{(0)},...,s_{e_n}^{(0)}\}) = \{s_v^{(0)} \,|\, v \in \Delta^{(0)}\}.$$

Apply Theorem 3.6. □

REMARKS 3.8. (i) The transitivity result in part (b) holds also for a bilinear lattice $(H_\mathbb{Z}, L)$ which is not necessarily unimodular, if it comes equipped with a triangular basis $\underline{e}$ with $L(e_i, e_j) \leq 0$ for $i>j$. This is the case of a generalized Cartan lattice (Remark 2.4 (v)). This is crucial in [**HK16**].

(ii) Especially in the case of a unimodular bilinear lattice $(H_{\mathbb{Z}}, L, \underline{e})$ with triangular basis $\underline{e}$ and matrix $S = L(\underline{e}^t, \underline{e})^t \in T_n^{uni}(\mathbb{Z})$ with $S_{ij} \leq -2$ for all $i < j$ we have $\Gamma^{(0)} = G^{fCox,n}$ with generators $s^{(0)}_{e_1}, ..., s^{(0)}_{e_n}$.

(iii) Theorem 6.11 (g) gives in the case $n = 3$ $\Gamma^{(0)} = G^{fCox,3}$ with generators $s^{(0)}_{e_1}, s^{(0)}_{e_2}, s^{(0)}_{e_3}$ also in the following cases: if $S_{ij} \geq 3$ for $i < j$ and if additionally

$$2S_{12} \leq S_{13}S_{23}, \quad 2S_{13} \leq S_{12}S_{23}, \quad 2S_{23} \leq S_{12}S_{13}.$$

(iv) Theorem 6.18 (g) gives in the situation of part (iii) also $\Gamma^{(1)} = G^{free,3}$ with generators $s^{(1)}_{e_1}, s^{(1)}_{e_2}, s^{(1)}_{e_3}$.

(v) Though in the situation of part (ii) there are cases with $\Gamma^{(1)} = G^{free,n}$, and there are cases with $\Gamma^{(1)} \neq G^{free,n}$. The odd cases are more complicated than the even cases. For the cases with $n = 3$ see the Remarks 4.17, Lemma 4.18 and Theorem 6.18.

3.2. Braid group action on tuples of cycles

Consider a unimodular bilinear lattice $(H_{\mathbb{Z}}, L)$ of some rank $n \geq 2$ and the groups $O^{(k)} = \mathrm{Aut}(H_{\mathbb{Z}}, I^{(k)})$ for $k \in \{0; 1\}$. Recall that here the set of roots $R^{(0)}$ is

$$R^{(0)} = \{\delta \in H_{\mathbb{Z}} \,|\, L(\delta, \delta) = 1\}.$$

In order to treat the even case $k = 0$ and the odd case $k = 1$ uniformly, we define

$$R^{(1)} := H_{\mathbb{Z}}.$$

The Hurwitz action of Br_n on $(O^{(k)})^n$ restricts because of

$$s^{(k)}_a s^{(k)}_b (s^{(k)}_a)^{-1} = s^{(k)}_{s^{(k)}_a(b)} \quad \text{for} \quad a, b \in R^{(k)} \tag{3.1}$$

(Lemma 2.2 (c)) to an action on the subset $(\{s^{(k)}_v \,|\, v \in R^{(k)}\})^n$.

It turns out that this action has a natural lift to an action of a certain semidirect product $\mathrm{Br}_n \ltimes \{\pm 1\}^n$ on the set $(R^{(k)})^n$. Here the sets $(R^{(k)})^n$ and $(\{s^{(k)}_v \,|\, v \in R^{(k)}\})^n$ are related by the map

$$\begin{array}{rcl} \pi^{(k)}_n : (R^{(k)})^n & \to & (\{s^{(k)}_v \,|\, v \in R^{(k)}\})^n \subset (O^{(k)})^n, \\ \underline{v} = (v_1, ..., v_n) & \mapsto & (s^{(k)}_{v_1}, ..., s^{(k)}_{v_n}). \end{array}$$

Recall also the map

$$\pi_n : (O^{(k)})^n \to O^{(k)}, \quad (g_1, ..., g_n) \mapsto g_1 ... g_n$$

which was defined for an arbitrary group before Definition 3.1.

Furthermore it turns out that both actions, for $k = 0$ and for $k = 1$, restrict to the same action on the set $\mathcal{B}^{tri}$ of triangular bases if this set is not empty. This is the action in which we are interested most. In the case of a unimodular bilinear lattice from singularity theory, it is well known [**Eb01**, 5.7] [**AGV88**, §1.9] and has been studied by A'Campo, Brieskorn, Ebeling, Gabrielov, Gusein-Zade, Krüger and others. In fact it works also for bilinear lattices with $\mathcal{B}^{tri} \neq \emptyset$ which are not necessarily unimodular, see Remark 3.14.

Finally, the actions induce actions of $\mathrm{Br}_n \ltimes \{\pm 1\}^n$ on several spaces of matrices. The purpose of this section is to fix all these well known actions.

Lemma 3.9 presents the semidirect product $\mathrm{Br}_n \ltimes \{\pm 1\}^n$. Lemma 3.10 gives its action on $(R^{(k)})^n$. Lemma 3.11 gives its action on $\mathcal{B}^{tri}$ if this set is not empty.

LEMMA 3.9. *Fix $n \in \mathbb{Z}_{\geq 2}$.*

(a) The multiplicative group $\{\pm 1\}^n$ is called sign group. It is generated by the elements $\delta_j = ((-1)^{\delta_{ij}})_{i=1,...,n} \in \{\pm 1\}^n$ (here δ_{ij} is the Kronecker symbol) for $j \in \{1,...,n\}$.

(b) The following relations define a semidirect product $\mathrm{Br}_n \ltimes \{\pm 1\}^n$ of Br_n and $\{\pm 1\}^n$ with $\{\pm 1\}^n$ as normal subgroup,

$$\begin{aligned} \sigma_j \delta_i \sigma_j^{-1} &= \delta_i \quad \text{for } i \in \{1,...,n\} - \{j, j+1\}, \\ \sigma_j \delta_j \sigma_j^{-1} &= \delta_{j+1}, \quad \sigma_j \delta_{j+1} \sigma_j^{-1} = \delta_j. \end{aligned}$$

In the following $\mathrm{Br}_n \ltimes \{\pm 1\}^n$ always means this semidirect product.

Proof: Part (a) is a notation. Part (b) requires a proof. We have the exact sequence

$$\{1\} \to \mathrm{Br}_n^{pure} \to \mathrm{Br}_n \to S_n \to \{1\}$$

where $\mathrm{Br}_n^{pure} \subset \mathrm{Br}_n$ is the normal subgroup of pure braids (and $\sigma_i \in \mathrm{Br}_n$ maps to the transposition $(i\ i+1) \in S_n$). The natural action of S_n on $\{\pm 1\}^n$,

$$S_n \ni \alpha : \underline{\varepsilon} = (\varepsilon_1, ..., \varepsilon_n) \mapsto (\varepsilon_{\alpha^{-1}(1)}, ..., \varepsilon_{\alpha^{-1}(n)}) =: \alpha.\underline{\varepsilon}$$

lifts to an action of Br_n on $\{\pm 1\}^n$, $\sigma : \underline{\varepsilon} \mapsto \sigma.\underline{\varepsilon}$. This action can be used to define a semidirect product of Br_n and $\{\pm 1\}^n$ by $\sigma \underline{\varepsilon} \sigma^{-1} := \sigma.\underline{\varepsilon}$. It is the semidirect product in part (b). □

LEMMA 3.10. *Let $(H_{\mathbb{Z}}, L)$ be a unimodular bilinear lattice of rank $n \geq 2$. Fix $k \in \{0, 1\}$.*

(a) The following formulas define an action of the semidirect product $\mathrm{Br}_n \ltimes \{\pm 1\}^n$ from Definition 3.9 (b) on the set $(R^{(k)})^n$,

$$\begin{aligned} \sigma_j(\underline{v}) &= (v_1, ..., v_{j-1}, s^{(k)}_{v_j}(v_{j+1}), v_j, v_{j+2}, ..., v_n), \\ \sigma_j^{-1}(\underline{v}) &= (v_1, ..., v_{j-1}, v_{j+1}, (s^{(k)}_{v_{j+1}})^{-1}(v_j), v_{j+2}, ..., v_n), \\ \delta_j(\underline{v}) &= (v_1, ..., v_{j-1}, -v_j, v_{j+1}, ..., v_n), \end{aligned}$$

for $\underline{v} = (v_1, ..., v_n) \in (R^{(k)})^n$.

(b) The map $\pi_n^{(k)} : (R^{(k)})^n \to (O^{(k)})^n$ *is compatible with the action of* $\mathrm{Br}_n \ltimes \{\pm 1\}^n$ *on* $(R^{(k)})^n$ *from part (a) and the Hurwitz action of* Br_n *on* $(O^{(k)})^n$*, so the diagram*

$$\begin{array}{ccc} (R^{(k)})^n & \xrightarrow{\sigma_j} & (R^{(k)})^n \\ {\scriptstyle \pi_n^{(k)}}\downarrow & & \downarrow{\scriptstyle \pi_n^{(k)}} \\ (O^{(k)})^n & \xrightarrow{\sigma_j} & (O^{(k)})^n \end{array}$$

commutes. Here the sign group $\{\pm 1\}^n$ *acts trivially on* $(O^{(k)})^n$*. Especially, each orbit in* $(R^{(k)})^n$ *is contained in one fiber of the projection* $\pi_n \circ \pi_n^{(k)} : (R^{(k)})^n \to O^{(k)}$.

Proof: (a) We denote the actions in part (a) by $\sigma_j^{(k)}, (\sigma_j^{-1})^{(k)}$ and $\delta_j^{(k)}$ (of course $\delta_j^{(0)} = \delta_j^{(1)}$). The identities

$$\begin{array}{rl} \sigma_j^{(k)}(\sigma_j^{-1})^{(k)} = (\sigma_j^{-1})^{(k)}\sigma_j^{(k)} = \mathrm{id} & \text{for } j \in \{1, ..., n-1\} \\ \sigma_i^{(k)}\sigma_j^{(k)} = \sigma_j^{(k)}\sigma_i^{(k)} & \text{for } |i-j| \geq 2, \\ \sigma_j^{(k)}\delta_i^{(k)}(\sigma_j^{-1})^{(k)} = \delta_i^{(i)} & \text{for } i \in \{1, ..., n\} - \{j, j+1\}, \\ \sigma_j^{(k)}\delta_j^{(k)}(\sigma_j^{-1})^{(k)} = \delta_{j+1}^{(k)} & \text{and} \\ \sigma_j^{(k)}\delta_{j+1}^{(k)}(\sigma_j^{-1})^{(k)} = \delta_j^{(k)} & \text{for } j \in \{1, ..., n-1\} \end{array}$$

are obvious or easy to see. The identities

$$\sigma_i^{(k)}\sigma_{i+1}^{(k)}\sigma_i^{(k)} = \sigma_{i+1}^{(k)}\sigma_i^{(k)}\sigma_{i+1}^{(k)} \qquad \text{for } i \in \{1, ..., n-2\}$$

are proved by the following calculation with $\underline{v} \in (R^{(k)})^n$,

$$\begin{array}{rl} & \sigma_i^{(k)}\sigma_{i+1}^{(k)}\sigma_i^{(k)}(\underline{v}) \\ = & \sigma_i^{(k)}\sigma_{i+1}^{(k)}(..., v_{i-1}, s_{v_i}^{(k)}(v_{i+1}), v_i, v_{i+2}, v_{i+3} ...) \\ = & \sigma_i^{(k)}(..., v_{i-1}, s_{v_i}^{(k)}(v_{i+1}), s_{v_i}^{(k)}(v_{i+2}), v_i, v_{i+3}, ...) \\ = & (..., v_{i-1}, s^{(k)}_{s_{v_i}^{(k)}(v_{i+1})}(s_{v_i}^{(k)}(v_{i+2})), s_{v_i}^{(k)}(v_{i+1}), v_i, v_{i+3}, ...) \\ \overset{(3.1)}{=} & (..., v_{i-1}, s_{v_i}^{(k)}s_{v_{i+1}}^{(k)}(s_{v_i}^{(k)})^{-1}(s_{v_i}^{(k)}(v_{i+2})), s_{v_i}^{(k)}(v_{i+1}), v_i, v_{i+3}, ...) \\ = & (..., v_{i-1}, s_{v_i}^{(k)}(s_{v_{i+1}}^{(k)}(v_{i+2})), s_{v_i}^{(k)}(v_{i+1}), v_i, v_{i+3}, ...) \\ = & \sigma_{i+1}^{(k)}(..., v_{i-1}, s_{v_i}^{(k)}(s_{v_{i+1}}^{(k)}(v_{i+2})), v_i, v_{i+1}, v_{i+3}, ...) \\ = & \sigma_{i+1}^{(k)}\sigma_i^{(k)}(..., v_{i-1}, v_i, s_{v_{i+1}}^{(k)}(v_{i+2}), v_{i+1}, v_{i+3}, ...) \\ = & \sigma_{i+1}^{(k)}\sigma_i^{(k)}\sigma_{i+1}^{(k)}(\underline{v}). \end{array}$$

The maps $\sigma_j^{(k)}$ and $\delta_i^{(k)}$ satisfy all relations between the generators σ_j and δ_i of the group $\mathrm{Br}_n \ltimes \{\pm 1\}^n$. Therefore the formulas in part (a) define an action of this group on the set $(R^{(k)})^n$.

(b) The actions are compatible because of (3.1). The sign group acts trivially on $(O^{(k)})^n$ because $s_v^{(k)} = s_{-v}^{(k)}$ for $v \in R^{(k)}$. Each orbit of the Hurwitz action on $(O^{(k)})^n$ is contained in one fiber of the map π_n, as was remarked in section 3.1. $\square$

LEMMA 3.11. *Let $(H_{\mathbb{Z}}, L)$ be a unimodular bilinear lattice with nonempty set $\mathcal{B}^{tri}$ of triangular bases. The actions in Lemma 3.10 of $\mathrm{Br}_n \ltimes \{\pm 1\}^n$ on $(R^{(0)})^n$ and on $(R^{(1)})^n$ both restrict to the same action on $\mathcal{B}^{tri}$. This action can also be written as follows,*

$$
\begin{aligned}
\sigma_j(\underline{v}) &= (v_1, ..., v_{j-1}, v_{j+1} - L(v_{j+1}, v_j)v_j, v_j, v_{j+2}, ..., v_n), \\
\sigma_j^{-1}(\underline{v}) &= (v_1, ..., v_{j-1}, v_{j+1}, v_j - L(v_{j+1}, v_j)v_{j+1}, v_{j+2}, ..., v_n), \\
\delta_j(\underline{v}) &= (v_1, ..., v_{j-1}, -v_j, v_{j+1}, ..., v_n),
\end{aligned}
$$

for $\underline{v} = (v_1, ..., v_n) \in \mathcal{B}^{tri}$.

Proof: $\underline{v} \in \mathcal{B}^{tri}$ implies $L(v_j, v_{j+1}) = 0$ and $2L(v_j, v_j) = 2 = I^{(0)}(v_j, v_j)$. Recall $I^{(0)} = L + L^t$ and $I^{(1)} = L^t - L$. Therefore

$$
\begin{aligned}
s_{v_j}^{(k)}(v_{j+1}) &= v_{j+1} - I^{(k)}(v_j, v_{j+1})v_j = v_{j+1} - L(v_{j+1}, v_j)v_j, \\
(s_{v_{j+1}}^{(k)})^{-1}(v_j) &= v_j - (-1)^k I^{(k)}(v_{j+1}, v_j)v_{j+1} = v_j - L(v_{j+1}, v_j)v_{j+1}.
\end{aligned}
$$

So $\sigma_j(\underline{v})$ and $\sigma_j^{-1}(\underline{v})$ are given by the formulas in Lemma 3.11. It remains to see that the images are again in $\mathcal{B}^{tri}$. They are in $(R^{(0)})^n$ because of the even case $k = 0$. They form $\mathbb{Z}$-bases of $H_{\mathbb{Z}}$ because $\underline{v}$ is a $\mathbb{Z}$-basis of $H_{\mathbb{Z}}$. They are triangular bases because

$$
\begin{aligned}
L(\sigma_j(\underline{v})_j, \sigma_j(\underline{v})_{j+1}) &= L(v_{j+1} - L(v_{j+1}, v_j)v_j, v_j) = 0, \\
L(\sigma_j^{-1}(\underline{v})_j, \sigma_j^{-1}(\underline{v})_{j+1}) &= L(v_{j+1}, v_j - L(v_{j+1}, v_j)v_{j+1}) = 0.
\end{aligned}
$$

Of course $\delta_j(\underline{v}) \in \mathcal{B}^{tri}$. $\square$

DEFINITION 3.12. Fix $n \in \mathbb{N}$ and $R \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}\}$.

(a) Recall the definition of the set $T_n^{uni}(R)$ of upper triangular $n \times n$ matrices with entries in R and 1's on the diagonal in Definition 2.3 (c). Additionally we define the sets of symmetric and skewsymmetric matrices

$$
\begin{aligned}
T_n^{(0)}(R) &:= \{A \in M_{n\times n}(R) \mid A^t = A, A_{ii} = 2\}, \\
T_n^{(1)}(R) &:= \{A \in M_{n\times n}(R) \mid A^t = -A\}.
\end{aligned}
$$

(b) For $a\in\mathbb{Z}$ define the $n\times n$ matrix

$$C_{n,j}(a) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & a & 1 & & \\ & & 1 & 0 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

which differs from the unit matrix only in the positions $(j,j),(j,j+1),(j+1,j),(j+1,j+1)$. Its inverse is

$$C_{n,j}^{-1}(a) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & 1 & & \\ & & 1 & -a & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

with $-a$ at the position $(j+1,j+1)$.

LEMMA 3.13. *Fix $n\in\mathbb{Z}_{\geq 2}$ and $R\in\{\mathbb{Z},\mathbb{Q},\mathbb{R},\mathbb{C}\}$.*

(a) The following formulas define an action of the semidirect product $\mathrm{Br}_n\ltimes\{\pm 1\}^n$ on each of the sets of matrices $T_n^{(0)}(R)$, $T_n^{(1)}(R)$ and $T_n^{uni}(R)$,

$$\begin{aligned} \sigma_j(A) &= C_{n,j}(-A_{j,j+1})\cdot A\cdot C_{n,j}(-A_{j,j+1}) \quad \text{for } j\in\{1,...,n-1\},\\ \sigma_j^{-1}(A) &= C_{n,j}^{-1}(A_{j,j+1})\cdot A\cdot C_{n,j}^{-1}(A_{j,j+1}) \quad \text{for } j\in\{1,...,n-1\},\\ \delta_j(A) &= \mathrm{diag}(((-1)^{\delta_{ij}})_{i=1,...,n})\cdot A\cdot \mathrm{diag}(((-1)^{\delta_{ij}})_{i=1,...,n})\\ &\qquad \text{for } j\in\{1,...,n\}. \end{aligned}$$

(b) Let $(H_\mathbb{Z},L)$ be a unimodular bilinear lattice of rank n.

(i) Fix $k\in\{0;1\}$. The map

$$(R^{(k)})^n\to T_n^{(k)}(\mathbb{Z}),\quad \underline{v}\mapsto I^{(k)}(\underline{v}^t,\underline{v}),$$

is compatible with the actions of $\mathrm{Br}_n\ltimes\{\pm 1\}^n$ on $(R^{(k)})^n$ and on $T_n^{(k)}(\mathbb{Z})$.

(ii) Suppose that $\mathcal{B}^{tri}$ is not empty. The map

$$\mathcal{B}^{tri}\to T_n^{uni}(\mathbb{Z}),\quad \underline{v}\mapsto L(\underline{v}^t,\underline{v})^t,$$

is compatible with the actions of $\mathrm{Br}_n\ltimes\{\pm 1\}^n$ on $\mathcal{B}^{tri}$ and on $T_n^{uni}(\mathbb{Z})$.

Proof: (a) For $A\in T_n^{(k)}(\mathbb{Z})$ there is a unique matrix $S\in T_n^{uni}(\mathbb{Z})$ with $A=S+(-1)^kS^t$. Consider a unimodular bilinear lattice $(H_\mathbb{Z},L,\underline{e})$

with a triangular basis $\underline{e}$ with matrix $S=L(\underline{e}^t,\underline{e})^t$. Then

$$\begin{aligned}
\sigma_j(\underline{e}) &= (...,e_{j-1},e_{j+1}-S_{j,j+1}e_j,e_j,e_{j+2},...)\\
&= \underline{e}\cdot C_{n,j}(-S_{j,j+1}),\\
\sigma_j^{-1}(\underline{e}) &= (...,e_{j-1},e_{j+1},e_j-S_{j,j+1}e_{j+1},e_{j+2},...)\\
&= \underline{e}\cdot C_{n,j}^{-1}(S_{j,j+1}),\\
\delta_j(\underline{e}) &= (...,e_{i-1},-e_i,e_{i+1},...)\\
&= \underline{e}\cdot \mathrm{diag}(((-1)^{\delta_{ij}})_{i=1,...,n}).
\end{aligned}$$

Observe that $C_{n,j}(a)$ for $a\in\mathbb{Z}$ is symmetric. Therefore for $k\in\{0;1\}$

$$\begin{aligned}
I^{(k)}(\sigma_j(\underline{e})^t,\sigma_j(\underline{e})) &= C_{n,j}(-S_{j,j+1})\cdot I^{(k)}(\underline{e}^t,\underline{e})\cdot C_{n,j}(-S_{j,j+1}),\\
I^{(k)}(\sigma_j^{-1}(\underline{e})^t,\sigma_j^{-1}(\underline{e})) &= C_{n,j}^{-1}(S_{j,j+1})\cdot I^{(k)}(\underline{e}^t,\underline{e})\cdot C_{n,j}^{-1}(S_{j,j+1}),
\end{aligned}$$

similarly for L instead of $I^{(k)}$, and also similarly for the action of δ_j. This shows part (a) for $R=\mathbb{Z}$. Changing the set of scalars does not change the matrix identities which say that the group $\mathrm{Br}_n\ltimes\{\pm 1\}^n$ acts.

(b) This follows from the proof of part (a). □

REMARKS 3.14. Let $(H_\mathbb{Z},L)$ be a bilinear lattice, not necessarily unimodular.

(i) The action of $\mathrm{Br}_n\ltimes\{\pm1\}^n$ on $(R^{(0)})^n$ in Lemma 3.10 (a) works also in this case. It restricts as in Lemma 3.11 to an action on $\mathcal{B}^{tri}$, if this set is not empty, though here for $\underline{v}\in\mathcal{B}^{tri}$

$$\begin{aligned}
\sigma_j(\underline{v}) &= (v_1,...,v_{j-1},v_{j+1}-\frac{L(v_{j+1},v_j)}{L(v_j,v_j)}v_j,v_j,v_{j+2},...,v_n),\\
\sigma_j^{-1}(\underline{v}) &= (v_1,...,v_{j-1},v_{j+1},v_j-\frac{L(v_{j+1},v_j)}{L(v_{j+1},v_{j+1})}v_{j+1},v_{j+2},...,v_n).
\end{aligned}$$

(ii) The action of $\mathrm{Br}_n\ltimes\{\pm1\}^n$ in Lemma 3.10 on $(R^{(1)})^n$ does not generalize. In Lemma 2.6 (a) (v) we defined in the case of a general bilinear lattice $s_a^{(1)}$ only for $a\in R^{(0)}$. We defined $s_a^{(1)}$ for any $a\in R^{(1)}$ in Lemma 2.6 (d) only in the case of a unimodular bilinear lattice.

(iii) On the other hand, part (a) (vii) of Lemma 2.6 says that the action in Lemma 3.10 for $k=1$ works for $\underline{v}\in\mathcal{B}^{tri}$. Though at the end this is just the action in (i) above.

(iv) The action in (i) on $\mathcal{B}^{tri}$ is compatible with an action on the set T_n^{tri} of matrices in Lemma 2.3 (c), which generalizes the action in Lemma 3.13 (a). Here $C_{n,j}(-S_{j,j+1})$ and $C_{n,j}^{-1}(S_{j,j+1})$ in Lemma 3.13 (a) have to be replaced by $C_{n,j}(-\frac{S_{j,j+1}}{S_{j,j}})$ and $C_{n,j}^{-1}(\frac{S_{j,j+1}}{S_{j+1,j+1}})$.

$\{\pm 1\}^n$ is the normal subgroup in the semidirect product $\mathrm{Br}_n \ltimes \{\pm 1\}^n$. Therefore, if $\mathrm{Br}_n \ltimes \{\pm 1\}^n$ acts on some set Σ, the group Br_n acts on the quotient $\Sigma/\{\pm 1\}^n$. Often it is good to consider this quotient and the action of Br_n on it.

LEMMA 3.15. *Let $(H_{\mathbb{Z}}, L)$ be a unimodular bilinear lattice of some rank $n \geq 2$. Fix $k \in \{0; 1\}$.*

(a) The map

$$\pi_n^{(k)} : (R^{(k)})^n \to (\{s_v^{(k)} \,|\, v \in R^{(k)}\})^n \subset (O^{(k)})^n, \quad \underline{v} \mapsto (s_{v_1}^{(k)}, ..., s_{v_n}^{(k)}),$$

factors into maps

$$(R^{(k)})^n \longrightarrow (R^{(k)})^n/\{\pm 1\}^n \xrightarrow{\pi_n^{(k)}/\{\pm 1\}^n} (\{s_v^{(k)} \,|\, v \in R^{(k)}\})^n.$$

Br_n acts on the quotient $(R^{(k)})^n/\{\pm 1\}^n$, and the second map $\pi_n^{(k)}/\{\pm 1\}^n$ is Br_n equivariant. The image of $\underline{v}$ in $(R^{(k)})^n/\{\pm 1\}^n$ is denoted by $\underline{v}/\{\pm 1\}^n$.

(b) The second map

$$\pi_n^{(k)}/\{\pm 1\}^n : (R^{(k)})^n/\{\pm 1\}^n \to (\{s_v^{(k)} \,|\, v \in R^{(k)}\})^n$$

in part (a) is a bijection if $k = 0$ or if $k = 1$ and $\mathrm{Rad}\, I^{(1)} = \{0\}$.

(c) Consider the case $k = 1$ and $\mathrm{Rad}\, I^{(1)} \supsetneqq \{0\}$. Consider a triangular basis $\underline{e} \in \mathcal{B}^{tri}$ and the induced set $\Delta^{(1)}$ of odd vanishing cycles. The second map restricts to a Br_n equivariant bijection

$$(\Delta^{(1)})^n/\{\pm 1\}^n \to (\{s_v^{(1)} \,|\, v \in \Delta^{(1)}\})^n, \quad \underline{v} \mapsto (s_{v_1}^{(1)}, ..., s_{v_n}^{(1)}),$$

if $(H_{\mathbb{Z}}, L, \underline{e})$ is either irreducible or reducible with at most one summand of type A_1.

Proof: Part (a) is trivial. (b) Suppose $k = 0$ or ($k = 1$ and $\mathrm{Rad}\, I^{(1)} = \{0\}$). If $k = 1$ and $v = 0$ then $s_v^{(1)} = \mathrm{id}$. If $k = 0$ and $v \in R^{(0)}$ or if $k = 1$ and $v \in R^{(1)} - \{0\}$ then $v \notin \mathrm{Rad}\, I^{(k)}$ and $s_v^{(k)} \neq \mathrm{id}$ for any $v \in R^{(k)}$. Then one can recover $\pm v$ from $s_v^{(k)}$, essentially because of

$$\{0\} \subsetneqq (s_v^{(k)} - \mathrm{id})(H_{\mathbb{Z}}) \subset \mathbb{Z}v.$$

(c) If $(H_{\mathbb{Z}}, L, \underline{e})$ is irreducible then $\Delta^{(1)} \cap \mathrm{Rad}\, I^{(1)} = \emptyset$, and the argument of part (b) holds. If $(H_{\mathbb{Z}}, L, \underline{e})$ is reducible with only one summand of type A_1 then for a unique $j \in \{1, ..., n\}$ $e_j \in \mathrm{Rad}\, I^{(1)}$, and then $\Delta^{(1)} \cap \mathrm{Rad}\, I^{(1)} = \{\pm e_j\}$. Then $v \in \Delta^{(1)}$ satisfies $s_v^{(1)} = \mathrm{id}$ if and only if $v = \pm e_j$. So also then one can recover $\pm v$ from $s_v^{(1)}$ for any $v \in \Delta^{(1)}$. $\square$

REMARKS 3.16. (i) Consider the action of Br_n on $T_n^{uni}(\mathbb{Z})$. The elementary braid σ_j maps $S=(S_{ij})\in T_n^{uni}(\mathbb{Z})$ to

$$\begin{aligned}\sigma_j(S) &= C_{n,j}(-S_{j,j+1})\cdot S\cdot C_{n,j}(-S_{j,j+1})\\ \text{with}\quad & \begin{pmatrix}\sigma_j(S)_{jj} & \sigma_j(S)_{j,j+1}\\ \sigma_j(S)_{j+1,j} & \sigma_j(S)_{j+1,j+1}\end{pmatrix}=\begin{pmatrix}1 & -S_{j,j+1}\\ 0 & 1\end{pmatrix}.\end{aligned}$$

(ii) Especially, in the case $n=2$ δ_1, δ_2 and σ_1 all map $S=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$ to $\begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}$, so the $\mathrm{Br}_2\ltimes\{\pm 1\}^2$ orbit equals the Br_2 orbit and the $\langle\delta_1\rangle$ orbit and consists only of $S=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}$.

(iii) Under rather special circumstances, also in higher rank n the sign group action is eaten up by the braid group action. Ebeling proved the following lemma.

LEMMA 3.17. *Let $(H_\mathbb{Z},L)$ be a unimodular bilinear lattice and $\underline{e}\in\mathcal{B}^{tri}$ a triangular basis with $S=L(\underline{e}^t,\underline{e})^t\in T_n^{uni}(\mathbb{Z})$.*

(a) [**Eb83**, proof of Prop. 2.2] *Suppose $S_{j,j+1}=\varepsilon\in\{\pm 1\}$ for some $j\in\{1,...,n-1\}$. Then*

$$\sigma_j^{3\varepsilon}(\underline{e})=\delta_j(\underline{e}),\quad \sigma_j^{-3\varepsilon}(\underline{e})=\delta_{j+1}(\underline{e}).$$

(b) [**Eb83**, Prop. 2.2] *Suppose $S_{ij}\in\{0,1,-1\}$ for all i,j and that $(H_\mathbb{Z},L)$ is irreducible. Then the orbit of $\underline{e}$ under Br_n coincides with the orbit of $\underline{e}$ under $\mathrm{Br}_n\ltimes\{\pm 1\}^n$.*

3.3. Distinguished bases

In this section we continue the discussion of the braid group action on $\mathcal{B}^{tri}$. Now we fix one triangular basis $\underline{e}$. Definition 3.18 gives notations. The Remarks 3.19 pose questions on the orbit of $\underline{e}$ under $\mathrm{Br}_n\ltimes\{\pm 1\}^n$. The questions will guide the work which will be done in chapter 7.

DEFINITION 3.18. Let $(H_\mathbb{Z},L)$ be a unimodular lattice of rank $n\geq 2$ with nonempty set $\mathcal{B}^{tri}$ of triangular bases.

Given a triangular basis $\underline{e} \in \mathcal{B}^{tri}$ and $k \in \{0,1\}$, we are interested in the following orbits:

$$\begin{array}{rcl} \text{the set } \mathcal{B}^{dist} & := & \mathrm{Br}_n \ltimes \{\pm 1\}^n(\underline{e}) \text{ of } \textit{distinguished bases}, \\ \text{the set } \mathcal{R}^{(k),dist} & := & \mathrm{Br}_n(\pi_n^{(k)}(\underline{e})) \text{ of } \textit{distinguished tuples of} \\ & & \textit{reflections or transvections}, \\ \text{the set } \mathcal{S}^{dist} & := & \mathrm{Br}_n \ltimes \{\pm 1\}^n(S) \text{ of } \textit{distinguished matrices}, \\ & & \text{where } S = L(\underline{e}^t, \underline{e})^t \in T_n^{uni}(\mathbb{Z}), \end{array}$$

the quotient sets $\mathcal{B}^{dist}/\{\pm 1\}^n$ and $\mathcal{S}^{dist}/\{\pm 1\}^n$, which are Br_n orbits. We are also interested in the stabilizers in Br_n of the points $\underline{e}/\{\pm 1\}^n \in \mathcal{B}^{dist}/\{\pm 1\}^n$ and $S/\{\pm 1\}^n \in \mathcal{S}^{dist}/\{\pm 1\}^n$, namely the groups

$$(\mathrm{Br}_n)_{\underline{e}/\{\pm 1\}^n} \subset (\mathrm{Br}_n)_{S/\{\pm 1\}^n} \subset \mathrm{Br}_n.$$

REMARKS 3.19. In the situation of Definition 3.18 the following constraints on the set $\mathcal{B}^{dist}$ of distinguished bases are clear from what has been said,

$$\begin{aligned} \mathcal{B}^{dist} \quad \subset \quad & \mathcal{B}^{tri} \cap (\Delta^{(0)})^n \cap (\Delta^{(1)})^n \cap \{\underline{v} \in (H_{\mathbb{Z}})^n \,|\, \sum_{i=1}^n \mathbb{Z}v_i = H_{\mathbb{Z}}\} \\ & \cap (\pi_n \circ \pi_n^{(0)})^{-1}(-M) \cap (\pi_n \circ \pi_n^{(1)})^{-1}(M), \end{aligned} \tag{3.2}$$

where $\pi_n \circ \pi_n^{(k)} : (R^{(k)})^n \to O^{(k)}, \quad \underline{v} \mapsto s_{v_1}^{(k)} ... s_{v_n}^{(k)}$.

An interesting problem is which - if any - of these constraints are sufficient to characterize the orbit $\mathcal{B}^{dist}$. We are most interested in the questions whether the inclusions

$$\mathcal{B}^{dist} \quad \subset \quad \{\underline{v} \in (\Delta^{(0)})^n \,|\, \pi_n \circ \pi_n^{(0)}(\underline{v}) = -M\}, \tag{3.3}$$

$$\mathcal{B}^{dist} \quad \subset \quad \{\underline{v} \in (\Delta^{(1)})^n \,|\, \pi_n \circ \pi_n^{(1)}(\underline{v}) = M\}, \tag{3.4}$$

are equalities. We will study this problem systematically in chapter 7 for $n = 2$ and $n = 3$. In this section 3.3 we give some examples.

REMARKS 3.20. Consider a unimodular bilinear lattice $(H_{\mathbb{Z}}, L)$ of rank $n \geq 2$ and $k \in \{0; 1\}$.

(i) Two basic invariants of the $\mathrm{Br}_n \ltimes \{\pm 1\}^n$ orbit of a tuple $\underline{v} \in (R^{(k)})^n$ are the product $(\pi_n \circ \pi_n^{(k)})(\underline{v})) = s_{v_1}^{(k)} ... s_{v_n}^{(k)} \in O^{(k)}$ and the sublattice $\sum_{i=1}^n \mathbb{Z}v_i \subset H_{\mathbb{Z}}$, namely

$$(\pi_n \circ \pi_n^{(k)})(\sigma_j(\underline{v})) = (\pi_n \circ \pi_n^{(k)})(\underline{v}) \quad \text{and} \quad \sum_{i=1}^n \mathbb{Z}\sigma_j(\underline{v})_i = \sum_{i=1}^n \mathbb{Z}v_i.$$

(ii) A triangular basis $\underline{e} \in \mathcal{B}^{tri}$ induces the even and odd monodromy groups $\Gamma^{(0)}$ and $\Gamma^{(1)}$ and the sets $\Delta^{(0)}$ and $\Delta^{(1)}$ of even and odd vanishing cycles. Each distinguished basis $\underline{v} \in \mathcal{B}^{dist}$ induces the same even and odd monodromy groups $\Gamma^{(0)}$ and $\Gamma^{(1)}$ and the same sets $\Delta^{(0)}$ and $\Delta^{(1)}$ of even and odd vanishing cycles. This is obvious from the action of $\mathrm{Br}_n \ltimes \{\pm 1\}^n$ on $\mathcal{B}^{dist}$, the Hurwitz action of Br_n on $\mathcal{R}^{(k),dist}$ and the definition of $\Gamma^{(k)}$ and $\Delta^{(k)}$. So they are invariants of the set $\mathcal{B}^{dist}$ of distinguished bases.

We did not mention the monodromy M, because it is by Theorem 2.7 an invariant of $(H_{\mathbb{Z}}, L)$ if $\mathcal{B}^{tri} \neq \emptyset$, so it does not depend on the choice of a $\mathrm{Br}_n \ltimes \{\pm 1\}^n$ orbit in $\mathcal{B}^{tri}$.

(iii) A matrix $S \in T_n^{uni}(\mathbb{Z})$ determines a triple $(H_{\mathbb{Z}}, L, \underline{e})$ with $(H_{\mathbb{Z}}, L)$ a unimodular bilinear lattice and $\underline{e}$ a triangular basis with $S = L(\underline{e}^t, \underline{e})^t$ up to isomorphism. For a second matrix $\widetilde{S}$ in the $\mathrm{Br}_n \ltimes \{\pm 1\}^n$ orbit of S then a triangular basis $\widetilde{\underline{e}}$ of $(H_{\mathbb{Z}}, L)$ with $\widetilde{S} = L(\widetilde{\underline{e}}^t, \widetilde{\underline{e}})^t$ exists (but it is not unique in general). Therefore the triple $(H_{\mathbb{Z}}, L, \mathcal{B}^{dist})$ and all induced data depend only on the $\mathrm{Br}_n \ltimes \{\pm 1\}^n$ orbit $\mathcal{S}^{dist}$ of S. These induced data comprise $R^{(0)}$, $\Gamma^{(0)}$, $\Gamma^{(1)}$, $\Delta^{(0)}$ and $\Delta^{(1)}$.

(iv) Choose a triangular basis $\underline{e}$. Then $s_\delta^{(k)} \in \Gamma^{(k)}$ for $\delta \in \Delta^{(k)}$. This follows from the definition of a vanishing cycle and from formula 3.2. It also implies that the set $(\Delta^{(k)})^n$ is invariant under the action of $\mathrm{Br}_n \ltimes \{\pm 1\}^n$ on $(R^{(k)})^n$.

Remarks 3.21. (i) Consider a unimodular bilinear lattice which is not a lattice of type A_1^n and a fixed triangular basis $\underline{e}$. Then $\Delta^{(0)} \subset R^{(0)}$, but $\Delta^{(1)} \not\subset R^{(0)}$ by Corollary 6.22 (a). Nevertheless $\mathcal{B}^{dist} \subset (\Delta^{(0)} \cap \Delta^{(1)})^n$, so many odd vanishing cycles do not turn up in bases in the braid group orbit of $\underline{e}$, i.e. in distinguished bases.

(ii) In all cases except A_1^n $\Delta^{(1)} \not\subset \Delta^{(0)}$ because $\Delta^{(1)} \not\subset R^{(0)}$. In some cases $\Delta^{(0)} \subset \Delta^{(1)}$, in many cases $\Delta^{(0)} \not\subset \Delta^{(1)}$. See Corollary 6.22.

Given a unimodular bilinear lattice $(H_{\mathbb{Z}}, L)$, any element $g \in O^{(k)}$ acts on $(R^{(k)})^n$ by $g(\underline{v}) := (g(v_1), ..., g(v_n))$. Part (a) of the next Lemma 3.22 says especially that this action commutes with the action of $\mathrm{Br}_n \ltimes \{\pm 1\}^n$ on $(R^{(k)})^n$. Part (b) gives implications, which will be used to construct the interesting Examples 3.23 (i) and (ii).

Lemma 3.22. *Let $(H_{\mathbb{Z}}, L)$ be a unimodular bilinear lattice. Fix $k \in \{0, 1\}$.*

(a) If $g \in O^{(k)}$ and $(\alpha, \varepsilon) \in \mathrm{Br}_n \ltimes \{\pm 1\}^n$ then for $\underline{v} \in (R^{(k)})^n$

$$\begin{aligned} g((\alpha, \varepsilon)(\underline{v})) &= (\alpha, \varepsilon)(g(\underline{v})), \\ (\pi_n \circ \pi_n^{(k)})(g(\underline{v})) &= g \circ (\pi_n \circ \pi_n^{(k)})(\underline{v}) \circ g^{-1}. \end{aligned}$$

Here $g(\underline{v})$ means $(g(v_1), ..., g(v_n))$, and similarly for $g((\alpha, \varepsilon)(\underline{v}))$.

(b) If $g\in G_{\mathbb Z}^{(k)}-G_{\mathbb Z}$ and $\underline{e}\in\mathcal B^{tri}$ then

$$\begin{array}{rcl} g(\underline{e}) & \notin & \mathcal B^{tri},\\ \text{so especially}\quad g(\underline{e}) & \notin & \mathrm{Br}_n\ltimes\{\pm 1\}^n(\underline{e}),\\ \text{but}\quad (\pi_n\circ\pi_n^{(k)})(g(\underline{e})) & = & (\pi_n\circ\pi_n^{(k)})(\underline{e})=(-1)^{k+1}M. \end{array}$$

Proof (a) $g((\mathrm{id},\varepsilon)(\underline{v}))=(\mathrm{id},\varepsilon)(g(\underline{v}))$ is trivial. Consider $(\alpha,\varepsilon)=(\sigma_j,(1,...,1))=\sigma_j$.

$$\begin{array}{rcl} g(\sigma_j(\underline{v})) & = & (g(v_1),...,g(v_{j-1}),gs_{v_j}^{(k)}(v_{j+1}),g(v_j),g(v_{j+2}),...,g(v_n))\\ & = & (g(v_1),...,g(v_{j-1}),s_{g(v_j)}^{(k)}(gv_{j+1}),g(v_j),g(v_{j+2}),...,g(v_n))\\ & = & \sigma_j(g(\underline{v})), \end{array}$$

because of $gs_{v_j}^{(k)}g^{-1}=s_{g(v_j)}^{(k)}$ (Lemma 2.2 (c)).

$$\begin{array}{rcl} (\pi_n\circ\pi_n^{(k)})(g(\underline{v})) & = & s_{g(v_1)}^{(k)}...s_{g(v_n)}^{(k)}=\big(gs_{v_1}^{(k)}g^{-1}\big)...\big(gs_{v_n}^{(k)}g^{-1}\big)\\ & = & gs_{v_1}^{(k)}...s_{v_n}^{(k)}g^{-1}=g\circ(\pi_n\circ\pi_n^{(k)})(\underline{v})\circ g^{-1}. \end{array}$$

(b) Suppose $g\in G_{\mathbb Z}^{(k)}-G_{\mathbb Z}$ and $\underline{e}\in\mathcal B^{tri}$. Then $gMg^{-1}=M$ and

$$\begin{array}{rcl} (\pi_n\circ\pi_n^{(k)})(g(\underline{e})) & \stackrel{(a)}{=} & g\circ(\pi_n\circ\pi_n^{(k)})(\underline{e})\circ g^{-1}\\ & = & g\big((-1)^{k+1}M\big)g^{-1}=(-1)^{k+1}M=(\pi_n\circ\pi_n^{(k)})(\underline{e}). \end{array}$$

Furthermore $S=L(\underline{e}^t,\underline{e})^t\in T_n^{uni}(\mathbb Z)$, $I^{(k)}=L^t+(-1)^kL$ and

$$\begin{array}{rcl} I^{(k)}(g(\underline{e})^t,g(\underline{e})) & = & I^{(k)}(\underline{e}^t,\underline{e})=S+(-1)^kS^t\quad\text{because } g\in G_{\mathbb Z}^{(k)},\\ L(g(\underline{e})^t,g(\underline{e}))^t & \neq & L(\underline{e}^t,\underline{e})^t=S\quad\text{because } g\notin G_{\mathbb Z},\\ \text{so } L(g(\underline{e})^t,g(\underline{e})) & \notin & T_n^{uni}(\mathbb Z), \end{array}$$

so $g(\underline{e})\notin\mathcal B^{tri}$, so $g(\underline{e})\notin\mathrm{Br}_n\ltimes\{\pm 1\}^n(\underline{e})$. □

Examples 3.23. Let $(H_{\mathbb Z},L)$ be a unimodular bilinear lattice of rank $n\geq 2$, $\underline{e}\in\mathcal B^{tri}$ a triangular basis, $S=L(\underline{e}^t,\underline{e})^t\in T_n^{uni}(\mathbb Z)$ and $k\in\{0,1\}$.

(i) $k=1$, $n=3$, $S=S(-3,3,-3)=S(\mathbb P^2)$, the odd case $\mathbb P^2$. To carry out this example we need two results which will be proved later, Theorem 5.14 (b) and Theorem 6.21 (h).

By Theorem 5.14 (b) (i) there is a root $M^{root}\in G_{\mathbb Z}$ of the monodromy with $(M^{root})^3=M$ and

$$\begin{array}{rcl} G_{\mathbb Z}^{(1)} & = & \{\pm(M^{root})^l(\mathrm{id}+a(M^{root}-\mathrm{id})^2)\,|\,l,a\in\mathbb Z\}\\ & \supsetneqq & G_{\mathbb Z}=\{\pm(M^{root})^l\,|\,l\in\mathbb Z\}. \end{array}$$

Also, here $\mathrm{Rad}\, I^{(1)} = \mathbb{Z}f_3$ with $f_3 = e_1 + e_2 + e_3$. The shape of M^{root} in Theorem 5.14 (b) shows

$$(M^{root} - \mathrm{id})^2(\underline{e}) = \underline{e}(\begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} - E_3)^2 = f_3(1, -2, 1).$$

For example $g := \mathrm{id} + (M^{root} - \mathrm{id})^2 \in G_{\mathbb{Z}}^{(1)} - G_{\mathbb{Z}}$ satisfies

$$\begin{array}{rcl} g(\underline{e}) & = & \underline{e} + f_3(1, -2, 1), \\ I^{(1)}(g(\underline{e})^t, g(\underline{e})) & = & I^{(1)}(\underline{e}^t, \underline{e}) = S - S^t = \begin{pmatrix} 0 & -3 & 3 \\ 3 & 0 & -3 \\ -3 & 3 & 0 \end{pmatrix}, \\ L(g(\underline{e})^t, g(\underline{e}))^t & = & \begin{pmatrix} 3 & -7 & 5 \\ -4 & 9 & -7 \\ 2 & -4 & 3 \end{pmatrix}, \\ s^{(1)}_{g(e_1)} s^{(1)}_{g(e_2)} s^{(1)}_{g(e_3)} & = & M, \\ g(e_1), g(e_2), g(e_3) & \notin & R^{(0)}, \quad \text{because } 3 \neq 1 \text{ and } 9 \neq 1, \\ g(e_1), g(e_2), g(e_3) & \notin & \Delta^{(1)}. \end{array}$$

The last claim $g(e_j) \notin \Delta^{(1)}$ holds because of $g(e_j) \neq e_j$, but $g(e_j) \in e_j + \mathbb{Z}f_3$, and because the projection $H_{\mathbb{Z}} \to H_{\mathbb{Z}}/\mathbb{Z}f_3$ restricts to an injective map $\Delta^{(1)} \to H_{\mathbb{Z}}/\mathbb{Z}f_3$ by Theorem 6.21 (h).

(ii) $k = 0$, $n = 3$, $S = S(-2, 2, -2) = S(\mathcal{H}_{1,2})$, the even case $\mathcal{H}_{1,2}$. To carry out this example we need two results which will be proved later, Theorem 5.14 (a) and Theorem 6.14 (e).

Recall Theorem 5.14 (b) (i),

$$\begin{array}{rcl} (H_{\mathbb{Z}}, L) & = & (H_{\mathbb{Z},1}, L_1) \oplus (H_{\mathbb{Z},2}, L_2) \\ \text{with } H_{\mathbb{Z},1} & = & \mathbb{Z}f_1 \oplus \mathbb{Z}f_2 = \ker \Phi_2(M), \quad H_{\mathbb{Z},2} = \mathbb{Z}f_3 = \ker \Phi_1(M), \\ (f_1, f_2, f_3) & = & \underline{e} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \\ G_{\mathbb{Z}}^{(0)} & = & G_{\mathbb{Z},1}^{(0)} \times G_{\mathbb{Z},2}^{(0)} \supsetneqq G_{\mathbb{Z}} = G_{\mathbb{Z},1} \times G_{\mathbb{Z},2} \\ \text{with } G_{\mathbb{Z},1}^{(0)} & = & \mathrm{Aut}(H_{\mathbb{Z},1}) \supsetneqq G_{\mathbb{Z},1} = \{g \in \mathrm{Aut}(H_{\mathbb{Z},1}) \,|\, \det g = 1\}, \\ G_{\mathbb{Z},2}^{(0)} & = & G_{\mathbb{Z},2} = \{\pm \mathrm{id}|_{\mathbb{Z}f_3}\}. \end{array}$$

For example $g := ((f_1,f_2,f_3)\mapsto(f_1,-f_2,f_3))\in G_{\mathbb{Z}}^{(0)}-G_{\mathbb{Z}}$ satisfies

$$\begin{aligned} g(\underline{e}) &= g(f_3-f_2,-f_3+f_1+f_2,f_3-f_1) \\ &= (f_3+f_2,-f_3+f_1-f_2,f_3-f_1) \\ &= \underline{e}+f_2(2,-2,0), \\ s^{(0)}_{g(e_1)}s^{(0)}_{g(e_2)}s^{(0)}_{g(e_3)} &= -M, \end{aligned}$$

$$\begin{aligned} I^{(0)}(g(\underline{e})^t,g(\underline{e})) &= I^{(0)}(\underline{e}^t,\underline{e}) = S+S^t, \\ L(g(\underline{e})^t,g(\underline{e}))^t &= \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix} = S^t \neq S = L(\underline{e}^t,\underline{e})^t, \\ g(\underline{e}) &\notin \mathcal{B}^{tri}, \text{ so } g(\underline{e})\notin\mathcal{B}^{dist}, \\ g(e_1),g(e_2),g(e_3) &\in \Delta^{(0)}. \end{aligned}$$

The last claim $g(e_j)\in\Delta^{(0)}$ holds because of Theorem 6.14 (e). Therefore

$$g(\underline{e})\in\{\underline{v}\in(\Delta^{(0)})^3\,|\,(\pi_3\circ\pi_3^{(0)})(\underline{v})=-M,\sum_{i=1}^3\mathbb{Z}v_i=H_{\mathbb{Z}}\}.$$

Especially, here the inclusion in (3.3) is not an equality. In a certain sense, this example is the worst case within all cases $S(\underline{x})$ with $k=0$, $n=3$ and eigenvalues of M unit roots. See Theorem 7.3 (b).

(iii) $n\in\mathbb{N}$, $S=E_n$, the even and odd case A_1^n. Compare Lemma 2.12.

$$\begin{aligned} \Delta^{(0)} &= R^{(0)}=\Delta^{(1)}=\{\pm e_1,...,\pm e_n\}, \\ M &= \mathrm{id}, \\ \mathcal{B}^{dist} &= \{(\varepsilon_1 e_{\sigma(1)},...,\varepsilon_n e_{\sigma(n)})\,|\,\varepsilon_1,...,\varepsilon_n\in\{\pm1\},\sigma\in S_n\}, \\ &= \{\underline{v}\in(\Delta^{(0)})^n\,|\,(\pi_n\circ\pi_n^{(0)})(\underline{e})=-M=-\mathrm{id}\} \end{aligned}$$

The last equality follows from $-M=-\mathrm{id}$ and

$$s^{(0)}_{e_i}|_{\mathbb{Z}e_i}=-\mathrm{id},\quad s^{(0)}_{e_i}|_{\sum_{j\neq i}\mathbb{Z}e_j}=\mathrm{id}\,.$$

Here the inclusion in (3.3) is an equality. On the contrary, the inclusion (3.4) is not an equality, the set

$$\{\underline{v}\in(\Delta^{(1)})^n\,|\,(\pi_n\circ\pi_n^{(1)})(\underline{v})=M=\mathrm{id}\}$$

is much bigger than $\mathcal{B}^{dist}$ if $n\geq 2$, it consists of many $\mathrm{Br}_n\ltimes\{\pm1\}^n$ orbits. Each of these orbits contains a unique one of the following tuples,

$$(e_{l(1)},e_{l(2)},...,e_{l(n)})\quad\text{with}\quad 1\leq l(1)\leq l(2)\leq...\leq l(n)\leq n.$$

This follows from $s^{(1)}_{e_i}=\textup{id}$.

The braids act just by permutations on the $\textup{Br}_n$ orbit $\mathcal{B}^{dist}/\{\pm 1\}^n$. Therefore the stabilizer of $\underline{e}/\{\pm 1\}^n$ is the group $\textup{Br}_n^{pure}$ of pure braids. The stabilizer of $S/\{\pm 1\}^n$ is the whole group $\textup{Br}_n$.

(iv) Reconsider Example 3.4, so a case where

$$\Gamma^{(k)}=\left\{\begin{array}{ll} G^{fCox,n} & \textup{with generators }s^{(0)}_{e_1},...,s^{(0)}_{e_n}\textup{ if }k=0,\\ G^{free,n} & \textup{with generators }s^{(1)}_{e_1},...,s^{(1)}_{e_n}\textup{ if }k=1.\end{array}\right.$$

In the notation of Definition 3.1 $\Delta(G^{fCox,n})=\{s^{(0)}_\delta\,|\,\delta\in\Delta^{(0)}\}$ if $k=0$ and $\Delta(G^{free,n})=\{s^{(1)}_\delta\,|\,\delta\in\Delta^{(1)}\}$ if $k=1$. By Theorem 3.2

$$\mathcal{R}^{dist,(k)}=\{(s^{(k)}_{v_1},...,s^{(k)}_{v_n})\,|\,\underline{v}\in(\Delta^{(k)})^n, s^{(k)}_{v_1}...s^{(k)}_{v_n}=(-1)^{k+1}M\}.$$

Furthermore, the shape of $\Gamma^{(k)}$ shows that $(H_{\mathbb{Z}},L,\underline{e})$ is irreducible. Lemma 3.15 (b) or (c) applies. Therefore

$$\mathcal{B}^{dist}=\{\underline{v}\in(\Delta^{(k)})^n\,|\,s^{(k)}_{v_1}...s^{(k)}_{v_n}=(-1)^{k+1}M\}.$$

So in this case only the constraints $\underline{v}\in(\Delta^{(k)})^n$ and $s^{(k)}_{v_1}...s^{(k)}_{v_n}=(-1)^{k+1}M$ in the Remarks 3.19 are needed in order to characterize the orbit $\mathcal{B}^{dist}$. The inclusions in (3.3) and (3.4) are here equalities.

By Theorem 3.2 the stabilizer of $\pi^{(k)}_n(\underline{e})$ and of $\underline{e}/\{\pm 1\}^n$ is $\{\textup{id}\}\subset\textup{Br}_n$. The size of the stabilizer $(\textup{Br}_n)_{S/\{\pm 1\}^n}$ depends on the case. Theorem 7.11 gives cases with $n=3$ where it is $\langle\sigma_2\sigma_1\rangle$ or $\langle\sigma_2\sigma_1^2\rangle$ or $\langle\sigma^{mon}\rangle$.

(v) Suppose $S_{ij}\leq 0$ for $i<j$, so $(H_{\mathbb{Z}},L,\underline{e})$ is a generalized Cartan lattice as in Theorem 3.7. By Theorem 3.7 (b)

$$\mathcal{R}^{dist,(0)} = \{(g_1,...,g_n)\in\bigl(\{s^{(0)}_\delta\,|\,\delta\in\Delta^{(0)}\}\bigr)^n\,|\,g_1...g_n=-M\}.$$

With Lemma 3.15 (b) this implies

$$\mathcal{B}^{dist} = \{\underline{v}\in(\Delta^{(0)})^n\,|\,s^{(k)}_{v_1}...s^{(k)}_{v_n}=-M\}.$$

Also here only the two constraints $\underline{v}\in(\Delta^{(0)})^n$ and $s^{(k)}_{v_1}...s^{(k)}_{v_n}=-M$ in the Remarks 3.19 are needed in order to characterize the orbit $\mathcal{B}^{dist}$. The inclusion in (3.3) is here an equality.

3.4. From $\textup{Br}_n\ltimes\{\pm 1\}^n$ to $G_{\mathbb{Z}}$

Definition 3.24 gives a map $Z:\textup{Br}_n\ltimes\{\pm 1\}^n\to\textup{Aut}(H_{\mathbb{Z}})$, which restricts to a group antihomomorphism $Z:(\textup{Br}_n\ltimes\{\pm 1\}^n)_S\to G_{\mathbb{Z}}$. The definition and the restriction to a group antihomomorphism are classical. Lemma 3.25 provides basic facts around this map. Also Theorem 3.26 (b) is classical. It states that $Z((\delta_n^{1-k}\sigma^{root})^n)=(-1)^{k+1}M$.

Theorem 3.26 (c) gives a condition when $Z(\delta_n^{1-k}\sigma^{root})\in G_{\mathbb{Z}}$. Then this is an n-th root of $(-1)^{k+1}M$. Theorem 3.26 (c) embraces Theorem

4.5 (a)+(b) in [**BH20**]. It gives more because in [**BH20**] no braids are considered. The braids allow a new and more elegant proof than the one in [**BH20**]. Theorem 3.26 (c) will be used in the discussion of the groups $G_{\mathbb{Z}}$ in many cases in chapter 5.

DEFINITION 3.24. Let $(H_{\mathbb{Z}}, L, \underline{e})$ be a unimodular bilinear lattice of rank $n \geq 2$ with a triangular basis $\underline{e}$. For $(\alpha, \varepsilon) \in \mathrm{Br}_n \ltimes \{\pm 1\}^n$ define an automorphism $Z((\alpha,\varepsilon)) \in Aut(H_{\mathbb{Z}})$ by the following action on the $\mathbb{Z}$-basis $\underline{e}$ of $H_{\mathbb{Z}}$,

$$\begin{aligned} Z : \mathrm{Br}_n \ltimes \{\pm 1\}^n &\to \mathrm{Aut}(H_{\mathbb{Z}}), \\ (\alpha,\varepsilon) &\mapsto Z((\alpha,\varepsilon)) = (\underline{e} \to (\alpha,\varepsilon)(\underline{e})). \end{aligned}$$

LEMMA 3.25. *Let $(H_{\mathbb{Z}}, L, \underline{e})$ be a unimodular bilinear lattice of rank $n \geq 2$ with a triangular basis $\underline{e}$.*

(a) For $(\alpha,\varepsilon) \in \mathrm{Br}_n \ltimes \{\pm 1\}^n$

$$Z((\alpha,\varepsilon)) \in G_{\mathbb{Z}} \iff Z((\alpha,\varepsilon)) \in G_{\mathbb{Z}}^{(0)} \iff Z((\alpha,\varepsilon)) \in G_{\mathbb{Z}}^{(1)}.$$

(b) The stabilizer of S in $\mathrm{Br}_n \ltimes \{\pm 1\}^n$ is

$$(\mathrm{Br}_n \ltimes \{\pm 1\}^n)_S = \{(\alpha,\varepsilon) \in \mathrm{Br}_n \ltimes \{\pm 1\}^n \,|\, Z((\alpha,\varepsilon)) \in G_{\mathbb{Z}}\}.$$

(c) The restriction of the map Z to the stabilizer $(\mathrm{Br}_n \ltimes \{\pm 1\}^n)_S$ is also denoted Z,

$$Z : (\mathrm{Br}_n \ltimes \{\pm 1\}^n)_S \to G_{\mathbb{Z}}.$$

It is a group antihomomorphism with kernel the stabilizer $(\mathrm{Br}_n \ltimes \{\pm 1\}^n)_{\underline{e}}$ of $\underline{e}$.

(d) The triple $(H_{\mathbb{Z}}, L, \mathcal{B}^{dist})$ with $\mathcal{B}^{dist} = \mathrm{Br}_n \ltimes \{\pm 1\}^n(\underline{e})$ the set of distinguished bases (Definition 3.18) gives rise to the subgroup $G_{\mathbb{Z}}^{\mathcal{B}}$ of $G_{\mathbb{Z}}$,

$$G_{\mathbb{Z}}^{\mathcal{B}} := \mathrm{Aut}(H_{\mathbb{Z}}, L, \mathcal{B}^{dist}) := \{g \in G_{\mathbb{Z}} \,|\, g(\mathcal{B}^{dist}) = \mathcal{B}^{dist}\} \subset G_{\mathbb{Z}}.$$

It does not depend on $\underline{e}$, but only on the triple $(H_{\mathbb{Z}}, L, \mathcal{B}^{dist})$. Then $G_{\mathbb{Z}}^{\mathcal{B}}$ is the image of $(\mathrm{Br}_n \ltimes \{\pm 1\}^n)_S$ under Z in $G_{\mathbb{Z}}$,

$$G_{\mathbb{Z}}^{\mathcal{B}} = Z((\mathrm{Br}_n \ltimes \{\pm 1\}^n)_S) \subset G_{\mathbb{Z}}.$$

(e) The subgroup $Z((\{\pm 1\}^n)_S)$ of $G_{\mathbb{Z}}^{\mathcal{B}}$ is a normal subgroup of $G_{\mathbb{Z}}^{\mathcal{B}}$, and the group antihomomorphism Z in part (c) induces a group antihomomorphism

$$\overline{Z} : (\mathrm{Br}_n)_{S/\{\pm 1\}^n} \to G_{\mathbb{Z}}^{\mathcal{B}} / Z((\{\pm 1\}^n)_S)$$

with kernel $(\mathrm{Br}_n)_{\underline{e}/\{\pm 1\}^n}$, which is isomorphic to $(\mathrm{Br}_n \ltimes \{\pm 1\}^n)_{\underline{e}}$.

(f) Suppose that $(H_{\mathbb{Z}}, L, \underline{e})$ is irreducible (Definition 2.10 (a)). Then

$$\begin{aligned}(\{\pm 1\}^n)_S &= \{(1,...,1),(-1,...,-1)\},\\ Z((-1,...,-1)) &= -\mathrm{id} \in G_{\mathbb{Z}},\\ Z((\{\pm 1\}^n)_S) &= \{\pm \mathrm{id}\}.\end{aligned}$$

$\{\pm \mathrm{id}\}$ *is a normal subgroup of* $G_{\mathbb{Z}}$. *The group antihomorphism* $\overline{Z}$ *in part (d) becomes*

$$\overline{Z} : (\mathrm{Br}_n)_{S/\{\pm 1\}^n} \to G_{\mathbb{Z}}/\{\pm \mathrm{id}\}$$

with kernel $(\mathrm{Br}_n)_{\underline{e}/\{\pm 1\}^n}$ *and image* $G_{\mathbb{Z}}^{\mathcal{B}}/\{\pm \mathrm{id}\}$.

Proof: (a) Fix $k \in \{0,1\}$ and $(\alpha,\varepsilon) \in \mathrm{Br}_n \ltimes \{\pm 1\}^n$. Then

$$Z((\alpha,\varepsilon)) \in G_{\mathbb{Z}}^{(k)} \iff I^{(k)}((\alpha,\varepsilon)(\underline{e})^t, (\alpha,\varepsilon)(\underline{e})) = S + (-1)^k S^t.$$

If this equality holds then $I^{(k)} = L^t + (-1)^k L$ and $L((\alpha,\varepsilon)(\underline{e})^t, (\alpha,\varepsilon)(\underline{e}))^t \in T_n^{uni}(\mathbb{Z})$ imply $L((\alpha,\varepsilon)(\underline{e})^t, (\alpha,\varepsilon)(\underline{e}))^t = S$, so $Z((\alpha,\varepsilon)) \in G_{\mathbb{Z}}$.

(b) Trivial with the compatibility of the actions of $\mathrm{Br}_n \ltimes \{\pm 1\}^n$ on $\mathcal{B}^{dist}$ and on $T_n^{uni}(\mathbb{Z})$ in Theorem 3.4 (d).

(c) The following calculation shows that the map $Z : (\mathrm{Br}_n \ltimes \{\pm 1\}^n)_S \to G_{\mathbb{Z}}$ is a group antihomomorphism,

$$\begin{aligned} Z((\alpha,\varepsilon)(\beta,\widetilde{\varepsilon}))(\underline{e}) &= (\alpha,\varepsilon)(\beta,\widetilde{\varepsilon})(\underline{e})\\ &= (\alpha,\varepsilon)(Z((\beta,\widetilde{\varepsilon}))(e_1),...,Z((\beta,\widetilde{\varepsilon}))(e_n))\\ &\stackrel{3.22\,(a)}{=} Z((\beta,\widetilde{\varepsilon}))(\alpha,\varepsilon)(\underline{e}) = Z((\beta,\widetilde{\varepsilon}))Z((\alpha,\varepsilon))(\underline{e}).\end{aligned}$$

It is trivial that the kernel of this map Z is $(\mathrm{Br}_n \ltimes \{\pm 1\}^n)_{\underline{e}}$.

(d) $G_{\mathbb{Z}} \subset Z((\mathrm{Br}_n \ltimes \{\pm 1\}^n)_S)$: Consider $g \in G_{\mathbb{Z}}^{\mathcal{B}}$. Then $g(\underline{e}) \in \mathcal{B}^{dist}$ comes with same matrix S as $\underline{e}$ because g respects L. There is a pair $(\alpha,\varepsilon) \in \mathrm{Br}_n \ltimes \{\pm 1\}^n$ with $Z((\alpha,\varepsilon))(\underline{e}) = (\alpha,\varepsilon)(\underline{e}) = g(\underline{e})$. Therefore $g = Z((\alpha,\varepsilon))$.

$G_{\mathbb{Z}} \supset Z((\mathrm{Br}_n \ltimes \{\pm 1\}^n)_S)$: Consider $(\alpha,\varepsilon) \in (\mathrm{Br}_n \ltimes \{\pm 1\}^n)_S$, $(\beta,\widetilde{\varepsilon}) \in \mathrm{Br}_n \ltimes \{\pm 1\}^n$ and $\underline{v} := (\beta,\widetilde{\varepsilon})(\underline{e})$. We have to show $Z((\alpha,\varepsilon))(\underline{v}) \in \mathcal{B}^{dist}$. This is rather obvious with the commutativity of the actions of $O^{(k)}$ and $\mathrm{Br}_n \ltimes \{\pm 1\}^n$ in Lemma 3.22 (a),

$$\begin{aligned} Z((\alpha,\varepsilon))(\underline{v}) &= Z((\alpha,\varepsilon))(\beta,\widetilde{\varepsilon})(\underline{e})\\ &\stackrel{3.22(a)}{=} (\beta,\widetilde{\varepsilon})Z((\alpha,\varepsilon))(\underline{e})\\ &= (\beta,\widetilde{\varepsilon})(\alpha,\varepsilon)(\underline{e}).\end{aligned}$$

(e) Elementary group theory gives the group isomorphisms

$$\begin{aligned}(\mathrm{Br}_n)_{S/\{\pm 1\}^n} &\cong \frac{(\mathrm{Br}_n \ltimes \{\pm 1\}^n)_S}{(\{\pm 1\}^n)_S},\\ (\mathrm{Br}_n)_{\underline{e}/\{\pm 1\}^n} &\cong \frac{(\mathrm{Br}_n \ltimes \{\pm 1\}^n)_{\underline{e}}}{(\{\pm 1\}^n)_{\underline{e}}} \cong (\mathrm{Br}_n \ltimes \{\pm 1\}^n)_{\underline{e}}\end{aligned}$$

Here use $(\{\pm 1\}^n)_{\underline{e}} = \{(1,...,1)\}$. $\{\pm 1\}^n$ is a normal subgroup of $\mathrm{Br}_n \ltimes \{\pm 1\}^n$. Therefore $(\{\pm 1\}^n)_S$ is a normal subgroup of $(\mathrm{Br}_n \ltimes \{\pm 1\}^n)_S$. Therefore $Z((\{\pm 1\}^n)_S)$ is a normal subgroup of $G_{\mathbb{Z}}^{\mathcal{B}}$. Therefore $\overline{Z}$ is well defined. Its kernel is still $(\mathrm{Br}_n)_{\underline{e}/\{\pm 1\}^n} \cong (\mathrm{Br}_n \ltimes \{\pm 1\}^n)_{\underline{e}}$ because $(\mathrm{Br}_n \ltimes \{\pm 1\}^n)_{\underline{e}} \cap (\{\pm 1\}^n)_S = (\{\pm 1\}^n)_{\underline{e}} = \{(1,...,1)\}$.

(f) If $(H_{\mathbb{Z}}, L, \underline{e})$ is irreducible, then the following graph is connected: its vertices are $e_1,...,e_n$, and it has an edge between e_i and e_j for $i < j$ if $S_{ij}(= L(e_j, e_i)) \neq 0$. Therefore then $(\{\pm 1\}^n)_S = \{(1,...,1),(-1,...,-1)\}$. Everything else follows from this and from part (d). □

The antihomomorphism $Z : (\mathrm{Br}_n \ltimes \{\pm 1\}^n)_S \to G_{\mathbb{Z}}$ is not always surjective, but in many cases. See Theorem 3.28 and the Remarks 3.29. Theorem 3.26 (b) writes $(-1)^{k+1}M$ as an image of a braid by Z. Theorem 3.26 (c) gives conditions when it has an n-th root which is also an image of a braid by Z.

Theorem 3.26. *Let $(H_{\mathbb{Z}}, L, \underline{e})$ be a unimodular bilinear lattice of rank $n \geq 2$ with a triangular basis $\underline{e}$ and matrix $S = L(\underline{e}^t, \underline{e})^t \in T_n^{uni}(\mathbb{Z})$. Fix $k \in \{0,1\}$. Recall from chapter 3.1*

$$\begin{aligned}\sigma^{root} &:= \sigma_{n-1}\sigma_{n-2}...\sigma_2\sigma_1 \in \mathrm{Br}_n,\\ \sigma^{mon} &:= (\sigma^{root})^n,\\ \mathrm{center}(\mathrm{Br}_n) &= \langle \sigma^{mon} \rangle.\end{aligned}$$

(a)

$$\begin{aligned}\delta_n^{1-k}\sigma^{root}(\underline{e}) &= Z(\delta_n^{1-k}\sigma^{root})(\underline{e})\\ &= (s_{e_1}^{(k)}(e_2), s_{e_1}^{(k)}(e_3), ..., s_{e_1}^{(k)}(e_n), s_{e_1}^{(k)}(e_1))\\ &= \underline{e} \cdot R\end{aligned}$$

$$\text{with } R := \left(\begin{array}{ccc|c} -q_{n-1} & \dots & -q_1 & -q_0 \\ \hline & & & \\ & E_{n-1} & & \\ & & & \end{array}\right)$$

$$\text{so } R_{ij} = \begin{cases} -q_{n-j} & \text{if } i = 1,\\ \delta_{i-1,j} & \text{if } i \geq 2,\end{cases}$$

where $q_0=(-1)^k$, $q_{n+1-j}=S_{1j}$ *for* $j\in\{2,...,n\}$.

(b)

$$\begin{array}{rcl} Z((\delta_n^{1-k}\sigma^{root})^n) &=& (-1)^{k+1}M,\\ \text{so especially } (\delta_n^{1-k}\sigma^{root})^n &\in& (\mathrm{Br}_n\ltimes\{\pm 1\}^n)_S. \end{array}$$

(c) Write $q_0=(-1)^k$, $q_{n+1-j}=S_{1j}$ *for* $j\in\{2,...,n\}$ *as in part (a) and additionally* $q_n:=1$. *Suppose* $q_{n-j}=q_0q_j$ *for* $j\in\{1,...,n-1\}$, *and suppose that* S *has the following shape,*

$$\begin{array}{rcl} S &:=& \begin{pmatrix} 1 & q_{n-1} & \dots & q_1\\ & \ddots & \ddots & \vdots\\ & & \ddots & q_{n-1}\\ & & & 1\end{pmatrix},\\ \text{so } S_{ij} &=& \begin{cases} 0 & \text{if } i>j,\\ q_{n-(j-i)} & \text{if } i\leq j,\end{cases} \end{array}$$

Then

$$\begin{array}{rcl} M^{root} &:=& Z(\delta_n^{1-k}\sigma^{root})\in G_{\mathbb{Z}}\\ \text{with } (M^{root})^n &=& (-1)^{k+1}M. \end{array}$$

M^{root} *is regular and cyclic and has the characteristic polynomial* $q(t):=\sum_{i=0}^n q_it^i\in\mathbb{Z}[t]$.

Proof: (a) The second line follows from the definition of the action of $\delta_n^{1-k}\sigma^{root}$ on $\underline{e}$. For the third line observe $s_{e_1}^{(k)}(e_j)=e_j-S_{1j}e_1$ for $j\geq 2$ and $s_{e_1}^{(k)}(e_1)=-q_0e_1$.

(b) Use part (a) and

$$s^{(k)}_{s^{(k)}_{e_1}(e_2)}(s^{(k)}_{e_1}(e_j))=s^{(k)}_{e_1}s^{(k)}_{e_2}(s^{(k)}_{e_1})^{-1}s^{(k)}_{e_1}(e_j)=s^{(k)}_{e_1}s^{(k)}_{e_2}(e_j)$$

to find

$$(\delta_n^{1-k}\sigma^{root})^2(\underline{e})=(s^{(k)}_{e_1}s^{(k)}_{e_2}(e_3),...,s^{(k)}_{e_1}s^{(k)}_{e_2}(e_n),s^{(k)}_{e_1}s^{(k)}_{e_2}(e_1),s^{(k)}_{e_1}s^{(k)}_{e_2}(e_2)).$$

One continues inductively and finds

$$(\delta_n^{1-k}\sigma^{root})^n(\underline{e})=(s^{(k)}_{e_1}...s^{(k)}_{e_n}(e_1),...,s^{(k)}_{e_1}...s^{(k)}_{e_n}(e_n))=(-1)^{k+1}M(\underline{e}),$$

so $Z((\delta_n^{1-k}\sigma^{root})^n)=(-1)^{k+1}M$.

(c) If S is as in part (c) then

$$\begin{array}{rcl} I^{(k)}(\delta_n^{1-k}\sigma^{root}(\underline{e})^t,\delta_n^{1-k}\sigma^{root}(\underline{e})) &\overset{(1)}{=}& I^{(k)}((e_2\ e_3\ ...\ e_n\ e_1)^t,(e_2\ e_3\ ...\ e_n\ e_1))\\ &\overset{(2)}{=}& S+(-1)^kS^t=I^{(k)}(\underline{e}^t,\underline{e}). \end{array}$$

Here $\stackrel{(1)}{=}$ uses $s^{(k)}_{e_1} \in O^{(k)}$, and $\stackrel{(2)}{=}$ uses that $I^{(k)}(\underline{e}^t, \underline{e}) = S + (-1)^k S^t$ and that S is as in part (c).

Therefore $M^{root} := Z(\delta_n^{1-k}\sigma^{root}) \in G_{\mathbb{Z}}^{(k)}$, so by Lemma 3.25 (a)

$$M^{root} \in G_{\mathbb{Z}} \quad \text{and} \quad \delta_n^{1-k}\sigma^{root} \in (\mathrm{Br}_n \ltimes \{\pm 1\}^n)_S.$$

Also

$$(M^{root})^n = (Z(\delta_n^{1-k}\sigma^{root}))^n = Z((\delta_n^{1-k}\sigma^{root})^n) = (-1)^{k+1}M.$$

Let $\underline{e}^*$ be the $\mathbb{Z}$-basis of $H_{\mathbb{Z}}$ which is left L-dual to the $\mathbb{Z}$-basis $\underline{e}$, so with $L((\underline{e}^*)^t, \underline{e}) = E_n$. Remark 4.8 in [**BH20**] says

$$M^{root}\underline{e}^* = \underline{e}^* R^{-t} = \underline{e}^* \cdot \left(\begin{array}{c|c} & -q_0 \\ \hline & -q_1 \\ E_{n-1} & \vdots \\ & -q_{n-1} \end{array}\right).$$

The matrix R^{-t} is the companion matrix of the polynomial $q(t)$. Therefore M^{root} is regular, cyclic with generating vector $c = e_1^*$ and has the characteristic polynomial $q(t)$. □

Remarks 3.27. The main part of part (c) of Theorem 3.26 has also the following matrix version: For $q(t) = \sum_{i=0}^n q_i t^i \in \mathbb{Z}[t]$ with $q_n = 1$, $q_0 = (-1)^k$ for some $k \in \{0; 1\}$ and $q_{n-j} = q_0 q_j$ the matrix R in part (a) and the matrix S in part (c) of Theorem 3.26 satisfy

$$R^n = (-1)^{k+1} S^{-1} S^t.$$

A proof using matrices of this version of part (c) of Theorem 3.26 was given in [**BH20**, Theorem 4.5 (a)+(b)]. The proof here with the braid group action is more elegant.

The antihomomorphism $Z : (\mathrm{Br}_n \ltimes \{\pm 1\}^n)_S \to G_{\mathbb{Z}}$ is not surjective in general. A simple example with $n = 4$ is given in the Remarks 3.29. But it is surjective in the case $n = 1$, in all cases with $n = 2$ and in almost all cases with $n = 3$. Theorem 3.28 gives precise statements. Its proof requires first a good control of the braid group action on $T_3^{uni}(\mathbb{Z})$, which is the subject of chapter 4 and second complete knowledge of the group $G_{\mathbb{Z}}$ for all cases with $n \leq 3$, which is the subject of chapter 5. Theorem 3.28 is proved within the theorems in chapter 5 which treat the different cases with $n \in \{1, 2, 3\}$, namely Lemma 5.4 (the cases A_1^n), Theorem 5.5 (the rank 2 cases), Theorem 5.13 (the reducible rank 3 cases), Theorem 5.14 (the irreducible rank 3 cases with all eigenvalues in S^1), Theorem 5.16 (some special other rank 3 cases), Theorem 5.18 (the rest of the rank 3 cases).

THEOREM 3.28. *Let $(H_{\mathbb{Z}}, L, \underline{e})$ be a unimodular bilinear lattice of rank $n \leq 3$ with triangular basis $\underline{e}$ and matrix $S = L(\underline{e}^t, \underline{e})^t \in T_n^{uni}(\mathbb{Z})$. The group antihomomorphism $Z : (\mathrm{Br}_n \ltimes \{\pm 1\}^n)_S \to G_{\mathbb{Z}}$ is not surjective in the four cases with $n = 3$ where S is in the $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbit of $S(\underline{x})$ with*

$$\underline{x} \in \{(3,3,4), (4,4,4), (5,5,5), (4,4,8)\},$$

so then $G_{\mathbb{Z}} \supsetneqq G_{\mathbb{Z}}^{\mathcal{B}}$. It is surjective in all other cases with $n \leq 3$, so then $G_{\mathbb{Z}} = G_{\mathbb{Z}}^{\mathcal{B}}$.

REMARKS 3.29. (i) Contrary to Theorem 3.28 for the cases $n = 1, 2, 3$, it is in the cases $n \geq 4$ easy to find matrices $S \in T_n^{uni}(\mathbb{Z})$ such that the group antihomomorphism $Z : (\mathrm{Br}_n \ltimes \{\pm 1\}^n)_S \to G_{\mathbb{Z}}$ is not surjective. Though the construction which we propose in part (ii) and carry out in one example in part (iii) leads to matrices which are rather particular. For a given matrix S it is in general not easy to see whether Z is surjective or not.

(ii) Consider a reducible unimodular bilinear lattice $(H_{\mathbb{Z}}, L, \underline{e})$ of rank n with triangular basis $\underline{e}$ and matrix $S = L(\underline{e}^t, \underline{e})^t \in T_n^{uni}(\mathbb{Z})$. There are an L-orthogonal decomposition $H_{\mathbb{Z}} = \bigoplus_{j=1}^l H_{\mathbb{Z},j}$ with $l \geq 2$ and $\operatorname{rank} H_{\mathbb{Z},j} \geq 1$ and a surjective map $\alpha : \{1, ..., n\} \to \{1, ..., l\}$ with $e_i \in H_{\mathbb{Z},\alpha(i)}$. Then for $k \in \{0, 1\}$

$$\Gamma^{(k)}\{e_i\} \subset H_{\mathbb{Z},\alpha(i)},$$

$$\text{so} \quad \Delta^{(k)} \subset \bigcup_{j=1}^l H_{\mathbb{Z},j}.$$

Especially any $g \in G_{\mathbb{Z}}^{\mathcal{B}} \subset G_{\mathbb{Z}}$ maps each e_i to an element of $\bigcup_{j=1}^l H_{\mathbb{Z},j}$. This does not necessarily hold for any $g \in G_{\mathbb{Z}}$. Part (iii) gives an example.

(iii) Consider the unimodular bilinear lattice $(H_{\mathbb{Z}}, L, \underline{e})$ of rank 4 with triangular basis $\underline{e}$ and matrix

$$S = L(\underline{e}^t, \underline{e})^t = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in T^{uni,4}(\mathbb{Z}).$$

Then $H_{\mathbb{Z}} = H_{\mathbb{Z},1} \oplus H_{\mathbb{Z},2}$ with $H_{\mathbb{Z},1} = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ and $H_{\mathbb{Z},2} = \mathbb{Z}e_3 \oplus \mathbb{Z}e_4$. The $\mathbb{Z}$-linear map $g : H_{\mathbb{Z}} \to H_{\mathbb{Z}}$ with

$$(g(e_1), g(e_2), g(e_3), g(e_4)) = (e_1 + (e_3 - e_4), e_2 + (e_3 - e_4),$$
$$e_3 + (e_1 - e_2), e_4 + (e_1 - e_2))$$

is not in $G_{\mathbb{Z}}^{\mathcal{B}}$ because $g(e_1), g(e_2), g(e_3), g(e_4) \notin H_{\mathbb{Z},1} \cup H_{\mathbb{Z},2}$. But $g \in G_{\mathbb{Z}}$ because

$$L(e_1 - e_2, e_1 - e_2) = 0 = L(e_3 - e_4, e_3 - e_4),$$

so $\quad L(g(e_i), g(e_j)) = L(e_i, e_j) \quad$ for $\{i,j\} \subset \{1,2\}$ or $\{i,j\} \subset \{3,4\}$,

and also

$L(g(e_i), g(e_j)) = L(e_i, e_j) \quad$ for $(i,j) \in (\{1,2\} \times \{3,4\}) \cup (\{3,4\} \times \{1,2\})$.

So here $g \in G_{\mathbb{Z}} - G_{\mathbb{Z}}^{\mathcal{B}}$, so $G_{\mathbb{Z}}^{\mathcal{B}} \subsetneqq G_{\mathbb{Z}}$.

CHAPTER 4

Braid group action on upper triangular 3×3 matrices

The subject of this chapter is the case $n = 3$ of the action in Lemma 3.13 of $\mathrm{Br}_n \ltimes \{\pm 1\}^n$ on the matrices in $T_n^{uni}(\mathbb{Z})$.

In section 4.1 the action on $T_3^{uni}(\mathbb{R})$ is made concrete. The (quotient) group of actions is given in new generators. It is

$$(G^{phi} \ltimes G^{sign}) \rtimes \langle \gamma \rangle \cong (G^{phi} \rtimes \langle \gamma \rangle) \ltimes G^{sign},$$

where G^{phi} is a free Coxeter group with three generators, G^{sign} is the group of actions in $T_3^{uni}(\mathbb{R})$ which the sign group $\{\pm 1\}^3$ induces, and γ acts cyclically of order 3. In fact, $G^{phi} \rtimes \langle \gamma \rangle \cong PSL_2(\mathbb{Z})$, so we have a nonlinear action of $PSL_2(\mathbb{Z})$, but this way to look at it is less useful than the presentation as $G^{phi} \rtimes \langle \gamma \rangle$.

The action on $T_3^{uni}(\mathbb{Z})$ had been studied already by Krüger [**Kr90**, §12] and by Cecotti-Vafa [**CV93**, Ch. 6.2]. Section 4.2 recovers and refines their results. Like them, it puts emphasis on the cases where the monodromy of a corresponding unimodular bilinear lattice has eigenvalues in S^1. Section 4.2 follows largely Krüger [**Kr90**, §12].

Section 4.3 uses *pseudo-graphs* to systematically study all cases, not only those where the monodromy has eigenvalues in S^1. This goes far beyond Krüger and Cecotti-Vafa.

The results of section 4.3 and the pseudo-graphs are used in section 4.4 to determine in all cases the stabilizer $(\mathrm{Br}_3 \ltimes \{\pm 1\}^3)_S$ respectively the stabilizer $(\mathrm{Br}_3)_{S/\{\pm 1\}^3}$. Section 7.4 will build on this and determine the stabilizer $(\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3}$ of a distingushed basis $\underline{e} \in \mathcal{B}^{dist}$ for any unimodular bilinear lattice of rank 3 with a fixed triangular basis.

Section 4.5 starts with the observation that a matrix $S \in T_n^{uni}(\mathbb{Z})$ and the matrix $\widetilde{S} \in T_n^{uni}(\mathbb{Z})$ with $\widetilde{S}_{ij} = -S_{ij}$ for $i < j$ lead to unimodular bilinear lattices with the same odd monodromy groups and the same odd vanishing cycles. This motivates to study the action on $T_n^{uni}(\mathbb{Z})$ which extends the action of $\mathrm{Br}_n \ltimes \{\pm 1\}^n$ by this global sign change. Section 4.5 carries this out in the case $n = 3$ and gives standard representatives for each orbit. Though examples show that the action is rather wild. Similar looking triples in $\mathbb{Z}^3$ are in the orbits of very different standard representatives.

4.1. Braid group action on real upper triangular 3×3 matrices

The action of $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ on $T_3^{uni}(\mathbb{Z})$ will be studied in the next sections. It extends to an action on $T_3^{uni}(\mathbb{R}) \cong \mathbb{R}^3$ which will be studied here. By Theorem 3.4 (d), σ_1 acts on $T_3^{uni}(\mathbb{Z})$ by

$$\sigma_1 : \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} -x_1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -x_1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & -x_1 & x_3 - x_1x_2 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

It extends to an action on $T_3^{uni}(\mathbb{R})$. With the isomorphism

$$T_3^{uni}(R) \xrightarrow{\cong} R^3, \quad \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} \mapsto (x_1, x_2, x_3) \quad \text{for } R \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}\}$$

this gives the action

$$\sigma_1^{\mathbb{R}} : \mathbb{R}^3 \to \mathbb{R}^3, \quad (x_1, x_2, x_3) \mapsto (-x_1, x_3 - x_1x_2, x_2).$$

Analogously

$$\begin{aligned}
(\sigma_1^{\mathbb{R}})^{-1} : \mathbb{R}^3 \to \mathbb{R}^3, &\quad (x_1, x_2, x_3) \mapsto (-x_1, x_3, x_2 - x_1x_3), \\
\sigma_2^{\mathbb{R}} : \mathbb{R}^3 \to \mathbb{R}^3, &\quad (x_1, x_2, x_3) \mapsto (x_2 - x_1x_3, x_1, -x_3), \\
(\sigma_2^{\mathbb{R}})^{-1} : \mathbb{R}^3 \to \mathbb{R}^3, &\quad (x_1, x_2, x_3) \mapsto (x_2, x_1 - x_2x_3, -x_3), \\
\delta_1^{\mathbb{R}} : \mathbb{R}^3 \to \mathbb{R}^3, &\quad (x_1, x_2, x_3) \mapsto (-x_1, -x_2, x_3), \\
\delta_2^{\mathbb{R}} : \mathbb{R}^3 \to \mathbb{R}^3, &\quad (x_1, x_2, x_3) \mapsto (-x_1, x_2, -x_3), \\
\delta_3^{\mathbb{R}} : \mathbb{R}^3 \to \mathbb{R}^3, &\quad (x_1, x_2, x_3) \mapsto (x_1, -x_2, -x_3).
\end{aligned}$$

One sees

$$\delta_3^{\mathbb{R}} = \delta_1^{\mathbb{R}}\delta_2^{\mathbb{R}} \quad \text{and} \quad G^{sign} := \langle \delta_1^{\mathbb{R}}, \delta_2^{\mathbb{R}} \rangle \cong \{\pm 1\}^2.$$

The group $\langle \sigma_1^{\mathbb{R}}, \sigma_2^{\mathbb{R}} \rangle \ltimes G^{sign} \subset \mathrm{Aut}_{pol}(\mathbb{R}^3)$ of polynomial automorphisms of $\mathbb{R}^3$ will be more transparent in other generators.

Definition 4.1. Define the polynomial automorphisms of $\mathbb{R}^3$

$$\begin{aligned}\varphi_1 : \mathbb{R}^3 \to \mathbb{R}^3, &\quad (x_1, x_2, x_3) \mapsto (x_2x_3 - x_1, x_3, x_2),\\ \varphi_2 : \mathbb{R}^3 \to \mathbb{R}^3, &\quad (x_1, x_2, x_3) \mapsto (x_3, x_1x_3 - x_2, x_1),\\ \varphi_3 : \mathbb{R}^3 \to \mathbb{R}^3, &\quad (x_1, x_2, x_3) \mapsto (x_2, x_1, x_1x_2 - x_3),\\ \gamma : \mathbb{R}^3 \to \mathbb{R}^3, &\quad (x_1, x_2, x_3) \mapsto (x_3, x_1, x_2),\end{aligned}$$

and the group $G^{phi} := \langle \varphi_1, \varphi_2, \varphi_3 \rangle \subset \mathrm{Aut}_{pol}(\mathbb{R}^3)$.

Theorem 4.2. *(a) The group G^{phi} is a free Coxeter group with the three generators $\varphi_1, \varphi_2, \varphi_3$, so $G^{phi} \cong G^{fCox,3}$.*
(b) $\langle \gamma \rangle \cong \mathbb{Z}/3\mathbb{Z} \cong A_3 \subset S_3$.
(c) $\langle \sigma_1^{\mathbb{R}}, \sigma_2^{\mathbb{R}} \rangle \ltimes G^{sign} = (G^{phi} \ltimes G^{sign}) \rtimes \langle \gamma \rangle$.

Proof: (a) $\varphi_1^2 = \varphi_2^2 = \varphi_3^2 = \mathrm{id}$ is obvious, and also that $(2, 2, 2) \in \mathbb{R}^3$ is a fixed point of G^{phi}. We will show that the group $\langle d_{(2,2,2)}\varphi_1, d_{(2,2,2)}\varphi_2, d_{(2,2,2)}\varphi_3 \rangle$ of induced actions on the tangent space $T_{(2,2,2)}\mathbb{R}^3$ is a free Coxeter group with three generators. This will imply $G^{phi} \cong G^{fCox,3}$.

Affine linear coordinates $(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)$ on $\mathbb{R}^3$ which vanish at $(2, 2, 2)$ with

$$(x_1, x_2, x_3) = (2 + \widetilde{x}_1, 2 + \widetilde{x}_2, 2 + \widetilde{x}_3) = (2, 2, 2) + (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)$$

are also linear coordinates on $T_{(2,2,2)}\mathbb{R}^3$. We have

$$\varphi_1((2, 2, 2) + (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)) = (2, 2, 2) + (\widetilde{x}_2\widetilde{x}_3 + 2\widetilde{x}_2 + 2\widetilde{x}_3 - \widetilde{x}_1, \widetilde{x}_3, \widetilde{x}_2),$$

$$d_{(2,2,2)}\varphi_1(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) \begin{pmatrix} -1 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix},$$

and analogously

$$d_{(2,2,2)}\varphi_2(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) \begin{pmatrix} 0 & 2 & 1 \\ 0 & -1 & 0 \\ 1 & 2 & 0 \end{pmatrix},$$

$$d_{(2,2,2)}\varphi_3(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 0 & 0 & -1 \end{pmatrix}.$$

The group G^{phi} respects the fibers of the map

$$r_{\mathbb{R}} : \mathbb{R}^3 \to \mathbb{R}, \quad (x_1, x_2, x_3) \mapsto x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3.$$

The group $\langle d_{(2,2,2)}\varphi_1, d_{(2,2,2)}\varphi_2, d_{(2,2,2)}\varphi_3 \rangle$ respects the tangent cone at $(2, 2, 2)$ of the fiber $r_{\mathbb{R}}^{-1}(4)$. This tangent cone is the zero set of the quadratic form

$$\begin{aligned} q_{\mathbb{R}^3,(2,2,2)} : \mathbb{R}^3 &\to \mathbb{R}, \\ (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) &\mapsto -\widetilde{x}_1^2 - \widetilde{x}_2^2 - \widetilde{x}_3^2 + 2\widetilde{x}_1\widetilde{x}_2 + 2\widetilde{x}_1\widetilde{x}_3 + 2\widetilde{x}_2\widetilde{x}_3. \end{aligned}$$

This quadratic form is indefinite with signature $(+,-,-)$. As in Theorem A.4, its cone of positive vectors is called $\mathcal{K}$. Consider the six vectors

$$v_1 = (1,1,0),\ v_2 = (1,0,1),\ v_3 = (0,1,1),$$
$$w_1 = v_1 + v_2,\ w_2 = v_1 + v_3,\ w_3 = v_2 + v_3.$$

Then

$$\begin{aligned} v_1, v_2, v_3 \in \partial\mathcal{K}, \quad w_1, w_2, w_3 \in \mathcal{K}, \\ d_{(2,2,2)}\varphi_1 : v_1 \leftrightarrow v_2,\ w_1 \mapsto w_1, \\ d_{(2,2,2)}\varphi_2 : v_1 \leftrightarrow v_3,\ w_2 \mapsto w_2, \\ d_{(2,2,2)}\varphi_3 : v_2 \leftrightarrow v_3,\ w_3 \mapsto w_3. \end{aligned}$$

Compare Theorem A.4. In the model $\mathcal{K}/\mathbb{R}^*$ of the hyperbolic plane, $d_{(2,2,2)}\varphi_i$ for $i \in \{1,2,3\}$ gives a rotation with angle π and elliptic fixed point $\mathbb{R}^* w_i$, which maps the hyperbolic line with euclidean boundary points $\mathbb{R}^*v_1 \& \mathbb{R}^*v_2$ respectively $\mathbb{R}^*v_1 \& \mathbb{R}^*v_3$ respectively $\mathbb{R}^*v_2 \& \mathbb{R}^*v_3$ to itself

Theorem A.2 (b) applies and shows $\langle d_{(2,2,2)}\varphi_i \,|\, i \in \{1,2,3\}\rangle \cong G^{fCox,3}$. Figure 4.1 illustrates this.

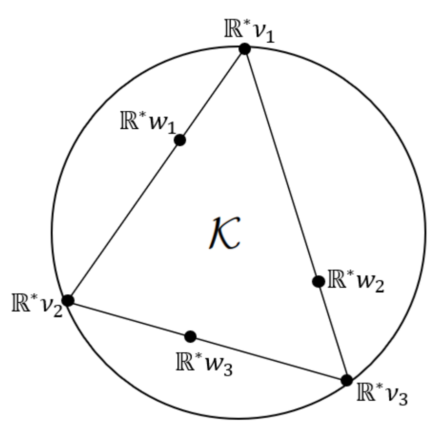


FIGURE 4.1. $G^{fCox,3}$ generated by 3 elliptic Möbius transformations, an application of Theorem A.2 (b)

(b) Trivial.

(c) The equality of groups

$$\langle\sigma_1^{\mathbb{R}},\sigma_2^{\mathbb{R}}\rangle\ltimes G^{sign}=\langle\varphi_1,\varphi_2,\varphi_3,\gamma,\delta_1^{\mathbb{R}},\delta_2^{\mathbb{R}}\rangle$$

follows from

$$\begin{aligned}\gamma &= \delta_3^{\mathbb{R}}\sigma_2^{\mathbb{R}}\sigma_1^{\mathbb{R}}, &(4.1)\\ \varphi_1 &= \delta_1^{\mathbb{R}}\gamma^{-1}(\sigma_2^{\mathbb{R}})^{-1}, &(4.2)\\ \varphi_2 &= \delta_1^{\mathbb{R}}\gamma\sigma_2^{\mathbb{R}}=\delta_3^{\mathbb{R}}\gamma^{-1}(\sigma_1^{\mathbb{R}})^{-1}, &(4.3)\\ \varphi_3 &= \delta_3^{\mathbb{R}}\gamma\sigma_1^{\mathbb{R}}. &(4.4)\end{aligned}$$

and

$$\sigma_1^{\mathbb{R}}=\gamma^{-1}\delta_3^{\mathbb{R}}\varphi_3,\quad \sigma_2^{\mathbb{R}}=\gamma^{-1}\delta_1^{\mathbb{R}}\varphi_2. \tag{4.5}$$

G^{phi} fixes $(2,2,2)$, and therefore $G^{phi}\cap G^{sign}=\{\mathrm{id}\}$. As G^{sign} is a normal subgroup of $\langle\sigma_1^{\mathbb{R}},\sigma_2^{\mathbb{R}}\rangle\ltimes G^{sign}$, it is also a normal subgroup of $\langle\varphi_1,\varphi_2,\varphi_3,\delta_1^{\mathbb{R}},\delta_2^{\mathbb{R}}\rangle$, so $\langle\varphi_1,\varphi_2,\varphi_3,\delta_1^{\mathbb{R}},\delta_2^{\mathbb{R}}\rangle=G^{phi}\ltimes G^{sign}$. More precisely

$$\varphi_i\delta_j^{\mathbb{R}}\varphi_i^{-1}=\delta_k^{\mathbb{R}}\quad\text{for } (i,j,k)\in\{(1,3,3),(2,2,2),(3,1,1),\\ (1,1,2),(1,2,1),(2,1,3),(2,3,1),(3,2,3),(3,3,2)\}.$$

We claim $\gamma\notin G^{phi}\ltimes G^{sign}$. If γ were in $G^{phi}\ltimes G^{sign}$ then $\gamma\in G^{phi}$ as γ fixes $(2,2,2)$. But all elements of finite order in $G^{phi}\cong G^{fCox,3}$ have order two, though γ has order three. Hence $\gamma\notin G^{phi}$ and $\gamma\notin G^{phi}\ltimes G^{sign}$.

The claim and

$$\gamma\varphi_1\gamma^{-1}=\varphi_2,\ \gamma\varphi_2\gamma^{-1}=\varphi_3,\ \gamma\varphi_3\gamma^{-1}=\varphi_1, \tag{4.6}$$

$$\gamma\delta_1^{\mathbb{R}}\gamma^{-1}=\delta_3^{\mathbb{R}},\ \gamma\delta_2^{\mathbb{R}}\gamma^{-1}=\delta_1^{\mathbb{R}},\ \gamma\delta_3^{\mathbb{R}}\gamma^{-1}=\delta_2^{\mathbb{R}}, \tag{4.7}$$

show

$$\langle\varphi_1,\varphi_2,\varphi_3,\delta_1^{\mathbb{R}},\delta_2^{\mathbb{R}},\gamma\rangle=(G^{phi}\ltimes G^{sign})\rtimes\langle\gamma\rangle.$$ □

4.2. Braid group action on integer upper triangular 3×3 matrices

In this section we will give a partial classification of the orbits of the action of $\mathrm{Br}_3\ltimes\{\pm1\}^3$ on $T_3^{uni}(\mathbb{Z})$. This refines results which were obtained independently by Krüger [**Kr90**, §12] and Cecotti-Vafa [**CV93**, Ch. 6.2] (building on Mordell [**Mo69**, p 106 ff]).

The refinement consists in the following. By Theorem 4.2 the action of $\mathrm{Br}_3\ltimes\{\pm1\}^3$ on $T_3^{uni}(\mathbb{Z})$ coincides with the action of $(G^{phi}\ltimes G^{sign})\rtimes\langle\gamma\rangle$. For reasons unknown to us, Krüger and Cecotti-Vafa considered the action of the slightly larger group $(G^{phi}\ltimes G^{sign})\rtimes\langle\gamma,\gamma_2\rangle$ with

$$\gamma_2:\mathbb{R}^3\to\mathbb{R}^3,\quad (x_1,x_2,x_3)\mapsto(x_2,x_1,x_3),\quad\text{so }\langle\gamma,\gamma_2\rangle\cong S_3.$$

Thus they obtained a slightly coarser classification. Nevertheless Theorem 4.6 is essentially due to them (and Mordell [**Mo69**, p 106 ff]). The following definition and lemma prepare it. They are due to Krüger [**Kr90**, §12].

Definition 4.3. [**Kr90**, Def. 12.2] For $\underline{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ we set as usual $\|\underline{x}\| := \sqrt{x_1^2 + x_2^2 + x_3^2}$. A tuple $\underline{x} \in \mathbb{R}^3$ is called a *local minimum* if

$$\|\underline{x}\| \leq \min(\|\sigma_1^{\mathbb{R}}(\underline{x})\|, \|(\sigma_1^{\mathbb{R}})^{-1}(\underline{x})\|, \|\sigma_2^{\mathbb{R}}(\underline{x})\|, \|(\sigma_2^{\mathbb{R}})^{-1}(\underline{x})\|).$$

This is obviously equivalent to

$$\|\underline{x}\| \leq \min(\|\varphi_1(\underline{x})\|, \|\varphi_2(\underline{x})\|, \|\varphi_3(\underline{x})\|).$$

Lemma 4.4. [**Kr90**, Lemma 12.3] *$\underline{x} \in \mathbb{R}^3$ is a local minimum if and only if it satisfies (i) or (ii),*

(i) $x_1x_2x_3 \leq 0$,
(ii) $x_1x_2x_3 > 0,\ 2|x_1| \leq |x_2x_3|,\ 2|x_2| \leq |x_1x_3|,\ 2|x_3| \leq |x_1x_2|$.

In the case (ii) also $|x_1| \geq 2$, $|x_2| \geq 2$ and $|x_3| \geq 2$ hold.

Proof: $\underline{x} \in \mathbb{R}^3$ is a local minimum if for all i, j, k with $\{i, j, k\} = \{1, 2, 3\}$

$$x_i^2 + x_j^2 + x_k^2 \leq x_i^2 + x_j^2 + (x_k - x_ix_j)^2$$

holds, which is equivalent to

$$2x_ix_jx_k \leq x_i^2x_j^2.$$

1st case, $x_1x_2x_3 \leq 0$: Then $\underline{x}$ is a local minimum.

2nd case, $x_1x_2x_3 > 0$: Then the condition $2x_ix_jx_k \leq x_i^2x_j^2$ is equivalent to

$$2|x_k| \leq |x_ix_j|.$$

These three conditions together imply

$$4|x_k| \leq 2|x_i||x_j| \leq |x_i||x_i||x_k|, \quad \text{so } 4 \leq |x_i|^2, \quad \text{so } 2 \leq |x_i|. \qquad \square$$

The square $\|.\|^2 : \mathbb{Z}^3 \to \mathbb{Z}_{\geq 0}$ of the norm has on $\mathbb{Z}^3$ values in $\mathbb{Z}_{\geq 0}$. Therefore each $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbit in $\mathbb{Z}^3$ has local minima. Krüger showed that the only $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbits in $\mathbb{R}^3$ without local minimal are of the following shape. We will not use this result, but we find it interesting enough to cite it.

THEOREM 4.5. [**Kr90**, Theorem 12.6] *Let* $\underline{x} \in \mathbb{R}^3$ *whose* $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ *orbit does not contain a local minimum. Then*

$$\begin{aligned} x_1x_2x_3 > 0, \quad 2 < \min(|x_1|, |x_2|, |x_3|), \\ 4 = r_{\mathbb{R}}(\underline{x})(= x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3). \end{aligned}$$

Furthermore, there is a sequence $(\psi_n)_{n\in\mathbb{N}}$ *with* $\psi_n \in \{\varphi_1, \varphi_2, \varphi_3\}$ *with* $\psi_n \neq \psi_{n+1}$ *such that the sequence* $(\underline{x}^{(n)})_{n\in\mathbb{N}\cup\{0\}}$ *with* $\underline{x}^{(0)} = \underline{x}$ *and* $\underline{x}^{(n+1)} = \psi_n(\underline{x}^{(n)})$ *satisfies*

$$\begin{aligned} \|\underline{x}^{(n+1)}\| < \|\underline{x}^{(n)}\| \quad \text{for all } n \in \mathbb{N} \cup \{0\}, \\ \lim_{n\to\infty}(|x_1^{(n)}|, |x_2^{(n)}|, |x_3^{(n)}|) = (2, 2, 2). \end{aligned}$$

Now we come to the classification of $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbits in $\mathbb{Z}^3$. The following result is except for its part (f) a refinement of [**Kr90**, Theorem 12.7] and of [**CV93**, Ch 6.2]. The proof below follows (except for the part (f)) the proof in [**Kr90**]. Recall that each $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbit in $\mathbb{R}^3$ is contained in one fiber of the map $r_{\mathbb{R}} : \mathbb{R}^3 \to \mathbb{R}$, $\underline{x} \mapsto x_1^2+x_2^2+x_3^2-x_1x_2x_3$.

THEOREM 4.6. *(a) Each fiber of*

$$r : \mathbb{Z}^3 \to \mathbb{Z},\ r(\underline{x}) = x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3$$

except the fiber $r^{-1}(4)$ *contains only finitely many local minima.*

(b) Each fiber of $r : \mathbb{Z}^3 \to \mathbb{Z}$ *except the fiber* $r^{-1}(4)$ *consists of only finitely many* $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ *orbits.*

(c) For $\rho \in \mathbb{Z}_{<0}$*, each local minimum* $\underline{x} \in r^{-1}(\rho)$ *satisfies* $x_1x_2x_3 > 0$ *and* $|x_1| \geq 3$, $|x_2| \geq 3$, $|x_3| \geq 3$.

(d) For $\rho \in \mathbb{N} - \{4\}$*, each local minimum* $\underline{x} \in r^{-1}(\rho)$ *satisfies* $x_1x_2x_3 \leq 0$.

(e) The following table gives all local minima in $r^{-1}(\{0, 1, 2, 3, 4\})$. *The local minima in one* $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ *orbit are in one line. The last entry in each line is one matrix in the corresponding orbit in* $T_3^{uni}(\mathbb{Z})$.

$r = 3$	$-$	$-$
$r = 0$	$(0, 0, 0)$	$S(A_1^3)$
$r = 0$	$(3, 3, 3), (-3, -3, 3), (-3, 3, -3), (3, -3, -3)$	$S(\mathbb{P}^2)$
$r = 1$	$(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$	$S(A_2A_1)$
$r = 2$	$(\pm 1, \pm 1, 0), (\pm 1, 0, \pm 1), (0, \pm 1, \pm 1)$	$S(A_3)$
$r = 4$	$(\pm 2, 0, 0), (0, \pm 2, 0), (0, 0, \pm 2)$	$S(\mathbb{P}^1 A_1)$
$r = 4$	$(-1, -1, -1), (1, 1, -1), (1, -1, 1), (-1, 1, 1)$	$S(\widehat{A}_2)$
$r = 4$	$(2, 2, 2), (-2, -2, 2), (-2, 2, -2), (2, -2, -2)$	$S(\mathcal{H}_{1,2})$
$\left\{ \begin{array}{c} r = 4 \\ l \in \mathbb{Z}_{\geq 3} \end{array} \right\}$	$\left\{ \begin{array}{l} (\varepsilon_1 2, \varepsilon_2 l, \varepsilon_3 l), (\varepsilon_1 l, \varepsilon_2 2, \varepsilon_3 l), (\varepsilon_1 l, \varepsilon_2 l, \varepsilon_3 2) \\ \text{for } \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{\pm 1\} \text{ with } \varepsilon_1\varepsilon_2\varepsilon_3 = 1 \end{array} \right\}$	$S(-l, 2, -l)$

So there are seven single $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ *orbits and one series with parameter* $l \in \mathbb{Z}_{\geq 3}$ *of* $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ *orbits with* $r \in \{0,1,2,3,4\}$*. These are the most interesting orbits as the monodromy matrix* $S(\underline{x})^{-1}S(\underline{x})^t$ *for* $\underline{x} \in \mathbb{R}^3$ *has eigenvalues in* S^1 *if and only if* $r(\underline{x}) \in [0,4]$*.*

(f) For a given local minimum $\underline{x} \in \mathbb{Z}^3$ *the set of all local minima in the* $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ *orbit of* $\underline{x}$ *is either the set* $G^{sign} \rtimes \langle \gamma \rangle (\underline{x})$ *or the set* $G^{sign} \rtimes \langle \gamma, \gamma_2 \rangle (\underline{x})$ *(see Theorem 4.13 (b) for details).*

Proof: (a) Fix $\rho \in \mathbb{Z} - \{4\}$. Let $\underline{x} \in \mathbb{Z}^3$ be a local minimum with $r(\underline{x}) = \rho$.

1st case, $x_1x_2x_3 \leq 0$: Then $\rho = r(\underline{x}) = x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 \geq \|\underline{x}\|^2$, so $\rho \geq 0$. The closed ball of radius $\sqrt{\rho}$ around 0 in $\mathbb{R}^3$ intersects $\mathbb{Z}^3$ only in finitely many points.

2nd case, $x_1x_2x_3 > 0$: We can suppose $x_i > 0$ for $i \in \{1,2,3\}$ because of the action of G^{sign} on $\mathbb{R}^3$ and $\mathbb{Z}^3$. Lemma 4.4 says $2x_1 \leq x_2x_3$, $2x_2 \leq x_1x_3$, $2x_3 \leq x_1x_2$, $x_i \geq 2$ for $i \in \{1,2,3\}$.

We can suppose $x_1 = \min(x_1,x_2,x_3)$ (the other cases are analogous). If $x_1 = 2$ then $4 \neq \rho = r(\underline{x}) = 4 + x_2^2 + x_3^2 - 2x_2x_3 = 4 + (x_2 - x_3)^2$, so $x_2 \neq x_3$, which is a contradiction to $2x_2 \leq x_1x_3 = 2x_3$, $2x_3 \leq x_1x_2 = 2x_2$. Therefore $x_1 \geq 3$.

We can suppose $x_1 \leq x_2 \leq x_3$ (the other cases are analogous).

$$\begin{aligned}
\rho &= r(\underline{x}) = x_1^2 + x_2^2 + (x_3 - \frac{1}{2}x_1x_2)^2 - \frac{1}{4}x_1^2x_2^2 \\
&\leq x_1^2 + x_2^2 + (x_2 - \frac{1}{2}x_1x_2)^2 - \frac{1}{4}x_1^2x_2^2 \quad \text{(because } x_2 \leq x_3 \leq \frac{1}{2}x_1x_2\text{)} \\
&= x_1^2 + 2x_2^2 - x_1x_2^2 = (x_1 - 2)(x_1 + 2 - x_2^2) + 4 \\
&\leq (x_1 - 2)(x_1 + 2 - x_1^2) + 4 = -(x_1 - 2)(x_1 - 2)(x_1 + 1) + 4 \\
&\leq \begin{cases} -(3-2)(3-2)(3+1) + 4 = 0, \\ -(x_1 - 2)^3 + 4. \end{cases}
\end{aligned}$$

This implies $\rho \leq 0$ and $x_1 \leq 2 + \sqrt[3]{4 - \rho}$, so x_1 is one of the finitely many values in $\mathbb{Z} \cap [3, 2 + \sqrt[3]{4 - \rho}]$.

The inequality $\rho \leq (x_1 - 2)(x_1 + 2 - x_2^2) + 4$ implies

$$x_2^2 \leq \frac{4 - \rho}{x_1 - 2} + x_1 + 2,$$

so x_2 is one of the finitely many values in $\mathbb{Z} \cap [x_1, \sqrt{\frac{4-\rho}{x_1-2} + x_1 + 2}]$.

Because of $x_3 \leq \frac{1}{2}x_1x_2$ also x_3 can take only finitely many values.

(b) Each $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbit in $\mathbb{Z}^3$ is mapped by $\|.\|^2$ to a subset of $\mathbb{Z}_{\geq 0}$. A preimage in this orbit of the minimum of this subset is a local minimum. Therefore (a) implies (b).

(c) Suppose $\rho < 0$ and $\underline{x} \in r^{-1}(\rho)$ is a local minimum. $0 > \rho = \|\underline{x}\|^2 - x_1x_2x_3$ implies $x_1x_2x_3 > 0$. Lemma 4.4 gives $|x_1| \geq 2$.

If $x_1 = 2\varepsilon$ with $\varepsilon \in \{\pm 1\}$ then $\rho = r(\underline{x}) = 4 + (x_2 - \varepsilon x_3)^2 \geq 4$, a contradiction. So $|x_1| \geq 3$. Analogously $|x_2| \geq 3$ and $|x_3| \geq 3$.

(d) Suppose $\rho \in \mathbb{N} - \{4\}$ and $\underline{x} \in r^{-1}(\rho)$ is a local minimum. In the second case $x_1x_2x_3 > 0$ in the proof of part (a) we concluded $\rho \leq 0$. Therefore we are in the first case in the proof of part (a), so $x_1x_2x_3 \leq 0$.

(e) Suppose $\rho \in \{0, 1, 2, 3, 4\}$, and $\underline{x} \in r^{-1}(\rho)$ is a local minimum.

In the cases $\rho \in \{1, 2, 3\}$ by part (d) $x_1x_2x_3 \leq 0$ and $\rho = r(\underline{x}) = x_1^2 + x_2^2 + x_3^2 + |x_1x_2x_3|$, so in these cases all $x_i \neq 0$ is impossible, so some $x_i = 0$, so $\rho = r(\underline{x}) = x_j^2 + x_k^2$ where $\{i, j, k\} = \{1, 2, 3\}$.

The case $\rho = 3$: $3 = x_j^2 + x_k^2$ is impossible, the case $\rho = 3$ is impossible, $r^{-1}(3) = \emptyset$.

The case $\rho = 1$: $1 = x_j^2 + x_k^2$ is solved only by $(x_j, x_k) \in \{(\pm 1, 0), (0, \pm 1)\}$. The six local minima $(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$ are in one orbit of $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ because $\gamma(1, 0, 0) = (0, 1, 0)$, $\gamma(0, 1, 0) = (0, 0, 1)$.

The case $\rho = 2$: $x_j^2 + x_k^2 = 2$ is solved only by $(x_j, x_k) \in \{(\pm 1, \pm 1)\}$. The twelve local minimal $(\pm 1, \pm 1, 0), (\pm 1, 0, \pm 1), (0, \pm 1, \pm 1)$ are in one orbit of $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ because $\gamma(1, 1, 0) = (0, 1, 1)$, $\gamma(0, 1, 1) = (1, 0, 1)$.

The case $\rho = 0$: We use the proof of part (a).

1st case, $x_1x_2x_3 \leq 0$: $0 = \rho = \|x\|^2 - x_1x_2x_3 \geq \|x\|^2$, so $\underline{x} = (0, 0, 0)$. Its $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbit consists only of $(0, 0, 0)$.

2nd case, $x_1x_2x_3 > 0$: We can suppose $x_i > 0$ for each $i \in \{1, 2, 3\}$. Suppose $x_i \leq x_j \leq x_k$ for $\{i, j, k\} = \{1, 2, 3\}$. The proof of part (a) gives

$$3 \leq x_i \leq 2 + \sqrt[3]{4 - \rho} = 2 + \sqrt[3]{4}, \text{ so } x_i = 3$$

and

$$3 = x_i \leq x_j \leq \sqrt{\frac{4 - \rho}{x_i - 2} + x_i + 2} = 3, \text{ so } x_j = 3.$$

$0 = r(\underline{x}) = 9 + 9 + x_k^2 - 9x_k = (x_k - 3)(x_k - 6)$ and $x_k \leq \frac{1}{2}x_ix_j = \frac{9}{2}$ show $x_k = 3$. The four local minima $(3, 3, 3)$, $(-3, -3, 3)$, $(-3, 3, -3)$ and $(3, -3, -3)$ are in one $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbit because of the action of G^{sign}.

The case $\rho = 4$:

1st case, some $x_i = 0$: Then with $\{i, j, k\} = \{1, 2, 3\}$ $4 = r(\underline{x}) = x_j^2 + x_k^2$. This is solved only by $(x_j, x_k) \in \{(\pm 2, 0), (0, \pm 2)\}$. The six local minima $(\pm 2, 0, 0)$, $(0, \pm 2, 0)$, $(0, 0, \pm 2)\}$ are in one $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbit because $\gamma((2, 0, 0)) = (0, 2, 0)$, $\gamma((0, 2, 0)) = (0, 0, 2)$.

2nd case, all $x_i \neq 0$ and $x_1x_2x_3 < 0$: $4 = r(\underline{x}) = x_1^2 + x_2^2 + x_3^2 + |x_1x_2x_3|$, so $(x_1, x_2, x_3) \in \{(-1,-1,-1),(1,1,-1),(1,-1,1),(-1,1,1)\}$. These four local minima are in one $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbit because of the action of G^{sign}.

3rd case, all $x_i \neq 0$ and $x_1x_2x_3 > 0$: We can suppose $x_i > 0$ for each $i \in \{1,2,3\}$ and $x_i \leq x_j \leq x_k$ for some i,j,k with $\{i,j,k\} = \{1,2,3\}$. As in the proof of part (a) we obtain the estimate

$$4 = \rho = r(\underline{x}) \leq -(x_i - 2)^3 + 4, \quad \text{so } x_i = 2,$$

and

$$4 = \rho = r(\underline{x}) = 4 + (x_j - x_k)^2, \quad \text{so } l := x_j = x_k \geq 2.$$

For $l = 2$ the four local minima

$$(2,2,2),(-2,-2,2),(-2,2,-2),(2,-2,-2)$$

and for $l \geq 3$ the 24 local minima

$$(\varepsilon_1 2, \varepsilon_2 l, \varepsilon_3 l),(\varepsilon_1 l, \varepsilon_2 2, \varepsilon_3 l),(\varepsilon_1 l, \varepsilon_2 l, \varepsilon_3 2) \\ \text{with } \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{\pm 1\}, \varepsilon_1\varepsilon_2\varepsilon_3 = 1,$$

are in one $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbit because of the action of G^{sign} and γ.

It remains to see that local minima in different lines in the list in part (e) are in different $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbits. One reason is part (f). Another way to argue is given in the Remarks 4.7.

(f) See Lemma 4.10 (e). □

Remarks 4.7. Part (f) of Theorem 4.6 is strong and allows easily to see when $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbits are separate. Nevertheless it is also interesting to find invariants of the orbits which separate them.

Now we discuss several invariants which help to prove the claim that local minima in different lines in the list in part (e) are in different $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbits.

The number $r(\underline{x}) \in \mathbb{Z}$ is such an invariant. Furthermore the set $\{(0,0,0)\}$ is a single orbit and thus different from the orbit of $S(\mathbb{P}^2)$. Therefore the claim is true for the lines with $r \in \{0,1,2,3\}$.

It remains to consider the $3 + \infty$ lines with $r = 4$. Certainly the reducible case $S(\mathbb{P}^1 A_1)$ is separate from the other cases, which are all irreducible. The signature of $I^{(0)}$ is an invariant. It is given in Lemma 5.7. It allows to see that the orbits of $S(\widehat{A}_2)$, $S(\mathcal{H}_{1,2})$ are different from one another and from the orbits of $S(-l,2,-l)$ for $l \geq 3$. In order to see that the orbits of $S(-l,2,-l)$ for $l \geq 3$ are pairwise different, we can offer Lemma 7.10, which in fact allows to separate all the lines with $r = 4$. It considers the induced monodromy on the quotient lattice $H_{\mathbb{Z}}/\operatorname{Rad} I^{(1)}$.

REMARKS 4.8. Krüger [**Kr90**, §12] and Cecotti-Vafa [**CV93**, Ch. 6.2] considered the action of the group $(G^{phi} \ltimes G^{sign}) \rtimes \langle\gamma,\gamma_2\rangle$ with $\gamma_2 : \mathbb{R}^3 \to \mathbb{R}^3$, $(x_1,x_2,x_3) \mapsto (x_2,x_1,x_3)$, so $\langle\gamma,\gamma_2\rangle \cong S_3$ which is slightly larger than $(G^{phi} \ltimes G^{sign}) \rtimes \langle\gamma\rangle$. Because of

$$\gamma_2\varphi_1\gamma_2^{-1} = \varphi_2,\ \gamma_2\varphi_2\gamma_2^{-1} = \varphi_1,\ \gamma_2\varphi_3\gamma_2^{-1} = \varphi_3,\ \gamma_2\gamma\gamma_2^{-1} = \gamma^{-1},$$

we have

$$\mathrm{Br}_3 \ltimes \{\pm 1\}^3\big(\gamma_2(\underline{x})\big) = \gamma_2\big(\mathrm{Br}_3 \ltimes \{\pm 1\}^3(\underline{x})\big).$$

Especially, the $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbit of $\underline{x}$ coincides with the $(G^{phi} \ltimes G^{sign}) \rtimes \langle\gamma,\gamma_2\rangle$ orbit of $\underline{x}$ in the following cases:

(i) if $x_i = x_j$ for some $i \neq j$,
(ii) if $x_i = 0$ for some i (observe $\delta_3^{\mathbb{R}}\gamma\varphi_1(x_1,x_2,0) = (x_2,x_1,0)$),
(iii) if $\underline{x} = (x_1,x_2,\frac{1}{2}x_1x_2)$ with $|x_i| \geq 3$ and $|x_1| \neq |x_2|$ (observe $\varphi_3(\underline{x}) = (x_2,x_1,\frac{1}{2}x_1x_2)$).

In Lemma 4.12 24 sets $C_1,...,C_{24}$ of local minima are considered. The only local minima $\underline{x} \in \bigcup_{i=1}^{24} C_i$ which satisfy none of the conditions (i), (ii) and (iii) are those in $C_{16} \cup C_{22} \cup C_{24}$. Theorem 4.13 (b) shows that in these cases the $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbits of $\underline{x}$ and of $\gamma_2(\underline{x})$ are indeed disjoint.

Especially all orbits in the fibers $r^{-1}(\rho)$ with $\rho \in \{0,1,2,3,4\}$ contain local minima which satify (i), (ii) or (iii), so there the classifications in Theorem 4.6 and the classification by Krüger and Cecotti-Vafa coincide.

We do not know whether for $\underline{x} \in C_{16} \cup C_{22} \cup C_{24}$ and $\gamma_2(\underline{x})$ the corresponding unimodular bilinear lattices with sets of distinguished bases are isomorphic or not.

For $\underline{x}$ and $\gamma_2(\underline{x})$ in one $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbit they are isomorphic, see Remark 3.20 (iii).

4.3. A classification of the $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbits in $\mathbb{Z}^3$

This section refines the results of section 4.2 on the braid group action on integer upper triangular 3×3 matrices. Using pseudo-graphs, it gives a classification of all orbits of Br_3 on $\mathbb{Z}^3/\{\pm 1\}^3$. Definition 4.9 makes precise what is meant here by a pseudo-graph, and it defines a pseudo-graph $\mathcal{G}(\underline{x})$ for any local minimum $\underline{x} \in \mathbb{Z}^3$.

As $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ and $(G^{phi} \ltimes \{\pm 1\}^3) \rtimes \langle\gamma\rangle$ are semidirect products with normal subgroups $\{\pm 1\}^3$, the groups Br_3 and $G^{phi} \rtimes \langle\gamma\rangle$ act on $\mathbb{Z}^3/\{\pm 1\}^3$.

Definition 4.9. (a) For any set $\mathcal{V}$, $\mathcal{P}(\mathcal{V})$ denotes its power set, so the set of all its subsets, and $\mathcal{P}_k(\mathcal{V})$ for some $k \in \mathbb{N}$ denotes the set of all subsets with k elements. We will use only $\mathcal{P}_1(\mathcal{V})$ and $\mathcal{P}_2(\mathcal{V})$.

(b) A *pseudo-graph* is here a tuple $\mathcal{G} = (\mathcal{V}, \mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, v_0, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_\gamma)$ with the following ingredients:

$\mathcal{V}$ is a non-empty finite or countably infinite set of vertices.

$\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2 \subset \mathcal{V}$ are pairwise disjoint subsets, $\mathcal{V}_0$ is not empty (the sets $\mathcal{V}_1$ and $\mathcal{V}_2$ may be empty, the union $\mathcal{V}_0 \cup \mathcal{V}_1 \cup \mathcal{V}_2$ can be equal to $\mathcal{V}$ or a proper subset of $\mathcal{V}$).

$v_0 \in \mathcal{V}_0$ is a distinguished vertex in $\mathcal{V}_0$.

$\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \subset \mathcal{P}_1(\mathcal{V}) \cup \mathcal{P}_2(\mathcal{V})$ are sets of undirected edges. A subset of $\mathcal{V}$ with two elements means an edge between the two vertices. A subset of $\mathcal{V}$ with one element means a loop from the vertex to itself.

$\mathcal{E}_\gamma = \{(v_0, v_1), (v_2, v_0)\}$ for some $v_1, v_2 \in \mathcal{V}_0$ is a set of two or one directed edges, one only if $v_1 = v_2 = v_0$, and then it is a directed loop.

(c) An isomorphism between two pseudo-graphs $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ is a bijection $\phi : \mathcal{V} \to \widetilde{\mathcal{V}}$ with $\phi(v_0) = \widetilde{v_0}$ which induces bijections $\phi : \mathcal{V}_i \to \widetilde{\mathcal{V}}_i$, $\phi : \mathcal{E}_j \to \widetilde{\mathcal{E}}_j$ and $\phi : \mathcal{E}_\gamma \to \widetilde{\mathcal{E}}_\gamma$.

(d) $\mathcal{G}|_{\mathcal{V}_0 \cup \mathcal{V}_1}$ denotes the restriction of a pseudo-graph $\mathcal{G}$ to the vertex set $\mathcal{V}_0 \cup \mathcal{V}_1$, so one deletes all vertices in $\mathcal{V} - (\mathcal{V}_0 \cup \mathcal{V}_1)$ and all edges with at least one end in $\mathcal{V} - (\mathcal{V}_0 \cup \mathcal{V}_1)$. Analogously, $\mathcal{G}|_{\mathcal{V}_0}$ denotes the restriction of a pseudo-graph $\mathcal{G}$ to the vertex set $\mathcal{V}_0$.

(e) Define

$$\begin{aligned} \mathcal{L}_0 &:= \{\underline{x}/\{\pm 1\}^3 \,|\, \underline{x} \in \mathbb{Z}^3 \text{ is a local minimum}\}, \\ \mathcal{L}_1 &:= \{\underline{y}/\{\pm 1\}^3 \,|\, \underline{y} \in \mathbb{Z}^3 \text{ with } |y_i| = 1 \text{ for some } i\} - \mathcal{L}_0, \\ \mathcal{L}_2 &:= \mathbb{Z}^3/\{\pm 1\}^3 - (\mathcal{L}_0 \cup \mathcal{L}_1). \end{aligned}$$

(f) A pseudo-graph $\mathcal{G}(\underline{x})$ is associated to a local minimum $\underline{x} \in \mathbb{Z}^3$ in the following way:

$$\begin{aligned} \mathcal{V} &:= \mathrm{Br}_3(\underline{x}/\{\pm 1\}^3) \subset \mathbb{Z}^3/\{\pm 1\}^3, \\ \mathcal{V}_0 &:= \mathcal{V} \cap \mathcal{L}_0 \text{ is the set of sign classes in } \mathcal{V} \text{ of local minima}, \\ \mathcal{V}_1 &:= \mathcal{V} \cap \mathcal{L}_1, \\ \mathcal{V}_2 &:= \{w \in \mathcal{V} \cap \mathcal{L}_2 \,|\, \text{an } i \in \{1,2,3\} \text{ with } \varphi_i(w) \in \mathcal{V}_0 \cup \mathcal{V}_1 \text{ exists}\}, \\ v_0 &:= \underline{x}/\{\pm 1\}^3, \\ \mathcal{E}_i &:= \{\{w, \varphi_i(w)\} \,|\, w \in \mathcal{V}\} \quad \text{for } i \in \{1,2,3\}, \\ \mathcal{E}_\gamma &:= \{(v_0, \gamma(v_0)), (\gamma^{-1}(v_0), v_0)\}. \end{aligned}$$

(g) An *infinite tree* $(\mathcal{W}, \mathcal{F})$ consists of a countably infinite set $\mathcal{W}$ of vertices and a set $\mathcal{F} \subset \mathcal{P}_2(\mathcal{W})$ of undirected edges such that the graph is connected and has no cycles. A $(2, \infty \times 3)$-tree is an infinite tree with a distinguished vertex with two neighbours such that any other vertex has three neighbours.

The next lemma gives already structural results about the pseudo-graphs $\mathcal{G}(\underline{x})$ for the local minima $\underline{x} \in \mathbb{Z}^3$. Theorem 4.13 and the Remarks 4.14 will give a complete classification of all isomorphism classes of pseudo-graphs $\mathcal{G}(\underline{x})$ for the local minima $\underline{x} \in \mathbb{Z}^3$.

LEMMA 4.10. *Let $\underline{x} \in \mathbb{Z}^3$ be a local minimum with pseudo-graph $\mathcal{G}(\underline{x}) = (\mathcal{V}, \mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, v_0, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_\gamma)$.*

(a) ($\underline{x}$ and $\mathcal{G}(\underline{x})$ are not used in part (a)) For $w \in \mathcal{L}_2$, there are i, j, k with $\{i, j, k\} = \{1, 2, 3\}$ and $\|\varphi_i(w)\| < \|w\|$, $\|\varphi_j(w)\| > \|w\|$, $\|\varphi_k(w)\| > \|w\|$, $\varphi_j(w) \in \mathcal{L}_2$, $\varphi_k(w) \in \mathcal{L}_2$ and $\varphi_j(w) \neq \varphi_k(w)$.

(b) Let $w \in \mathcal{V}_2$, so especially $w \in \mathcal{L}_2$. Choose i, j, k as in part (a). The edge which connects $w \in \mathcal{V}_2$ with $\mathcal{V}_0 \cup \mathcal{V}_1$ is in $\mathcal{E}_i$. After deleting this edge, the component of the remaining pseudo-graph which contains w is a $(2, \infty \times 3)$-tree with distinguished vertex w and all vertices in $\mathcal{L}_2$.

(c) The pseudo-graph $\mathcal{G}(\underline{x})$ is connected.

(d) The pseudo-graph $\mathcal{G}(\underline{x})|_{\mathcal{V}_0 \cup \mathcal{V}_1}$ is connected.

(e) The pseudo-graph $\mathcal{G}(\underline{x})|_{\mathcal{V}_0 \cup \mathcal{V}_1}$ is finite, and

$$\mathcal{V}_0 = \langle \gamma \rangle (v_0) \quad \text{or} \quad \mathcal{V}_0 = \langle \gamma, \gamma_2 \rangle (v_0).$$

Proof: (a) Consider $w = \underline{y}/\{\pm 1\}^3 \in \mathcal{L}_2$. Then $|y_i| \geq 2$ for $i \in \{1, 2, 3\}$ because $w \notin \mathcal{L}_0 \cup \mathcal{L}_1$. $w \notin \mathcal{L}_0$ implies $y_1 y_2 y_3 > 0$. We can suppose $y_1, y_2, y_3 \in \mathbb{Z}_{\geq 2}$. Observe for i, j, k with $\{i, j, k\} = \{1, 2, 3\}$

$$\begin{aligned} & \|\varphi_j(\underline{y})\|^2 - \|\varphi_i(\underline{y})\|^2 \\ = \; & (y_i^2 + (y_i y_k - y_j)^2 + y_k^2) - ((y_j y_k - y_i)^2 + y_j^2 + y_k^2) \\ = \; & (y_i^2 - y_j^2) y_k^2. \end{aligned}$$

Consider the case $2 \leq y_1 \leq y_2 \leq y_3$. The other cases are analogous. Then

$$\|\varphi_3(\underline{y})\| \leq \|\varphi_2(\underline{y})\| \leq \|\varphi_1(\underline{y})\|.$$

Because $w \notin \mathcal{L}_0$ $\|\varphi_3(\underline{y})\| < \|\underline{y}\|$. Also $\varphi_2(\underline{y}) = (y_3, y_1 y_3 - y_2, y_1)$ with

$$\varphi_2(\underline{y})_2 = y_1 y_3 - y_2 \geq (y_1 - 1) y_3 \begin{cases} \geq 2y_3 > y_2 & \text{if } y_1 > 2, \\ = y_3 > y_2 & \text{if } y_1 = 2. \end{cases}$$

Here $y_3 > y_2$ if $y_1 = 2$ because $\underline{y} = (2, y_2, y_3)$ is not a local minimum.

Therefore $\|\varphi_2(\underline{y})\| > \|\underline{y}\|$, and also $\|\varphi_1(\underline{y})\| \geq \|\varphi_2(\underline{y})\| > \|\underline{y}\|$. Especially $\varphi_2(\underline{y})$ and $\varphi_1(\underline{y})$ are not local minima, so $\varphi_2(w) \notin \mathcal{L}_0$ and $\varphi_1(w) \notin \mathcal{L}_0$.

Obviously $\varphi_2(\underline{y})_i \geq 2$ and $\varphi_1(\underline{y})_i \geq 2$ for $i \in \{1,2,3\}$, so $\varphi_2(w) \in \mathcal{L}_2$ and $\varphi_1(w) \in \mathcal{L}_2$. The inequality $\varphi_2(w) \neq \varphi_1(w)$ follows from

$$\begin{aligned} \varphi_2(\underline{y})_2 \geq 2y_3 > y_3 = \varphi_1(\underline{y})_2 \qquad & \text{if } y_1 > 2, \\ \varphi_2(\underline{y})_2 = y_1y_3 - y_2 > (y_1-1)y_3 = y_3 = \varphi_1(\underline{y})_2 \qquad & \text{if } y_1 = 2. \end{aligned}$$

(b) Because $\varphi_j(w), \varphi_k(w) \in \mathcal{L}_2$, the edge which connects w to $\mathcal{V}_0 \cup \mathcal{V}_1$ cannot be in $\mathcal{E}_j$ or $\mathcal{E}_k$, so it is in $\mathcal{E}_i$.

Using part (a) again and again, one sees that the component of $\mathcal{G}(\underline{x})-$ (this edge) which contains w is a $(2, \infty \times 3)$ tree.

(c) Any vertex $w \in \mathcal{V} = (G^{phi} \rtimes \langle\gamma\rangle)(v_0)$ is obtained from v_0 by applying an element $\psi\gamma^\xi$ with $\psi \in G^{phi}$ and $\xi \in \{0, \pm 1\}$. As G^{phi} is a free Coxeter group with generators φ_1, φ_2, φ_3, applying $\psi\gamma^\xi$ to v_0 yields a path in $\mathcal{G}(\underline{x})$ from v_0 to w.

(d) This follows from (b) and (c).

(e) Consider $w = \underline{y}/\{\pm 1\}^3 \in \mathcal{V}_1$. Because $w \notin \mathcal{V}_0$, $y_1y_2y_3 > 0$. We can suppose $y_1, y_2, y_3 \in \mathbb{N}$, and one of them is equal to 1. Suppose $1 = y_1 \leq y_2 \leq y_3$. Then

$$\begin{aligned} \varphi_3(\underline{y}) &= (y_2, 1, y_2 - y_3), \text{ so } y_2 \cdot 1 \cdot (y_2 - y_3) \leq 0, \\ &\quad \text{so } \varphi_3(w) \in \mathcal{V}_0, && (4.8) \\ \varphi_2(\underline{y}) &= (y_3, y_3 - y_2, 1), \\ &\quad \text{so } \varphi_2(w) \in \mathcal{V}_0 \cup \mathcal{V}_1 \text{ (in } \mathcal{V}_0 \text{ only if } y_3 = y_2), && (4.9) \\ \varphi_1\varphi_2(\underline{y}) &= (-y_2, 1, y_3 - y_2), \\ &\quad \text{so } \varphi_1\varphi_2(w) = \varphi_3(w) \in \mathcal{V}_0. && (4.10) \end{aligned}$$

Especially, each vertex in $\mathcal{V}_1$ is connected by an edge to a vertex in $\mathcal{V}_0$.

Therefore the main point is to show $\mathcal{V}_0 = \langle\gamma\rangle(v_0)$ or $\mathcal{V}_0 = \langle\gamma, \gamma_2\rangle(v_0)$. Then $\mathcal{V}_0$ and $\mathcal{V}_1$ are finite.

First case, the restricted pseudo-graph $\mathcal{G}(\underline{x})|_{\mathcal{V}_0}$ is connected: Then $\|w\| = \|v_0\|$ for each $w \in \mathcal{V}_0$. This easily implies $\mathcal{V}_0 = \langle\gamma\rangle(v_0)$ or $\mathcal{V}_0 = \langle\gamma, \gamma_2\rangle$.

Second case, the restricted pseudo-graph $\mathcal{G}(\underline{x})|_{\mathcal{V}_0}$ is not connected: We will lead this to a contradiction. Within all paths in $\mathcal{G}(\underline{x})|_{\mathcal{V}_0 \cup \mathcal{V}_1}$ which connect vertices in different components of $\mathcal{G}(\underline{x})|_{\mathcal{V}_0}$, consider a shortest path. It does not contain an edge in $\mathcal{E}_\gamma$, because else one could go over to a path of the same length with an edge in $\mathcal{E}_\gamma$ at one end and between the same vertices, but dropping that edge would lead to a shorter path. Because each vertex in $\mathcal{V}_1$ is connected by an edge to a vertex in $\mathcal{V}_0$, a shortest path contains either one or two vertices in $\mathcal{V}_1$. The observations (4.8)–(4.10) lead in both cases to the vertices at the end of the path being in the same component of $\mathcal{G}(\underline{x})|_{\mathcal{V}_0}$, so to a contradiction. $\square$

EXAMPLES 4.11. The following 14 figures show the pseudo-graphs $\mathcal{G}_1, ..., \mathcal{G}_{14}$ for 14 values $v_0 = \underline{x}/\{\pm 1\}^3$ with $\underline{x} \in \mathbb{Z}^3$ a local minimum. The ingredients of the figures have the following meaning.

$\bullet$ a vertex in $\mathcal{V}_0$,

$\otimes$ a vertex in $\mathcal{V}_1$,

$\curlyvee\!\!\circ$ a vertex in $\mathcal{V}_2$ together with the $(2, \infty \times 3)$ tree (compare Lemma 4.10 (b)),

$\overset{i}{\text{———}}$ an edge in $\mathcal{E}_i$,

$\overset{\gamma}{\rightarrow\!\!\!-}$ an edge in $\mathcal{E}_\gamma$.

The pseudo-graphs are enriched in the following way. Each vertex is labeled with a value $\underline{y}$ of its sign class $\underline{y}/\{\pm 1\}^3$. We have chosen $\underline{y} \in \mathbb{N}^3$ if $y_1 y_2 y_3 > 0$ (this holds for all $\underline{y}/\{\pm 1\}^3 \in \mathcal{V}_1 \cup \mathcal{V}_2$ and some $\underline{y}/\{\pm 1\}^3 \in \mathcal{V}_0$) and $\underline{y} \in \mathbb{Z}^3_{\leq 0}$ if $y_1 y_2 y_3 \leq 0$ (this holds for some $\underline{y}/\{\pm 1\}^3 \in \mathcal{V}_0$). The vertex v_0 can be recognized by the edges in $\mathcal{E}_\gamma$ leading to and from it. The sets C_i are defined in Lemma 4.12. The relations $\mathcal{G}_j : C_i$ are explained in Theorem 4.13.

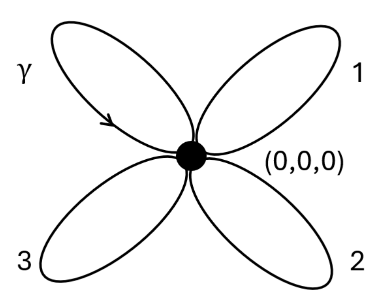


FIGURE 4.2. G_1 : C_1 (A_1^3), C_2 $(\mathcal{H}_{1,2})$. Here $\underline{x}$ = $(0, 0, 0) \in C_1$.

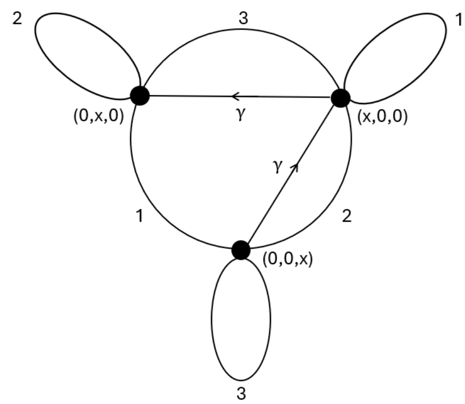


FIGURE 4.3. G_2 : C_3, C_4, C_5, so the reducible cases without A_1^3. Here $\underline{x} = (x, 0, 0) \in C_3 \cup C_4 \cup C_5$, $x < 0$.

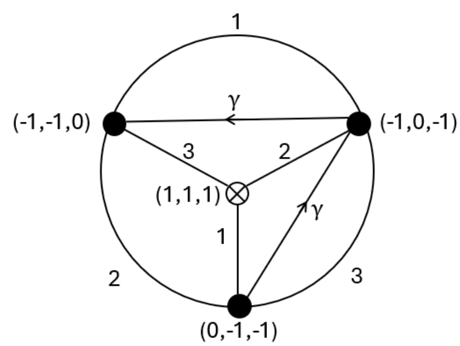


FIGURE 4.4. G_3 : C_6 (A_3). Here $\underline{x} = (-1, 0, -1)$.

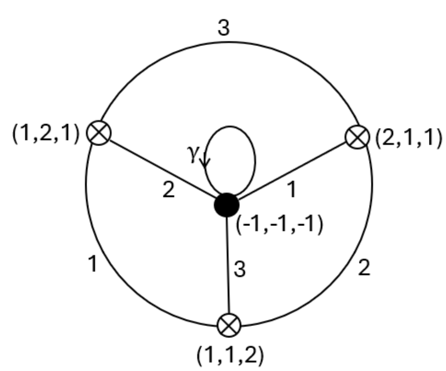


FIGURE 4.5. G_4 : C_7 $(\widehat{A}_2)$. Here $\underline{x} = (-1, -1, -1)$.

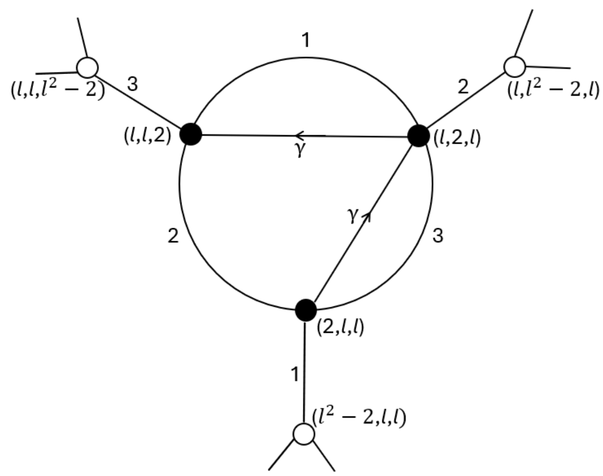


FIGURE 4.6. G_5 : C_8, C_9. Here $\underline{x} = (l, 2, l) \sim (-l, 2, -l)$ with $l \geq 3$.

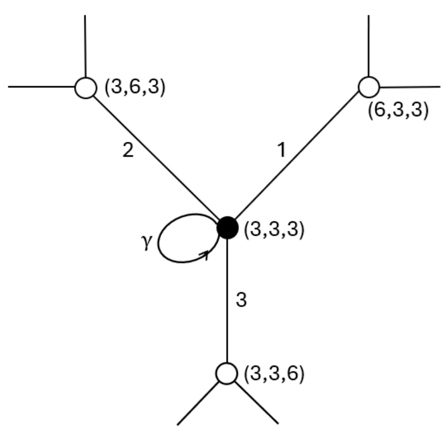


FIGURE 4.7. G_6 : C_{10}, C_{11}, C_{12}. Here $\underline{x} = (3, 3, 3) \in C_{10}$.

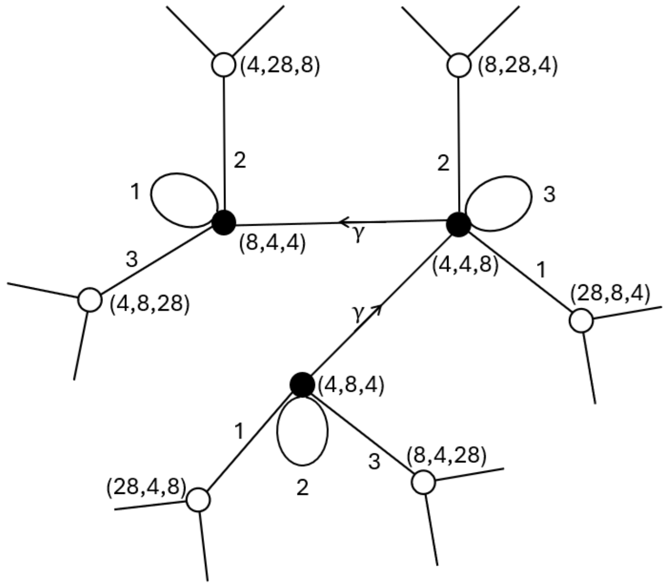


FIGURE 4.8. G_7 : C_{13}. Here $\underline{x} = (4, 4, 8)$.

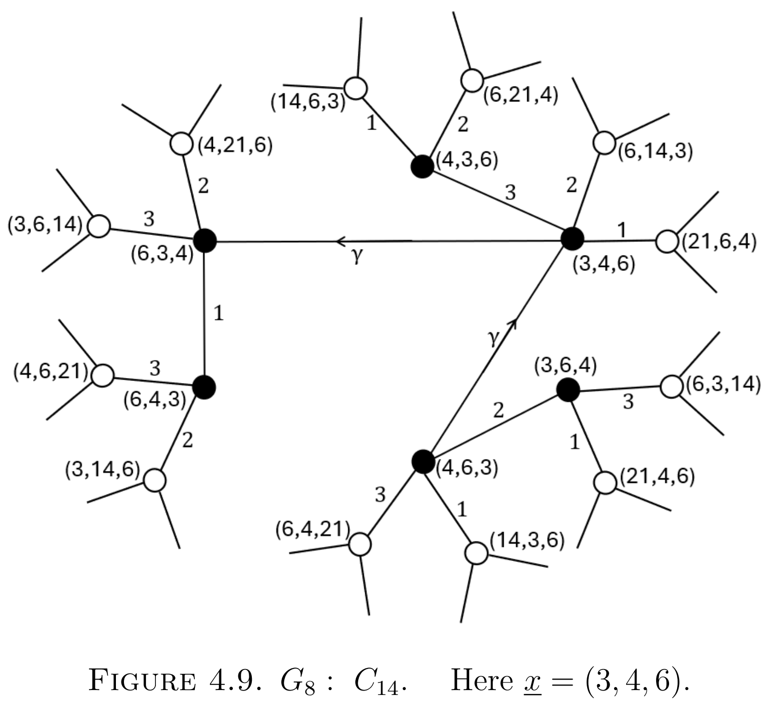


FIGURE 4.9. G_8 : C_{14}. Here $\underline{x} = (3, 4, 6)$.

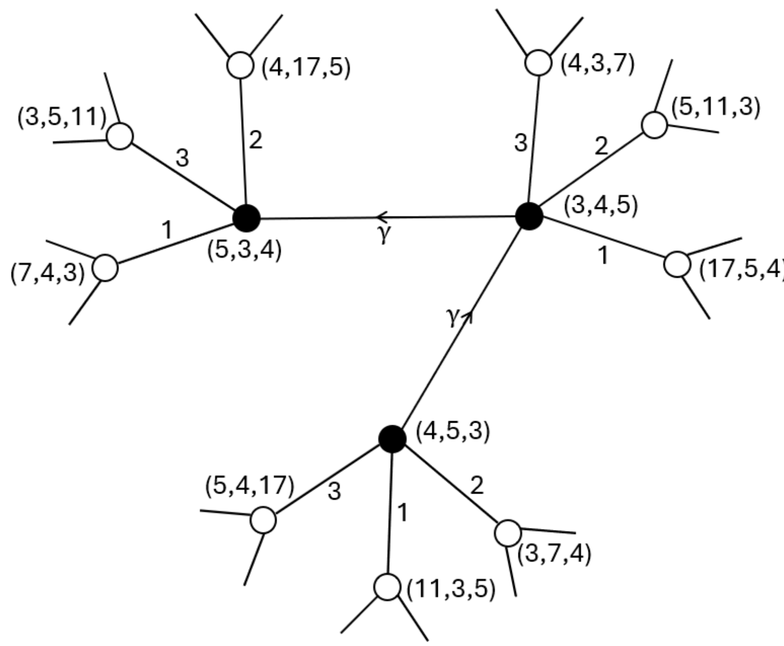


FIGURE 4.10. G_9 : $C_{15}, C_{16}, C_{23}, C_{24}$. Here $\underline{x} = (3,4,5) \in C_{16}$.

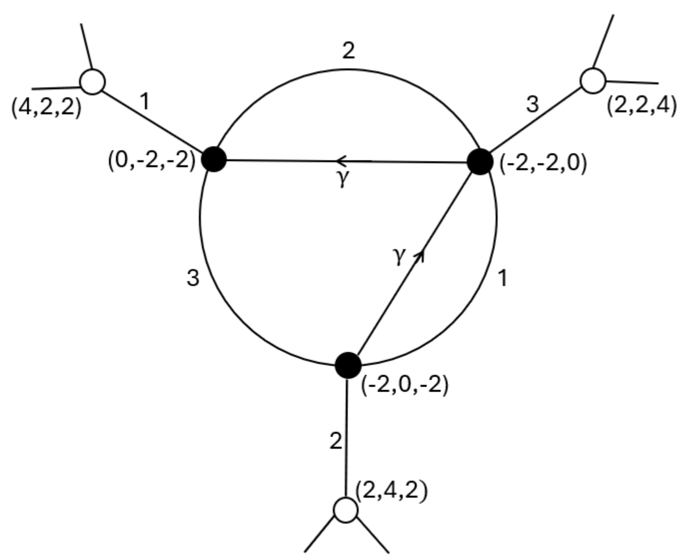


FIGURE 4.11. G_{10} : C_{17}. Here $\underline{x} = (-2,-2,0)$.

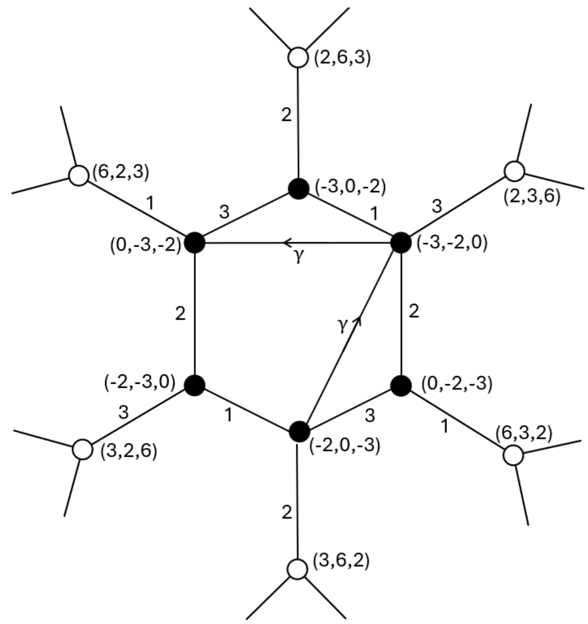


FIGURE 4.12. G_{11} : C_{18}. Here $\underline{x} = (-3, -2, 0)$.

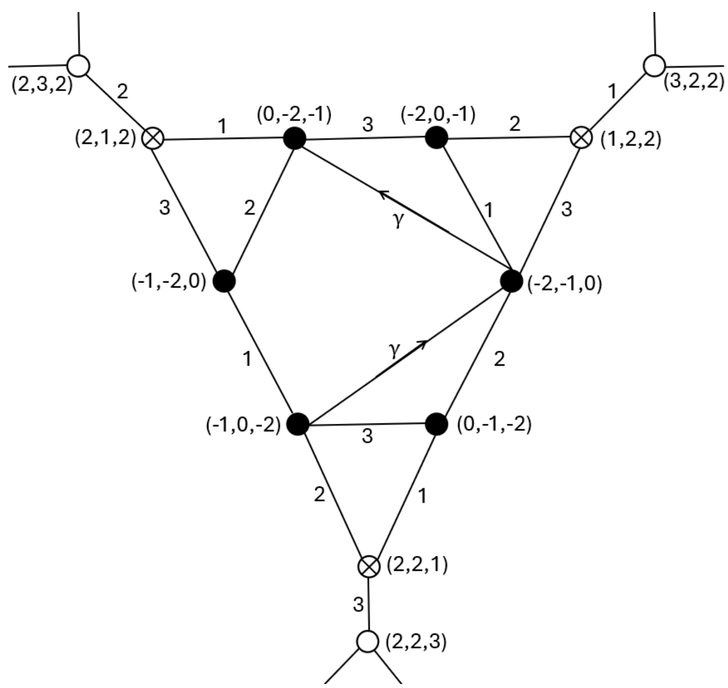


FIGURE 4.13. G_{12} : C_{19}. Here $\underline{x} = (-2, -1, 0)$.

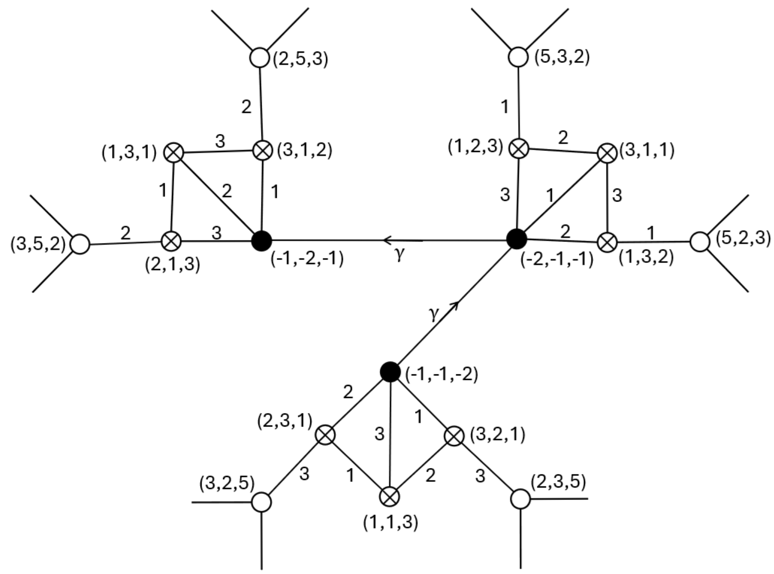


FIGURE 4.14. G_{13} : C_{20}. Here $\underline{x} = (-2, -1, -1)$.

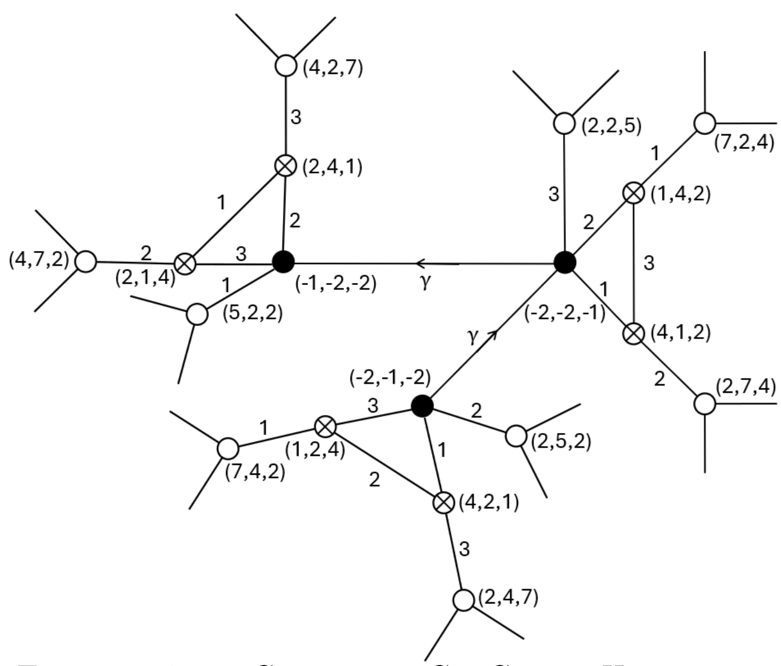


FIGURE 4.15. G_{14} : C_{21}, C_{22}. Here $\underline{x} = (-2, -2, -1) \in C_{21}$.

LEMMA 4.12. *Consider the following 24 sets* C_i, $i \in \{1,2,...,24\}$, *of triples in* $\mathbb{Z}^3$.

$$
\begin{array}{rcl}
C_1 &=& \{(0,0,0)\} \quad (A_1^3),\\
C_2 &=& \{(2,2,2)\} \quad (\mathcal{H}_{1,2}),\\
C_3 &=& \{(-1,0,0)\} \quad (A_2A_1),\\
C_4 &=& \{(-2,0,0)\} \quad (\mathbb{P}^1A_1),\\
C_5 &=& \{(x,0,0)\,|\,x\in\mathbb{Z}_{\leq -3}\}\\
 && \text{(the reducible cases without } A_1^3, A_2A_1, \mathbb{P}^1A_1),\\
C_6 &=& \{(-1,0,-1)\} \quad (A_3),\\
C_7 &=& \{(-1,-1,-1)\} \quad (\widehat{A}_2),\\
C_8 &=& \{(-l,2,-l)\,|\,l\in\mathbb{Z}_{\geq 3}\text{ odd}\},\\
C_9 &=& \{(-l,2,-l)\,|\,l\in\mathbb{Z}_{\geq 4}\text{ even}\},\\
C_{10} &=& \{(3,3,3)\} \quad (\mathbb{P}^2),\\
C_{11} &=& \{(x,x,x)\,|\,x\in\mathbb{Z}_{\geq 4}\},\\
C_{12} &=& \{(x,x,x)\,|\,x\in\mathbb{Z}_{\leq -2}\},\\
C_{13} &=& \{(2y,2y,2y^2)\,|\,y\in\mathbb{Z}_{\geq 2}\},\\
C_{14} &=& \{(x_1,x_2,\frac{1}{2}x_1x_2\,|\,3\leq x_1<x_2, x_1x_2\text{ even}\},\\
C_{15} &=& \{(x_1,x_1,x_3)\,|\,3\leq x_1<x_3<\frac{1}{2}x_1^2\}\\
 && \cup\,\{(x_1,x_2,x_2)\,|\,3\leq x_1<x_2\},\\
C_{16} &=& \{(x_1,x_2,x_3)\,|\,3\leq x_1<x_2<x_3<\frac{1}{2}x_1x_2\}\\
 && \cup\,\{(x_1,x_2,x_3)\,|\,3\leq x_2<x_1<x_3<\frac{1}{2}x_1x_2\},\\
C_{17} &=& \{(x,x,0)\,|\,x\in\mathbb{Z}_{\leq -2}\},\\
C_{18} &=& \{(x_1,x_2,0)\,|\,x_1<x_2\leq -2\},\\
C_{19} &=& \{(x,-1,0)\,|\,x\in\mathbb{Z}_{\leq -2}\},\\
C_{20} &=& \{(x,-1,-1)\,|\,x\in\mathbb{Z}_{\leq -2}\},\\
C_{21} &=& \{(x,x,-1)\,|\,x\in\mathbb{Z}_{\leq -2}\},\\
C_{22} &=& \{(x_1,x_2,-1)\,|\,x_1<x_2\leq -2\}\\
 && \cup\,\{(x_1,x_2,-1)\,|\,x_2<x_1\leq -2\},\\
C_{23} &=& \{(x_1,x_1,x_3)\,|\,x_1<x_3\leq -2\}\\
 && \cup\,\{(x_1,x_2,x_2)\,|\,x_1<x_2\leq -2\},\\
C_{24} &=& \{(x_1,x_2,x_3)\,|\,x_1<x_2<x_3\leq -2\}\\
 && \cup\,\{(x_1,x_2,x_3)\,|\,x_2<x_1<x_3\leq -2\}.
\end{array}
$$

(a) Each triple in $\bigcup_{i=1}^{24} C_i$ is a local minimum. All $\underline{x}$ in one set C_i have the value in the following table or satisfy the inequality in the following table,

ρ	i with $r(\underline{x})=\rho$ for $\underline{x}\in C_i$
0	1, 10
1	3
2	6
4	2, 4, 7, 8, 9
<0	11, 13, 14, 15, 16
>4	5, 12, 17, 18, 19, 20, 21, 22, 23, 24

(b) The following table makes statements about the $\langle\gamma\rangle$ orbits and the $\langle\gamma,\gamma_2\rangle$ orbits of $v_0=\underline{x}/\{\pm 1\}^3$ with $\underline{x}\in\bigcup_{i=1}^{24} C_i$,

$i\in\{1,2,7,10,11,12\}$	$\langle\gamma\rangle(v_0)$ is γ_2-invariant and has size 1
$i\in\{3,4,5,6,8,9,13,15,17,20,21,23\}$	$\langle\gamma\rangle(v_0)$ is γ_2-invariant and has size 3
$i\in\{14,16,18,19,22,24\}$	$\langle\gamma\rangle(v_0)$ is not γ_2-invariant and has size 3, $\langle\gamma,\gamma_2\rangle(v_0)$ has size 6

(c) The set of all local minima in $\mathbb{Z}^3$ is the following disjoint union,

$$\Big(\dot\bigcup_{i\in\{1,...,24\}-\{14,18,19\}}\dot\bigcup_{\underline{x}\in C_i}(G^{sign}\rtimes\langle\gamma\rangle)\{\underline{x}\}\Big)$$
$$\dot\cup\quad\Big(\dot\bigcup_{i\in\{14,18,19\}}\dot\bigcup_{\underline{x}\in C_i}(G^{sign}\rtimes\langle\gamma,\gamma_2\rangle)\{\underline{x}\}\Big).$$

Proof: Part (b) is trivial. The parts (a) and (c) follow with the characterization of local minima in Lemma 4.4 and Theorem 4.6 (c)–(e). $\square$

THEOREM 4.13. *(a) $\mathbb{Z}^3$ is the disjoint union*

$$\dot\bigcup_{i\in\{1,...,24\}}\dot\bigcup_{\underline{x}\in C_i}(\mathrm{Br}_3\ltimes\{\pm 1\}^3)(\underline{x}).$$

(b) For $v_0=\underline{x}/\{\pm 1\}^3$ with $\underline{x}\in\bigcup_{i=1}^{24} C_i$, the set $\mathcal{V}_0=\mathrm{Br}_3(v_0)\cap\mathcal{L}_0$ of sign classes of local minima in the $\mathrm{Br}_3\ltimes\{\pm 1\}^3$ orbit of $\underline{x}$ is as follows,

$$\mathcal{V}_0=\langle\gamma\rangle(v_0)\quad\text{if } i\in\{1,...,24\}-\{14,18,19\},$$
$$\mathcal{V}_0=\langle\gamma,\gamma_2\rangle(v_0)\quad\text{if } i\in\{14,18,19\}.$$

(c) The set $\{\mathcal{G}(\underline{x})\,|\,\underline{x}\in\bigcup_{i=1}^{24} C_i\}$ of pseudo-graphs $\mathcal{G}(\underline{x})$ for $\underline{x}\in\bigcup_{i=1}^{24} C_i$ consists of the 14 isomorphism classes $\mathcal{G}_1,...,\mathcal{G}_{14}$ in the Examples 4.11. All $\underline{x}$ in one set C_i have the same pseudo-graph. The first

and second column in the following table give for each of the 14 pseudo-graphs $\mathcal{G}_j$ the set or the sets C_i with $\mathcal{G}(\underline{x}) = \mathcal{G}_j$ for $\underline{x} \in C_i$. The third and fourth column in the following table are subject of Theorem 4.16.

	sets	$(G^{phi} \rtimes \langle\gamma\rangle)_{\underline{x}/\{\pm 1\}^3}$	$(\mathrm{Br}_3)_{\underline{x}/\{\pm 1\}^3}$
$\mathcal{G}_1$	$C_1\ (A_1^3),\ C_2\ (\mathcal{H}_{1,2})$	$G^{phi} \rtimes \langle\gamma\rangle$	Br_3
$\mathcal{G}_2$	C_3, C_4, C_5 (red. cases)	$\langle\varphi_1, \gamma^{-1}\varphi_3\rangle$	$\langle\sigma_1, \sigma_2^2\rangle$
$\mathcal{G}_3$	$C_6\ (A_3)$	$\langle\gamma\varphi_3, \varphi_2\varphi_1\varphi_3\rangle$	$\langle\sigma_1\sigma_2, \sigma_1^3\rangle$
$\mathcal{G}_4$	$C_7\ (\widehat{A}_2)$	$\langle\gamma, \varphi_2\varphi_1\varphi_3\rangle$	$\langle\sigma_2\sigma_1, \sigma_1^3\rangle$
$\mathcal{G}_5$	$C_8,\ C_9\ ((-l, 2, -l))$	$\langle\gamma^{-1}\varphi_1\rangle$	$\langle\sigma^{mon}, \sigma_1^{-1}\sigma_2^{-1}\sigma_1\rangle$
$\mathcal{G}_6$	$C_{10}(\mathbb{P}^2),\ C_{11},\ C_{12}$	$\langle\gamma\rangle$	$\langle\sigma_2\sigma_1\rangle$
$\mathcal{G}_7$	C_{13} (e.g. $(4, 4, 8)$)	$\langle\varphi_3\rangle$	$\langle\sigma_2\sigma_1^2\rangle$
$\mathcal{G}_8$	C_{14}(e.g. $(3, 4, 6)$)	$\langle\mathrm{id}\rangle$	$\langle\sigma^{mon}\rangle$
$\mathcal{G}_9$	$C_{15},\ C_{16},\ C_{23},\ C_{24}$	$\langle\mathrm{id}\rangle$	$\langle\sigma^{mon}\rangle$
$\mathcal{G}_{10}$	C_{17} (e.g. $(-2, -2, 0)$)	$\langle\gamma^{-1}\varphi_2\rangle$	$\langle\sigma^{mon}, \sigma_2\rangle$
$\mathcal{G}_{11}$	C_{18} (e.g. $(-3, -2, 0)$)	$\langle\gamma^{-1}\varphi_3\varphi_1\rangle$	$\langle\sigma^{mon}, \sigma_2^2\rangle$
$\mathcal{G}_{12}$	C_{19} (e.g. $(-2, -1, 0)$)	$\langle\gamma^{-1}\varphi_3\varphi_1, \varphi_3\varphi_2\varphi_1\rangle$	$\langle\sigma^{mon}, \sigma_2^2, \sigma_2\sigma_1^3\sigma_2^{-1}\rangle$
$\mathcal{G}_{13}$	C_{20} (e.g. $(-2, -1, -1)$)	$\langle\varphi_2\varphi_3\varphi_1, \varphi_3\varphi_2\varphi_1\rangle$	$\langle\sigma^{mon}, \sigma_2^3, \sigma_2\sigma_1^3\sigma_2^{-1}\rangle$
$\mathcal{G}_{14}$	$C_{21},\ C_{22}$ (e.g. $(-2, -2, -1)$)	$\langle\varphi_2\varphi_3\varphi_1\rangle$	$\langle\sigma^{mon}, \sigma_2^3\rangle$

Proof: We start with part (c). It can be seen rather easily for all $\underline{x}$ in one family C_i simultaneously. We do not give more details.

(b) The pseudo-graphs G_8, G_{11} and G_{12} are the only of the 14 pseudo-graphs with $|\mathcal{V}_0| = 6$. By inspection of them or by Lemma 4.10 (e), for them $\mathcal{V}_0 = \langle\gamma, \gamma_2\rangle(v_0)$. The table in part (c) gives the correspondence $G_8 \leftrightarrow C_{14}$, $G_{11} \leftrightarrow C_{18}$, $G_{12} \leftrightarrow C_{19}$. The other 11 pseudo-graphs satisfy $|\mathcal{V}_0| = 1$ or $|\mathcal{V}_0| = 3$, so in any case $\mathcal{V}_0 = \langle\gamma\rangle(v_0)$.

(a) Part (c) of Lemma 4.12 alone shows already that $\mathbb{Z}^3$ is the union given. Part (b) of Theorem 4.13 adds only the fact that this is a disjoint union. □

Remarks 4.14. (i) We have 14 pseudo-graphs $\mathcal{G}_1, ..., \mathcal{G}_{14}$, but 24 sets $C_1, ..., C_{24}$ because the separation into the sets shall be fine enough for the table in Theorem 4.13 (c) and the tables in Lemma 4.12 (a) and (b).

(ii) In the pseudo-graphs $\mathcal{G}_j$ with $|\mathcal{V}_0| = 3$ or $|\mathcal{V}_0| = 6$ one can choose another distinguished vertex $\widetilde{v_0} \in \mathcal{V}_0$ and change the set $\mathcal{E}_\gamma$ to a set $\widetilde{\mathcal{E}_\gamma}$ accordingly. This gives a pseudo-graph $\widetilde{\mathcal{G}_j}$ which is not equal to $\mathcal{G}_j$, but closely related.

The graphs $\mathcal{G}_5$ and $\mathcal{G}_{10}$ are related by such a change.

The following table shows for each $\mathcal{G}_j$ except $\mathcal{G}_{10}$ the number of isomorphism classes of pseudo-graphs obtained in this way (including the original pseudo-graphs). In the cases $\mathcal{G}_8$ and $\mathcal{G}_{11}$ there is $|\mathcal{V}_0| = 6$, but because of some symmetry of the pseudo-graph without $\mathcal{E}_\gamma$, there are only 3 related pseudo-graphs.

$\mathcal{G}_i,\ i$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	$\sum$
related pseudo-graphs	1	3	3	1	3	1	3	3	3	G_5	3	6	3	3	36

The total number 36 is the number of isomorphism classes of pseudo-graphs $\mathcal{G}(\underline{x})$ for $\underline{x}\in\mathbb{Z}^3$ a local minimum.

4.4. The stabilizers of upper triangular 3×3 matrices

The groups $G^{phi}\ltimes\langle\gamma\rangle$ and Br_3 act on $\mathbb{Z}^3/\{\pm 1\}^3$. The pseudo-graphs in the Examples 4.11 and Theorem 4.13 offer a convenient way to determine the stabilizers $(G^{phi}\ltimes\langle\gamma\rangle)_{v_0}$ and $(\text{Br}_3)_{v_0}$ for $v_0=\underline{x}/\{\pm 1\}^3$ with $\underline{x}\in\bigcup_{i=1}^{24}C_i$ a local minimum.

The stabilizers of v_0 depend only on the pseudo-graph G_j with $G_j=\mathcal{G}(\underline{x})$. The results are presented in Theorem 4.16. The Remarks 4.15 prepare this.

Remarks 4.15. (i) First we recall some well known facts about the groups $SL_2(\mathbb{Z})$ and $PSL_2(\mathbb{Z})$ and their relation to Br_3. The group $SL_2(\mathbb{Z})$ is generated by the matrices

$$A_1:=\begin{pmatrix}1&-1\\0&1\end{pmatrix}\quad\text{and}\quad A_2:=\begin{pmatrix}1&0\\1&1\end{pmatrix}.$$

Generating relations are

$$A_1A_2A_1=A_2A_1A_2\quad\text{and}\quad (A_2A_1)^6=E_2.$$

The group Br_3 is generated by the elementary braids σ_1 and σ_2. The only generating relation is $\sigma_1\sigma_2\sigma_1=\sigma_2\sigma_1\sigma_2$. Therefore there is a surjective group homomorphism

$$\text{Br}_3\to SL_2(\mathbb{Z}),\quad \sigma_1\mapsto A_1,\quad \sigma_2\mapsto A_2,$$

with kernel $\langle(\sigma_2\sigma_1)^6\rangle=\langle(\sigma^{mon})^2\rangle$. It induces a surjective group homomorphism

$$\text{Br}_3\to PSL_2(\mathbb{Z}),\quad \sigma_1\mapsto [A_1],\quad \sigma_2\mapsto [A_2],$$

with kernel $\langle\sigma^{mon}\rangle$ because $(A_2A_1)^3=-E_2$.

(ii) The action of $\text{Br}_3\ltimes\{\pm 1\}^3$ on $T_3^{uni}(\mathbb{Z})$ and on $\mathbb{Z}^3$ is fixed in the beginning of section 4.1. One sees that $\sigma^{mon}=(\sigma_2\sigma_1)^3$ acts trivially on $T_3^{uni}(\mathbb{Z})$ and $\mathbb{Z}^3$. This can be checked directly. Or it can be seen as a consequence of the following two facts.

(1) The action of $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ on $(\mathrm{Br}_3 \ltimes \{\pm 1\}^3)(\underline{x})$ for some $\underline{x} \in \mathbb{Z}^3$ is induced by the action of $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ on the set $\mathcal{B}^{dist}$ of distinguished bases of a triple $(H_{\mathbb{Z}}, L, \underline{e})$ with $L(\underline{e}^t, \underline{e})^t = S(\underline{x})$ by $S((\alpha,\varepsilon)(\underline{x})) = L((\alpha,\varepsilon)(\underline{e})^t, (\alpha,\varepsilon)(\underline{e}))^t$ for $(\alpha,\varepsilon) \in \mathrm{Br}_3 \ltimes \{\pm 1\}^3$.

(2) For $(\alpha,\varepsilon) = (\sigma^{mon}, (1,1,1))$ $(\alpha,\varepsilon)(\underline{e}) = Z((\alpha,\varepsilon))(\underline{e}) = M(\underline{e})$ by Theorem 3.10, and $L(M(\underline{e})^t, M(\underline{e})) = L(\underline{e}^t, \underline{e})$.

In any case the action of $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ on $\mathbb{Z}^3$ boils down to a nonlinear action of $PSL_2(\mathbb{Z}) \ltimes G^{sign}$ where $G^{sign} = \langle \delta_1^{\mathbb{R}}, \delta_2^{\mathbb{R}} \rangle$ catches the action of $\{\pm 1\}^3$ on $\mathbb{Z}^3$, see section 4.1.

(iii) The shape of this nonlinear action led us in Definition 4.1 and Theorem 4.2 to the group $(G^{phi} \ltimes G^{sign}) \rtimes \langle \gamma \rangle = (G^{phi} \rtimes \langle \gamma \rangle) \ltimes G^{sign}$. In fact, $G^{phi} \rtimes \langle \gamma \rangle \cong PSL_2(\mathbb{Z})$. This can be seen as follows.

The formulas (4.1)–(4.4) in the proof of Theorem 4.2 (c) give lifts to $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ of the generators $\varphi_1, \varphi_2, \varphi_3$ and γ of $G^{phi} \ltimes \langle \gamma \rangle$. Dropping the generators of the sign action in these lifts, we obtain the following lifts to Br_3,

$$\left.\begin{array}{l} l(\gamma) = \sigma_2\sigma_1, \quad l(\gamma^{-1}) = \sigma_1^{-1}\sigma_2^{-1}, \\ l(\varphi_1) = l(\gamma)^{-1}\sigma_2^{-1} = \sigma_1^{-1}\sigma_2^{-2}, \\ l(\varphi_2) = l(\gamma)\sigma_2 = \sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1, \\ l(\varphi_3) = l(\gamma)\sigma_1 = \sigma_2\sigma_1^2. \end{array}\right\} \quad (4.11)$$

The equality of groups in Theorem 4.2 (c) boils down after dropping the sign action to an equality of groups

$$\langle \sigma_1^{\mathbb{R}}, \sigma_2^{\mathbb{R}} \rangle \cong G^{phi} \rtimes \langle \gamma \rangle. \quad (4.12)$$

As $(\sigma_2^{\mathbb{R}}\sigma_1^{\mathbb{R}})^3 = \mathrm{id}$, we obtain a surjective group homomorphism $PSL_2(\mathbb{Z}) \to G^{phi} \rtimes \langle \gamma \rangle$ with $[A_i] \mapsto \sigma_i^{\mathbb{R}}$. The subgroup $\langle [A_1]^{-1}[A_2]^{-2}, [A_2][A_1][A_2], [A_2][A_1]^2 \rangle$ of $PSL_2(\mathbb{Z})$ is mapped to G^{phi}. One easily calculates that this subgroup is the free Coxeter group with three generators which was considered in Remark 6.12 (iv) and which has index three in $PSL_2(\mathbb{Z})$. As G^{phi} is also a free Coxeter group with three generators and has index 3 in $G^{phi} \rtimes \langle \gamma \rangle$, the map $PSL_2(\mathbb{Z}) \to G^{phi} \rtimes \langle \gamma \rangle$ is a group isomorphism.

(iv) For use in the proof of Theorem 4.16 we recall the formulas (4.6)

$$\left.\begin{array}{rclrclrcl} \gamma\varphi_1 &=& \varphi_2\gamma, & \gamma\varphi_2 &=& \varphi_3\gamma, & \gamma\varphi_3 &=& \varphi_1\gamma, \\ \varphi_1\gamma^{-1} &=& \gamma^{-1}\varphi_2, & \varphi_2\gamma^{-1} &=& \gamma^{-1}\varphi_3, & \varphi_3\gamma^{-1} &=& \gamma^{-1}\varphi_1, \end{array}\right\} (4.13)$$

from the proof of Theorem 4.2.

(v) The relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ is equivalent to each of the two relations

$$\sigma_1\sigma_2\sigma_1^{-1} = \sigma_2^{-1}\sigma_1\sigma_2 \quad \text{and} \quad \sigma_1^{-1}\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2^{-1}$$

and induces for any $m \in \mathbb{Z}$ the relations

$$\sigma_1\sigma_2^m\sigma_1^{-1} = \sigma_2^{-1}\sigma_1^m\sigma_2 \quad \text{and} \quad \sigma_1^{-1}\sigma_2^m\sigma_1 = \sigma_2\sigma_1^m\sigma_2^{-1}. \tag{4.14}$$

Also this will be useful in the proof of Theorem 4.16.

THEOREM 4.16. *Consider $v_0 := \underline{x}/\{\pm 1\}^3$ with $\underline{x} \in \bigcup_{i=1}^{24} C_i \subset \mathbb{Z}^3$ a local minimum, and consider the pseudo-graph $\mathcal{G}_j$ with $\mathcal{G}_j = \mathcal{G}(\underline{x})$. The entries in the third and fourth column in the table in Theorem 4.13, which are in the line of $\mathcal{G}_j$, give the stabilizers $(G^{phi} \rtimes \langle\gamma\rangle)_{v_0}$ and $(\mathrm{Br}_3)_{v_0}$ of v_0.*

Proof: First we treat the stabilizer $(G^{phi} \rtimes \langle\gamma\rangle)_{v_0}$.

The *total set* $|\mathcal{G}_j|$ of the pseudo-graph $\mathcal{G}_j$ means the union of vertices and edges in an embedding of the pseudo-graph in the real plane $\mathbb{R}^2$ as in the figures in the Examples 4.11. The fundamental group $\pi_1(|\mathcal{G}_j|, v_0)$ is a free group with 0, 1, 3, 4, 5 or 6 generators. The number of generators is the number of compact components in $\mathbb{R}^2 - |\mathcal{G}_j|$. A generator is the class of a closed path which starts and ends at v_0 and turns once around one of these compact components. Any such closed path induces a word in $\varphi_1, \varphi_2, \varphi_3, \gamma$ and γ^{-1}. This word gives an element of $(G^{phi} \rtimes \langle\gamma\rangle)_{v_0}$. We obtain a group homomorphism

$$\pi_1(|\mathcal{G}_j|, v_0) \to (G^{phi} \rtimes \langle\gamma\rangle)_{v_0}.$$

It is surjective because any element of $(G^{phi} \rtimes \langle\gamma\rangle)_{v_0}$ can be written as $\psi\gamma^\xi$ with $\psi \in G^{phi}$ and $\xi \in \{0, \pm 1\}$. The element $\psi\gamma^\xi$ leads to and comes from a closed path in $|\mathcal{G}_j|$ which starts and ends at v_0.

In fact, this shows that we could restrict to closed paths which run never or only once at the beginning through an edge in $\mathcal{E}_\gamma$. But we will not use this fact.

The following list gives for each of the 14 pseudo-graphs $\mathcal{G}_1, ..., \mathcal{G}_{14}$ in the first line one word in $\varphi_1, \varphi_2, \varphi_3, \gamma$ and γ^{-1} for each compact component of $\mathbb{R}^2 - |\mathcal{G}_j|$. In the following lines the relations (4.13) are used to show that all these words are generated in $G^{phi} \rtimes \langle\gamma\rangle$ by the generators in the third column in the table in Theorem 4.13. The generators are underlined.

$\mathcal{G}_1$: $\underline{\varphi_1}$, $\underline{\varphi_2}$, $\underline{\varphi_3}$, $\underline{\gamma}$.

$\mathcal{G}_2$: $\underline{\varphi_1}$, $\varphi_3\gamma$, $\gamma\varphi_2$, $\gamma^{-1}\varphi_2\gamma$, $\gamma\varphi_3\gamma^{-1}$, $\gamma\varphi_1\gamma$.

$$\begin{aligned}\gamma\varphi_1\gamma &= \gamma^2\varphi_3 = \underline{\gamma^{-1}\varphi_3},\\ \gamma\varphi_2 &= \varphi_3\gamma = (\gamma^{-1}\varphi_3)^{-1},\\ \gamma^{-1}\varphi_2\gamma &= \varphi_1,\\ \gamma\varphi_3\gamma^{-1} &= \varphi_1.\end{aligned}$$

$\mathcal{G}_3$: $\varphi_1\gamma$, $\varphi_2\varphi_3\gamma$, $\gamma\varphi_2\gamma$, $\gamma\varphi_1\varphi_2$, $\underline{\gamma\varphi_3}$.

$$\begin{aligned}\varphi_1\gamma &= \gamma\varphi_3,\\ \varphi_2\varphi_3\gamma &= \varphi_2\gamma\varphi_2 = \gamma\varphi_1\varphi_2 = (\gamma\varphi_3)(\underline{\varphi_2\varphi_1\varphi_3})^{-1},\\ \gamma\varphi_2\gamma &= \gamma^2\varphi_1 = \gamma^{-1}\varphi_1 = \varphi_3\gamma^{-1} = (\gamma\varphi_3)^{-1}.\end{aligned}$$

$\mathcal{G}_4$: $\underline{\gamma}$, $\underline{\varphi_2\varphi_1\varphi_3}$, $\varphi_3\varphi_2\varphi_1$, $\varphi_1\varphi_3\varphi_2$.

$$\begin{aligned}\gamma\varphi_2\varphi_1\varphi_3\gamma^{-1} &= \gamma\varphi_2\varphi_1\gamma^{-1}\varphi_1 = \gamma\varphi_2\gamma^{-1}\varphi_2\varphi_1 = \varphi_3\varphi_2\varphi_1,\\ \gamma^{-1}\varphi_2\varphi_1\varphi_3\gamma &= \gamma^{-1}\varphi_2\varphi_1\gamma\varphi_2 = \gamma^{-1}\varphi_2\gamma\varphi_3\varphi_2 = \varphi_1\varphi_3\varphi_2.\end{aligned}$$

$\mathcal{G}_5$: $\underline{\gamma^{-1}\varphi_1}$, $\gamma\varphi_2\gamma$, $\gamma\varphi_3$.

$$\begin{aligned}\gamma\varphi_2\gamma &= \gamma^2\varphi_1 = \gamma^{-1}\varphi_1,\\ \gamma\varphi_3 &= \varphi_1\gamma = (\gamma^{-1}\varphi_1)^{-1}.\end{aligned}$$

$\mathcal{G}_6$: $\underline{\gamma}$.
$\mathcal{G}_7$: $\underline{\varphi_3}$, $\gamma^{-1}\varphi_1\gamma$, $\gamma\varphi_2\gamma^{-1}$.

$$\begin{aligned}\gamma^{-1}\varphi_1\gamma &= \varphi_3,\\ \gamma\varphi_2\gamma^{-1} &= \varphi_3.\end{aligned}$$

$\mathcal{G}_8$: $\underline{\mathrm{id}}$.
$\mathcal{G}_9$: $\underline{\mathrm{id}}$.
$\mathcal{G}_{10}$: $\underline{\gamma^{-1}\varphi_2}$, $\gamma\varphi_3\gamma$, $\gamma\varphi_1$.

$$\begin{aligned}\gamma\varphi_3\gamma &= \gamma^2\varphi_2 = \gamma^{-1}\varphi_2,\\ \gamma\varphi_1 &= \varphi_2\gamma = (\gamma^{-1}\varphi_2)^{-1}.\end{aligned}$$

$\mathcal{G}_{11}$: $\underline{\gamma^{-1}\varphi_3\varphi_1}$, $\gamma\varphi_1\varphi_2\gamma$, $\varphi_2\varphi_3\gamma^{-1}$.

$$\begin{aligned}\gamma\varphi_1\varphi_2\gamma &= \gamma\varphi_1\gamma\varphi_1 = \gamma^2\varphi_3\varphi_1 = \gamma^{-1}\varphi_3\varphi_1,\\ \varphi_2\varphi_3\gamma^{-1} &= \varphi_2\gamma^{-1}\varphi_1 = \gamma^{-1}\varphi_3\varphi_1.\end{aligned}$$

$\mathcal{G}_{12}$: $\underline{\varphi_3\varphi_2\varphi_1}$, $\underline{\gamma^{-1}\varphi_3\varphi_1}$, $\gamma^{-1}\varphi_1\varphi_3\varphi_2\gamma$, $\gamma\varphi_1\varphi_2\gamma$, $\varphi_2\varphi_3\gamma^{-1}$, $\gamma\varphi_2\varphi_1\varphi_3\gamma^{-1}$.

$$\begin{aligned}
\gamma^{-1}\varphi_1\varphi_3\varphi_2\gamma &= \gamma^{-1}\varphi_1\varphi_3\gamma\varphi_1 = \gamma^{-1}\varphi_1\gamma\varphi_2\varphi_1 = \varphi_3\varphi_2\varphi_1, \\
\gamma\varphi_1\varphi_2\gamma &= \gamma\varphi_1\gamma\varphi_1 = \gamma^2\varphi_3\varphi_1 = \gamma^{-1}\varphi_3\varphi_1, \\
\varphi_2\varphi_3\gamma^{-1} &= \varphi_2\gamma^{-1}\varphi_1 = \gamma^{-1}\varphi_3\varphi_1, \\
\gamma\varphi_2\varphi_1\varphi_3\gamma^{-1} &= \gamma\varphi_2\varphi_1\gamma^{-1}\varphi_1 = \gamma\varphi_2\gamma^{-1}\varphi_2\varphi_1 = \varphi_3\varphi_2\varphi_1.
\end{aligned}$$

$\mathcal{G}_{13}$: $\underline{\varphi_2\varphi_3\varphi_1}$, $\underline{\varphi_3\varphi_2\varphi_1}$, $\gamma^{-1}\varphi_3\varphi_1\varphi_2\gamma$, $\gamma^{-1}\varphi_1\varphi_3\varphi_2\gamma$, $\gamma\varphi_1\varphi_2\varphi_3\gamma^{-1}$, $\gamma\varphi_2\varphi_1\varphi_3\gamma^{-1}$.

$$\begin{aligned}
\gamma^{-1}\varphi_3\varphi_1\varphi_2\gamma &= \gamma^{-1}\varphi_3\varphi_1\gamma\varphi_1 = \gamma^{-1}\varphi_3\gamma\varphi_3\varphi_1 = \varphi_2\varphi_3\varphi_1, \\
\gamma^{-1}\varphi_1\varphi_3\varphi_2\gamma &= \gamma^{-1}\varphi_1\varphi_3\gamma\varphi_1 = \gamma^{-1}\varphi_1\gamma\varphi_2\varphi_1 = \varphi_3\varphi_2\varphi_1, \\
\gamma\varphi_1\varphi_2\varphi_3\gamma^{-1} &= \gamma\varphi_1\varphi_2\gamma^{-1}\varphi_1 = \gamma\varphi_1\gamma^{-1}\varphi_3\varphi_1 = \varphi_2\varphi_3\varphi_1, \\
\gamma\varphi_2\varphi_1\varphi_3\gamma^{-1} &= \gamma\varphi_2\varphi_1\gamma^{-1}\varphi_1 = \gamma\varphi_2\gamma^{-1}\varphi_2\varphi_1 = \varphi_3\varphi_2\varphi_1.
\end{aligned}$$

$\mathcal{G}_{14}$: $\underline{\varphi_2\varphi_3\varphi_1}$, $\gamma^{-1}\varphi_3\varphi_1\varphi_2\gamma$, $\gamma\varphi_1\varphi_2\varphi_3\gamma^{-1}$.

$$\begin{aligned}
\gamma^{-1}\varphi_3\varphi_1\varphi_2\gamma &= \varphi_2\varphi_3\varphi_1 \quad (\text{see } \mathcal{G}_{13}), \\
\gamma\varphi_1\varphi_2\varphi_3\gamma^{-1} &= \varphi_2\varphi_3\varphi_1 \quad (\text{see } \mathcal{G}_{13}).
\end{aligned}$$

Therefore the stabilizer $(G^{phi} \rtimes \langle\gamma\rangle)_{v_0}$ is as claimed in the third column of the table in Theorem 4.13.

Now we treat the stabilizer $(\mathrm{Br}_3)_{v_0}$. It is the preimage in Br_3 of $(G^{phi} \rtimes \langle\gamma\rangle)_{v_0}$ under the surjective group homomorphism $\mathrm{Br}_3 \to G^{phi} \rtimes \langle\gamma\rangle$ with kernel $\langle\sigma^{mon}\rangle$. So if $(G^{phi} \rtimes \langle\gamma\rangle)_{v_0} = \langle g_1, ..., g_m\rangle$ and $h_1, ..., h_m$ are any lifts to Br_3 of $g_1, ..., g_m$ then $(\mathrm{Br}_3)_{v_0} = \langle\sigma^{mon}, h_1, ..., h_m\rangle$.

For any word in $\varphi_1, \varphi_2, \varphi_3, \gamma$ and γ^{-1} we use the lifts in (4.11) to construct a lift of this word.

The following list gives for each of the 14 pseudo-graphs $\mathcal{G}_1, ..., \mathcal{G}_{14}$ for each generator of $(G^{phi} \times \langle\gamma\rangle)_{v_0}$ in the third column of the table in Theorem 4.13 this lift and rewrites it using the relations (4.14). The generators in the fourth column of the table in Theorem 4.13 are underlined.

$$
\begin{array}{lll}
\mathcal{G}_1: & G^{phi} & \rightsquigarrow \mathrm{Br}_3, \\
\mathcal{G}_2: & \varphi_1 & \rightsquigarrow \sigma_1^{-1}\sigma_2^{-2} = \sigma_2(\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1})\sigma_2^{-1} = \sigma_2(\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1})\sigma_2^{-1} \\
 & & = (\sigma_2^2\sigma_1)(\sigma_1^{-1}\sigma_2^{-1})^3 = \underline{\sigma_2^2}\sigma_1(\sigma^{mon})^{-1}, \\
 & \gamma^{-1}\varphi_3 & \rightsquigarrow (\sigma_1^{-1}\sigma_2^{-1})(\sigma_2\sigma_1^2) = \underline{\sigma_1}, \\
\mathcal{G}_3: & \gamma\varphi_3 & \rightsquigarrow (\sigma_2\sigma_1)(\sigma_2\sigma_1^2) = (\sigma_2\sigma_1\sigma_2)\sigma_1^2 = (\underline{\sigma_1\sigma_2})\sigma_1^3, \\
 & \varphi_2\varphi_1\varphi_3 & \rightsquigarrow (\sigma_1\sigma_2\sigma_1)(\sigma_1^{-1}\sigma_2^{-2})(\sigma_2\sigma_1^2) = \underline{\sigma_1^3}, \\
\mathcal{G}_4: & \gamma & \rightsquigarrow \underline{\sigma_2\sigma_1}, \\
 & \varphi_2\varphi_1\varphi_3 & \rightsquigarrow \underline{\sigma_1^3}, \\
\mathcal{G}_5: & \gamma^{-1}\varphi_1 & \rightsquigarrow (\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_2^{-2}) = (\sigma_1^{-1}\sigma_2^{-1})^3\sigma_2\sigma_1\sigma_2^{-1} \\
 & & \overset{(4.14)}{=} (\sigma^{mon})^{-1}(\underline{\sigma_1^{-1}\sigma_2^{-1}\sigma_1})^{-1}, \\
\mathcal{G}_6: & \gamma & \rightsquigarrow \underline{\sigma_2\sigma_1}, \\
\mathcal{G}_7: & \varphi_3 & \rightsquigarrow \underline{\sigma_2\sigma_1^2},
\end{array}
$$

$$
\begin{array}{lll}
\mathcal{G}_8: & \mathrm{id} & \rightsquigarrow \underline{\mathrm{id}}, \\
\mathcal{G}_9: & \mathrm{id} & \rightsquigarrow \underline{\mathrm{id}}, \\
\mathcal{G}_{10}: & \gamma^{-1}\varphi_2 & \rightsquigarrow (\sigma_1^{-1}\sigma_2^{-1})(\sigma_2\sigma_1\sigma_2) = \underline{\sigma_2}, \\
\mathcal{G}_{11}: & \gamma^{-1}\varphi_3\varphi_1 & \rightsquigarrow (\sigma_1^{-1}\sigma_2^{-1})(\sigma_2\sigma_1^2)(\sigma_1^{-1}\sigma_2^{-2}) = \sigma_2^{-2} = (\underline{\sigma_2^2})^{-1}, \\
\mathcal{G}_{12}: & \gamma^{-1}\varphi_3\varphi_1 & \rightsquigarrow (\underline{\sigma_2^2})^{-1}, \\
 & \varphi_3\varphi_2\varphi_1 & \rightsquigarrow (\sigma_2\sigma_1^2)(\sigma_1\sigma_2\sigma_1)(\sigma_1^{-1}\sigma_2^{-2}) = \underline{\sigma_2\sigma_1^3\sigma_2^{-1}}, \\
\mathcal{G}_{13}: & \varphi_2\varphi_3\varphi_1 & \rightsquigarrow (\sigma_1\sigma_2\sigma_1)(\sigma_2\sigma_1^2)(\sigma_1^{-1}\sigma_2^{-2}) = (\sigma_1\sigma_2)^3\sigma_2^{-3} \\
 & & = \sigma^{mon}(\underline{\sigma_2^3})^{-1}, \\
 & \varphi_3\varphi_2\varphi_1 & \rightsquigarrow \underline{\sigma_2\sigma_1^3\sigma_2^{-1}}, \\
\mathcal{G}_{14}: & \varphi_2\varphi_3\varphi_1 & \rightsquigarrow \sigma^{mon}(\underline{\sigma_2^3})^{-1}.
\end{array}
$$

Observe

$$(\sigma_2\sigma_1)^3 = \sigma^{mon}, \quad (\sigma_2\sigma_1^2)^2 = \sigma^{mon}.$$

Therefore the stabilizer $(\mathrm{Br}_3)_{v_0}$ is as claimed in the fourth column of the table in Theorem 4.13. □

4.5. A global sign change, relevant for the odd case

REMARKS 4.17. (i) For $S \in T_n^{uni}(\mathbb{Z})$ consider a unimodular bilinear lattice $(H_\mathbb{Z}, L)$ with a triangular basis $\underline{e} = (e_1, ..., e_n)$ with $L(\underline{e}^t, \underline{e})^t = S$. Consider also the matrix $\widetilde{S} \in T_n^{uni}(\mathbb{Z})$ with $\widetilde{S}_{ij} = -S_{ij}$ for $i < j$.

On the same lattice $H_{\mathbb{Z}}$ and the same basis $\underline{e}$ we define a second unimodular bilinear form $\widetilde{L}$ by $\widetilde{L}(\underline{e}^t,\underline{e})^t = \widetilde{S}$ and denote all objects associated to $(H_{\mathbb{Z}},\widetilde{L},\underline{e})$ with a tilde, $\widetilde{I}^{(k)}$, $\widetilde{\mathcal{B}}^{tri}$, $\widetilde{\Gamma}^{(k)}$, $\widetilde{\Delta}^{(k)}$.

Most of them differ a lot from the objects associated to $(H_{\mathbb{Z}},L,\underline{e})$. But the odd intersection forms differ only by the sign. Therefore the monodromies M and $\widetilde{M}$ are different (in general), but the odd monodromy groups and the sets of odd vanishing cycles coincide:

$$\begin{aligned}
\widetilde{I}^{(1)} &= -I^{(1)},\\
\text{so } \widetilde{s}^{(1)}_{e_i} &= (s^{(1)}_{e_i})^{-1},\\
\widetilde{M} &= \widetilde{s}^{(1)}_{e_1}\circ ...\circ \widetilde{s}^{(1)}_{e_n} \overset{\text{in general}}{\neq} M = s^{(1)}_{e_1}\circ ...\circ s^{(1)}_{e_n},\\
\text{but } \widetilde{\Gamma}^{(1)} &= \langle \widetilde{s}^{(1)}_{e_1},...,\widetilde{s}^{(1)}_{e_n}\rangle = \langle s^{(1)}_{e_1},...,s^{(1)}_{e_n}\rangle = \Gamma^{(1)},\\
\widetilde{\Delta}^{(1)} &= \widetilde{\Gamma}^{(1)}\{\pm e_1,...,\pm e_n\} = \Gamma^{(1)}\{\pm e_1,...,\pm e_n\} = \Delta^{(1)}.
\end{aligned}$$

Because of $\widetilde{\Gamma}^{(1)}=\Gamma^{(1)}$ and $\widetilde{\Delta}^{(1)}=\Delta^{(1)}$ the global sign change from S to $\widetilde{S}$ is interesting.

(ii) In this section we will study the action on $T_3^{uni}(\mathbb{Z})$ of the extension of the action of $\mathrm{Br}_3\ltimes\{\pm 1\}^3$ by this global sign change. Define

$$\delta^{\mathbb{R}}:\mathbb{R}^3\to\mathbb{R}^3,\ (x_1,x_2,x_3)\mapsto(-x_1,-x_2,-x_3)$$

and

$$\widetilde{G}^{sign} := \langle \delta^{\mathbb{R}}_1,\delta^{\mathbb{R}}_2,\delta^{\mathbb{R}}\rangle \cong \{\pm 1\}^3.$$

It is easy to see that the double semidirect product $(G^{phi}\ltimes G^{sign})\rtimes\langle\gamma\rangle$ extends to the double semidirect product $(G^{phi}\ltimes \widetilde{G}^{sign})\rtimes\langle\gamma\rangle$.

LEMMA 4.18. *Each $(G^{phi}\ltimes \widetilde{G}^{sign})\rtimes\langle\gamma\rangle$ orbit in $\mathbb{Z}^3$ contains at least one local minimum of one of the following types:*

(a) $\underline{x}\in\mathbb{Z}^3_{\geq 3}$ with $2x_i\leq x_jx_k$ for $\{i,j,k\}=\{1,2,3\}$.
(b) $(-l,2,-l)$ for some $l\in\mathbb{Z}_{\geq 2}$.
(c) $(x_1,x_2,0)$ for some $x_1,x_2\in\mathbb{Z}_{\geq 0}$ with $x_1\geq x_2$.

Proof: In (c) we can restrict to $x_1\geq x_2$ because $\delta^{\mathbb{R}}_3\gamma\varphi_1(x_1,x_2,0)=(x_2,x_1,0)$.

Each $(G^{phi}\ltimes \widetilde{G}^{sign})\rtimes\langle\gamma\rangle$ orbit consists of one or several $(G^{phi}\ltimes G^{sign})\rtimes\langle\gamma\rangle$ orbits and thus contains local minima.

Suppose that $\underline{x}\in\mathbb{Z}^3$ is such a local minimum and is not obtained with G^{sign} from a local minimum in (a), (b) or (c). Then either $\underline{x}$ is a local minimum associated to $S(\widehat{A}_2)$, so $\underline{x}\in$

$\{(-1,-1,-1),(1,1,-1),(1,-1,1),(-1,1,1)\}$ or $x_1x_2x_3 < 0$ and $r(\underline{x}) > 4$.

In the first case $\delta^{\mathbb{R}}(-1,-1,-1) = (1,1,1) = \varphi_3(1,1,0)$, so the orbit contains a local minimum in (c).

In the second case $r(\delta^{\mathbb{R}}(\underline{x})) = r(-\underline{x}) = r(\underline{x}) + 2x_1x_2x_3 < r(\underline{x})$. We consider a local minimum $\underline{x}^{(1)}$ in the $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbit of $-\underline{x}$. If it is not obtained with G^{sign} from a local minimum in (a), (b) or (c), then $\underline{x}^{(1)}$ is a local minimum associated to $S(\widehat{A}_2)$ or $x_1^{(1)}x_2^{(1)}x_3^{(1)} < 0$ and $r(\underline{x}^{(1)}) > 4$.

We repeat this procedure until we arrive at a local minimum obtained with G^{sign} from one in (a), (b) or (c). This stops after finitely many steps because $r(\underline{x}^{(1)}) = r(-\underline{x}) < r(\underline{x})$. □

Remark 4.19. Corollary 6.23 will say that the $(G^{phi} \ltimes \widetilde{G}^{sign}) \rtimes \langle\gamma\rangle$ orbits of the local minima in the parts (b) and (c) of Lemma 4.18 are pairwise different and also different from the orbits of the local minima in part (a).

Examples 4.20. Given an element $\underline{x} \in \mathbb{Z}^3$, it is not obvious which local minimum of a type in (a), (b) or (c) is contained in the $(G^{phi} \ltimes \widetilde{G}^{sign}) \rtimes \langle\gamma\rangle$ orbit of $\underline{x}$. We give four families of examples. An arrow $\mapsto$ between two elements of $\mathbb{Z}^3$ means that these two elements are in the same orbit.

(i) Start with $(x_1, x_2, -1)$ with $x_1 \geq x_2 > 0$.

$$\begin{aligned} (x_1,x_2,-1) &\overset{\widetilde{G}^{sign}}{\mapsto} (x_1,x_2,1) \overset{\varphi_1}{\mapsto} (x_2-x_1,1,x_2) \overset{\widetilde{G}^{sign}}{\mapsto} (x_1-x_2,1,x_2) \\ &\mapsto \ldots \mapsto (\gcd(x_1,x_2),1,0). \end{aligned}$$

(ii) Start with $(x_1, x_2, -2)$ with $x_1 \geq x_2 \geq 2$.

$$\begin{aligned} (x_1,x_2,-2) &\overset{\widetilde{G}^{sign}}{\mapsto} (x_1,x_2,2) \overset{\varphi_1}{\mapsto} (2x_2-x_1,2,x_2) \mapsto \ldots \\ &\mapsto \begin{cases} (\gcd(x_1,x_2),2,0) & \text{if } x_1 \text{ and } x_2 \\ & \quad\text{contain different powers of 2,} \\ (-\gcd(x_1,x_2),2,-\gcd(x_1,x_2)) & \text{if } x_1 \text{ and } x_2 \\ & \quad\text{contain the same power of 2} \end{cases} \end{aligned}$$

In order to understand the case discussion, observe that $2x_2 - x_1$ and x_2 contain the same power of 2 if and only if x_1 and x_2 contain the same power of 2. Furthermore 0 is divisible by an arbitrarily large power of 2.

(iii) The examples in part (ii) lead in the special case $\gcd(x_1, x_2) = 1$ to the following,

$$\begin{aligned} (x_1, x_2, -2) \mapsto ... \mapsto \ &\mapsto \ (1, 2, 0) \text{ or } (-1, 2, -1) \\ &\mapsto \ (2, 1, 0) \text{ or } (1, 1, 0), \end{aligned}$$

again depending on whether x_1 and x_2 contain different powers of 2 or the same power of 2.

(iv) Start with $(-3, -3, -l)$ for some $l \in \mathbb{Z}_{\geq 2}$.

$$\begin{aligned} (-3, -3, -l) &\overset{\delta^{\mathbb{R}}}{\mapsto} (3, 3, l) \mapsto (3, 3, \pm(l - 9)) \\ &\mapsto (3, 3, \widetilde{l}) \text{ for some } \widetilde{l} \in \{0, 1, 2, 3, 4\} \\ &\mapsto (3, 3, 0),\ (3, 1, 0),\ (-3, 2, -3),\ (3, 3, 3) \text{ or } (4, 3, 3). \end{aligned}$$

$(3, 3, 0)$ and $(3, 1, 0)$ are in part (c), $(-3, 2, -3)$ is in part (b), and $(3, 3, 3)$ and $(4, 3, 3)$ are in part (a) of Lemma 4.18.

CHAPTER 5

Automorphism groups

A unimodular bilinear lattice $(H_{\mathbb{Z}}, L)$ comes equipped with four automorphism groups $G_{\mathbb{Z}}$, $G_{\mathbb{Z}}^{(0)}$, $G_{\mathbb{Z}}^{(1)}$ and $G_{\mathbb{Z}}^{M}$ of $H_{\mathbb{Z}}$, which all respect the monodromy M and possibly some bilinear form. They are the subject of this chapter.

Section 5.1 gives two basic observations which serve as general tools to control these groups under reasonable conditions. Another very useful tool is Theorem 3.26 (c), which gives in favourable situations an n-th root of $(-1)^{k+1}M$ in $G_{\mathbb{Z}}$. Section 5.1 treats also the cases A_1^n.

Section 5.2 takes care of the rank 2 cases. It makes use of some statements on quadratic units in Lemma B.1 (a) in Appendix B.

All further sections 5.3–5.7 are devoted to the rank 3 cases. Section 5.3 discusses the setting and the basic data. It also introduces the special automorphism $Q \in \mathrm{Aut}(H_{\mathbb{Q}}, L)$ which is id on $\ker(M - \mathrm{id})$ and $-\mathrm{id}$ on $\ker(M^2 - (2-r)M + \mathrm{id})$ and determines the (rather few) cases where Q is in $G_{\mathbb{Z}}$.

The treatment of the reducible rank 3 cases in section 5.4 builds on the rank 2 cases and is easy.

The irreducible rank 3 cases with all eigenvalues in S^1 form the four single cases A_3, $\widehat{A}_2$, $\mathbb{P}^2$, $\mathcal{H}_{1,2}$ and the series $S(-l, 2, -l)$ with $l \in \mathbb{Z}_{\geq 3}$. Here in section 5.5 third roots of the monodromy and in the series $S(-l, 2, -l)$ even higher roots of the monodromy turn up.

The sections 5.6 and 5.7 treat all irreducible rank 3 cases with eigenvalues not all in S^1 (1 is always one eigenvalue). Section 5.6 takes care of those families of cases where $G_{\mathbb{Z}} \supsetneqq \{\pm M^m \mid m \in \mathbb{Z}\}$, section 5.7 of all others.

Section 5.6 is rather long. Here again roots of the monodromy turn up, and statements on quadratic units in Lemma B.1 are used.

Section 5.7 is very long. The main result $G_{\mathbb{Z}} = \{\pm M^m \mid m \in \mathbb{Z}\}$ in these cases requires an extensive case discussion.

This chapter determines $G_{\mathbb{Z}}$ in all cases with rank $n \leq 3$. An application is a proof of Theorem 3.28, which says that in almost all cases with rank $n \leq 3$ the map $Z : (\mathrm{Br}_n \ltimes \{\pm 1\}^n)_S \to G_{\mathbb{Z}}$ is surjective, the exception being four cases in section 5.6.

5.1. Basic observations

Given a bilinear lattice $(H_{\mathbb{Z}}, L)$, the most important of the four automorphisms groups $G_{\mathbb{Z}}^M, G_{\mathbb{Z}}^{(0)}, G_{\mathbb{Z}}^{(1)}$ and $G_{\mathbb{Z}}$ in Definition 2.3 (b) (iv) and Lemma 2.6 (a) (iii) is the smallest group $G_{\mathbb{Z}}$. But the key to it is often the largest group $G_{\mathbb{Z}}^M$. We collect some elementary observations on these groups.

LEMMA 5.1. *Let $H_{\mathbb{Z}}$ be a $\mathbb{Z}$-lattice of some rank $n \in \mathbb{N}$ and let $M : H_{\mathbb{Z}} \to H_{\mathbb{Z}}$ be an automorphism of it.*

(a) The characteristic polynomial $p_{ch,M}(t) \in \mathbb{Z}[t]$ of the automorphism M is unitary. Each eigenvalue $\lambda \in \mathbb{C}$ of M is an algebraic integer and a unit in the ring $\mathcal{O}_{\mathbb{Q}[\lambda]} \subset \mathbb{Q}[\lambda]$ of algebraic integers in $\mathbb{Q}[\lambda]$, so in $\mathcal{O}^*_{\mathbb{Q}[\lambda]}$. Also $\lambda^{-1} \in \mathcal{O}^*_{\mathbb{Q}[\lambda]}$.

(b) Suppose that M is regular*, that means, $M : H_{\mathbb{C}} \to H_{\mathbb{C}}$ has for each eigenvalue only one Jordan block.*

(i) Then

$$\mathbb{Q}[M] = \bigoplus_{i=0}^{n-1} \mathbb{Q}M^i \stackrel{!}{=} \mathrm{End}(H_{\mathbb{Q}}, M) := \{g : H_{\mathbb{Q}} \to H_{\mathbb{Q}} \,|\, gM = Mg\}.$$

(ii) Consider a polynomial $p(t) = \sum_{i=0}^{n-1} p_i t^i \in \mathbb{Q}[t]$ and the endomorphism $g = p(M) \in \mathrm{End}(H_{\mathbb{Q}}, M)$ of $H_{\mathbb{Q}}$ and $H_{\mathbb{C}}$. Then g has the eigenvalue $p(\lambda)$ on the generalized eigenspace H_λ of M with eigenvalue λ. If $g \in \mathrm{End}(H_{\mathbb{Z}}, M)$ then $p(\lambda) \in \mathcal{O}_{\mathbb{Q}[\lambda]}$ for each eigenvalue λ of M. If $g \in G_{\mathbb{Z}}^M$ then $p(\lambda) \in \mathcal{O}^*_{\mathbb{Q}[\lambda]}$ for each eigenvalue λ of M.

(c) Suppose that M is the monodromy of a bilinear lattice $(H_{\mathbb{Z}}, L)$. Suppose that M is regular.

(i) Then

$$G_{\mathbb{Z}}^{(0)} \cup G_{\mathbb{Z}}^{(1)} \subset \{p(M) \,|\, p(t) = \sum_{i=0}^{n-1} p_i t^i \in \mathbb{Q}[t],\ p(M) \in \mathrm{End}(H_{\mathbb{Z}}),$$
$$p(\lambda)p(\lambda^{-1}) = 1 \text{ for each eigenvalue } \lambda \text{ of } M\}.$$

(ii) If M is semisimple then $G_{\mathbb{Z}} = G_{\mathbb{Z}}^{(0)} = G_{\mathbb{Z}}^{(1)}$, and this set is equal to the set on the right hand side of (i).

Proof: (a) Trivial.

(b) (i) As M is regular, one can choose a vector $c \in H_{\mathbb{Q}}$ with $H_{\mathbb{Q}} = \bigoplus_{i=0}^{n-1} \mathbb{Q}M^i c$. The inclusion $\mathbb{Q}[M] \subset \mathrm{End}(H_{\mathbb{Q}}, M)$ is clear. Suppose

$g \in \operatorname{End}(H_\mathbb{Q}, M)$. Write $gc = p(M)c$ for some polynomial $p(t) = \sum_{i=0}^{n-1} p_i t^i \in \mathbb{Q}[t]$. As

$$gM^k c = M^k gc = M^k p(M)c = p(M)M^k c$$

for each $k \in \{0, 1, ..., n-1\}$, $g = p(M)$. Thus $\mathbb{Q}[M] = \operatorname{End}(H_\mathbb{Q}, M)$.

(ii) Similar to part (a).

(c) (i) Suppose $g = p(M) \in G_\mathbb{Z}^{(k)}$ for some $k \in \{0; 1\}$ with $p(t) = \sum_{i=0}^{n-1} p_i t^i \in \mathbb{Q}[t]$. Recall $\operatorname{Rad} I^{(k)} = \ker(M - (-1)^{k+1} \operatorname{id} : H_\mathbb{Z} \to H_\mathbb{Z})$.

For $\lambda \neq (-1)^{k+1}$ the generalized eigenspaces H_λ and $H_{\lambda^{-1}}$ are dual to one another with respect to $I^{(k)}$, and g has eigenvalue $p(\lambda)$ on H_λ and eigenvalue $p(\lambda^{-1})$ on $H_{\lambda^{-1}}$. That g respects $I^{(k)}$ implies $p(\lambda)p(\lambda^{-1}) = 1$.

For $\lambda = (-1)^{k+1}$, g restricts to an automorphism of the sublattice $H_\lambda \cap H_\mathbb{Z}$ of $H_\mathbb{Z}$ with determinant $\pm 1 = \det(g|_{H_\lambda \cap H_\mathbb{Z}}) = p(\lambda)^{\dim H_\lambda}$, so $p(\lambda) = \pm 1$.

(ii) Suppose additionally that M is semisimple and that $g = p(M)$ with $p(t) = \sum_{i=0}^{n-1} p_i t^i \in \mathbb{Q}[t]$ satifies $g \in \operatorname{End}(H_\mathbb{Z})$ and $p(\lambda)p(\lambda^{-1}) = 1$ for each eigenvalue λ of M. As M is regular and semisimple, each eigenvalue has multiplicity 1. The (1-dimensional) eigenspaces H_λ and $H_{\lambda^{-1}}$ are dual to one another with respect to L. The conditions $p(\lambda)p(\lambda^{-1}) = 1$ imply that g respects L and also that $\det g = \prod_{\lambda \text{ eigenvalue}} p(\lambda) = \pm 1$. Together with $g \in \operatorname{End}(H_\mathbb{Z})$ this shows $g \in G_\mathbb{Z}$. $\square$

The situation in the following Lemma 5.2 arises surprisingly often. One reason is Theorem 3.26 (c). See the Remarks 5.3 below.

Lemma 5.2. *Let $H_\mathbb{Z}$ be a $\mathbb{Z}$-lattice of some rank $n \in \mathbb{N}$, let $M : H_\mathbb{Z} \to H_\mathbb{Z}$ and $M^{root} : H_\mathbb{Z} \to H_\mathbb{Z}$ be automorphisms of $H_\mathbb{Z}$, and let $l \in \mathbb{N}$ and $\varepsilon \in \{\pm 1\}$ be such that the following holds: M is regular,*

$$(M^{root})^l = \varepsilon M,$$

and M^{root} is cyclic, *that means, a vector $c \in H_\mathbb{Z}$ with $\bigoplus_{i=0}^{n-1} \mathbb{Z}(M^{root})^i c = H_\mathbb{Z}$ exists.*

(a) Then M^{root} is regular, and

$$\begin{aligned} \operatorname{End}(H_\mathbb{Z}, M) &= \operatorname{End}(H_\mathbb{Z}, M^{root}) = \mathbb{Z}[M^{root}], \\ \operatorname{Aut}(H_\mathbb{Z}, M) &= \operatorname{Aut}(H_\mathbb{Z}, M^{root}) = \{p(M^{root}) \,|\, p(t) = \sum_{i=0}^{n-1} p_i t^i \in \mathbb{Z}[t], \\ &\qquad p(\kappa) \in (\mathbb{Z}[\kappa])^* \text{ for each eigenvalue } \kappa \text{ of } M^{root}\}. \end{aligned}$$

(b) Suppose that M is the monodromy of a bilinear lattice $(H_\mathbb{Z}, L)$ and that the set of eigenvalues of M^{root} is invariant under inversion, that means, with κ an eigenvalue of M^{root} also κ^{-1} is an eigenvalue of M^{root}.

(i) Then $M^{root} \in G_{\mathbb{Z}}$ *and*

$$
\begin{aligned}
G^{(0)}_{\mathbb{Z}} \cup G^{(1)}_{\mathbb{Z}} \quad \subset \quad & \{p(M^{root}) \,|\, p(t) = \sum_{i=0}^{n-1} p_i t^i \in \mathbb{Z}[t], \\
& p(\kappa)p(\kappa^{-1}) = 1 \text{ for each eigenvalue } \kappa \text{ of } M^{root}\}.
\end{aligned}
$$

(ii) If M is semisimple then $G_{\mathbb{Z}} = G^{(0)}_{\mathbb{Z}} = G^{(1)}_{\mathbb{Z}}$*, and this set is equal to the set on the right hand side if (i).*

Proof: (a) M^{root} is regular as M is regular and $(M^{root})^l = \varepsilon M$. This equation also implies $\mathbb{Q}[M] \subset \mathbb{Q}[M^{root}]$. As these $\mathbb{Q}$-vector spaces have both dimension n,

$$\text{End}(H_{\mathbb{Q}}, M) = \mathbb{Q}[M] = \mathbb{Q}[M^{root}] = \text{End}(H_{\mathbb{Q}}, M^{root}).$$

Then $\text{End}(H_{\mathbb{Z}}, M) = \text{End}(H_{\mathbb{Z}}) \cap \mathbb{Q}[M^{root}]$. Consider $g \in \text{End}(H_{\mathbb{Z}}, M)$. Choose a cyclic generator $c \in H_{\mathbb{Z}}$ with $\bigoplus_{i=0}^{n-1} \mathbb{Z}(M^{root})^i c = H_{\mathbb{Z}}$. Write $g(c) = p(M^{root})c$ for some polynomial $p(t) = \sum_{i=0}^{n-1} p_i t^i \in \mathbb{Z}[t]$. As in the proof of Lemma 5.1, one finds $g = p(M^{root})$, so $\text{End}(H_{\mathbb{Z}}, M) = \mathbb{Z}[M^{root}]$. The element $g = p(M^{root})$ above is in $\text{Aut}(H_{\mathbb{Z}}, M)$ if and only if $\det g = \pm 1$, and this holds if and only if the algebraic integer $p(\kappa)$ is a unit in $\mathbb{Z}[\kappa]$ for each eigenvalue κ of M^{root}.

(b) (i) The main point is to show $M^{root} \in G_{\mathbb{Z}}$. As M and M^{root} are both regular, the map $\kappa \mapsto \varepsilon\kappa^l$ is a bijection from the set of eigenvalues of M^{root} to the set of eigenvalues of M. For $\lambda = \varepsilon\kappa^l$, the generalized eigenspaces H_λ and $H_{\lambda^{-1}}$ of M are the generalized eigenspaces of M^{root} with eigenvalues κ and κ^{-1}. These two spaces are dual to one another with respect to $I^{(0)}$ (if $\lambda \neq -1$), $I^{(1)}$ (if $\lambda \neq 1$) and L.

Consider the decomposition of M^{root} into the commuting semisimple part M^{root}_s and unipotent part M^{root}_u with nilpotent part N^{root} with $\exp(N^{root}) = M^{root}_u$, and also the decomposition $M = M_s M_u$ with nilpotent part N with $\exp N = M_u$.

M^{root}_s and M_s respect L because they have eigenvalue κ and λ on H_λ and eigenvalue κ^{-1} and λ^{-1} on $H_{\lambda^{-1}}$.

As M and M_s respect L, also M_u respects L. Therefore N is an infinitesimal isometry. Because $N = lN^{root}$, also N^{root} is an infinitesimal isometry. Therefore M^{root}_u respects L. Thus also $M^{root} = M^{root}_s M^{root}_u$ respects L, so $M^{root} \in G_{\mathbb{Z}}$.

Part (ii) and the rest of part (i) are proved as part (b) in Lemma 5.1. □

Remarks 5.3. (i) The pair $(H_{\mathbb{Z}}, M^{root})$ in Lemma 5.2 is an *Orlik block* if M^{root} is of finite order. They are important building blocks in the unimodular bilinear lattices $(H_{\mathbb{Z}}, L)$ for many isolated hypersurface singularities.

(ii) If the matrix $S = L(\underline{e}^t, \underline{e})^t$ of a unimodular bilinear lattice $(H_{\mathbb{Z}}, L)$ with a triangular basis $\underline{e}$ has the special shape in Theorem 3.26 (c), then the monodromy $(-1)^{k+1}M$ has by Theorem 3.26 (c) a specific n-th root $M^{root} \in G_{\mathbb{Z}}$. This situation is special, but it arises surprisingly often, in singularity theory and in the cases in part (ii).

(iii) Theorem 3.26 (c) applies to all matrices $S(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ with $x \in \mathbb{Z}$ and to all matrices $S = \begin{pmatrix} 1 & x & \varepsilon x \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$ with $x \in \mathbb{Z}$ and $\varepsilon \in \{\pm 1\}$.

It applies especially to the matrices $S(A_1^3)$, $S(\widehat{A}_2)$, $S(\mathcal{H}_{1,2})$ and $S(\mathbb{P}^2)$ in the Examples 1.1 and to the matrix $S(-1, 1, -1)$ in the $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbit of $S(A_3)$. Though it is not useful in the cases $S(A_1^3)$ and $S(\mathcal{H}_{1,2})$ because their monodromies are not regular. In the case $S(A_3)$ we will not use it as there the monodromy itself is cyclic.

The completely reducible cases A_1^n for $n \in \mathbb{N}$ can be treated easily, building on Lemma 2.12.

Lemma 5.4. *Fix $n \in \mathbb{N}$ and consider the case A_1^n with $S = S(A_1^n) = E_n$. Then*

$$\begin{aligned} G_{\mathbb{Z}} = G_{\mathbb{Z}}^{(0)} &\cong O_n(\mathbb{Z}) = \{A \in GL_n(\{0; \pm 1\}) \mid \exists\, \sigma \in S_n \\ &\qquad \exists\, \varepsilon_1, ..., \varepsilon_n \in \{\pm 1\} \text{ such that } A_{ij} = \varepsilon_i \delta_{i\sigma(j)}\}, \\ G_{\mathbb{Z}}^{(1)} = G_{\mathbb{Z}}^{M} &= \mathrm{Aut}(H_{\mathbb{Z}}) \cong GL_n(\mathbb{Z}). \end{aligned}$$

The map $Z : (\mathrm{Br}_n \ltimes \{\pm 1\}^n)_S = \mathrm{Br}_n \ltimes \{\pm 1\}^n \to G_{\mathbb{Z}}$ is surjective.

Proof: The groups $G_{\mathbb{Z}}^{(1)}$ and $G_{\mathbb{Z}}^{M}$ are as claimed because $M = \mathrm{id}$ and $I^{(1)} = 0$. The groups $G_{\mathbb{Z}}$ and $G_{\mathbb{Z}}^{(0)}$ map the set $R^{(0)} = \{\pm e_1, ..., \pm e_n\}$ to itself and are therefore also as claimed.

The stabilizer of $S = E_n$ is the whole group $\mathrm{Br}_n \ltimes \{\pm 1\}^n$. The subgroup $\{\pm 1\}^n$ gives all sign changes of the basis $\underline{e} = (e_1, ..., e_n)$. The subgroup Br_n gives under Z all permutations of the elements of the tuple $(e_1, ..., e_n)$. Therefore Z is surjective. □

5.2. The rank 2 cases

For $x \in \mathbb{Z}$ consider the matrix $S = S(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in T_2^{uni}(\mathbb{Z})$, and consider a unimodular bilinear lattice $(H_{\mathbb{Z}}, L)$ with a triangular basis

$\underline{e} = (e_1, e_2)$ with $L(\underline{e}^t, \underline{e})^t = S$. Then

$$\begin{aligned} M\underline{e} &= \underline{e}S^{-1}S^t = \underline{e}\begin{pmatrix} 1-x^2 & -x \\ x & 1 \end{pmatrix}, \\ p_{ch,M}(t) &= t^2 - (2-x^2)t + 1 \\ \text{with zeros } \lambda_{1/2} &= \frac{2-x^2}{2} \pm \frac{1}{2}x\sqrt{x^2-4}. \end{aligned}$$

Theorem 3.26 (c) applies with $(n, k, q_0, q_1) = (2, 0, 1, x)$, namely

$$\begin{aligned} \delta_2\sigma^{root} &= \delta_2\sigma_1 \in (\mathrm{Br}_2 \ltimes \{\pm 1\}^2)_S, \\ M^{root} &:= Z(\delta_2\sigma_1) \in G_{\mathbb{Z}} \\ \text{with } M^{root}\underline{e} &= \underline{e}\begin{pmatrix} -x & -1 \\ 1 & 0 \end{pmatrix} \\ \text{and } (M^{root})^2 &= -M. \end{aligned}$$

M^{root} is regular and cyclic,

$$\begin{aligned} p_{ch,M^{root}}(t) &= t^2 + xt + 1 \\ \text{with zeros } \kappa_{1/2} &= -\frac{x}{2} \pm \frac{1}{2}\sqrt{x^2-4} \\ \text{with } \kappa_i^2 &= -\lambda_i = -x\kappa_i - 1,\ \kappa_1 + \kappa_2 = -x,\ \kappa_1\kappa_2 = 1. \end{aligned}$$

M is regular if $x \neq 0$. M and M^{root} are semisimple if $x \neq \pm 2$. If $x = \pm 1$, M has eigenvalues $e^{\pm 2\pi i/6}$ and M^{root} has eigenvalues $e^{\pm 2\pi i/3}$ respectively $e^{\pm 2\pi i/6}$. If $|x| \geq 3$, M and M^{root} have real eigenvalues and infinite order. If $x = \pm 2$, they have a 2×2 Jordan block with eigenvalue -1 respectively $-\frac{x}{2}$.

THEOREM 5.5. *(a) If $x \neq 0$ then*

$$\begin{aligned} G_{\mathbb{Z}} &= G_{\mathbb{Z}}^{(0)} = G_{\mathbb{Z}}^{(1)} = \{\pm (M^{root})^l \mid l \in \mathbb{Z}\}, \\ G_{\mathbb{Z}}^M &= \begin{cases} G_{\mathbb{Z}} & \text{if } x \neq \pm 3, \\ \{\pm (M^{root} + \frac{x}{|x|}\,\mathrm{id})^l \mid l \in \mathbb{Z}\} & \text{if } x = \pm 3. \end{cases} \end{aligned}$$

If $x = 0$ then

$$G_{\mathbb{Z}} = G_{\mathbb{Z}}^{(0)} \cong O_2(\mathbb{Z}) = \{\begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix}, \begin{pmatrix} 0 & \varepsilon_1 \\ \varepsilon_2 & 0 \end{pmatrix} \mid \varepsilon_1, \varepsilon_2 \in \{\pm 1\}\},$$

$$G_{\mathbb{Z}}^{(1)} = G_{\mathbb{Z}}^M = \mathrm{Aut}(H_{\mathbb{Z}}) \cong GL_2(\mathbb{Z}).$$

In all cases $G_{\mathbb{Z}} = G_{\mathbb{Z}}^{\mathcal{B}}$, so $Z : (\mathrm{Br}_2 \ltimes \{\pm 1\}^2)_S \to G_{\mathbb{Z}}$ is surjective.

(b) Properties of $I^{(0)}$ and $I^{(1)}$:

$$\begin{array}{ll} x=0: & I^{(1)}=0,\ \operatorname{Rad} I^{(1)}=H_{\mathbb{Z}},\ L(\underline{e}^t,\underline{e})^t=E_2,\ I^{(0)}(\underline{e}^t,\underline{e})=2E_2.\\ x\neq 0: & \operatorname{Rad} I^{(1)}=\{0\}.\\ |x|\leq 1: & I^{(0)} \text{ is positive definite.}\\ |x|=2: & I^{(0)} \text{ is positive semi-definite, } \operatorname{Rad} I^{(0)}=\mathbb{Z}(e_1-\frac{x}{|x|}e_2).\\ |x|>2: & I^{(0)} \text{ is indefinite, } \operatorname{Rad} I^{(0)}=\{0\}.\end{array}$$

Proof: Part (b) is obvious.

The case $x=0$ is the case A_1^2. It is covered by Lemma 5.4.

Consider the cases $x\neq 0$ in part (a). We can restrict to $x<0$ because of $L((e_1,-e_2)^t,(e_1,-e_2))^t=\begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}$. So suppose $x<0$.

We know $\{\pm(M^{root})^l\,|\,l\in\mathbb{Z}\}\subset G_{\mathbb{Z}}\subset G_{\mathbb{Z}}^M$. By Lemma 5.2 (a)

$$G_{\mathbb{Z}}^M=\{p(M^{root})\,|\,p(t)=p_1t+p_0\in\mathbb{Z}[t],\ p(\kappa_1)p(\kappa_2)\in\{\pm 1\}\}.$$

The map

$$Q_2:\mathbb{Z}^2\to\mathbb{Z},\quad (p_1,p_0)\mapsto p(\kappa_1)p(\kappa_2)=p_1^2-p_1p_0x+p_0^2,$$

is a quadratic form. Lemma 5.2 (a) shows

$$G_{\mathbb{Z}}^M=\{p_1M^{root}+p_0\operatorname{id}\,|\,(p_1,p_0,\varepsilon_1)\in\mathbb{Z}^2\times\{\pm 1\},Q_2(p_1,p_0)=\varepsilon\}.$$

Lemma 5.2 (b) (ii) shows for $x\neq -2$

$$G_{\mathbb{Z}}=G_{\mathbb{Z}}^{(0)}=G_{\mathbb{Z}}^{(1)}=\{p_1M^{root}+p_0\operatorname{id}\,|\,(p_1,p_0)\in\mathbb{Z}^2,Q_2(p_1,p_0)=1\}.$$

For $x=-2$ it shows only

$$G_{\mathbb{Z}},G_{\mathbb{Z}}^{(0)},G_{\mathbb{Z}}^{(1)}\subset\{p_1M^{root}+p_0\operatorname{id}\,|\,(p_1,p_0)\in\mathbb{Z}^2,Q_2(p_1,p_0)=1\}.$$

We discuss the cases $x=-1$, $x=-2$ and $x\leq -3$ separately.

The case $x=-1$: Q_2 is positive definite, so $Q_2(p_1,p_0)=-1$ is impossible, and

$$\{(p_1,p_0)\,|\,Q_2(p_1,p_0)=1\}=\{\pm(0,1),\pm(1,0),\pm(1,-1)\},$$
$$G_{\mathbb{Z}}=G_{\mathbb{Z}}^{(0)}=G_{\mathbb{Z}}^{(1)}=G_{\mathbb{Z}}^M=\{\pm\operatorname{id},\pm M^{root},\pm(M^{root})^2\}.$$

Because of $M^2=-M^{root}$ this equals $\{\pm\operatorname{id},\pm M,\pm M^2\}$.

The case $x=-2$: This follows from Lemma 5.6 below.

Remark: Here Q_2 is positive semidefinite with $Q_2(p_1,p_0)=(p_1+p_0)^2$. The solution $(p_1,p_0)=(p_1,-p_1+\varepsilon_2)$ with $\varepsilon_2\in\{\pm 1\}$ of

$Q_2(p_1,p_0)=1$ corresponds to

$$\begin{aligned} p_1 M^{root}+(-p_1+\varepsilon_2)\,\mathrm{id} &= \varepsilon_2(\mathrm{id}+\varepsilon_2 p_1(M^{root}-\mathrm{id})) \\ &= \varepsilon_2(\mathrm{id}+(M^{root}-\mathrm{id}))^{\varepsilon_2 p_1} \\ &= \varepsilon_2(M^{root})^{\varepsilon_2 p_1}. \end{aligned}$$

The cases $x\leq -3$: The arguments above show

$$\begin{aligned} & G_{\mathbb{Z}}^M \cong (\mathbb{Z}[\kappa_1])^* \\ \supset\; & G_{\mathbb{Z}} \cong \{p_1\kappa_1+p_0\in\mathbb{Z}[\kappa_1]\,|\,(p_1\kappa_1+p_0)(p_1\kappa_2+p_0)=1\}. \end{aligned}$$

So we need to understand the unit group of $\mathbb{Z}[\kappa_1]$ and the subgroup of elements with norm 1. Both are treated in Lemma B.1 (a) in Appendix B.

It remains to show $G_{\mathbb{Z}}=G_{\mathbb{Z}}^{\mathcal{B}}$ in all cases $x\in\mathbb{Z}_{\leq -1}$. This follows from $G_{\mathbb{Z}}=\{\pm(M^{root})^l\,|\,l\in\mathbb{Z}\}$ and

$$Z(\delta_1\delta_2)=-\mathrm{id},\quad Z(\delta_2\sigma_1)=M^{root}. \qquad \square$$

Lemma 5.6. *Let $H_{\mathbb{Z}}$ be a $\mathbb{Z}$-lattice of rank 2, and let $\widetilde{M}:H_{\mathbb{Z}}\to H_{\mathbb{Z}}$ be an automorphism with a 2×2 Jordan block and eigenvalue $\lambda\in\{\pm 1\}$.*

(a) Then a cyclic automorphism $\widetilde{M}^{root}:H_{\mathbb{Z}}\to H_{\mathbb{Z}}$ with eigenvalue 1 and a number $l\in\mathbb{N}$ with $(\widetilde{M}^{root})^l=\lambda M$ exist. They are unique.

$$\mathrm{Aut}(H_{\mathbb{Z}},\widetilde{M})=\{\pm(\widetilde{M}^{root})^l\,|\,l\in\mathbb{Z}\}.$$

(b) If $\widetilde{I}:H_{\mathbb{Z}}\times H_{\mathbb{Z}}\to\mathbb{Z}$ is an $\widetilde{M}$-invariant bilinear form then it is also $\widetilde{M}^{root}$-invariant and

$$\mathrm{Aut}(H_{\mathbb{Z}},\widetilde{M},\widetilde{I})=\mathrm{Aut}(H_{\mathbb{Z}},\widetilde{M})=\{\pm(\widetilde{M}^{root})^l\,|\,l\in\mathbb{Z}\}.$$

Proof: (a) There is a $\mathbb{Z}$-basis $\underline{f}=(f_1,f_2)$ of $H_{\mathbb{Z}}$ and an $l\in\mathbb{N}$ with

$$\widetilde{M}\underline{f}=\underline{f}\lambda\begin{pmatrix}1 & l\\ 0 & 1\end{pmatrix}.$$

Here f_1 is a generator of the rank 1 $\mathbb{Z}$-lattice $\ker(\widetilde{M}-\lambda\,\mathrm{id})\subset H_{\mathbb{Z}}$. It is unique up to the sign. It is a primitive element of $H_{\mathbb{Z}}$. An element f_2 with $H_{\mathbb{Z}}=\mathbb{Z}f_1\oplus\mathbb{Z}f_2$ exists. It is unique up to sign and up to adding a multiple of f_1. The sign is fixed by l in the matrix above being positive. l is unique.

Define $\widetilde{M}^{root}:H_{\mathbb{Z}}\to H_{\mathbb{Z}}$ by

$$\widetilde{M}^{root}\underline{f}=\underline{f}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}.$$

Obviously $(\widetilde{M}^{root})^l=\lambda\widetilde{M}$.

Any $g\in \mathrm{Aut}(H_{\mathbb Z},\widetilde{M})$ must fix $\mathbb{Z}f_1=\ker(\widetilde{M}-\lambda\,\mathrm{id})$. Therefore it must be up to the sign a power of $\widetilde{M}^{root}$.

(b) That $\widetilde{M}^{root}$ respects $\widetilde{I}$ follows by the same arguments as $M^{root}\in G_{\mathbb Z}$ in the proof of Lemma 5.2 (b) (i) (but now the situation is simpler, as $\widetilde{M}$ and $\widetilde{M}^{root}$ have a single 2×2 Jordan block). The rest follows with part (a). $\square$

5.3. Generalities on the rank 3 cases

For $\underline{x}=(x_1,x_2,x_3)\in\mathbb{Z}^3$ consider the matrix $S=S(\underline{x})=\begin{pmatrix}1&x_1&x_2\\0&1&x_3\\0&0&1\end{pmatrix}\in T_3^{uni}(\mathbb{Z})$, and consider a unimodular bilinear lattice $(H_{\mathbb Z},L)$ with a triangular basis $\underline{e}=(e_1,e_2,e_3)$ with $L(\underline{e}^t,\underline{e})^t=S$. Then

$$\begin{aligned}
S^{-1} &= \begin{pmatrix}1&-x_1&x_1x_3-x_2\\0&1&-x_3\\0&0&1\end{pmatrix},\\
S^{-1}S^t &= \begin{pmatrix}1-x_1^2-x_2^2+x_1x_2x_3 & -x_1-x_2x_3+x_1x_3^2 & x_1x_3-x_2\\ x_1-x_2x_3 & 1-x_3^2 & -x_3\\ x_2 & x_3 & 1\end{pmatrix},\\
M\underline{e} &= \underline{e}S^{-1}S^t,
\end{aligned}$$

$$\begin{aligned}
I^{(0)}(\underline{e}^t,\underline{e}) &= S+S^t=\begin{pmatrix}2&x_1&x_2\\x_1&2&x_3\\x_2&x_3&2\end{pmatrix},\\
I^{(1)}(\underline{e}^t,\underline{e}) &= S-S^t=\begin{pmatrix}0&x_1&x_2\\-x_1&0&x_3\\-x_2&-x_3&0\end{pmatrix},\\
p_{ch,M}(t) &= (t-1)(t^2-(2-r(\underline{x}))t+1),
\end{aligned}$$

where

$$r:\mathbb{Z}^3\to\mathbb{Z},\quad \underline{x}=(x_1,x_2,x_3)\mapsto x_1^2+x_2^2+x_3^2-x_1x_2x_3.$$

For $(x_1,x_2,x_3)\neq(0,0,0)$ define

$$f_3 := \underline{e}\,\frac{1}{\gcd(x_1,x_2,x_3)}\begin{pmatrix}-x_3\\x_2\\-x_1\end{pmatrix}.$$

This is a primitive vector in $H_{\mathbb Z}$.

$$\mathrm{Rad}\,I^{(1)} \overset{2.5\,(a)(ii)}{=} \ker(M-\mathrm{id})=\begin{cases}\mathbb{Z}f_3 & \text{if } (x_1,x_2,x_3)\neq(0,0,0),\\ H_{\mathbb Z} & \text{if } (x_1,x_2,x_3)=(0,0,0).\end{cases}$$

Also

$$\begin{aligned} p_{ch,S+S^t}(t) &= t^3-6t^2+(12-x_1^2-x_2^2-x_3^2)t-2(4-r),\\ L(f_3,f_3) &= \frac{r(\underline{x})}{\gcd(x_1,x_2,x_3)^2}, \quad I^{(0)}(f_3,f_3)=2L(f_3,f_3). \end{aligned}$$

The eigenvalues of M are called

$$\lambda_{1/2}=\frac{2-r}{2}\pm\frac{1}{2}\sqrt{r(r-4)}, \quad \lambda_3=1$$

with $\lambda_1+\lambda_2=2-r$, $\lambda_1\lambda_2=1$.

The following Lemma gives implicitly precise information on $p_{ch,M}$ and $\operatorname{sign} I^{(0)}$ for all $\underline{x}\in\mathbb{Z}^3$. *Implicitly*, because one has to determine with the tools from section 4.2 in the cases $r(\underline{x})\in\{0,1,2,4\}$ in which $\mathrm{Br}_3\ltimes\{\pm 1\}^3$ orbit in Theorem 4.6 (e) the matrix $S(\underline{x})$ is.

LEMMA 5.7. *(a)* $r^{-1}(3l)=\emptyset$ *for* $l\in\mathbb{Z}-3\mathbb{Z}$.

(b) Consider $\underline{x}\in\mathbb{Z}^3$ *with* $r=r(\underline{x})<0$ *or* >4 *or with* $S(\underline{x})$ *one of the cases in Theorem 4.6 (e). Then* $p_{ch,M}$ *and* $\operatorname{sign} I^{(0)}$ *are as follows.*

	$p_{ch,M}$	$\operatorname{sign} I^{(0)}$	$S(\underline{x})$
$r<0$	$\lambda_1,\lambda_2>0$	$(+--)$	$S(\underline{x})$
$r=0$	Φ_1^3	$(+++)$	$S(A_1^3)$
$r=0$	Φ_1^3	$(+--)$	$S(\mathbb{P}^2)$
$r=1$	$\Phi_6\Phi_1$	$(+++)$	$S(A_2A_1)$
$r=2$	$\Phi_4\Phi_1$	$(+++)$	$S(A_3)$
$r=4$	$\Phi_2^2\Phi_1$	$(++\ 0)$	$S(\mathbb{P}^1A_1)$
$r=4$	$\Phi_2^2\Phi_1$	$(++\ 0)$	$S(\widehat{A}_2)$
$r=4$	$\Phi_2^2\Phi_1$	$(+\ 0\ 0)$	$S(\mathcal{H}_{1,2})$
$r=4$	$\Phi_2^2\Phi_1$	$(+\ 0\ -)$	$S(-l,2,-l)$ with $l\geq 3$
$r>4$	$\lambda_1,\lambda_2<0$	$(++-)$	$S(\underline{x})$

Proof: (a) If $(3|x_1,3|x_2,3|x_3)$ then $9|r$.
If $(3|x_1,3|x_2,3\nmid x_3)$ then $3|(r-1),3\nmid r$.
If $(3|x_1,3\nmid x_2,3\nmid x_3)$ then $3|(r-2),3\nmid r$.
If $(3\nmid x_1,3\nmid x_2,3\nmid x_3)$ then $3|(x_1^2+x_2^2+x_3^2),3\nmid r$.
(b) The statements on $p_{ch,M}$ are obvious.

$$\operatorname{Rad} I^{(0)} \overset{2.5\ \text{(a)(ii)}}{=} \ker(M+\mathrm{id})\supsetneqq\{0\}\iff\Phi_2|p_{ch,M}\iff r=4.$$

In the cases with $r=4$, one calculates $p_{ch,S+S^t}(t)$ and reads off $\operatorname{sign} I^{(0)}$ from the zeros of $p_{ch,S+S^t}(t)$. The case $S(A_1^3)=S(0,0,0)$ is trivial.

Consider the cases with $r\neq 4$ and $\underline{x}\neq(0,0,0)$. Then $I^{(0)}$ is nondegenerate. The product of the signs in the signature of $I^{(0)}$ is the sign of $\det(S+S^t)=2(4-r)$. Because of the 2's on the diagonal of $S+S^t$, $I^{(0)}$ cannot be negative definite. Also recall $I^{(0)}(f_3,f_3) = 2r(\gcd(x_1,x_2,x_3))^{-2}$. This shows $\operatorname{sign} I^{(0)} = (+--)$ for $r<0$, $\operatorname{sign} I^{(0)} = (++-)$ for $r>4$ and $\operatorname{sign} I^{(0)} = (+++)$ or $(+--)$ for $r\in\{0,1,2\}$.

The classification of $\mathrm{Br}_3 \ltimes \{\pm 1\}$ orbits in $T_3^{uni}(\mathbb{Z})$ in Theorem 4.6 (e) says that for each of the cases $r\in\{0,1,2\}$ there is only one orbit (with $\underline{x}\neq(0,0,0)$ in the case $r=0$), namely $S(\mathbb{P}^2)$, $S(A_2A_1)$ and $S(A_3)$. One checks the claims on $\operatorname{sign} I^{(0)} = \operatorname{sign}(S+S^t)$ immediately. □

REMARKS 5.8. (i) It is very remarkable that the fibers $r^{-1}(1)$ and $r^{-1}(2)\subset\mathbb{Z}^3$ of $r:\mathbb{Z}^3\to\mathbb{Z}$ consist each of only one orbit. If one looks at the fibers of the real map

$$r_{\mathbb{R}}:\mathbb{R}^3\to\mathbb{R},\quad \underline{x}\mapsto x_1^2+x_2^2+x_3^2-x_1x_2x_3,$$

this does not hold. Each real fiber $r_{\mathbb{R}}^{-1}(\rho)$ with $\rho\in(0,4)$ has five components, one compact (homeomorphic to a 2-sphere), four non-compact (homeomorphic to $\mathbb{R}^2$). The four non-compact components are related by the action of G^{sign}. It is remarkable that the fibers $r_{\mathbb{R}}^{-1}(1)$ and $r_{\mathbb{R}}^{-1}(2)\subset\mathbb{R}^3$ intersect $\mathbb{Z}^3$ only in the central piece.

(ii) By $p_{ch,M}=(t-1)(t^2-(2-r(\underline{x}))t+1)$, the monodromy matrix $S^{-1}S^t$ for $\underline{x}\in\mathbb{R}^3$ and $S=S(\underline{x})$ has all eigenvalues in S^1 if and only if $r_{\mathbb{R}}(\underline{x})\in[0,4]$.

(iii) The semialgebraic subvariety $r_{\mathbb{R}}^{-1}([0,4])\subset\mathbb{R}^3$ was studied in [**BH20**, 5.2]. It has a central piece which is G^{sign} invariant and which looks like a tetrahedron with smoothened edges and four other pieces which are permuted by G^{sign}. Each other piece is homeomorphic to $[0,1]\times\mathbb{R}^2$ and is glued in one point (its only singular point) to one of the vertices of the central piece. The four vertices are $(2,2,2),(2,-2,-2),(-2,2,-2),(-2,-2,2)$, so the elements of the $\mathrm{Br}_3\ltimes\{\pm1\}^3$ orbit of $(2,2,2)$.

For $\underline{x}$ inside the central piece $S+S^t$ is positive definite, on its boundary except the vertices $\operatorname{sign}(S+S^t)=(++0)$, at a vertex $\operatorname{sign}(S+S^t)=(+00)$. On those boundary components of the other four pieces which contain one of the vertices $\operatorname{sign}(S+S^t)=(+0-)$ (except at the vertex). On the interior of $r_{\mathbb{R}}^{-1}(-\infty,4]$ except the central piece $\operatorname{sign}(S+S^t)=(+--)$. On the exterior of $r_{\mathbb{R}}^{-1}(-\infty,4]$ $\operatorname{sign}(S+S^t)=(++-)$.

(iv) Due to Lemma 5.7 (b) and Theorem 4.6, the seven cases $S(A_1^3)$, $S(\mathbb{P}^2)$, $S(A_2A_1)$, $S(A_3)$, $S(\mathbb{P}^1A_1)$, $S(\widehat{A}_2)$, $S(\mathcal{H}_{1,2})$ and the series $S(-l,2,-l)$ for $l\geq 3$ give the only rank 3 unimodular bilinear lattices where all eigenvalues of the monodromy are unit roots. In the sections

5.4 and 5.5 we will focus on the reducible cases and these cases. In the sections 5.6 and 5.7 we will treat the other cases.

The following definition presents a special automorphism Q in $\text{Aut}(H_{\mathbb{Q}}, L)$. Theorem 5.11 will say in which cases Q is in $G_{\mathbb{Z}}$ and in which cases not. The determination of the group $G_{\mathbb{Z}}$ in all irreducible rank 3 cases in the sections 5.5–5.7 will build on this result. It is preceded by Lemma 5.10 which provides notations and estimates which will be used in the proof of Theorem 5.11 and also later.

Definition 5.9. Consider $\underline{x} \in \mathbb{Z}^3$ with $r(\underline{x}) \neq 0$. Then $H_{\mathbb{Q}} = H_{\mathbb{Q},1} \oplus H_{\mathbb{Q},2}$ with $H_{\mathbb{Q},1} := \ker(M^2 - (2 - r(\underline{x}))M + \text{id} : H_{\mathbb{Q}} \to H_{\mathbb{Q}})$ and $H_{\mathbb{Q},2} := \ker(M - \text{id} : H_{\mathbb{Q}} \to H_{\mathbb{Q}})$. This decomposition is left and right L-orthogonal. Then $Q : H_{\mathbb{Q}} \to H_{\mathbb{Q}}$ denotes the automorphism with $Q|_{H_{\mathbb{Q},1}} = -\text{id}$ and $Q|_{H_{\mathbb{Q},2}} = \text{id}$. It is in $\text{Aut}(H_{\mathbb{Q}}, L)$.

Lemma 5.10. *(a) For* $\underline{x} \in \mathbb{Z}^3 - \{(0,0,0)\}$ *write* $r := r(\underline{x})$ *and define*

$$\begin{aligned} g := g(\underline{x}) &:= \gcd(x_1, x_2, x_3) \in \mathbb{N}, \\ \widetilde{\underline{x}} := (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) &:= g^{-1}\underline{x} \in \mathbb{Z}^3. \end{aligned}$$

Then $f_3 = -\widetilde{x}_3 e_1 + \widetilde{x}_2 e_2 - \widetilde{x}_1 e_3$, $\gcd(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) = 1$ *and*

$$g^2 \,|\, r, \quad \frac{r}{g^2} = \widetilde{x}_1^2 + \widetilde{x}_2^2 + \widetilde{x}_3^2 - g\widetilde{x}_1\widetilde{x}_2\widetilde{x}_3. \tag{5.1}$$

(b) Consider a local minimum (Definition 4.3) $\underline{x} \in \mathbb{Z}^3_{\geq 3}$ *with* $x_i \leq x_j \leq x_k$ *for some* i, j, k *with* $\{i, j, k\} = \{1, 2, 3\}$. *Then*

$$\widetilde{x}_i \leq \frac{2 + (4 - r)^{1/3}}{g}, \tag{5.2}$$

$$x_j^2 \leq \frac{4 - r}{x_i - 2} + x_i + 2, \tag{5.3}$$

$$x_k \leq \frac{1}{2} x_i x_j \quad \text{and} \quad \widetilde{x}_k \leq \frac{g}{2} \widetilde{x}_i \widetilde{x}_j. \tag{5.4}$$

Proof: (a) Trivial.

(b) Lemma 4.4 shows $x_k \leq \frac{1}{2} x_i x_j$, which is (5.4). This is equivalent to $\widetilde{x}_k \leq \frac{g}{2} \widetilde{x}_i \widetilde{x}_j$. We also have $\widetilde{x}_i \leq \widetilde{x}_j \leq \widetilde{x}_k$, and we know $r \leq 0$ from Theorem 4.6.

The proof of (5.2) is similar to the second case in the proof of Theorem 4.6 (a).

$$\begin{aligned}
\frac{r}{g^2} &= \widetilde{x}_i^2+\widetilde{x}_j^2+(\widetilde{x}_k-\frac{g}{2}\widetilde{x}_i\widetilde{x}_j)^2-\frac{g^2}{4}\widetilde{x}_i^2\widetilde{x}_j^2 \\
&\leq \widetilde{x}_i^2+\widetilde{x}_j^2+(\widetilde{x}_j-\frac{g}{2}\widetilde{x}_i\widetilde{x}_j)^2-\frac{g^2}{4}\widetilde{x}_i^2\widetilde{x}_j^2 \quad (\text{because } \widetilde{x}_j\leq\widetilde{x}_k\leq\frac{g}{2}\widetilde{x}_i\widetilde{x}_j) \\
&= \widetilde{x}_i^2+2\widetilde{x}_j^2-g\widetilde{x}_i\widetilde{x}_j^2 \\
&= (\widetilde{x}_i-g\widetilde{x}_j^2+\frac{2}{g})(\widetilde{x}_i-\frac{2}{g})+\frac{4}{g^2}.
\end{aligned}$$

If $\widetilde{x}_i<\frac{2}{g}$, then (5.2) holds anyway. If $\widetilde{x}_i\geq\frac{2}{g}$ then we can further estimate the last formula using $-g\widetilde{x}_j^2\leq -g\widetilde{x}_i^2$. Then we obtain

$$\begin{aligned}
\frac{r}{g^2} &\leq (\widetilde{x}_i-g\widetilde{x}_i^2+\frac{2}{g})(\widetilde{x}_i-\frac{2}{g})+\frac{4}{g^2} \\
&= -g(\widetilde{x}_i-\frac{2}{g})^2(\widetilde{x}_i+\frac{1}{g})+\frac{4}{g^2} \\
&\leq -g(\widetilde{x}_i-\frac{2}{g})^3+\frac{4}{g^2}, \\
\text{so } (\widetilde{x}_i-\frac{2}{g})^3 &\leq \frac{4-r}{g^3}.
\end{aligned}$$

This shows (5.2). The inequality (5.3) was proved within the second case in the proof of Theorem 4.6 (a). □

THEOREM 5.11. *Consider $\underline{x}\in\mathbb{Z}^3$ with $r\neq 0$. The automorphism $Q\in\operatorname{Aut}(H_{\mathbb{Q}},L)$ which was defined in Definition 5.9 can be written in two interesting ways,*

$$\begin{aligned}
Q &= (\underline{e}\mapsto -\underline{e}+\frac{2g^2}{r}f_3(-\widetilde{x}_3,\widetilde{x}_2-g\widetilde{x}_1\widetilde{x}_3,-\widetilde{x}_1)), &(5.5)\\
Q &= \operatorname{id}+2(M-\operatorname{id})+\frac{2}{r}(M-\operatorname{id})^2. &(5.6)
\end{aligned}$$

We have

$$Q\in G_{\mathbb{Z}} \iff \frac{2g^2}{r}\in\mathbb{Z} \iff \frac{r}{g^2}\in\{\pm1,\pm2\}. \qquad (5.7)$$

This holds if and only if $\underline{x}$ *is in the* $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ *orbit of a triple in the following set:*

$$\begin{array}{ll} & \{(x,0,0)\,|\, x\in\mathbb{N}\} \quad \text{(these are the reducible cases except } A_1^3),\\ \cup & \{(x,x,0)\,|\, x\in\mathbb{N}\} \quad \text{(these cases include } A_3),\\ \cup & \{(-l,2,-l)\,|\, l\geq 2 \text{ even}\} \quad \text{(these cases include } \mathcal{H}_{1,2}),\\ \cup & \{(3,3,4),(4,4,4),(5,5,5),(4,4,8)\}.\end{array}$$

So, within the cases with $r\in\{0,1,2,3,4\}$*,* Q *is not defined for* A_1^3 *and* $\mathbb{P}^2$*, and* $Q\notin G_{\mathbb{Z}}$ *for* $\widehat{A}_2$ *and* $S(-l,2,-l)$ *with* $l\geq 3$ *odd.*

Proof: First we prove (5.5). The 2-dimensional subspace $H_{\mathbb{Q},1} = \ker(M^2-(2-r)M+\mathrm{id})\subset H_{\mathbb{Q}}$, on which Q is $-\mathrm{id}$, can also be characterized as the right L-orthogonal subspace $H_{\mathbb{Q},1} = (\mathbb{Q}f_3)^\perp$ to $\mathbb{Q}f_3$. For $b=\underline{e}\cdot\underline{y}^t\in H_{\mathbb{Q}}$ with $\underline{y}\in\mathbb{Q}^3$

$$\begin{aligned} L(f_3,b) &= (-\widetilde{x}_3,\widetilde{x}_2,-\widetilde{x}_1)\begin{pmatrix}1&0&0\\x_1&1&0\\x_2&x_3&1\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}\\ &= (-\widetilde{x}_3,\widetilde{x}_2-g\widetilde{x}_1\widetilde{x}_3,-\widetilde{x}_1)\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix},\end{aligned}$$

so

$$H_{\mathbb{Q},1} = \{\underline{e}\cdot\underline{y}^t\,|\,\underline{y}\in\mathbb{Q}^3, 0=(-\widetilde{x}_3,\widetilde{x}_2-g\widetilde{x}_1\widetilde{x}_3,-\widetilde{x}_1)\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}\}. \tag{5.8}$$

Denote the endomorphisms on the right hand sides of (5.5) and (5.6) by $Q^{(5.5)}$ respectively $Q^{(5.6)}$. The formulas (5.5) and (5.8) show $Q^{(5.5)}|_{H_{\mathbb{Q},1}} = -\mathrm{id}$. Also

$$Q^{(5.5)}(f_3) = -f_3 + f_3\frac{2g^2}{r}(\widetilde{x}_3^2 + (\widetilde{x}_2-g\widetilde{x}_1\widetilde{x}_3)\widetilde{x}_2+\widetilde{x}_1^2) = f_3.$$

Therefore $Q^{(5.5)}=Q$, so (5.5) holds.

Now we prove (5.6). Because $(M-\mathrm{id})(f_3)=0$, we have $Q^{(5.6)}(f_3) = f_3$. Consider $b\in H_{\mathbb{Q},1}$. Then

$$\begin{aligned} 0 &= (M^2-(2-r)M+\mathrm{id})(b), \quad \text{so}\\ (M-\mathrm{id})^2(b) &= -rM(b), \quad \text{so}\\ Q^{(5.6)}(b) &= (\mathrm{id}+2(M-\mathrm{id})+\frac{2}{r}(-rM))(b) = (-id)(b) = -b.\end{aligned}$$

Therefore $Q^{(5.6)}=Q$, so (5.6) holds.

(5.7) can be proved either with (5.6) and Lemma 5.17 below or with (5.5), which is easier and which we do now. Observe $\gcd(-\widetilde{x}_3, \widetilde{x}_2 - g\widetilde{x}_1\widetilde{x}_3, -\widetilde{x}_1) = \gcd(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) = 1$. Also, f_3 is a primitive vector in $H_{\mathbb{Z}}$. This shows that $Q(e_1), Q(e_2), Q(e_3)$ are all in $H_{\mathbb{Z}}$ if and only if $\frac{2g^2}{r} \in \mathbb{Z}$. This shows (5.7).

It is easy to see that all triples in the set in Theorem 5.11 satisfy $\frac{r}{g^2} \in \{\pm 1, \pm 2\}$:

$$\frac{r}{g^2}((x,0,0)) = \frac{x^2}{x^2} = 1, \quad \frac{r}{g^2}((x,x,0)) = \frac{2x^2}{x^2} = 2,$$

$$\text{for even } l \quad \frac{r}{g^2}((-l,2,-l)) = \frac{4}{4} = 1,$$

$$\frac{r}{g^2}((3,3,4)) = \frac{-2}{1} = -2, \quad \frac{r}{g^2}((4,4,4)) = \frac{-16}{16} = -1,$$
$$\frac{r}{g^2}((5,5,5)) = \frac{-50}{25} = -2, \quad \frac{r}{g^2}((4,4,8)) = \frac{-32}{16} = -2.$$

The difficult part is to see that there are no other $\underline{x} \in \mathbb{Z}^3$ with $r \neq 0$ and $\frac{r}{g^2} \in \{\pm 1, \pm 2\}$. It is sufficient to consider local minima (Definition 4.3). The calculations

$$\frac{r}{g^2}((-l,2,-l)) = \frac{4}{1} = 4 \quad \text{for odd } l \geq 3$$
$$\text{and} \quad \frac{r}{g^2}((-1,-1,-1)) = \frac{4}{1} = 4$$

deal with the other cases with $r \in \{1,2,3,4\}$, see Theorem 4.6 (e).

Consider $\underline{x} \in \mathbb{Z}^3_{\leq 0}$ with $x_i \leq x_j \leq x_k$ for some i,j,k with $\{i,j,k\} = \{1,2,3\}$. Then $\frac{r}{g^2} = \widetilde{x}_1^2 + \widetilde{x}_2^2 + \widetilde{x}_3^2 + g|\widetilde{x}_1\widetilde{x}_2\widetilde{x}_3|$ can be 1 or 2 only if $\widetilde{x}_k = 0$ and $\widetilde{x}_i = \widetilde{x}_j = -1$. Then $\underline{x} = (-g,-g,0)$. This is in the $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbit of $(g,g,0)$.

Consider a local minimum $\underline{x} \in \mathbb{Z}^3_{\geq 3}$ with $x_i \leq x_j \leq x_k$ for some i,j,k with $\{i,j,k\} = \{1,2,3\}$. Suppose $\frac{r}{g^2} \in \{-1,-2\}$. We have to show $\underline{x} \in \{(3,3,4),(4,4,4),(5,5,5),(4,4,8)\}$. Of course $\underline{x} \neq (3,3,3)$ because $r \neq 0$.

If $g = 1$ then $r \in \{-1,-2\}$. One sees easily that $r = -1$ is impossible and that $r = -2$ is only satisfied for $\underline{x} = (3,3,4)$.

From now on suppose $g \geq 2$. Write $\rho := |\frac{r}{g^2}| \in \{1,2\}$. (5.2) takes the shape

$$\widetilde{x}_i \leq \frac{2}{g} + (\frac{\rho}{g} + \frac{4}{g^3})^{1/3}.$$

The only pairs $(\widetilde{x}_i, g) \in \mathbb{N} \times \mathbb{Z}_{\geq 2}$ which satisfy this and $x_i = \widetilde{x}_i g \geq 3$ are in the following two tables,

$$\rho = 1 : \begin{array}{c|c|c|c|c} g & 2 & 3 & 4 & 5 \\ \hline \widetilde{x}_i & 2 & 1 & 1 & 1 \end{array}, \quad \rho = 2 : \begin{array}{c|c|c|c|c|c} g & 2 & 3 & 4 & 5 & 6 \\ \hline \widetilde{x}_i & 2 & 1 & 1 & 1 & 1 \end{array}$$

The following table discusses these nine cases. Three of them lead to $(4, 4, 4)$, $(5, 5, 5)$ and $(4, 4, 8)$, six of them are impossible. The symbol $\circledast$ denotes *impossible*. The inequalities $x_i \leq x_j \leq x_k$ and (5.3) and (5.4) are used, and also $x_i = g\widetilde{x}_i$ and $x_j, x_k \in g\mathbb{N}$.

ρ	g	$-r$	$\widetilde{x}_i$	x_i	$x_j^2 \leq \frac{4-r}{x_i-2} + x_i + 2$	x_j	$x_k \leq \frac{1}{2}x_i x_j$	x_k
1	2	4	2	4	$x_j^2 \leq 10$	$\circledast$		
1	3	9	1	3	$x_j^2 \leq 18$	3	$x_k \leq \frac{9}{2}$	3 $\circledast$
1	4	16	1	4	$x_j^2 \leq 16$	4	$x_k \leq 8$	4
1	5	25	1	5	$x_j^2 \leq 16 + \frac{2}{3}$	$\circledast$		
2	2	8	2	4	$x_j^2 \leq 12$	$\circledast$		
2	3	18	1	3	$x_j^2 \leq 27$	3	$x_k \leq \frac{9}{2}$	3 $\circledast$
2	4	32	1	4	$x_j^2 \leq 24$	4	$x_k \leq 8$	8
2	5	50	1	5	$x_j^2 \leq 25$	5	$x_k \leq 12 + \frac{1}{2}$	5
2	6	72	1	6	$x_j^2 \leq 27$	$\circledast$		

This finishes the proof of Theorem 5.11. □

5.4. The reducible rank 3 cases

Definition 2.10 proposed the notion of reducible triple $(H_{\mathbb{Z}}, L, \underline{e})$ where $(H_{\mathbb{Z}}, L)$ is a unimodular bilinear lattice and $\underline{e}$ is a triangular basis. The following Remarks propose the weaker notion when a unimodular bilinear lattice $(H_{\mathbb{Z}}, L)$ (without a triangular basis) is reducible. Then the groups $G_{\mathbb{Z}}, G_{\mathbb{Z}}^{(0)}, G_{\mathbb{Z}}^{(1)}$ and $G_{\mathbb{Z}}^M$ split accordingly if also the eigenvalues of M split in a suitable sense.

Remarks 5.12. (i) Suppose that a unimodular bilinear lattice $(H_{\mathbb{Z}}, L)$ splits into a direct sum $H_{\mathbb{Z},1} \oplus H_{\mathbb{Z},2}$ which is left and right L-orthogonal. Then the restrictions of $L, I^{(0)}, I^{(1)}$ and M to $H_{\mathbb{Z},i}$ are called $L_i, I_i^{(0)}, I_i^{(1)}$ and M_i for $i \in \{1; 2\}$. We say that $(H_{\mathbb{Z}}, L)$ is *reducible* and that it splits into the direct sum $(H_{\mathbb{Z},1}, L_1) \oplus (H_{\mathbb{Z},2}, L_2)$.

(ii) In the situation of (i), suppose that the eigenvalues of M_1 are pairwise different from the eigenvalues of M_2. Then any element of $G_{\mathbb{Z}}^M$ respects the splitting. For $\{i, j\} = \{1, 2\}$ write $G_{\mathbb{Z},i}^M := G_{\mathbb{Z}}^M(H_{\mathbb{Z},i}, L_i)$. Then

$$G_{\mathbb{Z}}^M = G_{\mathbb{Z},1}^M \times G_{\mathbb{Z},2}^M,$$

and with analogous notations

$$G_{\mathbb{Z}} = G_{\mathbb{Z},1} \times G_{\mathbb{Z},2},\ G_{\mathbb{Z}}^{(0)} = G_{\mathbb{Z},1}^{(0)} \times G_{\mathbb{Z},2}^{(0)},\ G_{\mathbb{Z}}^{(1)} = G_{\mathbb{Z},1}^{(1)} \times G_{\mathbb{Z},2}^{(1)}.$$

(iii) There is only one unimodular bilinear lattice of rank 1. We call it A_1-lattice and denote the matrix $S = S(A_1) = (1) \in M_{1\times 1}(\mathbb{Z})$. Here $G_{\mathbb{Z}} = G_{\mathbb{Z}}^{(0)} = G_{\mathbb{Z}}^{(1)} = G_{\mathbb{Z}}^{M} = \{\pm \mathrm{id}\}$.

(iv) Suppose that the characteristic polynomial $p_{ch,M}(t) \in \mathbb{Z}[t]$ of the monodromy M of a unimodular bilinear lattice $(H_{\mathbb{Z}}, L)$ splits into a product $p_{ch,M} = p_1 p_2$ of non-constant polynomials p_1 and p_2 with $\gcd(p_1, p_2) = 1$. Then $\ker p_1(M) \oplus \ker p_2(M)$ is a sublattice of finite index in $H_{\mathbb{Z}}$, and the summands are left and right L-orthogonal to one another. If the index is 1, we are in the situation of (ii). Theorem 5.14 will show that this applies to the cases $S(\mathcal{H}_{1,2})$ and $S(-l, 2, -l)$ with $l \geq 4$ even, but not to the cases $S(A_3)$, $S(\widehat{A}_2)$ and $S(-l, 2, -l)$ with $l \geq 3$ odd.

These remarks apply especially to the reducible 3×3 cases except A_1^3 (which is part of Lemma 5.4). This includes the two reducible cases A_2A_1 and $\mathbb{P}^1A_1$ with eigenvalues in S^1.

Theorem 5.13. *Consider $\underline{x} = (x, 0, 0) \in \mathbb{Z}^3$ with $x \neq 0$ and the unimodular bilinear lattice $(H_{\mathbb{Z}}, L, \underline{e})$ with triangular basis $\underline{e}$ with $L(\underline{e}^t, \underline{e})^t = S(\underline{x}) \in T_3^{uni}(\mathbb{Z})$.*

Then $(H_{\mathbb{Z}}, L, \underline{e})$ is reducible with the summands $(H_{\mathbb{Z},1}, L_1, (e_1, e_2))$ and $(H_{\mathbb{Z},2}, L_2, e_3)$ with $H_{\mathbb{Z},1} = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ and $H_{\mathbb{Z},3} = \mathbb{Z}e_3$. The first summand is an irreducible rank two unimodular bilinear lattice with triangular basis. Its groups $G_{\mathbb{Z},1}, G_{\mathbb{Z},1}^{(0)}, G_{\mathbb{Z},1}^{(1)}$ and $G_{\mathbb{Z},1}^{M}$ are treated in Theorem 5.5. The second summand is of type A_1. See Remark 5.12 (iii) for its groups.

The decompositions in Remark 5.12 (ii) hold for the groups $G_{\mathbb{Z}}$, $G_{\mathbb{Z}}^{(0)}$, $G_{\mathbb{Z}}^{(1)}$ and $G_{\mathbb{Z}}^{M}$. Here $G_{\mathbb{Z}}^{M} = G_{\mathbb{Z}}$ if $x \neq \pm 3$ and $G_{\mathbb{Z}} = G_{\mathbb{Z}}^{(0)} = G_{\mathbb{Z}}^{(1)}$ always.

The map $Z : (\mathrm{Br}_3 \ltimes \{\pm 1\}^3)_S \to G_{\mathbb{Z}}$ is surjective.

Proof: The first point to see is that Remark 5.12 (ii) applies. It does because the characteristic polynomials of the monodromies M_1 and M_2 of the two summands are $t^2 - (2 - x^2)t + 1$ and $t - 1$, and here $x \neq 0$, so that the eigenvalues of M_1 are not equal to the eigenvalue 1 of M_2.

The second point to see is the surjectivity of the map Z. This follows from the surjectivity of the map Z in the irreducible rank 2 cases in Theorem 5.5 and in the case A_1 in Lemma 5.4. □

5.5. The irreducible rank 3 cases with all eigenvalues in S^1

Theorem 5.14 is the only point in this section. It treats the irreducible rank 3 cases with all eigenvalues in S^1.

THEOREM 5.14. *Consider for each of the matrices $S(\mathbb{P}^2)$, $S(A_3)$, $S(\widehat{A}_2)$, $S(\mathcal{H}_{1,2})$ and $S(-l,2,-l)$ for $l\geq 3$ in the Examples 1.1 a unimodular bilinear lattice $(H_\mathbb{Z},L)$ with a triangular basis $\underline{e}$ with $L(\underline{e}^t,\underline{e})^t = S$.*

(a) The cases $S(\mathcal{H}_{1,2})$ and $S(-l,2,-l)$ for $l\geq 4$ even: Then $(H_\mathbb{Z},L)$ is reducible (in the sense of Remark 5.12 (i)), $H_\mathbb{Z}=H_{\mathbb{Z},1}\oplus H_{\mathbb{Z},2}$ with

$$\begin{aligned} H_{\mathbb{Z},1} &:= \ker(M+\mathrm{id})^2 \text{ of rank 2} \\ \text{and } H_{\mathbb{Z},2} &:= \ker(M-\mathrm{id}) \text{ of rank 1}. \end{aligned}$$

$(H_{\mathbb{Z},2},L_2)$ is an A_1-lattice. In all cases the decompositions in Remark 5.12 (ii) hold for the groups $G_\mathbb{Z}^M$, $G_\mathbb{Z}^{(0)}$, $G_\mathbb{Z}^{(1)}$ and $G_\mathbb{Z}$. The groups $G_{\mathbb{Z},1}^M$, $G_{\mathbb{Z},1}^{(0)}$, $G_{\mathbb{Z},1}^{(1)}$ and $G_{\mathbb{Z},1}$ are as follows.

(i) $S(\mathcal{H}_{1,2})$: $H_{\mathbb{Z},1}$ has a $\mathbb{Z}$-basis $\underline{f}=(f_1,f_2)$ with

$$L(\underline{f}^t,\underline{f})^t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad I^{(0)}(\underline{f}^t,\underline{f}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
$$I^{(1)}(\underline{f}^t,\underline{f}) = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \quad M\underline{f} = \underline{f}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\begin{aligned} G_{\mathbb{Z},1}^M &= G_{\mathbb{Z},1}^{(0)} = \mathrm{Aut}(H_{\mathbb{Z},1}) \cong GL_2(\mathbb{Z}), \\ G_{\mathbb{Z},1} &= G_{\mathbb{Z},1}^{(1)} = \{g\in \mathrm{Aut}(H_{\mathbb{Z},1})\,|\,\det g = 1\} \cong SL_2(\mathbb{Z}). \end{aligned}$$

(ii) $S(-l,2,-l)$ for $l\geq 4$ even: $H_{\mathbb{Z},1}$ has a $\mathbb{Z}$-basis $\underline{f}=(f_1,f_2)$ with

$$L(\underline{f}^t,\underline{f})^t = \begin{pmatrix} 0 & -1 \\ 1 & 1-\frac{l^2}{4} \end{pmatrix}, \quad I^{(0)}(\underline{f}^t,\underline{f}) = \begin{pmatrix} 0 & 0 \\ 0 & 2-\frac{l^2}{2} \end{pmatrix},$$
$$I^{(1)}(\underline{f}^t,\underline{f}) = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \quad M\underline{f} = \underline{f}\begin{pmatrix} -1 & 2-\frac{l^2}{2} \\ 0 & -1 \end{pmatrix},$$

Define $M_1^{root}\in\mathrm{Aut}(H_{\mathbb{Z},1})$ by $M_1^{root}\underline{f} = \underline{f}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then

$$(M_1^{root})^{l^2/2-2} = -M_1 \qquad \text{and}$$
$$G_{\mathbb{Z},1} = G_{\mathbb{Z},1}^{(0)} = G_{\mathbb{Z},1}^{(1)} = G_{\mathbb{Z},1}^M = \{\pm (M_1^{root})^m\,|\, m\in\mathbb{Z}\}.$$

(b) The cases $S(\mathbb{P}^2)$, $S(A_3)$, $S(\widehat{A}_2)$, $S(-l,2,-l)$ with $l\geq 3$ odd: Then $(H_\mathbb{Z},L)$ is irreducible.

$$G_\mathbb{Z}=G_\mathbb{Z}^{(0)}=\{\pm(M^{root})^m\,|\,m\in\mathbb{Z}\}.$$

Here M^{root} is defined by

$$M^{root} := Z(\sigma^{root}) \quad \text{in the case } S(\mathbb{P}^2), \tag{5.9}$$

$$M^{root} := M=Z(\sigma^{mon}) \quad \text{in the case } S(A_3), \tag{5.10}$$

$$M^{root} := Z(\delta_3\sigma^{root}) \quad \text{in the case } S(\widehat{A}_2), \tag{5.11}$$

$$M^{root} := (-M)\circ Z(\delta_3\sigma_1^{-1}\sigma_2^{-1}\sigma_1)^{(5-l^2)/2} \quad \text{in the case } S(-l,2,-l) \text{ for odd } l\geq 3. \tag{5.12}$$

It satisfies

$$M^{root}\underline{e}=\underline{e}M^{root,mat} \quad\text{and}\quad (M^{root})^m=\varepsilon M$$

where $M^{root,mat}$, m and ε are as follows:

	$S(\mathbb{P}^2)$	$S(A_3)$
$M^{root,mat}$	$\begin{pmatrix}3&-3&1\\1&0&0\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\-1&0&1\\0&-1&1\end{pmatrix}$
(m,ε)	$(3,1)$	$(1,1)$
	$S(\widehat{A}_2)$	$S(-l,2,-l)$ with $l\geq 3$ odd
$M^{root,mat}$	$\begin{pmatrix}1&1&-1\\1&0&0\\0&1&0\end{pmatrix}$	$\frac12\begin{pmatrix}1-l^2&l^3-l&-1-l^2\\-2l&2l^2-2&-2l\\1-l^2&l^3-3l&3-l^2\end{pmatrix}$
(m,ε)	$(3,-1)$	$(l^2-4,-1)$

M and M^{root} are regular. M^{root} is cyclic. In the cases $S(A_3)$, $S(\widehat{A}_2)$ and $S(-l,2,-l)$ for $l\geq 3$ odd

$$G_\mathbb{Z}=G_\mathbb{Z}^{(0)}=G_\mathbb{Z}^{(1)}=G_\mathbb{Z}^M=\{\pm(M^{root})^m\,|\,m\in\mathbb{Z}\}.$$

Some additional information:

(i) $S(\mathbb{P}^2)$: $\operatorname{sign} I^{(0)}=(+--)$, $p_{ch,M}=p_{ch,M^{root}}=\Phi_1^3$, *$M$ and M^{root} have a 3×3 Jordan block,*

$$G_\mathbb{Z}^{(1)}=G_\mathbb{Z}^M=\{\pm(M^{root})^m(\mathrm{id}+a(M^{root}-\mathrm{id})^2)\,|\,m,a\in\mathbb{Z}\}\supsetneqq G_\mathbb{Z}.$$

(ii) $S(A_3)$: $\operatorname{sign} I^{(0)}=(+++)$, $p_{ch,M}=\Phi_4\Phi_1$, $M=M^{root}$, $|G_\mathbb{Z}|=8$.

(iii) $S(\widehat{A}_2)$: $\operatorname{sign} I^{(0)}=(++\,0)$, $p_{ch,M}=\Phi_2^2\Phi_1$, $p_{ch,M^{root}}=\Phi_1^2\Phi_2$, *$M$ and M^{root} have a 2×2 Jordan block with eigenvalue -1 respectively 1.*

(iv) $S(-l,2,-l)$ with $l\geq 3$ odd: $\operatorname{sign} I^{(0)} = (+\,0\,-)$, *$p_{ch,M} = \Phi_2^2\Phi_1$, $p_{ch,M^{root}} = \Phi_1^2\Phi_2$, M and M^{root} have a 2×2 Jordan block with eigenvalue -1 respectively 1.*

(c) In all cases in this theorem the map $Z : (\mathrm{Br}_3 \ltimes \{\pm 1\}^3)_S \to G_{\mathbb{Z}}$ is surjective, so $G_{\mathbb{Z}} = G_{\mathbb{Z}}^{\mathcal{B}}$.

Proof: (a) Recall

$$H_{\mathbb{Z},2} = \ker(M-\mathrm{id}) = \mathrm{Rad}\, I^{(1)} = \mathbb{Z}f_3,\ f_3 = -\widetilde{x}_3e_1 + \widetilde{x}_2e_2 - \widetilde{x}_1e_3.$$

We will choose a $\mathbb{Z}$-basis $\underline{f} = (f_1,f_2)$ of $H_{\mathbb{Z},1} := \ker(M+\mathrm{id})^2$. Denote $\underline{\widetilde{f}} := (f_1,f_2,f_3)$. In all cases it will be easy to see that $\underline{\widetilde{f}}$ is a $\mathbb{Z}$-basis of $H_{\mathbb{Z}}$. Therefore in all cases $H_{\mathbb{Z}} = H_{\mathbb{Z},1}\oplus H_{\mathbb{Z},2}$.

(i) $S(\mathcal{H}_{1,2})$: Recall $p_{ch,M} = \Phi_2^2\Phi_1$,

$$S = \begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix},\ S^{-1}S^t = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -3 & 2 \\ 2 & -2 & 1 \end{pmatrix},\ f_3 = \underline{e}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Define

$$f_1 := \underline{e}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},\ f_2 := \underline{e}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Then

$$L(\underline{\widetilde{f}}^t,\underline{\widetilde{f}})^t = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ M\underline{\widetilde{f}} = \underline{\widetilde{f}}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ H_{\mathbb{Z},1} = \mathbb{Z}f_1\oplus\mathbb{Z}f_2.$$

The claims on the groups $G_{\mathbb{Z},1}^M, G_{\mathbb{Z},1}^{(1)}, G_{\mathbb{Z},1}^{(0)}$ and $G_{\mathbb{Z},1}$ follow from the shape of the matrices of $M, I^{(1)}, I^{(0)}$ and L with respect to the basis $\underline{f}$ of $H_{\mathbb{Z},1}$.

(ii) $S(-l,2,-l)$ with $l\geq 4$ even: Recall $p_{ch,M} = \Phi_2^2\Phi_1$,

$$S = \begin{pmatrix} 1 & -l & 2 \\ 0 & 1 & -l \\ 0 & 0 & 1 \end{pmatrix},\ S^{-1}S^t = \begin{pmatrix} l^2-3 & -l^3+3l & l^2-2 \\ l & -l^2+1 & l \\ 2 & -l & 1 \end{pmatrix},\ f_3 = \underline{e}\begin{pmatrix} \frac{l}{2} \\ 1 \\ \frac{l}{2} \end{pmatrix}.$$

Define

$$f_1 := \underline{e}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},\ f_2 := \underline{e}\begin{pmatrix} \frac{l^2-2}{2} \\ \frac{l}{2} \\ 0 \end{pmatrix}.$$

Then

$$L(\underline{\widetilde{f}}^t,\underline{\widetilde{f}})^t=\begin{pmatrix}0&-1&0\\1&1-\frac{l^2}{4}&0\\0&0&1\end{pmatrix},\ M\underline{\widetilde{f}}=\underline{\widetilde{f}}\begin{pmatrix}-1&2-\frac{l^2}{2}&0\\0&-1&0\\0&0&1\end{pmatrix},$$

$$H_{\mathbb{Z},1}=\mathbb{Z}f_1\oplus\mathbb{Z}f_2,\ H_{\mathbb{Z}}=H_{\mathbb{Z},1}\oplus H_{\mathbb{Z},2},$$

$M_1=M|_{H_{\mathbb{Z},1}}$ and M_1^{root} have each a 2×2 Jordan block, M_1^{root} is cyclic and $(M_1^{root})^{l^2/2-2}=-M_1$. Lemma 5.6 shows

$$G^M_{\mathbb{Z},1}=G^{(0)}_{\mathbb{Z},1}=G^{(1)}_{\mathbb{Z},1}=G_{\mathbb{Z},1}=\{\pm(M_1^{root})^m\,|\,m\in\mathbb{Z}\}.$$

(b) Recall

	$S(\mathbb{P}^2)$	$S(A_3)$
$S^{-1}S^t$	$\begin{pmatrix}10&-15&6\\6&-8&3\\3&-3&1\end{pmatrix}$	$\begin{pmatrix}0&0&1\\-1&0&1\\0&-1&1\end{pmatrix}$
$p_{ch,M}$	Φ_1^3	$\Phi_4\Phi_1$
	$S(\widehat{A}_2)$	$S(-l,2,-l)$ with $l\geq 3$ odd
$S^{-1}S^t$	$\begin{pmatrix}-2&-1&2\\-2&0&1\\-1&-1&1\end{pmatrix}$	$\begin{pmatrix}l^2-3&-l^3+3l&l^2-2\\l&-l^2+1&l\\2&-l&1\end{pmatrix}$
$p_{ch,M}$	$\Phi_2^2\Phi_1$	$\Phi_2^2\Phi_1$

The case $S(-l,2,-l)$ with $l\geq 3$ odd will be treated separately below. Theorem 3.26 (c) applies in the case $S(\mathbb{P}^2)$ with $k=1$ and in the case $S(\widehat{A}_2)$ with $k=0$. It shows in these cases $(M^{root})^3=\varepsilon M$. By Theorem 3.26 (c) the matrices $M^{root,mat}$ are as claimed in the cases $S(\mathbb{P}^2)$ and $S(\widehat{A}_2)$. In the case A_3 by definition $M^{root}=M$.

In all three cases $S(\mathbb{P}^2)$, $S(A_3)$ and $S(\widehat{A}_2)$ M^{root} is cyclic with cyclic generator e_1. In the cases $S(\mathbb{P}^2)$ and $S(A_3)$ $p_{ch,M^{root}}=p_{ch,M}$. In the case $S(\widehat{A}_2)$ $p_{ch,M^{root}}=\phi_1^2\phi_2$ and $p_{ch,M}=\phi_2^2\phi_1$. Lemma 5.2 (a) shows in all three cases

$$\begin{aligned}G^M_{\mathbb{Z}}\ &=\ \{p(M^{root})\,|\,p(t)=\sum_{i=0}^2 p_it^i\in\mathbb{Z}[t],\\ &\qquad p(\kappa)\in(\mathbb{Z}[\kappa])^*\text{ for each eigenvalue }\kappa\text{ of }M^{root}\}.\end{aligned}$$

(i) $S(\mathbb{P}^2)$: $M^{root}-\mathrm{id}$ is nilpotent with $(M^{root}-\mathrm{id})^2\neq 0$, $(M^{root}-\mathrm{id})^3=0$. An element of $\mathbb{Z}[M^{root}]$ can be written in the form

$$q_0\,\mathrm{id}+q_1(M^{root}-\mathrm{id})+q_2(M^{root}-\mathrm{id})^2\text{ with }q_0,q_1,q_2\in\mathbb{Z}.$$

It is in $\mathrm{Aut}(H_\mathbb{Z})$ if and only if $q_0\in\{\pm 1\}$. Then it can be written as

$$q_0(M^{root})^{q_0q_1}(\mathrm{id}+\widetilde{q}_2(M^{root}-\mathrm{id})^2) \text{ for some } \widetilde{q}_2\in\mathbb{Z}.$$

Therefore $G_\mathbb{Z}^M$ is as claimed.

Because $(M^{root}-\mathrm{id})^3=0$, $(M^{root}-\mathrm{id})^2(H_\mathbb{Z})\subset\ker(M^{root}-\mathrm{id})=\mathrm{Rad}\, I^{(1)}$. Therefore $\mathrm{id}+\widetilde{q}_2(M^{root}-\mathrm{id})^2$ and thus any element of $G_\mathbb{Z}^M$ respects $I^{(1)}$, so $G_\mathbb{Z}^M=G_\mathbb{Z}^{(1)}$.

On the other hand, one easily checks that $\mathrm{id}+\widetilde{q}_2(M^{root}-\mathrm{id})^2$ respects $I^{(0)}$ only if $\widetilde{q}_2=0$. Therefore

$$G_\mathbb{Z}=G_\mathbb{Z}^{(0)}=\{\pm(M^{root})^m\,|\,m\in\mathbb{Z}\}\subsetneqq G_\mathbb{Z}^{(1)}=G_\mathbb{Z}^M.$$

(ii) $S(A_3)$: For $p(t)=\sum_{i=0}^2 p_it^i\in\mathbb{Z}[t]$ write $\mu_j:=p(\lambda_j)$ for $j\in\{1,2,3\}$ for the eigenvalues of the element $p(M)\in\mathbb{Z}[M]=\mathrm{End}(H_\mathbb{Z},M)$ where $\lambda_1=i,\lambda_2=-i,\lambda_3=1$ are the eigenvalues of M. Because of $(\mathbb{Z}[i])^*=\{\pm 1,\pm i\}$, one can multiply a given element of $G_\mathbb{Z}^M$ with a suitable power of M and obtain an element with $\mu_1=\mu_2=1$. Therefore

$$\begin{aligned} G_\mathbb{Z}^M=\{M^{m_1}(\mathrm{id}+m_2\Phi_4(M))\quad|\quad & m_1\in\{0,1,2,3\},m_2\in\mathbb{Z},\\ & 1+m_2\Phi_4(1)\in\{\pm 1\}\}.\end{aligned}$$

This forces $m_2\in\{0,-1\}$. The case $m_2=-1$ gives $\mathrm{id}+(-1)(M^2+\mathrm{id})=-M^2$. Therefore

$$\{\pm M^m\,|\,m\in\{0,1,2,3\}\}=\{\pm M^m\,|\,m\in\mathbb{Z}\}=G_\mathbb{Z}^M=G_\mathbb{Z}^{(0)}=G_\mathbb{Z}^{(1)}=G_\mathbb{Z}.$$

(iii) $S(\widehat{A}_2)$: Write $\underline{f}=(f_1,f_2):=\underline{e}\begin{pmatrix}1&2\\1&1\\1&0\end{pmatrix}$. Then

$$\begin{aligned} M^{root}\underline{f} &= \underline{f}\begin{pmatrix}1&1\\0&1\end{pmatrix},\\ H_{\mathbb{Z},1} &= \ker\Phi_2^2(M)=\ker\Phi_1^2(M^{root})=\mathbb{Z}f_1\oplus\mathbb{Z}f_2.\end{aligned}$$

Lemma 5.6 implies

$$\begin{aligned} \{g|_{H_{\mathbb{Z},1}}\,|\,g\in G_\mathbb{Z}^M\}&=\{\pm(M^{root}|_{H_{\mathbb{Z},1}})^m\,|\,m\in\mathbb{Z}\},\\ G_\mathbb{Z}^M&=\{\pm(M^{root})^m\,|\,m\in\mathbb{Z}\}\times\{g\in G_\mathbb{Z}^M\,|\,g|_{H_{\mathbb{Z},1}}=\mathrm{id}\}.\end{aligned}$$

But $p(t)=1+q\Phi_1^2(t)$ with $q\in\mathbb{Z}$ satisfies $p(-1)=1+q\cdot 2^2\in\{\pm 1\}$ only if $q=0$. Therefore

$$\{\pm(M^{root})^m\,|\,m\in\mathbb{Z}\}=G_\mathbb{Z}^M=G_\mathbb{Z}^{(0)}=G_\mathbb{Z}^{(1)}=G_\mathbb{Z}.$$

(iv) $S(-l,2,-l)$ with $l\geq 3$ odd: Recall f_3 and define $(f_1,\widetilde{f}_2)$ and $\underline{d}=(d_1,d_2,d_3)$ with

$$(f_1,\widetilde{f}_2,f_3)=\underline{e}\begin{pmatrix}1 & l^2-2 & l\\ 0 & l & 2\\ -1 & 0 & l\end{pmatrix},\quad \underline{d}=\underline{e}\begin{pmatrix}\frac{l^2-l-2}{2} & \frac{l^2-1}{2} & \frac{l^2-l}{2}\\ \frac{l-1}{2} & \frac{l+1}{2} & \frac{l-1}{2}\\ 0 & \frac{l-1}{2} & -1\end{pmatrix}.$$

The matrix which expresses $(f_1,\widetilde{f}_2,f_3)$ with $\underline{e}$ has determinant 4, the matrix which expresses $\underline{d}$ with $\underline{e}$ has determinant 1. Therefore $\mathbb{Z}f_1\oplus\mathbb{Z}\widetilde{f}_2\oplus\mathbb{Z}f_3$ is a sublattice of index 4 in $H_\mathbb{Z}$, and $\underline{d}$ is a $\mathbb{Z}$-basis of $H_\mathbb{Z}$. One calculates

$$M(f_1,\widetilde{f}_2,f_3) = (f_1,\widetilde{f}_2,f_3)\begin{pmatrix}-1 & -l^2+4 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Especially

$$\mathbb{Z}f_1\oplus\mathbb{Z}\widetilde{f}_2=\ker(M^2+\mathrm{id})^2\supset\mathbb{Z}f_1=\ker(M+\mathrm{id})=\mathrm{Rad}\, I^{(0)}.$$

Observe

$$\delta_3\sigma_1^{-1}\sigma_2^{-1}\sigma_1\in(\mathrm{Br}_3\ltimes\{\pm 1\}^3)_S. \tag{5.13}$$

Define

$$\widetilde{M}:=Z(\delta_3\sigma_1^{-1}\sigma_2^{-1}\sigma_1)\in G_\mathbb{Z}. \tag{5.14}$$

Then $M^{root}=(-M)\circ\widetilde{M}^{(5-l^2)/2}\in G_\mathbb{Z}$. One calculates

$$\widetilde{M}\underline{e} = \underline{e}+f_1(-1,l,-1), \tag{5.15}$$

$$\widetilde{M}(f_1,\widetilde{f}_2,f_3) = (f_1,\widetilde{f}_2,f_3)\begin{pmatrix}1 & 2 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{5.16}$$

$$\begin{aligned} M^{root}(f_1,\widetilde{f}_2,f_3) &= (f_1,\widetilde{f}_2,f_3)\begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix},\\ (M^{root})^{l^2-4} &= -M,\\ (M^{root})^2 &= \widetilde{M}. \end{aligned}$$

Finally, one calculates

$$M^{root}\underline{d}=\underline{d}\begin{pmatrix}0 & 0 & -1\\ 1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}.$$

Therefore M^{root} is cyclic with cyclic generator d_1 and regular. Lemma 5.2 (a) shows

$$G_{\mathbb{Z}}^M = \{p(M^{root})\,|\,p(t) = \sum_{i=0}^{2} p_i t^i \in \mathbb{Z}[t], p(1), p(-1) \in \{\pm 1\}\}.$$

As in the case $S(\widehat{A}_2)$ one finds with Lemma 5.6

$$\{g|_{H_{\mathbb{Z},1}}\,|\,g \in G_{\mathbb{Z}}^M\} = \{\pm(M^{root}|_{H_{\mathbb{Z},1}})^m\,|\,m \in \mathbb{Z}\},$$
$$G_{\mathbb{Z}}^M = \{\pm(M^{root})^m\,|\,m \in \mathbb{Z}\} \times \{g \in G_{\mathbb{Z}}^M\,|\,g|_{H_{\mathbb{Z},1}} = \mathrm{id}\}.$$

But $p(t) = 1 + q\Phi_1^2(t)$ with $q \in \mathbb{Z}$ satisfies $p(-1) = 1 + q \cdot 2^2 \in \{\pm 1\}$ only if $q = 0$. Therefore

$$\{\pm(M^{root})^m\,|\,m \in \mathbb{Z}\} = G_{\mathbb{Z}}^M = G_{\mathbb{Z}}^{(0)} = G_{\mathbb{Z}}^{(1)} = G_{\mathbb{Z}}.$$

(c) Of course $Z(\delta_1\delta_2\delta_3) = -\mathrm{id}$. In the cases in part (b) $G_{\mathbb{Z}} = \{\pm(M^{root})^m\,|\,m \in \mathbb{Z}\}$. The definitions (5.9)–(5.12) show that Z is surjective.

The case $\mathcal{H}_{1,2}$: $G_{\mathbb{Z}} \cong SL_2(\mathbb{Z}) \times \{\pm 1\}$. The group $G_{\mathbb{Z}}$ is generated by $-\mathrm{id}$, h_1 and h_2 with

$$h_1 := (\underline{\widetilde{f}} \mapsto \underline{\widetilde{f}} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}) = (\underline{e} \mapsto \underline{e} \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}) = Z(\delta_2\sigma_1),$$
$$h_2 := (\underline{\widetilde{f}} \mapsto \underline{\widetilde{f}} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}) = (\underline{e} \mapsto \underline{e} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 0 \end{pmatrix}) = Z(\delta_3\sigma_2).$$

The cases $S(-l, 2, -l)$ with $l \geq 4$ even: (5.13)–(5.16) hold also for even l. With respect to the $\mathbb{Z}$-basis $\underline{\widetilde{f}} = (f_1, f_2, f_3) = (f_1, \frac{1}{2}\widetilde{f}_2, f_3)$ of $H_{\mathbb{Z}}$

$$\widetilde{M}\underline{\widetilde{f}} = \underline{\widetilde{f}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

One sees

$$G_{\mathbb{Z}} \;=\; \langle -\mathrm{id}, \widetilde{M}, Q\rangle$$

with

$$\begin{aligned} Q &= (\underline{\widetilde{f}} \mapsto \underline{\widetilde{f}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}) = M \circ \widetilde{M}^{2-l^2/2} \\ &= Z(\sigma^{mon}) \circ Z(\delta_3\sigma_1^{-1}\sigma_2^{-1}\sigma_1)^{2-l^2/2}. \qquad \square \end{aligned}$$

$\square$

5.6. Special rank 3 cases with eigenvalues not all in S^1

This section starts with a general lemma for all rank 3 cases with eigenvalues not all in S^1. It gives coarse information on the four groups $G_{\mathbb{Z}}^M, G_{\mathbb{Z}}^{(0)}, G_{\mathbb{Z}}^{(1)}$ and $G_{\mathbb{Z}}$. Afterwards Theorem 5.16 determines these groups precisely for three series of cases and one exceptional case. Theorem 5.18 in section 5.7 will treat the other irreducible cases with eigenvalues not all in S^1.

LEMMA 5.15. *Fix $\underline{x} \in \mathbb{Z}^3$ with $r(\underline{x}) < 0$ or $r(\underline{x}) > 4$. Then*

$$G_{\mathbb{Z}}^M \overset{(2:1) \text{ or } (1:1)}{\supset} G_{\mathbb{Z}} = G_{\mathbb{Z}}^{(0)} = G_{\mathbb{Z}}^{(1)} \overset{(\text{finite}:1)}{\supset} \{\pm M^l \,|\, l \in \mathbb{Z}\}.$$

Proof: For $p(t) = \sum_{i=0}^2 p_i t^i \in \mathbb{Q}[t]$ write $\mu_j := p(\lambda_j)$ for $j \in \{1,2,3\}$. Then μ_1, μ_2 and μ_3 are the eigenvalues of $p(M) \in \operatorname{End}(H_{\mathbb{Q}})$. The monodromy is because of $r(\underline{x}) \in \mathbb{Z} - \{0,1,2,3,4\}$ and Lemma 5.7 semisimple and regular. Lemma 5.1 (c) (i) and (ii) applies and gives

$$\begin{aligned} G_{\mathbb{Z}}^M &= \{p(M) \,|\, p(t) = \sum_{i=0}^2 p_i t^i \in \mathbb{Q}[t], \\ &\qquad p(M) \in \operatorname{End}(H_{\mathbb{Z}}), \mu_1\mu_2 \in \{\pm 1\}, \mu_3 \in \{\pm 1\}\} && (5.17) \\ \supset G_{\mathbb{Z}} &= G_{\mathbb{Z}}^{(0)} = G_{\mathbb{Z}}^{(1)} = \{p(M) \in G_{\mathbb{Z}}^M \,|\, \mu_1\mu_2 = 1\} && (5.18) \end{aligned}$$

Especially $h \in G_{\mathbb{Z}}^M \Rightarrow h^2 \in G_{\mathbb{Z}}$. Also, $\operatorname{End}(H_{\mathbb{Q}}, M) = \mathbb{Q}[M]$, and the map

$$\operatorname{End}(H_{\mathbb{Q}}, M) = \mathbb{Q}[M] \to \mathbb{Q}[\lambda_1] \times \mathbb{Q}, \quad p(M) \mapsto (\mu_1, \mu_3)$$

is an isomorphism of $\mathbb{Q}$-algebras. This is a special case of the chinese remainder theorem. Q is mapped to $(-1, 1)$.

Observe $(-\operatorname{id}) \in G_{\mathbb{Z}} \subset G_{\mathbb{Z}}^M$. Therefore the subgroup $\{h \in G_{\mathbb{Z}} \,|\, \mu_3 = 1\}$ has index 2 in $G_{\mathbb{Z}}$, and the subgroup $\{h \in G_{\mathbb{Z}}^M \,|\, \mu_3 = 1\}$ has index 2 in $G_{\mathbb{Z}}^M$. The map

$$\{h \in G_{\mathbb{Z}}^M \,|\, \mu_3 = 1\} \to \mathcal{O}_{\mathbb{Q}[\lambda_1]}^*, \quad h \mapsto \mu_1, \qquad (5.19)$$

is injective. The element $-1 \in \mathcal{O}_{\mathbb{Q}[\lambda_1]}^*$ is in the image of the map in (5.19) if and only if $Q \in G_{\mathbb{Z}}^M$, and then it is the image of Q. By Dirichlet's unit theorem [**BSh73**, Ch. 2 4.3 Theorem 5] the group $\mathcal{O}_{\mathbb{Q}[\lambda_1]}^*$ is isomorphic to the group $\{\pm 1\} \times \mathbb{Z}$. Therefore $\{h \in G_{\mathbb{Z}}^M \,|\, \mu_3 = 1\}$ is isomorphic to $\{\pm 1\} \times \mathbb{Z}$ if $Q \in G_{\mathbb{Z}}^M$ and to $\mathbb{Z}$ if $Q \notin G_{\mathbb{Z}}^M$. If

$Q\in G_{\mathbb Z}^M$ then because of $Q\in \mathrm{Aut}(H_{\mathbb Q},L)$ also $Q\in G_{\mathbb Z}$. This and the implication $h\in G_{\mathbb Z}^M\Rightarrow h^2\in G_{\mathbb Z}$ show

$$[G_{\mathbb Z}^M:G_{\mathbb Z}]=[\{h\in G_{\mathbb Z}^M\,|\,\mu_3=1\}:\{h\in G_{\mathbb Z}\,|\,\mu_3=1\}]\in\{1,2\}.$$

The group $\{\pm M^l\,|\,l\in\mathbb Z\}\subset G_{\mathbb Z}\subset G_{\mathbb Z}^M$ is isomorphic to $\{\pm 1\}\times\mathbb Z$, so it has a free part of rank 1, just as $G_{\mathbb Z}$ and $G_{\mathbb Z}^M$. Therefore $[G_{\mathbb Z}:\{\pm M^l\,|\,l\in\mathbb Z\}]<\infty$. □

Later we will see precisely how much bigger $G_{\mathbb Z}^M$ and $G_{\mathbb Z}$ are than $\{\pm M^l\,|\,l\in\mathbb Z\}$. In the majority of the cases they are not bigger, but $G_{\mathbb Z}^M=G_{\mathbb Z}=\{\pm M^l\,|\,l\in\mathbb Z\}$.

The following theorem determines the groups $G_{\mathbb Z}$ and $G_{\mathbb Z}^M\supset G_{\mathbb Z}$ for three series of triples and the exceptional case $(3,3,4)$. In Theorem 5.18 in section 5.7 we will see that the $\mathrm{Br}_3\ltimes\{\pm 1\}^3$ orbits of these three series and of the triple $(3,3,4)$ are the only triples $\underline{x}$ with $r(\underline{x})\in\mathbb Z_{<0}\cup\mathbb Z_{>4}$, $(H_{\mathbb Z},L,\underline{e})$ irreducible and *not* $G_{\mathbb Z}^M=G_{\mathbb Z}=\{\pm M^l\,|\,l\in\mathbb Z\}$.

THEOREM 5.16. *For each $\underline{x}\in\mathbb Z^3$ below fix also the associated triple $(H_{\mathbb Z},L,\underline{e})$.*

(a) Consider $\underline{x}=(x,x,x)$ with $x\in\mathbb Z-\{-1,0,1,2,3\}$ and $S=S(\underline{x})$. Then $\delta_3\sigma_2\sigma_1\in(\mathrm{Br}_3\ltimes\{\pm1\}^3)_S$ and by Theorem 3.26 (c) (with $k=0$)

$$M^{root,3}:=Z(\delta_3\sigma_2\sigma_1)\in G_{\mathbb Z}\quad\text{with}\quad (M^{root,3})^3=-M.$$

$M^{root,3}$ is cyclic with $M^{root,3}(f_3)=-f_3$. In the case $x=4$ define also

$$M^{root,6}:=-(M^{root,3})^2-2M^{root,3}.$$

Then

$$\begin{array}{rcl} G_{\mathbb Z} &=& G_{\mathbb Z}^M=\{\pm(M^{root,3})^l\,|\,l\in\mathbb Z\}\quad \text{if } x\notin\{4,5\},\\ G_{\mathbb Z} &=& G_{\mathbb Z}^M=\{\mathrm{id},Q\}\times\{\pm(M^{root,3})^l\,|\,l\in\mathbb Z\}\quad\text{if } x=5,\\ G_{\mathbb Z} &=& G_{\mathbb Z}^{(0)}=G_{\mathbb Z}^{(1)}=\{\mathrm{id},Q\}\times\{\pm(M^{root,3})^l\,|\,l\in\mathbb Z\}\\ &\overset{1:2}{\subset}& G_{\mathbb Z}^M=\{\mathrm{id},Q\}\times\{\pm(M^{root,6})^l\,|\,l\in\mathbb Z\}\quad\text{if } x=4. \end{array}$$

If $x=4$ then $(M^{root,6})^2=-M^{root,3}$.

(b) Consider $\underline{x}=(2y,2y,2y^2)$ with $y\in\mathbb Z_{\geq 2}$ and $S=S(\underline{x})$. Then $\sigma_2\sigma_1^2\in(\mathrm{Br}_3\ltimes\{\pm1\}^3)_S$. By Lemma 3.25 (b)

$$M^{root,2}:=Z(\sigma_2\sigma_1^2)\in G_{\mathbb Z}.$$

It satisfies $(M^{root,2})^2=M$ and $M^{root,2}(f_3)=-f_3$. In the case $y=2$ define also

$$M^{root,4}:=-\frac14 M-2M^{root,2}-\frac34\,\mathrm{id}\,.$$

Then

$$\begin{aligned} G_{\mathbb{Z}} &= G_{\mathbb{Z}}^M = \{\pm (M^{root,2})^l \,|\, l\in\mathbb{Z}\} \quad \text{if } y\geq 3,\\ G_{\mathbb{Z}} &= G_{\mathbb{Z}}^{(0)} = G_{\mathbb{Z}}^{(1)} = \{\mathrm{id}, Q\}\times\{\pm (M^{root,2})^l \,|\, l\in\mathbb{Z}\}\\ &\stackrel{1:2}{\subset} G_{\mathbb{Z}}^M = \{\mathrm{id}, Q\}\times\{\pm (M^{root,4})^l \,|\, l\in\mathbb{Z}\} \quad \text{if } y=2. \end{aligned}$$

If $y=2$ then $(M^{root,4})^2 = -M^{root,2}$.

(c) Consider $\underline{x}=(x,x,0)$ with $x\in\mathbb{Z}_{\geq 2}$ and $S=S(\underline{x})$. In the case $x=2$ define also

$$M^{root,2} := \frac{1}{2}M + \frac{1}{2}\,\mathrm{id}\,.$$

Then

$$\begin{aligned} G_{\mathbb{Z}} &= G_{\mathbb{Z}}^M = \{\mathrm{id}, Q\}\times\{\pm M^l \,|\, l\in\mathbb{Z}\} \quad \text{if } x\geq 3,\\ G_{\mathbb{Z}} &= G_{\mathbb{Z}}^{(0)} = G_{\mathbb{Z}}^{(1)} = \{\mathrm{id}, Q\}\times\{\pm M^l \,|\, l\in\mathbb{Z}\}\\ &\stackrel{1:2}{\subset} G_{\mathbb{Z}}^M = \{\mathrm{id}, Q\}\times\{\pm (M^{root,2})^l \,|\, l\in\mathbb{Z}\} \quad \text{if } x=2. \end{aligned}$$

If $x=2$ then $(M^{root,2})^2 = M$.

(d) Consider $\underline{x}=(3,3,4)$. Then

$$G_{\mathbb{Z}} = G_{\mathbb{Z}}^M = \{\mathrm{id}, Q\}\times\{\pm M^l \,|\, l\in\mathbb{Z}\}.$$

(e) In all cases in (a)–(d) except for the four cases

$$\underline{x}\in\{(4,4,4),(5,5,5),(4,4,8),(3,3,4)\}$$

the map $Z:(\mathrm{Br}_3\ltimes\{\pm 1\}^3)_S\to G_{\mathbb{Z}}$ is surjective so $G_{\mathbb{Z}}=G_{\mathbb{Z}}^{\mathcal{B}}$. In the four exceptional cases $Q\in G_{\mathbb{Z}}-G_{\mathbb{Z}}^{\mathcal{B}}$.

Proof: In all cases in this theorem $r(\underline{x})<0$ or $r(\underline{x})>4$, so Lemma 5.15 applies, so $G_{\mathbb{Z}}=G_{\mathbb{Z}}^{(0)}=G_{\mathbb{Z}}^{(1)}$.

(a) S is as in Theorem 3.26 (c) with $k=0$. Therefore $\delta_3\sigma_2\sigma_1$ is in the stabilizer of S and $M^{root,3}$ is in $G_{\mathbb{Z}}$, it is cyclic, and it satisfies $(M^{root,3})^3=-M$. Explicitly (by Theorem 3.26 (a))

$$M^{root,3}(\underline{e}) = \underline{e}\cdot\begin{pmatrix} -x & -x & -1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}.$$

One sees $M^{root,3}(f_3)=-f_3$ where $f_3=-e_1+e_2-e_3$, so its third eigenvalue is $\kappa_3=-1$. The other two eigenvalues κ_1 and κ_2 are determined by the trace $-x=\kappa_1+\kappa_2-1$ and the determinant $-1=\kappa_1\kappa_2(-1)$ of $M^{root,3}$. The eigenvalues are

$$\kappa_{1/2} = \frac{1-x}{2}\pm\frac{1}{2}\sqrt{x^2-2x-3},\quad \kappa_3=-1.$$

Because M and $M^{root,3}$ are regular, Lemma 5.1 applies. It gives an isomorphism of $\mathbb{Q}$-algebras

$$\begin{aligned} &\mathrm{End}(H_{\mathbb{Q}}, M) = \mathrm{End}(H_{\mathbb{Q}}, M^{root,3}) \\ &= \{p(M^{root,3}) \,|\, p(t) = \sum_{i=0}^{2} p_i t^i \in \mathbb{Q}[t]\} \quad \to \quad \mathbb{Q}[\kappa_1] \times \mathbb{Q} \\ &\qquad\qquad\qquad\qquad p(M^{root,3}) \quad \mapsto \quad (p(\kappa_1), p(-1)). \end{aligned}$$

The image of $-\mathrm{id}$ is $(-1,-1)$, the image of Q is $(-1,1)$, the image of $M^{root,3}$ is $(\kappa_1,-1)$. The image of $G_{\mathbb{Z}}^M$ is a priori a subgroup of $\mathcal{O}^*_{\mathbb{Q}[\kappa_1]} \times \{\pm 1\}$. We have to find out which one. By Theorem 5.11 $Q \in G_{\mathbb{Z}}^M$ (and then also $Q \in G_{\mathbb{Z}}$) only for $x \in \{4,5\}$.

Lemma 5.2 applies because $M^{root,3}$ is cyclic. It shows

$$\begin{aligned} G_{\mathbb{Z}}^M &= \{p(M^{root,3}) \,|\, p(t) = p_2t^2 + p_1t + p_0 \in \mathbb{Z}[t] \text{ with} \\ &\qquad (p(\kappa_1), p(-1)) \in \mathbb{Z}[\kappa_1]^* \times \{\pm 1\}\}, \\ G_{\mathbb{Z}} &= \{p(M^{root,3}) \,|\, p(t) = p_2t^2 + p_1t + p_0 \in \mathbb{Z}[t] \text{ with} \\ &\qquad (p(\kappa_1), p(-1)) \in \mathbb{Z}[\kappa_1]^* \times \{\pm 1\} \text{ and } p(\kappa_1)p(\kappa_2) = 1\}. \end{aligned}$$

Now Lemma B.1 (a) is useful. It says

$$\mathbb{Z}[\kappa_1]^* = \begin{cases} \{\pm \kappa_1^l \,|\, l \in \mathbb{Z}\} & \text{for } x \notin \{4,-2\}, \\ \{\pm (\kappa_1 - 1)^l \,|\, l \in \mathbb{Z}\} & \text{for } x = -2, \\ \{\pm (\kappa_1 + 1)^l \,|\, l \in \mathbb{Z}\} & \text{for } x = 4. \end{cases}$$

with

$$\begin{aligned} &(\kappa_1 - 1)(\kappa_2 - 1) = -1 \quad \text{and} \quad (\kappa_1 - 1)^2 = \kappa_1 \quad \text{for} \quad x = -2, \\ &(\kappa_1 + 1)(\kappa_2 + 1) = -1 \quad \text{and} \quad (\kappa_1 + 1)^2 = -\kappa_1 \quad \text{for} \quad x = 4. \end{aligned}$$

For $x \notin \{4,5,-2\}$ the facts $-\mathrm{id}, M^{root,3} \in G_{\mathbb{Z}}$ and $Q \notin G_{\mathbb{Z}}$ show that the image of $G_{\mathbb{Z}}$ and $G_{\mathbb{Z}}^M$ in $\mathbb{Z}[\kappa_1]^* \times \{\pm 1\} = \{\pm \kappa_1^l \,|\, l \in \mathbb{Z}\} \times \{\pm 1\}$ has index 2. Therefore then $G_{\mathbb{Z}} = G_{\mathbb{Z}}^M$ is as claimed.

For $x = 5$ the facts $-\mathrm{id}, M^{root,3}, Q \in G_{\mathbb{Z}}$ show that the image of $G_{\mathbb{Z}}$ and $G_{\mathbb{Z}}^M$ is $\mathbb{Z}[\kappa_1]^* \times \{\pm 1\} = \{\pm \kappa_1^l \,|\, l \in \mathbb{Z}\} \times \{\pm 1\}$. Therefore then $G_{\mathbb{Z}} = G_{\mathbb{Z}}^M$ is as claimed.

Consider the case $x = -2$. If an automorphism $p(M^{root,3})$ is in $G_{\mathbb{Z}}^M$ which corresponds to a pair $(\kappa_1 - 1, \pm 1)$, then $p(\kappa_1) = \kappa_1 - 1$ means $p(t) = t - 1 + l_2(t^2 - 3t + 1)$ for some $l_2 \in \mathbb{Z}$. But then $p(-1) = -2 + l_2 \cdot 5 \notin \{\pm 1\}$. So $G_{\mathbb{Z}}^M$ does not contain such an automorphism. $G_{\mathbb{Z}}^M$ and $G_{\mathbb{Z}}$ are as claimed, because $Q \notin G_{\mathbb{Z}}^M$.

Consider the case $x = 4$. The polynomial $p(t) = -t^2 - 2t$ satisfies $p(-1) = 1$, $p(\kappa_1) = -\kappa_1^2 - 2\kappa_1 = -(-3\kappa_1 - 1) - 2\kappa_1 = \kappa_1 + 1$. Therefore

$M^{root,6} = p(M^{root,3}) \in G_{\mathbb{Z}}^M$. Because κ_1+1 has norm -1, $M^{root,6}$ is not in $G_{\mathbb{Z}}$. The groups $G_{\mathbb{Z}}^M$ and $G_{\mathbb{Z}}$ are as claimed.

(b) Compare the sections 4.1 and 3.2 for the actions of $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ on $\mathbb{Z}^3$ and on $\mathcal{B}^{dist}$.

$$\begin{aligned} & \sigma_2\sigma_1^2(2y,2y,2y^2) \\ = \ & \sigma_2\sigma_1(-2y, 2y^2-2y\cdot 2y, 2y) = \sigma_2\sigma_1(-2y,-2y^2,2y) \\ = \ & \sigma_2(2y, 2y-(-2y)(-2y^2), -2y^2) = \sigma_2(2y, 2y-4y^3, -2y^2) \\ = \ & ((2y-4y^3)-2y\cdot(-2y^2), 2y, 2y^2) = (2y,2y,2y^2), \end{aligned}$$

so $\sigma_2\sigma_1^2$ is in the stabilizer of $(2y,2y,2y^2)$, so $M^{root,2} := Z(\sigma_2\sigma_1^2) \in G_{\mathbb{Z}}$.

$$\begin{aligned} & M^{root,2}(\underline{e}) = \sigma_2\sigma_1^2(\underline{e}) \\ = \ & \sigma_2\sigma_1(s^{(0)}_{e_1}(e_2), e_1, e_3) \\ = \ & \sigma_2\sigma_1(e_2-2ye_1, e_1, e_3) \\ = \ & \sigma_2(s^{(0)}_{e_2-2ye_1}(e_1), e_2-2ye_1, e_3) \\ = \ & \sigma_2(e_1+2y(e_2-2ye_1), e_2-2ye_1, e_3) \\ = \ & ((1-4y^2)e_1+2ye_2, s^{(0)}_{e_2-2ye_1}(e_3), e_2-2ye_1) \\ = \ & ((1-4y^2)e_1+2ye_2, e_3+2y^2(e_2-2ye_1), e_2-2ye_1) \\ = \ & \underline{e}\cdot \begin{pmatrix} 1-4y^2 & -4y^3 & -2y \\ 2y & 2y^2 & 1 \\ 0 & 1 & 0 \end{pmatrix} =: \underline{e}\cdot M^{root,2,mat}. \end{aligned}$$

The map $Z : (\mathrm{Br}_3 \ltimes \{\pm 1\}^3)_S \to G_{\mathbb{Z}}$ in Lemma 3.25 is a group antihomomorphism. By Theorem 3.26 $Z(\sigma^{mon}) = M$. Therefore

$$\begin{aligned} (M^{root,2})^2 &= Z(\sigma_2\sigma_1^2)Z(\sigma_2\sigma_1^2) = Z(\sigma_2\sigma_1(\sigma_1\sigma_2\sigma_1)\sigma_1) \\ &= Z(\sigma_2\sigma_1(\sigma_2\sigma_1\sigma_2)\sigma_1) = Z((\sigma_2\sigma_1)^3) = Z(\sigma^{mon}) = M. \end{aligned}$$

One sees $M^{root,2}(f_3) = -f_3$, where $f_3 = -ye_1+e_2-e_3$, so its third eigenvalue is $\kappa_3 = -1$. The other two eigenvalues κ_1 and κ_2 are determined by the trace $1-2y^2 = \kappa_1+\kappa_2-1$ and the product $\kappa_1\kappa_2 = 1$, which holds because of $M^{root,2} \in G_{\mathbb{Z}}$ (or because $\det M^{root,2} = -1$). The eigenvalues are

$$\kappa_{1,2} = (1-y^2) \pm y\sqrt{y^2-2}, \quad \kappa_3 = -1.$$

Because M and $M^{root,2}$ are regular, Lemma 5.1 applies. It gives an isomorphism of $\mathbb{Q}$-algebras

$$\begin{aligned} &\mathrm{End}(H_\mathbb{Q}, M) = \mathrm{End}(H_\mathbb{Q}, M^{root,2}) \\ &= \{p(M^{root,2}) \,|\, p(t) = \sum_{i=0}^{2} p_i t^i \in \mathbb{Q}[t]\} \quad \to \quad \mathbb{Q}[\kappa_1] \times \mathbb{Q} \\ &\qquad\qquad\qquad\qquad p(M^{root,2}) \quad \mapsto \quad (p(\kappa_1), p(-1)). \end{aligned}$$

The image of $-\mathrm{id}$ is $(-1,-1)$, the image of Q is $(-1,1)$, the image of $M^{root,2}$ is $(\kappa_1,-1)$. The image of $G_\mathbb{Z}^M$ is a priori a subgroup of $\mathcal{O}^*_{\mathbb{Q}[\kappa_1]} \times \{\pm 1\}$. We have to find out which one. By Theorem 5.11 $Q \in G_\mathbb{Z}^M$ (and then also $Q \in G_\mathbb{Z}$) only for $y = 2$.

Consider the decomposition $H_\mathbb{Q} = H_{\mathbb{Q},1} \oplus H_{\mathbb{Q},2}$ as in Definition 5.9 and the primitive sublattices $H_{\mathbb{Z},1} = H_{\mathbb{Q},1} \cap H_\mathbb{Z}$ and $H_{\mathbb{Z},2} = H_{\mathbb{Q},2} \cap H_\mathbb{Z} = \mathbb{Z} f_3$ in $H_\mathbb{Z}$. The sublattice $H_{\mathbb{Z},1}$ is the right L-orthogonal subspace $(\mathbb{Z} f_3)^\perp \subset H_\mathbb{Z}$ of $\mathbb{Z} f_3$, see (5.8):

$$\begin{aligned} H_{\mathbb{Z},1} &= \{\underline{e} \cdot \underline{z}^t \,|\, \underline{z} \in \mathbb{Z}^3, 0 = (-y, 1-2y^2, -1) \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \} \\ &= \langle f_1, f_2 \rangle_\mathbb{Z} \quad \text{with} \quad f_1 = e_1 - y e_3, \quad f_2 = e_2 + (1-2y^2) e_3. \end{aligned}$$

Write $\underline{f} = (f_1, f_2, f_3) = \underline{e} \cdot M(\underline{e}, \underline{f})$,

$$M^{root,2}(\underline{e}) = \underline{e} \cdot M^{root,2,mat}, \quad M^{root,2}(\underline{f}) = \underline{f} M^{root,2,mat,\underline{f}}.$$

Then

$$\begin{aligned} M(\underline{e}, \underline{f}) &= \begin{pmatrix} 1 & 0 & -y \\ 0 & 1 & 1 \\ -y & 1-2y^2 & -1 \end{pmatrix}, \\ M(\underline{e}, \underline{f})^{-1} &= \frac{1}{y^2-2} \begin{pmatrix} 2y^2-2 & 2y^3-y & y \\ -y & -y^2-1 & -1 \\ y & 2y^2-1 & 1 \end{pmatrix}, \end{aligned}$$

$$M^{root,2,mat,\underline{f}} = M(\underline{e}, \underline{f})^{-1} M^{root,2,mat} M(\underline{e}, \underline{f}) = \begin{pmatrix} -2y^2+1 & -2y & 0 \\ y & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Any element $h = p(M) \in G_\mathbb{Z}^M$ with $p(t) \in \mathbb{Q}[t]$ restricts to an automorphism of $H_{\mathbb{Z},1}$ which commutes with $M^{root,2}|_{H_{\mathbb{Z},1}}$, so its restriction to $H_{\mathbb{Z},1}$ has the shape $h|_{H_{\mathbb{Z},1}} = a\,\mathrm{id} + b M^{root,2}|_{H_{\mathbb{Z},1}}$ with $a, b \in \mathbb{Q}$ with

$$a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} -2y^2+1 & -2y \\ y & 1 \end{pmatrix} \in GL_2(\mathbb{Z}).$$

This implies $by\in\mathbb{Z}$ and $a+b\in\mathbb{Z}$. The eigenvalue $p(\kappa_1)$ is

$$\begin{aligned} & a+b\kappa_1 = a+b(1-y^2+y\sqrt{y^2-2}) \\ = & \ (a+b)+by(-y+\sqrt{y^2-2})\in\mathbb{Z}[\sqrt{y^2-2}]^*. \end{aligned}$$

Now Lemma B.1 (b) is useful. It says

$$\mathbb{Z}[\sqrt{y^2-2}]^* = \begin{cases} \{\pm\kappa_1^l\,|\,l\in\mathbb{Z}\} & \text{if } y\geq 3, \\ \{\pm(1+\sqrt{2})^l\,|\,l\in\mathbb{Z}\} & \text{if } y=2. \end{cases}$$

Furthermore κ_1 has norm 1, $1+\sqrt{2}$ has norm -1, and $(1+\sqrt{2})^2=-\kappa_2$ if $y=2$.

Consider the cases $y\geq 3$. The map

$$G_{\mathbb{Z}}^M\to\mathcal{O}^*_{\mathbb{Q}[\kappa_1]}\times\{\pm 1\},\quad p(M^{root,2})\mapsto (p(\kappa_1),p(-1)),$$

has because of $Q\notin G_{\mathbb{Z}}^M$ as image the index 2 subgroup of $\{\pm\kappa_1^l\,|\,l\in\mathbb{Z}\}\times\{\pm 1\}$ which is generated by $(\kappa_1,-1)$ and $(-1,-1)$, because $Q\notin G_{\mathbb{Z}}^M$. Therefore then $G_{\mathbb{Z}}^M=G_{\mathbb{Z}}=\{\pm(M^{root,2})^l\,|\,l\in\mathbb{Z}\}$.

Consider the case $y=2$. Then $Q\in G_{\mathbb{Z}}\subset G_{\mathbb{Z}}^M$. Therefore then $G_{\mathbb{Z}}=\{\mathrm{id},Q\}\times\{\pm(M^{root,2})^l\,|\,l\in\mathbb{Z}\}$. The question remains whether $1+\sqrt{2}$ arises as eigenvalue $p(\kappa_1)$ for an element $p(M^{root,2})\in G_{\mathbb{Z}}^M$. It does. $M^{root,4}$ has the first eigenvalue

$$-\frac{1}{4}(-3+2\sqrt{2})^2-2(-3+2\sqrt{2})-\frac{3}{4}=1-\sqrt{2}$$

with $(1-\sqrt{2})^2=3-2\sqrt{2}=-\kappa_1$ and the third eigenvalue $-\frac{1}{4}-2(-1)-\frac{3}{4}=1$. Therefore $(M^{root,4})^2=-M^{root,2}$. $M^{root,4}$ is in $G_{\mathbb{Z}}^M$ because

$$\begin{aligned} M^{root,4}(\underline{e}) &= \underline{e}\left(-\frac{1}{4}\begin{pmatrix} 97 & 220 & 28 \\ -28 & -63 & -8 \\ 4 & 8 & 1 \end{pmatrix} - 2\begin{pmatrix} -15 & -32 & -4 \\ 4 & 8 & 1 \\ 0 & 1 & 0 \end{pmatrix} - \frac{3}{4}E_3\right) \\ &= \underline{e}\begin{pmatrix} 5 & 9 & 1 \\ -1 & -1 & 0 \\ -1 & -4 & -1 \end{pmatrix}. \end{aligned}$$

Therefore then $G_{\mathbb{Z}}^M=\{\mathrm{id},Q\}\times\{\pm(M^{root,4})^l\,|\,l\in\mathbb{Z}\}$.

(c) Observe $r=2x^2$. M has the eigenvalues $\lambda_1,\lambda_2,\lambda_3$ with

$$\lambda_{1/2}=(1-x^2)\pm x\sqrt{x^2-2},\quad \lambda_3=1.$$

Because M is regular, Lemma 5.1 applies. It gives an isomorphism of $\mathbb{Q}$-algebras

$$\begin{aligned} & \mathrm{End}(H_{\mathbb{Q}},M) \\ = \{p(M)\,|\,p(t)=p_2t^2+p_1t+p_0\in\mathbb{Q}[t]\} & \to \mathbb{Q}[\lambda_1]\times\mathbb{Q} \\ p(M) & \mapsto (p(\lambda_1),p(1)). \end{aligned}$$

The image of $-\mathrm{id}$ is $(-1,-1)$, the image of Q is $(-1,1)$, the image of M is $(\lambda_1, 1)$. The image of $G_{\mathbb{Z}}^M$ is a priori a subgroup of $\mathcal{O}^*_{\mathbb{Q}[\lambda_1]}\times\{\pm 1\}$. We have to find out which one. By Theorem 5.11 $Q\in G_{\mathbb{Z}}\subset G_{\mathbb{Z}}^M$.

Consider the decomposition $H_{\mathbb{Q}}=H_{\mathbb{Q},1}\oplus H_{\mathbb{Q},2}$ as in Definition 5.9 and the primitive sublattices $H_{\mathbb{Z},1}=H_{\mathbb{Q},1}\cap H_{\mathbb{Z}}$ and $H_{\mathbb{Z},2}=H_{\mathbb{Q},2}\cap H_{\mathbb{Z}}=\mathbb{Z}f_3$ in $H_{\mathbb{Z}}$. The sublattice $H_{\mathbb{Z},1}$ is the right L-orthogonal subspace $(\mathbb{Z}f_3)^\perp\subset H_{\mathbb{Z}}$ of $\mathbb{Z}f_3$, see (5.8):

$$\begin{aligned} H_{\mathbb{Z},1} &= \{\underline{e}\cdot \underline{z}^t\,|\,\underline{z}\in\mathbb{Z}^3, 0=(0,1,-1)\begin{pmatrix} z_1\\ z_2\\ z_3\end{pmatrix}\}\\ &= \langle f_1,f_2\rangle_{\mathbb{Z}} \quad\text{with}\quad f_1=e_1,\quad f_2=e_2+e_3. \end{aligned}$$

Write $\underline{f}=(f_1,f_2,f_3)=\underline{e}\cdot M(\underline{e},\underline{f})$,

$$M(\underline{e})=\underline{e}\cdot M^{mat},\quad M(\underline{f})=\underline{f}M^{mat,\underline{f}}.$$

Then

$$\begin{aligned} M(\underline{e},\underline{f}) &= \begin{pmatrix}1&0&0\\0&1&1\\0&1&-1\end{pmatrix},\quad M(\underline{e},\underline{f})^{-1}=\frac{1}{2}\begin{pmatrix}2&0&0\\0&1&1\\0&1&-1\end{pmatrix},\\ M^{mat} &= \begin{pmatrix}1-2x^2&-x&-x\\x&1&0\\x&0&1\end{pmatrix},\\ M^{mat,\underline{f}} &= M(\underline{e},\underline{f})^{-1}M^{mat}M(\underline{e},\underline{f})=\begin{pmatrix}1-2x^2&-2x&0\\x&1&0\\0&0&1\end{pmatrix}. \end{aligned}$$

The upper left 2×2-matrix in $M^{mat,\underline{f}}$ coincides after identification of x and y with the upper left 2×2-matrix in $M^{root,2,mat,\underline{f}}$ in the proof of part (b). Therefore we can argue exactly as in the proof of part (b). Lemma B.1 (b) applies in the same way.

We obtain for $x\geq 3$ $G_{\mathbb{Z}}^M=G_{\mathbb{Z}}=\{\mathrm{id},Q\}\times\{\pm M^l\,|\,l\in\mathbb{Z}\}$ and for $x=2$ $G_{\mathbb{Z}}=\{\mathrm{id},Q\}\times\{\pm M^l\,|\,l\in\mathbb{Z}\}$.

Consider the case $x=2$. Then $M^{root,2}$ has the first eigenvalue

$$\frac{1}{2}\lambda_1+\frac{1}{2}=-1+\sqrt{2}$$

with $(-1+\sqrt{2})^2=3-2\sqrt{2}=-\lambda_1$ and the third eigenvalue $\frac{1}{2}+\frac{1}{2}=1$. Therefore $(M^{root,2})^2=QM$. $M^{root,2}$ is in $G_{\mathbb{Z}}^M$ because

$$M^{root,2}(\underline{e}) = \underline{e}\left(\frac{1}{2}\begin{pmatrix}-7&-2&-2\\2&1&0\\2&0&1\end{pmatrix}+\frac{1}{2}E_3\right)=\underline{e}\begin{pmatrix}-3&-1&-1\\1&1&0\\1&0&1\end{pmatrix}.$$

Therefore $G_{\mathbb{Z}}^M = \{\mathrm{id}, Q\} \times \{\pm (M^{root,2})^l \,|\, l \in \mathbb{Z}\}$ for $x = 2$.

(d) Here $r = -2$. M has the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with

$$\lambda_{1/2} = 2 \pm \sqrt{3}, \quad \lambda_3 = 1.$$

It is well known and can be seen easily either elementarily or with Theorem B.6 that

$$\mathcal{O}^*_{\mathbb{Q}[\lambda_1]} = \{\pm \lambda_1^l \,|\, l \in \mathbb{Z}\}.$$

$Q \in G_{\mathbb{Z}} \subset G_{\mathbb{Z}}^M$ by Theorem 5.11. Recall the proof of Lemma 5.15. The restriction of the map in (5.19) to the map

$$\{\mathrm{id}, Q\} \times \{M^l \,|\, l \in \mathbb{Z}\} \to \mathcal{O}^*_{\mathbb{Q}[\lambda_1]}$$

is an isomorphism. Therefore the map in (5.19) is an isomorphism and

$$G_{\mathbb{Z}}^M = G_{\mathbb{Z}} = \{\mathrm{id}, Q\} \times \{\pm M^l \,|\, l \in \mathbb{Z}\}.$$

(e) Observe in part (c)

$$\begin{aligned} Z(\sigma_2)(\underline{e}) &= \sigma_2(\underline{e}) = (e_1, e_3, e_2), \\ \text{so } Z(\sigma_2)(f_1, f_2, f_3) &= (f_1, f_2, -f_3), \\ \text{so } Z(\sigma_2) &= -Q. \end{aligned}$$

Now in all cases

$$-\mathrm{id} = Z(\delta_1\delta_2\delta_3), \quad M = Z(\sigma^{mon})$$

and

$$\text{in part (a): } \begin{cases} M^{root,3} = Z(\delta_3\sigma_2\sigma_1), \\ G_{\mathbb{Z}} = \{\pm (M^{root,3})^l \,|\, l \in \mathbb{Z}\} \text{ for } x \notin \{4,5\}, \end{cases}$$

$$\text{in part (b): } \begin{cases} M^{root,2} = Z(\sigma_2\sigma_1^2), \\ G_{\mathbb{Z}} = \{\pm (M^{root,2})^l \,|\, l \in \mathbb{Z}\} \text{ for } y \neq 2, \end{cases}$$

$$\text{in part (c): } \begin{cases} Q = Z(\delta_1\delta_2\delta_3\sigma_2), \\ G_{\mathbb{Z}} = \{\mathrm{id}, Q\} \times \{\pm M^l \,|\, l \in \mathbb{Z}\}. \end{cases}$$

This shows $G_{\mathbb{Z}} = G_{\mathbb{Z}}^{\mathcal{B}}$ in all but the four cases $\underline{x} \in \{(4,4,4), (5,5,5), (4,4,8), (3,3,4)\}$. In these four cases $Q \in G_{\mathbb{Z}}$. It remains to see $Q \notin G_{\mathbb{Z}}^{\mathcal{B}}$. We offer two proofs.

First proof: It uses that in these four cases the stabilizer of $\underline{e}$ in $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ is $\{\mathrm{id}\}$, which will be proved as part of Theorem 7.11. It also follows from $\Gamma^{(1)} = G^{free,3}$ in Theorem 6.18 (g) or $\Gamma^{(0)} = G^{fCox,3}$ in Theorem 6.11 (g) and from Example 3.4 (respectively Theorem 3.2 (a) or (b)). This implies that here $Z : (\mathrm{Br}_3 \ltimes \{\pm 1\}^3)_S \to G_{\mathbb{Z}}$ is injective. Observe furthermore $Q^2 = \mathrm{id}$. If $Q = Z(\beta)$ for some braid β, then $\beta^2 = \mathrm{id}$ as Z is a group antihomomorphism. But there is no braid of order two.

Second proof: By formula (5.5) in Theorem 5.11 in the four cases

$$\begin{aligned}
Q(\underline{e}) &= -\underline{e} + 2f_3(1,3,1) \quad \text{in the case } \underline{x} = (4,4,4),\\
Q(\underline{e}) &= -\underline{e} + f_3(1,4,1) \quad \text{in the case } \underline{x} = (5,5,5),\\
Q(\underline{e}) &= -\underline{e} + f_3(2,7,1) \quad \text{in the case } \underline{x} = (4,4,8),\\
Q(\underline{e}) &= -\underline{e} + f_3(4,9,3) \quad \text{in the case } \underline{x} = (3,3,4).
\end{aligned}$$

By Theorem 6.21 (g) the restriction to $\Delta^{(1)}$ of the projection $\mathrm{pr}^{H,(1)} : H_{\mathbb{Z}} \to \overline{H_{\mathbb{Z}}}^{(1)}$ is injective. Therefore in all four cases $Q(e_i) \notin \Delta^{(1)}$ for $i \in \{1,2,3\}$. But any automorphism in $G_{\mathbb{Z}}^{\mathcal{B}}$ maps each e_i to an odd vanishing cycle. Thus $Q \notin G_{\mathbb{Z}}^{\mathcal{B}}$. □

5.7. General rank 3 cases with eigenvalues not all in S^1

Theorem 5.18 below will show $G_{\mathbb{Z}} = G_{\mathbb{Z}}^M = \{\pm M^l \,|\, l \in \mathbb{Z}\}$ in all irreducible rank 3 with eigenvalues not all in S^1 which have not been treated in Theorem 5.16. This result is simple to write down, but the proof is long. It is a case discussion with many subcases. It builds on part (b) of the technical Lemma 5.17 which gives necessary and sufficient conditions when an endomorphism in $\mathrm{End}(H_{\mathbb{Q}})$ of a certain shape is in $\mathrm{End}(H_{\mathbb{Z}})$.

Any element of $\{h \in G_{\mathbb{Z}}^M \,|\, \mu_3 = 1\}$ can be written as $h = q(M)$ with $q(t) = 1 + q_0(t-1) + q_1(t-1)^2$ with unique coefficients $q_0, q_1 \in \mathbb{Q}$, but not all values $q_0, q_1 \in \mathbb{Q}$ give such an element. Part (b) of the following lemma says which integrality conditions on q_0 and q_1 are necessary for $q(M) \in \mathrm{End}(H_{\mathbb{Z}})$. Part (a) is good to know in this context.

LEMMA 5.17. *Fix $\underline{x} \in \mathbb{Z}^3 - \{(0,0,0)\}$ and the associated triple $(H_{\mathbb{Z}}, L, \underline{e})$. Recall $g = \gcd(x_1,x_2,x_3)$ and $\widetilde{x}_i = g^{-1}x_i$. Define*

$$\begin{aligned}
g_1 &:= \gcd(2x_1 - x_2x_3, 2x_2 - x_1x_3, 2x_3 - x_1x_2) \in \mathbb{N} \cup \{0\},\\
g_2 &:= \frac{g_1}{g} = \gcd(2\widetilde{x}_1 - g\widetilde{x}_2\widetilde{x}_3, 2\widetilde{x}_2 - g\widetilde{x}_1\widetilde{x}_3, 2\widetilde{x}_3 - g\widetilde{x}_1\widetilde{x}_2) \in \mathbb{N} \cup \{0\}.
\end{aligned} \tag{5.20}$$

(a) We separate three cases.
(i) Case (three or two of x_1, x_2, x_3 are odd): Then g and g_2 are odd and

$$\gcd(g_2, \widetilde{x}_i) = \gcd(g_2, g) = 1, \quad g_2^2 \,|\, (r-4).$$

(ii) Case (exactly one of x_1, x_2, x_3 is odd): Then g is odd, $g_2 \equiv 2(4)$ and

$$\gcd(\frac{g_2}{2}, \widetilde{x}_i) = \gcd(\frac{g_2}{2}, g) = 1, \quad (\frac{g_2}{2})^2 \,|\, (r-4).$$

(iii) Case (none of x_1, x_2, x_3 is odd): Then g and g_2 are even. More precisely, $g_2 \equiv 0(4)$ only if $\frac{g}{2}$ and $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3$ are odd. Else $g_2 \equiv 2(4)$. Always

$$\gcd(\frac{g_2}{2}, \widetilde{x}_i) = \gcd(\frac{g_2}{2}, \frac{g}{2}) = 1, \quad g_2^2 \,|\, (r-4).$$

(b) Consider $q_0, q_1 \in \mathbb{Q}$, $q(t) := 1 + q_0(t-1) + q_1(t-1)^2 \in \mathbb{Q}[t]$ and $h = q(M) \in \mathbb{Q}[M]$. Define $q_2 := q_0 - 2q_1 \in \mathbb{Q}$.

Then $h \in \mathrm{End}(H_{\mathbb{Z}}, M)$ if and only if the following integrality conditions (5.21)–(5.24) are satisfied.

$$q_2 \cdot g^2 \in \mathbb{Z}, \tag{5.21}$$

$$q_1 \cdot g g_1 \in \mathbb{Z}, \tag{5.22}$$

$$q_0 x_i - q_1 x_j x_k \in \mathbb{Z} \quad \text{for} \quad \{i,j,k\} = \{1,2,3\}, \tag{5.23}$$

$$q_1(x_i^2 - x_j^2) \in \mathbb{Z} \quad \text{for} \quad \{i,j,k\} = \{1,2,3\}. \tag{5.24}$$

If these conditions hold, then also the following holds,

$$q_0 \cdot g_1 \in \mathbb{Z}. \tag{5.25}$$

(c) In part (b) the eigenvalue of $q(M)$ on $H_{\mathbb{C},\lambda_1}$ is

$$\begin{aligned} \mu_1 &:= q(\lambda_1) = (1 - rq_1) + (q_0 - rq_1)(\lambda_1 - 1) \\ &= (1 - q_0) + (q_0 - rq_1)\lambda_1. \end{aligned}$$

Proof: (a) g is odd in the cases (i) and (ii) and even in case (iii). Therefore g_2 is odd in case (i) and even in the cases (ii) and (iii), and furthermore $\frac{g_2}{2}$ is odd in case (ii). Also $\frac{g_2}{2}$ odd in case (iii) almost always, namely except when $\frac{g}{2}$ and $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3$ are odd, as can be seen from the definition (5.20) of g_2. Here observe that at least one of $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3$ is odd because $\gcd(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) = 1$.

Now we consider first case (iii). A common divisor of $\frac{g_2}{2}$ and $\widetilde{x}_1$ would be odd. Because of the second term $2\widetilde{x}_2 - g\widetilde{x}_1\widetilde{x}_3$ and the third term $2\widetilde{x}_3 - g\widetilde{x}_1\widetilde{x}_2$ in (5.20) it would also divide $\widetilde{x}_2$ and $\widetilde{x}_3$. This is impossible because of $\gcd(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) = 1$. Therefore $\gcd(\frac{g_2}{2}, \widetilde{x}_1) = 1$. Analogously for $\widetilde{x}_2$ and $\widetilde{x}_3$.

A common divisor of $\frac{g_2}{2}$ and $\frac{g}{2}$ would be odd. Because of all three terms in (5.20) it would divide $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3$. Therefore $\gcd(\frac{g_2}{2}, \frac{g}{2}) = 1$.

For i,j,k with $\{i,j,k\} = \{1,2,3\}$ observe

$$2(2\widetilde{x}_i - g\widetilde{x}_j\widetilde{x}_k) + g\widetilde{x}_k(2\widetilde{x}_j - g\widetilde{x}_i\widetilde{x}_k) = \widetilde{x}_i(4 - x_k^2), \tag{5.26}$$

$$4(r-4) = g^2(2\widetilde{x}_i - g\widetilde{x}_j\widetilde{x}_k)^2 - (4 - x_j^2)(4 - x_k^2). \tag{5.27}$$

(5.26) and $\gcd(\frac{g_2}{2}, \widetilde{x}_i) = 1$ imply in case (iii) that $\frac{g_2}{2}$ divides $4^{-1}(4 - x_k^2)$. This and (5.27) imply that $(\frac{g_2}{2})^2$ divides $\frac{r-4}{4}$, so g_2^2 divides $r-4$.

The claims for the cases (i) and (ii) follow similarly.

(b) Recall the shape of $M^{mat} \in M_{3\times 3}(\mathbb{Z})$ with $M\underline{e} = \underline{e}M^{mat}$ from the beginning of Section 5.3. It gives

$$M^{mat} - E_3 = \begin{pmatrix} -x_1^2 - x_2^2 + x_1x_2x_3 & -x_1 - x_2x_3 + x_1x_3^2 & x_1x_3 - x_2 \\ x_1 - x_2x_3 & -x_3^2 & -x_3 \\ x_2 & x_3 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} (M^{mat} - E_3) = \begin{pmatrix} 0 & -x_1 & -x_2 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{pmatrix},$$

$$(M^{mat} - E_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_2 & -x_3 & 1 \end{pmatrix} = \begin{pmatrix} -x_1^2 & -x_1 & x_1x_3 - x_2 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{pmatrix}.$$

Now

$$q(M) \in \mathrm{End}(H_\mathbb{Z}) \iff q(M^{mat}) - E_3 \in M_{3\times 3}(\mathbb{Z}),$$

and this is equivalent to the following matrix being in $M_{3\times 3}(\mathbb{Z})$,

$$\begin{aligned} &\begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} (q(M^{mat}) - E_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_2 & -x_3 & 1 \end{pmatrix} \\ = &\begin{pmatrix} 0 & -x_1 & -x_2 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{pmatrix} \\ &\cdot \left[q_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_2 & -x_3 & 1 \end{pmatrix} + q_1 \begin{pmatrix} -x_1^2 & -x_1 & x_1x_3 - x_2 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{pmatrix} \right] \\ = \; & q_0 \begin{pmatrix} x_2^2 & -x_1 + x_2x_3 & -x_2 \\ x_1 + x_2x_3 & x_3^2 & -x_3 \\ x_2 & x_3 & 0 \end{pmatrix} \\ + \; & q_1 \begin{pmatrix} -x_1^2 - x_2^2 & -x_2x_3 & x_1x_3 \\ -x_1^3 - x_2x_3 & -x_1^2 - x_3^2 & x_1^2x_3 - x_1x_2 \\ -x_1^2x_2 + x_1x_3 & -x_1x_2 & x_1x_2x_3 - x_2^2 - x_3^2 \end{pmatrix}. \end{aligned}$$

This gives nine scalar conditions, which we denote by their place $[a, b]$ with $a, b \in \{1, 2, 3\}$ in the matrix, so for example $[2, 1]$ is the condition $q_0(x_1 + x_2x_3) + q_1(-x_1^3 - x_2x_3) \in \mathbb{Z}$. These nine conditions are sufficient and necessary for $q(M) \in \mathrm{End}(H_\mathbb{Z})$.

The following trick allows an easy derivation of implied conditions. Recall the cyclic action $\gamma : \mathbb{Z}^3 \to \mathbb{Z}^3, \underline{x} \mapsto (x_3, x_1, x_2)$. It lifts to an action of $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ on triangular bases of $(H_\mathbb{Z}, L)$. Therefore together with $M^{mat} = S(\underline{x})^{-1}S(\underline{x})^t$ also the matrix $\widetilde{M}^{mat} := S(\gamma(\underline{x}))^{-1}S(\gamma(\underline{x}))^t$

is a monodromy matrix. Integrality of $q(M^{mat})$ is equivalent to integrality of $q(\widetilde{M}^{mat})$. Therefore if the nine conditions hold, also the conditions hold which are obtained from the nine conditions by replacing (x_1,x_2,x_3) by (x_3,x_1,x_2) or by (x_2,x_3,x_1). In the following $[a,b]$ denotes all three so obtained conditions, so for example $[2,1]$ denotes the conditions

$$q_0(x_i+x_jx_k)+q_1(-x_i^3-x_jx_k)\in\mathbb{Z}$$
$$\text{for } (i,j,k)\in\{(1,2,3),(3,1,2),(2,3,1)\}.$$

We have to show the following equivalence:

$$\text{the conditions } [a,b] \text{ for } a,b\in\{1,2,3\} \iff \text{the conditions (5.21)} - \text{(5.24)}.$$

$\Longrightarrow$: $[1,3]$ and $[3,2]$ are equivalent to one another and to (5.23).

$[3,3]$ says $q_1(x_i^2-r)\in\mathbb{Z}$. One derives $q_1(x_i^2-x_j^2)\in\mathbb{Z}$, which is (5.24).

$[1,1]$ and (5.24) give $q_2x_i^2\in\mathbb{Z}$, so $q_2\gcd(x_1^2,x_2^2,x_3^2)=q_2g^2\in\mathbb{Z}$ which is (5.21).

The derivation of $q_1gg_1\in\mathbb{Z}$ is laborious and goes as follows:
$[3,1]\&[1,3]$ imply $q_1x_i(2x_k-x_ix_j)\in\mathbb{Z}$.
$[3,2]\&[2,3]$ imply $q_1x_i(2x_j-x_ix_k)\in\mathbb{Z}$.
$[3,3]\&(5.24)$ imply $q_1x_i(2x_i-x_jx_k)\in\mathbb{Z}$.
One sees $q_1x_ig_1\in\mathbb{Z}$ and then $q_1gg_1\in\mathbb{Z}$.

$\Longleftarrow$: (5.23) gives $[1,3]$ and $[3,2]$.

(5.21) and (5.24) give $[1,1]$ and $[2,2]$.

(5.23) reduces $[1,2]$, $[2,1]$, $[2,3]$ and $[3,1]$ to $q_2x_jx_k$, $q_0x_jx_k-q_1x_i^3$, $q_1x_i(x_ix_k-2x_j)$ and $q_1x_i(-x_ix_j+2x_k)$. The first follows from (5.21), the third and fourth follow from (5.22). The second reduces with (5.24) to $x_j(q_0x_k-q_1x_ix_j)$, which follows from (5.23).

$[3,3]$ reduces with (5.24) to $q_1x_j(x_ix_k-2x_j)$ which follows from (5.22).

The equivalence of the conditions $[a,b]$ with the conditions (5.21)–(5.24) is shown.

It remains to show how (5.21)-(5.24) imply (5.25). One combines two times (5.23), $2q_0x_i-2q_1x_jx_k\in\mathbb{Z}$, with (5.21), $(q_0-2q_1)x_jx_k\in\mathbb{Z}$, and obtains $q_0(2x_i-x_jx_k)\in\mathbb{Z}$.

(c) Recall

$$(t-\lambda_1)(t-\lambda_2)=t^2-(2-r)t+1,$$
$$\text{so}\quad \lambda_1+\lambda_2=2-r,\quad \lambda_1\lambda_2=1,$$
$$\lambda_1^2=(2-r)\lambda_1-1,\quad (\lambda_1-1)^2=(-r)\lambda_1,$$

so

$$\begin{aligned}\mu_1 &= 1 + q_0(\lambda_1 - 1) + q_1(\lambda_1 - 1)^2 \\ &= (1 - rq_1) + (q_0 - rq_1)(\lambda_1 - 1). \qquad \square\end{aligned}$$

THEOREM 5.18. *Consider a triple* $\underline{x} \in \mathbb{Z}^3$ *with* $r(\underline{x}) \in \mathbb{Z}_{<0} \cup \mathbb{Z}_{>4}$ *which is neither reducible nor in the* $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ *orbit of a triple in Theorem 5.16. More explicitly, the triple* $\underline{x}$ *is any triple in* $\mathbb{Z}^3$ *which is not in the* $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ *orbits of the triples in the following set,*

$$\begin{aligned}&\{(x,0,0)\,|\,x \in \mathbb{Z}\} \cup \{(x,x,x)\,|\,x \in \mathbb{Z}\} \cup \{(2y,2y,2y^2\,|\,y \in \mathbb{Z}_{\geq 2}\} \\ &\quad \cup \{(x,x,0)\,|\,x \in \mathbb{Z}\} \cup \{(-l,2,-l)\,|\,l \in \mathbb{Z}_{\geq 3}\} \cup \{(3,3,4)\}.\end{aligned}$$

Consider the associated triple $(H_{\mathbb{Z}}, L, \underline{e})$ *with* $L(\underline{e}^t, \underline{e})^t = S(\underline{x})$*. Then*

$$G_{\mathbb{Z}} = G_{\mathbb{Z}}^M = \{\pm M^l \,|\, l \in \mathbb{Z}\}.$$

The map $Z : (\mathrm{Br}_3 \ltimes \{\pm 1\}^3)_S \to G_{\mathbb{Z}}$ *is surjective, so* $G_{\mathbb{Z}} = G_{\mathbb{Z}}^{\mathcal{B}}$*.*

Proof: The surjectivity of Z follows from $G_{\mathbb{Z}} = \{\pm M^l \,|\, l \in \mathbb{Z}\}$, $-\mathrm{id} = Z(\delta_1\delta_2\delta_3)$ and $M = Z(\sigma^{mon})$. The main point is to prove $G_{\mathbb{Z}}^M = \{\pm M^l \,|\, l \in \mathbb{Z}\}$.

Theorem 5.11 says for which $\underline{x}$ the automorphism Q of $H_{\mathbb{Q}}$ in Definition 5.9 is in $G_{\mathbb{Z}}^M$. They are all excluded here. So here $Q \notin G_{\mathbb{Z}}^M$.

We use the notation $g = g(\underline{x})$ in Lemma 5.10 and the notations from the beginning of section 5.3. Especially $r := r(\underline{x}) \in \mathbb{Z}_{<0} \cup \mathbb{Z}_{>4}$, and $\lambda_3 = 1$ and $\lambda_{1/2} = \frac{2-r}{2} \pm \sqrt{r(r-4)}$ are the eigenvalues of the monodromy.

The proof of Lemma 5.15 gives a certain control on $G_{\mathbb{Z}}^M$ and $G_{\mathbb{Z}}$. Recall the notations there. For $p(t) = \sum_{i=0}^2 p_i t^i \in \mathbb{Q}[t]$ write $\mu_j := p(\lambda_j)$ for the eigenvalues of $p(M) \in \mathrm{End}(H_{\mathbb{Q}})$. Recall (5.17), (5.18), the isomorphism of $\mathbb{Q}$-algebras

$$\begin{aligned}\mathrm{End}(H_{\mathbb{Q}}, M) = \{p(M)\,|\,p(t) = \sum_{i=0}^2 p_i t^i \in \mathbb{Q}[t]\} &\to \mathbb{Q}[\lambda_1] \times \mathbb{Q}, \\ p(M) &\mapsto (p(\lambda_1), p(1)),\end{aligned}$$

and its restriction in (5.19), the injective group homomorphism

$$\{h \in G_{\mathbb{Z}}^M \,|\, \mu_3 = 1\} \to \mathcal{O}_{\mathbb{Q}[\lambda_1]}^*, \quad h = p(M) \mapsto \mu_1 = p(\lambda_1). \qquad (5.28)$$

The image of $Q \in \mathrm{End}(H_{\mathbb{Q}}, M)$ in $\mathbb{Q}[\lambda_1] \times \mathbb{Q}$ is $(-1, 1)$. Because $Q \notin G_{\mathbb{Z}}^M$, the image in (5.28) does not contain -1, so it is a cyclic group. It contains λ_1 which is the image of M. Therefore the group $\{h \in G_{\mathbb{Z}}^M \,|\, \mu_3 = 1\}$ is cyclic. It has two generators which are inverse to one another. We denote by h_{gen} the generator such that a positive power of it is M, namely $(h_{gen})^{l_{gen}} = M$ for a unique number $l_{gen} \in \mathbb{N}$. We have to prove $l_{gen} = 1$.

We will argue indirectly. We will assume the existence of a root $h = p(M) \in G_{\mathbb{Z}}^M$ with $h^l = M$ for some $l \geq 2$, first eigenvalue $\mu_1 = p(\lambda_1)$ and third eigenvalue $\mu_3 = p(1) = 1$. Then $\mu_1^l = \lambda_1$ and $\mu_1 = (1 - q_0) + (q_0 - rq_1)\lambda_1$ for certain $q_0, q_1 \in \mathbb{Q}$ which must satisfy the properties in (5.21)–(5.24). We will come to a contradiction.

We can restrict to $\underline{x}$ in the following set, as the $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbits of the elements in this set are all $\underline{x}$ which we consider in this theorem:

Consider the two sets Y_I and $Y_{II} \subset \mathbb{Z}^3$,

$$\begin{aligned} Y_I &:= \{\underline{x} \in \mathbb{Z}^3_{\leq 0} \,|\, x_1 \leq x_2 \leq x_3\} \\ &- \big[\{(x,0,0) \,|\, x \in \mathbb{Z}_{<0}\} \cup \{(x,x,x) \,|\, x \in \mathbb{Z}_{\leq 0}\} \\ &\quad \cup \{(x,x,0) \,|\, x \in \mathbb{Z}_{<0}\}\big], \\ Y_{II} &:= \{\underline{x} \in \mathbb{Z}^3_{\geq 3} \,|\, x_1 \leq x_2 \leq x_3, 2x_3 \leq x_1 x_2\} \\ &- \big[\{(x,x,x) \,|\, x \in \mathbb{Z}_{\geq 3}\} \\ &\quad \cup \{(2y, 2y, 2y^2) \,|\, y \in \mathbb{Z}_{\geq 2}\} \cup \{(3,3,4)\}\big]. \end{aligned}$$

All triples $\underline{x}$ in this theorem are in the $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbits of the triples in $Y_I \cup Y_{II} \cup \{\underline{x} \,|\, (x_2, x_1, x_3) \in Y_I \cup Y_{II}\}$. We will restrict to $\underline{x} \in Y_I \cup Y_{II}$. For $\underline{x}$ with $(x_2, x_1, x_3) \in Y_I \cup Y_{II}$, one can copy the following proof and exchange x_1 and x_2.

Because the triples (x,x,x) are excluded, $x_1 < x_3$ and $\widetilde{x}_1 < \widetilde{x}_3$.

We will assume the existence of a unit $\mu_1 \in \mathcal{O}^*_{\mathbb{Q}[\lambda_1]}$ with $\mu_1^l = \lambda_1$ for some $l \geq 2$ and some norm $\mathcal{N}(\mu_1) \in \{\pm 1\}$. The integrality conditions in Lemma 5.17 (b) for $q_0, q_1, q_2 \in \mathbb{Q}$ with $\mu_1 = (1 - q_0) + (q_0 - rq_1)\lambda_1$ and $q_2 = q_0 - 2q_1$ will lead to a contradiction. The proof is a case discussion. The cases split as follows.

Case I: $\underline{x} \in Y_I$.
- Subcase I.1: $l \geq 3$ odd.
- Subcase I.2: $l = 2$.

Case II: $\underline{x} \in Y_{II}$.
- Subcase II.1: $l \geq 3$ odd.
- Subcase II.2: $l = 2$.
  - Subcase II.2.1: $\mathcal{N}(\mu_1) = 1$:
    - Subcase II.2.1.1: $\mu_1 = \kappa_a$ for some $a \in \mathbb{Z}_{\geq 3}$.
    - Subcase II.2.1.2: $\mu_1 = -\kappa_a$ for some $a \in \mathbb{Z}_{\geq 3}$.
  - Subcase II.2.2: $\mathcal{N}(\mu_1) = -1$.

The treatment of the cases I, II.1 and II.2.2 will be fairly short. The treatment of the cases II.2.1 will be laborious.

Lemma C.2 prepares all cases with $\mathcal{N}(\mu_1) = 1$. Consider such a case. Suppose $\mu_1 = \kappa_a$ for some $a \in \mathbb{Z}_{\leq -3} \cup \mathbb{Z}_{\geq 3}$. Compare Lemma C.2 (c) and Lemma 5.17 (c):

$$q_0 = q_{0,l}(a), \quad q_1 = q_{1,l}(a), \quad q_2 = q_{2,l}(a), \quad r = r_l(a).$$

The integrality condition (5.21) $q_2 g^2 \in \mathbb{Z}$ together with (C.6) and (C.5) tells that $r/(2-a) = r_l(a)/(2-a)$ divides g^2.

For $l \geq 3$ odd $r/(2-a)$ is itself a square by (C.4), and g can be written as $g = \gamma_1\gamma_3$ with $\gamma_1, \gamma_3 \in \mathbb{N}$ and $\gamma_1^2 = r/(2-a)$. For $l = 2$ $r/(2-a) = a+2$, and g can be written as $g = \gamma_1\gamma_2\gamma_3$ with $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{N}$, γ_2 squarefree, and $a + 2 = \gamma_1^2\gamma_2$, so $g^2 = (a+2)\gamma_2\gamma_3^2$.

On the other hand g^2 divides $r = r_l(a)$ by Lemma 1.3. (5.1) takes the shape

$$\widetilde{x}_1^2 + \widetilde{x}_2^2 + \widetilde{x}_3^2 - g\widetilde{x}_1\widetilde{x}_2\widetilde{x}_3 = \frac{r}{g^2} = \begin{cases} \frac{2-a}{\gamma_3^2} & \text{if } l \geq 3 \text{ is odd,} \\ \frac{2-a}{\gamma_2\gamma_3^2} & \text{if } l = 2. \end{cases} \tag{5.29}$$

This equation will be the key to contradictions in the cases discussed below. The absolute value of the left hand side will be large, the absolute value of the right hand side will be small. Now we start the case discussion.

Case I.1, $\underline{x} \in Y_I$, $l \geq 3$ odd: $\mathcal{N}(\mu_1)^l = \mathcal{N}(\lambda_1) = 1$ and l odd imply $\mathcal{N}(\mu_1) = 1$. Here $\lambda_1 \in (-1, 0)$, so $\mu_1 \in (-1, 0)$, so $\mu_1 = \kappa_a$ for some $a \in \mathbb{Z}_{\leq -3}$. By the discussion above $g = \gamma_1\gamma_3$, $\gamma_1 = |b_{(l+1)/2} + b_{(l-1)/2}|$, and (5.29) holds.

Case I.1.1, all $x_i < 0$: We excluded the triples (x, x, x). Therefore $\widetilde{x}_1 \leq -2$. For $l \geq 3$ $\gamma_1 = |b_{(l+1)/2} + b_{(l-1)/2}| \geq |b_2 + b_1| = |a| - 1$ by Lemma C.2 (b). Now by (5.29)

$$|a| + 2 \geq \frac{2-a}{\gamma_3^2} \geq 2g + 4 + 1 + 1 = 2\gamma_1\gamma_3 + 6 \geq 2(|a| - 1) + 6,$$

a contradiction.

Case I.1.2, $x_3 = 0$: The integrality conditions (5.23) and (5.24) say here

$$q_0x_1, q_0x_2, q_1x_1x_2 \in \mathbb{Z}, \quad q_1x_1^2, q_1x_2^2 \in \mathbb{Z}, \quad \text{so} \quad q_1g^2 \in \mathbb{Z}$$

(which is a bit stronger than (5.22)). $q_1g^2 \in \mathbb{Z}$ means

$$\frac{(b_l(a) - b_{l-1}(a) - 1)g^2}{rb_l(a)} = \frac{b_l(a) - b_{l-1}(a) - 1}{b_l(a)(2-a)/\gamma_3^2} \in \mathbb{Z}.$$

But $\frac{b_l(a)-b_{l-1}(a)-1}{b_l(a)} \in (1, 2)$ because of Lemma C.2 (b), and $\frac{2-a}{\gamma_3^2} \in \mathbb{N}$, a contradiction.

Case I.2, $\underline{x} \in Y_I$, $l = 2$: Here $\lambda_1 < 0$. Therefore $\mu_1^2 = \lambda_1$ is impossible.

Case II.1, $\underline{x} \in Y_{II}$, $l \geq 3$ odd: $\mathcal{N}(\mu_1)^l = \mathcal{N}(\lambda_1) = 1$ and l odd imply $\mathcal{N}(\mu_1) = 1$. Here $\lambda_1 > 1$, so $\mu_1 > 1$, so $\mu_1 = \kappa_a$ for some $a \in \mathbb{Z}_{\geq 3}$. By the discussion above $g = \gamma_1\gamma_3$, $\gamma_1 = b_{(l+1)/2} + b_{(l-1)/2}$, and (5.29) holds. The proof of Lemma 5.10 (b) gives the first inequality below,

$$\begin{aligned}
\frac{a-2}{\gamma_3^2} &= \frac{|r|}{g^2} \geq g\widetilde{x}_1\widetilde{x}_2^2 - \widetilde{x}_1^2 - 2\widetilde{x}_2^2 \geq \widetilde{x}_2^2(g\widetilde{x}_1 - 3), \\
\text{so} \quad a-2 &\geq \gamma_3^2\widetilde{x}_2^2(\gamma_1\gamma_3\widetilde{x}_1 - 3), \\
\text{so} \quad (\gamma_3^2\widetilde{x}_2^2 - 1)3 &\geq (\gamma_3^2\widetilde{x}_2^2 - 1)\gamma_1\gamma_3\widetilde{x}_1 + \gamma_1\gamma_3\widetilde{x}_1 - (a+1). \quad (5.30)
\end{aligned}$$

Observe with Lemma C.2 (b)

$$\gamma_1 = b_{(l+1)/2} + b_{(l-1)/2} \geq b_2 + b_1 = a + 1 \geq 4.$$

Therefore the inequality (5.30) can only hold if $\gamma_3 = \widetilde{x}_2 = \widetilde{x}_1 = 1$ and $\gamma_1 = a + 1$, so $l = 3$. Then also $g = \gamma_1\gamma_3 = a + 1$.

$$\begin{aligned}
a - 2 &= \frac{a-2}{\gamma_3^2} = \frac{|r|}{g^2} = g\widetilde{x}_3 - \widetilde{x}_3^2 - 1 - 1 = (a + 1 - \widetilde{x}_3)\widetilde{x}_3 - 2, \\
0 &= (\widetilde{x}_3 - 1)(\widetilde{x}_3 - a).
\end{aligned}$$

We excluded the triples (x, x, x), so $\widetilde{x}_3 > 1$. But also $\widetilde{x}_3 \leq \frac{g}{2}\widetilde{x}_1\widetilde{x}_2 = \frac{a+1}{2} < a$. A contradiction.

Case II.2, $\underline{x} \in Y_{II}$, $l = 2$: Then $\mathcal{N}(\mu_1) = \varepsilon_1$ for some $\varepsilon_1 \in \{\pm 1\}$. Also $\lambda_1 > 1$ and $\varepsilon_2\mu_1 > 1$ for some $\varepsilon_2 \in \{\pm 1\}$. Then μ_1 is a zero of a polynomial $t^2 - \varepsilon_2 at + \varepsilon_1$, namely

$$\mu_1 = \varepsilon_2\left(\frac{a}{2} + \frac{1}{2}\sqrt{a^2 - 4\varepsilon_1}\right) \quad \text{with} \begin{cases} a \in \mathbb{Z}_{\geq 3} & \text{if } \varepsilon_1 = 1, \\ a \in \mathbb{N} & \text{if } \varepsilon_1 = -1, \end{cases}$$

$$\mu_1 + \mu_1^{conj} = \varepsilon_2 a, \quad \mu_1\mu_1^{conj} = \varepsilon_1, \quad \mu_1^2 = \varepsilon_2 a\mu_1 - \varepsilon_1.$$

Comparison with

$$\begin{aligned}
\lambda_1 &= \frac{2-r}{2} + \frac{1}{2}\sqrt{r(r-4)} \\
&= \mu_1^2 = \frac{a^2 - 2\varepsilon_1}{2} + \frac{a}{2}\sqrt{a^2 - 4\varepsilon_1} \\
&= \varepsilon_2 a\mu_1 - \varepsilon_1
\end{aligned}$$

shows

$$\begin{aligned}
r &= -a^2 + 2(\varepsilon_1 + 1), \\
\mu_1 &= \frac{\varepsilon_1\varepsilon_2}{a} + \frac{\varepsilon_2}{a}\lambda_1 \\
&= (1 - q_0) + (q_0 - rq_1)\lambda_1,
\end{aligned}$$

with

$$q_0 = \frac{a-\varepsilon_1\varepsilon_2}{a},$$
$$q_1 = \frac{\varepsilon_2(\varepsilon_1+1)-a}{a(a^2-2(\varepsilon_1+1))}.$$

Case II.2.1, $\mathcal{N}(\mu_1)=1$: Then $\varepsilon_1=1$ and

$$\begin{aligned} r &= -a^2+4=(2-a)(2+a), \\ q_0 &= \frac{a-\varepsilon_2}{a}, \\ q_1 &= \frac{2\varepsilon_2-a}{a(a^2-4)} = \frac{-1}{a(a+2\varepsilon_2)}, \\ q_2 &= q_0-2q_1 = \frac{a+\varepsilon_2}{a+2\varepsilon_2}. \end{aligned}$$

Write $a+2\varepsilon_2=\gamma_1^2\gamma_2$ with $\gamma_1,\gamma_2\in\mathbb{N}$ and γ_2 squarefree. The integrality condition (5.21) $q_2g^2\in\mathbb{Z}$ tells

$$g=\gamma_1\gamma_2\gamma_3 \quad \text{with} \quad a+2\varepsilon_2=\gamma_1^2\gamma_2, \quad g^2=(a+2\varepsilon_2)\gamma_2\gamma_3^2, \tag{5.31}$$

for some $\gamma_3\in\mathbb{N}$. The conditions $r<0$ and $g^2|r$ tell

$$\frac{a-2\varepsilon_2}{\gamma_2\gamma_3^2} = \frac{-r}{g^2} = g\widetilde{x}_1\widetilde{x}_2\widetilde{x}_3-\widetilde{x}_1^2-\widetilde{x}_2^2-\widetilde{x}_3^2\in\mathbb{N}. \tag{5.32}$$

γ_2 divides $a+2\varepsilon_2$ and $a-2\varepsilon_2$, so it divides 4. As it is squarefree, $\gamma_2\in\{1,2\}$. Also $\gcd(g,a)=\gcd(\gamma_1\gamma_2\gamma_3,a)\in\{1,2\}$ as $a+2\varepsilon_2=\gamma_1^2\gamma_2$ and $\gamma_2\gamma_3^3$ divides $a-2\varepsilon_2$.

The integrality conditions (5.23), $q_0x_i-q_1x_jx_k\in\mathbb{Z}$ tell

$$\frac{a-\varepsilon_2}{a}x_i+\frac{1}{a(a+2\varepsilon_2)}x_jx_k = x_i+\frac{1}{a}(-g\varepsilon_2\widetilde{x}_i+\gamma_2\gamma_3^2\widetilde{x}_j\widetilde{x}_k)\in\mathbb{Z},$$

so after multiplying with γ_1 $\quad \frac{\gamma_3}{a}(-(a+2\varepsilon_2)\varepsilon_2\widetilde{x}_i+g\widetilde{x}_j\widetilde{x}_k)\in\mathbb{Z},$

$$\text{so} \quad \frac{\gamma_3}{a}(-2\widetilde{x}_i+g\widetilde{x}_j\widetilde{x}_k)\in\mathbb{Z},$$

$$\text{so} \quad \frac{\gamma_3}{a}g_2\in\mathbb{Z}.$$

This is a bit stronger than the integrality condition (5.22) $q_1g^2g_2\in\mathbb{Z}$ which says $\frac{\gamma_2\gamma_3^2}{a}g_2\in\mathbb{Z}$. We can improve it even more, to

$$\frac{g_2}{a}\in\mathbb{Z}, \quad \frac{1}{a}(-2\widetilde{x}_i+g\widetilde{x}_j\widetilde{x}_k)\in\mathbb{Z}, \tag{5.33}$$

by the following case discussion: If $a\equiv 1(2)$, then $\gamma_3\equiv 1(2)$, so $\gcd(a,\gamma_3)=1$, so (5.33) holds. It $a\equiv 0(4)$, then $a-2\varepsilon_2\equiv 2(4)$, so $\gamma_3\equiv 1(2)$, so $\gcd(a,\gamma_3)=1$, so (5.33) holds. If $a\equiv 2(4)$, then a priori $\frac{2}{a}g_2 \in \mathbb{Z}$. But then $\frac{a}{2} \equiv 1(2)$ and $g \equiv 0(2)$, so $g_2 \equiv 0(2)$, so (5.33) holds.

Finally, the integrality conditions (5.24) $q_1(x_i^2 - x_j^2)$ say

$$\frac{\gamma_2\gamma_3^2}{a}(\widetilde{x}_i^2 - \widetilde{x}_j^2) \in \mathbb{Z}. \tag{5.34}$$

The following estimate which arises from (5.2) will also be useful:

$$\begin{aligned} \widetilde{x}_1^2 &\leq \frac{(2+(4-r)^{1/3})^2}{g^2} = \frac{(2+a^{2/3})^2}{g^2} = \frac{4+4a^{2/3}+a^{4/3}}{g^2} \\ &\leq \frac{4+2a^{1/3}+2a+a^{4/3}}{g^2} = \frac{(a+2)(2+a^{1/3})}{(a+2\varepsilon_2)\gamma_2\gamma_3^2}. \end{aligned} \tag{5.35}$$

Case II.2.1.1, $\varepsilon_2 = 1$, $\mu_1 = \kappa_a$ for some $a \in \mathbb{Z}_{\geq 3}$: The estimate (5.35) says

$$\widetilde{x}_1^2 \leq \lfloor \frac{2+a^{1/3}}{\gamma_2\gamma_3^2} \rfloor \leq \lfloor 2+a^{1/3} \rfloor \leq a \quad (\text{recall } a \in \mathbb{Z}_{\geq 3}). \tag{5.36}$$

Recall that $\underline{x}$ is a local minimum, so $2\widetilde{x}_3 \leq g\widetilde{x}_1\widetilde{x}_2$ and that $\widetilde{x}_1 \leq \widetilde{x}_2 \leq \widetilde{x}_3$ and $1 \leq \widetilde{x}_1 < \widetilde{x}_3$ (as (x,x,x) is excluded).

Case II.2.1.1.1, $2\widetilde{x}_3 < g\widetilde{x}_1\widetilde{x}_2$: Then (5.33) gives the existence of $\alpha \in \mathbb{N}$ with $g\widetilde{x}_1\widetilde{x}_2 = \alpha a + 2\widetilde{x}_3$. With this we go into (5.32),

$$\begin{aligned} \frac{a-2}{\gamma_2\gamma_3^2} &= \alpha a\widetilde{x}_3 - \widetilde{x}_1^2 + (\widetilde{x}_3^2 - \widetilde{x}_2^2) \\ &\overset{(5.36)}{\geq} \alpha a\widetilde{x}_3 - a + 0 \overset{\widetilde{x}_3 \geq 2}{\geq} a, \end{aligned}$$

a contradiction.

Case II.2.1.1.2, $2\widetilde{x}_3 = g\widetilde{x}_1\widetilde{x}_2$, $\widetilde{x}_1 \geq 2$: (5.32) takes the shape

$$\begin{aligned} \frac{a-2}{\gamma_2\gamma_3^2} &= \widetilde{x}_3^2 - \widetilde{x}_1^2 - \widetilde{x}_2^2 = g^2(\frac{\widetilde{x}_1}{2})^2\widetilde{x}_2^2 - \widetilde{x}_1^2 - \widetilde{x}_2^2 \\ &\geq (g^2-2)\widetilde{x}_2^2 \geq ((a+2)-2)\widetilde{x}_2^2 = a\widetilde{x}_2^2 \geq a, \end{aligned}$$

a contradiction.

Case II.2.1.1.3, $2\widetilde{x}_3 = g\widetilde{x}_1\widetilde{x}_2$, $\widetilde{x}_1 = 1$: Write $\gamma_4 := \gamma_2\gamma_3^2$. Then (5.32) takes the shape

$$\begin{aligned} \frac{a-2}{\gamma_4} &= \widetilde{x}_3^2 - 1 - \widetilde{x}_2^2 = (\frac{1}{4}(a+2)\gamma_4 - 1)\widetilde{x}_2^2 - 1, \\ &\overset{\widetilde{x}_2 \geq 1}{\geq} \frac{1}{4}(a+2)\gamma_4 - 2, \\ (\gamma_4^2 - 4)a &\leq -2(\gamma_4^2 - 4) + 8(\gamma_4 - 2). \end{aligned}$$

If $\gamma_4 > 2$ then $a \leq -2 + \frac{8}{\gamma_4+2} \leq -2 + \frac{8}{5}$, a contradiction. Therefore $\gamma_4 \in \{1,2\}$. If $\gamma_4 = 1$ then (5.32) becomes

$$a - 2 \;=\; \frac{a-2}{4}\widetilde{x}_2^2 - 1, \quad \text{so} \quad 4 = (a-2)(\widetilde{x}_2^2 - 4),$$

which has no solution $(a, \widetilde{x}_2) \in \mathbb{Z}_{\geq 3} \times \mathbb{N}$, so a contradiction.

If $\gamma_4 = 2$ then (5.32) is solved with $\widetilde{x}_2 = 1$ and $a \in \mathbb{Z}_{\geq 3}$ arbitrary. Then $\widetilde{x}_1 = 1$, $\gamma_2 = 2$, $\gamma_3 = 1$, $g = 2\gamma_1$, $\widetilde{x}_3 = \frac{g}{2} = \gamma_1$, $\underline{x} = (2\gamma_1, 2\gamma_1, 2\gamma_1^2)$. These cases are excluded.

Case II.2.1.2, $\varepsilon_2 = -1$, $\mu_1 = -\kappa_a$ for some $a \in \mathbb{Z}_{\geq 3}$: The estimates (5.35) say here

$$\widetilde{x}_1^2 \;\leq\; \lfloor \frac{(2+a^{2/3})^2}{(a-2)\gamma_2\gamma_3^2} \rfloor \leq \lfloor \frac{(a+2)(2+a^{1/3})}{(a-2)\gamma_2\gamma_3^2} \rfloor. \tag{5.37}$$

This implies

$$\widetilde{x}_1^2 < a \quad \text{if} \quad a \geq 8. \tag{5.38}$$

We treat small a first. Recall $\gamma_2 \in \{1,2\}$ and $a = \gamma_1^2\gamma_2 + 2$. So if $a \leq 9$ then $a \in \{3,4,6\}$. The following table lists constraints for $a \in \{3,4,6\}$. For $\widetilde{x}_1$ (5.37) gives an upper bound and $\frac{3}{g} \leq \frac{x_1}{g} = \widetilde{x}_1$ gives a lower bound. Recall the conditions $a - 2 = \gamma_1^2\gamma_2$ and $\frac{a+2}{\gamma_2\gamma_3^2} \in \mathbb{N}$.

a	3	4	6
$(\gamma_1,\gamma_2,\gamma_3)$	$(1,1,1)$	$(1,2,1)$	$(2,1,1)$ or $(2,1,2)$
$\gamma_2\gamma_3^2$	1	2	1 or 4
g	1	2	2 or 4
$\frac{3}{g}$	3	$\frac{3}{2}$	$\frac{3}{2}$ or $\frac{3}{4}$
$\frac{(2+a^{2/3})^2}{a-2}$	$16,64..$	$10,21..$	$7,03..$
$\widetilde{x}_1$	3 or 4	2	2 or 1

This gives five cases $(a, \widetilde{x}_1) \in \{(3,3),(3,4),(4,2),(6,2),(6,1)\}$ with $a \leq 9$. We treat these cases first and then all cases with $a \geq 10$. Because of (5.33) a number $\alpha \in \mathbb{Z}_{\geq 0}$ with $g\widetilde{x}_1\widetilde{x}_2 = \alpha a + 2\widetilde{x}_3$ exists. Then (5.32) becomes

$$\frac{a+2}{\gamma_2\gamma_3^2} = \alpha a \widetilde{x}_3 - \widetilde{x}_1^2 + (\widetilde{x}_3^2 - \widetilde{x}_2^2). \tag{5.39}$$

Also recall (5.34) $\frac{\gamma_2\gamma_3^2}{a}(\widetilde{x}_i^2 - \widetilde{x}_j^2) \in \mathbb{Z}$.

Case II.2.1.2.1, $(a, \gamma_1, \gamma_2, \gamma_3, g, \widetilde{x}_1) = (3,1,1,1,1,3)$: (5.39) says

$$5 = 3\alpha\widetilde{x}_3 - 9 + (\widetilde{x}_3^2 - \widetilde{x}_2^2).$$

(5.34) says that 3 divides $\widetilde{x}_3^2 - \widetilde{x}_2^2$. A contradiction.

Case II.2.1.2.2, $(a,\gamma_1,\gamma_2,\gamma_3,g,\widetilde{x}_1)=(3,1,1,1,1,4)$: (5.39) says

$$5=3\alpha\widetilde{x}_3-16+(\widetilde{x}_3^2-\widetilde{x}_2^2).$$

$g\widetilde{x}_1\widetilde{x}_2=4\widetilde{x}_2=\alpha a+2\widetilde{x}_3$ implies that α is even. This and $\widetilde{x}_3>\widetilde{x}_1=4$ and (5.39) show $\alpha=0$, so $2\widetilde{x}_2=\widetilde{x}_3$, so $5=0-16+3\widetilde{x}_2^2$, a contradiction.

Case II.2.1.2.3, $(a,\gamma_1,\gamma_2,\gamma_3,g,\widetilde{x}_1)=(4,1,2,1,2,2)$: (5.39) says

$$3=4\alpha\widetilde{x}_3-4+(\widetilde{x}_3^2-\widetilde{x}_2^2).$$

(5.34) says that 2 divides $\widetilde{x}_3^2-\widetilde{x}_2^2$. A contradiction.

Case II.2.1.2.4, $(a,\gamma_1,\gamma_2,\gamma_3,g,\widetilde{x}_1)=(6,2,1,1,2,2)$: (5.39) says

$$8=6\alpha\widetilde{x}_3-4+(\widetilde{x}_3^2-\widetilde{x}_2^2).$$

$\widetilde{x}_3>\widetilde{x}_1=2$ and (5.39) imply $\alpha=0$, so $12=\widetilde{x}_3^2-\widetilde{x}_2^2$. Only $\widetilde{x}_2=2$ and $\widetilde{x}_3=4$ satisfy this. But then $\gcd(\widetilde{x}_1,\widetilde{x}_2,\widetilde{x}_3)=2\neq 1$, a contradiction.

Case II.2.1.2.5, $(a,\gamma_1,\gamma_2,\gamma_3,g,\widetilde{x}_1)=(6,2,1,2,4,1)$: (5.39) says

$$2=6\alpha\widetilde{x}_3-1+(\widetilde{x}_3^2-\widetilde{x}_2^2).$$

It implies $\alpha=0$ and $\widetilde{x}_2=1$, $\widetilde{x}_3=2$, so $\underline{x}=(4,4,8)$. This case was excluded in Theorem 5.18.

Case II.2.1.2.6, $a\geq 10$:

Case II.2.1.2.6.1, $\alpha>0$: (5.38) gives $\widetilde{x}_1^2<a$. This and (5.39) and $\widetilde{x}_3>\widetilde{x}_1\geq 1$ show $\alpha=1$, $\widetilde{x}_3=2$, $\widetilde{x}_1=1$, $\gamma_2=\gamma_3=1$, so $a+2=2a-1+(4-\widetilde{x}_2^2)$. A contradiction to $a\geq 10$.

Case II.2.1.2.6.2, $\alpha=0$, $\widetilde{x}_1\geq 2$: (5.39) says

$$\begin{aligned}\frac{a+2}{\gamma_2\gamma_3^2} &= \widetilde{x}_3^2-\widetilde{x}_1^2-\widetilde{x}_2^2=((a-2)\gamma_2\gamma_3^2\frac{1}{4}\widetilde{x}_1^2-1)\widetilde{x}_2^2-\widetilde{x}_1^2,\\ \text{so}\quad a+2 &\geq ((a-2)-1)\widetilde{x}_2^2-\widetilde{x}_1^2\geq((a-2)-2)4,\\ \text{so}\quad 3a &\leq 18,\quad a\leq 6,\end{aligned}$$

a contradiction.

Case II.2.1.2.6.3, $\alpha=0$, $\widetilde{x}_1=1$: Write $\gamma_4:=\gamma_2\gamma_3^2$. Then (5.39) says

$$\begin{aligned}\frac{a+2}{\gamma_4} &= \widetilde{x}_3^2-1-\widetilde{x}_2^2=((a-2)\gamma_4\frac{1}{4}-1)\widetilde{x}_2^2-1,\\ \text{so}\quad \widetilde{x}_2^2 &= \frac{4}{\gamma_4^2}\cdot\frac{a+2+\gamma_4}{a-2-\frac{4}{\gamma_4}}.\end{aligned}\tag{5.40}$$

The right hand side must be ≥ 1. This means

$$\begin{aligned}\gamma_4^2(a-2-\frac{4}{\gamma_4}) &\leq 4(a+2+\gamma_4),\\ (\gamma_4^2-4)a &\leq 2(\gamma_4^2-4)+8(\gamma_4+2).\end{aligned}$$

If $\gamma_4 > 2$ then $a \leq 2 + \frac{8}{\gamma_4 - 2}$, which is in contradiction to $a \geq 10$, as $\gamma_4 = 3$ would mean $\gamma_2 = 3$, which is impossible. Therefore $\gamma_4 \in \{1, 2\}$.

If $\gamma_4 = 1$ then (5.40) says $\widetilde{x}_2^2 = 4\frac{a+3}{a-6} = 4 + \frac{36}{a-6}$. But the right hand side is not a square for any $a \geq 10$, a contradiction.

If $\gamma_4 = 2$ then (5.40) says $\widetilde{x}_2^2 = \frac{a+4}{a-4}$, which is also not a square for any $a \geq 10$, a contradiction.

Case II.2.2, $\mathcal{N}(\mu_1) = \varepsilon_1 = -1$: Recall the formulas for r, q_0 and q_1 at the beginning of case II.2. Now

$$\begin{aligned} r &= -a^2, \\ q_0 &= \frac{a + \varepsilon_2}{a}, \\ q_1 &= \frac{-1}{a^2}, \\ q_2 &= \frac{a^2 + \varepsilon_2 a + 2}{a^2}. \end{aligned}$$

The integrality condition (5.21) $q_2 g^2 \in \mathbb{Z}$ says $\frac{a}{2} \,|\, g$ if $a \equiv 2(4)$ and $a \,|\, g$ if $a \equiv 1(2)$ or $a \equiv 0(4)$. Also $g^2 \,|\, r = -a^2$. Therefore $g = a$ or $g = \frac{a}{2}$, and $g = \frac{a}{2}$ only if $a \equiv 2(4)$. The case $g = a$ means $\frac{r}{g^2} = -1$ which implies by Lemma 5.11 $Q \in G_{\mathbb{Z}}$. But all such cases are excluded in Theorem 5.18.

Therefore $g = \frac{a}{2}$ and $a \equiv 2(4)$. The integrality condition (5.22) $q_1 g^2 g_2 \in \mathbb{Z}$ says $\frac{g_2}{4} \in \mathbb{Z}$. But $g_2 \equiv 0(4)$ and $g \equiv 1(2)$ are together impossible in view of the definition $g_2 = \gcd(2\widetilde{x}_1 - g\widetilde{x}_2\widetilde{x}_3, 2\widetilde{x}_2 - g\widetilde{x}_1\widetilde{x}_3, 2\widetilde{x}_3 - g\widetilde{x}_1\widetilde{x}_2)$ and $\gcd(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) = 1$. A contradiction.

Therefore in all cases the assumption that a nontrivial root μ_1 of λ_1 exists which satisfies the integrality conditions (5.21)–(5.24) leads to a contradiction. Theorem 5.18 is proved □

Remark 5.19. The results in this chapter give complete results on $G_{\mathbb{Z}}$ and $G_{\mathbb{Z}}^M \supset G_{\mathbb{Z}}$ for all unimodular bilinear lattices of rank 3.

The reducible cases: Lemma 5.4, Theorem 5.13.

The irreducible cases with $r \in \{0, 1, 2, 4\}$: Theorem 5.14

The irreducible cases with $r \in \mathbb{Z}_{<0} \cup \mathbb{Z}_{>4}$ and $G_{\mathbb{Z}}^M \supsetneqq \{\pm M^l \,|\, l \in \mathbb{Z}\}$: Theorem 5.16.

The irreducible cases with $r \in \mathbb{Z}_{<0} \cup \mathbb{Z}_{>4}$ and $G_{\mathbb{Z}}^M = \{\pm M^l \,|\, l \in \mathbb{Z}\}$: Theorem 5.18.

CHAPTER 6

Monodromy groups and vanishing cycles

This chapter studies the monodromy groups $\Gamma^{(0)}$ and $\Gamma^{(1)}$ of the unimodular bilinear lattices $(H_{\mathbb{Z}}, L, \underline{e})$ with triangular basis $\underline{e}$ which have rank 2 or 3. In rank 3 the even as well as the odd cases split into many different case studies. They make the chapter long.

Section 6.1 considers for $k \in \{0; 1\}$ the quotient lattice $\overline{H_{\mathbb{Z}}}^{(1)} := H_{\mathbb{Z}}/\operatorname{Rad} I^{(k)}$ and the induced bilinear form $\overline{I}^{(k)}$ on it. Because $\Gamma^{(k)}$ acts trivially on $\operatorname{Rad} I^{(k)}$, it acts on this quotient lattice and respects $\overline{I}^{(k)}$. The homomorphism $\Gamma^{(k)} \to \operatorname{Aut}(\overline{H_{\mathbb{Z}}}^{(1)}, \overline{I}^{(k)})$ has an image $\Gamma_s^{(k)}$, the *simple part* of $\Gamma^{(k)}$ and a kernel $\Gamma_u^{(k)}$, the *unipotent part* of $\Gamma^{(k)}$. There is the exact sequence

$$\{\mathrm{id}\} \to \Gamma_u^{(k)} \to \Gamma^{(k)} \to \Gamma_s^{(k)} \to \{\mathrm{id}\}.$$

We will study $\Gamma^{(k)}$ together with $\Gamma_s^{(k)}$ and $\Gamma_u^{(k)}$. Also the natural homomorphism $j^{(k)} : H_{\mathbb{Z}} \to H_{\mathbb{Z}}^\sharp := \operatorname{Hom}_{\mathbb{Z}}(H_{\mathbb{Z}}, \mathbb{Z})$ and in the even case the spinor norm will be relevant. Section 6.1 fixes more or less well known general facts.

Section 6.2 treats the rank 2 cases. The even cases A_1^2 and A_2 are classical and easy. In the other even cases $\Gamma^{(0)} \cong G^{fCox,2}$. There we can characterize $\Delta^{(0)}$ arithmetically and geometrically. In the odd case A_2 $\Gamma^{(1)} \cong SL_2(\mathbb{Z})$. In the other irreducible odd cases $\Gamma^{(1)} \cong G^{free,2}$. The matrix group $\Gamma^{(1),mat} \subset SL_2(\mathbb{Z})$ is a Fuchsian group of the second kind, but has infinite index in $SL_2(\mathbb{Z})$ in most cases. We do not have a characterization of $\Delta^{(1)}$ which is as nice as in the even cases.

The long Theorem 6.11 in section 6.3 states our results on the even monodromy group $\Gamma^{(0)}$ in the rank 3 cases. The results are detailed except for the local minima $\underline{x} \in \mathbb{Z}^3_{\geq 3}$ with $r(\underline{x}) \leq 0$ where we only state $\Gamma^{(0)} \cong \Gamma_s^{(0)} \cong G^{fCox,3}$ and $\Gamma_u^{(0)} = \{\mathrm{id}\}$. It is followed by Theorem 6.14 which gives the set $\Delta^{(0)}$ of even vanishing cycles in many, but not all cases. Especially in the cases of the local minima $\underline{x} \in \mathbb{Z}^3_{\geq 3}$ with $r(\underline{x}) \leq 0$ we know little and only state $\Delta^{(0)} = R^{(0)}$ in the case $(3, 3, 3)$, but $\Delta^{(0)} \subsetneqq R^{(0)}$ in the four cases $(3, 3, 4)$, $(4, 4, 4)$, $(5, 5, 5)$ and $(4, 4, 8)$.

The result $\Delta^{(0)} = R^{(0)}$ in the case $(3,3,3)$ seems to be new . Its proof is rather laborious.

Section 6.4 treats the odd monodromy group $\Gamma^{(1)}$ and the set of odd vanishing cycles $\Delta^{(1)}$ in the rank 3 cases. The long Theorem 6.18 fixes the results on $\Gamma^{(1)}$. The even longer Theorem 6.21 fixes the results on $\Delta^{(1)}$. Also their proofs are long. They are preceded by two technical lemmas, the second one helps to control $\Gamma_u^{(1)}$. Similar to the even rank 3 cases, in the case of a local minimum $\underline{x} \in \mathbb{Z}^3_{\geq 3}$ with $r(\underline{x}) \leq 0$ $\Gamma^{(1)} \cong \Gamma_s^{(1)} \cong G^{free,3}$ and $\Gamma_u^{(1)} = \{\text{id}\}$. In the same case, interestingly, the map $\Delta^{(1)} \to \overline{H_{\mathbb{Z}}}^{(1)}$ is injective. This leads in this case to the problem how to recover an odd vanishing cycle from its image in $\overline{H_{\mathbb{Z}}}^{(1)}$. One solution is offered in the most important case $\underline{x} = (3,3,3)$ in Lemma 6.26. One general application of the Theorems 6.18 and 6.21 is given in Corollary 6.23. It allows to separate many of the orbits of the bigger group $(G^{phi} \ltimes \widetilde{G}^{sign}) \rtimes \langle\gamma\rangle$ which acts on $T_3^{uni}(\mathbb{Z})$ and $\mathbb{Z}^3$ in Lemma 4.18.

6.1. Basic observations

Let $(H_{\mathbb{Z}}, L, \underline{e})$ be a unimodular bilinear lattice of rank $n \in \mathbb{N}$ with a triangular basis $\underline{e}$. Definition 2.8 gave two monodromy groups $\Gamma^{(0)}$ and $\Gamma^{(1)}$ and two sets $\Delta^{(0)}$ and $\Delta^{(1)}$ of vanishing cycles. Later in this chapter they shall be studied rather systematically in essentially all cases with $n = 2$ or $n = 3$. For that we need some notations and basic facts, which are collected here. Everything in this section is well known. Most of it is stated in the even case in [Eb84] and in the odd case in [Ja83].

Definition 6.1. Let $(H_{\mathbb{Z}}, L)$ be a unimodular bilinear lattice of rank $n \in \mathbb{N}$. In the following $k \in \{0;1\}$. Denote

$$\begin{aligned}
O^{(k)} &:= \text{Aut}(H_{\mathbb{Z}}, I^{(k)}) \quad \text{the group of automorphisms of } H_{\mathbb{Z}} \\
&\qquad \text{which respect } I^{(k)}. \\
H_{\mathbb{Z}}^{\sharp} &:= \text{Hom}(H_{\mathbb{Z}}, \mathbb{Z}) \quad \text{the dual lattice.} \\
j^{(k)} &: H_{\mathbb{Z}} \to H_{\mathbb{Z}}^{\sharp}, \quad a \mapsto (b \mapsto I^{(k)}(a,b)). \\
t^{(k)} &: O^{(k)} \to \text{Aut}(H_{\mathbb{Z}}^{\sharp}), \quad g \mapsto (l \mapsto l \circ g^{-1}).
\end{aligned}$$

$$\begin{array}{rcl}
\overline{H_{\mathbb{Z}}}^{(k)} & := & H_{\mathbb{Z}}/\operatorname{Rad} I^{(k)}, \quad \overline{H_{\mathbb{R}}}^{(k)} := H_{\mathbb{R}}/\operatorname{Rad}_{\mathbb{R}} I^{(k)}. \\
\operatorname{pr}^{H,(k)} & = & \overline{(.)}^{(k)} : H_{\mathbb{Z}} \to \overline{H_{\mathbb{Z}}}^{(k)}, \quad a \mapsto \overline{a}^{(k)}, \quad \text{the projection.} \\
\overline{I}^{(k)} & : & \overline{H_{\mathbb{Z}}}^{(k)} \times \overline{H_{\mathbb{Z}}}^{(k)} \to \mathbb{Z} \quad \text{the bilinear form on } \overline{H_{\mathbb{Z}}}^{(k)} \\
 & & \text{which is induced by } I^{(k)}, \\
\overline{H_{\mathbb{Z}}}^{(k),\sharp} & := & \operatorname{Hom}(\overline{H_{\mathbb{Z}}}^{(k)}, \mathbb{Z}) \quad \text{the dual lattice.} \\
O^{(k),Rad} & := & \{g \in O^{(k)} \,|\, g|_{\operatorname{Rad} I^{(k)}} = \operatorname{id}\}. \\
\operatorname{pr}^{A,(k)} & = & \overline{(.)} : O^{(k),Rad} \to \operatorname{Aut}(\overline{H_{\mathbb{Z}}}^{(k)}, \overline{I}^{(k)}), \quad g \mapsto \overline{g}, \\
 & & \text{the natural map to the set of induced automorphisms.}
\end{array}$$

For any subgroup $G^{(k)} \subset O^{(k),Rad}$ define

$$\begin{array}{rcl}
G_s^{(k)} & := & \operatorname{pr}^{A,(k)}(G^{(k)}) \subset \operatorname{Aut}(\overline{H_{\mathbb{Z}}}^{(k)}, \overline{I}^{(k)}), \\
G_u^{(k)} & := & \ker(\operatorname{pr}^{A,(k)} : G^{(k)} \to \operatorname{Aut}(\overline{H_{\mathbb{Z}}}^{(k)}, \overline{I}^{(k)})).
\end{array}$$

$G_s^{(k)}$ is called the *simple part* of $G^{(k)}$, and $G_u^{(k)}$ is called the *unipotent part* of $G^{(k)}$.

LEMMA 6.2. *Let $(H_{\mathbb{Z}}, L, \underline{e})$ be a unimodular bilinear lattice of rank $n \in \mathbb{N}$ with a triangular basis $\underline{e}$.*

(a) The map $t^{(k)} : O^{(k)} \to \operatorname{Aut}(H_{\mathbb{Z}}^\sharp)$ is a group homomorphism. For $g \in O^{(k)}$, $t^{(k)}(g)$ maps $j^{(k)}(H_{\mathbb{Z}}) \subset H_{\mathbb{Z}}^\sharp$ to itself, so it induces an automorphism $\tau^{(k)}(g) \in \operatorname{Aut}(H_{\mathbb{Z}}^\sharp/j^{(k)}(H_{\mathbb{Z}}))$. The map

$$\tau^{(k)} : O^{(k)} \to \operatorname{Aut}(H_{\mathbb{Z}}^\sharp/j^{(k)}(H_{\mathbb{Z}}))$$

is a group homomorphism.

(b) For $a \in R^{(0)}$ if $k = 0$ and for $a \in H_{\mathbb{Z}}$ if $k = 1$ the reflection or transvection $s_a^{(k)} \in O^{(k)}$ is in $\ker \tau^{(k)}$. Therefore $\Gamma^{(k)} \subset \ker \tau^{(k)}$.

(c) $\ker \tau^{(k)} \subset O^{(k),Rad}$.

(d) The horizontal lines of the following diagram are exact sequences.

$$\begin{array}{ccccccccc}
\{\operatorname{id}\} & \to & \Gamma_u^{(k)} & \to & \Gamma^{(k)} & \to & \Gamma_s^{(k)} & \to & \{\operatorname{id}\} \\
\| & & \cap & & \cap & & \cap & & \| \\
\{\operatorname{id}\} & \to & (\ker \tau^{(k)})_u & \to & \ker \tau^{(k)} & \to & (\ker \tau^{(k)})_s & \to & \{\operatorname{id}\} \\
\| & & \cap & & \cap & & \cap & & \| \\
\{\operatorname{id}\} & \to & O_u^{(k),Rad} & \to & O^{(k),Rad} & \to & O_s^{(k),Rad} & \to & \{\operatorname{id}\}
\end{array}$$

The second and third exact sequence split non-canonically.

$$O_s^{(k),Rad} = \operatorname{Aut}(\overline{H_{\mathbb{Z}}}^{(k)}, \overline{I}^{(k)}).$$

(e) The map

$$\begin{aligned} T: \overline{H_{\mathbb{Z}}}^{(k),\sharp} \otimes \operatorname{Rad} I^{(k)} &\to O_u^{(k),Rad}, \\ \sum_{i\in I} l_i \otimes r_i &\mapsto \big(a \mapsto a + \sum_{i\in I} l_i(\overline{a}^{(k)}) r_i\big), \\ \text{shorter: } h &\mapsto \big(a \mapsto a + h(\overline{a}^{(k)})\big), \end{aligned}$$

is an isomorphism between abelian groups with

$$T(h_1 + h_2) = T(h_1) \circ T(h_2), \quad T(h)^{-1} = T(-h).$$

It restricts to an isomorphism

$$T: \overline{j}^{(k)}(\overline{H_{\mathbb{Z}}}^{(k)}) \otimes \operatorname{Rad} I^{(k)} \to (\ker \tau^{(k)})_u,$$

where $\overline{j}^{(k)} : \overline{H_{\mathbb{Z}}}^{(k)} \to \overline{H_{\mathbb{Z}}}^{(k),\sharp}$ *is the map*

$$a \mapsto \big(b \mapsto \overline{I}^{(k)}(a,b)\big) \quad \text{for an arbitrary } b \in \overline{H_{\mathbb{Z}}}^{(k)}.$$

(f) For $g \in O^{(k),Rad}$, $a \in \overline{H_{\mathbb{Z}}}^{(k)}$, $r \in \operatorname{Rad} I^{(k)}$

$$g \circ T(\overline{j}^{(k)}(a) \otimes r) \circ g^{-1} = T(\overline{j}^{(k)}(\overline{g}(a)) \otimes r).$$

(g) Analogously to $t^{(k)}$ *and* $\tau^{(k)}$ *there are the group homomorphisms*

$$\begin{aligned} \overline{t}^{(k)} : O_s^{(k),Rad} = \operatorname{Aut}(\overline{H_{\mathbb{Z}}}^{(k)}, \overline{I}^{(k)}) &\to \operatorname{Aut}(\overline{H_{\mathbb{Z}}}^{(k),\sharp}),\ g \mapsto (l \mapsto l \circ g^{-1}), \\ \text{and} \quad \overline{\tau}^{(k)} : O_s^{(k),Rad} &\to \operatorname{Aut}(\overline{H_{\mathbb{Z}}}^{(k),\sharp} / \overline{j}^{(k)}(\overline{H_{\mathbb{Z}}}^{(k)})). \end{aligned}$$

$\tau^{(k)}$ *and* $\overline{\tau}^{(k)}$ *satisfy*

$$(\ker \tau^{(k)})_s = \ker \overline{\tau}^{(k)}.$$

Proof: (a) The map $t^{(k)}$ is a group homomorphism because

$$t^{(k)}(g_1 g_2)(l) = l \circ (g_1 g_2)^{-1} = l \circ g_2^{-1} \circ g_1^{-1} = t^{(k)}(g_1) t^{(k)}(g_2)(l).$$

$t^{(k)}(g)$ maps $j^{(k)}(H_{\mathbb{Z}})$ to itself because

$$\begin{aligned} t^{(k)}(g)(j^{(k)}(a)) &= j^{(k)}(a) \circ g^{-1} = I^{(k)}(a, g^{-1}(.)) \\ &= I^{(k)}(g(a), (.)) = j^{(k)}(g(a)). \end{aligned}$$

$\tau^{(k)}$ is a group homomorphism because $t^{(k)}$ is one.

(b) Choose $l\in H_{\mathbb Z}^\sharp$ and $b\in H_{\mathbb Z}$. Then

$$\begin{aligned}(t^{(k)}(s_a^{(k)})(l)-l)(b) &= l\circ (s_a^{(k)})^{-1}(b)-l(b)\\ &= l(b-(-1)^kI^{(k)}(a,b)a)-l(b)\\ &= (-1)^{k+1}I^{(k)}(a,b)l(a)\\ &= j^{(k)}((-1)^{k+1}l(a)a)(b),\\ \text{so } t^{(k)}(s_a^{(k)})(l)-l &= j^{(k)}((-1)^{k+1}l(a)a)\in j^{(k)}(H_{\mathbb Z}),\\ \text{so } \tau^{(k)}(s_a^{(k)}) &= \mathrm{id},\end{aligned}$$

so $s_a^{(k)}\in\ker\tau^{(k)}$.

(c) Let $g\in\ker\tau^{(k)}$ and let $r\in\mathrm{Rad}\, I^{(k)}$. Also $g^{-1}\in\ker\tau^{(k)}$. Choose $l\in H_{\mathbb Z}^\sharp$. Now $\tau^{(k)}(g^{-1})=\mathrm{id}$ implies

$$t^{(k)}(g^{-1})(l)-l=j^{(k)}(a)\quad\text{for some } a\in H_{\mathbb Z},$$
$$0=I^{(k)}(a,r)=j^{(k)}(a)(r)=\big(t^{(k)}(g^{-1})(l)-l\big)(r)=l((g-\mathrm{id})(r)).$$

Because l is arbitrary, $(g-\mathrm{id})(r)=0$, so $g(r)=r$, so $g\in O^{(k),Rad}$.

(d) The exact sequences are obvious. Choose an arbitrary splitting of $H_{\mathbb Z}$ as $\mathbb Z$-module into $\mathrm{Rad}\, I^{(k)}$ and a suitably chosen $\mathbb Z$-module $\widetilde{H_{\mathbb Z}}^{(k)}$,

$$H_{\mathbb Z}=\mathrm{Rad}\, I^{(k)}\oplus\widetilde{H_{\mathbb Z}}^{(k)}.$$

The projection $\mathrm{pr}^{H,(k)}:H_{\mathbb Z}\to\overline{H_{\mathbb Z}}^{(k)}$ restricts to an isomorphism

$$\mathrm{pr}^{H,(k)}:(\widetilde{H_{\mathbb Z}}^{(k)},I^{(k)}|_{\widetilde{H_{\mathbb Z}}^{(k)}})\to(\overline{H_{\mathbb Z}}^{(k)},\overline{I}^{(k)}).$$

Via this isomorphism, any element of $\mathrm{Aut}(\overline{H_{\mathbb Z}}^{(k)},\overline{I}^{(k)})$ lifts to an element of $O^{(k),Rad}$. This shows $O_s^{(k),Rad}=\mathrm{Aut}(\overline{H_{\mathbb Z}}^{(k)},\overline{I}^{(k)})$, and it gives a non-canonical splitting of the third exact sequence. The end of the proof of part (g) will show that this splitting restricts to a non-canonical splitting of the second exact sequence.

(e) The fact $\overline{r}^{(k)}=0$ for $r\in\mathrm{Rad}\, I^{(k)}$ easily implies that T is a group homomorphism with $T(h_1+h_2)=T(h_1)T(h_2)$ and with image in $O_u^{(k),Rad}$.

Consider $g\in O_u^{(k),Rad}$. Then $g|_{\mathrm{Rad}\, I^{(k)}}=\mathrm{id}$ and $(g-\mathrm{id})(a)\in\mathrm{Rad}\, I^{(k)}$ for any $a\in H_{\mathbb Z}$. If $b\in a+\mathrm{Rad}\, I^{(k)}$ then $(g-\mathrm{id})(a-b)=0$, so $(g-\mathrm{id})(b)=(g-\mathrm{id})(a)$. Thus there is an element $h\in\overline{H_{\mathbb Z}}^{(k),\sharp}\otimes\mathrm{Rad}\, I^{(k)}$ with $h(\overline{a}^{(k)})=(g-\mathrm{id})(a)$ for any $a\in H_{\mathbb Z}$, so $T(h)(a)=a+h(\overline{a}^{(k)})=g(a)$, so $T(h)=g$.

Choose a $\mathbb Z$-basis $r_1,...,r_m$ of $\mathrm{Rad}\, I^{(k)}$ and linear forms $l_1,...,l_m\in H_{\mathbb Z}^\sharp$ with $l_i(r_j)=\delta_{ij}$. Then any $r\in\mathrm{Rad}\, I^{(k)}$ satisfies $r=\sum_{i=1}^m l_i(r)r_i$.

Consider $h \in \overline{H_{\mathbb{Z}}}^{(k),\sharp} \otimes \operatorname{Rad} I^{(k)}$ with $T(h) \in (\ker \tau^{(k)})_u$. Then

$$\begin{aligned} t^{(k)}(T(h))(l_i) - l_i &= j^{(k)}(a_i) \quad \text{for some } a_i \in H_{\mathbb{Z}}, \text{ and also} \\ t^{(k)}(T(h))(l_i) - l_i &= l_i \circ T(h)^{-1} - l_i = l_i \circ T(-h) - l_i \\ &= l_i \circ (\operatorname{id} - h(\overline{(.)}^{(k)})) - l_i = -l_i(h(\overline{(.)}^{(k)})). \end{aligned}$$

For $b \in H_{\mathbb{Z}}$ $h(\overline{b}^{(k)}) \in \operatorname{Rad} I^{(k)}$, so

$$\begin{aligned} h(\overline{b}^{(k)}) &= \sum_{i=1}^{m} l_i(h(\overline{b}^{(k)}))r_i = -\sum_{i=1}^{m} j^{(k)}(a_i)(b)r_i, \\ \text{so } h &\in \overline{j}^{(k)}(\overline{H_{\mathbb{Z}}}^{(k)}) \otimes \operatorname{Rad} I^{(k)}. \end{aligned}$$

Going backwards through these arguments, one sees that any $h \in \overline{j}^{(k)}(\overline{H_{\mathbb{Z}}}^{(k)}) \otimes \operatorname{Rad} I^{(k)}$ satisfies $T(h) \in (\ker \tau^{(k)})_u$.

(f) For $b \in H_{\mathbb{Z}}$

$$\begin{aligned} (g \circ T(\overline{j}^{(k)}(a) \otimes r) \circ g^{-1})(b) &= g(g^{-1}(b) + \overline{I}^{(k)}(a, \overline{g^{-1}(b)}^{(k)})r) \\ &= b + \overline{I}^{(k)}(\overline{g}(a), \overline{b}^{(k)})r \\ &= T(\overline{j}^{(k)}(\overline{g}(a)) \otimes r)(b). \end{aligned}$$

(g) The projection $\operatorname{pr}^{H,(k)} = \overline{()}^{(k)} : H_{\mathbb{Z}} \to \overline{H_{\mathbb{Z}}}^{(k)}$ induces the embedding

$$\begin{aligned} i^{(k)} : \overline{H_{\mathbb{Z}}}^{(k),\sharp} &\hookrightarrow H_{\mathbb{Z}}^{\sharp},\ l \mapsto l \circ \operatorname{pr}^{H,(k)}, \\ \text{with} \qquad \operatorname{Im}(i^{(k)}) &= \{l \in H_{\mathbb{Z}}^{\sharp} \,|\, l|_{\operatorname{Rad} I^{(k)}} = 0\} \\ \text{and} \qquad j^{(k)}(H_{\mathbb{Z}}) &= i^{(k)}(\overline{j}^{(k)}(\overline{H_{\mathbb{Z}}}^{(k)})) \subset \operatorname{Im}(i^{(k)}). \end{aligned}$$

The three lattices

$$H_{\mathbb{Z}}^{\sharp} \supset \operatorname{Im}(i^{(k)}) \supset j^{(k)}(H_{\mathbb{Z}})$$

have ranks n, $n - \operatorname{rk} \operatorname{Rad} I^{(k)}$, $n - \operatorname{rk} \operatorname{Rad} I^{(k)}$ and are for each $g \in O^{(k),\mathrm{Rad}}$ invariant under the map $t^{(k)}(g) = (l \mapsto l \circ g^{-1})$.

This map acts trivially on the quotient $H_{\mathbb{Z}}^{\sharp}/\operatorname{Im}(i^{(k)})$. It acts trivially on the quotient $\operatorname{Im}(i^{(k)})/j^{(k)}(H_{\mathbb{Z}})$ if and only if $\overline{g}^{(k)} \in \ker \overline{\tau}^{(k)}$. It acts trivially on the quotient $H_{\mathbb{Z}}^{\sharp}/j^{(k)}(H_{\mathbb{Z}})$ if and only if $g \in \ker \tau^{(k)}$. Therefore $(\ker \tau^{(k)})_s \subset \overline{\tau}^{(k)}$. It remains to find for each $\widetilde{g} \in \ker \overline{\tau}^{(k)}$ an element $g \in \ker \tau^{(k)}$ with $\overline{g}^{(k)} = \widetilde{g}$.

Choose a $\mathbb{Z}$-basis $r_1, ..., r_n$ of $H_{\mathbb{Z}}$ such that $r_1, ..., r_m$ (with $m = \operatorname{rk} \operatorname{Rad} I^{(k)}$) is a $\mathbb{Z}$-basis of $\operatorname{Rad} I^{(k)}$. Then $H_{\mathbb{Z}} = \operatorname{Rad} I^{(k)} \oplus \widetilde{H_{\mathbb{Z}}}^{(k)}$ with $\widetilde{H_{\mathbb{Z}}}^{(k)} = \bigoplus_{j=m+1}^{n} \mathbb{Z} \cdot r_j$ is a splitting of $H_{\mathbb{Z}}$ with $\widetilde{H_{\mathbb{Z}}}^{(k)} \cong \overline{H_{\mathbb{Z}}}^{(k)}$.

Consider the dual $\mathbb{Z}$-basis $l_1, ..., l_n$ of $H_\mathbb{Z}^\sharp$ with $l_i(r_j) = \delta_{ij}$. Then $\mathrm{Im}(i^{(k)}) = \bigoplus_{j=m+1}^n \mathbb{Z} \cdot l_j \supset j^{(k)}(H_\mathbb{Z})$.

An element $\widetilde{g} \in O_s^{(k),Rad}$ has a unique lift to an element $g \in O^{(k),Rad}$ with $g(\widetilde{H_\mathbb{Z}}^{(k)}) = \widetilde{H_\mathbb{Z}}^{(k)}$. This splitting of the third exact sequence in part (d) was used already in the proof of part (d). We claim that $g \in \ker \tau^{(k)}$ if $\widetilde{g} \in \ker \overline{\tau}^{(k)}$. We have

$$\begin{aligned} l_j - l_j \circ g^{-1} &\in j^{(k)}(H_\mathbb{Z}) \quad \text{for } j \in \{m+1, ..., n\} \\ &\qquad \text{because of } \widetilde{g} \in \ker \overline{\tau}^{(k)}, \\ l_i - l_i \circ g^{-1} &= 0 \quad \text{for } i \in \{1, ..., m\}. \end{aligned}$$

This shows the claim. Therefore $(\ker \tau^{(k)})_s = \ker \overline{\tau}^{(k)}$. The claim also shows that the non-canonical splitting of the third exact sequence in part (d) restricts to a non-canonical splitting of the second exact sequence in part (d). □

Remarks 6.3. (i) The exact sequence $\{\mathrm{id}\} \to \Gamma_u^{(k)} \to \Gamma^{(k)} \to \Gamma_s^{(k)} \to \{\mathrm{id}\}$ splits sometimes, sometimes not. When it splits and when $\Gamma_u^{(k)}$, $\Gamma_s^{(k)}$ and the splitting are known, then also $\Gamma^{(k)}$ is known.

(ii) Suppose that one has a presentation of $\Gamma_s^{(k)}$, namely an isomorphism

$$\begin{aligned} \Gamma_s^{(k)} &\xrightarrow{\cong} \langle g_1, ..., g_n \,|\, w_1(g_1, ..., g_m), ..., w_m(g_1, ..., g_n)\rangle, \\ \overline{s_{e_i}^{(k)}} &\mapsto g_i, \end{aligned}$$

where $w_1(g_1...., g_n), ..., w_m(g_1, ..., g_n)$ are certain words in $g_1^{\pm 1}, ..., g_n^{\pm 1}$. Then the group $\Gamma_u^{(k)}$ is the normal subgroup of $\Gamma^{(k)}$ generated by the elements $w_1(s_{e_1}^{(k)}, ..., s_{e_n}^{(k)})$,..., $w_m(s_{e_1}^{(k)}, ..., s_{e_n}^{(k)})$. In many of the cases with $n = 2$ or $n = 3$ we have such a presentation.

(iii) The symmetric bilinear form $\overline{I}^{(0)}$ on $\overline{H_\mathbb{Z}}^{(0)}$ is nondegenerate. It is well known that for any $g \in \mathrm{Aut}(\overline{H_\mathbb{R}}^{(0)}, \overline{I}^{(0)})$ some $m \in \mathbb{N}$ and elements $a_1, ..., a_m \in \overline{H_\mathbb{R}}^{(0)}$ with $\overline{I}^{(0)}(a_i, a_i) \in \mathbb{R}^*$ and $g = \overline{s}_{a_1}^{(0)} ... \overline{s}_{a_m}^{(0)}$ exist and that the sign

$$\overline{\sigma}(g) := \prod_{i=1}^m \mathrm{sign}(\overline{I}^{(0)}(a_i, a_i)) \in \{\pm 1\}$$

is independent of m and $a_1, ..., a_m$. This sign $\overline{\sigma}(g) \in \{\pm 1\}$ is the *spinor norm* of g. The map $\overline{\sigma} : \mathrm{Aut}(\overline{H_\mathbb{R}}^{(0)}, \overline{I}^{(0)}) \to \{\pm 1\}$ is obviously a group homomorphism.

Definition 6.4. Keep the situation of Definition 6.1. Define the *spinor norm homomorphism*

$$\sigma : O^{(0),Rad} \to \{\pm 1\}, \quad \sigma(g) := \overline{\sigma}(\overline{g}).$$

Define the subgroup $O^{(k),*}$ of $O^{(k),Rad}$

$$O^{(k),*} \quad := \quad \begin{cases} \ker \tau^{(1)} & \text{if } k=1, \\ \ker \tau^{(0)} \cap \ker \sigma & \text{if } k=0. \end{cases}$$

Remarks 6.5. (i) For $a \in R^{(0)}$ of course $\sigma(s_a^{(0)}) = 1$. Therefore $\Gamma^{(0)} \subset \ker \sigma$. Thus

$$\Gamma^{(k)} \subset O^{(k),*} \quad \text{for } k \in \{0;1\}.$$

(ii) For $g \in (\ker \tau^{(0)})_u$ $\sigma(g) = 1$ because $\overline{g} = \mathrm{id}$. Therefore

$$O_u^{(k),*} = (\ker \tau^{(k)})_u \quad \text{for } k \in \{0;1\}.$$

Remarks 6.6. Finally we make some comments on the sets of vanishing cycles in a unimodular bilinear lattice $(H_{\mathbb{Z}}, L, \underline{e})$ with a triangular basis.

(i) Let $H_{\mathbb{Z}}^{prim}$ denote the set of primitive vectors in $H_{\mathbb{Z}}$, i.e. vectors $a \in H_{\mathbb{Z}} - \{0\}$ with $\mathbb{Z}a = \mathbb{Q}a \cap H_{\mathbb{Z}}$, and analogously $\overline{H_{\mathbb{Z}}}^{(k),prim}$. Then

$$\Delta^{(0)} \subset R^{(0)} \subset H_{\mathbb{Z}}^{prim}, \quad \Delta^{(1)} \subset H_{\mathbb{Z}}^{prim},$$
$$\overline{\Delta}^{(0)} \subset \overline{R}^{(0)} \subset \overline{H_{\mathbb{Z}}}^{(0),prim}, \quad \text{where } \overline{\Delta}^{(k)} := \mathrm{pr}^{H,(k)}(\Delta^{(k)}).$$

Here $R^{(0)} \subset H_{\mathbb{Z}}^{prim}$ and $\overline{R}^{(0)} \subset \overline{H_{\mathbb{Z}}}^{(0),prim}$ because of $2 = I^{(0)}(a,a) = \overline{I}^{(0)}(\overline{a}^{(0)}, \overline{a}^{(0)})$ for $a \in R^{(0)}$. Furthermore

$$\overline{e_i}^{(1)} \in \overline{H_{\mathbb{Z}}}^{(1),prim} \quad \Longleftrightarrow \quad \Gamma_s^{(1)}\{\overline{e_i}^{(1)}\} \subset \overline{H_{\mathbb{Z}}}^{(1),prim}.$$

Whether or not $\overline{e_i}^{(1)} \in \overline{H_{\mathbb{Z}}}^{(1),prim}$, that depends on the situation. $\overline{\Delta}^{(1)} \subset \overline{H_{\mathbb{Z}}}^{(1),prim}$ may hold or not.

(ii) In general, an element $a \in H_{\mathbb{Z}}$ satisfies

$$\overline{a}^{(k)} \in \overline{H_{\mathbb{Z}}}^{(k),prim} \quad \Longleftrightarrow \quad a + \mathrm{Rad}\, I^{(k)} \subset H_{\mathbb{Z}}^{prim}.$$

(iii) The set $\overline{\Delta}^{(k)} \subset \overline{H_{\mathbb{Z}}}^{(k)}$ is often simpler to describe than the set $\Delta^{(k)}$. The control of $\overline{\Delta}^{(k)}$ is a step towards the control of $\Delta^{(k)}$.

(iv) Given $a \in \Delta^{(k)}$, it is interesting to understand the three sets

$$(a + \mathrm{Rad}\, I^{(k)}) \cap \Delta^{(k)} \overset{(1)}{\supset} (a + \mathrm{Rad}\, I^{(k)}) \cap \Gamma^{(k)}\{a\} \overset{(2)}{\supset} \Gamma_u^{(k)}\{a\}.$$

The next lemma makes comments on the inclusions $\overset{(1)}{\supset}$ and $\overset{(2)}{\supset}$.

Lemma 6.7. *Keep the situation in Remark 6.6 (iv).*

(a) In $\overset{(1)}{\supset}$ *equality holds if and only if the image in* $\overline{H_{\mathbb{Z}}}^{(k)}$ *of any* $\Gamma^{(k)}$ *orbit different from* $\Gamma^{(k)}\{a\}$ *is different from* $\Gamma_s^{(k)}\{\overline{a}^{(k)}\}$.

(b) The following is an inclusion of groups,

$$\mathrm{Stab}_{\Gamma^{(k)}}(\overline{a}^{(k)}) \overset{(3)}{\supset} \Gamma_u^{(k)} \cdot \mathrm{Stab}_{\Gamma^{(k)}}(a).$$

In $\overset{(2)}{\supset}$ *equality holds if and only if in* $\overset{(3)}{\supset}$ *equality holds.*

Proof: Trivial. □

6.2. The rank 2 cases

For $x \in \mathbb{Z}-\{0\}$ consider the matrix $S = S(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in T_2^{uni}(\mathbb{Z})$, and consider a unimodular bilinear lattice $(H_{\mathbb{Z}}, L)$ with a triangular basis $\underline{e} = (e_1, e_2)$ with $L(\underline{e}^t, \underline{e})^t = S$. Recall the formulas and the results in section 5.2, especially $M^{root} : H_{\mathbb{Z}} \to H_{\mathbb{Z}}$ and its eigenvalues $\kappa_{1/2} = \frac{-x}{2} \pm \frac{1}{2}\sqrt{x^2-4}$.

We can restrict to $x < 0$ because of $L((e_1, -e_2)^t, (e_1, -e_2))^t = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$. So suppose $x < 0$.

First we consider the even cases. Then $\Gamma^{(0)}$ is a Coxeter group, and $\Gamma^{(0)}$ and $\Delta^{(0)}$ are well known. Still we want to document and derive the facts in our way.

Theorem 6.8. *(a) We have*

$$\begin{aligned} s_{e_i}^{(0)}\underline{e} &= \underline{e} \cdot s_{e_i}^{(0),mat} \quad \text{with} \\ s_{e_1}^{(0),mat} &= \begin{pmatrix} -1 & -x \\ 0 & 1 \end{pmatrix}, \quad s_{e_2}^{(0),mat} = \begin{pmatrix} 1 & 0 \\ -x & -1 \end{pmatrix}, \\ \Gamma^{(0)} &\cong \Gamma^{(0),mat} := \langle s_{e_1}^{(0),mat}, s_{e_2}^{(0),mat} \rangle \subset GL_2(\mathbb{Z}), \\ R^{(0)} &= \{y_1 e_1 + y_2 e_2 \in H_{\mathbb{Z}} \,|\, 1 = y_1^2 + x y_1 y_2 + y_2^2\}. \end{aligned}$$

(b) The case $x = -1$*:* $(H_{\mathbb{Z}}, I^{(0)})$ *is the* A_2 *root lattice.* $\Gamma^{(0)} \cong D_6$ *is a dihedral group with six elements, the identity, three reflections and two rotations,*

$$\begin{aligned} \Gamma^{(0)} &= \langle \mathrm{id},\ s_{e_1}^{(0)},\ s_{e_2}^{(0)},\ s_{e_1}^{(0)} s_{e_2}^{(0)} s_{e_1}^{(0)},\ s_{e_1}^{(0)} s_{e_2}^{(0)},\ s_{e_2}^{(0)} s_{e_1}^{(0)} \rangle \cong \Gamma^{(0),mat} \\ &= \langle E_2, \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \rangle. \end{aligned}$$

The set $\Delta^{(0)}$ *of vanishing cycles coincides with the set* $R^{(0)}$ *of roots and is*

$$\Delta^{(0)} = R^{(0)} = \{\pm e_1, \pm e_2, \pm(e_1+e_2)\}.$$

The following picture shows the action of $\Gamma^{(0)}$ *on* $\Delta^{(0)}$*. One sees the action of* D_6 *on the vertices of a regular 6-gon.*

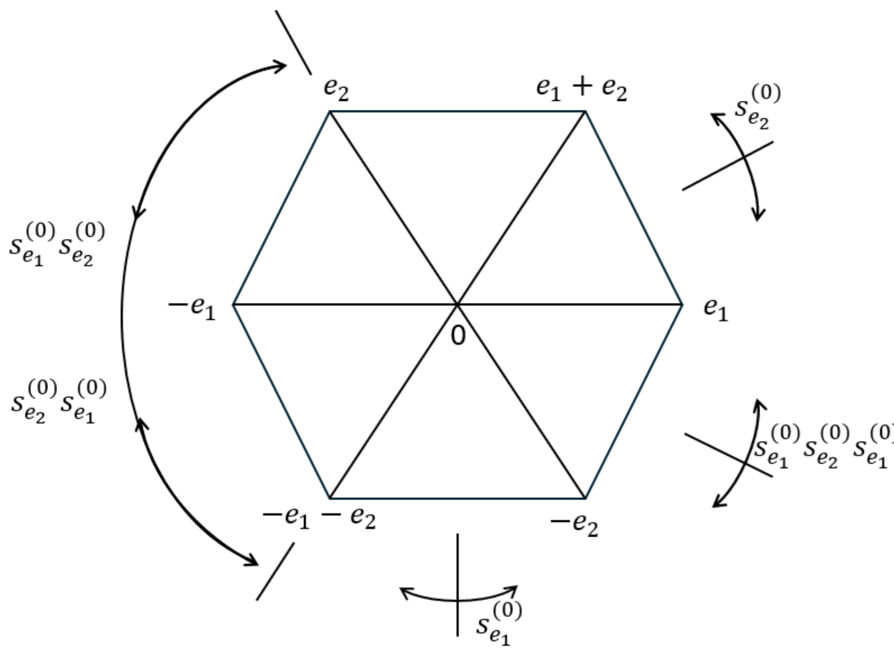


FIGURE 6.1. A regular 6-gon, actions of D_6 and D_{12}

One sees also the action of D_{12} *on the regular 6-gon. This fits to the following.*

$$D_6 \cong \Gamma^{(0)} = \ker \tau^{(0)} = O^{(0),*} \overset{1:2}{\subset} O^{(0)} \cong D_{12}.$$

(c) The case $x = -2$*:* $\Gamma^{(0)} \cong G^{fCox,2}$ *is a free Coxeter group with the two generators* $s_{e_1}^{(0)}$ *and* $s_{e_2}^{(0)}$*. Here*

$$\mathrm{Rad}\, I^{(0)} = \mathbb{Z} f_1 \quad \text{with} \quad f_1 = e_1 + e_2.$$

The set $\Delta^{(0)}$ *of vanishing cycles coincides with the set* $R^{(0)}$ *of roots and is*

$$\begin{aligned} \Delta^{(0)} = R^{(0)} &= \{y_1 e_1 + y_2 e_2 \in H_{\mathbb{Z}} \mid 1 = (y_1 - y_2)^2\} \\ &= (e_1 + \mathbb{Z} f_1) \,\dot\cup\, (e_2 + \mathbb{Z} f_1). \end{aligned}$$

It splits into the two disjoint orbits

$$\Gamma^{(0)}\{e_1\} = \Gamma^{(0)}\{-e_1\} = (e_1 + \mathbb{Z}2f_1) \dot\cup (-e_1 + \mathbb{Z}2f_1),$$
$$\Gamma^{(0)}\{e_2\} = \Gamma^{(0)}\{-e_2\} = (e_2 + \mathbb{Z}2f_1) \dot\cup (-e_2 + \mathbb{Z}2f_1).$$

$s^{(0)}_{e_1}$ *acts on* $\Delta^{(0)}$ *by permuting vanishing cycles horizontally, so by adding* $\pm 2e_1$. $s^{(0)}_{e_2}$ *acts on* $\Delta^{(0)}$ *by permuting vanishing cycles vertically, so by adding* $\pm 2e_2$*, see the following formulas and Figure 6.2. For* $\varepsilon \in \{\pm 1\}$ *and* $m \in \mathbb{Z}$

$$s^{(0)}_{e_1}(\varepsilon e_1 + mf_1) = -\varepsilon e_1 + mf_1, \quad s^{(0)}_{e_1}(\varepsilon e_2 + mf_1) = -\varepsilon e_2 + (m+2\varepsilon)f_1,$$
$$s^{(0)}_{e_2}(\varepsilon e_1 + mf_1) = -\varepsilon e_1 + (m+2\varepsilon)f_1, \quad s^{(0)}_{e_2}(\varepsilon e_2 + mf_1) = -\varepsilon e_2 + mf_1.$$

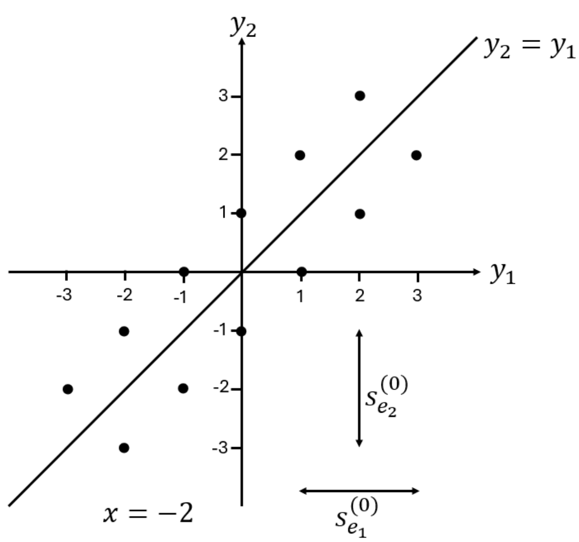


FIGURE 6.2. Even vanishing cycles in the case $S = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$

The matrix group $\Gamma^{(0),mat}$ *is given by the following formulas for* $m \in \mathbb{Z}$,

$$(s^{(0)}_{e_1} s^{(0)}_{e_2})^m(\underline{e}) = \underline{e}\begin{pmatrix} 2m+1 & -2m \\ 2m & -2m+1 \end{pmatrix},$$
$$(s^{(0)}_{e_1} s^{(0)}_{e_2})^m s^{(0)}_{e_1}(\underline{e}) = \underline{e}\begin{pmatrix} -2m-1 & 2m+2 \\ -2m & 2m+1 \end{pmatrix}.$$

(d) The cases $x \leq -3$*:* $\Gamma^{(0)} \cong G^{fCox,2}$ *is a free Coxeter group with the two generators* $s^{(0)}_{e_1}$ *and* $s^{(0)}_{e_2}$*. The set* $\Delta^{(0)}$ *of vanishing cycles coincides with the set* $R^{(0)}$ *of roots. More information on* $\Delta^{(0)}$*:*

(i) Recall Lemma B.1 (a). The map

$$\begin{aligned} u:\Delta^{(0)} &\to \{\text{units in } \mathbb{Z}[\kappa_1] \text{ with norm } 1\} = \{\pm\kappa_1^l \,|\, l\in\mathbb{Z}\} \\ y_1e_1+y_2e_2 &\mapsto y_1-\kappa_1 y_2, \end{aligned}$$

is well defined and a bijection with

$$s_{e_1}^{(0)}(u^{-1}(\varepsilon\kappa_1^l)) = u^{-1}(-\varepsilon\kappa_1^{-l}), \quad s_{e_2}^{(0)}(u^{-1}(\varepsilon\kappa_1^l)) = u^{-1}(-\varepsilon\kappa_1^{2-l}),$$

for $\varepsilon\in\{\pm1\}$, $l\in\mathbb{Z}$. *Especially,* $\Delta^{(0)}$ *splits into the two disjoint orbits* $\Gamma^{(0)}\{e_1\}$ *and* $\Gamma^{(0)}\{e_2\}$.

(ii) The matrix group $\Gamma^{(0),mat}$ *is given by the following formulas for* $m\in\mathbb{Z}$,

$$\begin{aligned} (s_{e_1}^{(0)}s_{e_2}^{(0)})^m(\underline{e}) &= \underline{e}\begin{pmatrix} y_1 & -y_2 \\ y_2 & y_1+xy_2\end{pmatrix} = (u^{-1}(\kappa_1^{-2m}), u^{-1}(-\kappa_1^{-2m+1})), \\ & \quad \text{where } u^{-1}(\kappa_1^{-2m}) = y_1-\kappa_1 y_2, \\ (s_{e_1}^{(0)}s_{e_2}^{(0)})^m s_{e_1}^{(0)}(\underline{e}) &= \underline{e}\begin{pmatrix} y_1 & xy_1+y_2 \\ y_2 & -y_1\end{pmatrix} = (u^{-1}(-\kappa_1^{-2m}), u^{-1}(\kappa_1^{-2m-1})) \\ & \quad \text{where } u^{-1}(-\kappa_1^{-2m}) = y_1-\kappa_1 y_2. \end{aligned}$$

(iii) $\Delta^{(0)}\subset H_\mathbb{R}\cong\mathbb{R}^2$ *is part of the hyperbola* $\{y_1e_1+y_2e_2\in H_\mathbb{R}\,|\,(y_1-\kappa_1y_2)(y_1-\kappa_2y_2)=1\}$ *with asymptotic lines* $y_2=\kappa_2y_1$ *and* $y_2=\kappa_1y_1$. *Both branches of this hyperbola are strictly monotonously increasing. The lower right branch is concave and contains the points* $u^{-1}(\kappa_1^l)$ *for* $l\in\mathbb{Z}$, *the upper left branch is convex and contains the points* $u^{-1}(-\kappa^l)$ *for* $l\in\mathbb{Z}$. *The horizontal respectively vertical line through a vanishing cycle* $a\in\Delta^{(0)}$ *intersects the other branch of the hyperbola in* $s_{e_1}^{(0)}(a)$ *respectively* $s_{e_2}^{(0)}(a)$. *See Figure 6.3.*

(iv) Denote by $\widetilde{s}_{e_i}^{(0)}$, $\widetilde{\Gamma}^{(0)}$ *and* $\widetilde{\Delta}^{(0)}$ *the objects for* $x=-2$ *and as usual by* $s_{e_i}^{(0)}$, $\Gamma^{(0)}$ *and* $\Delta^{(0)}$ *the objects for an* $x\leq-3$.

The map $\widetilde{s}_{e_i}^{(0)}\mapsto s_{e_i}^{(0)}$ *extends to a group isomorphism* $\widetilde{\Gamma}^{(0)}\to\Gamma^{(0)}$. *The map*

$$\begin{aligned} &\widetilde{\Delta}^{(0)}\to\Delta^{(0)}, \\ &\underline{e}\begin{pmatrix}1-l\\-l\end{pmatrix}\mapsto u^{-1}(\kappa_1^l), \quad \underline{e}\begin{pmatrix}l-1\\l\end{pmatrix}\mapsto u^{-1}(-\kappa_1^l) \quad \text{ for } l\in\mathbb{Z}, \end{aligned}$$

is a bijection. The bijections $\widetilde{\Gamma}^{(0)}\to\Gamma^{(0)}$ *and* $\widetilde{\Delta}^{(0)}\to\Delta^{(0)}$ *are compatible with the action of* $\widetilde{\Gamma}^{(0)}$ *on* $\widetilde{\Delta}^{(0)}$ *and of* $\Gamma^{(0)}$ *on* $\Delta^{(0)}$.

(e) More on the cases $x\leq-2$: *The automorphism* $g_{1,2}:H_\mathbb{Z}\to H_\mathbb{Z}$ *with* $g_{1,2}:e_1\leftrightarrow e_2$ *is in* $O^{(0)}$. *The set* $\{\pm\,\mathrm{id},\pm g_{1,2}\}$ *is a subgroup of*

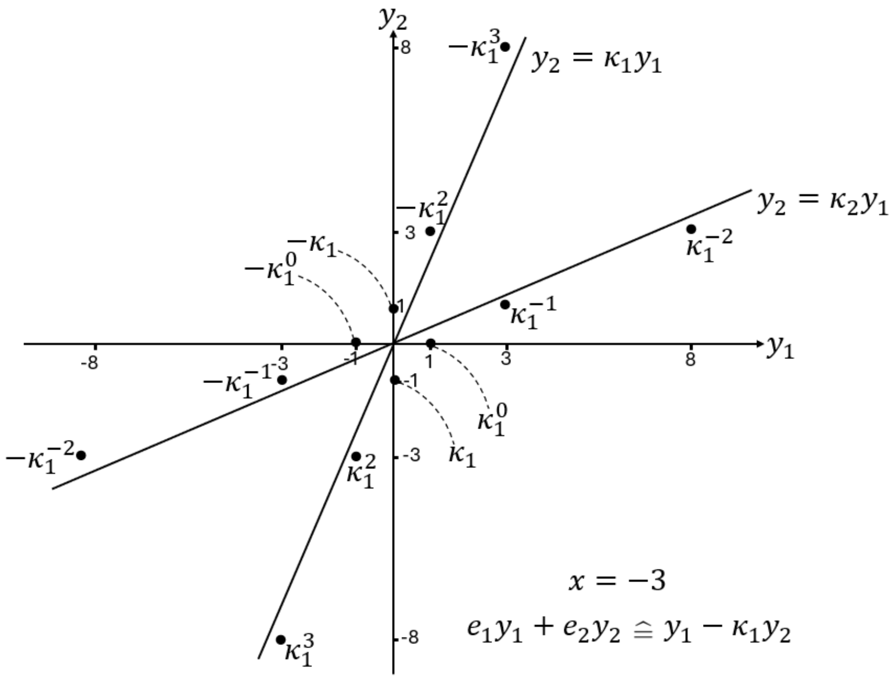


FIGURE 6.3. Even vanishing cycles in the case $S = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$

$O^{(0)}$ *with*

$$\Gamma^{(0)} = \ker \tau^{(0)} = O^{(0),*} \overset{1:4}{\subset} O^{(0)} = \Gamma^{(0)} \rtimes \{\pm \mathrm{id}, \pm g_{1,2}\}.$$

Proof: (a) Everything except possibly the shape of $R^{(0)}$ is obvious.

$$\begin{aligned} R^{(0)} &= \{y_1 e_1 + y_2 e_2 \in H_{\mathbb{Z}} \,|\, 2 = I^{(0)}(y_1 e_1 + y_2 e_2, y_1 e_1 + y_2 e_2)\} \\ &= \{y_1 e_1 + y_2 e_2 \in H_{\mathbb{Z}} \,|\, 2 = \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} 2 & x \\ x & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \} \\ &= \{y_1 e_1 + y_2 e_2 \in H_{\mathbb{Z}} \,|\, 1 = y_1^2 + x y_1 y_2 + y_2^2\}. \end{aligned}$$

(b) This is classical and elementary. $R^{(0)} = \{\pm e_1, \pm e_2, \pm(e_1 + e_2)\}$. The actions of $s_{e_1}^{(0)}$ and $s_{e_2}^{(0)}$ on this set extend to the action of the dihedral group D_6 on the vertices of a regular 6-gon. Therefore $\Delta^{(0)} = R^{(0)}$ and $\Gamma^{(0)} \cong D_6$.

$O^{(0)} \cong D_{12}$ is obvious as the vanishing cycles form the vertices of a regular 6-gon in $(H_{\mathbb{Z}}, I^{(0)})$. It remains to show for some element $g \in O^{(0)} - \Gamma^{(0)}$ $g \notin \ker \tau^{(0)}$.

Consider the reflection $g \in O^{(0)}$ with $g(\underline{e}) = (e_1, -e_1 - e_2)$ and the linear form $l : H_{\mathbb{Z}} \to \mathbb{Z}$ with $l(\underline{e}) = (1, 0)$. Then

$$t^{(0)}(g)(l) = l \circ g^{-1}, \quad (l \circ g^{-1})(\underline{e}) = (l(e_1), l(-e_1 - e_2)) = (1, -1),$$
$$(l - l \circ g^{-1})(\underline{e}) = (0, 1),$$
$$l - l \circ g^{-1} \notin j^{(0)}(H_{\mathbb{Z}}) = \langle(\underline{e} \mapsto (2, -1)), (\underline{e} \mapsto (-1, 2))\rangle,$$

so $g \notin \ker \tau^{(0)}$.

(c) and (d) The group $\Gamma^{(0)}$ for $x \leq -2$: Recall the Remarks and Notations A.1. The matrices $s_{e_1}^{(0),mat} = \begin{pmatrix} -1 & -x \\ 0 & 1 \end{pmatrix}$ and $s_{e_2}^{(0),mat} = \begin{pmatrix} 1 & 0 \\ -x & -1 \end{pmatrix}$ have the eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ respectively $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with eigenvalue -1 and the eigenvectors $\begin{pmatrix} -x/2 \\ 1 \end{pmatrix}$ respectively $\begin{pmatrix} -2/x \\ 1 \end{pmatrix}$ with eigenvalue 1

Therefore $\mu(s_{e_1}^{(0),mat})$ and $\mu(s_{e_2}^{(0),mat}) \in \text{Isom}(\mathbb{H})$ are reflections along the hyperbolic line $A(\infty, -\frac{x}{2})$ respectively $A(0, -\frac{2}{x})$. As $x \leq -2$, we have $-\frac{2}{x} \leq -\frac{x}{2}$, so $A(0, -\frac{2}{x}) \cap A(-\frac{x}{2}, \infty) = \emptyset$, see the pictures in Figure 6.4.

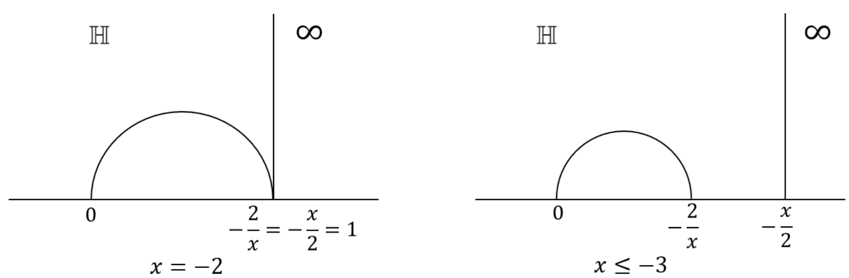


FIGURE 6.4. Fundamental domain in $\mathbb{H}$ of $\Gamma^{(0),mat}$ for $x = -2$ and $x \leq -3$

Theorem A.2 (a) applies and shows that $\langle \mu(s_{e_1}^{(0),mat}), \mu(s_{e_2}^{(0),mat}) \rangle \subset \text{Isom}(\mathbb{H})$ is a free Coxeter group with the two given generators. Therefore also $\Gamma^{(0)}$ is a free Coxeter group with the two generators $s_{e_1}^{(0)}$ and $s_{e_2}^{(0)}$.

(c) The set $\Delta^{(0)}$ for $x = -2$:

$$\begin{aligned} R^{(0)} &= \{y_1 e_1 + y_2 e_2 \in H_{\mathbb{Z}} \,|\, 1 = (y_1 - y_2)^2\} \\ &= (e_1 + \mathbb{Z} f_1) \,\dot\cup\, (e_2 + \mathbb{Z} f_1). \end{aligned}$$

For $m\in\mathbb{Z}$, $\varepsilon\in\{\pm 1\}$,

$$\begin{array}{rcl} s_{e_1}^{(0)}(\varepsilon e_1+mf_1) &=& -\varepsilon e_1+mf_1,\\ s_{e_2}^{(0)}(\varepsilon e_2+mf_1) &=& -\varepsilon e_2+mf_1,\\ s_{e_1}^{(0)}s_{e_2}^{(0)}(e_1+mf_1,e_2+mf_1) &=& (e_1+(m+2)f_1,e_2+(m-2)f_1). \end{array}$$

This shows all claims on $\Delta^{(0)}$ in part (c).

(d) (i) Recall $\kappa_1+\kappa_2=-x$, $\kappa_1\kappa_2=1$, $0=\kappa_i^2+x\kappa_i+1$, $\kappa_2=\kappa_1^{-1}=-\kappa_1-x$, $\kappa_1=-\kappa_2-x$.

Because of $R^{(0)}=\{y_1e_1+y_2e_2\in H_\mathbb{Z}\,|\,1=y_1^2+xy_1y_2+y_2^2\}$ the map

$$\begin{array}{rcl} u:R^{(0)} &\to& \{\text{the units with norm 1 in } \mathbb{Z}[\kappa_1]\}\\ y_1e_1+y_2e_2 &\mapsto& y_1-\kappa_1y_2 \end{array}$$

is well defined and a bijection. Because of Lemma B.1 (a)

$$\{\text{the units with norm 1 in } \mathbb{Z}[\kappa_1]\}=\{\pm\kappa_1^l\,|\,l\in\mathbb{Z}\}.$$

Now

$$s_{e_1}^{(0)}(u^{-1}(\varepsilon\kappa_1^l))=u^{-1}(-\varepsilon\kappa_1^{-l})\quad\text{and}\quad s_{e_2}^{(0)}(u^{-1}(\varepsilon\kappa_1^l))=u^{-1}(-\varepsilon\kappa_1^{2-l})$$

follow from

$$\begin{array}{c} s_{e_1}^{(0)}\underline{e}\begin{pmatrix}y_1\\y_2\end{pmatrix}=\underline{e}\begin{pmatrix}-1&-x\\0&1\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}=\underline{e}\begin{pmatrix}-y_1-xy_2\\y_2\end{pmatrix},\\ (-y_1-xy_2)-\kappa_1y_2=-(y_1-\kappa_2y_2)=-(y_1-\kappa_1y_2)^{-1},\\ s_{e_2}^{(0)}\underline{e}\begin{pmatrix}y_1\\y_2\end{pmatrix}=\underline{e}\begin{pmatrix}1&0\\-x&-1\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}=\underline{e}\begin{pmatrix}y_1\\-xy_1-y_2\end{pmatrix},\\ y_1-\kappa_1(-xy_1-y_2)=\kappa_1((\kappa_2+x)y_1+y_2)=\kappa_1(-\kappa_1y_1+y_2)\\ =-\kappa_1^2(y_1-\kappa_2y_2)=-\kappa_1^2(y_1-\kappa_1y_2)^{-1}. \end{array}$$

This shows $\Delta^{(0)}=R^{(0)}$ and that $\Delta^{(0)}$ splits into the two disjoint orbits $\Gamma^{(0)}\{e_1\}$ and $\Gamma^{(0)}\{e_2\}$.

(ii) The formulas in part (i) show immediately $(s_{e_1}^{(0)}s_{e_2}^{(0)})^m(u^{-1}(\varepsilon\kappa_1^l))=u^{-1}(\varepsilon\kappa_1^{l-2m})$ for $l,m\in\mathbb{Z}$, $\varepsilon\in\{\pm1\}$. Together with $u(e_1)=1$ and $u(e_2)=-\kappa_1$ this implies the formulas in part (ii).

(iv) This follows from the formulas for the action of $\widetilde{s}_{e_i}^{(0)}$ on $\widetilde{R}^{(0)}$ and of $s_{e_i}^{(0)}$ on $R^{(0)}$.

(iii) First we consider the lower right branch of the hyperbola. There $y_1-\kappa_1y_2>0$ and $y_1-\kappa_2y_2>0$. We consider y_2 as an implicit function in y_1. The equation

$$1=y_1^2+xy_1y_2+y_2^2=(y_1-\kappa_1y_2)(y_1-\kappa_2y_2)$$

implies

$$\begin{array}{rcl} 0 &=& (y_1-\kappa_1y_2)(1-\kappa_2y_2')+(y_1-\kappa_2y_2)(1-\kappa_1y_2') \\ &=& [(y_1-\kappa_1y_2)+(y_1-\kappa_2y_2)]-[(y_1-\kappa_1y_2)\kappa_2+(y_1-\kappa_2y_2)\kappa_1]y_2', \\ \text{so} && y_2'>0 \quad\text{and}\quad (1-\kappa_1y_2')(1-\kappa_2y_2')<0, \\ 0 &=& (y_1-\kappa_1y_2)(-\kappa_2y_2'')+2(1-\kappa_1y_2')(1-\kappa_2y_2') \\ && +(y_1-\kappa_2y_2)(-\kappa_1y_2'') \\ &=& -[(y_1-\kappa_1y_2)\kappa_2+(y_1-\kappa_2y_2)\kappa_1]y_2''+2(1-\kappa_1y_2')(1-\kappa_2y_2'), \\ \text{so} && y_2''<0. \end{array}$$

Therefore the lower right branch of the hyperbola is strictly monotonously increasing and concave. The upper left branch is obtained from the lower right branch by the reflection $H_\mathbb{R} \to H_\mathbb{R}$, $(y_1,y_2)\mapsto(y_2,y_1)$, along the diagonal. Therefore it is strictly monotonously increasing and convex.

By definition $s^{(0)}_{e_1}$ maps each horizontal line in $H_\mathbb{R}$ to itself, and $s^{(0)}_{e_2}$ maps each vertical line in $H_\mathbb{R}$ to itself. As they map $\Delta^{(0)}=R^{(0)}$ to itself, this shows all statements in (iii).

(e) Obviously $g_{1,2}\in O^{(0)}$ and $\{\pm\operatorname{id},\pm g_{1,2}\}$ is a subgroup of $O^{(0)}$. Recall

$$\begin{aligned} H^\sharp_\mathbb{Z}\supset j^{(0)}(H_\mathbb{Z}) &= \langle j^{(0)}(e_1), j^{(0)}(e_2)\rangle \\ \text{with}\quad j^{(0)}(e_1)(\underline{e}) &= (2,x), j^{(0)}(e_2)(\underline{e})=(x,2). \end{aligned}$$

Define $l\in H^\sharp_\mathbb{Z}$ with $l(\underline{e})=(1,0)$. Then

$$\begin{aligned} (l-l\circ(-\operatorname{id})^{-1})(\underline{e}) &= 2l(\underline{e})=(2,0), \\ \text{so}\quad l-l\circ(-\operatorname{id})^{-1}\notin j^{(0)}(H_\mathbb{Z}), \text{so} &\quad -\operatorname{id}\notin\ker\tau^{(0)}. \\ (l-l\circ g_{1,2}^{-1})(\underline{e}) &= (1,-1), \\ \text{so}\quad l-l\circ g_{1,2}^{-1}\notin j^{(0)}(H_\mathbb{Z}), \text{so} &\quad g_{1,2}\notin\ker\tau^{(0)}. \end{aligned}$$

Denote for a moment by $\widetilde{O}^{(0)}$ the subgroup of $O^{(0)}$ which is generated by $\{\pm\operatorname{id},\pm g_{1,2}\}$ and $\Gamma^{(0)}$. We just saw

$$\begin{aligned} (\ker\tau^{(0)})\cap\widetilde{O}^{(0)} &= \Gamma^{(0)}, \\ \text{so}\quad \widetilde{O}^{(0)} &= \Gamma^{(0)}\rtimes\{\pm\operatorname{id},\pm g_{1,2}\}. \end{aligned}$$

It remains to show that this subgroup is $O^{(0)}$.

By the parts (c) and (d) (iii) the set $\Delta^{(0)}=R^{(0)}$ splits into two $\Gamma^{(0)}$ orbits, $\Gamma^{(0)}\{e_1\}$ and $\Gamma^{(0)}\{e_2\}$. The element $g_{1,2}$ interchanges them, so $\Delta^{(0)}=R^{(0)}$ is a single $\widetilde{O}^{(0)}$ orbit.

Therefore each element of $O^{(0)}$ can be written as a product of an element in $\widetilde{O}^{(0)}$ and an element $g\in O^{(0)}$ with $g(e_1)=e_1$. It is sufficient to show $g \in \widetilde{O}^{(0)}$. Observe that the set $\{v \in H_{\mathbb{Z}} \,|\, I^{(0)}(v,v) > 0\}$ consists of two components, with e_1 in one component and e_2 in the other component. Because of $g(e_1) = e_1$, $g(e_2)$ is in the same component as e_2. Now $H_{\mathbb{Z}} = \mathbb{Z}e_1 + \mathbb{Z}g(e_2)$ shows $g(e_2) \in \{e_2, s^{(0)}_{e_1}(-e_2)\}$ and $g \in \{\mathrm{id}, -s^{(0)}_{e_1}\}$. Therefore $\widetilde{O}^{(0)} = O^{(0)}$. □

REMARKS 6.9. (i) By part (iv) of Theorem 6.8 (d) the pairs $(\Gamma^{(0)}, \Delta^{(0)})$ with the action of $\Gamma^{(0)}$ on $\Delta^{(0)}$ are isomorphic for all $x \leq -2$. This is interesting as in the case $x = -2$ the set $\Delta^{(0)}$ and this action could be written down in a very simple way.

The parts (i) and (iii) of Theorem 6.8 (d) offered two ways to control the set $\Delta^{(0)}$ and this action also for $x \leq -3$, a number theoretic way and a geometric way. But both ways are less simple than $\Delta^{(0)}$ in the case $x = -2$.

(ii) In the odd cases the situation will be partly similar, partly different. The pairs $(\Gamma^{(1)}, \Delta^{(1)})$ with the action of $\Gamma^{(1)}$ on $\Delta^{(1)}$ are isomorphic for all $x \leq -2$. In the case $x = -2$ the set $\Delta^{(1)}$ and this action can be written down in a fairly simple way. But we lack analoga of the parts (i) and (iii) in Theorem 6.8 (d). We do not have a good control on the sets $\Delta^{(1)}$ for $x \leq -3$.

(iii) For each $x \leq -2$ and $i \in \{1,2\}$ $\mathrm{Stab}_{\Gamma^{(0)}}(e_i) = \{\mathrm{id}\}$, so the map $\Gamma^{(0)} \to \Gamma^{(0)}\{e_i\}$, $\gamma \mapsto \gamma(e_i)$, is a bijection. The action of $\Gamma^{(0)}$ on $\Gamma^{(0)}\{e_i\}$ shows again immediately that $\Gamma^{(0)}$ is a free Coxeter group with generators $s^{(0)}_{e_1}$ and $s^{(0)}_{e_2}$.

Now we come to the odd cases. As before we restrict to $x \in \mathbb{Z}_{<0}$.

THEOREM 6.10. *(a) We have*

$$\begin{aligned} O^{(1)} &\cong SL_2(\mathbb{Z}), \\ \ker \tau^{(1)} &\cong \Gamma(x) := \{A \in SL_2(\mathbb{Z}) \,|\, A \equiv E_2 \bmod x\}, \end{aligned}$$

$$\begin{aligned} s^{(1)}_{e_i}\underline{e} &= \underline{e} \cdot s^{(1),mat}_{e_i} \quad \text{with} \\ s^{(1),mat}_{e_1} &= \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}, \quad s^{(1),mat}_{e_2} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \\ \Gamma^{(1)} &\cong \Gamma^{(1),mat} := \langle s^{(1),mat}_{e_1}, s^{(1),mat}_{e_2} \rangle \subset SL_2(\mathbb{Z}), \end{aligned}$$

The map from $\Delta^{(1)}$ to its image in $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ under the composition $C : \Delta^{(1)} \to \widehat{\mathbb{R}}$ of maps

$$\begin{array}{rcl} \Delta^{(1)} & \to & (\mathbb{R}^*\Delta^{(1)})/\mathbb{R}^* = \{\text{lines through vanishing cycles}\} \\ & \hookrightarrow & (H_{\mathbb{R}} - \{0\})/\mathbb{R}^* = \{\text{lines in } H_{\mathbb{R}}\} \stackrel{\cong}{\longrightarrow} \widehat{\mathbb{R}}, \\ & & \mathbb{R}^*\underline{e}\begin{pmatrix} y_1 \\ 1 \end{pmatrix} \mapsto y_1, \quad \mathbb{R}^* e_1 \mapsto \infty, \end{array}$$

is two-to-one.

(b) The case $x = -1$: $\Gamma^{(1),mat} = SL_2(\mathbb{Z})$.

$$\Delta^{(1)} = \Gamma^{(1)}\{e_1\} = \{y_1e_1 + y_2e_2 \in H_{\mathbb{Z}} \,|\, \gcd(y_1, y_2) = 1\} = H_{\mathbb{Z}}^{prim},$$

where $H_{\mathbb{Z}}^{prim}$ denotes the set of primitive vectors in $H_{\mathbb{Z}}$. The image $C(\Delta^{(1)}) \subset \widehat{\mathbb{R}}$ is $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$.

(c) The case $x = -2$: $\Gamma^{(1)} \cong G^{free,2}$ is a free group with the two generators $s^{(1)}_{e_1}$ and $s^{(1)}_{e_2}$. The (isomorphic) matrix group $\Gamma^{(1),mat}$ is

$$\Gamma^{(1),mat} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,|\, a \equiv d \equiv 1(4),\ b \equiv c \equiv 0(2)\}.$$

It is a subgroup of index 2 in the principal congruence subgroup

$$\Gamma(2) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \,|\, a \equiv d \equiv 1(2),\ b \equiv c \equiv 0(2)\}$$

of $SL_2(\mathbb{Z})$. The set $\Delta^{(1)} = \Gamma^{(1)}\{\pm e_1, \pm e_2\}$ is

$$\Delta^{(1)} = \{y_1e_1 + y_2e_2 \in H_{\mathbb{Z}}^{prim} \,|\, y_1 + y_2 \equiv 1(2)\}.$$

It splits into the four disjoint orbits

$$\begin{array}{rcl} \Gamma^{(1)}\{e_1\} & = & \{y_1e_1 + y_2e_2 \in H_{\mathbb{Z}}^{prim} \,|\, y_1 \equiv 1(4), y_2 \equiv 0(2)\} \\ \Gamma^{(1)}\{-e_1\} & = & \{y_1e_1 + y_2e_2 \in H_{\mathbb{Z}}^{prim} \,|\, y_1 \equiv 3(4), y_2 \equiv 0(2)\} \\ \Gamma^{(1)}\{e_2\} & = & \{y_1e_1 + y_2e_2 \in H_{\mathbb{Z}}^{prim} \,|\, y_1 \equiv 0(2), y_2 \equiv 1(4)\} \\ \Gamma^{(1)}\{-e_2\} & = & \{y_1e_1 + y_2e_2 \in H_{\mathbb{Z}}^{prim} \,|\, y_1 \equiv 0(2), y_2 \equiv 3(4)\}. \end{array}$$

The set $H_{\mathbb{Z}}^{prim}$ of primitive vectors is the disjoint union of $\Delta^{(1)}$ and the set

$$\{y_1e_1 + y_2e_2 \in H_{\mathbb{Z}}^{prim} \,|\, y_1 \equiv y_2 \equiv 1(2)\}.$$

The image $C(\Delta^{(1)}) \subset \widehat{\mathbb{R}}$ is

$$\{\infty\} \cup \{\frac{a}{b} \,|\, a \in \mathbb{Z}, b \in \mathbb{N}, \gcd(a, b) = 1, a \equiv 0(2) \text{ or } b \equiv 0(2)\} \subset \widehat{\mathbb{Q}},$$

and is dense in $\widehat{\mathbb{R}}$.

(d) The cases $x \leq -3$.

(i) $\Gamma^{(1)} \cong G^{free,2}$ *is a free group with the two generators* $s^{(1)}_{e_1}$ *and* $s^{(1)}_{e_2}$.

(ii) The matrix group $\Gamma^{(1),mat}$ *is a Fuchsian group of the second kind. It has infinite index in the group*

$$\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,|\, a \equiv d \equiv 1(x^2), b \equiv c \equiv 0(x)\},$$

which has finite index in $SL_2(\mathbb{Z})$.

(iii) The image $C(\Delta^{(1)}) \subset \widehat{\mathbb{R}}$ *is a subset of* $\widehat{\mathbb{Q}}$ *which is nowhere dense in* $\widehat{\mathbb{R}}$.

(iv) Denote by $\widetilde{s}^{(1)}_{e_i}$, $\widetilde{\Gamma}^{(1)}$ *and* $\widetilde{\Delta}^{(1)}$ *the objects for* $x = -2$ *and as before by* $s^{(1)}_{e_i}$, $\Gamma^{(1)}$ *and* $\Delta^{(1)}$ *the objects for an* $x \leq -3$.

The map $\widetilde{s}^{(1)}_{e_i} \mapsto s^{(1)}_{e_i}$ *extends to a group isomorphism* $\widetilde{\Gamma}^{(1)} \to \Gamma^{(1)}$, *which maps the stabilizer* $\langle \widetilde{s}^{(1)}_{e_i} \rangle$ *of* e_i *in* $\widetilde{\Gamma}^{(1)}$ *to the stabilizer* $\langle s^{(1)}_{e_i} \rangle$ *of* e_i *in* $\Gamma^{(1)}$. *The induced map*

$$\widetilde{\Delta}^{(1)} \to \Delta^{(1)},\ \widetilde{\gamma}(\varepsilon e_i) \mapsto \gamma(\varepsilon e_i) \quad \text{for } \varepsilon \in \{\pm 1\}, \widetilde{\gamma} \mapsto \gamma,$$

is a bijection. The bijections $\widetilde{\Gamma}^{(1)} \to \Gamma^{(1)}$ *and* $\widetilde{\Delta}^{(1)} \to \Delta^{(1)}$ *are compatible with the actions of* $\widetilde{\Gamma}^{(1)}$ *on* $\widetilde{\Delta}^{(1)}$ *and of* $\Gamma^{(1)}$ *on* $\Delta^{(1)}$. *Especially,* $\Delta^{(1)}$ *splits into the four disjoint orbits* $\Gamma^{(1)}\{e_1\}$, $\Gamma^{(1)}\{-e_1\}$, $\Gamma^{(1)}\{e_2\}$, $\Gamma^{(1)}\{-e_2\}$.

(e) In the case A_1^2 $\Delta^{(0)} = \Delta^{(1)} = \{\pm e_1, \pm e_2\}$. *In all other rank 2 cases* $\Delta^{(0)} \subsetneqq \Delta^{(1)}$.

Proof: (a) Because of $I^{(1)}(\underline{e}^t, \underline{e}) = \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}$ we have $O^{(1)} \cong SL_2(\mathbb{Z})$. In order to see $\ker \tau^{(1)} \cong \Gamma(x)$, consider the generators $l_1, l_2 \in H^\sharp_{\mathbb{Z}}$ of $H^\sharp_{\mathbb{Z}}$ with $l_1(\underline{e}) = (1, 0)$ and $l_2(\underline{e}) = (0, 1)$. Observe first

$$j^{(1)}(e_1)(\underline{e}) = (0, x),\ j^{(1)}(e_2)(\underline{e}) = (-x, 0), \quad \text{so } j^{(1)}(H_{\mathbb{Z}}) = xH^\sharp_{\mathbb{Z}},$$

and second that $g \in O^{(1)}$ with $g^{-1}(\underline{e}) = \underline{e} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies

$$\begin{aligned} (l_1 - l_1 \circ g^{-1})(\underline{e}) &= (1 - a, b), \\ (l_2 - l_2 \circ g^{-1})(\underline{e}) &= (-c, 1 - d), \end{aligned}$$

so $g \in \ker \tau^{(1)}$ if and only if $\begin{pmatrix} a & b \\ c & e \end{pmatrix} \equiv E_2 \bmod x$.

It remains to prove that the line $\mathbb{R} \cdot \delta \subset H_{\mathbb{R}}$ through a vanishing cycle $\delta \in \Delta^{(1)}$ intersects $\Delta^{(1)}$ only in $\pm\delta$. To prove this we can restrict to $\delta = e_i$. There it follows from the fact that any matrix $A \in SL_2(\mathbb{Z})$ with a zero in an entry A_{ij} has entries $A_{i,j+1(2)}$, $A_{i+1(2),j} \in \{\pm 1\}$.

(b) $\Gamma^{(1),mat} = SL_2(\mathbb{Z})$ is well known. The standard arguments for this are as follows. The group $\mu(\Gamma^{(1),mat}) \subset \text{Isom}(\mathbb{H})$ is generated by

$$\begin{aligned} \mu(s_{e_1}^{(1),mat}) = \mu(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) &= (z \mapsto z+1), \\ \mu(s_{e_1}^{(1),mat} s_{e_2}^{(1),mat} s_{e_1}^{(1),mat}) = \mu(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) &= (z \mapsto -z^{-1}). \end{aligned}$$

One sees almost immediately that it acts transitively on $\widehat{\mathbb{Q}}$ and that the stabilizer $\langle \mu(s_{e_1}^{(1),mat})\rangle$ of ∞ in $\mu(\Gamma^{(1),mat})$ coincides with the stabilizer of ∞ in $\mu(SL_2(\mathbb{Z}))$. Therefore $\mu(\Gamma^{(1),mat}) = \mu(SL_2(\mathbb{Z}))$. But $-E_2 \in \Gamma^{(1),mat}$ because of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = -E_2$. Therefore $\Gamma^{(1),mat} = SL_2(\mathbb{Z})$.

The fact that $\mu(SL_2(\mathbb{Z}))$ acts transitively on $\widehat{\mathbb{Q}}$ shows $C(\Delta^{(1)}) = \widehat{\mathbb{Q}}$. Together with $-E_2 \in SL_2(\mathbb{Z})$ this shows

$$\Delta^{(1)} = \Gamma^{(1)}\{e_1\} = H_{\mathbb{Z}}^{prim}.$$

(c) and (d) The group $\Gamma^{(1)}$ for $x \leq -2$: Recall the Remarks and Notations A.1. The elements

$$\begin{aligned} \mu(s_{e_1}^{(1),mat}) = \mu(\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}) &= (z \mapsto z - x) \quad \text{and} \\ \mu(s_{e_2}^{(1),mat}) = \mu(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}) &= (z \mapsto \frac{z}{xz+1}) \end{aligned}$$

of $\text{Isom}(\mathbb{H})$ are parabolic with fixed points ∞ respectively 0 on $\widehat{\mathbb{R}}$. Observe

$$\begin{aligned} \mu(s_{e_1}^{(1),mat})^{-1}(1) = 1+x, \quad \mu(s_{e_2}^{(1),mat})(1) = (1+x)^{-1}, \\ (1+x)^{-1} \geq 1+x \quad \text{for } x \leq -2. \end{aligned}$$

Therefore $\mu(s_{e_1}^{(1),mat})^{-1}(A(\infty,1)) = A(\infty, 1+x)$ and $\mu(s_{e_2}^{(1),mat})(A(0,1)) = A(0,(1+x)^{-1})$ do not intersect. See Figure 6.5

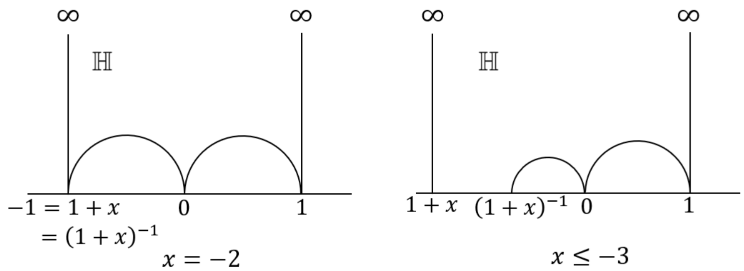


FIGURE 6.5. Fundamental domain in $\mathbb{H}$ of $\Gamma^{(1),mat}$ for $x=-2$ and $x\leq -3$

Theorem A.2 (c) applies and shows that $\mu(\Gamma^{(1),mat})$ is a free group with the two generators $\mu(s_{e_1}^{(1),mat})$ and $\mu(s_{e_2}^{(1),mat})$. Therefore also $\Gamma^{(1)}$ is a free group with the two generators $s_{e_1}^{(1)}$ and $s_{e_2}^{(1)}$. As therefore the map $\Gamma^{(1),mat}\to\mu(\Gamma^{(1),mat})$ is an isomorphism, $-E_2\notin\Gamma^{(1),mat}$ and $-\mathrm{id}\notin\Gamma^{(1)}$.

Theorem A.2 (c) says also that the contractible open set $\mathcal{F}$ whose hyperbolic boundary consists of the four hyperbolic lines which were used above, $A(1+x,\infty)$, $A(\infty,1)$, $A(1,0)$, $A(0,(1+x)^{-1})$, is a fundamental domain for $\mu(\Gamma^{(1),mat})$. If $x=-2$ then its euclidean boundary in $\widehat{\mathbb{C}}$ consists of these four hyperbolic lines and the four points ∞, 1, 0, $-1=1+x=(1+x)^{-1}$. If $x\leq -3$ then its euclidean boundary consists of these four hyperbolic lines, the three points ∞, 1, 0, and the interval $[1+x,(1+x)^{-1}]$.

(c) $\Gamma^{(1),mat}$ and $\Delta^{(1)}$ for $x=-2$: The following facts together imply $\mu(\Gamma^{(1),mat})=\mu(\Gamma(2))$:

$$\Gamma^{(1),mat}\subset\Gamma(2),\quad [SL_2(\mathbb{Z}):\Gamma(2)]=6,\quad -E_2\in\Gamma(2),$$
$$(\text{hyperbolic area of the fundamental domain } \mathcal{F} \text{ of } \mu(\Gamma^{(1),mat}))=2\pi,$$
$$(\text{hyperbolic area of a fundamental domain of } \mu(SL_2(\mathbb{Z})))=\frac{\pi}{3}.$$

Therefore either $\Gamma^{(1),mat}=\Gamma(2)$ or $\Gamma^{(1),mat}$ is a subgroup of index 2 in $\Gamma(2)$. But $\Gamma^{(1),mat}$ is certainly a subgroup of the subgroup

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})\,|\,a\equiv d\equiv 1(4), b\equiv c\equiv 0(2)\}$$

of $\Gamma(2)$ and does not contain $-E_2$. Therefore $\Gamma^{(1),mat}$ coincides with this subgroup of $\Gamma(2)$ and has index 2 in $\Gamma(2)$.

Therefore the orbits $\Gamma^{(1)}\{e_1\}$, $\Gamma^{(1)}\{-e_1\}$, $\Gamma^{(1)}\{e_2\}$, $\Gamma^{(1)}\{-e_2\}$ are contained in the right hand sides of the equations in part (c) which describe them and are disjoint. It remains to show equality.

We restrict to $\Gamma^{(1)}\{e_1\}$. The argument for $\Gamma^{(1)}\{e_2\}$ is analogous, and the equations for $\Gamma^{(1)}\{-e_1\}$ and $\Gamma^{(1)}\{-e_2\}$ follow immediately.

Suppose $y_1, y_3 \in \mathbb{Z}$ with $y_1 \equiv 1(4)$, $y_3 \equiv 0(2)$, $\gcd(y_1, y_3) = 1$. We have to show $\underline{e}\begin{pmatrix} y_1 \\ y_3 \end{pmatrix} \in \Gamma^{(1)}\{e_1\}$. For that we have to find $y_2, y_4 \in \mathbb{Z}$ with $\begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \in \Gamma^{(1)}$, so with $1 = y_1y_4 - y_2y_3$, $y_4 \equiv 1(4)$, $y_2 \equiv 0(2)$.

The condition $\gcd(y_1, y_3)$ implies existence of $\widetilde{y}_2, \widetilde{y}_4 \in \mathbb{Z}$ with $1 = y_1\widetilde{y}_4 - \widetilde{y}_2y_3$.

1st case, $\widetilde{y}_2 \equiv 0(2)$: Then $1 = y_1\widetilde{y}_4 - \widetilde{y}_2y_3$ shows $\widetilde{y}_4 \equiv 1(4)$, and $(y_2, y_4) = (\widetilde{y}_2, \widetilde{y}_4)$ works.

2nd case, $\widetilde{y}_2 \equiv 1(2)$: Then $(y_2, y_4) = (\widetilde{y}_2 + y_1, \widetilde{y}_4 + y_3)$ satisfies $1 = y_1y_4 - y_2y_3$ and $y_2 \equiv 0(2)$, so we are in the 1st case, so (y_2, y_4) works.

Therefore the orbit $\Gamma^{(1)}\{e_1\}$ is as claimed the set $\{y_1e_1 + y_2e_2 \in H_{\mathbb{Z}}^{prim} \,|\, y_1 \equiv 1(4), y_2 \equiv 0(2)\}$.

The statements on $H_{\mathbb{Z}}^{prim}$ and $C(\Delta^{(1)})$ are clear now, too.

(d) $\Gamma^{(1),mat}$ and $\Delta^{(1)}$ for $x \leq -3$: Part (i) was shown above.

(ii) The euclidean boundary of the fundamental domain $\mathcal{F}$ above of $\mu(\Gamma^{(1),mat})$ contains the real interval $[(1+x)^{-1}, 1+x]$. Therefore its hyperbolic area is ∞, $\Gamma^{(1),mat}$ is a Fuchsian group of the second kind, and the index of $\Gamma^{(1),mat}$ in $SL_2(\mathbb{Z})$ is ∞ (see e.g. [**Fo51**, 34.] [**Be83**, §5.3, §8.1]).

(iii) The set $C(\Delta^{(1)}) \subset \widehat{\mathbb{R}}$ is the union of the $\mu(\Gamma^{(1),mat})$ orbits of ∞ and 0 in $\widehat{\mathbb{R}}$. Because $\Gamma^{(1),mat}$ is a Fuchsian group of the second kind, these two orbits are nowhere dense in $\widehat{\mathbb{R}}$ (see e.g. [**Fo51**] [**Be83**]). For example, they contain no point of the open interval $((1+x)^{-1}, (1+x))$ and of its $\mu(\Gamma^{(1),mat})$ orbit.

(iv) The groups $\widetilde{\Gamma}^{(1)}$ and $\Gamma^{(1)}$ are free groups with generators $\widetilde{s}_{e_1}^{(1)}$, $\widetilde{s}_{e_2}^{(1)}$ and $s_{e_1}^{(1)}$, $s_{e_2}^{(1)}$. Therefore there is a unique isomorphism $\widetilde{\Gamma}^{(1)} \to \Gamma^{(1)}$ with $\widetilde{s}_{e_i}^{(1)} \mapsto s_{e_i}^{(1)}$. The other statements follow immediately.

(e) The case $x = 0$, so A_1^2:

$$\Delta^{(0)} = \Delta^{(1)} = \{\pm e_1, \pm e_2\}.$$

The case $x = -1$, so A_2:

$$\Delta^{(0)} = \{\pm e_1, \pm e_2, \pm(e_1 + e_2)\} \subsetneqq H_{\mathbb{Z}}^{prim} = \Delta^{(1)}.$$

The case $x=-2$, so $\mathbb{P}^1A_1$:

$$\begin{aligned} \Delta^{(0)} &= (e_1+\mathbb{Z}f_1)\,\dot\cup\,(e_2+\mathbb{Z}f_1) \\ &\subsetneqq \{y_1e_1+y_2e_2\in H_\mathbb{Z}^{prim}\,|\,y_1+y_2\equiv 1(2)\}=\Delta^{(1)}. \end{aligned}$$

The cases $x\leq -3$: Recall $s_{e_1}^{(0)}s_{e_2}^{(0)}=-M=-s_{e_1}^{(1)}s_{e_2}^{(1)}$. Therefore for $m\in\mathbb{Z},\varepsilon\in\{\pm 1\}$

$$\begin{aligned} (s_{e_1}^{(0)}s_{e_2}^{(0)})^m(\underline{e}) &= (-s_{e_1}^{(1)}s_{e_2}^{(1)})^m(\underline{e})\in(\Delta^{(1)})^2, \\ (s_{e_1}^{(0)}s_{e_2}^{(0)})^m s_{e_1}^{(0)}(e_1) &= -(-s_{e_1}^{(1)}s_{e_2}^{(1)})^m(e_1)\in\Delta^{(1)}, \\ (s_{e_1}^{(0)}s_{e_2}^{(0)})^m s_{e_1}^{(0)}(e_2) &= (s_{e_1}^{(0)}s_{e_2}^{(0)})^{m+1} s_{e_2}^{(0)}(e_2)\in\Delta^{(1)}, \\ &= -(-s_{e_1}^{(1)}s_{e_2}^{(1)})^{m+1}(e_2)\in\Delta^{(1)}. \end{aligned}$$

This shows $\Delta^{(0)}\subset\Delta^{(1)}$. For example $(s_{e_1}^{(1)})^{-1}(e_2)=xe_1+e_2$ satisfies

$$L(xe_1+e_2,xe_1+e_2)=\begin{pmatrix}x & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}\begin{pmatrix}x\\ 1\end{pmatrix}=2x^2+1\neq 1,$$

so $\Delta^{(0)}\subsetneqq\Delta^{(1)}$. □

6.3. The even rank 3 cases

For $\underline{x}=(x_1,x_2,x_3)\in\mathbb{Z}^3$ consider the matrix $S=S(\underline{x})=\begin{pmatrix}1 & x_1 & x_2\\ 0 & 1 & x_3\\ 0 & 0 & 1\end{pmatrix}\in T_3^{uni}(\mathbb{Z})$, and consider a unimodular bilinear lattice $(H_\mathbb{Z},L)$ with a triangular basis $\underline{e}=(e_1,e_2,e_3)$ with $L(\underline{e}^t,\underline{e})^t=S$. In this section we will determine in all cases the even monodromy group $\Gamma^{(0)}=\langle s_{e_1}^{(0)},s_{e_2}^{(0)},s_{e_3}^{(0)}\rangle$ and in many, but not all, cases the set $\Delta^{(0)}=\Gamma^{(0)}\{\pm e_1,\pm e_2,\pm e_3\}$ of even vanishing cycles. The cases where we control $\Delta^{(0)}$ well contain all cases with $r(\underline{x})\in\{0,1,2,4\}$ (3 does not turn up).

The group $\mathrm{Br}_3\ltimes\{\pm 1\}^3$ acts on the set $\mathcal{B}^{tri}$ of triangular bases of $(H_\mathbb{Z},L)$, but this action does not change $\Gamma^{(0)}$ and $\Delta^{(0)}$. Therefore the analysis of the action of $\mathrm{Br}_3\ltimes\{\pm 1\}^3$ on $T_3^{uni}(\mathbb{Z})$ in Theorem 4.6 allows to restrict to the matrices $S(\underline{x})$ with $\underline{x}$ in the following list:

$$\begin{aligned} &S(\underline{x})\text{ with }\underline{x}\in\mathbb{Z}^3_{\leq 0}\text{ and }r(\underline{x})>4, \\ &S(A_1^3),\ S(\mathbb{P}^2),\ S(A_2A_1),\ S(A_3),\ S(\mathbb{P}^1A_1),\ S(\widehat{A}_2),\ S(\mathcal{H}_{1,2}), \\ &S(-l,2,-l)\text{ for }l\geq 3, \\ &\left\{\begin{array}{l} S(\underline{x})\text{ with }\underline{x}\in\mathbb{Z}^3_{\geq 3}\text{ and }r(\underline{x})<0\text{ and} \\ \quad x_i\leq\frac{1}{2}x_jx_k\text{ for }\{i,j,k\}=\{1,2,3\}. \end{array}\right. \end{aligned}$$

The following of these matrices satisfy $\underline{x} \in \mathbb{Z}^3_{\leq 0}$:

$$\begin{gathered} S(\underline{x}) \text{ with } \underline{x} \in \mathbb{Z}^3_{\leq 0} \text{ and } r(\underline{x}) > 4, \\ S(A_1^3),\ S(A_2A_1),\ S(A_3),\ S(\mathbb{P}^1A_1),\ S(\widehat{A}_2). \end{gathered}$$

These are all Coxeter matrices. Their even monodromy groups $\Gamma^{(0)}$ are Coxeter groups and are well known. The cases with $\underline{x} \in \{0,1,2\}^3$ are classical, the extension to all $\underline{x} \in \mathbb{Z}^3_{\leq 0}$ has been done by Vinberg [**Vi71**, Prop. 6, Thm. 1, Thm. 2, Prop. 17]. If we write $(x_1,x_2,x_3) = (S_{12},S_{13},S_{23})$ then the following holds [**Hu90**, 5.3+5.4] [**Vi71**] [**BB05**, 4.1+4.2]: All relations in $\Gamma^{(0)} = \langle s^{(0)}_{e_1}, s^{(0)}_{e_2}, s^{(0)}_{e_3}\rangle$ are generated by the relations

$$\begin{aligned}
(s^{(0)}_{e_i})^2 &= \mathrm{id} && \text{for } i \in \{1,2,3\}, && (6.1)\\
(s^{(0)}_{e_i}s^{(0)}_{e_j})^2 &= \mathrm{id} && \text{for } \{i,j,k\} = \{1,2,3\} \text{ with } s_{ij} = 0 && (6.2)\\
&\text{[equivalent:} && s^{(0)}_{e_i} \text{ and } s^{(0)}_{e_j} \text{ commute]}, && \\
(s^{(0)}_{e_i}s^{(0)}_{e_j})^3 &= \mathrm{id} && \text{for } \{i,j,k\} = \{1,2,3\} \text{ with } s_{ij} = -1, && (6.3)\\
&\text{no relation} && \text{for } \{i,j,k\} = \{1,2,3\} \text{ with } s_{ij} \leq -2. && (6.4)
\end{aligned}$$

Especially, $\Gamma^{(0)} \cong G^{fCox,3}$ is a free Coxeter group with three generators if $\underline{x} \in \mathbb{Z}^3_{\leq -2}$.

In Theorem 6.11 we recover this result, we say more about the Coxeter groups with $r \in \{0,1,2,4\}$, so the cases $S(A_1^3), S(A_2A_1), S(A_3), S(\mathbb{P}^1A_1), S(\widehat{A}_2)$, and we treat also the other cases where $\Gamma^{(0)}$ is not a Coxeter group.

The only cases where $\mathrm{Rad}\, I^{(0)} \supsetneqq \{0\}$ are the cases with $r(\underline{x}) = 4$, so the cases $S(\mathbb{P}^1A_1)$, $S(\widehat{A}_2)$, $S(\mathcal{H}_{1,2})$ and $S(-l,2,-l)$ with $l \geq 3$. In these cases, we have the exact sequence

$$\{1\} \to \Gamma^{(0)}_u \to \Gamma^{(0)} \to \Gamma^{(0)}_s \to \{1\} \tag{6.5}$$

in Lemma 6.2 (d).

Theorem 6.11. *(a) We have*

$$\begin{aligned}
s^{(0)}_{e_i}\underline{e} &= \underline{e}\cdot s^{(0),mat}_{e_i} \quad\text{with}\quad s^{(0),mat}_{e_1} = \begin{pmatrix} -1 & -x_1 & -x_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\\
s^{(0),mat}_{e_2} &= \begin{pmatrix} 1 & 0 & 0 \\ -x_1 & -1 & -x_3 \\ 0 & 0 & 1 \end{pmatrix},\ s^{(0),mat}_{e_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_2 & -x_3 & -1 \end{pmatrix},\\
\Gamma^{(0)} &\cong \Gamma^{(0),mat} := \langle s^{(0),mat}_{e_1}, s^{(0),mat}_{e_2}, s^{(0),mat}_{e_3}\rangle \subset GL_3(\mathbb{Z}),\\
R^{(0)} &= \{y_1e_1+y_2e_2+y_3e_3\in H_{\mathbb{Z}}\,|\\
&\quad 1 = y_1^2+y_2^2+y_3^2+x_1y_1y_2+x_2y_1y_3+x_3y_2y_3\}.
\end{aligned}$$

(b) In the cases $S(\underline{x})$ with $\underline{x}\in\mathbb{Z}^3_{\leq 0}$ and $r(\underline{x})>4$ and in the reducible cases $S(A_1^3), S(A_2A_1), S(\mathbb{P}^1A_1)$, all relations in $\Gamma^{(0)}$ are generated by the relations in (6.1)–(6.4). *Especially*

$$\begin{aligned}
\Gamma^{(0)}(A_1^3) &\cong \Gamma^{(0)}(A_1)\times\Gamma^{(0)}(A_1)\times\Gamma^{(0)}(A_1) \cong (G^{fCox,1})^3 \cong \{\pm 1\}^3,\\
\Gamma^{(0)}(A_2A_1) &\cong \Gamma^{(0)}(A_2)\times\Gamma^{(0)}(A_1) \cong D_6\times\{\pm 1\} \cong S_3\times\{\pm 1\},\\
\Gamma^{(0)}(\mathbb{P}^1A_1) &\cong \Gamma^{(0)}(\mathbb{P}^1)\times\Gamma^{(0)}(A_1) \cong G^{fCox,2}\times\{\pm 1\}.
\end{aligned}$$

(c) In the case $S(A_3)$ the group $\Gamma^{(0)}$ is the Weyl group of the root system A_3, so $\Gamma^{(0)} = \ker\tau^{(0)} = O^{(0),*}\cong S_4$.

(d) In the case $S(\widehat{A}_2)$ the group $\Gamma^{(0)}$ is the Weyl group of the affine root system $\widehat{A}_2$. More concretely, the following holds.

$$\begin{aligned}
\operatorname{Rad} I^{(0)} &= \mathbb{Z}f_1 \text{ with } f_1 = e_1+e_2+e_3,\\
\overline{H_{\mathbb{Z}}}^{(0)} &= \mathbb{Z}\overline{e_1}^{(0)}\oplus\mathbb{Z}\overline{e_2}^{(0)},\\
\Gamma^{(0)}_u &= \big(\ker\tau^{(0)}\big)_u = T(\overline{j}^{(0)}(\overline{H_{\mathbb{Z}}}^{(0)})\otimes\mathbb{Z}f_1)\\
&= \langle T(\overline{j}^{(0)}(\overline{e_1}^{(0)})\otimes f_1), T(\overline{j}^{(0)}(\overline{e_2}^{(0)})\otimes f_1)\rangle \cong \mathbb{Z}^2 \quad\text{with}\\
&\quad T(\overline{j}^{(0)}(\overline{e_1}^{(0)})\otimes f_1)(\underline{e}) = \underline{e}+f_1(2,-1,-1),\\
&\quad T(\overline{j}^{(0)}(\overline{e_2}^{(0)})\otimes f_1)(\underline{e}) = \underline{e}+f_1(-1,2,-1),\\
\Gamma^{(0)}_u &= \big(\ker\tau^{(0)}\big)_u \overset{1:3}{\subset} O^{(0),Rad}_u = T(\overline{H_{\mathbb{Z}}}^{(0),\sharp}\otimes\mathbb{Z}f_1),\\
\Gamma^{(0)}_s &= (\ker\tau^{(0)})_s \cong \Gamma^{(0)}(A_2)\cong D_6\cong S_3,\\
\Gamma^{(0)}_s &\overset{1:2}{\subset} O^{(0),Rad}_s = \operatorname{Aut}(\overline{H_{\mathbb{Z}}}^{(0)},\overline{I}^{(0)})\cong D_{12},\\
\Gamma^{(0)} &= \ker\tau^{(0)} = O^{(0),*}\overset{1:6}{\subset} O^{(0),Rad}.
\end{aligned}$$

The exact sequence (6.5) *splits non-canonically with $\Gamma^{(0)}_s\cong\langle s^{(0)}_{e_1}, s^{(0)}_{e_2}\rangle\subset\Gamma^{(0)}$ (for example).*

(e) *The case* $S(\mathcal{H}_{1,2})$*: The following holds.*

$$\begin{aligned}
H_\mathbb{Z} &= \mathbb{Z}f_3 \oplus \mathrm{Rad}\, I^{(0)} \text{ with } f_3 = e_1+e_2+e_3,\\
\overline{H_\mathbb{Z}}^{(0)} &= \mathbb{Z}\overline{f_3}^{(0)},\\
\mathrm{Rad}\, I^{(0)} &= \mathbb{Z}f_1 \oplus \mathbb{Z}f_2 \text{ with } f_1 = e_1+e_2,\ f_2 = e_2+e_3,\\
\Gamma_u^{(0)} &= (\ker\tau^{(0)})_u = T(\overline{j}^{(0)}(\overline{H_\mathbb{Z}}^{(0)}) \otimes \mathrm{Rad}\, I^{(0)})\\
&= \langle T(\overline{j}^{(0)}(\overline{f_3}^{(0)})\otimes f_1), T(\overline{j}^{(0)}(\overline{f_3}^{(0)})\otimes f_2)\rangle \cong \mathbb{Z}^2 \quad \text{with}\\
&\ T(\overline{j}^{(0)}(\overline{f_3}^{(0)})\otimes f_1)(f_1,f_2,f_3) = (f_1,f_2,f_3+2f_1),\\
&\ T(\overline{j}^{(0)}(\overline{f_3}^{(0)})\otimes f_2)(f_1,f_2,f_3) = (f_1,f_2,f_3+2f_2),\\
\Gamma_u^{(0)} &\overset{1:4}{\subset} O_u^{(0),\mathrm{Rad}} = T(\overline{H_\mathbb{Z}}^{(0),\sharp} \otimes \mathrm{Rad}\, I^{(0)}),\\
\Gamma_s^{(0)} &= (\ker\tau^{(0)})_s = O_s^{(0),Rad} \cong \Gamma^{(0)}(A_1) \cong \{\pm 1\},\\
\Gamma^{(0)} &= \ker\tau^{(0)} = O^{(0),*} \overset{1:4}{\subset} O^{(0),Rad}.
\end{aligned}$$

The exact sequence (6.5) *splits non-canonically with* $\Gamma_s^{(0)} \cong \langle -M\rangle \subset \Gamma^{(0)}$ *and* $-M(f_1,f_2,f_3) = (f_1,f_2,-f_3)$*. Therefore*

$$\Gamma^{(0)} = \{(f_1,f_2,f_3) \mapsto (f_1,f_2,\varepsilon f_3 + 2\beta_1 f_1 + 2\beta_2 f_2)\,|\,\varepsilon\in\{\pm 1\}, \beta_1,\beta_2\in\mathbb{Z}\}.$$

(f) *The cases* $S(-l,2,-l)$ *with* $l\geq 3$*: The following holds.*

$$\begin{aligned}
\mathrm{Rad}\, I^{(0)} &= \mathbb{Z}f_1 \text{ with } f_1 = e_1 - e_3,\\
\overline{H_\mathbb{Z}}^{(0)} &= \mathbb{Z}\overline{e_1}^{(0)} \oplus \mathbb{Z}\overline{e_2}^{(0)},\\
&\ \overline{I}^{(0)}((\overline{e_1}^{(0)},\overline{e_2}^{(0)})^t, (\overline{e_1}^{(0)},\overline{e_2}^{(0)})) = \begin{pmatrix} 2 & -l\\ -l & 2\end{pmatrix},\\
\Gamma_u^{(0)} &= \langle T(\overline{j}^{(0)}(\overline{e_1}^{(0)})\otimes f_1), T(\overline{j}^{(0)}(l\overline{e_2}^{(0)})\otimes f_1)\rangle \cong \mathbb{Z}^2 \quad\text{with}\\
&\ T(\overline{j}^{(0)}(\overline{e_1}^{(0)})\otimes f_1)(\underline{e}) = \underline{e} + f_1(2,-l,2),\\
&\ T(\overline{j}^{(0)}(l\overline{e_2}^{(0)})\otimes f_1)(\underline{e}) = \underline{e} + f_1(-l^2,2l,-l^2),\\
\Gamma_u^{(0)} &\overset{1:l}{\subset} (\ker\tau^{(0)})_u \overset{1:(l^2-4)}{\subset} O_u^{(0),Rad} \cong \mathbb{Z}^2,\\
\Gamma_s^{(0)} &\cong \Gamma^{(0)}(S(-l)) \cong G^{fCox,2},\\
\Gamma_s^{(0)} &= (\ker\tau^{(0)})_s \cap \ker\overline{\sigma} \overset{1:4}{\subset} O_s^{(0),Rad},\\
\Gamma^{(0)} &\overset{1:l}{\subset} O^{(0),*} \overset{1:4(l^2-4)}{\subset} O^{(0),Rad}.
\end{aligned}$$

The exact sequence (6.5) *splits non-canonically with* $\Gamma_s^{(0)} \cong \langle s_{e_1}^{(0)}, s_{e_2}^{(0)}\rangle \subset \Gamma^{(0)}$ *(for example).*

(g) The case $S(\mathbb{P}^2)$ and the cases $S(\underline{x})$ with $\underline{x}\in\mathbb{Z}^3_{\geq 3}$, $r(\underline{x})<0$ and $x_i\leq \frac{1}{2}x_jx_k$ for $\{i,j,k\}=\{1,2,3\}$: $\Gamma^{(0)}\cong G^{fCox,3}$ is a free Coxeter group with the three generators $s^{(0)}_{e_1},s^{(0)}_{e_2},s^{(0)}_{e_3}$.

Proof: (a) This follows from the definitions in Lemma 2.6 (a) and in Definition 2.8. Especially

$$\begin{aligned} R^{(0)} &= \{\underline{e}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}\in H_{\mathbb{Z}}\,|\,2=I^{(0)}((\underline{e}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix})^t,\underline{e}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix})\\ &= \{\underline{e}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}\in H_{\mathbb{Z}}\,|\,2=(y_1\ y_2\ y_3)\begin{pmatrix}2&x_1&x_2\\x_1&2&x_3\\x_2&x_3&2\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}\}. \end{aligned}$$

(b) First we consider the cases $S(\underline{x})$ with $\underline{x}\in\mathbb{Z}^3_{\leq 0}$ and $r(\underline{x})>4$. By Lemma 5.7 (b) $\operatorname{sign} I^{(0)}=(++-)$. We will apply Theorem A.4 with $I^{[0]}:=-I^{(0)}$, which has signature $\operatorname{sign} I^{[0]}=(+--)$.

The vectors e_1,e_2,e_3 are negative with respect to $I^{[0]}$. By Theorem A.4 (c) (vi) the reflection $s^{(0)}_{e_i}$ acts on the model $\mathcal{K}/\mathbb{R}^*$ of the hyperbolic plane as reflection along the hyperbolic line $((\mathbb{R}e_i)^\perp\cap\mathcal{K})/\mathbb{R}^*\subset\mathcal{K}/\mathbb{R}^*$. The corresponding three planes $(\mathbb{R}e_1)^\perp$, $(\mathbb{R}e_2)^\perp$ and $(\mathbb{R}e_3)^\perp$ in $H_{\mathbb{R}}$ intersect pairwise in the following three lines

$$\begin{aligned} (\mathbb{R}e_1)^\perp\cap(\mathbb{R}e_2)^\perp=\mathbb{R}y^{[1]},\quad & y^{[1]}=(-2x_2+x_1x_3,-2x_3+x_1x_2,4-x_1^2),\\ (\mathbb{R}e_1)^\perp\cap(\mathbb{R}e_3)^\perp=\mathbb{R}y^{[2]},\quad & y^{[2]}=(-2x_1+x_2x_3,4-x_2^2,-2x_3+x_1x_2),\\ (\mathbb{R}e_2)^\perp\cap(\mathbb{R}e_3)^\perp=\mathbb{R}y^{[3]},\quad & y^{[3]}=(4-x_3^2,-2x_1+x_2x_3,-2x_2+x_1x_3). \end{aligned}$$

$\underline{x}\in\mathbb{Z}^3_{\leq 0}$ and $y^{[1]}=0$ would imply $x_2=x_3=0$, $x_1=-2$, $r(\underline{x})=4$. But $r(\underline{x})>4$ by assumption. Therefore $y^{[1]}\neq 0$. Analogously $y^{[2]}\neq 0$ and $y^{[3]}\neq 0$.

One calculates

$$\begin{aligned} I^{[0]}(y^{[i]},y^{[i]}) &= 2(4-x_i^2)(r(\underline{x})-4)\quad\text{for } i\in\{1,2,3\}\\ &\begin{cases}\leq 0 & \text{for } x_i\leq -2,\\ >0 & \text{for } x_i\in\{0,-1\}.\end{cases} \end{aligned}$$

Therefore two of the three hyperbolic lines $((\mathbb{R}e_j)^\perp\cap\mathcal{K})/\mathbb{R}^*$ ($j\in\{1,2,3\}$) intersect in $\mathcal{K}/\mathbb{R}^*$ if and only if the corresponding x_i is 0 or -1.

Claim: *If $x_i\in\{0,-1\}$ then the angle between the two hyperbolic lines at the intersection point $\mathbb{R}^*y^{[i]}\in\mathcal{K}/\mathbb{R}^*$ is $\frac{\pi}{2}$ if $x_i=0$ and $\frac{\pi}{3}$ if $x_i=-1$.*

We prove the claim in an indirect way. Observe in general

$$\begin{aligned}
x_1=0\Rightarrow \big(s_{e_1}^{(0),mat}s_{e_2}^{(0),mat}\big)^2 &= (\begin{pmatrix}-1&0&-x_2\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&-1&-x_3\\0&0&1\end{pmatrix})^2\\
&= \begin{pmatrix}-1&0&-x_2\\0&-1&-x_3\\0&0&1\end{pmatrix}^2 = E_3, \qquad (6.6)\\
x_1=-1\Rightarrow \big(s_{e_1}^{(0),mat}s_{e_2}^{(0),mat}\big)^3 &= (\begin{pmatrix}-1&1&-x_2\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&-1&-x_3\\0&0&1\end{pmatrix})^3\\
&= \begin{pmatrix}-1&-1&-x_2-x_3\\0&-1&-x_3\\0&0&1\end{pmatrix}^3 = E_3, \qquad (6.7)
\end{aligned}$$

and analogously for x_2 and x_3. Therefore the angle between the hyperbolic lines must be $\frac{\pi}{2}$ if $x_i=0$ and $\frac{\pi}{3}$ or $\frac{2\pi}{3}$ if $x_i=-1$. But in the case $x_1=x_2=x_3=-1$ the three intersection points are the vertices of a hyperbolic triangle, so then the angles are all $\frac{\pi}{3}$. Deforming x_2 and x_3 does not change the angle at $\mathbb{R}^*y^{[1]}$, so it is $\frac{\pi}{3}$ if $x_1=-1$. This proves the Claim. ($\Box$)

Now Theorem A.2 (a) shows that in the group of automorphisms of $\mathcal{K}/\mathbb{R}^*$ which is induced by $\Gamma^{(0)}$ all relations are generated by the relations in (6.1)–(6.4). Therefore this holds also for $\Gamma^{(0)}$ itself.

Now we consider the three reducible cases $S(A_1^3)$, $S(A_2A_1)$ and $S(\mathbb{P}^1A_1)$. Lemma 2.11 gives the first isomorphisms in part (b) for $\Gamma^{(0)}(A_1^3)$, $\Gamma^{(0)}(A_2A_1)$ and $\Gamma^{(0)}(\mathbb{P}^1A_1)$. Lemma 2.12 (for A_1) and Theorem 6.8 (b) and (c) give the second isomorphisms in part (b). The isomorphisms show in these three cases that all relations in $\Gamma^{(0)}$ are generated by the relations in (6.1)–(6.4).

(c) It is classical that in the case of the A_3 root lattice the monodromy group $\Gamma^{(0)}$ is the Weyl group and is $\ker\tau^{(0)}\cong S_4$.

(d) The proof of Theorem 5.14 (b) (iii) shows

$$\begin{aligned}
\operatorname{Rad} I^{(0)} &= \ker\Phi_2(M)=\ker\Phi_1(M^{root})=\mathbb{Z}f_1,\\
\overline{H_{\mathbb{Z}}}^{(0)} &= \mathbb{Z}\overline{e_1}^{(0)}\oplus\mathbb{Z}\overline{e_2}^{(0)},\\
&\quad \overline{I}^{(0)}((\overline{e_1}^{(0)},\overline{e_2}^{(0)})^t,(\overline{e_1}^{(0)},\overline{e_2}^{(0)}))=\begin{pmatrix}2&-1\\-1&2\end{pmatrix},
\end{aligned}$$

so $(\overline{H_{\mathbb{Z}}}^{(0)},\overline{I}^{(0)})$ is an A_2 root lattice. This was treated in Theorem 6.8 (b). We have

$$\begin{aligned}
O_s^{(0),Rad} &= \mathrm{Aut}(\overline{H_{\mathbb{Z}}}^{(0)},\overline{I}^{(0)}) \cong D_{12},\\
\overline{e_3}^{(0)} &= -\overline{e_1}^{(0)} - \overline{e_2}^{(0)},\\
R^{(0)}(\overline{H_{\mathbb{Z}}}^{(0)},\overline{I}^{(0)}) &= \{\pm\overline{e_1}^{(0)},\pm\overline{e_2}^{(0)},\pm\overline{e_3}^{(0)}\},\\
\overline{\Gamma^{(0)}} = \Gamma_s^{(0)} &= \langle \overline{s_{e_1}^{(0)}}, \overline{s_{e_2}^{(0)}}\rangle \cong \Gamma^{(0)}(A_2) \cong D_6 \cong S_3,\\
\Gamma_s^{(0)} &= (\ker\tau^{(0)})_s \overset{1:2}{\subset} O_s^{(0),Rad}.
\end{aligned}$$

Observe also

$$\begin{aligned}
& s_{e_2}^{(0)} s_{e_1}^{(0)} s_{e_2}^{(0)} s_{e_3}^{(0)}(\underline{e})\\
=\ & \underline{e}\begin{pmatrix}1&0&0\\1&-1&1\\0&0&1\end{pmatrix}\begin{pmatrix}-1&1&1\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\1&-1&1\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\1&1&-1\end{pmatrix}\\
=\ & \underline{e} + f_1(1,1,-2) = T(\overline{j}^{(0)}(-\overline{e_3}^{(0)})\otimes f_1)(\underline{e}),
\end{aligned}$$

so $T(j^{(0)}(-\overline{e_3}^{(0)}) \otimes f_1) \in \Gamma_u^{(0)}$. Compare Lemma 6.2 (f) and recall that $\Gamma_s^{(0)}$ acts transitively on $\{\pm\overline{e_1}^{(0)},\pm\overline{e_2}^{(0)},\pm\overline{e_3}^{(0)}\}$. Therefore $T(\overline{j}^{(0)}(\overline{e_j}^{(0)})\otimes f_1) \in \Gamma_u^{(0)}$ for $j\in\{1,2,3\}$ with

$$\begin{aligned}
T(\overline{j}^{(0)}(\overline{e_1}^{(0)})\otimes f_1)(\underline{e}) &= \underline{e} + f_1(2,-1,-1),\\
T(\overline{j}^{(0)}(\overline{e_2}^{(0)})\otimes f_1)(\underline{e}) &= \underline{e} + f_1(-1,2,-1),
\end{aligned}$$

so

$$\begin{aligned}
\Gamma_u^{(0)} &= T(\overline{j}^{(0)}(\overline{H_{\mathbb{Z}}}^{(0)})\otimes f_1) = (\ker\tau^{(0)})_u\\
&\subset T(\overline{H_{\mathbb{Z}}}^{(0),\sharp}\otimes f_1) = O_u^{(0),Rad} \cong \mathbb{Z}^2.
\end{aligned}$$

$T(\overline{H_{\mathbb{Z}}}^{(0),\sharp}\otimes f_1)$ is generated by

$$(\underline{e}\mapsto \underline{e}+f_1(1,-1,0)) \quad\text{and}\quad (\underline{e}\mapsto\underline{e}+f_1(0,1,-1)).$$

Therefore $\Gamma_u^{(0)} \overset{1:3}{\subset} O_u^{(0),Rad}$.

Together $\Gamma_s^{(0)} = (\ker\tau^{(0)})_s \overset{1:2}{\subset} \mathcal{O}_s^{(0),Rad}$ and $\Gamma_u^{(0)} = (\ker\tau^{(0)})_u \overset{1:3}{\subset} \mathcal{O}_u^{(0),Rad}$ show

$$\Gamma^{(0)} = \ker\tau^{(0)} \overset{1:6}{\subset} O^{(0),Rad}$$

(6.7) and $x_1=-1$ show $\langle s_{e_1}^{(0)}, s_{e_2}^{(0)}\rangle \cong D_6$, so $\Gamma_s^{(0)} \cong \langle s_{e_1}^{(0)}, s_{e_2}^{(0)}\rangle \subset \Gamma^{(0)}$, so the exact sequence (6.5) splits non-canonically.

(e) Recall from the proof of Theorem 5.14 (a) (i) that

$$\underline{f} = (f_1, f_2, f_3) := \underline{e}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

is a $\mathbb{Z}$-basis of $H_\mathbb{Z}$ and

$$\begin{array}{rcl} \mathrm{Rad}\, I^{(0)} &=& \mathbb{Z}f_1 \oplus \mathbb{Z}f_2, \\ H_\mathbb{Z} &=& \mathbb{Z}f_3 \oplus \mathrm{Rad}\, I^{(0)}, \\ \overline{H_\mathbb{Z}}^{(0)} &=& \mathbb{Z}\overline{f_3}^{(0)}. \end{array}$$

Also observe

$$\begin{array}{rcl} s^{(0)}_{e_j}|_{\mathrm{Rad}\, I^{(0)}} &=& \mathrm{id} \quad \text{ for } i \in \{1,2,3\}, \\ s^{(0)}_{e_1}(f_3) &=& -f_3 + 2f_2, \\ s^{(0)}_{e_2}(f_3) &=& -f_3 + 2f_1 + 2f_2, \\ s^{(0)}_{e_3}(f_3) &=& -f_3 + 2f_1, \\ \overline{\Gamma^{(0)}} = \Gamma^{(0)}_s &=& \{\pm\,\mathrm{id}\} = (\ker\tau^{(0)})_s = O^{(0),Rad}_s \cong \Gamma^{(0)}(A_1) \cong \{\pm 1\}. \end{array}$$

Therefore

$$\Gamma^{(0)}_u \ni s^{(0)}_{e_1}s^{(0)}_{e_2} = (\underline{f} \mapsto \underline{f} + 2f_1(0,0,1)) = T(\overline{j}^{(0)}(\overline{f_3}^{(0)}) \otimes f_1),$$

$$\Gamma^{(0)}_u \ni s^{(0)}_{e_3}s^{(0)}_{e_2} = (\underline{f} \mapsto \underline{f} + 2f_2(0,0,1)) = T(\overline{j}^{(0)}(\overline{f_3}^{(0)}) \otimes f_2),$$

so

$$\begin{array}{rcl} \Gamma^{(0)}_u &=& (\ker\tau^{(0)})_u = T(\overline{j}^{(0)}(\overline{H_\mathbb{Z}}^{(0)}) \otimes \mathrm{Rad}\, I^{(0)}) \\ &\stackrel{1:4}{\subset}& O^{(0),Rad} = T(\overline{H_\mathbb{Z}}^{(0),\sharp} \otimes \mathrm{Rad}\, I^{(0)}) \\ &=& \langle (\underline{f} \mapsto \underline{f} + f_1(0,0,1)), (\underline{f} \mapsto \underline{f} + f_2(0,0,1)) \rangle \cong \mathbb{Z}^2. \end{array}$$

Together the statements on $\Gamma^{(0)}_s$ and $\Gamma^{(0)}_u$ imply

$$\Gamma^{(0)} = \ker\tau^{(0)} \stackrel{1:4}{\subset} O^{(0),Rad}.$$

The exact sequence (6.5) splits non-canonically with $\Gamma^{(0)}_s = \{\pm id\} \cong \langle -M \rangle \subset \Gamma^{(0)}$ (for example).

(f) Recall from the proof of Theorem 5.14 (a) (ii) and (b) (iv)

$$\begin{array}{rcl} \mathrm{Rad}\, I^{(0)} &=& \mathbb{Z}f_1 \quad \text{with} \quad f_1 = e_1 - e_3, \quad \text{so} \\ \overline{H_\mathbb{Z}}^{(0)} &=& \mathbb{Z}\overline{e_1}^{(0)} \oplus \mathbb{Z}\overline{e_2}^{(0)}, \\ && \overline{I}^{(0)}((\overline{e_1}^{(0)}, \overline{e_2}^{(0)})^t, (\overline{e_1}^{(0)}, \overline{e_2}^{(0)})) = \begin{pmatrix} 2 & -l \\ -l & 2 \end{pmatrix}. \end{array}$$

Observe

$$s_{e_1}^{(0),mat} = \begin{pmatrix} -1 & l & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ s_{e_2}^{(0),mat} = \begin{pmatrix} 1 & 0 & 0 \\ l & -1 & l \\ 0 & 0 & 1 \end{pmatrix},$$

$$s_{e_3}^{(0),mat} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & l & -1 \end{pmatrix},$$

$$\overline{s_{e_1}^{(0)}}(\overline{e_1}^{(0)}, \overline{e_2}^{(0)}) = \overline{s_{e_3}^{(0)}}(\overline{e_1}^{(0)}, \overline{e_2}^{(0)}) = (\overline{e_1}^{(0)}, \overline{e_2}^{(0)}) \begin{pmatrix} -1 & l \\ 0 & 1 \end{pmatrix},$$

$$\overline{s_{e_2}^{(0)}}(\overline{e_1}^{(0)}, \overline{e_2}^{(0)}) = \begin{pmatrix} 1 & 0 \\ l & -1 \end{pmatrix}.$$

Theorem 6.8 (d) shows

$$\overline{\Gamma^{(0)}} = \Gamma_s^{(0)} \cong \Gamma^{(0)}(S(-l)) \cong G^{fCox,2}.$$

Therefore with respect to the generators $\overline{s_{e_1}^{(0)}}, \overline{s_{e_2}^{(0)}}$ and $\overline{s_{e_3}^{(0)}}$, all relations in $\Gamma_s^{(0)}$ are generated by the relations

$$(\overline{s_{e_1}^{(0)}})^2 = (\overline{s_{e_2}^{(0)}})^2 = \overline{s_{e_1}^{(0)}}\ \overline{s_{e_3}^{(0)}} = \mathrm{id}\,.$$

Therefore $\Gamma_u^{(0)}$ is generated by the set $\{g s_{e_1}^{(0)} s_{e_3}^{(0)} g^{-1} \,|\, g \in \Gamma^{(0)}\}$ of conjugates of $s_{e_1}^{(0)} s_{e_3}^{(0)}$. Observe

$$\begin{aligned} s_{e_1}^{(0)} s_{e_3}^{(0)}(\underline{e}) &= \underline{e} \begin{pmatrix} -1 & l & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & l & -1 \end{pmatrix} \\ &= \underline{e} \begin{pmatrix} 3 & -l & 2 \\ 0 & 1 & 0 \\ -2 & l & -1 \end{pmatrix} = \underline{e} + f_1(2, -l, 2), \\ \text{so } s_{e_1}^{(0)} s_{e_3}^{(0)} &= T(\overline{j}^{(0)}(\overline{e_1}^{(0)}) \otimes f_1), \end{aligned}$$

and recall Lemma 6.2 (f). The $\mathbb{Z}$-lattice generated by the $\Gamma^{(0)}$ orbit of e_1 is $\mathbb{Z}e_1 \oplus \mathbb{Z}le_2 \oplus \mathbb{Z}\gcd(2,l)e_3$. Therefore

$$\Gamma_u^{(0)} = \langle T(\overline{j}^{(0)}(\overline{e_1}^{(0)}) \otimes f_1), T(\overline{j}^{(0)}(l\overline{e_2}^{(0)}) \otimes f_1)\rangle \cong \mathbb{Z}^2 \text{ with}$$
$$T(\overline{j}^{(0)}(l\overline{e_2}^{(0)})) \otimes f_1)(\underline{e}) = \underline{e} + f_1(-l^2, 2l, -l^2).$$

Compare

$$\begin{aligned}(\ker\tau^{(0)})_u &= T(\overline{j}^{(0)}(\overline{H_{\mathbb{Z}}}^{(0)})\otimes f_1)\\ &= \langle T(\overline{j}^{(0)}(\overline{e_1}^{(0)})\otimes f_1), T(\overline{j}^{(0)}(\overline{e_2}^{(0)})\otimes f_1)\rangle,\\ O^{(0),Rad} &= T(\overline{H_{\mathbb{Z}}}^{(0),\sharp}\otimes f_1)\\ &= \langle(\underline{e}\mapsto\underline{e}+f_1(1,0,1)),(\underline{e}\mapsto\underline{e}+f_1(0,1,0))\rangle.\end{aligned}$$

Therefore

$$\Gamma_u^{(0)}\stackrel{1:l}{\subset}(\ker\tau^{(0)})_u\stackrel{1:(l^2-4)}{\subset}O_u^{(0),Rad}\cong\mathbb{Z}^2.$$

Theorem 6.8 (e) shows also

$$\Gamma_s^{(0)}=(\ker\tau^{(0)})_s\cap\ker\overline{\sigma}\stackrel{1:4}{\subset}O_s^{(0),Rad}.$$

Therefore

$$\Gamma^{(0)}\stackrel{1:l}{\subset}O^{(0),*}\stackrel{1:4(l^2-4)}{\subset}O^{(0),Rad}.$$

(g) By Lemma 5.7 (b) $\operatorname{sign} I^{(0)}=(+--)$. We will apply Theorem A.4 with $I^{[0]}=I^{(0)}$ and Theorem A.2 (b). The vectors e_1, e_2 and e_3 are positive. By Theorem A.4 (c) (vii) the reflection $s_{e_i}^{(0)}$ acts on the model $\mathcal{K}/\mathbb{R}^*$ of the hyperbolic plane as an elliptic element of order 2 with fixed point $\mathbb{R}^*e_i\in\mathcal{K}/\mathbb{R}^*$.

Consider the three vectors $v_1,v_2,v_3\in H_{\mathbb{Z}}\subset H_{\mathbb{R}}$

$$\begin{aligned}v_1 &:= -x_3e_1+x_2e_2+x_1e_3,\\ v_2 &:= x_3e_1-x_2e_2+x_1e_3,\\ v_3 &:= x_3e_1+x_2e_2-x_1e_3,\end{aligned}$$

and observe

$$v_1+v_2=2x_1e_3,\ v_1+v_3=2x_2e_2,\ v_2+v_3=2x_3e_1,\qquad I^{(0)}(v_i,v_i)=r(\underline{x})\le 0.$$

The three planes $\mathbb{R}v_1\oplus\mathbb{R}v_2$, $\mathbb{R}v_1\oplus\mathbb{R}v_3$ and $\mathbb{R}v_2\oplus\mathbb{R}v_3$ contain the lines $\mathbb{R}e_3$, $\mathbb{R}e_2$ respectively $\mathbb{R}e_1$. Two of the three planes intersect in one of the lines $\mathbb{R}v_1$, $\mathbb{R}v_2$ and $\mathbb{R}v_3$, and these three lines do not meet $\mathcal{K}$. Therefore the three hyperbolic lines $((\mathbb{R}v_1\oplus\mathbb{R}v_2)\cap\mathcal{K})/\mathbb{R}^*$, $((\mathbb{R}v_1\oplus\mathbb{R}v_3)\cap\mathcal{K})/\mathbb{R}^*$ and $((\mathbb{R}v_2\oplus\mathbb{R}v_3)\cap\mathcal{K})/\mathbb{R}^*$ in $\mathcal{K}/\mathbb{R}^*$ contain the points $\mathbb{R}^*e_3$, $\mathbb{R}^*e_2$ respectively $\mathbb{R}^*e_1$ and do not meet.

Now Theorem A.2 (b) shows that the group of automorphisms of $\mathcal{K}/\mathbb{R}^*$ which is induced by $\Gamma^{(0)}$ is isomorphic to $G^{fCox,3}$. Therefore also $\Gamma^{(0)}$ itself is isomorphic to $G^{fCox,3}$. □

REMARKS 6.12. (i) In part (g) of Theorem 6.12 we have less informations than in the other cases. We do not even know in which cases in part (g) $\Gamma^{(0)}=\mathcal{O}^{(0),*}$ respectively $\Gamma^{(0)}\subsetneqq\mathcal{O}^{(0),*}$ holds.

(ii) In the case of $S(\mathbb{P}^2)$, the proof of Theorem 6.11 (g) gave three hyperbolic lines in the model $\mathcal{K}/\mathbb{R}^*$ which form a degenerate hyperbolic triangle, so with vertices on the euclidean boundary of the hyperbolic plane. These vertices are the lines $\mathbb{R}^*v_1$, $\mathbb{R}^*v_2$, $\mathbb{R}^*v_3$, which are isotropic in the case of $S(\mathbb{P}^2)$ because there $I^{(0)}(v_i,v_i)=r(\underline{x})=0$. The reflections $s^{(0)}_{e_1}$, $s^{(0)}_{e_2}$, $s^{(0)}_{e_3}$ act as elliptic elements of order 2 with fixed points on these hyperbolic lines. Therefore the degenerate hyperbolic triangle is a fundamental domain of this action of $\Gamma^{(0)}$.

(iii) Milanov [**Mi19**, 4.1] had a different point of view on $\Gamma^{(0)}$ in the case of $S(\mathbb{P}^2)$. He gave an isomorphism $\Gamma^{(0)}\cong U$ to a certain subgroup U of index 3 in $PSL_2(\mathbb{Z})$. First we describe U in (iv), then we present our way to see this isomorphism in (v).

(iv) The class in $PSL_2(\mathbb{Z})$ of a matrix $A\in SL_2(\mathbb{Z})$ is denoted by $[A]$. It is well known that there is an isomorphism of the free product of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ with $PSL_2(\mathbb{Z})$,

$$\langle\alpha\,|\,\alpha^2=e\rangle*\langle\beta\,|\,\beta^3=e\rangle\to PSL_2(\mathbb{Z}),$$
$$\alpha\mapsto[\begin{pmatrix}0&-1\\1&0\end{pmatrix}],\ \beta\mapsto[\begin{pmatrix}-1&-1\\1&0\end{pmatrix}].$$

Consider the character

$$\chi:\langle\alpha\,|\,\alpha^2=e\rangle*\langle\beta\,|\,\beta^3=e\rangle\to\{1,e^{2\pi i/3},e^{2\pi i2/3}\},\ \alpha\mapsto 1,\ \beta\mapsto e^{2\pi i/3},$$

and the corresponding character $\widetilde{\chi}$ on $PSL_2(\mathbb{Z})$. Then

$$\begin{array}{rcl} U &=& \ker\widetilde{\chi}\overset{1:3}{\subset}PSL_2(\mathbb{Z}),\\ \ker\chi &=& \langle\alpha,\beta\alpha\beta^2,\beta^2\alpha\beta\rangle\\ \text{with} && \alpha\simeq[F_1],\beta\alpha\beta^2\simeq[F_2],\beta^2\alpha\beta\simeq[F_3]\text{ and}\\ F_1 &=& \begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad F_2=\begin{pmatrix}-1&-2\\1&1\end{pmatrix},\quad F_3=\begin{pmatrix}-1&-1\\2&1\end{pmatrix}.\end{array}$$

It is easy to see $\ker\chi\cong G^{fCox,3}$, with the three generators α, $\beta\alpha\beta^2$, $\beta^2\alpha\beta$. It is well known and easy to see that

$$\langle F_1,F_2,F_3\rangle=\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z})\,|\,a\equiv d\equiv 0\bmod 3\text{ or }b\equiv c\bmod 3\}.$$

The Möbius transformations $\mu(F_1)$, $\mu(F_2)$, $\mu(F_3)$ are elliptic of order 2 with fixed points $z_1=i$, $z_2=-1+i$, $z_3=-\frac{1}{2}+\frac{1}{2}i$.

The hyperbolic lines $\widetilde{l_1} := A(\infty,0)$ $\widetilde{l_2} := A(-1,\infty)$, $\widetilde{l_3} := A(0,-1)$ (notations from the Remarks and Notations A.1 (iii)) form a degenerate hyperbolic triangle, and $\widetilde{l_i}$ contains z_i.

(v) Consider the matrices

$$B := \begin{pmatrix} z_1\overline{z_1} & -z_2\overline{z_2} & 2z_3\overline{z_3} \\ \mathrm{Re}(z_1) & -\mathrm{Re}(z_2) & 2\,\mathrm{Re}(z_3) \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix} \quad \text{and}$$

$$B^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 3 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}.$$

One checks

$$B^t \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix} B = \begin{pmatrix} 2 & -3 & 3 \\ -3 & 2 & -3 \\ 3 & -3 & 2 \end{pmatrix} = S(\mathbb{P}^2) + S(\mathbb{P}^2)^t,$$

$$B^{-1}\Theta(F_i)B = -s_{e_i}^{(0),mat} \quad \text{for } i \in \{1,2,3\}$$

(see Theorem A.4 (i) for Θ). The $\mathbb{Z}$-basis $\underline{e}$ of $H_\mathbb{Z}$ and the $\mathbb{R}$-basis $\underline{f}$ in Theorem A.4 of $H_\mathbb{R}$ are related by $\underline{e} = \underline{f}\cdot B$. The tuple (v_1,v_2,v_3) in the proof of Theorem 6.11 (g) is

$$\begin{aligned}(v_1,v_2,v_3) &= \underline{e}\begin{pmatrix} 3 & -3 & -3 \\ 3 & -3 & 3 \\ -3 & -3 & 3 \end{pmatrix} \\ &= \underline{f}\cdot B\cdot \begin{pmatrix} 3 & -3 & -3 \\ 3 & -3 & 3 \\ -3 & -3 & 3 \end{pmatrix} = \underline{f}\cdot(-6)\begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.\end{aligned}$$

Finally observe that $\vartheta : \mathbb{H} \to \mathcal{K}/\mathbb{R}^*$ in Theorem A.4 extends to the euclidean boundary with

$$(\vartheta(-1),\vartheta(0),\vartheta(\infty)) = \mathbb{R}^*\cdot\underline{f}\cdot\begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

So the points $-1,0,\infty$ are mapped to the points $\mathbb{R}^*v_1$, $\mathbb{R}^*v_2$, $\mathbb{R}^*v_3$. The groups $\Gamma^{(0),mat} = \langle s_{e_i}^{(0),mat}\,|\,i\in\{1,2,3\}\rangle$ and $\langle -s_{e_i}^{(0),mat}\,|\,i\in\{1,2,3\}\rangle$ are isomorphic because $\Gamma^{(0),mat}$ does not contain $-E_3$ because else it would not be a free Coxeter group with three generators.

Therefore the group $U = \langle [F_1],[F_2],[F_3]\rangle \subset PSL_2(\mathbb{Z})$ is isomorphic to the group $\langle B^{-1}\Theta(F_i)B\,|\,i\in\{1,2,3\}\rangle = \langle -s_{e_i}^{(0),mat}\,|\,i\in\{1,2,3\}\rangle$ and to the groups $\Gamma^{(0),mat}$ and $\Gamma^{(0)}$.

Now we turn to the study of the set $R^{(0)}$ of roots and the subset $\Delta^{(0)} \subset R^{(0)}$ of vanishing cycles. For the set $R^{(0)}$ Theorem 6.11 (a) gave the general formula $R^{(0)} = \{y_1e_1+y_2e_2+y_3e_3 \in H_{\mathbb{Z}} \,|\, 1 = Q_3(y_1,y_2,y_3)\}$ with the quadratic form

$$Q_3 : \mathbb{Z}^3 \to \mathbb{Z}, \quad (y_1,y_2,y_3) \mapsto y_1^2+y_2^2+y_3^2+x_1y_1y_2+x_2y_1y_3+x_3y_2y_3.$$

It gave also a good control on $\Gamma^{(0)}$ for all cases $S(\underline{x})$ with $\underline{x} \in \mathbb{Z}^3$.

With respect to $\Delta^{(0)}$ and $R^{(0)}$ we know less. We have a good control on them for the cases with $r(\underline{x}) \in \{0,1,2,4\}$ and the reducible cases, but not for all other cases. Theorem 6.14 treats all cases except those in the Remarks 6.13 (ii).

Remarks 6.13. (i) The cases $S(\mathcal{H}_{1,2})$, $S(-l,2,-l)$ for $l \geq 3$ and the four cases $S(3,3,4)$, $S(4,4,4)$, $S(5,5,5)$, $S(4,4,8)$ (more precisely their $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbits) are the only cases in rank 3 where we know $\Delta^{(0)} \subsetneqq R^{(0)}$.

(ii) We do not know whether $\Delta^{(0)} = R^{(0)}$ or $\Delta^{(0)} \subsetneqq R^{(0)}$ in the following cases:

- (a) All cases $S(\underline{x})$ with $r(\underline{x}) < 0$ except the four cases $S(3,3,4)$, $S(4,4,4)$, $S(5,5,5)$, $S(4,4,8)$. With the action of $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ and Theorem 4.6 (c) they can be reduced to the cases $S(\underline{x})$ with $\underline{x} \in \mathbb{Z}^3_{\geq 3}$, $r(\underline{x}) < 0$, $x_i \leq \frac{1}{2}x_jx_k$ for $\{i,j,k\} = \{1,2,3\}$.
- (b) The irreducible cases $S(\underline{x})$ with $\underline{x} \in \mathbb{Z}^3_{\leq 0}$, $r(\underline{x}) > 4$ and $\underline{x} \notin \{0,-1,-2\}^3$.

Theorem 6.14. *(a) Consider the reducible cases (these include $S(A_1^3)$, $S(A_2A_1)$, $S(\mathbb{P}^1A_1)$). More precisely, suppose that $\underline{x} = (x_1,0,0)$. Then the tuple $(H_{\mathbb{Z}}, L, \underline{e})$ splits into the two tuples $(\mathbb{Z}e_1 \oplus \mathbb{Z}e_2, L|_{\mathbb{Z}e_1\oplus\mathbb{Z}e_2}, (e_1,e_2))$ and $(\mathbb{Z}e_3, L|_{\mathbb{Z}e_3}, e_3)$ with sets $\Delta_1^{(0)} = R_1^{(0)} \subset \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ and $\Delta_2^{(0)} = R_2^{(0)} = \{\pm e_3\} \subset \mathbb{Z}e_3$ of vanishing cycles and roots, and*

$$\Delta^{(0)} = \Delta_1^{(0)} \,\dot\cup\, \Delta_2^{(0)} = R^{(0)} = R_1^{(0)} \,\dot\cup\, R_2^{(0)}.$$

$\Delta_1^{(0)} = R_1^{(0)}$ is given in Theorem 6.8.

(b) Consider $S(\underline{x})$ with $\underline{x} \in \{0,-1,-2\}^3$ and $r(\underline{x}) > 4$. Then $\Delta^{(0)} = R^{(0)}$.

(c) The case $S(A_3)$ is classical. There

$$\Delta^{(0)} = R^{(0)} = \{\pm e_1, \pm e_2, \pm e_3, \pm(e_1+e_2), \pm(e_2+e_3), \pm(e_1+e_2+e_3)\}.$$

(d) The case $S(\widehat{A}_2)$: Recall $\operatorname{Rad} I^{(0)} = \mathbb{Z}f_1$ *with* $f_1 = e_1 + e_2 + e_3$. *There*

$$\begin{aligned} \Delta^{(0)} &= R^{(0)} = \Gamma^{(0)}\{e_1\} \\ &= (\pm e_1 + \mathbb{Z}f_1) \,\dot\cup\, (\pm e_2 + \mathbb{Z}f_1) \,\dot\cup\, (\pm e_3 + \mathbb{Z}f_1). \end{aligned}$$

(e) The case $S(\mathcal{H}_{1,2})$: Recall $(f_1, f_2, f_3) = \underline{e}\begin{pmatrix}1 & 0 & 1\\ 1 & 1 & 1\\ 0 & 1 & 1\end{pmatrix}$ *and* $\operatorname{Rad} I^{(0)} = \mathbb{Z}f_1 \oplus \mathbb{Z}f_2$. *The set of roots is*

$$R^{(0)} = \pm e_1 + \operatorname{Rad} I^{(0)} = (e_1 + \operatorname{Rad} I^{(0)}) \,\dot\cup\, (-e_1 + \operatorname{Rad} I^{(0)}),$$

with

$$e_1 + \operatorname{Rad} I^{(0)} = -e_2 + \operatorname{Rad} I^{(0)} = e_3 + \operatorname{Rad} I^{(0)} = f_3 + \operatorname{Rad} I^{(0)}.$$

It splits into the four $\Gamma^{(0)}$ orbits

$$\begin{aligned} &\Gamma^{(0)}\{e_1\} = \pm e_1 + 2\operatorname{Rad} I^{(0)}, \qquad \Gamma^{(0)}\{e_2\} = \pm e_2 + 2\operatorname{Rad} I^{(0)},\\ &\Gamma^{(0)}\{e_3\} = \pm e_3 + 2\operatorname{Rad} I^{(0)}, \qquad \Gamma^{(0)}\{f_3\} = \pm f_3 + 2\operatorname{Rad} I^{(0)}. \end{aligned}$$

The set $\Delta^{(0)}$ of vanishing cycles consists of the first three of these sets,

$$\Delta^{(0)} = \Gamma^{(0)}\{e_1\} \,\dot\cup\, \Gamma^{(0)}\{e_2\} \,\dot\cup\, \Gamma^{(0)}\{e_3\},$$

so $\Delta^{(0)} \subsetneqq R^{(0)}$.

(f) The cases $S(-l, 2, -l)$ with $l \geq 3$: Recall $\operatorname{Rad} I^{(0)} = \mathbb{Z}f_1$ *with* $f_1 = e_1 - e_3$. *As the tuple $(\overline{H_\mathbb{Z}}^{(0)}, \overline{I}^{(0)}, (\overline{e_1}^{(0)}, \overline{e_2}^{(0)}))$ is isomorphic to the corresponding tuple from the 2×2 matrix* $S(-l) = \begin{pmatrix}1 & -l\\ 0 & 1\end{pmatrix}$, *its sets of roots and its set of even vanishing cycles coincide because of Theorem 6.8. These sets are called $R^{(0)}(S(-l))$. Then*

$$R^{(0)} = \{\widetilde{y}_1 e_1 + \widetilde{y}_2 e_2 \in H_\mathbb{Z}^{(0)} \,|\, \widetilde{y}_1\overline{e_1}^{(0)} + \widetilde{y}_2\overline{e_2}^{(0)} \in R^{(0)}(S(-l))\} + \operatorname{Rad} I^{(0)}.$$

The cases with l even: $R^{(0)}$ splits into the following $l+2$ $\Gamma^{(0)}$ orbits,

$$\Gamma^{(0)}\{e_1\}, \quad \Gamma^{(0)}\{e_3\}, \quad \Gamma^{(0)}\{e_2 + mf_1\} \text{ for } m \in \{0, 1, ..., l-1\}.$$

The set $\Delta^{(0)}$ of vanishing cycles consists of the first three $\Gamma^{(0)}$ orbits,

$$\Delta^{(0)} = \Gamma^{(0)}\{e_1\} \,\dot\cup\, \Gamma^{(0)}\{e_3\} \,\dot\cup\, \Gamma^{(0)}\{e_2\}.$$

The cases with l odd: $R^{(0)}$ splits into the following $l+1$ $\Gamma^{(0)}$ orbits,

$$\Gamma^{(0)}\{e_1\} = \Gamma^{(0)}\{e_3\}, \quad \Gamma^{(0)}\{e_2 + mf_1\} \text{ for } m \in \{0, 1, ..., l-1\}.$$

The set $\Delta^{(0)}$ of vanishing cycles consists of the first two $\Gamma^{(0)}$ orbits,

$$\Delta^{(0)} = \Gamma^{(0)}\{e_1\} \,\dot\cup\, \Gamma^{(0)}\{e_2\}.$$

In both cases $\Delta^{(0)} \subsetneqq R^{(0)}$.

(g) The case $S(\mathbb{P}^2)$. Then $\Delta^{(0)} = R^{(0)}$, and $R^{(0)}$ splits into three $\Gamma^{(0)}$ orbits,

$$\begin{aligned} R^{(0)} &= \Gamma^{(0)}\{e_1\} \,\dot\cup\, \Gamma^{(0)}\{e_2\} \,\dot\cup\, \Gamma^{(0)}\{e_3\} \quad \text{with} \\ \Gamma^{(0)}\{e_i\} &\subset \pm e_i + 3H_{\mathbb{Z}} \quad \text{for } i \in \{1,2,3\} \end{aligned}$$

(but we would like to have a better control on $R^{(0)}$).

(h) The cases $S(3,3,4)$, $S(4,4,4)$, $S(5,5,5)$ and $S(4,4,8)$. Then

$$\Delta^{(0)} \subsetneqq R^{(0)}.$$

Proof: (a) The splittings $\Delta^{(0)} = \Delta_1^{(0)} \,\dot\cup\, \Delta_2^{(0)}$ and $R^{(0)} = R_1^{(0)} \,\dot\cup\, R_2^{(0)}$ are part of Lemma 2.11. Lemma 2.12 gives for A_1 $\Delta_2^{(0)} = R_2^{(0)} = \{\pm e_3\}$. Theorem 6.8 gives for any rank 2 case $\Delta_1^{(0)} = R_1^{(0)}$.

(b) The cases where $(H_{\mathbb{Z}}, L, \underline{e})$ is reducible are covered by part (a). In any irreducible case, the bilinear lattice $(H_{\mathbb{Z}}, L, \underline{e})$ with triangular basis is *hyperbolic* in the sense of the definition before Theorem 3.12 in [**HK16**], because $I^{(0)}$ is indefinite, but the submatrices (2), $\begin{pmatrix} 2 & x_1 \\ x_1 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & x_2 \\ x_2 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & x_3 \\ x_3 & 2 \end{pmatrix}$ of the matrix $I^{(0)}(\underline{e}^t, \underline{e})$ are positive definite or positive semidefinite. Theorem 3.12 in [**HK16**] applies and gives $\Delta^{(0)} = R^{(0)}$.

(c) This is classical. It follows also with

$$\begin{aligned} Q_3(y_1,y_2,y_3) &= 2(y_1^2 + y_2^2 + y_3^2 - y_1y_2 - y_2y_3) \\ &= y_1^2 + (y_1 - y_2)^2 + (y_2 - y_3)^2 + y_3^2 \end{aligned}$$

and the transitivity of the action of $\Gamma^{(0)}$ on $R^{(0)}$.

(d) The quotient lattice $(\overline{H_{\mathbb{Z}}}^{(0)}, \overline{I}^{(0)})$ is an A_2 lattice with set of roots $\{\pm\overline{e_1}^{(0)}, \pm\overline{e_2}^{(0)}, \pm\overline{e_3}^{(0)}\}$. Therefore

$$R^{(0)} = (\pm e_1 + \mathbb{Z}f_1) \,\dot\cup\, (\pm e_2 + \mathbb{Z}f_1) \,\dot\cup\, (\pm e_3 + \mathbb{Z}f_1).$$

(One can prove this also using $2Q_3 = (y_1-y_2)^2 + (y_1-y_3)^2 + (y_2-y_3)^2$.) $\Gamma_s^{(0)} \cong D_6$ acts transitively on the set $\{\pm\overline{e_1}^{(0)}, \pm\overline{e_2}^{(0)}, \pm\overline{e_3}^{(0)}\}$. The group $\Gamma_u^{(0)} \cong \mathbb{Z}^2$ contains the elements

$$(\underline{e} \mapsto \underline{e} + f_1(2,-1,-1)) \quad \text{and} \quad (\underline{e} \mapsto \underline{e} + f_1(-1,2,-1)).$$

Therefore it acts transitively on each of the six sets $\varepsilon e_i + \mathbb{Z}f_1$ with $\varepsilon \in \{\pm 1\}$, $i \in \{1,2,3\}$. Thus $\Gamma^{(0)}$ acts transitively on $R^{(0)}$, so $\Delta^{(0)} = R^{(0)} = \Gamma^{(0)}\{e_1\}$.

(e) The quotient lattice $(\overline{H_{\mathbb{Z}}}^{(0)},\overline{I}^{(0)})$ is an A_1 lattice with set of roots $\{\pm\overline{e_1}^{(0)}\}$. Therefore

$$R^{(0)}=\pm e_1+\operatorname{Rad} I^{(0)}=(e_1+\operatorname{Rad} I^{(0)})\ \dot\cup\ (-e_1+\operatorname{Rad} I^{(0)}).$$

(One can prove this also using $Q_3=(y_1-y_2+y_3)^2$.) $s_{e_i}^{(0)}$ exchanges e_i and $-e_i$, and $s_{e_1}^{(0)}$ maps f_3 to $-f_3+2f_2$. $\Gamma_u^{(0)}\cong\mathbb{Z}^2$ is generated by the elements

$$((f_1,f_2,f_3)\mapsto(f_1,f_2,f_3+2f_1))=(\underline{e}\mapsto\underline{e}+f_1(2,-2,2)),$$
$$((f_1,f_2,f_3)\mapsto(f_1,f_2,f_3+2f_2))=(\underline{e}\mapsto\underline{e}+f_2(2,-2,2)).$$

Therefore $R^{(0)}$ splits into the four $\Gamma^{(0)}$ orbits

$$\begin{aligned}\Gamma^{(0)}\{e_1\}&=\pm e_1+2\operatorname{Rad} I^{(0)},\quad \Gamma^{(0)}\{e_2\}=\pm e_2+2\operatorname{Rad} I^{(0)},\\ \Gamma^{(0)}\{e_3\}&=\pm e_3+2\operatorname{Rad} I^{(0)},\quad \Gamma^{(0)}\{f_3\}=\pm f_3+2\operatorname{Rad} I^{(0)}.\end{aligned}$$

$\Delta^{(0)}$ consists of the first three of them.

(f) The set of roots of the quotient lattice $(\overline{H_{\mathbb{Z}}}^{(0)},\overline{I}^{(0)})$ is called $R^{(0)}(S(-l))$. Theorem 6.8 describes it. Therefore

$$R^{(0)}=\{\widetilde{y}_1e_1+\widetilde{y}_2e_2\in H_{\mathbb{Z}}\,|\,\widetilde{y}_1\overline{e_1}^{(0)}+\widetilde{y}_2\overline{e_2}^{(0)}\in R^{(0)}(S(-l))\}+\operatorname{Rad} I^{(0)}.$$

By Theorem 6.8 (d) (iv) and (c), $R^{(0)}(S(-l))$ splits into the two $\Gamma_s^{(0)}$ orbits $\Gamma_s^{(0)}\{\overline{e_1}^{(0)}\}$ and $\Gamma_s^{(0)}\{\overline{e_2}^{(0)}\}$, and the action of $\Gamma_s^{(0)}$ on each of these two orbits is simply transitive. $\Gamma_u^{(0)}\cong\mathbb{Z}^2$ is generated by the elements

$$(\underline{e}\mapsto\underline{e}+f_1(2,-l,2))\text{ and }(\underline{e}\mapsto\underline{e}+f_1(-l^2,2l,-l^2)).$$

Therefore for $m\in\{0,1,...,l-1\}$

$$\Gamma^{(0)}\{e_2+mf_1\}\cap(e_2+\mathbb{Z}f_1)=e_2+mf_1+\mathbb{Z}lf_1.$$

If l is odd then $1=\gcd(2,l^2)$ and

$$\Gamma^{(0)}\{e_1\}\supset e_1+\mathbb{Z}f_1\ni e_3=e_1-f_1,\quad\text{so } \Gamma^{(0)}\{e_1\}=\Gamma^{(0)}\{e_3\}.$$

If l is even then $2=\gcd(2,l^2)$ and

$$\Gamma^{(0)}\{e_1\}\cap(e_1+2\mathbb{Z}f_1)=e_1+\mathbb{Z}2f_1\not\ni e_3,\quad\text{so } \Gamma^{(0)}\{e_1\}\cap\Gamma^{(0)}\{e_3\}=\emptyset.$$

This shows all claims.

(g) The matrices $s_{e_i}^{(0),mat}\in M_{3\times 3}(\mathbb{Z})$ with $s_{e_i}^{(0)}(\underline{e})=\underline{e}\cdot s_{e_i}^{(0),mat}$ are

$$s_{e_1}^{(0),mat}=\begin{pmatrix}-1&3&-3\\0&1&0\\0&0&1\end{pmatrix},$$

$$s_{e_2}^{(0),mat}=\begin{pmatrix}1&0&0\\3&-1&3\\0&0&1\end{pmatrix},\quad s_{e_3}^{(0),mat}=\begin{pmatrix}1&0&0\\0&1&0\\-3&3&-1\end{pmatrix}.$$

One sees $\Gamma^{(0)}\{e_i\} \subset \pm e_i + 3H_{\mathbb{Z}}$ for $i \in \{1,2,3\}$. Therefore $\Delta^{(0)}$ splits into three $\Gamma^{(0)}$ orbits,

$$\Delta^{(0)} = \Gamma^{(0)}\{e_1\} \,\dot\cup\, \Gamma^{(0)}\{e_2\} \,\dot\cup\, \Gamma^{(0)}\{e_3\}.$$

It remains to show $\Delta^{(0)} = R^{(0)}$. Write $\underline{\widetilde{e}} = (e_1, -e_2, e_3)$, so that

$$L(\underline{\widetilde{e}}^t, \underline{\widetilde{e}})^t = \widetilde{S} = \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad I^{(0)}(\underline{\widetilde{e}}^t, \underline{\widetilde{e}})^t = \widetilde{S} + \widetilde{S}^t = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}.$$

The quadratic form $\widetilde{Q}_3 : \mathbb{Z}^3 \to \mathbb{Z}$ with

$$\begin{aligned} \widetilde{Q}_3(\underline{y}) &= \frac{1}{2} I^{(0)}((\underline{\widetilde{e}} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix})^t, \underline{\widetilde{e}} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}) = \frac{1}{2}(y_1\ y_2\ y_3) \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\ &= y_1^2 + y_2^2 + y_3^2 + 3(y_1y_2 + y_1y_3 + y_2y_3) \end{aligned}$$

can also be written in the following two ways which will be useful below,

$$\widetilde{Q}_3(\underline{y}) = (y_1 + y_2)(y_1 + y_3) + (y_1 + y_2)(y_2 + y_3) + (y_1 + y_3)(y_2 + y_3), \tag{6.8}$$

$$\widetilde{Q}_3(\underline{y}) = \frac{3}{2}(y_1 + y_2 + y_3)^2 - \frac{1}{2}(y_1^2 + y_2^2 + y_3^2). \tag{6.9}$$

We have $R^{(0)} = \{y_1\widetilde{e_1} + y_2\widetilde{e_2} + y_3\widetilde{e_3} \in H_{\mathbb{Z}} \,|\, \widetilde{Q}_3(\underline{y}) = 1\}$. Define

$$\|a\| := \sqrt{y_1^2 + y_2^2 + y_3^2} \quad \text{for} \quad a = \sum_{i=1}^{3} y_i e_i \in H_{\mathbb{Z}}.$$

Claim: *For any $a \in R^{(0)} - \{\pm e_1, \pm e_2, \pm e_3\}$ an index $i \in \{1,2,3\}$ with*

$$\|s_{e_i}^{(0)}(a)\| < \|a\|$$

exists.

The Claim implies $\Delta^{(0)} = R^{(0)}$ because it says that any $a \in R^{(0)}$ can be mapped by a suitable sequence of reflections in $\{s_{e_1}^{(0)}, s_{e_2}^{(0)}, s_{e_3}^{(0)}\}$ to e_1 or e_2 or e_3. It remains to prove the Claim.

Proof of the Claim: Suppose $a \in R^{(0)} - \{\pm e_1, \pm e_2, \pm e_3\}$ satisfies $\|s_{e_i}^{(0)}(a)\| \geq a$ for any $i \in \{1,2,3\}$. Write $a = y_1\widetilde{e_1} + y_2\widetilde{e_2} + y_3\widetilde{e_3}$. For j

and k with $\{i,j,k\}=\{1,2,3\}$

$$\begin{array}{rl} & \|s_{e_i}^{(0)}(a)\| \geq \|a\| \iff \|s_{e_i}^{(0)}(a)\|^2 \geq \|a\|^2 \\ \iff & (-y_i-3y_j-3y_k)^2+y_j^2+y_k^2 \geq y_i^2+y_j^2+y_k^2 \\ \iff & 6y_i(y_j+y_k)+9(y_j+y_k)^2 \geq 0 \\ \iff & \begin{cases} 3(y_j+y_k)\geq -2y_i & \text{if } y_j+y_k>0, \\ 3(y_j+y_k)\leq -2y_i & \text{if } y_j+y_k<0, \\ \text{no condition} & \text{if } y_j+y_k=0. \end{cases} \end{array}$$

$y_j+y_k=0$ is impossible because else by formula (6.8)

$$1=\widetilde{Q}_3(\underline{y})=(y_i+y_j)(y_i+y_k)=(y_i+y_j)(y_i-y_j)=y_i^2-y_j^2, \qquad \text{so } y_i=\pm 1, y_j=y_k=0,$$

which is excluded by $a\in R^{(0)}-\{\pm e_1,\pm e_2,\pm e_3\}$. Also $(y_1+y_2>0, y_1+y_3>0, y_2+y_3>0)$ and $(y_1+y_2<0, y_1+y_3<0, y_2+y_3<0)$ are impossible because of $1=\widetilde{Q}_3(\underline{y})$ and (6.8).

We can suppose

$$y_1+y_2>0,\quad y_1+y_3>0,\quad y_2+y_3<0,\quad y_1\geq y_2\geq y_3.$$

Then

$$\begin{array}{c} y_1>0,\quad y_3\in\mathbb{Z}\cap[-y_1+1,-1],\quad y_2\in[-y_3-1,y_3], \\ 3(y_1+y_2)\geq -2y_3 \quad \text{because of } y_1+y_2>0, \\ 3(y_2+y_3)\leq -2y_1 \quad \text{because of } y_2+y_3<0, \end{array}$$

so

$$\begin{array}{rcl} y_1 & \geq & 3(y_1+y_2+y_3)\geq y_3\geq -y_1+1>-y_1, \\ |y_1| & \geq & 3|y_1+y_2+y_3|, \\ y_1^2 & \geq & 9(y_1+y_2+y_3)^2, \\ \widetilde{Q}_3(\underline{y}) & \overset{(6.9)}{=} & \frac{3}{2}(y_1+y_2+y_3)^2-\frac{1}{2}(y_1^2+y_2^2+y_3^2) \\ & \leq & \frac{1}{6}y_1^2-\frac{1}{2}(y_1^2+y_2^2+y_3^2)\leq 0, \end{array}$$

a contradiction. Therefore an $a\in R^{(0)}-\{\pm e_1,\pm e_2,\pm e_3\}$ with $\|s_{e_i}^{(0)}(a)\|\geq\|a\|$ for each $i\in\{1,2,3\}$ does not exist. The Claim is proved.

(h) By Theorem 6.11 (g) $\Gamma^{(0)}$ is a free Coxeter group with generators $s_{e_1}^{(0)}$, $s_{e_2}^{(0)}$ and $s_{e_3}^{(0)}$. By Example 3.23 (iv) equality holds in (3.3), so

$$\mathcal{B}^{dist}=\{\underline{v}\in(\Delta^{(0)})^3 \mid s_{v_1}^{(0)}s_{v_2}^{(0)}s_{v_3}^{(0)}=-M\}$$

(see also Theorem 7.2 (a)). By Theorem 5.16 (a)+(b)+(d)+(e) $Q \in G_{\mathbb{Z}} - G_{\mathbb{Z}}^{\mathcal{B}}$. Lemma 3.22 (a) and $QMQ^{-1} = M$ give

$$s^{(0)}_{Q(e_1)} s^{(0)}_{Q(e_2)} s^{(0)}_{Q(e_3)} = Q s^{(0)}_{e_1} s^{(0)}_{e_2} s^{(0)}_{e_3} Q^{-1} = Q(-M)Q^{-1} = -M.$$

If $Q(e_1), Q(e_2), Q(e_3)$ were all in $\Delta^{(0)}$ then equality in (3.3) would imply $(Q(e_1), Q(e_2), Q(e_3)) \in \mathcal{B}^{dist}$ and $Q \in G_{\mathbb{Z}}^{\mathcal{B}}$, a contradiction. So $Q(e_1), Q(e_2), Q(e_3)$ are not all in $\Delta^{(0)}$. But of course they are in $R^{(0)}$. $\square$

Remarks 6.15. (i) In view of Remark 6.13 (ii) it would be desirable to extend the proof of $\Delta^{(0)} = R^{(0)}$ in part (g) of Theorem 6.11 to other cases. The useful formulas (6.8) and (6.9) generalize as follows:

$$\begin{aligned} (x_1 + x_2 + x_3) Q_3(\underline{y}) = (x_1 + x_2 + x_3)(y_1^2 + y_2^2 + y_3^2) \\ -x_1 x_2 y_1^2 - x_1 x_3 y_2^2 - x_2 x_3 y_3^2 + (x_1 y_2 + x_2 y_3)(x_1 y_1 + x_3 y_3) \\ +(x_1 y_2 + x_2 y_3)(x_2 y_1 + x_3 y_2) + (x_1 y_1 + x_3 y_3)(x_2 y_1 + x_3 y_2). \end{aligned}$$

If $\underline{x} \in (\mathbb{Z} - \{0\})^3$ then

$$\begin{aligned} 2Q_3(\underline{y}) &= x_1 x_2 x_3 \Big(\frac{y_1}{x_3} + \frac{y_2}{x_2} + \frac{y_3}{x_1}\Big)^2 - (x_1 x_2 x_3 - 2x_3^2)\Big(\frac{y_1}{x_3}\Big)^2 \\ &- (x_1 x_2 x_3 - 2x_2^2)\Big(\frac{y_2}{x_2}\Big)^2 - (x_1 x_2 x_3 - 2x_1^2)\Big(\frac{y_3}{x_1}\Big)^2. \end{aligned}$$

Also the rephrasing in the proof of part (g) of the inequality $\|s^{(0)}_{e_i}(a)\| \geq \|a\|$ generalizes naturally. But the further arguments do not seem to generalize easily.

(ii) In view of Theorem 6.14, we know the following for $n = 3$:

$\Delta^{(0)} = R^{(0)}$	in the cases $S(\underline{x})$ with $\underline{x} \in \{0, -1, -2\}^3$ with $r(\underline{x}) > 4$, in all reducible cases and in the cases $A_3, \widehat{A}_2, \mathbb{P}^2$.
$\Delta^{(0)} \subsetneqq R^{(0)}$	in the cases $\mathcal{H}_{1,2},\ S(-l, 2, -l)$ with $l \geq 3$, $S(3,3,3),\ S(4,4,4),\ S(5,5,5)$ and $S(4,4,8)$.

In the following cases with $n = 3$ we do not know whether $\Delta^{(0)} = R^{(0)}$ or $\Delta^{(0)} \subsetneqq R^{(0)}$ holds: All cases $\underline{x} \in \mathbb{Z}^3$ with $r(\underline{x}) < 0$ except four cases and many cases $\underline{x} \in \mathbb{Z}^3$ with $r(\underline{x}) > 4$.

6.4. The odd rank 3 cases

For $\underline{x} \in \mathbb{Z}^3$ consider the matrix $S = S(\underline{x}) = \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} \in T_3^{uni}(\mathbb{Z})$, and consider a unimodular bilinear lattice $(H_{\mathbb{Z}}, L)$ with a triangular basis $\underline{e} = (e_1, e_2, e_3)$ with $L(\underline{e}^t, \underline{e})^t = S$.

In this section we will determine in all cases the odd monodromy group $\Gamma^{(1)} = \langle s^{(1)}_{e_1}, s^{(1)}_{e_2}, s^{(1)}_{e_3}\rangle \subset O^{(1)}$ and in many, but not all cases the set $\Delta^{(1)} = \Gamma^{(1)}\{\pm e_1, \pm e_2, \pm e_3\}$ of odd vanishing cycles.

Recall Remark 4.17. The group $\Gamma^{(1)}$ and the set $\Delta^{(1)}$ are determined by the triple $(H_{\mathbb{Z}}, I^{(1)}, \underline{e})$, and here $I^{(1)}$ is needed only up to the sign. By Remark 4.17 and Lemma 4.18 we can restrict to $\underline{x}$ in the union of the following three families. It will be useful to split each of the first two families into the three subfamilies on the right hand side.

$$(x_1, x_2, 0) \text{ with } x_1 \geq x_2 \geq 0, \quad \begin{cases} (x_1, 0, 0): & \text{reducible cases,} \\ (1, 1, 0): & A_3 \text{ and } \widehat{A}_2, \\ (x_1, x_2, 0) & \text{with } 2 \leq x_1 \geq x_2 > 0, \end{cases}$$

$$(-l, 2, -l) \text{ with } l \geq 2, \quad \begin{cases} (-l, 2, -l) \text{ with } l \equiv 0(4), \\ (-l, 2, -l) \text{ with } l \equiv 2(4) \text{ (this includes } \mathcal{H}_{1,2}), \\ (-l, 2, -l) \text{ with } l \equiv 1(2), \end{cases}$$

$$(x_1, x_2, x_3) \in \mathbb{Z}^3_{\geq 3} \text{ with } 2x_i \leq x_j x_k \text{ for } \{i, j, k\} = \{1, 2, 3\}$$
$$\text{(this includes } \mathbb{P}^2).$$

Recall

$$\underline{\widetilde{x}} = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) := \gcd(x_1, x_2, x_3)^{-1}(x_1, x_2, x_3) \quad \text{for } \underline{x} \neq (0, 0, 0).$$

Recall from section 5.3 the definition

$$f_3 := -\widetilde{x}_3 e_1 + \widetilde{x}_2 e_2 - \widetilde{x}_1 e_3 \in H_{\mathbb{Z}}^{prim} \quad \text{for } \underline{x} \neq (0, 0, 0)$$

and the fact

$$\operatorname{Rad} I^{(1)} = \begin{cases} \mathbb{Z} f_3 & \text{if } \underline{x} \neq (0, 0, 0), \\ H_{\mathbb{Z}} & \text{if } \underline{x} = (0, 0, 0). \end{cases}$$

Therefore in all cases except $\underline{x} = (0, 0, 0)$ the exact sequence

$$\{1\} \to \Gamma_u^{(1)} \to \Gamma^{(1)} \to \Gamma_s^{(1)} \to \{1\} \tag{6.10}$$

in Lemma 6.2 (d) is interesting.

Lemma 6.16. *Suppose $x_1 \neq 0$ (this holds in the three families above except for the case $\underline{x} = (0, 0, 0)$).*

(a) The sublattice $\mathbb{Z}\overline{e_1}^{(1)} + \mathbb{Z}\overline{e_2}^{(1)} \subset \overline{H_{\mathbb{Z}}}^{(1)}$ has index $\widetilde{x}_1$ in $\overline{H_{\mathbb{Z}}}^{(1)}$.

(b) $\overline{I}^{(1)}$ is nondegenerate. For each $\mathbb{Z}$-basis $\underline{b}=(b_1,b_2)$ of $\overline{H_\mathbb{Z}}^{(1)}$

$$\overline{I}^{(1)}(\underline{b}^t,\underline{b})=\varepsilon\gcd(x_1,x_2,x_3)\begin{pmatrix}0&-1\\1&0\end{pmatrix}$$

for some $\varepsilon\in\{\pm 1\}$. Also $O_s^{(1),Rad}\cong SL_2(\mathbb{Z})$.

(c)

$$\overline{e_1}^{(1)}\in\gcd(\widetilde{x}_1,\widetilde{x}_2)\overline{H_\mathbb{Z}}^{(1),prim},\quad \overline{e_2}^{(1)}\in\gcd(\widetilde{x}_1,\widetilde{x}_3)\overline{H_\mathbb{Z}}^{(1),prim},$$
$$\overline{e_3}^{(1)}\in\gcd(\widetilde{x}_2,\widetilde{x}_3)\overline{H_\mathbb{Z}}^{(1),prim}\text{ if }(\widetilde{x}_2,\widetilde{x}_3)\neq(0,0),\quad \text{else }\overline{e_3}^{(1)}=0.$$

Proof: (a)

$$\begin{aligned}\overline{H_\mathbb{Z}}^{(1)}&=\mathbb{Z}\overline{e_1}^{(1)}+\mathbb{Z}\overline{e_2}^{(1)}+\mathbb{Z}\overline{e_3}^{(1)}\\&=\mathbb{Z}\overline{e_1}^{(1)}+\mathbb{Z}\overline{e_2}^{(1)}+\mathbb{Z}\frac{1}{\widetilde{x}_1}(-\widetilde{x}_3\overline{e_1}^{(1)}+\widetilde{x}_2\overline{e_2}^{(1)})\\&=\mathbb{Z}\overline{e_1}^{(1)}+\mathbb{Z}\overline{e_2}^{(1)}+\mathbb{Z}\frac{\xi}{\widetilde{x}_1}h_2\quad\text{with}\\ \xi&:=\gcd(\widetilde{x}_2,\widetilde{x}_3),\quad h_2:=-\frac{\widetilde{x}_3}{\xi}\overline{e_1}^{(1)}+\frac{\widetilde{x}_2}{\xi}\overline{e_2}^{(1)}.\end{aligned}$$

The element h_2 is in $(\mathbb{Z}\overline{e_1}^{(1)}+\mathbb{Z}\overline{e_2}^{(1)})^{prim}$. One can choose a second element $h_1\in\mathbb{Z}\overline{e_1}^{(1)}+\mathbb{Z}\overline{e_2}^{(1)}$ with $\mathbb{Z}\overline{e_1}^{(1)}\oplus\mathbb{Z}\overline{e_2}^{(1)}=\mathbb{Z}h_1\oplus\mathbb{Z}h_2$. Then

$$\mathbb{Z}h_2+\mathbb{Z}\frac{\xi}{\widetilde{x}_1}h_2=\mathbb{Z}\frac{1}{\widetilde{x}_1}h_2$$

because $\gcd(\widetilde{x}_1,\xi)=1$. Therefore

$$\overline{H_\mathbb{Z}}^{(1)}=(\mathbb{Z}h_1+\mathbb{Z}h_2)+\mathbb{Z}\frac{\xi}{\widetilde{x}_1}h_2=\mathbb{Z}h_1\oplus\mathbb{Z}\frac{1}{\widetilde{x}_1}h_2\overset{\widetilde{x}_1:1}{\supset}\mathbb{Z}h_1\oplus\mathbb{Z}h_2.$$

(b) $\overline{I}^{(1)}(\overline{e_1}^{(1)},\overline{e_2}^{(1)})=x_1\neq 0$. With part (a) one sees

$$I^{(1)}(b_1,b_2)=\pm\frac{x_1}{\widetilde{x}_1}=\pm\gcd(x_1,x_2,x_3).$$

A rank two lattice with a nondegenerate skew-symmetric bilinear form has an automorphism group isomorphic to $SL_2(\mathbb{Z})$.

(c) The proof of part (a) and $1=\gcd(\widetilde{x}_3,\gcd(\widetilde{x}_1,\widetilde{x}_2))$ show

$$\mathbb{Q}\overline{e_1}^{(1)}\cap\overline{H_\mathbb{Z}}^{(1)}=\mathbb{Z}\overline{e_1}^{(1)}+\mathbb{Z}\frac{-\widetilde{x}_3}{\gcd(\widetilde{x}_1,\widetilde{x}_2)}\overline{e_1}^{(1)}=\mathbb{Z}\frac{1}{\gcd(\widetilde{x}_1,\widetilde{x}_2)}\overline{e_1}^{(1)}.$$

This shows $\overline{e_1}^{(1)}\in\gcd(\widetilde{x}_1,\widetilde{x}_2)\overline{H_\mathbb{Z}}^{(1),prim}$. Analogously $\overline{e_2}^{(1)}\in\gcd(\widetilde{x}_1,\widetilde{x}_3)\overline{H_\mathbb{Z}}^{(1),prim}$.

If $(\widetilde{x}_2,\widetilde{x}_3)=(0,0)$ then $-\widetilde{x}_1\overline{e_3}^{(1)}=\overline{f_3}^{(1)}=0$, so $\overline{e_3}^{(1)}=0$. If $\widetilde{x}_2\neq 0$ or $\widetilde{x}_3\neq 0$, formulas as in the proof of part (a) hold also for

$\mathbb{Z}\overline{e_1}^{(1)} + \mathbb{Z}\overline{e_3}^{(1)}$ respectively $\mathbb{Z}\overline{e_2}^{(1)} + \mathbb{Z}\overline{e_3}^{(1)}$. In both cases one shows $\overline{e_3}^{(1)} \in \gcd(\widetilde{x}_2, \widetilde{x}_3)\overline{H_{\mathbb{Z}}}^{(1),prim}$ as above. □

In Theorem 6.18 we will consider in many cases the three groups $\Gamma_u^{(1)} \subset (\ker \tau^{(1)})_u \subset O_u^{(1),Rad}$. Their descriptions in Lemma 6.2 (e) simplify because now $\mathrm{Rad}\, I^{(1)} = \mathbb{Z}f_3$ if $\underline{x} \neq (0,0,0)$. In the cases $\underline{x} = (-l, 2, -l)$ with $l \equiv 2(4)$ also the larger group

$$O_{\pm}^{(1),Rad} := \{g \in O^{(1),Rad} \,|\, \overline{g} = \pm\,\mathrm{id}\} \overset{2:1}{\supset} O_u^{(1),Rad}$$

will be considered. The following Lemma fixes notations and gives a description of $O_{\pm}^{(1),Rad}$ similar to the one for $O_u^{(1),Rad}$ in Lemma 6.2 (e). It makes also $O_u^{(1),Rad}$ and $(\ker \tau^{(1)})_u$ more explicit, and, under some condition, $\Gamma_u^{(1)}$ and $\Gamma^{(1)} \cap O_{\pm}^{(1),Rad}$.

Lemma 6.17. *Suppose $x_1 \neq 0$. Denote*

$$\begin{array}{lll} \mathrm{Hom}_0(H_{\mathbb{Z}}, \mathbb{Z}) & := & \{\lambda : H_{\mathbb{Z}} \to \mathbb{Z} \,|\, \lambda \text{ is } \mathbb{Z}\text{-linear}, \lambda(f_3) = 0\}, \\ \mathrm{Hom}_2(H_{\mathbb{Z}}, \mathbb{Z}) & := & \{\lambda : H_{\mathbb{Z}} \to \mathbb{Z} \,|\, \lambda \text{ is } \mathbb{Z}\text{-linear}, \lambda(f_3) = 2\}, \\ t_\lambda^+ : H_{\mathbb{Z}} \to H_{\mathbb{Z}} & & \text{with } t_\lambda^+(a) = a + \lambda(a) f_3 \quad \text{for } \lambda \in \mathrm{Hom}_0(H_{\mathbb{Z}}, \mathbb{Z}), \\ t_\lambda^- : H_{\mathbb{Z}} \to H_{\mathbb{Z}} & & \text{with } t_\lambda^-(a) = -a + \lambda(a) f_3 \quad \text{for } \lambda \in \mathrm{Hom}_2(H_{\mathbb{Z}}, \mathbb{Z}). \end{array}$$

(a) Then $t_\lambda^+ \in O_u^{(1),Rad}$ for $\lambda \in \mathrm{Hom}_0(H_{\mathbb{Z}}, \mathbb{Z})$, $t_\lambda^- \in O_{\pm}^{(1),Rad} - O_u^{(1),\mathrm{Rad}}$ for $\lambda \in \mathrm{Hom}_2(H_{\mathbb{Z}}, \mathbb{Z})$. The maps

$$\begin{array}{lll} \mathrm{Hom}_0(H_{\mathbb{Z}}, \mathbb{Z}) & \to & O_u^{(1),Rad}, \quad \lambda \mapsto t_\lambda^+, \\ \mathrm{Hom}_2(H_{\mathbb{Z}}, \mathbb{Z}) & \to & O_{\pm}^{(1),Rad} - O_u^{(1),Rad}, \quad \lambda \mapsto t_\lambda^-, \end{array}$$

are bijections, and the first one is a group isomorphism. For $\lambda_1, \lambda_2 \in \mathrm{Hom}_0(H_{\mathbb{Z}}, \mathbb{Z})$ and $\lambda_3, \lambda_4 \in \mathrm{Hom}_2(H_{\mathbb{Z}}, \mathbb{Z})$

$$\begin{array}{lll} t_{\lambda_2}^+ \circ t_{\lambda_1}^+ & = & t_{\lambda_2+\lambda_1}^+, \quad t_{\lambda_3}^- \circ t_{\lambda_1}^+ = t_{\lambda_3+\lambda_1}^-, \\ t_{\lambda_1}^+ \circ t_{\lambda_3}^- & = & t_{-\lambda_1+\lambda_3}^-, \quad t_{\lambda_4}^- \circ t_{\lambda_3}^- = t_{-\lambda_4+\lambda_3}^+, \\ & & \text{and especially } (t_{\lambda_3}^-)^2 = \mathrm{id}\,. \end{array}$$

(b)

$$(\ker \tau^{(1)})_u = \{t_\lambda^+ \,|\, \lambda(\underline{e}) \in \langle (0, x_1, x_2), (-x_1, 0, x_3), (x_2, x_3, 0)\rangle_{\mathbb{Z}}\}.$$

(c) If $\Gamma_u^{(1)}$ is the normal subgroup generated by $t_{\lambda_1}^+$ for some $\lambda_1 \in \mathrm{Hom}_0(H_{\mathbb{Z}}, \mathbb{Z})$, then $\Gamma_u^{(1)} = \{t_\lambda^+ \,|\, \lambda(\underline{e}) \in L\}$ where $L \subset \mathbb{Z}^3$ is the smallest sublattice with $\lambda_1(\underline{e}) \in L$ and $L \cdot (s_{e_i}^{(1),mat})^{\pm 1} \subset L$ for $i \in \{1, 2, 3\}$.

(d) If $\Gamma^{(1)}\cap O_\pm^{(1),Rad}$ *is the normal subgroup generated by* $t^-_{\lambda_1}$ *for some* $\lambda_1\in \mathrm{Hom}_2(H_\mathbb{Z},\mathbb{Z})$*, then*

$$\begin{aligned}\Gamma_u^{(1)} &= \{t_\lambda^+\,|\,\lambda(\underline{e})\in L\}\quad\text{and}\\ \Gamma^{(1)}\cap O_\pm^{(1),Rad}-\Gamma_u^{(1)} &= \{t_\lambda^-\,|\,\lambda(\underline{e})\in \lambda_1(\underline{e})+L\}\end{aligned}$$

where $L\subset\mathbb{Z}^3$ *is the smallest sublattice with* $\lambda_1(\underline{e})-\lambda_1(\underline{e})\cdot(s_{e_i}^{(1),mat})^{\pm 1}\in L$ *and* $L\cdot(s_{e_i}^{(1),mat})^{\pm1}\subset L$ *for* $i\in\{1,2,3\}$*.*

Proof: (a) By definition $t_\lambda^+=T(\overline{\lambda}\otimes f_3)$ where $\overline{\lambda}\in\overline{H_\mathbb{Z}}^{(1),\sharp}$ denotes the element which is induced by λ. The map $\mathrm{Hom}_0(H_\mathbb{Z},\mathbb{Z})\to O_u^{(1),Rad}$ is an isomorphism by Lemma 6.2 (e). The proofs of the other statements are similar or easy.

(b) The row vectors $(0,x_1,x_2),(-x_1,0,x_3),(-x_2,-x_3,0)$ are the rows of the matrix $I^{(1)}(\underline{e}^t,\underline{e})$. Because of this and Lemma 6.2 (e) $(\ker\tau^{(1)})_u$ is as claimed.

(c) This follows from

$$\begin{aligned}\Gamma_u^{(1)} &= \langle g^{-1}\circ t^+_{\lambda_1}\circ g\,|\,g\in\Gamma^{(1)}\rangle,\\ g^{-1}\circ t_\lambda^+\circ g &= t^+_{\lambda\circ g},\\ \text{and }\lambda\circ(s_{e_i}^{(1)})^{\pm1}(\underline{e}) &= \lambda(\underline{e})\cdot(s_{e_i}^{(1),mat})^{\pm1},\end{aligned}$$

for $\lambda\in\mathrm{Hom}_0(H_\mathbb{Z},\mathbb{Z})$.

(d) Similar to the proof of part (c). □

The following theorem gives $\Gamma^{(1)}$ for $\underline{x}$ in one of the three families above and thus via Remark 4.17 and Lemma 4.18 in principle for all $\underline{x}\in\mathbb{Z}^3$. Though recall that it is nontrivial to find for a given $\underline{x}\in\mathbb{Z}^3$ an element in one of the three families above which is in the $(G^{phi}\ltimes\widetilde{G}^{sign})\rtimes\langle\gamma\rangle$ orbit of $\underline{x}$.

Theorem 6.18. *(a) We have*

$$s_{e_i}^{(1)}\underline{e} = \underline{e}\cdot s_{e_i}^{(1),mat}\quad\text{with}\quad s_{e_1}^{(1),mat}=\begin{pmatrix}1&-x_1&-x_2\\0&1&0\\0&0&1\end{pmatrix},$$

$$s_{e_2}^{(1),mat}=\begin{pmatrix}1&0&0\\x_1&1&-x_3\\0&0&1\end{pmatrix},\quad s_{e_3}^{(1),mat}=\begin{pmatrix}1&0&0\\0&1&0\\x_2&x_3&1\end{pmatrix},$$

$$\Gamma^{(1)}\cong\Gamma^{(1),mat}=\langle s_{e_1}^{(1),mat},s_{e_2}^{(1),mat},s_{e_3}^{(1),mat}\rangle\subset SL_3(\mathbb{Z}).$$

(b) In each reducible case $\underline{x}=(x_1,0,0)$

$$\Gamma^{(1)}\cong\Gamma^{(1)}(S(-x_1))\times\Gamma^{(1)}(A_1)\cong\Gamma^{(1)}(S(-x_1))\times\{1\},$$

and $\Gamma^{(1)}(S(-x_1))$ *is given in Theorem 6.10, with*

$$\begin{aligned}
\Gamma^{(1)}(S(0)) &\cong \Gamma^{(1)}(A_1^2)\cong\{1\},\\
\Gamma^{(1)}(S(-1)) &\cong \Gamma^{(1)}(A_2)\cong SL_2(\mathbb{Z}),\\
\Gamma^{(1)}(S(-x_1)) &\cong G^{free,2}\quad \text{for } x_1\geq 2.
\end{aligned}$$

Also $\Gamma^{(1)}\cong\Gamma_s^{(1)}$ *and* $\Gamma_u^{(1)}=\{\mathrm{id}\}$.

(c) The case $\underline{x}=(1,1,0)$*: (This is the case of* A_3 *and* $\widehat{A}_2$*.)*

$$\begin{aligned}
\mathrm{Rad}\, I^{(1)} &= \mathbb{Z}f_3\quad\text{with}\quad f_3=e_2-e_3,\\
\overline{H_{\mathbb{Z}}}^{(1)} &= \mathbb{Z}\overline{e_1}^{(1)}\oplus\mathbb{Z}\overline{e_2}^{(1)},\\
\Gamma_u^{(1)} &= (\ker\tau^{(1)})_u=O_u^{(1),Rad}=\{t_\lambda^+\,|\,\lambda\in\langle\lambda_1,\lambda_2\rangle_{\mathbb{Z}}\}\cong\mathbb{Z}^2,\\
& \text{with}\quad \lambda_1(\underline{e})=(1,0,0),\quad \lambda_2(\underline{e})=(0,1,1),\\
\Gamma_s^{(1)} &= (\ker\tau^{(1)})_s=O_s^{(1),Rad}\cong SL_2(\mathbb{Z}),\\
\Gamma^{(1)} &= \ker\tau^{(1)}=O^{(1),Rad}.
\end{aligned}$$

The exact sequence (6.10) *splits non-canonically with* $\Gamma_s^{(1)}\cong\langle s_{e_1}^{(1)},s_{e_2}^{(1)}\rangle\subset\Gamma^{(1)}$ *(for example).*

(d) The cases $\underline{x}=(x_1,x_2,0)$ *with* $2\leq x_1\geq x_2>0$*: Write*

$$x_{12}:=\gcd(x_1,x_2)=\frac{x_1}{\widetilde{x}_1}=\frac{x_2}{\widetilde{x}_2}.$$

Then

$$\begin{aligned}
\mathrm{Rad}\, I^{(1)} &= \mathbb{Z}f_3\quad\text{with}\quad f_3=\widetilde{x}_2e_2-\widetilde{x}_1e_3,\\
\overline{H_{\mathbb{Z}}}^{(1)} &= \mathbb{Z}\overline{e_1}^{(1)}\oplus\mathbb{Z}g_2\quad\text{with } g_2:=\frac{1}{\widetilde{x}_1}\overline{e_2}^{(1)}=\frac{1}{\widetilde{x}_2}\overline{e_3}^{(1)}\in\overline{H_{\mathbb{Z}}}^{(1)},\\
\Gamma_u^{(1)} &= \{t_\lambda^+\,|\,\lambda\in\langle\lambda_1,\lambda_2\rangle_{\mathbb{Z}}\}\cong\mathbb{Z}^2\quad\text{with}\\
& \lambda_1(\underline{e})=x_{12}\widetilde{x}_1\widetilde{x}_2(1,0,0),\quad \lambda_2(\underline{e})=x_1x_2(0,\widetilde{x}_1,\widetilde{x}_2),\\
(\ker\tau^{(1)})_u &= \{t_\lambda^+\,|\,\lambda(\underline{e})\in\langle x_{12}(1,0,0),(0,x_1,x_2)\rangle_{\mathbb{Z}}\},\\
O_u^{(1),Rad} &= \{t_\lambda^+\,|\,\lambda(\underline{e})\in\langle(1,0,0),(0,\widetilde{x}_1,\widetilde{x}_2)\rangle_{\mathbb{Z}}\},
\end{aligned}$$

$$\begin{aligned}
\Gamma_s^{(1)} &\cong \Gamma^{(1)}(S(-x_{12}))\cong\langle\begin{pmatrix}1 & -x_{12}\\ 0 & 1\end{pmatrix},\begin{pmatrix}1 & 0\\ x_{12} & 1\end{pmatrix}\rangle\\
&\cong \begin{cases} G^{free,2} & \text{if } x_{12}>1\\ SL_2(\mathbb{Z}) & \text{if } x_{12}=1.\end{cases}
\end{aligned}$$

This matrix group has finite index in $SL_2(\mathbb{Z})$ *if and only if* $x_{12}\in\{1,2\}$*. The exact sequence* (6.10) *splits non-canonically.*

(e) The cases $\underline{x}=(-l,2,-l)$ *with* $l\geq 2$ *even:*

(This includes the case $\underline{x}=(-2,2,-2)$ which is the case of $\mathcal{H}_{1,2}$.)

$$\begin{aligned} \mathrm{Rad}\, I^{(1)} &= \mathbb{Z} f_3 \quad \text{with} \quad f_3=\frac{l}{2}(e_1+e_3)+e_2, \\ \overline{H_\mathbb{Z}}^{(1)} &= \mathbb{Z}\overline{e_1}^{(1)}\oplus\mathbb{Z}\overline{e_3}^{(1)}, \quad \overline{e_2}^{(1)}=-\frac{l}{2}(\overline{e_1}^{(1)}+\overline{e_3}^{(1)}), \end{aligned}$$

$$\begin{aligned} \langle s^{(1)}_{e_1}, s^{(1)}_{e_3}\rangle &\cong \langle \overline{s^{(1)}_{e_1}}, \overline{s^{(1)}_{e_3}}\rangle \cong \langle \begin{pmatrix}1&-2\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\2&1\end{pmatrix}\rangle \overset{1:2}{\subset} \Gamma(2), \\ \langle s^{(1)}_{e_1}, s^{(1)}_{e_3}\rangle &\cong G^{free,2}, \end{aligned}$$

$$\begin{aligned} (\ker\tau^{(1)})_u &= \{t^+_\lambda\,|\,\lambda(\underline{e})\in\langle(-2,0,2),(-2,l,0)\rangle_\mathbb{Z}\}, \\ O^{(1),Rad}_u &= \{t^+_\lambda\,|\,\lambda(\underline{e})\in\langle(-1,0,1),(-1,\frac{l}{2},0)\rangle_\mathbb{Z}\}. \end{aligned}$$

(i) The cases with $l\equiv 0(4)$: $\Gamma^{(1)}_s \cong \langle\overline{s^{(1)}_{e_1}},\overline{s^{(1)}_{e_3}}\rangle \cong G^{free,2}$. The isomorphism $\Gamma^{(1)}_s\cong\langle s^{(1)}_{e_1}, s^{(1)}_{e_3}\rangle\subset\Gamma^{(1)}$ gives a splitting of the exact sequence (6.10). *Here* $-\mathrm{id}\notin\Gamma^{(1)}_s$.

$$\begin{aligned} \Gamma^{(1)}_u &= \{t^+_\lambda\,|\,\lambda\in\langle\lambda_1,\lambda_2\rangle_\mathbb{Z}\}\cong\mathbb{Z}^2 \quad \text{with} \\ & \lambda_1(\underline{e})=(-l,0,l), \quad \lambda_2(\underline{e})=(2l,-l^2,0). \end{aligned}$$

(ii) The cases with $l\equiv 2(4)$: Here $-\mathrm{id}\in\Gamma^{(1)}_s$.

$$\begin{aligned} \Gamma^{(1)}_s &\cong \langle\overline{s^{(1)}_{e_1}},\overline{s^{(1)}_{e_3}},-\mathrm{id}\rangle\cong\langle\begin{pmatrix}1&-2\\0&1\end{pmatrix},\begin{pmatrix}1&0\\2&1\end{pmatrix},\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\rangle=\Gamma(2) \\ &\cong G^{free,2}\times\{\pm 1\}. \end{aligned}$$

The isomorphism $\Gamma^{(1)}_s/\{\pm\mathrm{id}\}\cong\langle s^{(1)}_{e_1}, s^{(1)}_{e_2}\rangle\subset\Gamma^{(1)}$ gives a splitting of the exact sequence

$$\{1\}\to\Gamma^{(1)}\cap O^{(1),Rad}_\pm\to\Gamma^{(1)}\to\Gamma^{(1)}_s/\{\pm\mathrm{id}\}\to\{1\}.$$

$$\begin{aligned} \Gamma^{(1)}_u &= \{t^+_\lambda\,|\,\lambda\in\langle 2\lambda_1,\lambda_2\rangle_\mathbb{Z}\}\cong\mathbb{Z}^2 \quad \text{with} \\ & \lambda_1(\underline{e})=(-l,0,l), \quad \lambda_2(\underline{e})=(2l,-l^2,0), \\ \Gamma^{(1)}\cap O^{(1),Rad}_\pm-\Gamma^{(1)}_u &= \{t^-_\lambda\,|\,\lambda\in\lambda_3+\langle 2\lambda_1,\lambda_2\rangle_\mathbb{Z}\} \quad \text{with} \\ & \lambda_3(\underline{e})=(-l,2,l), \quad \lambda_3(f_3)=2. \end{aligned}$$

(f) The cases $\underline{x}=(-l,2,-l)$ with $l\geq 3$ odd:

$$\begin{aligned} \mathrm{Rad}\, I^{(1)} &= \mathbb{Z}f_3 \quad \text{with} \quad f_3=l(e_1+e_3)+2e_2, \\ \overline{H_\mathbb{Z}}^{(1)} &= \mathbb{Z}\overline{e_1}^{(1)}\oplus\mathbb{Z}\overline{g_2}^{(1)} \quad \text{with} \quad g_2:=\frac{1}{2}(e_1+e_3)-\frac{l}{2}f_3\in H_\mathbb{Z}, \\ \underline{\widetilde{e}} &:= (e_1,g_2,f_3) \quad \text{is a } \mathbb{Z}\text{-basis of } H_\mathbb{Z}. \end{aligned}$$

Consider

$$s_4 \ := \ \big(s_{e_3}^{(1)}s_{e_1}^{(1)}\big)^{\frac{l^2-1}{4}}s_{e_2}^{(1)} \in \Gamma^{(1)}.$$

Then

$$\Gamma_s^{(1)} \ \cong \ \langle\overline{s_{e_1}^{(1)}},\overline{s_4}\rangle \cong \langle s_{e_1}^{(1)},s_4\rangle \cong SL_2(\mathbb{Z}),$$

and the isomorphism $\Gamma_s^{(1)} \cong \langle s_{e_1}^{(1)},s_4\rangle \subset \Gamma^{(1)}$ *gives a splitting of the exact sequence* (6.10)*.*

$$\begin{aligned}\Gamma_u^{(1)} \ &= \ \{t_\lambda^+ \,|\, \lambda\in\langle\lambda_1,\lambda_2\rangle_\mathbb{Z}\}\cong\mathbb{Z}^2 \quad \text{with} \\ &\qquad \lambda_1(\underline{e})=(-l,0,l), \quad \lambda_2(\underline{e})=(2l,-l^2,0), \\ (\ker\tau^{(1)})_u \ &= \ O_u^{(1),Rad}=\{t_\lambda^+\,|\,\lambda(\underline{e})\in\langle(-1,0,1),(2,-l,0)\rangle_\mathbb{Z}\}.\end{aligned}$$

(g) The cases $\underline{x}\in\mathbb{Z}^3_{\geq 3}$ *with* $2x_i\leq x_jx_k$ *for* $\{i,j,k\}=\{1,2,3\}$*:*
(This includes the case $\underline{x}=(3,3,3)$ *which is the case of* $\mathbb{P}^2$*.)*

$$\begin{aligned}\operatorname{Rad} I^{(1)} \ &= \ \mathbb{Z}f_3 \quad \text{with} \quad f_3=-\widetilde{x}_3e_1+\widetilde{x}_2e_2-\widetilde{x}_1e_3, \\ \Gamma_u^{(1)} \ &= \ \{\text{id}\}\subsetneqq(\ker\tau^{(1)})_u\cong\mathbb{Z}^2, \\ \Gamma^{(1)} \ &\cong \ \Gamma_s^{(1)}\cong G^{free,3},\end{aligned}$$

$\Gamma^{(1)}$ *and* $\Gamma_s^{(1)}$ *are free groups with the three generators* $s_{e_1}^{(1)}$*,* $s_{e_2}^{(1)}$*,* $s_{e_3}^{(1)}$ *respectively* $\overline{s_{e_1}^{(1)}}$*,* $\overline{s_{e_2}^{(1)}}$*,* $\overline{s_{e_3}^{(1)}}$*.*

Proof: (a) This follows from the definitions in Lemma 2.6 (a) and in Definition 2.8.

(b) This follows from Lemma 2.11 and Lemma 2.12.

(c)–(f) In (c)–(f) $(\ker\tau^{(1)})_u$ and $O_u^{(1),Rad}$ are calculated with Lemma 6.17 (a) and (b).

(c) The first statements $\operatorname{Rad} I^{(1)}=\mathbb{Z}f_3$ and $\overline{H_\mathbb{Z}}^{(1)}=\mathbb{Z}\overline{e_1}^{(1)}\oplus\mathbb{Z}\overline{e_2}^{(1)}$ are known respectively obvious.

Also (e_1,e_2,f_3) is a $\mathbb{Z}$-basis of $H_\mathbb{Z}$. With respect to this basis

$$s_{e_1}^{(1)}(e_1,e_2,f_3)=(e_1,e_2,f_3)\begin{pmatrix}1&-1&0\\0&1&0\\0&0&1\end{pmatrix},$$

$$s_{e_2}^{(1)}(e_1,e_2,f_3)=(e_1,e_2,f_3)\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}.$$

This shows $\langle s_{e_1}^{(1)},s_{e_2}^{(1)}\rangle\cong\langle\overline{s_{e_1}^{(1)}},\overline{s_{e_2}^{(1)}}\rangle\cong SL_2(\mathbb{Z})$. Together with Lemma 6.16 (b) and Lemma 6.2 (d) we obtain

$$\Gamma_s^{(1)}=\langle\overline{s_{e_1}^{(1)}},\overline{s_{e_2}^{(1)}}\rangle=(\ker\tau^{(1)})_s=O_s^{(1),Rad}\cong SL_2(\mathbb{Z}),$$

and that the exact sequence (6.10) splits non-canonically with $\Gamma_s^{(1)} \cong \langle s_{e_1}^{(1)}, s_{e_2}^{(1)}\rangle \subset \Gamma^{(1)}$.

From the actions of $s_{e_1}^{(1)}$ and $s_{e_2}^{(1)}$ on (e_1, e_2, f_3) and from

$$s_{e_3}^{(1)}((e_1, e_2, f_3)) = (e_1, e_2, f_3) \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix},$$

one sees that the map

$$\left((e_1, e_2, f_3) \mapsto (e_1, e_2, f_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\right) = t^+_{-\lambda_1}$$

is in $\Gamma_u^{(1)}$ and that

$$\Gamma^{(1)} = \langle s_{e_1}^{(1)}, s_{e_2}^{(1)}, t^+_{\lambda_1}\rangle.$$

Also

$$(s_{e_1}^{(1)})^{-1} \circ t^+_{\lambda_1} \circ s_{e_1}^{(1)} = t^+_{\lambda_1 \circ s_{e_1}^{(1)}}, \quad \text{with} \quad t^+_{\lambda_1 \circ s_{e_1}^{(1)}}(\underline{e}) = (1, -1, -1),$$

and therefore $t^+_{\lambda_2} = t^+_{\lambda_1} - t^+_{\lambda_1 \circ s_{e_1}^{(1)}} \in \Gamma_u^{(1)}$. But $O_u^{(1),Rad} = \langle t^+_{\lambda_1}, t^+_{\lambda_2}\rangle_{\mathbb{Z}}$, so

$$\Gamma_u^{(1)} = (\ker \tau^{(1)})_u = O_u^{(1),Rad} = \langle t^+_{\lambda_1}, t^+_{\lambda_2}\rangle_{\mathbb{Z}}$$

In the diagram of exact sequences in Lemma 6.2 (d) the inclusions in the second and fourth columns are bijections, so also the inclusions in the middle column,

$$\Gamma^{(1)} = \ker \tau^{(1)} = O^{(1),Rad}.$$

(d) $f_3 = \widetilde{x}_2 e_2 - \widetilde{x}_1 e_3$ implies $\widetilde{x}_2 \overline{e_2}^{(1)} = \widetilde{x}_1 \overline{e_3}^{(1)}$. Lemma 6.16 (c) gives $\overline{e_2}^{(1)} \in \widetilde{x}_1 \overline{H_{\mathbb{Z}}}^{(1),prim}$, so $g_2 := \frac{1}{\widetilde{x}_1}\overline{e_2}^{(1)} = \frac{1}{\widetilde{x}_2}\overline{e_3}^{(1)}$ is in $\overline{H_{\mathbb{Z}}}^{(1),prim}$. Thus

$$\overline{H_{\mathbb{Z}}}^{(1)} = \mathbb{Z}\overline{e_1}^{(1)} + \mathbb{Z}\overline{e_2}^{(1)} + \mathbb{Z}\overline{e_3}^{(1)} = \mathbb{Z}\overline{e_1}^{(1)} \oplus \mathbb{Z} g_2.$$

First we consider $\Gamma_s^{(1)}$. Define $\underline{g} := (\overline{e_1}^{(1)}, g_2)$. One sees

$$\overline{s_{e_1}^{(1)}}\underline{g} = \underline{g}\begin{pmatrix} 1 & -x_{12} \\ 0 & 1 \end{pmatrix}, \quad \overline{s_{e_2}^{(1)}}\underline{g} = \underline{g}\begin{pmatrix} 1 & 0 \\ x_1\widetilde{x}_1 & 1 \end{pmatrix}, \quad \overline{s_{e_3}^{(1)}}\underline{g} = \underline{g}\begin{pmatrix} 1 & 0 \\ x_2\widetilde{x}_2 & 1 \end{pmatrix}.$$

Choose $y_1, y_2 \in \mathbb{Z}$ with $1 = y_1\widetilde{x}_1^2 + y_2\widetilde{x}_2^2$ and define

$$s_4 := (s_{e_2}^{(1)})^{y_1}(s_{e_3}^{(1)})^{y_2} \in \Gamma^{(1)}.$$

Then

$$\overline{s_4}\,\underline{g} = \underline{g}\begin{pmatrix} 1 & 0 \\ x_{12} & 1 \end{pmatrix}, \quad s_4\,\underline{e} = \underline{e} \cdot s_4^{mat} \text{ with } s_4^{mat} = \begin{pmatrix} 1 & 0 & 0 \\ y_1x_1 & 1 & 0 \\ y_2x_2 & 0 & 1 \end{pmatrix}.$$

$\overline{s^{(1)}_{e_3}}$ and $\overline{s^{(1)}_{e_2}}$ are powers of $\overline{s_4}$, so $\Gamma^{(1)}_s=\langle\overline{s^{(1)}_{e_1}},\overline{s_4}\rangle$, so

$$\Gamma^{(1)}_s\cong\langle\begin{pmatrix}1&-x_{12}\\0&1\end{pmatrix},\begin{pmatrix}1&0\\x_{12}&1\end{pmatrix}\rangle=\Gamma^{(1),mat}(S(-x_{12}))\subset SL_2(\mathbb{Z}).$$

The matrix group $\Gamma^{(1),mat}(S(-x_{12}))$ was treated in Theorem 6.10 (b)–(d):

$$\Gamma^{(1),mat}(S(-x_{12}))\cong\langle\begin{pmatrix}1&-x_{12}\\0&1\end{pmatrix},\begin{pmatrix}1&0\\x_{12}&1\end{pmatrix}\rangle\begin{cases}\cong G^{free,2}&\text{if } x_{12}>1,\\=SL_2(\mathbb{Z})&\text{if } x_{12}=1.\end{cases}$$

The case $x_{12}>1$: Then $\Gamma^{(1)}_s=\langle\overline{s^{(1)}_{e_1}},\overline{s_4}\rangle$ and $\langle s^{(1)}_{e_1},s_4\rangle\subset\Gamma^{(1)}$ are free groups with the two given generators. Then $\langle s^{(1)}_{e_1},s_4\rangle\subset\Gamma^{(1)}$ gives a splitting of the exact sequence (6.10). The generating relations in $\Gamma^{(1)}_s$ with respect to the four elements $\overline{s^{(1)}_{e_1}}$, $\overline{s^{(1)}_{e_2}}$, $\overline{s^{(1)}_{e_3}}$ and $\overline{s_4}$ are

$$\overline{s^{(1)}_{e_2}}=(\overline{s_4})^{\widetilde{x}_1^2},\quad \overline{s^{(1)}_{e_3}}=(\overline{s_4})^{\widetilde{x}_2^2}. \tag{6.11}$$

Therefore $\Gamma^{(1)}_u$ is the normal subgroup of $\Gamma^{(1)}$ generated by the elements $s_4^{\widetilde{x}_1^2}(s^{(1)}_{e_2})^{-1}$ and $s_4^{\widetilde{x}_2^2}(s^{(1)}_{e_3})^{-1}$.

The case $x_{12}=1$: Then $\Gamma^{(1)}_s=\langle\overline{s^{(1)}_{e_1}},\overline{s_4}\rangle\cong SL_2(\mathbb{Z})$

Claim: Also $\langle s^{(1)}_{e_1},s_4\rangle\cong SL_2(\mathbb{Z})$.

Proof of the Claim: The generating relations in $SL_2(\mathbb{Z})$ with respect to the generators $\overline{s^{(1)}_{e_1}}^{mat}=\begin{pmatrix}1&-1\\0&1\end{pmatrix}$ and $\overline{s_4}^{mat}=\begin{pmatrix}1&0\\1&1\end{pmatrix}$ are

$$\overline{s^{(1)}_{e_1}}^{mat}\,\overline{s_4}^{mat}\,\overline{s^{(1)}_{e_1}}^{mat}=\overline{s_4}^{mat}\,\overline{s^{(1)}_{e_1}}^{mat}\,\overline{s_4}^{mat}\quad\text{and}\quad E_2=\big(\overline{s^{(1)}_{e_1}}^{mat}\,\overline{s_4}^{mat}\big)^6.$$

One checks with calculations that they lift to $\Gamma^{(1),mat}$,

$$s^{(1),mat}_{e_1}s^{mat}_4s^{(1),mat}_{e_1}=s^{mat}_4s^{(1),mat}_{e_1}s^{mat}_4\quad\text{and}\quad E_3=\big(s^{(1),mat}_{e_1}s^{mat}_4\big)^6.\quad(\Box)$$

Because of the Claim, $\langle s^{(1)}_{e_1},s_4\rangle\subset\Gamma^{(1)}$ gives a splitting of the exact sequence (6.10). The generating relations in $\Gamma^{(1)}_s$ with respect to the four elements $\overline{s^{(1)}_{e_1}}$, $\overline{s^{(1)}_{e_2}}$, $\overline{s^{(1)}_{e_3}}$ and $\overline{s_4}$ are the relations between $\overline{s^{(1)}_{e_1}}$ and $\overline{s_4}$ in the proof of the Claim and the relations in (6.11). The relations in the proof of the Claim lift to $\Gamma^{(1)}$. Therefore again $\Gamma^{(1)}_u$ is the normal subgroup of $\Gamma^{(1)}$ generated by the elements $s_4^{\widetilde{x}_1^2}(s^{(1)}_{e_2})^{-1}$ and $s_4^{\widetilde{x}_2^2}(s^{(1)}_{e_3})^{-1}$.

Back to both cases $x_{12}\geq 1$ together: We have to determine these two elements of $\Gamma^{(1)}_u$. The first one is given by the following calculation,

$$\begin{aligned}
& s_4^{\widetilde{x}_1^2}(s_{e_2}^{(1)})^{-1}(\underline{e}) \\
= \ & \underline{e}\begin{pmatrix} 1 & 0 & 0 \\ y_1x_1\widetilde{x}_1^2 & 1 & 0 \\ y_2x_2\widetilde{x}_1^2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -x_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underline{e}\begin{pmatrix} 1 & 0 & 0 \\ -x_1+y_1x_1\widetilde{x}_1^2 & 1 & 0 \\ y_2x_2\widetilde{x}_1^2 & 0 & 1 \end{pmatrix} \\
= \ & \underline{e}\begin{pmatrix} 1 & 0 & 0 \\ -y_2x_1\widetilde{x}_2^2 & 1 & 0 \\ y_2x_2\widetilde{x}_1^2 & 0 & 1 \end{pmatrix} = \underline{e} + f_3(-y_2x_{12}\widetilde{x}_1\widetilde{x}_2, 0, 0), \\
& \text{so } s_4^{\widetilde{x}_1^2}(s_{e_2}^{(1)})^{-1}(\underline{e}) = t^+_{-y_2\lambda_1}.
\end{aligned}$$

A similar calculation gives

$$s_4^{\widetilde{x}_2^2}(s_{e_3}^{(1)})^{-1} = t^+_{y_1\lambda_1}.$$

Observe $\gcd(y_1, y_2) = 1$. Therefore $\Gamma_u^{(1)}$ is the normal subgroup of $\Gamma^{(1)}$ generated by $t^+_{\lambda_1}$. Lemma 6.17 (c) and the calculations

$$\begin{aligned}
\lambda_1 \circ s_{e_1}^{(1)}(\underline{e}) &= x_{12}\widetilde{x}_1\widetilde{x}_2(1, -x_1, -x_2), \\
\text{so } \langle \lambda_1, \lambda_1 \circ s_{e_1}^{(1)} \rangle_{\mathbb{Z}} &= \langle \lambda_1, \lambda_2 \rangle_{\mathbb{Z}}, \\
\lambda_1 \circ (s_{e_i}^{(1)})^{\pm 1}, \lambda_2 \circ (s_{e_i}^{(1)})^{\pm 1} &\in \langle \lambda_1, \lambda_2 \rangle_{\mathbb{Z}},
\end{aligned}$$

give

$$\Gamma_u^{(1)} = \{t^+_\lambda \,|\, \lambda \in \langle \lambda_1, \lambda_2 \rangle_{\mathbb{Z}}\}.$$

(e) The first statements $\operatorname{Rad} I^{(1)} = \mathbb{Z}f_3,\ f_3 = \frac{l}{2}(e_1 + e_3) + e_2$ and $\overline{H_{\mathbb{Z}}}^{(1)} = \mathbb{Z}\overline{e_1}^{(1)} \oplus \mathbb{Z}\overline{e_3}^{(1)}$ are obvious, also

$$\overline{s_{e_i}^{(1)}}(\overline{e_1}^{(1)}, \overline{e_3}^{(1)}) = (\overline{e_1}^{(1)}, \overline{e_3}^{(1)})\overline{s_{e_i}^{(1)}}^{mat} \quad \text{with} \quad \overline{s_{e_1}^{(1)}}^{mat} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix},$$

$$\overline{s_{e_2}^{(1)}}^{mat} = \begin{pmatrix} 1+\frac{l^2}{2} & -\frac{l^2}{2} \\ \frac{l^2}{2} & 1-\frac{l^2}{2} \end{pmatrix}, \quad \overline{s_{e_3}^{(1)}}^{mat} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

By Theorem 6.10 (c) $\langle \overline{s_{e_1}^{(1)}}^{mat}, \overline{s_{e_3}^{(1)}}^{mat} \rangle \cong G^{free,2}$ and therefore

$$\langle s_{e_1}^{(1)}, s_{e_3}^{(1)} \rangle \cong G^{free,2}.$$

We have

$$\begin{aligned}
\overline{s_{e_2}^{(1)}}^{mat} &\equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 4 \quad \text{if } l \equiv 0(4), \\
\overline{s_{e_2}^{(1)}}^{mat} &\equiv \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \bmod 4 \quad \text{if } l \equiv 2(4).
\end{aligned}$$

If $l\equiv 0(4)$ then by Theorem 6.10 (c) $\overline{s_{e_2}^{(1)}}^{mat}\in\langle\overline{s_{e_1}^{(1)}}^{mat},\overline{s_{e_3}^{(1)}}^{mat}\rangle$, so then $\Gamma_s^{(1)}=\langle\overline{s_{e_1}^{(1)}},\overline{s_{e_3}^{(1)}}\rangle$, and the isomorphism $\Gamma_s^{(1)}\cong\langle s_{e_1}^{(1)},s_{e_3}^{(1)}\rangle\subset\Gamma^{(1)}$ gives a splitting of the exact sequence (6.10).

If $l\equiv 2(4)$ then by Theorem 6.10 (c) $\overline{s_{e_2}^{(1)}}^{mat}\notin\langle\overline{s_{e_1}^{(1)}}^{mat},\overline{s_{e_3}^{(1)}}^{mat}\rangle$, so then $\langle\overline{s_{e_i}^{(1)}}^{mat}\,|\,i\in\{1,2,3\}\rangle=\Gamma^{(2)}\cong G^{free,2}\times\{\pm 1\}$, $-\mathrm{id}\in\Gamma_s^{(1)}$, and the isomorphism $\Gamma_s^{(1)}/\{\pm\mathrm{id}\}\cong\langle s_{e_1}^{(1)},s_{e_3}^{(1)}\rangle\subset\Gamma^{(1)}$ gives a splitting of the exact sequence in part (ii).

Claim: $\Gamma_u^{(1)}$ if $l\equiv 0(4)$ and $\Gamma^{(1)}\cap O_{\pm}^{(1),Rad}$ if $l\equiv 2(4)$ is the normal subgroup generated by $s_4:=\big(s_{e_3}^{(1)}s_{e_1}^{(1)}\big)^{l^2/4}s_{e_2}^{(1)}$.

Proof of the Claim: Consider the $\mathbb{Z}$-basis $\widetilde{\underline{e}}:=(e_1,e_1+e_3,f_3)$ of $H_{\mathbb{Z}}$.

$$s_{e_1}^{(1)}\widetilde{\underline{e}}=\widetilde{\underline{e}}\begin{pmatrix}1&-2&0\\0&1&0\\0&0&1\end{pmatrix},\ s_{e_2}^{(1)}\widetilde{\underline{e}}=\widetilde{\underline{e}}\begin{pmatrix}1&0&0\\\frac{l^2}{2}&1&0\\-l&0&1\end{pmatrix},\ s_{e_3}^{(1)}\widetilde{\underline{e}}=\widetilde{\underline{e}}\begin{pmatrix}-1&-2&0\\2&3&0\\0&0&1\end{pmatrix},$$

$$\begin{aligned}s_4\,\widetilde{\underline{e}}&=\widetilde{\underline{e}}\begin{pmatrix}-1&0&0\\2&-1&0\\0&0&1\end{pmatrix}^{l^2/4}\begin{pmatrix}1&0&0\\\frac{l^2}{2}&1&0\\-l&0&1\end{pmatrix}\\&=\widetilde{\underline{e}}\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}^{l/2}\begin{pmatrix}1&0&0\\-\frac{l^2}{2}&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\\frac{l^2}{2}&1&0\\-l&0&1\end{pmatrix}\\&=\widetilde{\underline{e}}\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}^{l/2}\begin{pmatrix}1&0&0\\0&1&0\\-l&0&1\end{pmatrix}=\widetilde{\underline{e}}\begin{pmatrix}(-1)^{l/2}&0&0\\0&(-1)^{l/2}&0\\-l&0&1\end{pmatrix},\end{aligned}$$

$$s_4\,\underline{e}=(-1)^{l/2}\underline{e}+f_3(-l,1-(-1)^{l/2},l),$$
$$s_4=\begin{cases}t^+_{\lambda_1}\in\Gamma_u^{(1)}&\text{if }l\equiv 0(4),\\ t^-_{\lambda_3}\in\Gamma^{(1)}\cap O_{\pm}^{(1),Rad}&\text{if }l\equiv 2(4).\end{cases}\tag{6.12}$$

The splittings above of the exact sequences give semidirect products

$$\Gamma^{(1)}=\begin{cases}\Gamma_u^{(1)}\rtimes\langle s_{e_1}^{(1)},s_{e_3}^{(1)}\rangle&\text{if }l\equiv 0(4),\\ \Gamma^{(1)}\cap O_{\pm}^{(1),Rad}\rtimes\langle s_{e_1}^{(1)},s_{e_3}^{(1)}\rangle&\text{if }l\equiv 2(4).\end{cases}$$

$s_{e_2}^{(1)}$ turns up linearly in s_4. Therefore $\Gamma^{(1)}=\langle s_{e_1}^{(1)},s_{e_3}^{(1)},s_4\rangle$. Together these facts show that $\Gamma_u^{(1)}$ respectively $\Gamma^{(1)}\cap O_{\pm}^{(1),Rad}$ is the normal subgroup generated by s_4. This finishes the proof of the Claim. $(\Box)$

Now we can apply Lemma 6.17 (c) if $l \equiv 0(4)$ and Lemma 6.17 (d) if $l \equiv 2(4)$. The following calculations show the claims on $\Gamma_u^{(1)}$ and (in the case $l \equiv 2(4)$) $\Gamma^{(1)} \cap O_\pm^{(1),Rad}$.

The case $l \equiv 0(4)$:

$$\begin{aligned} \lambda_1 \circ s_{e_1}^{(1)}(\underline{e}) &= (-l, -l^2, 3l), \\ \text{so } \lambda_1 \circ s_{e_1}^{(1)} &= 3\lambda_1 + \lambda_2, \\ \lambda_1 \circ (s_{e_i}^{(1)})^{\pm 1}, \lambda_2 \circ (s_{e_i}^{(1)})^{\pm 1} &\in \langle \lambda_1, \lambda_2 \rangle_{\mathbb{Z}}. \end{aligned}$$

The case $l \equiv 2(4)$:

$$\begin{aligned} \lambda_3 \circ s_{e_1}^{(1)}(\underline{e}) &= (-l, 2-l^2, 3l), \\ \text{so } \lambda_3 \circ s_{e_1}^{(1)} &= \lambda_3 + 2\lambda_1 + \lambda_2, \\ \lambda_3 \circ (s_{e_2}^{(1)})^{-1}(\underline{e}) &= (l, 2, -l), \\ \text{so } \lambda_3 \circ (s_{e_2}^{(1)})^{-1} &= \lambda_3 - 2\lambda_1, \\ \lambda_3 \circ (s_{e_i}^{(1)})^{\pm 1} &\in \lambda_3 + \langle 2\lambda_1, \lambda_2 \rangle_{\mathbb{Z}}, \\ 2\lambda_1 \circ (s_{e_i}^{(1)})^{\pm 1}, \lambda_2 \circ (s_{e_i}^{(1)})^{\pm 1} &\in \langle 2\lambda_1, \lambda_2 \rangle_{\mathbb{Z}}. \end{aligned}$$

(f) $\mathrm{Rad}\, I^{(1)} = \mathbb{Z} f_3$ is known. $e_2 = -\frac{l}{2}(e_1 + e_3) + \frac{1}{2} f_3$ shows $\overline{H_{\mathbb{Z}}}^{(1)} = \mathbb{Z}\overline{e_1}^{(1)} \oplus \mathbb{Z}(\frac{1}{2}(\overline{e_1}^{(1)} + \overline{e_3}^{(1)}))$. We have

$$\underline{\widetilde{e}} = \underline{e} \begin{pmatrix} 1 & \frac{1-l^2}{2} & l \\ 0 & -l & 2 \\ 0 & \frac{1-l^2}{2} & l \end{pmatrix}, \quad \underline{e} = \underline{\widetilde{e}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & -l & 2 \\ 0 & \frac{1-l^2}{2} & l \end{pmatrix},$$

so $\underline{\widetilde{e}}$ is a $\mathbb{Z}$-basis of $H_{\mathbb{Z}}$.

First we calculate s_4 with respect to the $\mathbb{Q}$-basis $\underline{\widetilde{f}} := (e_1, \frac{1}{2}(e_1 + e_3), f_3)$ of $H_{\mathbb{Q}}$,

$$\begin{aligned} s_4(\underline{\widetilde{f}}) &= \underline{\widetilde{f}} \Big(\begin{pmatrix} -1 & -1 & 0 \\ 4 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Big)^{\frac{l^2-1}{4}} \begin{pmatrix} 1 & 0 & 0 \\ l^2 & 1 & 0 \\ -\frac{l}{2} & 0 & 1 \end{pmatrix} \\ &= \underline{\widetilde{f}} \begin{pmatrix} -1 & 0 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\frac{l^2-1}{4}} \begin{pmatrix} 1 & 0 & 0 \\ l^2 & 1 & 0 \\ -\frac{l}{2} & 0 & 1 \end{pmatrix} \\ &= \underline{\widetilde{f}} \begin{pmatrix} 1 & 0 & 0 \\ 1-l^2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ l^2 & 1 & 0 \\ -\frac{l}{2} & 0 & 1 \end{pmatrix} = \underline{\widetilde{f}} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -\frac{l}{2} & 0 & 1 \end{pmatrix}. \end{aligned}$$

Therefore

$$s_4(\underline{\widetilde{e}}) = \underline{\widetilde{e}} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad s_{e_1}^{(1)}(\underline{\widetilde{e}}) = \underline{\widetilde{e}} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.13}$$

Thus

$$\Gamma_s^{(1)} = \langle \overline{s_{e_1}^{(1)}}, \overline{s_4} \rangle \cong \langle s_{e_1}^{(1)}, s_4 \rangle \subset \Gamma^{(1)},$$

and this isomorphism gives a splitting of the exact sequence (6.10).

(6.13) shows that the group $\langle s_{e_1}^{(1)}, s_4 \rangle$ contains an element s_5 with

$$s_5(\underline{\widetilde{e}}) = \underline{\widetilde{e}} \begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

thus

$$\begin{aligned} s_5 s_{e_3}^{(1)}(\underline{\widetilde{e}}) &= \underline{\widetilde{e}} \begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 & 0 \\ 4 & 3 & 0 \\ 2l & l & 1 \end{pmatrix} = \underline{\widetilde{e}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2l & l & 1 \end{pmatrix}, \\ s_5 s_{e_3}^{(1)}(\underline{e}) &= \underline{e} + f_3(2l, -l^2, 0), \\ s_5 s_{e_3}^{(1)} &= t_{\lambda_2}^+ \quad \text{with} \quad \lambda_2(\underline{e}) = (2l, -l^2, 0). \end{aligned}$$

Also

$$\Gamma^{(1)} = \langle s_{e_1}^{(1)}, s_{e_2}^{(1)}, s_{e_3}^{(1)} \rangle = \langle s_4, s_{e_1}^{(1)}, s_{e_3}^{(1)} \rangle = \langle s_4, s_{e_1}^{(1)}, t_{\lambda_2}^+ \rangle,$$

so $\Gamma_u^{(1)}$ is the normal subgroup generated by $t_{\lambda_2}^+$.

Now we can apply Lemma 6.17 (c). The following formulas show $\Gamma_u^{(1)} = \{t_\lambda^+ \mid \lambda \in \langle \lambda_1, \lambda_2 \rangle_{\mathbb{Z}}\}$.

$$\begin{aligned} (2l, -l^2, 0) s_{e_1}^{(1),mat} = (2l, l^2, -4l) &= (2l, -l^2, 0) + (0, 2l^2, -4l), \\ 2(2l, -l^2, 0) + (0, 2l^2, -4l) &= (4l, 0, -4l), \\ (2l, -l^2, 0) s_{e_2}^{(1),mat} &= (2l, -l^2, 0) + (l^3, 0, -l^3), \\ \gcd(4l, l^3) &= l, \quad \text{so } \lambda_1 \in \Gamma_u^{(1)}, \\ \lambda_1 \circ (s_{e_i}^{(1)})^{\pm 1}, \lambda_2 \circ (s_{e_i}^{(1)})^{\pm 1} &\in \langle \lambda_1, \lambda_2 \rangle_{\mathbb{Z}}. \end{aligned}$$

(g) Because of the cyclic action of $\gamma \in (G^{phi} \ltimes \widetilde{G}^{sign}) \rtimes \langle \gamma \rangle$ on $\mathbb{Z}^3$, we can suppose $x_1 = \max(x_1, x_2, x_3)$. With respect to the $\mathbb{Q}$-basis $(\overline{e_1}^{(1)}, \overline{e_2}^{(1)})$ of $\overline{H_{\mathbb{Q}}}^{(1)}$, $\overline{s_{e_1}^{(1)}}$, $\overline{s_{e_2}^{(1)}}$ and $\overline{s_{e_3}^{(1)}}$ take the shape $\overline{s_{e_i}^{(1)}}(\overline{e_1}^{(1)}, \overline{e_2}^{(1)}) = (\overline{e_1}^{(1)}, \overline{e_2}^{(1)}) B_i$ with

$$B_1 = \begin{pmatrix} 1 & -x_1 \\ 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 \\ x_1 & 1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 1 - \frac{x_3 x_2}{x_1} & -\frac{x_3^2}{x_1} \\ \frac{x_2^2}{x_1} & 1 + \frac{x_3 x_2}{x_1} \end{pmatrix}.$$

We will show that the group $\langle\mu_1,\mu_2,\mu_3\rangle$ of Möbius transformations $\mu_i := \mu(B_i)$ is a free group with the three generators μ_1,μ_2,μ_3. This implies first $\Gamma_s^{(1)} \cong G^{free,3}$ and then $\Gamma^{(1)} \cong \Gamma_s^{(1)} \cong G^{free,3}$ and $\Gamma_u^{(1)} = \{\mathrm{id}\}$.

We will apply Theorem A.2 (c). μ_1,μ_2 and μ_3 are parabolic,

$$\begin{aligned}
\mu_1 &= \big(z\mapsto z-x_1\big) \quad \text{with fixed point } \infty,\\
\mu_2 &= \Big(z\mapsto \frac{z}{x_1z+1}\Big) \quad \text{with fixed point } 0,\\
\mu_3 &= \Big(z\mapsto \frac{(1-\frac{x_3x_2}{x_1})z-\frac{x_3^2}{x_1}}{\frac{x_2^2}{x_1}z+(1+\frac{x_3x_2}{x_1})} = \frac{(x_1-x_2x_3)z-x_3^2}{x_2^2z+(x_1+x_2x_3)}\Big)\\
&\qquad\qquad \text{with fixed point } -\frac{x_3}{x_2}.
\end{aligned}$$

Consider

$$r_1 := \mu_1(1) = 1-x_1,\quad r_2 := \mu_2^{-1}(1) = \frac{1}{1-x_1},$$
$$r_3 := \mu_3(r_1) = \frac{(x_1-x_2x_3)(1-x_1)-x_3^2}{x_2^2(1-x_1)+(x_1+x_2x_3)}.$$

It is sufficient to show the inequalities

$$-\infty < r_1 < -\frac{x_3}{x_2} < r_3 \le r_2 < 0 < 1 < \infty. \tag{6.14}$$

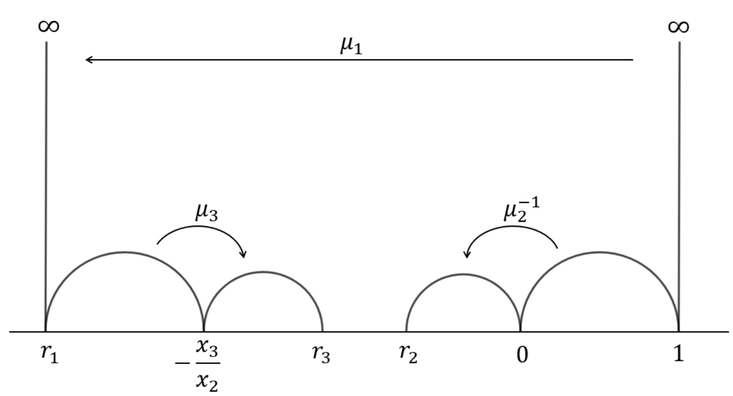


FIGURE 6.6. $G^{free,3}$ generated by three parabolic Möbius transformations, an application of Theorem A.2 (c)

Then Theorem A.2 (c) applies. Compare Figure 6.6.

We treat the case $\underline{x} = (3,3,3)$ separately and first. Then

$$(r_1, -\frac{x_3}{x_2}, r_3, r_2) = (-2, -1, -\frac{1}{2}, -\frac{1}{2}),$$

so then (6.14) holds.

From now on we suppose $x_1 \geq 4$. The inequality $r_2 < 0$ is trivial. Consider the number

$$r_4 := -\frac{x_1 + x_2x_3}{x_2^2} = -\frac{x_1}{x_2^2} - \frac{x_3}{x_2} < -\frac{x_3}{x_2}, \quad \text{with } \mu_3(r_4) = \infty.$$

We will show in this order the following claims:

(i) $r_1 < r_4$, which implies $r_1 < -\frac{x_3}{x_2}$.
(ii) $r_3 \in (-\frac{x_3}{x_2}, \infty)$.
(iii) The numerator $(x_2x_3 - x_1)(x_1 - 1) - x_3^2$ of r_3 is positive.
(iv) The denominator $x_2^2(1 - x_1) + (x_1 + x_2x_3)$ of r_3 is negative.
(v) $r_3 \leq r_2$.

Together (i)–(v) and $r_2 < 0$ show (6.14).

Two of the three inequalities $2x_i \leq x_jx_k$ for $\{i,j,k\} = \{1,2,3\}$ in the assumption on $\underline{x}$ can be improved,

$$x_1x_2 \geq 3x_1 \geq 3x_3, \quad x_1x_3 \geq 3x_1 \geq 3x_2, \quad \text{we keep} \quad x_2x_3 \geq 2x_1.$$

(i) holds because

$$r_1 < r_4 \iff 1 < x_1 - \frac{x_1}{x_2^2} - \frac{x_3}{x_2} \quad \text{and}$$
$$x_1 - \frac{x_1}{x_2^2} - \frac{x_3}{x_2} \geq x_1 - \frac{x_1}{9} - \frac{x_1}{3} = \frac{5x_1}{9} \geq \frac{20}{9}.$$

(ii) $r_1 < r_4$ and $\mu_3(r_4) = \infty$ imply $r_3 = \mu_3(r_1) \in (-\frac{x_3}{x_2}, \infty)$.

(iii) holds because

$$(x_2x_3 - x_1)(x_1 - 1) - x_3^2 > 0 \iff x_1 - 1 > \frac{x_3^2}{x_2x_3 - x_1} = \frac{x_3}{x_2}\frac{1}{1 - \frac{x_1}{x_2x_3}}$$
$$\text{and } \frac{x_3}{x_2}\frac{1}{1 - \frac{x_1}{x_2x_3}} \leq \frac{x_1}{3}\frac{1}{1 - \frac{1}{2}} = \frac{2x_1}{3}, \quad x_1 - 1 > \frac{2x_1}{3} \Leftarrow x_1 \geq 4.$$

(iv) holds because

$$x_2^2(x_1 - 1) - (x_1 + x_2x_3) > 0 \iff x_1 - 1 > \frac{x_1 + x_2x_3}{x_2^2} = \frac{x_3}{x_2}(1 + \frac{x_1}{x_2x_3})$$
$$\text{and } \frac{x_3}{x_2}(1 + \frac{x_1}{x_2x_3}) \leq \frac{x_1}{3}(1 + \frac{1}{2}) = \frac{x_1}{2}, \quad x_1 - 1 > \frac{x_1}{2} \Leftarrow x_1 \geq 4.$$

(ii)–(iv) show $r_3 \in (-\frac{x_3}{x_2}, 0)$. Now

$$r_3 \leq r_2 \iff \begin{array}{l}\big[(x_2x_3 - x_1)(x_1-1) - x_3^2\big](x_1-1) \\ -\big[x_2^2(x_1-1) - (x_1+x_2x_3)\big] \geq 0.\end{array}$$

The right hand side is

$$g(x_1,x_2,x_3) := (x_2x_3 - x_1)(x_1-1)^2 - (x_2^2+x_3^2)(x_1-1) + (x_1+x_2x_3).$$

This is symmetric and homogeneous of degree 2 in x_2 and x_3.

$$\frac{\partial g}{\partial x_2}(\underline{x}) = (x_1-1)^2x_3 - 2(x_1-1)x_2 + x_3,$$

so for fixed x_1 and x_3 $g(\underline{x})$ takes its maximum in

$$x_2^0 = \frac{(x_1-1)^2x_3 + x_3}{2(x_1-1)} = \frac{x_3(x_1-1)}{2} + \frac{x_3}{2(x_1-1)} > \frac{3(x_1-1)}{2} > x_1,$$

and is monotonously increasing left of x_2^0.

Because of the symmetry we can restrict to the cases $x_1 \geq x_2 \geq x_3$. Then $g(\underline{x})$ takes for fixed x_1 and x_3 its minimum in $x_2 = x_3$.

$$\widetilde{g}(x_1,x_3) := g(x_1,x_3,x_3) = x_3^2(x_1-2)^2 - (x_1-1)^2x_1 + x_1.$$

$\widetilde{g}$ takes for fixed x_1 its minimum at the minimal possible x_3, which is $x_3^0 := \max(3, \sqrt{2x_1})$ because of $2x_1 \leq x_2x_3 = x_3^2$.

$$\widetilde{g}(x_1,x_3^0) = \begin{cases} \widetilde{g}(4,3) = 4 > 0 & \text{if } x_1 = 4, \\ \widetilde{g}(x_1,\sqrt{2x_1}) = x_1^3 - 6x_1^2 + 8x_1 & \\ \qquad = x_1(x_1-2)(x_1-4) > 0 & \text{if } x_1 \geq 5. \end{cases}$$

Therefore $r_3 < r_2$ and (6.14) is proved. □

Remarks 6.19. (i) In the proof of part (g) of Theorem 6.18 the hyperbolic polygon P whose relative boundary is the union of the six arcs

$$A(\infty, r_1),\ A(r_1, -\frac{x_3}{x_2}),\ A(-\frac{x_3}{x_2}, r_3),\ A(r_2, 0),\ A(0,1),\ A(1,\infty),$$

is a fundamental domain for the action of the group $\langle \mu_1,\mu_2,\mu_3\rangle$ on the upper half plane $\mathbb{H}$.

(ii) In part (g) of Theorem 6.18 the case $\underline{x} = (3,3,3)$ is especially interesting. It is the only case within part (g) where $r_3 = r_2$, so the only case in part (g) where the hyperbolic polygon P has finite hyperbolic area. In this case

$$\langle B_1,B_2,B_3\rangle = \langle \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} -2 & -3 \\ 3 & 4 \end{pmatrix} \rangle = \Gamma(3).$$

Now we turn to the study of the set $\Delta^{(1)}$ of odd vanishing cycles. The shape of $\Delta^{(1)}$ and our knowledge on it are very different for different $\underline{x}\in\mathbb{Z}^3$.

REMARKS 6.20. (i) In the cases in the parts (c)–(f) of Theorem 6.21 we will give $\Delta^{(1)}$ rather explicitly. There $\overline{\Delta^{(1)}}$ is known, and we can give a subset of $\Delta^{(1)}$ explicitly which maps bijectively to $\overline{\Delta^{(1)}}$.

Furthermore $\Gamma_u^{(1)}\cong\mathbb{Z}^2$, and for $\varepsilon\in\{\pm 1\}$ and $i\in\{1,2,3\}$ there is a subset $F_{\varepsilon,i}\subset\mathbb{Z}f_3$ with

$$\begin{array}{rcl}\Delta^{(1)}\cap(\varepsilon e_i+\mathbb{Z}f_3) &=& \varepsilon e_i+F_{\varepsilon,i}\quad\text{and}\\ \Delta^{(1)}\cap(a+\mathbb{Z}f_3) &=& a+F_{\varepsilon,i}\quad\text{for any } a\in\Gamma^{(1)}\{\varepsilon e_i\}.\end{array}$$

In many, but not all cases $F_{\varepsilon,i}\subset\mathbb{Z}f_3$ is a sublattice.

(ii) On the contrary, in the cases in part (g) of Theorem 6.21 we know rather little. There $\Gamma_u^{(1)}=\{\mathrm{id}\}$, and remarkably the projection $\Delta^{(1)}\to\overline{\Delta^{(1)}}\subset\overline{H_{\mathbb{Z}}}^{(1)}$ is a bijection. In the case $\underline{x}=(3,3,3)$ we know $\overline{\Delta^{(1)}}$. But the lift $a\in\Delta^{(1)}$ of an element $\overline{a}^{(1)}\in\overline{\Delta^{(1)}}$ is difficult to determine. See Lemma 6.26.

THEOREM 6.21. *(a) (Empty: (b)–(g) shall correspond to (b)–(g) in Theorem 6.18.)*

(b) In each reducible case $\underline{x}=(x_1,0,0)$

$$\begin{array}{rcl}\Delta^{(1)} &=& \Delta^{(1)}\cap(\mathbb{Z}e_1\oplus\mathbb{Z}e_2)\ \dot\cup\ \{\pm e_3\}\\ \text{with} && \Delta^{(1)}\cap(\mathbb{Z}e_1\oplus\mathbb{Z}e_2)\cong\Delta^{(1)}(S(-x_1)),\end{array}$$

and $\Delta^{(1)}(S(-x_1))$ *is given in Theorem 6.10.*

(c) The case $\underline{x}=(1,1,0)$*:*

$$\begin{array}{rcl}\overline{\Delta^{(1)}} &=& \overline{H_{\mathbb{Z}}}^{(1),prim},\\ \Delta^{(1)} &=& (pr^{H,(1)})^{-1}(\overline{\Delta^{(1)}})=(\mathbb{Z}e_1\oplus\mathbb{Z}e_2)^{prim}+\mathbb{Z}f_3\subset H_{\mathbb{Z}}^{prim},\\ \Delta^{(1)} &=& \Gamma^{(1)}\{e_1\}.\end{array}$$

$\Delta^{(1)}$ *and* $\overline{\Delta^{(1)}}$ *consist each of one orbit.*

(d) The cases $\underline{x}=(x_1,x_2,0)$ *with* $2\leq x_1\geq x_2>0$*: Recall* $x_{12}:=\gcd(x_1,x_2)$*.*

(i) The cases with $x_{12}=1$ *and* $x_2>1$*: Then* $x_1=\widetilde{x}_1$*,* $x_2=\widetilde{x}_2$ *and* $x_1>x_2>1$*. Choose* $y_1,y_2\in\mathbb{Z}$ *with* $1=y_1x_1^2+y_2x_2^2$*.* $\overline{\Delta^{(1)}}$ *consists of the three orbits*

$$\begin{array}{rcl}\Gamma_s^{(1)}\{\overline{e_1}^{(1)}\} &=& \overline{H_{\mathbb{Z}}}^{(1),prim},\\ \Gamma_s^{(1)}\{\overline{e_2}^{(1)}\} &=& x_1\overline{H_{\mathbb{Z}}}^{(1),prim},\\ \Gamma_s^{(1)}\{\overline{e_3}^{(1)}\} &=& x_2\overline{H_{\mathbb{Z}}}^{(1),prim}.\end{array}$$

$\Delta^{(1)}$ *consists of five orbits:*

$$\Gamma^{(1)}\{e_1\}=\Gamma^{(1)}\{\pm e_1\},\quad \Gamma^{(1)}\{\varepsilon e_2\},\quad \Gamma^{(1)}\{\varepsilon e_3\}\quad \text{with } \varepsilon\in\{\pm 1\}.$$

Here

$$\begin{aligned}
&\Delta^{(1)}\cap(e_1+\mathbb{Z}f_3)=\Gamma^{(1)}\{e_1\}\cap(e_1+\mathbb{Z}f_3)\\
&\quad =\Gamma^{(1)}_u\{e_1\}=e_1+\mathbb{Z}x_1x_2f_3,\\
&\Delta^{(1)}\cap(\varepsilon e_2+\mathbb{Z}f_3)\\
&\quad =\Bigl(\Gamma^{(1)}\{\varepsilon e_2\}\cap(\varepsilon e_2+\mathbb{Z}f_3)\Bigr)\ \dot\cup\ \Bigl(\Gamma^{(1)}\{-\varepsilon e_2\}\cap(\varepsilon e_2+\mathbb{Z}f_3)\Bigr)\\
&\quad =\Bigl(\varepsilon e_2+\mathbb{Z}x_1^2x_2f_3\Bigr)\ \dot\cup\ \Bigl(\varepsilon e_2-\varepsilon 2y_2x_2f_3+\mathbb{Z}x_1^2x_2f_3\Bigr),\\
&\Delta^{(1)}\cap(\varepsilon e_3+\mathbb{Z}f_3)\\
&\quad =\Bigl(\Gamma^{(1)}\{\varepsilon e_3\}\cap(\varepsilon e_3+\mathbb{Z}f_3)\Bigr)\ \dot\cup\ \Bigl(\Gamma^{(1)}\{-\varepsilon e_3\}\cap(\varepsilon e_3+\mathbb{Z}f_3\Bigr)\\
&\quad =\Bigl(\varepsilon e_3+\mathbb{Z}x_1x_2^2f_3)\Bigr)\ \dot\cup\ \Bigl(\varepsilon e_3+\varepsilon 2y_1x_1f_3+\mathbb{Z}x_1x_2^2f_3\Bigr).
\end{aligned}$$

(ii) The cases with $x_2=1$: Then $x_1=\widetilde{x}_1>x_2=\widetilde{x}_2=x_{12}=1$. $\overline{\Delta^{(1)}}$ *consists of the two orbits*

$$\begin{aligned}
\Gamma^{(1)}_s\{\overline{e_1}^{(1)}\} &= \Gamma^{(1)}_s\{\pm\overline{e_1}^{(1)},\pm\overline{e_3}^{(1)}\}=\overline{H_{\mathbb{Z}}}^{(1),prim},\\
\Gamma^{(1)}_s\{\overline{e_2}^{(1)}\} &= \Gamma^{(1)}_s\{\pm\overline{e_2}^{(1)}\}=x_1\overline{H_{\mathbb{Z}}}^{(1),prim}.
\end{aligned}$$

$\Delta^{(1)}$ *consists of three orbits,*

$$\Gamma^{(1)}\{e_1\}=\Gamma^{(1)}\{\pm e_1,\pm e_3\},\quad \Gamma^{(1)}\{\varepsilon e_2\},\quad \text{with } \varepsilon\in\{\pm 1\}.$$

Here

$$\begin{aligned}
&\Delta^{(1)}\cap(e_1+\mathbb{Z}f_3)=\Gamma^{(1)}\{e_1\}\cap(e_1+\mathbb{Z}f_3)=\Gamma^{(1)}_u\{e_1\}=e_1+\mathbb{Z}x_1f_3,\\
&\Delta^{(1)}\cap(\varepsilon e_2+\mathbb{Z}f_3)\\
&\quad =\Bigl(\Gamma^{(1)}\{\varepsilon e_2\}\cap(\varepsilon e_2+\mathbb{Z}f_3)\Bigr)\ \dot\cup\ \Bigl(\Gamma^{(1)}\{-\varepsilon e_2\}\cap(\varepsilon e_2+\mathbb{Z}f_3)\Bigr)\\
&\quad =\Bigl(\varepsilon e_2+\mathbb{Z}x_1^2f_3\Bigr)\ \dot\cup\ \Bigl(\varepsilon e_2-\varepsilon 2f_3+\mathbb{Z}x_1^2f_3\Bigr).
\end{aligned}$$

(iii) The cases with $x_{12}>1$ and $x_1>x_2>1$: Then $\widetilde{x}_1>\widetilde{x}_2\geq 1$. Recall from Theorem 6.18 (s) $g_2=\frac{1}{\widetilde{x}_1}\overline{e_2}^{(1)}=\frac{1}{\widetilde{x}_2}\overline{e_3}^{(1)}\in\overline{H_{\mathbb{Z}}}^{(1)}$. Here $\overline{\Delta^{(1)}}\subset\overline{H_{\mathbb{Z}}}^{(1)}$ consists of the six orbits (with $\varepsilon\in\{\pm 1\}$)

$$\begin{aligned}
\Gamma^{(1)}_s\{\varepsilon\overline{e_1}^{(1)}\} &\subset \overline{H_{\mathbb{Z}}}^{(1),prim},\\
\Gamma^{(1)}_s\{\varepsilon\overline{e_2}^{(1)}\} &= \widetilde{x}_1\Gamma^{(1)}_s\{\varepsilon g_2\}\subset\widetilde{x}_1\overline{H_{\mathbb{Z}}}^{(1),prim},\\
\Gamma^{(1)}_s\{\varepsilon\overline{e_3}^{(1)}\} &= \widetilde{x}_2\Gamma^{(1)}_s\{\varepsilon g_2\}\subset\widetilde{x}_2\overline{H_{\mathbb{Z}}}^{(1),prim}
\end{aligned}$$

Theorem 6.10 (c) and (d) describe the orbits $\Gamma_s^{(1)}\{\overline{e_1}^{(1)}\}$ *and* $\Gamma_s^{(1)}\{g_2\}$. *Also* $\Delta^{(1)}$ *consists of six orbits.*

$$\begin{aligned} \Delta^{(1)}\cap(\varepsilon e_1+\mathbb{Z}f_3) &= \varepsilon e_1+\mathbb{Z}x_{12}\widetilde{x}_1\widetilde{x}_2 f_3, \\ \Delta^{(1)}\cap(\varepsilon e_2+\mathbb{Z}f_3) &= \varepsilon e_2+\mathbb{Z}x_1x_2\widetilde{x}_1 f_3, \\ \Delta^{(1)}\cap(\varepsilon e_3+\mathbb{Z}f_3) &= \varepsilon e_3+\mathbb{Z}x_1x_2\widetilde{x}_2 f_3. \end{aligned}$$

(iv) The cases with $x_1=x_2\geq 2$*: Then* $x_{12}=x_1=x_2$, $\widetilde{x}_1=\widetilde{x}_2=1$, $\overline{e_2}^{(1)}=\overline{e_3}^{(1)}=g_2$. *Then* $\overline{\Delta^{(1)}}\subset\overline{H_{\mathbb{Z}}}^{(1)}$ *consists of the four orbits (with* $\varepsilon\in\{\pm 1\}$*)*

$$\begin{aligned} \Gamma_s^{(1)}\{\varepsilon\overline{e_1}^{(1)}\} &\subset \overline{H_{\mathbb{Z}}}^{(1),prim}, \\ \Gamma_s^{(1)}\{\varepsilon\overline{e_2}^{(1)}\} &= \Gamma_s^{(1)}\{\varepsilon g_2\}\subset\overline{H_{\mathbb{Z}}}^{(1),prim}. \end{aligned}$$

Theorem 6.10 (c) and (d) describe these orbits. $\Delta^{(1)}$ *consists of six orbits.*

$$\begin{aligned} &\Delta^{(1)}\cap(\varepsilon e_1+\mathbb{Z}f_3)=\Gamma^{(1)}\{\varepsilon e_1\}\cap(\varepsilon e_1+\mathbb{Z}f_3)=\Gamma_u^{(1)}\{\varepsilon e_1\} \\ &\quad =\varepsilon e_1+\mathbb{Z}x_1 f_3, \\ &\Delta^{(1)}\cap(\varepsilon e_2+\mathbb{Z}f_3)=\Delta^{(1)}\cap(\varepsilon e_3+\mathbb{Z}f_3) \\ &\quad =\Big(\Gamma^{(1)}\{\varepsilon e_2\}\cap(\varepsilon e_2+\mathbb{Z}f_3)\Big)\ \dot\cup\ \Big(\Gamma^{(1)}\{\varepsilon e_3\}\cap(\varepsilon e_2+\mathbb{Z}f_3)\Big) \\ &\quad =\Big(\varepsilon e_2+\mathbb{Z}x_1^2 f_3\Big)\ \dot\cup\ \Big(\varepsilon e_2-\varepsilon f_3+\mathbb{Z}x_1^2 f_3\Big). \end{aligned}$$

(e) The cases $\underline{x}=(-l,2,-l)$ *with* $l\geq 2$ *even: Consider* $\varepsilon\in\{\pm 1\}$.
(i) The cases with $l\equiv 0(4)$*:* $\Delta^{(1)}$ *consists of six orbits,*

$$\begin{aligned} \Gamma^{(1)}\{\varepsilon e_1\} &= \{y_1e_1+y_2e_3\in H_{\mathbb{Z}}^{prim}\,|\,y_1\equiv\varepsilon(4),y_2\equiv 0(2)\}+\mathbb{Z}lf_3, \\ \Gamma^{(1)}\{\varepsilon e_3\} &= \{y_1e_1+y_2e_3\in H_{\mathbb{Z}}^{prim}\,|\,y_1\equiv 0(2),y_2\equiv\varepsilon(4)\}+\mathbb{Z}lf_3, \\ \Gamma^{(1)}\{\varepsilon e_2\} &= \frac{l}{2}\{y_1e_1+y_2e_3\in H_{\mathbb{Z}}^{prim}\,|\,y_1\equiv 1(2),y_2\equiv 1(2)\} \\ &\quad +\varepsilon f_3+\mathbb{Z}l^2 f_3. \end{aligned}$$

$\overline{\Delta^{(1)}}$ *consists of five orbits,* $\Gamma_s^{(1)}\{\overline{e_2}^{(1)}\}=\Gamma_s^{(1)}\{-\overline{e_2}^{(1)}\}$.

(ii) The cases with $l\equiv 2(4)$: $\Delta^{(1)}$ consists of six orbits,

$$\Gamma^{(1)}\{\varepsilon e_1\} = \Big(\{y_1e_1+y_2e_3\in H_{\mathbb{Z}}^{prim}\,|\,y_1\equiv\varepsilon(4),y_2\equiv 0(2)\}+\mathbb{Z}2lf_3\Big)$$
$$\dot\cup\Big(\{y_1e_1+y_2e_3\in H_{\mathbb{Z}}^{prim}\,|\,y_1\equiv-\varepsilon(4),y_2\equiv 0(2)\}+lf_3+\mathbb{Z}2lf_3\Big),$$
$$\Gamma^{(1)}\{\varepsilon e_3\} = \Big(\{y_1e_1+y_2e_3\in H_{\mathbb{Z}}^{prim}\,|\,y_1\equiv 0(2),y_2\equiv \varepsilon(4)\}+\mathbb{Z}2lf_3\Big)$$
$$\dot\cup\Big(\{y_1e_1+y_2e_3\in H_{\mathbb{Z}}^{prim}\,|\,y_1\equiv 0(2),y_2\equiv -\varepsilon(4)\}+lf_3+\mathbb{Z}2lf_3\Big),$$
$$\Gamma^{(1)}\{\varepsilon e_2\} = \frac{l}{2}\{y_1e_1+y_2e_3\in H_{\mathbb{Z}}^{prim}\,|\,y_1\equiv 1(2),y_2\equiv 1(2)\}$$
$$+\varepsilon f_3+\mathbb{Z}l^2f_3.$$

Especially

$$\begin{array}{rcl}\Delta^{(1)}\cap(\varepsilon e_1+\mathbb{Z}f_3) &=& \varepsilon e_1+\mathbb{Z}lf_3\\ \supsetneqq \Gamma^{(1)}\{\varepsilon e_1\}\cap(\varepsilon e_1+\mathbb{Z}f_3) &=& \varepsilon e_1+\mathbb{Z}2lf_3,\end{array}$$

and similarly for εe_3. $\overline{\Delta^{(1)}}$ consists of three orbits, $\Gamma_s^{(1)}\{\overline{e_i}^{(1)}\} = \Gamma_s^{(1)}\{-\overline{e_i}^{(1)}\}$ for $i\in\{1,2,3\}$.

(f) The cases $\underline{x}=(-l,2,-l)$ with $l\geq 3$ odd: $\Delta^{(1)}$ consists of three orbits (with $\varepsilon\in\{\pm 1\}$)

$$\begin{array}{rcl}\Gamma^{(1)}\{e_1\} &=& \Gamma^{(1)}\{\pm e_1,\pm e_3\} = (\mathbb{Z}e_1\oplus\mathbb{Z}g_2)^{prim}+\mathbb{Z}lf_3,\\ \Gamma^{(1)}\{\varepsilon e_2\} &=& l(\mathbb{Z}e_1\oplus\mathbb{Z}g_2)^{prim}+\varepsilon\dfrac{1-l^2}{2}f_3+\mathbb{Z}l^2f_3.\end{array}$$

$\overline{\Delta^{(1)}}$ consists of two orbits, $\Gamma_s^{(1)}\{\overline{e_2}^{(1)}\}=\Gamma_s^{(1)}\{-\overline{e_2}^{(1)}\}$.

(g) The cases $\underline{x}\in\mathbb{Z}^3_{\geq 3}$ with $2x_i\leq x_jx_k$ for $\{i,j,k\}=\{1,2,3\}$: $\Delta^{(1)}$ and $\overline{\Delta^{(1)}}$ consist each of six orbits,

$$\Gamma^{(1)}\{\varepsilon e_i\}\quad\text{respectively}\quad\Gamma_s^{(1)}\{\varepsilon\overline{e_i}^{(1)}\}\quad\text{for }(\varepsilon,i)\in\{\pm 1\}\times\{1,2,3\}.$$

The projection $\Delta^{(1)}\to\overline{\Delta^{(1)}}$ is a bijection.

(h) The case $\underline{x}=(3,3,3)$: In this subcase of (g) the statements in (g) hold, and

$$\Gamma^{(1)}\{\varepsilon e_i\}\subset\varepsilon e_i+3H_{\mathbb{Z}},\quad \Gamma_s^{(1)}\{\varepsilon\overline{e_i}^{(1)}\}=(\varepsilon\overline{e_i}^{(1)}+3\overline{H_{\mathbb{Z}}}^{(1)})^{prim}$$

for $\varepsilon\in\{\pm 1\}$, $i\in\{1,2,3\}$.

Proof: (b) The splitting of $\Delta^{(1)}$ follows with Lemma 2.11. The first and second subset are treated by Theorem 6.10 and Lemma 2.12.

(c) $\overline{\Delta^{(1)}}=\overline{H_{\mathbb{Z}}}^{prim}$ follows with $\Gamma_s^{(1)}\cong SL_2(\mathbb{Z})$ and Theorem 6.10 (b). $\Delta^{(1)}$ is the full preimage in $H_{\mathbb{Z}}$, because $\Gamma_u^{(1)}=O_u^{(1),Rad}=\{t_\lambda^+\,|\,\lambda(\underline{e})\in\langle(1,0,0),(0,1,1)\rangle_{\mathbb{Z}}\}$.

(d) (i) All statements on $\overline{\Delta^{(1)}}$ follow from $\Gamma_s^{(1)} \cong SL_2(\mathbb{Z})$ and $\overline{e_2}^{(1)} = x_1 g_2$, $\overline{e_3}^{(1)} = x_2 g_2$.

Recall the definition and the properties of $s_4 \in \Gamma^{(1)}$ in the proof of Theorem 6.18.

For $\Delta^{(1)}$ we have to use Lemma 6.7.

Claim 1: For $a \in \{\pm e_1, \pm e_2, \pm e_3\}$ equality in $\overset{(3)}{\supset}$ in Lemma 6.7 holds.

Proof of Claim 1: Suppose $h \in \mathrm{Stab}_{\Gamma^{(1)}}(\overline{a}^{(1)})$. Then

$$\overline{h} \in \mathrm{Stab}_{\Gamma_s^{(1)}}(\overline{a}^{(1)}) = \begin{cases} \langle \overline{s_{e_1}^{(1)}} \rangle & \text{if } a \in \{\pm e_1\}, \\ \langle \overline{s_4} \rangle & \text{if } a \in \{\pm e_2, \pm e_3\}, \end{cases}$$

so for a suitable $l \in \mathbb{Z}$

$$\overline{h}(\overline{s_{e_1}^{(1)}})^l = \mathrm{id} \quad \text{and} \quad h(s_{e_1}^{(1)})^l \in \Gamma_u^{(1)} \quad \text{for } a \in \{\pm e_1\},$$
$$\overline{h}(\overline{s_4})^l = \mathrm{id} \quad \text{and} \quad h(s_4)^l \in \Gamma_u^{(1)} \quad \text{for } a \in \{\pm e_2, \pm e_3\}.$$

As $s_{e_1}^{(1)} \in \mathrm{Stab}_{\Gamma^{(1)}}(\varepsilon e_1)$ and $s_4 \in \mathrm{Stab}_{\Gamma^{(1)}}(\varepsilon e_2) = \mathrm{Stab}_{\Gamma^{(1)}}(\varepsilon e_3)$

$$h \in \Gamma_u^{(1)} \mathrm{Stab}_{\Gamma^{(1)}}(a).$$

($\Box$)

By Lemma 6.7 (b)

$$\Gamma^{(1)}\{\varepsilon e_i\} \cap (\varepsilon e_i + \mathbb{Z} f_3) = \Gamma_u^{(1)}\{\varepsilon e_i\}. \tag{6.15}$$

Claim 2: $\big(s_{e_1}^{(1)} s_4 s_{e_1}^{(1)}\big)^2 = t_{\lambda_3}^-$ with $\lambda_3 \in \mathrm{Hom}_2(H_{\mathbb{Z}}, \mathbb{Z})$, $\lambda_3(\underline{e}) = (0, 2y_2x_2, -2y_1x_1)$.

Proof of Claim 2: This is a straightforward calculation with $s_{e_1}^{(1),mat}$ and s_4^{mat}. ($\Box$)

Therefore

$$\begin{aligned} \Gamma^{(1)}\{e_1\} &= \Gamma^{(1)}\{-e_1\}, \\ \Delta^{(1)} \cap (e_1 + \mathbb{Z} f_3) &= \Gamma^{(1)}\{e_1\} \cap (e_1 + \mathbb{Z} f_3) = \Gamma_u^{(1)}\{e_1\} \\ &= e_1 + \mathbb{Z} x_1 x_2 f_3. \end{aligned}$$

Also

$$\Gamma^{(1)}\{e_2\} \ni -e_2 + 2y_2x_2f_3, \quad \Gamma^{(1)}\{e_3\} \ni -e_3 - 2y_1x_1f_3.$$

This together with (6.15) and the shape of $\Gamma_u^{(1)}$ shows the statements on $\Delta^{(1)} \cap (\varepsilon e_2 + \mathbb{Z} f_3)$ and $\Delta^{(1)} \cap (\varepsilon e_3 + \mathbb{Z} f_3)$.

(ii) All statements on $\overline{\Delta^{(1)}}$ follow from $\Gamma_s^{(1)} \cong SL_2(\mathbb{Z})$ and $\overline{e_2}^{(1)} = x_1 g_2$ and $\overline{e_3}^{(1)} = g_2$.

For $\Delta^{(1)}$ we have to use Lemma 6.7. Claim 1 in the proof of part (i), its proof and the implication (6.15) still hold. Now we can choose $(y_1, y_2) = (0, 1)$, so $s_4 = s^{(1)}_{e_3}$. One calculates

$$s^{(1)}_{e_1} s^{(1)}_{e_3} s^{(1)}_{e_1} \underline{e} = \underline{e} \begin{pmatrix} 0 & -x_1 & -1 \\ 0 & 1 & 0 \\ 1 & -x_1 & 0 \end{pmatrix}.$$

Therefore $\Gamma^{(1)}\{e_1\} = \Gamma^{(1)}\{\pm e_1, \pm e_3\}$.

Claim 2 in the proof of part (i) still holds. It gives $(s^{(1)}_{e_1} s^{(1)}_{e_3} s^{(1)}_{e_1})^2 = t^-_{\lambda_3}$ with $\lambda_3 \in \mathrm{Hom}_2(H_\mathbb{Z}, \mathbb{Z})$, $\lambda_3(\underline{e}) = (0, 2, 0)$. Especially $t^-_{\lambda_3}(-e_2) = e_2 - 2f_3$.

This fact, (6.15) and the shape of $\Gamma^{(1)}_u$ imply the statements on $\Delta^{(1)} \cap (e_1 + \mathbb{Z} f_3)$ and $\Delta^{(1)} \cap (\varepsilon e_2 + \mathbb{Z} f_3)$.

(iii) and (iv) By Theorem 6.18 (d) $\Gamma^{(1)}_s \cong \Gamma^{(1)}(S(-x_{12}))$. By Theorem 6.10 (c) and (d) $\Gamma^{(1)}_s\{\pm \overline{e_1}^{(1)}, \pm g_2\}$ consists of the four orbits $\Gamma^{(1)}_s\{\varepsilon \overline{e_1}^{(1)}\}$ and $\Gamma^{(1)}_s\{\varepsilon g_2\}$ with $\varepsilon \in \{\pm 1\}$. Therefore if $x_1 > x_2$ then $\overline{\Delta^{(1)}}$ consists of the six orbits in (iii), and if $x_1 = x_2$ then $\overline{\Delta^{(1)}}$ consists of the four orbits $\Gamma^{(1)}_s\{\varepsilon \overline{e_1}^{(1)}\}$ and $\Gamma^{(1)}_s\{\varepsilon \overline{e_2}^{(1)}\} = \Gamma^{(1)}_s\{\varepsilon \overline{e_3}^{(1)}\} = \Gamma^{(1)}_s\{\varepsilon g_2\}$.

For $\Delta^{(1)}$ we have to use Lemma 6.7. Claim 1 in the proof of part (i), its proof and the implication (6.15) still hold.

In the case $x_1 > x_2$, $\overline{\Delta^{(1)}}$ consists of six orbits. Part (a) of Lemma 6.7, (6.15) and the shape of $\Gamma^{(1)}_u$ imply the statements on $\Delta^{(1)} \cap (\varepsilon e_i + \mathbb{Z} f_3)$ in part (iii).

In the case $x_1 = x_2$, $\overline{\Delta^{(1)}}$ consists of four orbits. Then $e_3 = e_2 - f_3$, (6.15) and the shape of $\Gamma^{(1)}_u$ imply the statements on $\Delta^{(1)} \cap (\varepsilon e_i + \mathbb{Z} f_3)$ in part (iv).

(e) Theorem 6.10 (c) and

$$s^{(1)}_{e_1}(e_1, e_3) = (e_1, e_3) \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \quad s^{(1)}_{e_3}(e_1, e_3) = (e_1, e_3) \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$

imply

$$\begin{aligned} \langle s^{(1)}_{e_1}, s^{(1)}_{e_3} \rangle \{\varepsilon e_1\} &= \{y_1 e_1 + y_2 e_3 \in H^{prim}_\mathbb{Z} \,|\, y_1 \equiv \varepsilon(4), y_2 \equiv 0(2)\}, \\ \langle s^{(1)}_{e_1}, s^{(1)}_{e_3} \rangle \{\varepsilon e_3\} &= \{y_1 e_1 + y_2 e_3 \in H^{prim}_\mathbb{Z} \,|\, y_1 \equiv 0(2)), y_2 \equiv \varepsilon(4)\}. \end{aligned}$$

In the case (i), the semidirect product $\Gamma^{(1)} = \Gamma^{(1)}_u \rtimes \langle s^{(1)}_{e_1}, s^{(1)}_{e_3} \rangle$ and the shape of $\Gamma^{(1)}_u$ show that $\Gamma^{(1)}\{\varepsilon e_1\}$ and $\Gamma^{(1)}\{\varepsilon e_3\}$ are as claimed.

In the case (ii), the semidirect product $\Gamma^{(1)} = \Gamma^{(1)} \cap O^{(1),Rad}_\pm \rtimes \langle s^{(1)}_{e_1}, s^{(1)}_{e_3} \rangle$ and the shape of $\Gamma^{(1)} \cap O^{(1),Rad}_\pm$ show that $\Gamma^{(1)}\{\varepsilon e_1\}$ and $\Gamma^{(1)}\{\varepsilon e_3\}$ are as claimed.

The following fact was not mentioned in Theorem 6.10 (c):

$$\{y_1e_1+y_2e_3\in H_{\mathbb{Z}}^{prim}\,|\,y_1\equiv y_2\equiv 1(2)\}=\langle s_{e_1}^{(1)},s_{e_3}^{(1)}\rangle\{e_1+e_3\},$$

so this set is a single $\langle s_{e_1}^{(1)},s_{e_3}^{(1)}\rangle$ orbit. We skip its proof (it contains the observation $s_{e_3}^{(1)}s_{e_1}^{(1)}(e_1+e_3)=-e_1-e_3$).

This fact, the semidirect products above of $\Gamma^{(1)}$, the shape of $\Gamma_u^{(1)}$ in case (i) and of $\Gamma^{(1)}\cap O_{\pm}^{(1),Rad}$ in case (ii), and $e_2=-\frac{l}{2}(e_1+e_3)+f_3$ show in case (i) and case (ii) that $\Gamma^{(1)}\{\varepsilon e_2\}$ is as claimed.

(f) Theorem 6.10 (b), $\underline{\widetilde{e}}=(e_1,g_2,f_3)$,

$$\begin{aligned} s_{e_1}^{(1)}(\underline{\widetilde{e}}) &= \underline{\widetilde{e}}\begin{pmatrix}1&-1&0\\0&1&0\\0&0&1\end{pmatrix},\quad s_4(\underline{\widetilde{e}})=\underline{\widetilde{e}}\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix},\\ e_3 &= -e_1+2g_2+lf_3\quad\text{and}\quad e_2=-lg_2+\frac{1-l^2}{2}f_3\end{aligned}$$

imply

$$\begin{aligned}\langle s_{e_1}^{(1)},s_4\rangle(e_1) &= (\mathbb{Z}e_1\oplus\mathbb{Z}g_2)^{prim},\\ \langle s_{e_1}^{(1)},s_4\rangle(e_3) &= (\mathbb{Z}e_1\oplus\mathbb{Z}g_2)^{prim}+lf_3,\\ \langle s_{e_1}^{(1)},s_4\rangle(e_2) &= l(\mathbb{Z}e_1\oplus\mathbb{Z}g_2)^{prim}+\frac{1-l^2}{2}f_3.\end{aligned}$$

The semidirect product $\Gamma^{(1)}=\Gamma_u^{(1)}\rtimes\langle s_{e_1}^{(1)},s_4\rangle$ and the shape of $\Gamma_u^{(1)}$ show (with $\varepsilon\in\{\pm1\}$)

$$\begin{aligned}\Gamma^{(1)}\{e_1\} &= \Gamma^{(1)}\{\pm e_1,\pm e_3\}=(\mathbb{Z}e_1\oplus\mathbb{Z}g_2)^{prim}+\mathbb{Z}lf_3,\\ \Gamma^{(1)}\{\varepsilon e_2\} &= l(\mathbb{Z}e_1\oplus\mathbb{Z}g_2)^{prim}+\varepsilon\frac{1-l^2}{2}f_3+\mathbb{Z}l^2f_3.\end{aligned}$$

(g) In the proof of part (g) of Theorem 6.18 the hyperbolic polygon P with the six arcs in Remark 6.19 was used. It is a fundamental polygon of the action of the group $\langle\mu_1,\mu_2,\mu_3\rangle$ on $\mathbb{H}$. Here μ_1,μ_2,μ_3 are parabolic Möbius transformations with fixed points $\infty,0$ and $-\frac{x_3}{x_2}$. These fixed points are cusps of P. This geometry implies

$$\begin{aligned}\mathrm{Stab}_{\langle\mu_1,\mu_2,\mu_3\rangle}(\infty) &= \langle\mu_1\rangle,\\ \mathrm{Stab}_{\langle\mu_1,\mu_2,\mu_3\rangle}(0) &= \langle\mu_2\rangle,\\ \mathrm{Stab}_{\langle\mu_1,\mu_2,\mu_3\rangle}(-\frac{x_3}{x_2}) &= \langle\mu_3\rangle.\end{aligned}$$

As $\langle\mu_1,\mu_2,\mu_3\rangle\cong\Gamma_s^{(1)}\big(\cong G^{free,3}\big)$ with $\mu_i\sim\overline{s_{e_i}^{(1)}}$, this implies

$$\mathrm{Stab}_{\Gamma_s^{(1)}}(\{\pm\overline{e_i}^{(1)}\})=\langle\overline{s_{e_i}^{(1)}}\rangle=\mathrm{Stab}_{\Gamma_s^{(1)}}(\overline{e_i}^{(1)})\quad\text{for}\quad i\in\{1,2,3\}.$$

Especially $-\overline{e_i}^{(1)} \notin \Gamma_s^{(1)}(\overline{e_i}^{(1)})$, so the orbits $\Gamma_s^{(1)}\{\overline{e_i}^{(1)}\}$ and $\Gamma_s^{(1)}\{-\overline{e_i}^{(1)}\}$ are disjoint.

The cusps ∞, 0 and $-\frac{x_3}{x_2}$ of the fundamental domain P of the group $\langle\mu_1,\mu_2,\mu_3\rangle$ are in disjoint orbits of $\langle\mu_1,\mu_2,\mu_3\rangle$. Therefore the sets $\Gamma_s^{(1)}\{\pm\overline{e_1}^{(1)}\}$, $\Gamma_s^{(1)}\{\pm\overline{e_2}^{(1)}\}$ and $\Gamma_s^{(1)}\{\pm\overline{e_3}^{(1)}\}$ are disjoint. Therefore $\overline{\Delta^{(1)}}$ consists of the six disjoint orbits $\Gamma_s^{(1)}\{\varepsilon\overline{e_i}^{(1)}\}$ with $(\varepsilon,i)\in\{\pm 1\}\times\{1,2,3\}$. Therefore also $\Delta^{(1)}$ consists of the six orbits $\Gamma^{(1)}\{\varepsilon e_i\}$ with $(\varepsilon,i)\in\{\pm 1\}\times\{1,2,3\}$.

Claim: For $a\in\{\pm e_1,\pm e_2,\pm e_3\}$ equality in $\overset{(1)}{\supset}$ and $\overset{(2)}{\supset}$ before Lemma 6.7 holds.

Proof of the Claim: Equality in $\overset{(1)}{\supset}$ holds because of Lemma 6.7 (a) and because $\Delta^{(1)}$ and $\overline{\Delta^{(1)}}$ each consist of six orbits.

Equality in $\overset{(2)}{\supset}$ is by Lemma 6.7 (b) equivalent to equality in $\overset{(3)}{\supset}$.

Here $\Gamma^{(1)}\cong\Gamma_s^{(1)}(\cong G^{free,3})$, $\Gamma_u^{(1)}=\{\mathrm{id}\}$, the lift $s_{e_i}^{(1)}\in\Gamma^{(1)}$ of $\overline{s_{e_i}^{(1)}}\in\Gamma_s^{(1)}$ is in $\mathrm{Stab}_{\Gamma^{(1)}}(\varepsilon e_i)$, and therefore

$$\mathrm{Stab}_{\Gamma^{(1)}}(\varepsilon\overline{e_i}^{(1)})=\langle s_{e_i}^{(1)}\rangle=\mathrm{Stab}_{\Gamma^{(1)}}(\varepsilon e_i)=\Gamma_u^{(1)}\cdot\mathrm{Stab}_{\Gamma^{(1)}}(\varepsilon e_i),$$

so equality in $\overset{(3)}{\supset}$ holds. The Claim is proved. ($\square$)

The Claim and $\Gamma_u^{(1)}=\{\mathrm{id}\}$ show

$$(\varepsilon e_i+\mathbb{Z}f_3)\cap\Delta^{(1)}=\varepsilon e_i.$$

Therefore the projection $\Delta^{(1)}\to\overline{\Delta^{(1)}}$ is a bijection.

(h) In the case $\underline{x}=(3,3,3)$

$$\widetilde{\underline{x}}=(1,1,1),\ f_3=-e_1+e_2-e_3,\ \overline{e_3}^{(1)}=-\overline{e_1}^{(1)}+\overline{e_2}^{(1)},$$
$$\overline{H_{\mathbb{Z}}}^{(1)}=\mathbb{Z}\overline{e_1}^{(1)}\oplus\mathbb{Z}\overline{e_2}^{(1)}.$$

Recall that the matrices B_1,B_2,B_3 in the proof of Theorem 6.18 (g) with $\overline{s_{e_i}^{(1)}}(\overline{e_1}^{(1)},\overline{e_2}^{(1)})=(\overline{e_1}^{(1)},\overline{e_2}^{(1)})B_i$ are here

$$B_1=\begin{pmatrix}1&-3\\0&1\end{pmatrix},\ B_1=\begin{pmatrix}1&0\\3&1\end{pmatrix},\ B_3=\begin{pmatrix}-2&-3\\3&4\end{pmatrix},$$

and generate $\Gamma(3)$. Because $s_{e_i}^{(1),mat}\equiv E_3\bmod 3$ and $B_i\equiv E_2\bmod 3$,

$$\Gamma^{(1)}\{\varepsilon e_i\}\subset\varepsilon e_i+3H_{\mathbb{Z}}\quad\text{and}\quad\Gamma_s^{(1)}\{\varepsilon\overline{e_i}^{(1)}\}\subset\varepsilon\overline{e_i}^{(1)}+3\overline{H_{\mathbb{Z}}}^{(1)}.$$

It remains to show $\Gamma_s^{(1)}\{\overline{e_i}^{(1)}\}=(\overline{e_i}^{(1)}+3\overline{H_{\mathbb{Z}}}^{(1)})^{prim}$. This is equivalent to the following three statements:

(i) For $(a_1,a_3)\in\mathbb{Z}^2$ with $(a_1,a_3)\equiv(1,0)\bmod 3$ and $\gcd(a_1,a_3)=1$ a pair $(a_2,a_4)\in\mathbb{Z}^2$ with $\begin{pmatrix}a_1 & a_2\\ a_3 & a_4\end{pmatrix}\in\Gamma(3)$ exists.

(ii) For $(a_2,a_4)\in\mathbb{Z}^2$ with $(a_2,a_4)\equiv(0,1)\bmod 3$ and $\gcd(a_2,a_4)=1$ a pair $(a_1,a_3)\in\mathbb{Z}^2$ with $\begin{pmatrix}a_1 & a_2\\ a_3 & a_4\end{pmatrix}\in\Gamma(3)$ exists.

(iii) For $(b_1,b_2)\in\mathbb{Z}^2$ with $(b_1,b_2)\equiv(-1,1)\bmod 3$ and $\gcd(b_1,b_2)=1$ a matrix $\begin{pmatrix}a_1 & a_2\\ a_3 & a_4\end{pmatrix}\in\Gamma(3)$ with $\begin{pmatrix}-a_1+a_2\\ -a_3+a_4\end{pmatrix}=\begin{pmatrix}b_1\\ b_2\end{pmatrix}$ exists.

(i) is proved as follows. There exist $(\widetilde{a_2},\widetilde{a_4})\in\mathbb{Z}^2$ with $1=a_1\widetilde{a_4}-a_3\widetilde{a_2}$. Thus $\widetilde{a_4}\equiv 1(3)$. Let $\widetilde{a_2}\equiv r(3)$ with $r\in\{0,1,2\}$. Choose $(a_2,a_4):=(\widetilde{a_2}-ra_1,\widetilde{a_4}-ra_3)$. The proofs of (ii) and (iii) are similar.

The proof of Theorem 6.21 (h) is finished. □

In quite some cases $\Delta^{(0)}\subset\Delta^{(1)}$, but nevertheless in general $\Delta^{(0)}\not\subset\Delta^{(1)}$. Corollary 6.22 gives some details.

COROLLARY 6.22. *(a) $\Delta^{(0)}=\Delta^{(1)}$ holds only in the cases A_1^n, so the cases with $S=E_n$ for some $n\in\mathbb{N}$. In all other cases $\Delta^{(1)}\not\subset R^{(0)}$.*

(b) $\Delta^{(0)}\subsetneqq\Delta^{(1)}$ in the cases $n=2$ except A_1^2, in the reducible cases with $n=3$ except A_1^3 and in the case A_3.

(c) $\Delta^{(0)}\not\subset\Delta^{(1)}$ holds in the following cases with $n=3$: $\widehat{A}_2$, $\mathcal{H}_{1,2}$, $S(-l,2,-l)$ with $l\geq 3$ and $\mathbb{P}^2$.

Proof: (a) In the cases A_1^n $\Delta^{(0)}=\Delta^{(1)}=\{\pm e_1,...,\pm e_n\}$ by Lemma 2.12. In a case with $S\in T_n^{uni}(\mathbb{Z})-\{E_n\}$ there is an entry $S_{ij}\neq 0$ for some $i<j$, so $L(e_j,e_i)\neq 0$. We can restrict to the rank 2 unimodular bilinear lattice $(\mathbb{Z}e_i+\mathbb{Z}e_j, L|_{\mathbb{Z}e_i+\mathbb{Z}e_j},(e_i,e_j))$ with triangular basis (e_i,e_j). Part (e) of Theorem 6.10 shows that it has odd vanishing cycles which are not roots. They are also odd vanishing cycles of $(H_\mathbb{Z},L,\underline{e})$, so then $\Delta^{(1)}\not\subset R^{(0)}\supset\Delta^{(0)}$.

(b) For the cases $n=2$ see Theorem 6.10 (e). The reducible cases with $n=3$ follow from the case A_1 and the cases with $n=2$. In the case A_3 the twelve elements of $\Delta^{(0)}$ are given in Theorem 6.14 (c). The set $\Delta^{(1)}$ is by Theorem 6.21 (c)

$$(\mathrm{pr}^{H,(1)})^{-1}(\overline{H_\mathbb{Z}}^{(1),prim})+\mathbb{Z}f_3.$$

which contains $\Delta^{(0)}$ as a strict subset.

(c) The case $\widehat{A}_2$: With $\underline{x}=(-1,-1,-1)$ we have $f_1=e_1+e_2+e_3$ and $f_3=e_1-e_2+e_3$. By Theorem 6.14 (d)

$$\Delta^{(0)}=(\pm e_1+\mathbb{Z}f_1)\,\dot\cup\,(\pm e_2+\mathbb{Z}f_1)\,\dot\cup\,(\pm e_3+\mathbb{Z}f_1).$$

By Theorem 6.21 (c)

$$\Delta^{(1)} = (\mathrm{pr}^{H,(1)})^{-1}(\overline{H_{\mathbb{Z}}}^{(1),prim}) + \mathbb{Z}f_3.$$

Here for example for $m \in \mathbb{Z} - \{0, -1\}$

$$\Delta^{(0)} \ni e_2 + mf_1 = (2m+1)e_2 + mf_3 \notin \Delta^{(1)}.$$

The case $\mathcal{H}_{1,2}$: With $\underline{x} = (-2, 2, -2)$ we have $\mathrm{Rad}\, I^{(0)} = \mathbb{Z}(e_1 + e_2) \oplus \mathbb{Z}(e_2 + e_3)$ and $f_3 = e_1 + e_2 + e_3$. By Theorem 6.14 (e)

$$\Delta^{(0)} = (\pm e_1 + 2\,\mathrm{Rad}\, I^{(0)}) \dot\cup (\pm e_2 + 2\,\mathrm{Rad}\, I^{(0)}) \dot\cup (\pm e_3 + 2\,\mathrm{Rad}\, I^{(0)}).$$

By Theorem 6.21 (e) (ii)

$$\Delta^{(1)} \subset (\mathbb{Z}e_1 + \mathbb{Z}e_3)^{prim} + \mathbb{Z}f_3.$$

Here for example

$$\begin{array}{rcl} \Delta^{(0)} & \ni & e_1 + 6(e_1 + e_2) + 4(e_2 + e_3) \\ & = & (-3)e_1 + (-6)e_3 + 10f_3 \notin (\mathbb{Z}e_1 + \mathbb{Z}e_3)^{prim} + \mathbb{Z}f_3. \end{array}$$

This element is not contained in $\Delta^{(1)}$ because by Theorem 6.21 (e)

$$\Delta^{(1)} \subset \Big((\mathbb{Z}e_1 + \mathbb{Z}e_3)^{prim} + \mathbb{Z}f_3\Big) \dot\cup \Big(\frac{l}{2}(\mathbb{Z}e_1 + \mathbb{Z}e_3)^{prim} + \mathbb{Z}f_3\Big).$$

The cases $S(-l, 2, -l)$: Recall $\mathrm{Rad}\, I^{(0)} = \mathbb{Z}f_1$, $f_1 = e_1 - e_3$,

$$f_3 = \frac{1}{2}(le_1 + 2e_2 + le_3) \quad \text{if } l \geq 3 \text{ is even},$$

$$\left.\begin{array}{l} f_3 = le_1 + 2e_2 + le_3 \\ g_2 = \frac{1}{2}(e_1 + e_3) - \frac{l}{2}e_2 \end{array}\right\} \text{ if } l \geq 3 \text{ is odd}.$$

Consider the element

$$\begin{array}{rcl} h_1(e_1) - la_1f_1 & = & h_1(e_1 - la_1f_1) \in H_{\mathbb{Z}} \quad \text{with} \\ h_1 & := & (s^{(0)}_{e_1} s^{(0)}_{e_2})^3 \in \Gamma^{(0)}, \\ a_1 & := & \frac{1}{2}(l^5 - 4l^3 + 3l) \in \mathbb{Z} \end{array}$$

By Theorem 6.11 (f) $T(\overline{j}^{(0)}(\overline{e_1}^{(0)}) \otimes f_1)$ and $T(\overline{j}^{(0)}(l\overline{e_2}^{(0)}) \otimes f_1) \in \Gamma^{(0)}_u$ with

$$\begin{array}{rcl} T(\overline{j}^{(0)}(\overline{e_1}^{(0)}) \otimes f_1)(e_1) & = & e_1 + 2f_1, \\ T(\overline{j}^{(0)}(l\overline{e_2}^{(0)}) \otimes f_1)(e_1) & = & e_1 - l^2 f_1, \end{array}$$

so

$$\Delta^{(0)} \supset \left\{ \begin{array}{ll} e_1 + 2\mathbb{Z}f_1 & \text{if } l \text{ is even}, \\ e_1 + \mathbb{Z}f_1 & \text{if is odd}. \end{array} \right.$$

Therefore $h_1(e_1)-la_1f_1 = h_1(e_1-la_1f_1)\in\Delta^{(0)}$. One calculates

$$\begin{aligned}
h_1(e_1)-la_1f_1 &= (s^{(0)}_{e_1}s^{(0)}_{e_2})^3(e_1)-la_1f_1 \\
&= \underline{e}(\begin{pmatrix}-1 & l & 2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ l & -1 & l\\ 0 & 0 & 1\end{pmatrix})^2\begin{pmatrix}1\\0\\0\end{pmatrix}-la_1f_1 \\
&= \underline{e}\begin{pmatrix}l^2-1 & -l & l^2-2\\ l & -1 & l\\ 0 & 0 & 1\end{pmatrix}^3\begin{pmatrix}1\\0\\0\end{pmatrix}-la_1f_1 \\
&= \underline{e}\begin{pmatrix}l^6-5l^4+6l^2-1\\ l^5-4l^3+3l\\ 0\end{pmatrix}-la_1f_1 \\
&= (-l^4+3l^2-1)e_1+\begin{cases}2a_1f_3 & \text{if } l \text{ is even,}\\ a_1f_3 & \text{if } l \text{ is odd.}\end{cases}
\end{aligned}$$

This element is not contained in $\Delta^{(1)}$ because $(-l^4+3l^2-1)\notin\{\pm 1,\pm\frac{l}{2},\pm l\}$ and because by Theorem 6.21 (e) and (f)

$$\begin{aligned}
\Delta^{(1)} &\subset \Big((\mathbb{Z}e_1+\mathbb{Z}e_3)^{prim}+\mathbb{Z}f_3\Big)\dot\cup\Big(\frac{l}{2}(\mathbb{Z}e_1+\mathbb{Z}e_3)^{prim}+\mathbb{Z}f_3\Big)\\
&\quad \text{if } l \text{ is even,}\\
\Delta^{(1)} &\subset \Big((\mathbb{Z}e_1+\mathbb{Z}g_2)^{prim}+\mathbb{Z}f_3\Big)\dot\cup\Big(l(\mathbb{Z}e_1+\mathbb{Z}g_2)^{prim}+\mathbb{Z}f_3\Big)\\
&\quad \text{if } l \text{ is odd.}
\end{aligned}$$

The case $\mathbb{P}^2$: With $\underline{x}=(-3,3,-3)$ we have $f_3=e_1+e_2+e_3$.

$$\begin{aligned}
\Delta^{(1)} &\ni s^{(1)}_{e_3}(s^{(1)}_{e_2})^{-2}(e_1)\\
&= \underline{e}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 3 & -3 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 3 & 1 & -3\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 3 & 1 & -3\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=\underline{e}\begin{pmatrix}1\\6\\-15\end{pmatrix}.
\end{aligned}$$

By Theorem 6.21 (g) the projection $\Delta^{(1)}\to\overline{\Delta^{(1)}}$ is a bijection. Therefore

$$\Delta^{(1)}\not\ni (e_1+6e_2-15e_3)+9f_3 = 10e_1+15e_2-6e_3.$$

On the other hand

$$\begin{aligned}
&L(10e_1+15e_2-6e_3, 10e_1+15e_2-6e_3)\\
&= \begin{pmatrix}10 & 15 & -6\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ -3 & 1 & 0\\ 3 & -3 & 1\end{pmatrix}\begin{pmatrix}10\\15\\-6\end{pmatrix}=1,
\end{aligned}$$

so $10e_1+15e_2-6e_3\in R^{(0)}=\Delta^{(0)}$. □

Consider a triple $\underline{x} \in \mathbb{Z}^3$ and a corresponding unimodular bilinear lattice $(H_\mathbb{Z}, L, \underline{e})$ with a triangular basis $\underline{e}$ with $L(\underline{e}^t, \underline{e})^t = S(\underline{x})$. The Remarks 4.17 explained that the tuple $(H_\mathbb{Z}, \pm I^{(1)}, \Gamma^{(1)}, \Delta^{(1)})$ depends only on the $(G^{phi} \ltimes \widetilde{G}^{sign}) \rtimes \langle\gamma\rangle$ orbit of $\underline{x} \in \mathbb{Z}^3$. Lemma 4.18 gave at least one element of each orbit of this group in $\mathbb{Z}^3$. Theorem 6.18 and Theorem 6.21 gave detailed information on the tuple $(H_\mathbb{Z}, \pm I^{(1)}, \Gamma^{(1)}, \Delta^{(1)})$ for the elements in Lemma 4.18 (b)+(c) and rather coarse information for the elements in Lemma 4.18 (a).

The next corollary uses this information to conclude that the $(G^{phi} \ltimes \widetilde{G}^{sign}) \rtimes \langle\gamma\rangle$ orbits of the elements in Lemma 4.18 (b)+(c) are pairwise different and also different from the orbits of the elements in Lemma 4.18 (a), because the corresponding tuples $(H_\mathbb{Z}, \pm I^{(1)}, \Gamma^{(1)}, \Delta^{(1)})$ are not isomorphic. As Theorem 6.18 and Theorem 6.21 give only coarse information on the cases in Lemma 4.18 (a), also Corollary 6.23 is vague about them.

The set of local minima in Lemma 4.18 (b)+(c) and $(3,3,3)$ is called Λ_1, the set of local minima in Lemma 4.18 (a) without $(3,3,3)$ is called Λ_2,

$$\begin{aligned}
\Lambda_1 &:= \{(3,3,3)\} \cup \{(-l,2,-l) \,|\, l \geq 2\} \\
&\quad \cup \{(x_1,x_2,0) \,|\, x_1,x_2 \in \mathbb{Z}_{\geq 0}, x_1 \geq x_2\}, \\
\Lambda_2 &:= \{\underline{x} \in \mathbb{Z}^3_{\geq 3} \,|\, 2x_i \leq x_j x_k \text{ for } \{i,j,k\} = \{1,2,3\}\} - \{(3,3,3)\}.
\end{aligned}$$

Corollary 6.23. *Consider $\underline{x}$ and $\underline{\widetilde{x}} \in \Lambda_1$ or $\underline{x} \in \Lambda_1$ and $\underline{\widetilde{x}} \in \Lambda_2$. Suppose $\underline{x} \neq \underline{\widetilde{x}}$. Then the tuples $(H_\mathbb{Z}, \pm I^{(1)}, \Gamma^{(1)}, \Delta^{(1)})$ of $\underline{x}$ and $\underline{\widetilde{x}}$ are not isomorphic. Consequently, the $(G^{phi} \ltimes \widetilde{G}^{sign}) \rtimes \langle\gamma\rangle$ orbits of $\underline{x}$ and $\underline{\widetilde{x}}$ are disjoint.*

Proof: In the following, (b), (c), (d), (d)(i), (d)(ii), (d)(iii), (d)(iv), (e), (e)(i), (e)(ii), (f), (g), (h)($\subset$ (g)) mean the corresponding families of cases in Theorem 6.21. Of course, (c) and (h) are single cases. We will first discuss how to separate the families by properties of the isomorphism classes of the tuples $(H_\mathbb{Z}, \pm I^{(1)}, \Gamma^{(1)}, \Delta^{(1)})$, and then how to separate the cases within one family.

The pair $(\Gamma^{(1)}_u, \Gamma^{(1)}_s)$ gives the following incomplete separation of families,

$\Gamma^{(1)}_u \cong ?$	$\Gamma^{(1)}_s \cong ?$	families
$\{\text{id}\}$	$G^{free,3}$	(g)
$\{\text{id}\}$	$\not\cong G^{free,3}$	(b)
$\mathbb{Z}^2$	$SL_2(\mathbb{Z})$	$(c), (d)(i)+(ii), (f)$
$\mathbb{Z}^2$	$SL_2(\mathbb{Z}) \times \{\pm 1\}$	$(e)(ii)$
$\mathbb{Z}^2$	$G^{free,2}$	$(d)(iii)+(iv), (e)(i)$

The fundamental polygon P in Remark 6.19 (ii) has finite area in the case (h) (i.e. $\underline{x}=(3,3,3)$) and infinite area in the other cases in (g). So it separates the case (h) from the other cases in (g).

The number of $\mathrm{Br}_3\ltimes\{\pm 1\}^3$ orbits in $\overline{\Delta^{(1)}}$ separates the families (c),(d),(e)(i),(f) almost completely:

$\lvert\{\text{orbits in }\overline{\Delta^{(1)}}\}\rvert$	1	2	3	4	5	6
families	(c)	$(d)(ii),(f)$	$(d)(i)$	$(d)(iv)$	$(e)(i)$	$(d)(iii)$

The separation of the family (d)(ii) from the family (f) is more difficult and can be done as follows. In both families of cases $\Delta^{(1)}$ consists of three orbits, and $\overline{\Delta^{(1)}}$ consists of two orbits. The two orbits $\Gamma^{(1)}\{e_2\}$ and $\Gamma^{(1)}\{-e_2\}$ unite to a single orbit $\Gamma_s^{(1)}\{\overline{e_2}^{(1)}\}=\Gamma_s^{(1)}\{-\overline{e_2}^{(1)}\}$. The set

$$\{n\in\mathbb{N}\quad|\quad\text{there exists }a_1\in\Gamma^{(1)}\{e_2\}\text{ and }\varepsilon\in\{\pm 1\}\\ \text{with }a_1+\varepsilon n f_3\in\Gamma^{(1)}\{-e_2\}\}$$

is well defined. Its minimum is 2 in each case in (d)(ii) because there $x_1>x_2=1$ and

$$\Gamma^{(1)}\{e_2\}\cap(e_2+\mathbb{Z}f_3)=e_2+\mathbb{Z}x_1^2f_3,$$
$$\Gamma^{(1)}\{-e_2\}\cap(e_2+\mathbb{Z}f_3)=e_2-2f_3+\mathbb{Z}x_1^2f_3.$$

Its minimum is 1 in each case in (f) because there

$$\begin{aligned}\Gamma^{(1)}\{e_2\}\cap(le_1+\mathbb{Z}f_3) &= le_1+\frac{1+l^2}{2}f_3+\mathbb{Z}l^2f_3,\\ \Gamma^{(1)}\{-e_2\}\cap(le_1+\mathbb{Z}f_3) &= le_1+\frac{-1+l^2}{2}+\mathbb{Z}l^2f_3.\end{aligned}$$

It remains to separate within each family (b), (d), (e) and (f) the cases ((c) and (h) are single cases). The pair $(\overline{H_{\mathbb{Z}}}^{(1)},\pm\overline{I}^{(1)})$ and Lemma 6.16 (b) allow to recover $\gcd(x_1,x_2,x_3)$ which is as follows in these families,

family of cases	(b)	(d)	(e)	(f)
$\gcd(x_1,x_2,x_3)$	x_1	x_{12}	2	1

Within the family (b) this separates the cases. For the family (d) we need additionally the pair $(\widetilde{x}_1,\widetilde{x}_2)$ because $(x_1,x_2)=(x_{12}\widetilde{x}_1,x_{12}\widetilde{x}_2)$. The pair $(\widetilde{x}_1,\widetilde{x}_2)$ can be read off from $\overline{\Delta}^{(1)}\subset\overline{H_{\mathbb{Z}}}^{(1)}$, more precisely, from the relation of the $\Gamma_s^{(1)}$ orbits in $\overline{\Delta^{(1)}}$ to the set $\overline{H_{\mathbb{Z}}}^{(1),prim}\subset\overline{H_{\mathbb{Z}}}^{(1)}$. In the family (e) one can read off $\frac{l}{2}$, and in the family (f) one can read off l from the relation of the $\Gamma_s^{(1)}$ orbits in $\overline{\Delta}^{(1)}$ to the set $\overline{H_{\mathbb{Z}}}^{(1),prim}\subset\overline{H_{\mathbb{Z}}}^{(1)}$.
□

REMARKS 6.24. Let $(H_{\mathbb{Z}}, L)$ be a unimodular bilinear lattice of rank $n \geq 2$, and let $\underline{e}$ be a triangular basis with matrix $S = L(\underline{e}^t, \underline{e})^t \in T_n^{uni}(\mathbb{Z})$. Recall Theorem 3.7 (a). If $S_{ij} \leq 0$ for $i < j$ then $(\Gamma^{(0)}, \{s_{e_1}^{(0)}, ..., s_{e_n}^{(0)}\})$ is a Coxeter system, and the presentation in Definition 3.15 of the Coxeter group $\Gamma^{(0)}$ is determined by S. Especially $\Gamma^{(0)} \cong G^{fCox,n}$ if $S_{ij} \leq -2$ for $i < j$.

One might hope for a similar easy control of $\Gamma^{(1)}$ if $S_{ij} \leq 0$ for $i < j$. In the cases with $n = 2$ this works by Lemma 2.12 and Theorem 6.10:

$$\Gamma^{(1)} \cong \begin{cases} \{\mathrm{id}\} & \text{if } x = 0, \\ SL_2(\mathbb{Z}) & \text{if } x = -1, \\ G^{free,2} & \text{if } x \leq -2. \end{cases}$$

But in the cases with $n = 3$ this fails. The Remarks 4.17 show $\Gamma^{(1)}(S(\underline{x})) \cong \Gamma^{(1)}(S(-\underline{x}))$ for any $\underline{x} \in \mathbb{Z}^3$. The cases $S(\widetilde{\underline{x}})$ with $\widetilde{\underline{x}} \in \mathbb{Z}^3_{\geq 0}$ lead by the action of $(G^{phi} \ltimes \widetilde{G}^{sign}) \rtimes \langle \gamma \rangle$ to all cases in Theorem 6.18.

Especially, the cases $S(\widetilde{\underline{x}})$ with $\widetilde{\underline{x}} \in \mathbb{Z}^3_{\geq 2}$ contain the nice cases in Theorem 6.18 (g) with $\Gamma^{(1)} \cong G^{free,3}$, but also many other cases. Compare the family $\{(3,3,l) \,|\, l \geq 2\}$ in the Examples 4.20 (iv) or the case $S = S(2,2,2) \sim S(-2,-2,-2) \sim S(\mathcal{H}_{1,2})$ with $\Gamma^{(1)}$ far from $G^{free,3}$.

REMARKS 6.25. In the cases $\underline{x} \in \mathbb{Z}^3$ in Lemma 4.18 (a), so $\underline{x} \in \mathbb{Z}^3_{\geq 2}$ with $2x_i \leq x_j x_k$ for $\{i,j,k\} = \{1,2,3\}$, Theorem 6.18 and Theorem 6.21 give rather coarse information,

$$\begin{aligned} \Gamma_u^{(1)} = \{\mathrm{id}\} \quad \text{and} \quad \Gamma^{(1)} \cong \Gamma_s^{(1)} \cong G^{free,3} \quad & \text{by Theorem 6.18,} \\ \Delta^{(1)} \to \overline{\Delta^{(1)}} \quad \text{is a bijection} \quad & \text{by Theorem 6.21.} \end{aligned}$$

But it is nontrivial to determine the unique preimage in $\Gamma^{(1),mat}$ of an element of $\Gamma_s^{(1)}$ and the unique preimage in $\Delta^{(1)}$ of an element of $\overline{\Delta^{(1)}}$. This holds especially for the case $\underline{x} = (3,3,3)$ where $\Gamma_s^{(1)} \cong \Gamma(3)$ and $\overline{\Delta^{(1)}}$ are known. Part (c) of the following lemma gives for $\underline{x} = (3,3,3)$ at least an inductive way to determine the preimage in $\Gamma^{(1),mat}$ of a matrix in $\Gamma(3) \cong \Gamma_s^{(1)}$.

LEMMA 6.26. *Consider the case* $\underline{x} = (3,3,3)$. *Denote by* $L_{\mathbb{P}^2} : \Gamma(3) \to \Gamma^{(1),mat}$ *the inverse of the natural group isomorphism*

$$\begin{array}{ccccccc} \Gamma^{(1),mat} & \longrightarrow & \Gamma^{(1)} & \longrightarrow & \Gamma_s^{(1)} & \longrightarrow & \Gamma(3), \\ s_{e_i}^{(1),mat} & \longrightarrow & s_{e_i}^{(1)} & \longrightarrow & \overline{s_{e_i}^{(1)}} & \longrightarrow & B_i \\ g^{mat} & \longleftarrow & g & \longrightarrow & \overline{g} & \longrightarrow & \overline{g}^{mat} \end{array}$$

with

$$\begin{aligned} g(\underline{e}) &= \underline{e} \cdot g^{mat} \qquad \text{and} \\ \overline{g}(\overline{e_1}^{(1)}, \overline{e_2}^{(1)}) &= (\overline{e_1}^{(1)}, \overline{e_2}^{(1)}) \cdot \overline{g}^{mat} \end{aligned}$$

for $g \in \Gamma^{(1)}$. *Define the subgroup of* $SL_3(\mathbb{Z})$

$$G^{(3,3,3)} := \{F \in SL_3(\mathbb{Z}) \,|\, F \equiv E_3 \bmod 3, F \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}\}.$$

Define the map (st for standard)

$$L_{st} : \Gamma(3) \to M_{3\times 3}(\mathbb{Z}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b & 1-a+b \\ c & d & -1-c+d \\ 0 & 0 & 1 \end{pmatrix},$$

and the three matrices

$$\begin{aligned} K_1 &:= \begin{pmatrix} 3 & 0 & -3 \\ -3 & 0 & 3 \\ 3 & 0 & -3 \end{pmatrix}, \quad K_2 := \begin{pmatrix} 0 & 3 & 3 \\ 0 & -3 & -3 \\ 0 & 3 & 3 \end{pmatrix}, \\ K_3 &:= K_1 + K_2 = \begin{pmatrix} 3 & 3 & 0 \\ -3 & -3 & 0 \\ 3 & 3 & 0 \end{pmatrix}. \end{aligned}$$

(a) L_{st} *is an injective group homomorphism* $L_{st} : \Gamma(3) \to G^{(3,3,3)}$,

$$\begin{aligned} K_i K_j = 0 \quad &\text{for } i, j \in \{1, 2, 3\}, \\ L_{st}(C) K_i = K_i \quad &\text{for } C \in \Gamma(3),\ i \in \{1, 2, 3\}, \\ G^{(3,3,3)} = \{L_{st}(C) + \alpha K_1 + \beta K_2 \,&|\, C \in \Gamma(3), \alpha, \beta \in \mathbb{Z}\}. \end{aligned}$$

The following sequence is an exact sequence of group homomorphisms,

$$\begin{array}{ccccccccc} \{1\} & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & G^{(3,3,3)} & \longrightarrow & \Gamma(3) & \longrightarrow & \{1\} \\ & & (\alpha, \beta) & \longrightarrow & E_3 + \alpha K_1 + \beta K_2 & & & & \\ & & & & L_{st}(C) + \alpha K_1 + \beta K_2 & \longrightarrow & C & & \end{array}$$

L_{st} *is a splitting of this exact sequence.*

(b) $\Gamma^{(1),mat} \subset G^{(3,3,3)}$*, and* $L_{\mathbb{P}^2} : \Gamma(3) \to \Gamma^{(1),mat}$ *is another splitting of the exact sequence in part (b). It satisfies*

$$\begin{aligned}
L_{\mathbb{P}^2}(B_1) &= L_{st}(B_1) = s^{(1),mat}_{e_1} = \begin{pmatrix} 1 & -3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for } B_1 = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}, \\
L_{\mathbb{P}^2}(B_2) &= L_{st}(B_2) = s^{(1),mat}_{e_2} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for } B_2 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \\
L_{\mathbb{P}^2}(B_3) &= L_{st}(B_3) + K_3 = s^{(1),mat}_{e_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 3 & 1 \end{pmatrix} \quad \text{for } B_3 = \begin{pmatrix} -2 & -3 \\ 3 & 4 \end{pmatrix}, \\
L_{\mathbb{P}^2}(B_3^{-1}) &= L_{st}(B_3^{-1}) - K_3.
\end{aligned}$$

(c) An arbitrary element $C \in \Gamma(3)$ *can be written in a unique way as a product*

$$C = C_1 B_3^{\varepsilon_1} C_2 B_3^{\varepsilon_2} C_3 ... C_m B_3^{\varepsilon_m} C_{m+1}$$
$$\text{with} \quad m \in \mathbb{Z}_{\geq 0},\ C_1, ..., C_{m+1} \in \langle B_1^{\pm 1}, B_2^{\pm 1} \rangle,\ \varepsilon_1, ..., \varepsilon_m \in \{\pm 1\}.$$

Then

$$\begin{aligned}
L_{\mathbb{P}^2}(C) &= L_{st}(C) + K_3 \Big(\varepsilon_1 L_{st}(C_2 B_3^{\varepsilon_2} C_3 ... C_m B_3^{\varepsilon_m} C_{m+1}) \\
&\qquad + \varepsilon_2 L_{st}(C_3 B_3^{\varepsilon_3} C_4 ... C_m B_3^{\varepsilon_m} C_{m+1}) \\
&\qquad + ... + \varepsilon_m L_{st}(C_{m+1}) \Big).
\end{aligned}$$

Proof: The parts (a) and (b) are easy.

(c) By part (b) $L_{\mathbb{P}^2}(C_j) = L_{st}(C_j)$ and $L_{\mathbb{P}^2}(B_3^{\varepsilon_j}) = L_{st}(B_3^{\varepsilon_j}) + \varepsilon_j K_3$, so

$$\begin{aligned}
L_{\mathbb{P}^2}(C) &= L_{\mathbb{P}^2}(C_1) L_{\mathbb{P}^2}(B_3^{\varepsilon_1}) L_{\mathbb{P}^2}(C_2) ... L_{\mathbb{P}^2}(C_m) L_{\mathbb{P}^2}(B_3^{\varepsilon_m}) L_{\mathbb{P}^2}(C_{m+1}) \\
&= L_{st}(C_1)(L_{st}(B_3^{\varepsilon_1}) + \varepsilon_1 K_3) L_{st}(C_2) ... \\
&\qquad L_{st}(C_m)(L_{st}(B_3^{\varepsilon_m}) + \varepsilon_m K_3) L_{st}(C_{m+1}).
\end{aligned}$$

Observe $K_3 L_{st}(\widetilde{C}) K_3 = K_3 K_3 = 0$ for $\widetilde{C} \in \Gamma(3)$. Therefore if one writes the product above as a sum of 2^m terms, only the $1+m$ terms do not vanish in which K_3 turns up at most once. This leads to the claimed formula for $L_{\mathbb{P}^2}(C)$. □

CHAPTER 7

Distinguished bases in the rank 2 and rank 3 cases

In section 3.3 we introduced the set of distinguished bases of a unimodular bilinear lattice $(H_{\mathbb{Z}}, L, \underline{e})$ with a triangular basis. It is the orbit $\mathcal{B}^{dist} = \text{Br}_n \ltimes \{\pm 1\}^n(\underline{e})$ of $\underline{e}$ under the group $\text{Br}_n \ltimes \{\pm 1\}^n$. We also posed the question when this set can be characterized in an easy way, more precisely, when the inclusions in (3.3) or (3.4) are equalities,

$$\mathcal{B}^{dist} \subset \{\underline{v} \in (\Delta^{(0)})^n \mid s^{(0)}_{v_1}...s^{(0)}_{v_n} = -M\}, \qquad (3.3)$$

$$\mathcal{B}^{dist} \subset \{\underline{v} \in (\Delta^{(1)})^n \mid s^{(1)}_{v_1}...s^{(1)}_{v_n} = M\}. \qquad (3.4)$$

Theorem 3.2 (a) and (b) imply that (3.4) is an equality if $\Gamma^{(1)}$ is a free group with generators $s^{(1)}_{e_1}, ..., s^{(1)}_{e_n}$ and that (3.3) is an equality if $\Gamma^{(0)}$ is a free Coxeter group with generators $s^{(0)}_{e_1}, ..., s^{(0)}_{e_n}$, see the Examples (3.23) (iv). More generally, if $(\Gamma^{(0)}, s^{(0)}_{e_1}, ..., s^{(0)}_{e_n})$ is a Coxeter system (Definition 3.5) then by Theorem 3.6 (3.3) is an equality, see the Examples 3.23 (v). It is remarkable that the property $\sum_{i=1}^n \mathbb{Z}v_i = H_{\mathbb{Z}}$, which each distinguished basis $\underline{v} \in \mathcal{B}^{dist}$ satisfies is not needed in the characterization in these cases.

These are positive results. In the sections 3.1–3.3 we study systematically all cases of rank 2 and 3 and find also negative results.

In rank 2 in section 6.2 (3.3) is always an equality, and (3.4) is an equality in all cases except the case A_1^2.

In the even rank 3 cases in section 7.2 (3.3) is in all cases except the case $\mathcal{H}_{1,2}$ an equality. In the case $\mathcal{H}_{1,2}$ the set on the right hand side contains $\text{Br}_3 \ltimes \{\pm 1\}^3$ orbits of tuples $\underline{v}$ with arbitrary large finite index $[H_{\mathbb{Z}} : \sum_{i=1}^3 \mathbb{Z}v_i]$ and two orbits with index 1, $\mathcal{B}^{dist}$ and one other orbit.

In the odd rank 3 cases in section 7.3 we understand the set $B_1 \cup B_2$ of triples $\underline{x} \in \mathbb{Z}^3$ such that (3.4) is an equality, and we also know a set $B_3 \cup B_4$ of triples $\underline{x} \in \mathbb{Z}^3$ such that (3.4) becomes an equality if one adds the condition $H_{\mathbb{Z}} = \sum_{i=1}^3 \mathbb{Z}v_i$. But for $\underline{x} \in \mathbb{Z}^3 - \cup_{j=1}^4 B_j$, we know little.

Section 7.4 builds on section 4.4 where for a unimodular bilinear lattice $(H_{\mathbb{Z}}, L, \underline{e})$ the stabilizer $(\text{Br}_3)_{\underline{x}/\{\pm 1\}^3}$ had been determined. It determines the stabilizer $(\text{Br}_3)_{\underline{e}/\{\pm 1\}^3}$. It uses the systematic results in chapter 5 on the group $G_{\mathbb{Z}}$ and on the map $Z : (\mathrm{Br}_n \ltimes \{\pm 1\}^n)_S \to G_{\mathbb{Z}}$ in the rank 3 cases.

In the sections 4.3 and 4.4 the pseudo-graph $\mathcal{G}(\underline{x})$ with vertex set an orbit $\mathrm{Br}_3(\underline{x}/\{\pm 1\})$ and edge set from generators of the group $G^{phi} \rtimes \langle\gamma\rangle$ had been crucial. In section 7.4 we introduce a variant with the same vertex set, but different edge set, namely (now) oriented edges coming from the elementary braids $\sigma_i^{\pm 1}$. We also define the much larger σ-pseudo-graph with vertex set a set $\mathcal{B}^{dist}/\{\pm 1\}^3$ of distinguished bases up to signs and oriented edges coming from the elementary braids $\sigma_i^{\pm 1}$. We consider especially the examples where the set $\mathrm{Br}_3(\underline{x}/\{\pm 1\}^3)$ is finite.

7.1. Distinguished bases in the rank 2 cases

In the rank 2 cases the inclusion (3.3) is always an equality, and the inclusion (3.4) is almost always an equality, namely in all cases except the case A_1^2.

Theorem 7.1. *Let $(H_{\mathbb{Z}}, L, \underline{e})$ be a unimodular bilinear lattice of rank 2 with a triangular basis $\underline{e} = (e_1, e_2)$ with matrix $S = S(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = L(\underline{e}^t, \underline{e})^t$ with $x \in \mathbb{Z}$. Fix $k \in \{0, 1\}$.*

(a) The inclusion (3.3) respectively (3.4) in Remark 3.19 is an equality in all cases except the odd case A_1^2, so the case $(k, x) = (1, 0)$. In that case the right hand side in (3.4) splits into the orbits of the three pairs (e_1, e_1), (e_1, e_2), (e_2, e_2).

(b) The stabilizers in Br_2 of $S/\{\pm 1\}^2$ and of $\underline{e}/\{\pm 1\}^2$ are

$$\begin{aligned} (\mathrm{Br}_2)_{S/\{\pm 1\}^2} &= \mathrm{Br}_2 \quad \text{and} \\ (\mathrm{Br}_2)_{\underline{e}/\{\pm 1\}^2} &= \begin{cases} \langle \sigma_1^2 \rangle & \text{if } x = 0, \\ \langle \sigma_1^3 \rangle & \text{if } x \in \{\pm 1\}, \\ \{\mathrm{id}\} & \text{if } |x| \geq 2. \end{cases} \end{aligned}$$

Proof: (a) The even and odd cases A_1^2: See the Examples 3.23 (iii).

The cases with $|x| \geq 2$: Theorem 6.8 (c)+(d) and Theorem 6.10 (c)+(d) show

$$\Gamma^{(k)} \cong \begin{cases} G^{fCox,2} & \text{with generators } s_{e_1}^{(0)}, s_{e_2}^{(0)} \text{ if } k = 0, \\ G^{free,2} & \text{with generators } s_{e_1}^{(1)}, s_{e_2}^{(1)} \text{ if } k = 1. \end{cases}$$

The Examples 3.23 (iv) apply and give equality in (3.3) and (3.4).

The cases with $x = \pm 1$: We can restrict to the case $x = -1$. The even case is a simple case of Example 3.23 (v) (in the Remarks 7.2 we will offer an elementary proof for the even case).

It remains to show equality in (3.4) in the odd case $(k,x)=(1,-1)$. Consider $\underline{v}\in(\Delta^{(1)})^2$ with $s^{(1)}_{v_1}s^{(1)}_{v_2}=M$. Let $b:=I^{(1)}(v_1,v_2)\in\mathbb{Z}$. If $b=0$ then $v_2=\pm v_1$ and $M=(s^{(1)}_{v_1})^2$ would have an eigenvalue 1, a contradiction. Therefore $b\neq 0$ and $\mathbb{Z}v_1+\mathbb{Z}v_2$ has rank 2. Then

$$\begin{aligned} M(\underline{v}) &= s^{(1)}_{v_1}s^{(1)}_{v_2}(\underline{v})=s^{(1)}_{v_1}(v_1+bv_2,v_2)\\ &= (v_1+bv_2-b^2v_1,v_2-bv_1)=\underline{v}\begin{pmatrix}1-b^2 & -b\\ b & 1\end{pmatrix}, \end{aligned}$$

$1=\operatorname{tr}M=(1-b^2)+1$, so $b=\pm 1$. By possibly changing the sign of v_2, we can suppose $b=-1=x$. Then

$$I^{(1)}(\underline{v}^t,\underline{v})=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

Therefore $\underline{v}$ is a $\mathbb{Z}$-basis of $H_\mathbb{Z}$, and the automorphism $g\in\operatorname{Aut}(H_\mathbb{Z})$ with $(g(e_1),g(e_2))=(v_1,v_2)$ is in $O^{(1)}$. By Lemma 3.22 (a)

$$gMg^{-1}=g\circ((\pi_2\circ\pi_2^{(1)})(\underline{e}))\circ g^{-1}=(\pi_2\circ\pi_2^{(1)})(\underline{v})=s^{(1)}_{v_1}s^{(1)}_{v_2}=M,$$

so $gMg^{-1}=M$, so $g\in G_\mathbb{Z}^{(1)}=G_\mathbb{Z}^M\cap O^{(1)}$. Theorem 5.5 can be applied and gives $\stackrel{(*)}{=}$,

$$G_\mathbb{Z}^{(1)}\stackrel{(*)}{=}G_\mathbb{Z}\stackrel{(*)}{=}\{\pm(M^{root})^l\,|\,l\in\mathbb{Z}\}\stackrel{(*)}{=}Z(\mathrm{Br}_2\ltimes\{\pm 1\}^2).$$

Therefore $\underline{v}\in\mathcal{B}^{dist}$. This shows equality in (3.4).

(b) Because of $\sigma_1\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}$, the stabilizer $(\mathrm{Br}_2)_{S/\{\pm 1\}^2}$ is the whole group $\mathrm{Br}_2=\langle\sigma_1\rangle$. If $x=0$,

$$(e_1,e_2)\stackrel{\sigma_1}{\mapsto}(e_2,e_1)\stackrel{\sigma_1}{\mapsto}(e_1,e_2),\quad\text{so }(\mathrm{Br}_2)_{\underline{e}/\{\pm 1\}^2}=\langle\sigma_1^2\rangle.$$

If $x=-1$,

$$\begin{aligned}(e_1,e_2)\stackrel{\sigma_1}{\mapsto}(e_1+e_2,e_1)\stackrel{\sigma_1}{\mapsto}(-e_2,e_1+e_2)\stackrel{\sigma_1}{\mapsto}(e_1,-e_2),\\ \text{so }(\mathrm{Br}_2)_{\underline{e}/\{\pm 1\}^2}=\langle\sigma_1^3\rangle.\end{aligned}$$

If $|x|\geq 2$ Theorem 3.2 (a) or (b) and $\Gamma^{(1)}\cong G^{free,2}$ or $\Gamma^{(0)}\cong G^{fCox,2}$ show $(\mathrm{Br}_2)_{\underline{e}/\{\pm 1\}^2}=\{\mathrm{id}\}$. □

Remarks 7.2. (i) A direct elementary proof of equality in (3.3) for the even case A_2, so $(k,x)=(0,-1)$, is instructive. Recall from Theorem 6.8 (b) that $\Delta^{(0)}=\{\pm e_1,\pm e_2,\pm(e_1+e_2)\}$. The map $\pi_2\circ\pi_2^{(0)}:$

$(\Delta^{(0)})^2 \to \Gamma^{(0)}$ has the three values $-M$, M^2 and id and the three fibers

$$\begin{aligned}
(\pi_2\circ\pi_2^{(0)})^{-1}(-M) &= \{(\pm e_1,\pm e_2),(\pm(e_1+e_2),\pm e_1),(\pm e_2,\pm(e_1+e_2)\}\\
&= \mathcal{B}^{dist},\\
(\pi_2\circ\pi_2^{(0)})^{-1}(M^2) &= \{(\pm e_2,\pm e_1),(\pm(e_1+e_2),\pm e_2),(\pm e_1,\pm(e_1+e_2)\}\\
&= \mathrm{Br}_2\ltimes\{\pm 1\}^2(e_2,e_1),\\
(\pi_2\circ\pi_2^{(0)})^{-1}(\mathrm{id}) &= \{(\pm e_1,\pm e_1),(\pm e_2,\pm e_2),(\pm(e_1+e_2),\pm(e_1+e_2)\}.
\end{aligned}$$

This gives equality in (3.3) in the case $(k,x)=(0,-1)$.

(ii) Also in the cases $(k=0,x\leq -2)$ a direct elementary proof of equality in (3.3) is instructive. Equality in (3.3) for $(k,x)=(0,-2)$ and Theorem 6.8 (d) (iv) imply equality in (3.3) for $(k=0,x\leq -3)$. Therefore we restrict to the case $(k,x)=(0,-2)$. Recall

$$\mathrm{Rad}\, I^{(0)}=\mathbb{Z}f_1 \quad\text{with}\quad f_1=e_1+e_2.$$

By Theorem 6.8 (c)

$$\Delta^{(0)}=(e_1+\mathbb{Z}f_1)\,\dot\cup\,(-e_1+\mathbb{Z}f_1)=(e_1+\mathbb{Z}f_1)\,\dot\cup\,(e_2+\mathbb{Z}f_1).$$

One easily sees for $b_1,b_2\in\mathbb{Z}$

$$s^{(0)}_{e_1+b_1f_1}s^{(0)}_{e_2+b_2f_1}=-M \iff b_1+b_2=0,$$

thus

$$\{\underline{v}\in(\Delta^{(0)})^2\,|\,s^{(0)}_{v_1}s^{(0)}_{v_2}=-M\}=\{\pm(e_1+bf_1),\pm(e_2-bf_1)\,|\,b\in\mathbb{Z}\}.$$

This set is a single $\mathrm{Br}_2\ltimes\{\pm1\}^2$ orbit because of

$$\delta_2\sigma_1(e_1+bf_1,e_2-bf_1)=(e_1+(b+1)f_1,e_2-(b+1)f_1). \qquad\Box$$

7.2. Distinguished bases in the even rank 3 cases

In the even cases with $n=3$ we have complete results on the question when the inclusion in (3.3) is an equality. It is one in all cases except the case $\mathcal{H}_{1,2}$.

THEOREM 7.3. *Let $(H_\mathbb{Z},L,\underline{e})$ be a unimodular bilinear lattice of rank 3 with a triangular basis $\underline{e}$ with matrix $S=S(\underline{x})=L(\underline{e}^t,\underline{e})^t\in T_3^{uni}(\mathbb{Z})$.*

(a) Suppose $S\notin(\mathrm{Br}_3\ltimes\{\pm1\}^3)(S(\mathcal{H}_{1,2}))$. Then the inclusion in (3.3) is an equality.

(b) Suppose $S=S(\mathcal{H}_{1,2})=S(-2,2,-2)$. Recall the basis $(f_1,f_2,f_3)=\underline{e}\begin{pmatrix}1&0&1\\1&1&1\\0&1&1\end{pmatrix}$ of $H_\mathbb{Z}$ with $\mathrm{Rad}\,I^{(1)}=\mathbb{Z}f_1\oplus\mathbb{Z}f_2$ and $\mathrm{Rad}\,I^{(0)}=\mathbb{Z}f_3$. The set $\{\underline{v}\in(\Delta^{(0)})^3\,|\,(\pi_3\circ\pi_3^{(0)})(\underline{v})=-M\}$ splits

into countably many orbits. The following list gives one representative for each orbit,

$$(f_3-g_1,-f_3+g_1+g_2,f_3-g_2) \quad \text{with} \quad \begin{pmatrix} g_1\\ g_2\end{pmatrix} = \begin{pmatrix} 0 & c_2\\ c_1 & c_3\end{pmatrix}\begin{pmatrix} f_1\\ f_2\end{pmatrix},$$
$$c_1\in\mathbb{N} \text{ odd},\ c_2\in\mathbb{Z}\text{ odd},\ c_3\in\{0,1,...,|c_2|-1\}.$$

The sublattice $\langle f_3-g_1,-f_3+g_1+g_2,f_3-g_2\rangle = \langle f_3,g_1,g_2\rangle\subset H_{\mathbb{Z}}$ *has finite index* $c_1\cdot|c_2|$ *in* $H_{\mathbb{Z}}$*. It is* $H_{\mathbb{Z}}$ *in the following two cases:*

$$\begin{aligned} &\underline{e}=(f_3-f_2,-f_3+f_1+f_2,f_3-f_1),\\ \text{so}\quad &(g_1,g_2,c_1,c_2,c_3)=(f_2,f_1,1,1,0),\\ &(f_3+f_2,-f_3+f_1-f_2,f_3-f_1),\\ \text{so}\quad &(g_1,g_2,c_1,c_2,c_3)=(-f_2,f_1,1,-1,0)\end{aligned}$$

(see also Example 3.23 (ii)) for the second case).

Proof: (a) We can replace $\underline{e}$ by an arbitrary element $\widetilde{\underline{e}}\in\mathcal{B}^{dist}$. By Theorem 4.6 the following cases exhaust all $\text{Br}_3\ltimes\{\pm1\}^3$ orbits except that of $\mathcal{H}_{1,2}$:

(A) $(H_{\mathbb{Z}},L,\underline{e})$ is irreducible with $\underline{x}\in\mathbb{Z}^3_{\leq 0}$.
(B) $r(\underline{x})\leq 0$ and $\underline{x}\neq(0,0,0)$.
(C) $\underline{x}=(x_1,0,0)$ with $x_1\in\mathbb{Z}_{\leq 0}$, so $(H_{\mathbb{Z}},L,\underline{e})$ is reducible (this includes the case A_1^3).
(D) $\underline{x}=(-l,2,-l)$ with $l\geq 3$.

The cases (A): $\Gamma^{(0)}$ is a Coxeter group by Theorem 3.7 (a). Theorem 3.7 (b) applies.

The cases (B): By Theorem 6.11 (g) $\Gamma^{(0)}$ is a free Coxeter group with generators $s^{(0)}_{e_1},s^{(0)}_{e_2},s^{(0)}_{e_3}$. Theorem 3.7 (b) or Example 3.23 (iv) can be used.

The cases (C): Consider a triple $\underline{v}\in(\Delta^{(0)})^3$ with $s^{(0)}_{v_1}s^{(0)}_{v_2}s^{(0)}_{v_3}=-M$. The set $\Delta^{(0)}$ splits into the subsets $\Delta^{(0)}\cap(\mathbb{Z}e_1+\mathbb{Z}e_2)$ and $\{\pm e_3\}$. Compare $-M|_{\mathbb{Z}e_3}=-\text{id}|_{\mathbb{Z}e_3}$ with $s^{(0)}_{e_3}|_{\mathbb{Z}e_3}=-\text{id}|_{\mathbb{Z}e_3}$ and $s^{(0)}_a|_{\mathbb{Z}e_3}=\text{id}|_{\mathbb{Z}e_3}$ for $a\in\Delta^{(0)}\cap(\mathbb{Z}e_1+\mathbb{Z}e_2)$. All three $v_i\in\{\pm e_3\}$ is impossible because $(s^{(0)}_{e_3})^3\neq -M$. Therefore there are i,j,k with $\{i,j,k\}=\{1,2,3\}$, $i<j$, $v_i,v_j\in\Delta^{(0)}\cap(\mathbb{Z}e_1+\mathbb{Z}e_2)$ and $v_k\in\{\pm e_3\}$. The reflection $s^{(0)}_{v_k}$ acts trivially on $\mathbb{Z}e_1+\mathbb{Z}e_2$ and commutes with $s^{(0)}_{v_i}$ and $s^{(0)}_{v_j}$. Therefore

$$s^{(0)}_{v_i}s^{(0)}_{v_j}s^{(0)}_{v_k}=s^{(0)}_{v_1}s^{(0)}_{v_2}s^{(0)}_{v_3}=(\pi_3\circ\pi_3^{(0)})(\underline{v})=-M=s^{(0)}_{e_1}s^{(0)}_{e_2}s^{(0)}_{e_3},$$
$$\text{so}\quad s^{(0)}_{v_i}s^{(0)}_{v_j}=s^{(0)}_{e_1}s^{(0)}_{e_2}.$$

The reflections $s^{(0)}_{v_i}$, $s^{(0)}_{v_j}$, $s^{(0)}_{e_1}$ and $s^{(0)}_{e_2}$ act trivially on $\mathbb{Z}e_3$. The inclusion in (3.3) is an equality because of Theorem 7.1 (a) for the rank 2 cases.

The cases (D): The proof of these cases is prepared by Lemma 7.4 and Lemma 7.5. The proof comes after the proof of Lemma 7.5.

(b) The proof of part (b) is prepared by Lemma 7.6 and comes after the proof of Lemma 7.6. ($\Box$)

The following lemma is related to t_λ^+ in Lemma 6.17. Recall also $j^{(k)}:H_{\mathbb Z}\to H_{\mathbb Z}^\sharp$, $a\mapsto I^{(k)}(a,.)$, in Definition 6.1.

LEMMA 7.4. *Let $(H_{\mathbb Z},L)$ be a unimodular bilinear lattice of rank $n\in\mathbb N$. Fix $k\in\{0,1\}$. Suppose* $\operatorname{Rad} I^{(k)}\neq\{0\}$ *and choose an element* $f\in\operatorname{Rad} I^{(k)}-\{0\}$. *Denote*

$$\begin{aligned}\operatorname{Hom}_{0,f}(H_{\mathbb Z},\mathbb Z)&:=\{\lambda:H_{\mathbb Z}\to\mathbb Z\,|\,\lambda\text{ is }\mathbb Z\text{-linear},\lambda(f)=0\},\\ t_\lambda:H_{\mathbb Z}\to H_{\mathbb Z}&\text{ with }t_\lambda(a)=a+\lambda(a)f\text{ for }\lambda\in\operatorname{Hom}_{0,f}(H_{\mathbb Z},\mathbb Z).\end{aligned}$$

Then $t_\lambda\in O_u^{(k),Rad}$. The map

$$\operatorname{Hom}_{0,f}(H_{\mathbb Z},\mathbb Z)\to O_u^{(k),Rad},\quad \lambda\mapsto t_\lambda,$$

is an injective group homomorphism. For $b\in R^{(k)}$ (with $R^{(1)}=H_{\mathbb Z}$, see 3.9 (i)) and $a\in\mathbb Z$

$$\begin{aligned}s_{b+af}^{(k)}&=s_b^{(k)}\circ t_{-aj^{(k)}(b)}=t_{(-1)^kaj^{(k)}(b)}\circ s_b^{(k)},\\ s_b^{(k)}\circ t_\lambda&=t_{\lambda-(-1)^k\lambda(b)j^{(k)}(b)}\circ s_b^{(k)}.\end{aligned}$$

Proof: The proof is straightforward. We skip the details. $\Box$

The following lemma studies the Hurwitz action of Br_3 on triples of reflections in $G^{fCox,2}$. It is related to Theorem 3.2 (b).

LEMMA 7.5. *As in Definition 3.1, $G^{fCox,2}$ denotes the free Coxeter group with two generators z_1 and z_2, so generating relations are $z_1^2=z_2^2=1$.*

(a) Its set of reflections is

$$\Delta(G^{fCox,2})=\bigcup_{i=1}^2\{wz_iw^{-1}\,|\,w\in G^{fCox,2}\}=\{(z_1z_2)^mz_1\,|\,m\in\mathbb Z\}.$$

The complement of this set is

$$G^{fCox,2}-\Delta(G^{fCox,2})=\{(z_1z_2)^m\,|\,m\in\mathbb Z\}.$$

$\Delta(G^{fCox,2})$ respectively its complement consists of the elements which can be written as words of odd respectively even length in z_1 and z_2.

(b) The set

$$\{(w_1,w_2,w_3)\in(\Delta(G^{fCox,2})^3\,|\,w_1w_2w_3=z_1z_2z_1\}$$

is the disjoint union of the following Br_3 *orbits:*

$$\dot{\bigcup}_{m\in\mathbb{Z}_{\geq 0}} \mathrm{Br}_3\big((z_1z_2z_1, (z_1z_2)^{1-m}z_1, (z_1z_2)^{1-m}z_1)\big).$$

Proof: (a) Clear.

(b) The map

$$\begin{array}{rcl} \{(w_1,w_2,w_3)\in(\Delta(G^{fCox,2}))^3 \,|\, w_1w_2w_3=z_1z_2z_1\} & \to & M_{2\times 1}(\mathbb{Z}), \\ (w_1,w_2,w_3) & \mapsto & (m_1,m_2)^t \\ \text{with } w_1w_2=(z_1z_2)^{m_1},\ w_2w_3=(z_1z_2)^{m_2}, & & \end{array}$$

is a bijection because

$$\begin{array}{rcl} z_1z_2z_1 & = & (w_1w_2)w_2(w_2w_3), \quad \text{so} \\ w_2 & = & (w_1w_2)^{-1}z_1z_2z_1(w_2w_3)^{-1}, \\ w_1 & = & (w_1w_2)w_2, \\ w_3 & = & w_2(w_2w_3), \end{array}$$

so a given column vector $(m_1,m_2)^t$ has a unique preimage.

The Hurwitz action of Br_3 on the set on the left hand side of the bijection above translates as follows to an action on $M_{2\times 1}(\mathbb{Z})$.

$$\begin{array}{rcl} \sigma_1(w_1,w_2,w_3) & = & (w_1w_2w_1, w_1, w_3), \\ w_1w_2w_1\cdot w_1 & = & w_1w_2=(z_1z_2)^{m_1}, \\ w_1w_3 & = & (w_1w_2)(w_2w_3)=(z_1z_2)^{m_1+m_2}, \\ \sigma_1\begin{pmatrix} m_1 \\ m_2\end{pmatrix} & = & \begin{pmatrix} m_1 \\ m_1+m_2\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1\end{pmatrix}\begin{pmatrix} m_1 \\ m_2\end{pmatrix}, \\ \sigma_2(w_1,w_2,w_3) & = & (w_1, w_2w_3w_2, w_2), \\ w_1\cdot w_2w_3w_2=(w_1w_2)(w_2w_3)^{-1} & = & (z_1z_2)^{m_1-m_2}, \\ w_2w_3w_2\cdot w_2 & = & w_2w_3=(z_1z_2)^{m_2}, \\ \sigma_2\begin{pmatrix} m_1 \\ m_2\end{pmatrix} & = & \begin{pmatrix} m_1-m_2 \\ m_2\end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1\end{pmatrix}\begin{pmatrix} m_1 \\ m_2\end{pmatrix}. \end{array}$$

So Br_3 acts as multiplication with matrices in $SL_2(\mathbb{Z})$ from the left on $M_{2\times 1}(\mathbb{Z})$. Each orbit has a unique element of the shape $\begin{pmatrix} m \\ 0\end{pmatrix}$ with $m\in\mathbb{Z}_{\geq 0}$. This element corresponds to

$$(z_1z_2z_1, (z_1z_2)^{1-m}z_1, (z_1z_2)^{1-m}z_1). \qquad \square$$

Proof of Theorem 7.3 (a) in the cases (D), $\underline{x} = (-l, 2, -l)$ with $l \geq 3$: Recall from Theorem 6.11 (f)

$$\begin{aligned} \mathrm{Rad}\, I^{(0)} &= \mathbb{Z} f_1 \quad \text{with} \quad f_1 = e_1 - e_3, \\ \Gamma_s^{(0)} &\cong G^{fCox,2} \quad \text{with generators } z_1 = \overline{s_{e_1}^{(0)}} = \overline{s_{e_3}^{(0)}},\ z_2 = \overline{s_{e_2}^{(0)}}. \end{aligned}$$

Suppose $\underline{v} \in (\Delta^{(0)})^3$ with $s_{v_1}^{(0)} s_{v_2}^{(0)} s_{v_3}^{(0)} = -M$. We want to show $\underline{v} \in \mathcal{B}^{dist}$ or equivalently $(s_{v_1}^{(0)}, s_{v_2}^{(0)}, s_{v_3}^{(0)}) \in \mathcal{R}^{(0),dist}$.

First we look at the images in $\Gamma_s^{(0)}$: $\overline{s_{v_1}^{(0)} s_{v_2}^{(0)} s_{v_3}^{(0)}} = z_1 z_2 z_1$. Because of Lemma 7.5 (b), we can make a suitable braid group action and then suppose

$$(\overline{s_{v_1}^{(0)}}, \overline{s_{v_2}^{(0)}}, \overline{s_{v_3}^{(0)}}) = (z_1 z_2 z_1, r, r) \text{ with } r = (z_1 z_2)^{1-m} z_1 \text{ for some } m \in \mathbb{Z}_{\geq 0}.$$

Write $\widetilde{e_2} := s_{e_1}^{(0)}(e_2) = e_2 + l e_1$ and observe

$$z_1 z_2 z_1 = \overline{s_{e_1}^{(0)} s_{e_2}^{(0)} s_{e_3}^{(0)}} = \overline{s_{\widetilde{e_2}}^{(0)}}.$$

After possibly changing the signs of v_1 and v_3, $\overline{s_{v_1}^{(0)}} = \overline{s_{\widetilde{e_2}}^{(0)}}$ and $\overline{s_{v_2}^{(0)}} = r = \overline{s_{v_3}^{(0)}}$ imply

$$v_1 = \widetilde{e_2} + a_1 f_1 \quad \text{and} \quad v_3 = v_2 + a_2 f_1 \quad \text{for some } a_1, a_2 \in \mathbb{Z}.$$

With Lemma 7.4 and $f_1 = (f \text{ in Lemma 7.4})$ we calculate

$$\begin{aligned} -M &= s_{e_1}^{(0)} s_{e_2}^{(0)} s_{e_3}^{(0)} = s_{e_1}^{(0)} s_{e_2}^{(0)} s_{e_1 - f_1}^{(0)} \\ &= s_{e_1}^{(0)} s_{e_2}^{(0)} s_{e_1}^{(0)} t_{j^{(0)}(e_1)} = s_{\widetilde{e_2}}^{(0)} t_{j^{(0)}(e_1)}, \\ -M &= s_{v_1}^{(0)} s_{v_2}^{(0)} s_{v_3}^{(0)} = s_{\widetilde{e_2}+a_1 f_1}^{(0)} s_{v_2}^{(0)} s_{v_2 + a_2 f_1}^{(0)} \\ &= s_{\widetilde{e_2}}^{(0)} t_{-a_1 j^{(0)}(\widetilde{e_2})} s_{v_2}^{(0)} s_{v_2}^{(0)} t_{-a_2 j^{(0)}(v_2)} \\ &= s_{\widetilde{e_2}}^{(0)} t_{-a_1 j^{(0)}(\widetilde{e_2}) - a_2 j^{(0)}(v_2)}, \end{aligned}$$

so

$$j^{(0)}(e_1) = -a_1 j^{(0)}(\widetilde{e_2}) - a_2 j^{(0)}(v_2).$$

Write

$$\overline{v_2}^{(0)} = b_1 \overline{e_1}^{(0)} + b_2 \overline{e_2}^{(0)} \quad \text{with } b_1, b_2 \in \mathbb{Z}.$$

By Theorem 6.14 (f) the tuple $(\overline{H_{\mathbb{Z}}}^{(0)}, \overline{I}^{(0)}, (\overline{e_1}^{(0)}, \overline{e_2}^{(0)}))$ is isomorphic to the corresponding tuple from the 2×2 matrix $S(-l) = \begin{pmatrix} 1 & -l \\ 0 & 1 \end{pmatrix}$.

The set of roots of this tuple is called $R^{(0)}(S(-l))$. It contains $\overline{v_2}^{(0)}$, $\overline{e_1}^{(0)}$, $\overline{e_1}^{(0)}$. By Theorem 6.8 (d)(i) the map

$$\begin{aligned} R^{(0)}(S(-l)) &\to \{\text{units in } \mathbb{Z}[\kappa_1] \text{ with norm } 1\} = \{\pm\kappa_1^m \,|\, m \in \mathbb{Z}\} \\ y_1\overline{e_1}^{(0)} + y_2\overline{e_2}^{(0)} &\mapsto y_1 - \kappa_1 y_2, \end{aligned}$$

is a bijection, where $\kappa_1 = \frac{l}{2} + \frac{1}{2}\sqrt{l^2-4}$. The norm of $b_1 - b_2\kappa_1$ is

$$1 = b_1^2 - lb_1b_2 + b_2^2.$$

Now

$$\begin{aligned} (2,-l) &= j^{(0)}(e_1)(e_1,e_2) = (-a_1 j^{(0)}(\widetilde{e_2}) - a_2 j^{(0)}(v_2))(e_1,e_2) \\ &= -a_1((-l,2) + l(2,-l)) - a_2(b_1(2,-l) + b_2(-l,2)) \\ &= (-a_1 l - a_2 b_1)(2,-l) + (-a_1 - a_2 b_2)(-l,2). \end{aligned}$$

so

$$a_1 = -a_2b_2, \quad 1 = -a_1 l - a_2 b_1 = a_2(b_2 l - b_1),$$
$$a_2 = \pm 1 \quad \text{and} \quad b_1 = -a_2 + b_2 l.$$

Calculate

$$\begin{aligned} 0 &= -1 + b_1^2 - lb_1b_2 + b_2^2 = -1 + (-a_2 + b_2 l)(-a_2) + b_2^2 \\ &= b_2(b_2 - a_2 l). \end{aligned}$$

We obtain the four solutions

$$\begin{aligned} (a_1,a_2,b_1,b_2) \in \{&(0,1,-1,0),(0,-1,1,0), \\ &(-l,1,l^2-1,l),(-l,-1,1-l^2,-l)\}. \end{aligned}$$

In the case of the third solution

$$\begin{aligned} \overline{v_2}^{(0)} &= (l^2-1)\overline{e_1}^{(0)} + l\overline{e_2}^{(0)} = \overline{s_{e_1}^{(0)} s_{e_2}^{(0)}(e_1)}, \\ r &= \overline{s_{v_2}^{(0)}} = \overline{s^{(0)}_{s_{e_1}^{(0)} s_{e_2}^{(0)}(e_1)}} = \overline{s_{e_1}^{(0)} s_{e_2}^{(0)} s_{e_1}^{(0)} s_{e_2}^{(0)} s_{e_1}^{(0)}} \\ &= (z_1z_2)^2 z_1 = (z_1z_2)^{1-m} z_1 \quad \text{with } m = -1. \end{aligned}$$

As $m = -1$ is not in the set $\mathbb{Z}_{\geq 0}$, we can discard the third solution. In fact, $(z_1z_2z_1, (z_1z_2)^2z_1, (z_1z_2)^2z_1)$ is in the Br_3 orbit of $(z_1z_2z_1, z_1, z_1)$ because $\begin{pmatrix}-1\\0\end{pmatrix}$ is in the $SL_2(\mathbb{Z})$ orbit of $\begin{pmatrix}1\\0\end{pmatrix}$. We can discard also the fourth solution because its vector $\overline{v_2}^{(0)}$ differs from the vector $\overline{v_2}^{(0)}$ in the third solution only by the sign.

Also the vector $\overline{v_2}^{(0)}$ in the first solution differs from the vector $\overline{v_2}^{(0)}$ in the second solution only by the sign.

The second solution gives $\overline{v_2}^{(0)} = \overline{e_1}^{(0)}$ and thus for some $b_3 \in \mathbb{Z}$

$$\underline{v} = (\widetilde{e_2}, e_1 + b_3 f_1, e_1 + b_3 f_1 - f_1) = (\widetilde{e_2}, e_1 + b_3 f_1, e_3 + b_3 f_1).$$

The observation

$$\begin{aligned} \delta_2\sigma_2(\underline{v}) &= \delta_2(\widetilde{e_2}, e_3+b_3f_1-2(e_1+b_3f_1), e_1+b_3f_1) \\ &= (\widetilde{e_2}, e_1+(b_3+1)f_1, e_3+(b_3+1)f_1) \end{aligned}$$

shows $\underline{v} \in \mathrm{Br}_3 \ltimes \{\pm 1\}^3(\widetilde{e_2}, e_1, e_3)$. This orbit is $\mathcal{B}^{dist}$ because $(\widetilde{e_2}, e_1, e_3) = \sigma_1(\underline{e})$. □

Lemma 7.6 states some facts which arise in the proof of part (b) of Theorem 7.3 and which are worth to be formulated explicitly.

LEMMA 7.6. *Let $(H_{\mathbb{Z}}, L, \underline{e})$ be the unimodular bilinear lattice of rank 3 with triangular basis $\underline{e}$ with matrix $S = S(\mathcal{H}_{1,2}) = S(-2,2,-2) = L(\underline{e}^t, \underline{e})^t$. Recall $\mathrm{Rad}\, I^{(0)} = \mathbb{Z}f_1 + \mathbb{Z}f_2$ and $R^{(0)} = \pm f_3 + \mathrm{Rad}\, I^{(0)}$ (Theorem 6.14 (f)).*

(a) For $g_1, g_2, g_3 \in \mathrm{Rad}\, I^{(0)}$

$$s^{(0)}_{f_3-g_1} s^{(0)}_{f_3-g_2} s^{(0)}_{f_3-g_3} = -M \iff g_2 = g_1 + g_3.$$

(b) The map

$$\begin{aligned} \Phi : M_{2\times 2}(\mathbb{Z}) &\to \{\underline{v} \in (R^{(0)})^3 \,|\, (\pi_3 \circ \pi_3^{(0)})(\underline{v}) = -M\}/\{\pm 1\}^3, \\ A &\mapsto (f_3-g_1, f_3-g_1-g_3, f_3-g_3)/\{\pm 1\}^3 \\ &\qquad \text{with } \begin{pmatrix} g_1 \\ g_3 \end{pmatrix} = A \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \end{aligned}$$

is a bijection. The action of Br_3 on the right hand side translates to the following action on the left hand side,

$$\sigma_1(A) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} A, \quad \sigma_2(A) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} A.$$

(c) $\underline{v} \in (R^{(0)})^3$ with $(\pi_3 \circ \pi_3^{(0)})(\underline{v}) = -M$ satisfies either (i) or (ii):

(i) There exists a permutation $\sigma \in S_3$ with $v_i \in \Gamma^{(0)}\{e_{\sigma(i)}\}$ for $i \in \{1,2,3\}$.

(ii) Either $v_1, v_2, v_3 \in \Gamma^{(0)}\{f_3\}$ or there exists a permutation $\sigma \in S_3$ and an $l \in \{1,2,3\}$ with $v_{\sigma(1)} \in \Gamma^{(0)}\{f_3\}$ and $v_{\sigma(2)}, v_{\sigma(3)} \in \Gamma^{(0)}\{e_l\}$.

(i) holds if and only if $\Phi^{-1}(\underline{v}/\{\pm 1\}^3)$ has an odd determinant.

(d) Let $SL_2(\mathbb{Z})$ act by multiplication from the left on $\{A \in M_{2\times 2}(\mathbb{Z}) \,|\, \det A \text{ is odd}\}$. Each orbit has a unique representative of the shape

$$\begin{pmatrix} 0 & c_2 \\ c_1 & c_3 \end{pmatrix} \quad \text{with} \quad c_1 \in \mathbb{N} \text{ odd}, c_2 \in \mathbb{Z} \text{ odd}, c_3 \in \{0,1,...,|c_2|-1\}.$$

Proof: (a) For $g_1, g_2, g_3 \in \operatorname{Rad} I^{(0)}$

$$s^{(0)}_{f_3-g_1}|_{\operatorname{Rad} I^{(0)}} = \mathrm{id}, \quad s^{(0)}_{f_3-g_1}(f_3+g_2) = -(f_3-g_2-2g_1),$$
$$s^{(0)}_{f_3-g_1}s^{(0)}_{f_3-g_2}s^{(0)}_{f_3-g_3}(f_3) = -f_3+2(g_1-g_2+g_3).$$

Compare $-M|_{\operatorname{Rad} I^{(0)}} = \mathrm{id}$, $-M(f_3) = -f_3$.

(b) Φ is a bijection because of $R^{(0)} = \pm f_3 + \operatorname{Rad} I^{(0)}$ and part (a). The action of Br_3 on the right hand side translates to the claimed action on the left hand side because of the following,

$$\begin{aligned}
\delta_1\sigma_1(f_3-g_1, f_3-g_1-g_3, f_3-g_3) &= (f_3-g_1+g_3, f_3-g_1, f_3-g_3), \\
\begin{pmatrix} g_1-g_3 \\ g_3 \end{pmatrix} &= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} g_1 \\ g_3 \end{pmatrix}, \\
\delta_2\sigma_2(f_3-g_1, f_3-g_1-g_3, f_3-g_3) &= (f_3-g_1, f_3-2g_1-g_3, f_3-g_1-g_3), \\
\begin{pmatrix} g_1 \\ g_1+g_3 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} g_1 \\ g_3 \end{pmatrix}.
\end{aligned}$$

(c) Recall that by Theorem 6.14 (e)

$$\Gamma^{(0)}\{e_i\} = \pm e_i + 2\operatorname{Rad} I^{(0)}, \quad \Gamma^{(0)}\{f_3\} = \pm f_3 + 2\operatorname{Rad} I^{(0)},$$
$$(e_1, e_2, e_3) = (f_3-f_2, -f_3+f_1+f_2, f_3-f_1),$$
$$\Delta^{(0)} = \Gamma^{(0)}\{e_1\} \dot\cup \Gamma^{(0)}\{e_2\} \dot\cup \Gamma^{(0)}\{e_3\}, \quad R^{(0)} = \Delta^{(0)} \dot\cup \Gamma^{(0)}\{f_3\}.$$

Observe that $g_1, g_2, g_3 \in \operatorname{Rad} I^{(0)}$ with $g_2 = g_1 + g_3$ satisfy either (i)' or (ii)',

- (i)' $g_1, g_2, g_3 \notin 2\operatorname{Rad} I^{(0)}$,
- (ii)' There exists a permutation $\sigma \in S_3$ with $g_{\sigma(1)} \in 2\operatorname{Rad} I^{(0)}$ and $g_{\sigma(2)} - g_{\sigma(3)} \in 2\operatorname{Rad} I^{(0)}$.

$\underline{v} = (f_3-g_1, f_3-g_2, f_3-g_3)$ satisfies (i) if (i)' holds, and it satisfies (ii) if (ii)' holds. If $\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = A \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ then (i)' holds if and only if $\det A$ is odd.

(d) This is elementary. We skip the details. □

Proof of Theorem 7.3 (b): $\underline{v} \in (\Delta^{(0)})^3$ with $(\pi_3 \circ \pi_3^{(0)})(\underline{v}) = -M$ satisfies property (i) in Lemma 7.6 (c) because it does not satisfy property (ii) in Lemma 7.6 (c). The parts (b) and (d) of Lemma 7.6 show that the set $(\Delta^{(0)})^3 \cap (\pi_3 \circ \pi_3^{(0)})^{-1}(-M)$ consists of countably many orbits. The parts (b) and (d) of Lemma 7.6 also give the claimed representative in each orbit. The rest is obvious. □

7.3. Distinguished bases in the odd rank 3 cases

Also in the odd cases with $n=3$ we have complete results on the question when the inclusion (3.4) is an equality. It is one if and only if $\underline{x}\in B_1\cup B_2$ where

$$\begin{aligned} B_1 &= \{\underline{x}\in\mathbb{Z}^3-\{(0,0,0)\}\,|\\ &\qquad ((G^{phi}\ltimes \widetilde{G}^{sign})\rtimes\langle\gamma\rangle)(\underline{x})\cap r^{-1}(\mathbb{Z}_{\leq 0})\neq\emptyset\},\\ B_2 &= \{\underline{x}\in\mathbb{Z}^3-\{(0,0,0)\}\,|\,S(\underline{x})\text{ is reducible, i.e. there are }i,j,k\\ &\qquad \text{with }\{i,j,k\}=\{1,2,3\}\text{ and }x_i\neq 0=x_j=x_k\},\\ B_3 &:= \{(0,0,0)\},\\ B_4 &:= \{\underline{x}\in\mathbb{Z}^3\,|\,S(\underline{x})\in(\mathrm{Br}_3\ltimes\{\pm 1\}^3)\Big(\{S(A_3),S(\widehat{A}_2),S(\mathcal{H}_{1,2})\}\\ &\qquad \cup\{S(-l,2,-l)\,|\,l\geq 3\}\Big)\},\\ B_5 &:= \mathbb{Z}^3-(B_1\cup B_2\cup B_3\cup B_4). \end{aligned}$$

B_2 is the set of $\underline{x}\neq(0,0,0)$ which give reducible cases. B_1 contains $r^{-1}(\mathbb{Z}_{\leq 0})-\{(0,0,0)\}$, but is bigger. $\underline{x}\in B_1$ if and only if the $(G^{phi}\ltimes \widetilde{G}^{sign})\rtimes\langle\gamma\rangle$ orbit of $\underline{x}$ contains a triple $\underline{\widetilde{x}}\in\mathbb{Z}^3$ as in Lemma 4.18 (a), so with $\underline{\widetilde{x}}\in\mathbb{Z}^3_{\geq 3}$ and $2\widetilde{x}_i\leq\widetilde{x}_j\widetilde{x}_k$ for $\{i,j,k\}=\{1,2,3\}$. The Examples 4.20 show that it is not so easy to describe B_1 more explicitly.

In Theorem 7.7 we show $B_4\subset\mathbb{Z}^3-(B_1\cup B_2\cup B_3)$. For $\underline{x}\in B_3\cup B_4$ the inclusion in (3.4) is not an equality, but we can add the constraint $\sum_{i=1}^3\mathbb{Z}v_i=H_\mathbb{Z}$ to (3.4) and obtain an equality. For $\underline{x}\in B_5$ we do not know whether (3.4) with the additional constraint $\sum_{i=1}^3\mathbb{Z}v_i=H_\mathbb{Z}$ becomes an equality.

THEOREM 7.7. *Let $(H_\mathbb{Z},L,\underline{e})$ be a unimodular bilinear lattice of rank 3 with a triangular basis $\underline{e}$ with matrix $L(\underline{e}^t,\underline{e})^t=S(\underline{x})\in T_3^{uni}(\mathbb{Z})$ for some $\underline{x}\in\mathbb{Z}^3$.*

(a) $(H_\mathbb{Z},L,\underline{e})$ is reducible if and only if $\underline{x}\in B_2\cup B_3$. Then $\Gamma_u^{(1)}=\{\mathrm{id}\}$.

(b) The following conditions are equivalent:

(i) $\underline{x}\in B_1$.

(ii) $\Gamma^{(1)}\cong G^{free,3}$.

(iii) $(H_\mathbb{Z},L,\underline{e})$ is irreducible and $\Gamma_u^{(1)}=\{\mathrm{id}\}$.

(c) $\mathbb{Z}^3=\dot{\bigcup}_{i\in\{1,2,3,4,5\}}B_i$.

(d) The inclusion in (3.4) is an equality $\iff$ $\underline{x}\in B_1\cup B_2$.

(e) Consider $\underline{x}=(0,0,0)$. The set $\{\underline{v}\in(\Delta^{(1)})^3\,|\,(\pi_3\circ\pi_3^{(1)})(\underline{v})=M\}$ is $(\Delta^{(1)})^3$. It consists of ten $\mathrm{Br}_3\ltimes\{\pm 1\}^3$ orbits, the orbit $\mathcal{B}^{dist}$ of

$\underline{e} = (e_1, e_2, e_3)$ *and the orbits of the nine triples*

$$(e_1, e_1, e_1), (e_1, e_1, e_2), (e_1, e_2, e_2), (e_2, e_2, e_2),$$
$$(e_1, e_1, e_3), (e_1, e_3, e_3), (e_3, e_3, e_3), (e_2, e_2, e_3), (e_2, e_3, e_3).$$

(f) Consider $\underline{x} \in B_4 \cup B_5$. *Then* $\Gamma_u^{(1)} \cong \mathbb{Z}^2$. *The map*

$$\begin{aligned} \Psi : \{\underline{v} \in (\Delta^{(1)})^3 \,|\, (\pi_3 \circ \pi_3^{(1)})(\underline{v}) = M\} &\to \mathbb{N} \cup \{\infty\}, \\ \underline{v} &\mapsto \Bigl(\text{index of } \sum_{i=1}^3 \mathbb{Z} v_i \text{ in } H_{\mathbb{Z}}\Bigr), \end{aligned}$$

has infinitely many values. The set $\{\underline{v} \in (\Delta^{(1)})^3 \,|\, (\pi_3 \circ \pi_3^{(1)})(\underline{v}) = M\}$ *contains besides* $\mathcal{B}^{dist}$ *infinitely many* $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ *orbits.*

(g) For $\underline{x} \in B_3 \cup B_4$

$$\mathcal{B}^{dist} = \{\underline{v} \in (\Delta^{(1)})^3 \,|\, (\pi_3 \circ \pi_3^{(1)})(\underline{v}) = M, \sum_{i=1}^3 \mathbb{Z} v_i = H_{\mathbb{Z}}\}.$$

Proof: (a) Compare Definition 2.10 in the case $n = 3$. For $\Gamma_u^{(1)} = \{\mathrm{id}\}$ see Theorem 6.18 (b).

(b) By the Remarks 4.17 the tuple $(H_{\mathbb{Z}}, \pm I^{(1)}, \Gamma^{(1)}, \Delta^{(1)})$ depends up to isomorphism only on the $(G^{phi} \ltimes \widetilde{G}^{sign}) \rtimes \langle \gamma \rangle$ orbit of $\underline{x}$. Lemma 4.18 gives representatives of each such orbit. Theorem 6.18 studies their groups $\Gamma^{(1)}$. Theorem 6.18 (b) treats $\underline{x} \in B_2 \cup B_3$. Theorem 6.18 (g) treats $\underline{x} \in B_1$. Theorem 6.18 (c)–(f) treats $\underline{x} \in \mathbb{Z}^3 - (B_1 \cup B_2 \cup B_3)$. One sees

$$\begin{aligned} \Gamma^{(1)} \cong G^{free,3} &\iff \underline{x} \in B_1, \\ \Gamma_u^{(1)} = \{\mathrm{id}\} &\iff \underline{x} \in B_1 \cup B_2 \cup B_3, \\ \Gamma_u^{(1)} \cong \mathbb{Z}^2 &\iff \underline{x} \in \mathbb{Z}^3 - (B_1 \cup B_2 \cup B_3). \end{aligned}$$

(c) $B_1 \cap B_2 = \emptyset$, $B_2 \cap B_4 = \emptyset$ and $(0,0,0) \notin B_1 \cup B_2 \cup B_4$ are clear. $B_1 \cap B_4 = \emptyset$ follows from $\Gamma^{(1)}(\underline{x}) \not\cong G^{free,3}$ for $\underline{x} \in B_4$.

(d) The parts (e) and (f) will give $\Longrightarrow$. Here we show $\Longleftarrow$, first for $\underline{x} \in B_1$, then for $\underline{x} \in B_2$.

Let $\underline{x} \in B_1$. Then by the Remarks 4.17 and Theorem 6.18 (g) $\Gamma^{(1)}$ is a free group with generators $s_{e_1}^{(1)}$, $s_{e_2}^{(1)}$ and $s_{e_3}^{(1)}$. Example 3.23 (iv) applies.

Let $\underline{x} \in B_2$. Because of the actions of γ and $\widetilde{G}^{sign}$ (here G^{sign} is sufficient) on B_2 we can suppose $\underline{x} = (x_1, 0, 0)$ with $x_1 \in \mathbb{Z}_{<0}$. Then $e_3 \in \mathrm{Rad}\, I^{(1)}$, $s_{e_3}^{(1)} = \mathrm{id}$, $\Gamma^{(1)} = \langle s_{e_1}^{(1)}, s_{e_2}^{(1)} \rangle$ and by Theorem 6.21 (b) $\Delta^{(1)} = \Delta^{(1)} \cap (\mathbb{Z} e_1 + \mathbb{Z} e_2) \;\cup\; \{\pm e_3\}$. The monodromy M has the characteristic polynomial $(t-1)(t^2-(2-r(\underline{x}))t+1) = (t-1)(t^2-(2-x_1^2)t+1)$, so three different eigenvalues.

Consider $\underline{v} \in (\Delta^{(1)})^3$ with $s^{(1)}_{v_1} \circ s^{(1)}_{v_2} \circ s^{(1)}_{v_3} = M$. Now $v_1, v_2, v_3 \in \{\pm e_3\}$ is impossible because $M \neq \mathrm{id}$. Two of v_1, v_2, v_3 in $\{\pm e_3\}$ cannot be because then M would have the eigenvalue 1 with multiplicity 3.

Claim: All three $v_1, v_2, v_3 \in \Delta^{(1)} \cap (\mathbb{Z}e_1 + \mathbb{Z}e_2)$ is impossible.

Proof of the Claim: Suppose $v_1, v_2, v_3 \in \Delta^{(1)} \cap (\mathbb{Z}e_1 + \mathbb{Z}e_2)$. First we consider a case with $x_1 \leq -2$. By Theorem 6.18 (b) and Theorem 6.10 (c)+(d) $\Gamma^{(1)} \cong G^{free,2}$ with generators $s^{(1)}_{e_1}$ and $s^{(1)}_{e_2}$. There is a unique group homomorphism

$$\Gamma^{(1)} \to \{\pm 1\} \quad \text{with} \quad s^{(1)}_{e_1} \mapsto -1,\ s^{(1)}_{e_2} \mapsto -1.$$

Each $s^{(1)}_{v_i}$ is conjugate to $s^{(1)}_{e_1}$ or $s^{(1)}_{e_2}$ and thus has image -1. Also their product $s^{(1)}_{v_1} \circ s^{(1)}_{v_2} \circ s^{(1)}_{v_3}$ has image -1. But $M = s^{(1)}_{e_1} \circ s^{(1)}_{e_2}$ has image 1, a contradiction.

Now consider the case $x_1 = -1$. By Theorem 6.18 (b) and Theorem 6.10 (a)+(b) $\Gamma^{(1)} \cong SL_2(\mathbb{Z})$ with $s^{(1)}_{e_1} \sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $s^{(1)}_{e_2} \sim \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. It is well known that the group $SL_2(\mathbb{Z})$ is isomorphic to the group with the presentation

$$\langle x_1, x_2 \,|\, x_1x_2x_1 = x_2x_1x_2,\ 1 = (x_1x_2)^6 \rangle$$

where $x_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $x_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. The differences of the lengths of the words in $x_1^{\pm 1}$ and $x_2^{\pm 1}$ which are connected by these relations are $3-3$ and $12-0$, so even. Therefore also in this situation there is a unique group homomorphism

$$\Gamma^{(1)} \to \{\pm 1\} \quad \text{with} \quad s^{(1)}_{e_1} \to -1,\ s^{(1)}_{e_2} \to -1.$$

The argument in the case $\Gamma^{(1)} \cong G^{free,2}$ goes here through, too. The Claim is proved. ($\Box$)

Therefore a permutation $\sigma \in S_3$ with $v_{\sigma(1)}, v_{\sigma(2)} \in \Delta^{(1)} \cap (\mathbb{Z}e_1 + \mathbb{Z}e_2)$, $\sigma(1) < \sigma(2)$ and $v_{\sigma(3)} \in \{\pm e_3\}$ exists. Then $s^{(1)}_{v_{\sigma(3)}} = \mathrm{id}$ and

$$s^{(1)}_{e_1} s^{(1)}_{e_2} = M = s^{(1)}_{v_1} s^{(1)}_{v_2} s^{(1)}_{v_3} = s^{(1)}_{v_{\sigma(1)}} s^{(1)}_{v_{\sigma(2)}}.$$

Because of Theorem 7.1 $(v_{\sigma(1)}, v_{\sigma(2)})$ is in the $\mathrm{Br}_2 \ltimes \{\pm 1\}^2$ orbit of (e_1, e_2). Therefore $\underline{v}$ is in $(\mathrm{Br}_3 \ltimes \{\pm 1\}^3)(\underline{e}) = \mathcal{B}^{dist}$.

(e) See the Examples 3.23 (iii).

(f) $\Gamma^{(1)}_u \cong \mathbb{Z}^2$ for $\underline{x} \in B_4 \cup B_5$ follows from Theorem 6.21 (c)–(f), the Remarks 4.17, Lemma 4.18 and the definition of B_4 and B_5. The last statement in (f) follows from the middle statement because the sublattice $\sum_{i=1}^3\mathbb{Z}v_i\subset H_\mathbb{Z}$ and its index in $H_\mathbb{Z}$ are invariants of the $\mathrm{Br}_3\ltimes\{\pm 1\}^3$ orbit of $\underline{v}$.

For the middle statement we consider

$$\underline{v}=(e_1+a(-\widetilde{x}_3)f_3,e_2+a(\widetilde{x}_2-\widetilde{x}_1x_3)f_3,e_3+a(-\widetilde{x}_1)f_3)\quad\text{with } a\in\mathbb{Z}.$$

The next Lemma 7.8 implies

$$\begin{aligned}(\pi_3\circ\pi_3^{(1)})(\underline{v}) &= M,\\ \Big(\text{index of }\sum_{i=1}^3\mathbb{Z}v_i\text{ in }H_\mathbb{Z}\Big) &= \Big|1+a\frac{r(\underline{x})}{\gcd(x_1,x_2,x_3)^2}\Big|\end{aligned}$$

and that for a suitable $a_0\in\mathbb{N}$ and any $a\in\mathbb{Z}a_0$ $\underline{v}\in(\Delta^{(1)})^3$. As $r(\underline{x})\neq 0$ for $\underline{x}\in B_4\cup B_5$, this shows that the map Ψ has countably many values.

(g) Part (g) will be prepared by Lemma 7.10 and will be proved after the proof of Lemma 7.10. □

LEMMA 7.8. *Let $(H_\mathbb{Z},L,\underline{e})$ be a unimodular bilinear lattice of rank 3 with a triangular basis $\underline{e}$ with matrix $L(\underline{e}^t,\underline{e})^t=S(\underline{x})$ for some $\underline{x}\in\mathbb{Z}^3-\{(0,0,0)\}$. Recall* $\mathrm{Rad}\,I^{(1)}=\mathbb{Z}f_3$ *with $f_3=-\widetilde{x}_3e_1+\widetilde{x}_2e_2-\widetilde{x}_1e_3$ and $(\widetilde{x}_1,\widetilde{x}_2,\widetilde{x}_3)=\gcd(x_1,x_2,x_3)^{-1}(x_1,x_2,x_3)$.*

(a) For $\underline{a}=(a_1,a_2,a_3)\in\mathbb{Z}^3$

$$\begin{aligned}&s^{(1)}_{e_1+a_1f_3}\circ s^{(1)}_{e_2+a_2f_3}\circ s^{(1)}_{e_3+a_3f_3}=M\\ \iff\ &(a_1,a_2,a_3)\in\mathbb{Z}(-\widetilde{x}_3,\widetilde{x}_2-\widetilde{x}_1x_3,-\widetilde{x}_1).\end{aligned}$$

(b) For $\underline{a}=(a_1,a_2,a_3)=a(-\widetilde{x}_3,\widetilde{x}_2-\widetilde{x}_1x_3,-\widetilde{x}_1)$ with $a\in\mathbb{Z}$, the index of $\sum_{i=1}^3\mathbb{Z}(e_i+a_if_3)$ in $H_\mathbb{Z}$ is $|1+a\frac{r(\underline{x})}{\gcd(x_1,x_2,x_3)^2}|$.

(c) If $\Gamma_u^{(1)}\cong\mathbb{Z}^2$ then there is a number $a_0\in\mathbb{N}$ with

$$(e_1+a(-\widetilde{x}_3)f_3,e_2+a(\widetilde{x}_2-\widetilde{x}_1x_3)f_3,e_3+a(-\widetilde{x}_1)f_3)\in(\Delta^{(1)})^3\text{ for }a\in\mathbb{Z}a_0.$$

Proof: (a) With Lemma 7.4 and $f_3=(f$ in Lemma 7.4$)$ one calculates

$$\begin{aligned}&s^{(1)}_{e_1+a_1f_3}\circ s^{(1)}_{e_2+a_2f_3}\circ s^{(1)}_{e_3+a_3f_3}\circ M^{-1}\\ =\ &t_{-a_1j^{(1)}(e_1)}\circ s^{(1)}_{e_1}\circ t_{-a_2j^{(1)}(e_2)}\circ s^{(1)}_{e_2}\circ t_{-a_3j^{(1)}(e_3)}\circ s^{(1)}_{e_3}\circ M^{-1}\\ =\ &t_{-a_1j^{(1)}(e_1)}\circ t_{-a_2j^{(1)}(e_2)-a_2j^{(1)}(e_2)(e_1)j^{(1)}(e_1)}\circ s^{(1)}_{e_1}\\ &\circ t_{-a_3j^{(1)}(e_3)-a_3j^{(1)}(e_3)(e_2)j^{(1)}(e_2)}\circ s^{(1)}_{e_2}\circ s^{(1)}_{e_3}\circ M^{-1}\\ =\ &t_{-A}\end{aligned}$$

with

$$\begin{aligned} A &= a_1 j^{(1)}(e_1) + a_2 j^{(1)}(e_2) + a_2 I^{(1)}(e_2,e_1) j^{(1)}(e_1) \\ &\quad + a_3 j^{(1)}(e_3) + a_3 I^{(1)}(e_3,e_2) j^{(1)}(e_2) + a_3 I^{(1)}(e_3,e_1) j^{(1)}(e_1) \\ &\quad + a_3 I^{(1)}(e_3,e_2) I^{(1)}(e_2,e_1) j^{(1)}(e_1) \\ &= j^{(1)}\Big((a_1 - a_2x_1 - a_3x_2 + a_3x_1x_3)e_1 + (a_2 - a_3x_3)e_2 + a_3e_3\Big). \end{aligned}$$

$t_{-A} = \mathrm{id}$ holds if and only if $A = 0$, so if and only if

$$(a_1 - a_2x_1 - a_3x_2 + a_3x_1x_3)e_1 + (a_2 - a_3x_3)e_2 + a_3e_3 \in \mathrm{Rad}\, I^{(1)} = \mathbb{Z}f_3.$$

The ansatz that it is $af_3 = a(-\widetilde{x}_3e_1 + \widetilde{x}_2e_2 - \widetilde{x}_1e_3)$ with $a \in \mathbb{Z}$ gives

$$-a\widetilde{x}_1 = a_3,\ a\widetilde{x}_2 = a_2 - a_3x_3,\ -a\widetilde{x}_3 = a_1 - a_2x_1 - a_3x_2 + a_3x_1x_3,$$
$$\text{so} \quad (a_1,a_2,a_3) = a(-\widetilde{x}_3, \widetilde{x}_2 - \widetilde{x}_1x_3, -\widetilde{x}_1).$$

(b) Write $\underline{a} = \widetilde{a}\gcd(x_1,x_2,x_3)(-x_3, x_2 - x_1x_3, -x_1)$ with $\widetilde{a} = \gcd(x_1,x_2,x_3)^{-2}a \in \gcd(x_1,x_2,x_3)^{-2}\mathbb{Z}$. Then

$$\begin{aligned} &(e_1 + a_1f_3, e_2 + a_2f_3, e_3 + a_3f_3) \\ = &\ \underline{e}\begin{pmatrix} 1+\widetilde{a}(-x_3)(-x_3) & \widetilde{a}(x_2 - x_1x_3)(-x_3) & \widetilde{a}(-x_1)(-x_3) \\ \widetilde{a}(-x_3)x_2 & 1+\widetilde{a}(x_2-x_1x_3)x_2 & \widetilde{a}(-x_1)x_2 \\ \widetilde{a}(-x_3)(-x_1) & \widetilde{a}(x_2-x_1x_3)(-x_1) & 1+\widetilde{a}(-x_1)(-x_1) \end{pmatrix}. \end{aligned}$$

The determinant of this matrix is $1 + \widetilde{a}r(\underline{x})$. The index of the lattice $\sum_{i=1}^3 \mathbb{Z}(e_i + a_if_3)$ in $H_\mathbb{Z}$ is the absolute value of this determinant.

(c) Suppose $\Gamma_u^{(1)} \cong \mathbb{Z}^2$. Compare Lemma 6.17. The set

$$\Lambda := \{\lambda \in \mathrm{Hom}_0(H_\mathbb{Z}, \mathbb{Z}) \,|\, t_\lambda^+ \in \Gamma_u^{(1)}\}$$

is a sublattice of rank 2 in the lattice $\mathrm{Hom}_0(H_\mathbb{Z},\mathbb{Z})$ of rank 2. For $i \in \{1,2,3\}$

$$\Gamma_u^{(1)}\{e_i\} = \{e_i + \lambda(e_i)f_3 \,|\, \lambda \in \Lambda\} \subset (e_i + \mathbb{Z}f_3) \cap \Delta^{(1)}.$$

The triple $(H_\mathbb{Z}, L, \underline{e})$ is irreducible because of $\Gamma_u^{(1)} \cong \mathbb{Z}^2$ and Theorem 7.7 (a). Therefore it is not reducible with a summand of type A_1, and thus $\{e_1,e_2,e_3\} \cap \mathrm{Rad}\, I^{(1)} = \emptyset$. Because of this and because Λ has finite index in $\mathrm{Hom}_0(H_\mathbb{Z},\mathbb{Z})$, there is a number $b_i \in \mathbb{N}$ with

$$\mathbb{Z}b_i = \{b \in \mathbb{Z} \,|\, e_i + bf_3 \in \Gamma_u^{(1)}\{e_i\}\}.$$

For each $\underline{a} = (a_1,a_2,a_3)$ with $a_i \in \mathbb{Z}b_i$ $e_i + a_if_3 \in \Delta^{(1)}$. Any number $a_0 \in \mathbb{N}$ (for example the smallest one) with

$$a_0\widetilde{x}_3 \in \mathbb{Z}b_1,\ a_0(\widetilde{x}_2 - \widetilde{x}_1x_3) \in \mathbb{Z}b_2,\ a_0\widetilde{x}_1 \in \mathbb{Z}b_3$$

works. □

REMARKS 7.9. If $\underline{x} \in B_1 \cup B_2$ then the map $\Delta^{(1)} \to \overline{\Delta^{(1)}}$ is a bijection by Theorem 6.18 (b)+(g). Therefore then

$$\underline{v} = (e_1 + a(-\widetilde{x}_3)f_3, e_2 + a(\widetilde{x}_2 - \widetilde{x}_1 x_3)f_3, e_3 + a(-\widetilde{x}_1)f_3) \in H_{\mathbb{Z}}^3$$

for $a \in \mathbb{Z} - \{0\}$ satisfies $(\pi_3 \circ \pi_3^{(1)})(\underline{v}) = M$, but $\underline{v} \notin (\Delta^{(1)})^3$. This fits to Theorem 7.7 (d).

LEMMA 7.10. *The* $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ *orbits in* $r^{-1}(4) \subset \mathbb{Z}^3$ *are classified in Theorem 4.6 (e). They are separated by the isomorphism classes of the pairs* $(\overline{H_{\mathbb{Z}}}^{(1)}, \overline{M})$ *for corresponding unimodular bilinear lattices* $(H_{\mathbb{Z}}, L, \underline{e})$ *with triangular bases* $\underline{e}$ *with* $L(\underline{e}^t, \underline{e})^t = S(\underline{x})$ *and* $r(\underline{x}) = 4$. *More precisely,* $\overline{H_{\mathbb{Z}}}^{(1)}$ *has a* $\mathbb{Z}$*-basis* (c_1, c_2) *with* $\overline{M}(c_1, c_2) = (c_1, c_2)\begin{pmatrix} -1 & \gamma \\ 0 & -1 \end{pmatrix}$ *with a unique* $\gamma \in \mathbb{Z}_{\geq 0}$, *which is as follows:*

$S(\underline{x})$	$S(\mathcal{H}_{1,2})$	$S(\mathbb{P}^1 A_1)$	$S(\widehat{A}_2)$	$S(-l, 2, -l)$ with $l \equiv 0(2)$	$S(-l, 2, -l)$ with $l \equiv 1(2)$
γ	0	2	3	$\frac{l^2}{2} - 2$	$l^2 - 4$

The numbers γ *in this table are pairwise different.*

Proof: $S(\mathcal{H}_{1,2})$: See Theorem 5.14 (a) (i).

$S(\mathbb{P}^1 A_1)$: By Theorem 5.13 $(\overline{H_{\mathbb{Z}}}^{(1)}, \overline{M}) \cong (H_{\mathbb{Z},1}, M_1)$ which comes from $S(\mathbb{P}^1) = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ with $S(\mathbb{P}^1)^{-1} S(\mathbb{P}^1)^t = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}$. This monodromy matrix is conjugate to $\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$ with respect to $GL_2(\mathbb{Z})$.

$S(\widehat{A}_2)$: Compare Theorem 5.14 (b) (iii) and its proof:

$$\begin{aligned}
f_3 &= e_1 - e_2 + e_3, \\
\overline{H_{\mathbb{Z}}}^{(1)} &= \mathbb{Z}\overline{e_1}^{(1)} + \mathbb{Z}\overline{e_2}^{(1)}, \\
M\underline{e} &= \underline{e} \begin{pmatrix} -2 & -1 & 2 \\ -2 & 0 & 1 \\ -1 & -1 & 1 \end{pmatrix}, \\
\overline{M}(\overline{e_1}^{(1)}, \overline{e_2}^{(1)}) &= (\overline{e_1}^{(1)}, \overline{e_2}^{(1)}) \begin{pmatrix} -1 & 0 \\ -3 & -1 \end{pmatrix}.
\end{aligned}$$

This monodromy matrix is conjugate to $\begin{pmatrix} -1 & 3 \\ 0 & -1 \end{pmatrix}$ with respect to $GL_2(\mathbb{Z})$.

$S(-l, 2, -l)$ with $l \geq 4, l \equiv 0(2)$: See Theorem 5.14 (a) (ii). Here $(\overline{H_{\mathbb{Z}}}^{(1)}, \overline{M}) \cong (H_{\mathbb{Z},1}, M_1)$.

$S(-l,2,-l)$ with $l\geq 3, l\equiv 1(2)$: Compare Theorem 5.14 (b) (iv) and its proof. Define elements $a_1,a_2\in H_{\mathbb{Z}}$,

$$\begin{aligned} a_1 &:= \frac{l+1}{2}e_1+e_2+\frac{l+1}{2}e_3=\frac{1}{2}f_1+\frac{1}{2}f_3,\\ a_2 &:= -e_1=\frac{1}{2}\widetilde{f_2}-\frac{l^2}{4}f_1-\frac{l}{4}f_3. \end{aligned}$$

The triple (a_1,a_2,f_3) is a $\mathbb{Z}$-basis of $H_{\mathbb{Z}}$. The equality in the proof of Theorem 5.14 (b) (iv),

$$M(f_1,\widetilde{f_2})=(f_1,\widetilde{f_2})\begin{pmatrix}-1 & l^2-4\\ 0 & -1\end{pmatrix},$$

implies

$$M(\overline{a_1}^{(1)},\overline{a_2}^{(1)})=(\overline{a_2}^{(1)},\overline{a_2}^{(1)})\begin{pmatrix}-1 & l^2-4\\ 0 & -1\end{pmatrix}. \qquad \square$$

Proof of Theorem 7.7 (g): The case $(0,0,0)$ is treated first and separately. Compare part (e). Of the ten triples listed there, only the triple $\underline{v}=(e_1,e_2,e_3)$ satisfies $\sum_{i=1}^3\mathbb{Z}v_i=H_{\mathbb{Z}}$. This shows part (g) in the case $\underline{x}=(0,0,0)$.

Now consider $\underline{x}\in B_4$. We can suppose

$$\underline{x}\in\{(-1,0,-1),(-1,-1,-1),(-2,2,-2)\}\cup\{(-l,2,-l)\,|\,l\geq 3\},$$

which are the cases $S(\underline{x})\in\{S(A_3),S(\widehat{A}_2),S(\mathcal{H}_{1,2})\}\cup\{S(-l,2,-l)\,|\,l\geq 3\}$. Consider $\underline{v}\in(\Delta^{(1)})^3$ with $(\pi_3\circ\pi_3^{(1)})(\underline{v})=M$ and $\underline{v}$ a $\mathbb{Z}$-basis of $H_{\mathbb{Z}}$. We want to show $\underline{v}\in\mathcal{B}^{dist}$. We have

$$I^{(1)}(\underline{v}^t,\underline{v})=\begin{pmatrix}0 & y_1 & y_2\\ -y_1 & 0 & y_3\\ -y_2 & -y_3 & 0\end{pmatrix}=S(\underline{y})-S(\underline{y})^t \quad\text{for some } \underline{y}\in\mathbb{Z}^3.$$

Define a new Seifert form $\widetilde{L}:H_{\mathbb{Z}}\times H_{\mathbb{Z}}\to\mathbb{Z}$ by $\widetilde{L}(\underline{v}^t,\underline{v})^t=S(\underline{y})$ (only at the end of the proof it will turn out that $\widetilde{L}=L$). Then

$$\widetilde{L}^t-\widetilde{L}=I^{(1)}=L^t-L.$$

$\underline{v}$ is a triangular basis with respect to $(H_{\mathbb{Z}},\widetilde{L})$. By Theorem 2.7 for $(H_{\mathbb{Z}},\widetilde{L})$ (alternatively, one can calculate the product of the matrices of $s_{v_1}^{(1)}$, $s_{v_2}^{(1)}$ and $s_{v_3}^{(1)}$ with respect to $\underline{v}$)

$$M\underline{v}=(\pi_3\circ\pi_3^{(1)})(\underline{v})=(s_{v_1}^{(1)}\circ s_{v_2}^{(1)}\circ s_{v_3}^{(1)})(\underline{v})=\underline{v}S(\underline{y})^{-1}S(\underline{y})^t.$$

Then

$$3-r(\underline{x})=\operatorname{tr}(M)=\operatorname{tr}(S(\underline{y})^{-1}S(\underline{y})^t)=3-r(\underline{y}),$$

so $r(\underline{x}) = r(\underline{y})$. In the case of A_3, $r^{-1}(2)$ is a unique $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbit. In the cases of $\widehat{A}_2$, $\mathcal{H}_{1,2}$ and $\underline{x} \in \{(-l, 2, -l) \,|\, l \geq 3\}$, Lemma 7.10 and $M\underline{v} = \underline{v}S(\underline{y})^{-1}S(\underline{y})^t$ show that $\underline{y}$ is in the same $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ orbit as $\underline{x}$. Therefore in any case there is an element of $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ which maps $\underline{v}$ to a $\mathbb{Z}$-basis $\underline{w}$ of $H_\mathbb{Z}$ with

$$\widetilde{L}(\underline{w}^t, \underline{w})^t = S(\underline{x}).$$

Then

$$I^{(1)}(\underline{w}^t, \underline{w}) = S(\underline{x}) - S(\underline{x})^t = I^{(1)}(\underline{e}^t, \underline{e}).$$

Define an automorphism $g \in O^{(1)}$ by $g(\underline{e}) = \underline{w}$. Because of

$$\begin{aligned} M &= s^{(1)}_{v_1} s^{(1)}_{v_2} s^{(1)}_{v_2} = s^{(1)}_{w_1} s^{(1)}_{w_2} s^{(1)}_{w_3} \\ &= s^{(1)}_{g(e_1)} s^{(1)}_{g(e_2)} s^{(1)}_{g(e_3)} = g s^{(1)}_{e_1} s^{(1)}_{e_2} s^{(1)}_{e_3} g^{-1} = gMg^{-1} \end{aligned}$$

g is in $G^{(1)}_\mathbb{Z}$. But for the considered cases of $\underline{x}$

$$G^{(1)}_\mathbb{Z} \overset{\text{Theorem 5.14}}{=} G_\mathbb{Z} \overset{\text{Theorem 3.28}}{=} Z(\mathrm{Br}_3 \ltimes \{\pm 1\}^3).$$

Therefore there is an element of $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ which maps $\underline{e}$ to $\underline{w}$. Altogether $\underline{v} \in \mathrm{Br}_3 \ltimes \{\pm 1\}^3(\underline{e}) = \mathcal{B}^{dist}$. (Now also $L(\underline{v}^t, \underline{v})^t \in T^{uni}_3(\mathbb{Z})$ and thus $L = \widetilde{L}$ are clear.) □

7.4. The stabilizers of distinguished bases in the rank 3 cases

Let $(H_\mathbb{Z}, L, \underline{e})$ be a unimodular bilinear lattice of rank 3 with a triangular basis $\underline{e}$ with $L(\underline{e}^t, \underline{e})^t = S(\underline{x}) \in T^{uni}_3(\mathbb{Z})$ for some $x \in \mathbb{Z}^3$. We are interested in the stabilizer $(\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3}$. The surjective map

$$\begin{aligned} \mathcal{B}^{dist} = (\mathrm{Br}_3 \ltimes \{\pm 1\}^3)(\underline{e}) &\to (\mathrm{Br}_3 \ltimes \{\pm 1\}^3)(\underline{x}) \\ \underline{\widetilde{e}} &\mapsto \underline{\widetilde{x}} \text{ with } L(\underline{\widetilde{e}}^t, \underline{\widetilde{e}})^t = S(\underline{\widetilde{x}}), \end{aligned}$$

is $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ equivariant. By Theorem 4.13 (a)

$$\mathbb{Z}^3 = \dot{\bigcup}_{\underline{x} \in \bigcup_{i=1}^{24} C_i} (\mathrm{Br}_3 \ltimes \{\pm 1\}^3)(\underline{x}).$$

Therefore we can and will restrict to $\underline{x} \in \bigcup_{i=1}^{24} C_i$.

The stabilizer $(\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3}$ is by Lemma 3.25 (e) the kernel of the group antihomomorphism

$$\overline{Z} : (\mathrm{Br}_3)_{\underline{x}/\{\pm 1\}^3} \to G^{\mathcal{B}}_\mathbb{Z} / Z((\{\pm 1\}^3)_{\underline{x}}).$$

Here this simplifies to

$$\overline{Z} : (\mathrm{Br}_3)_{\underline{x}/\{\pm 1\}^3} \to G_\mathbb{Z} / Z((\{\pm 1\}^3)_{\underline{x}})$$

because $G_{\mathbb{Z}}^{\mathcal{B}} = G_{\mathbb{Z}}$ in the reducible cases (and also in most irreducible cases) and $Z((\{\pm 1\}^3)_{\underline{x}}) = \{\pm \mathrm{id}\}$, which is a normal subgroup of $G_{\mathbb{Z}}$, in the irreducible cases by Lemma 3.25 (f).

Theorem 4.16 gives the stabilizer $(\mathrm{Br}_3)_{\underline{x}/\{\pm 1\}^3}$ in all cases. The following Theorem 7.11 gives the stabilizer $(\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3}$ in all cases.

THEOREM 7.11. *Consider a local minimum $\underline{x} \in C_i \subset \mathbb{Z}^3$ for some $i \in \{1, ..., 24\}$ and the pseudo-graph $\mathcal{G}_j$ with $\mathcal{G}_j = \mathcal{G}(\underline{x})$. In the following table, the entry in the fourth column and in the line of C_i is the stabilizer $(\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3}$. The first, second and third column are copied from the table in Theorem 4.13.*

	sets	$(\mathrm{Br}_3)_{\underline{x}/\{\pm 1\}^3}$	$(\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3}$
$\mathcal{G}_1$	$C_1\ (A_1^3)$	Br_3	Br_3^{pure}
$\mathcal{G}_1$	$C_2\ (\mathcal{H}_{1,2})$	Br_3	$\langle (\sigma^{mon})^2 \rangle$
$\mathcal{G}_2$	$C_3\ (A_2 A_1)$	$\langle \sigma_1, \sigma_2^2 \rangle$	$\langle \sigma_2^2, (\sigma^{mon})^{-1}\sigma_1^2, \sigma^{mon}\sigma_1 \rangle$
			$= \langle \sigma_2^2, \sigma_1\sigma_2^2\sigma_1^{-1}, \sigma_1^3 \rangle$
$\mathcal{G}_2$	$C_4\ (\mathbb{P}^1 A_1), C_5$	$\langle \sigma_1, \sigma_2^2 \rangle$	$\langle \sigma_2^2, (\sigma^{mon})^{-1}\sigma_1^2 \rangle$
			$= \langle \sigma_2^2, \sigma_1\sigma_2^2\sigma_1^{-1} \rangle$
$\mathcal{G}_3$	$C_6\ (A_3)$	$\langle \sigma_1\sigma_2, \sigma_1^3 \rangle$	$\langle (\sigma_1\sigma_2)^4, \sigma_1^3 \rangle$
$\mathcal{G}_4$	$C_7\ (\widehat{A}_2)$	$\langle \sigma_2\sigma_1, \sigma_1^3 \rangle$	$\langle \sigma_1^3, \sigma_2^3, \sigma_2\sigma_1^3\sigma_2^{-1} \rangle$
$\mathcal{G}_5$	$C_8,\ C_9\ ((-l, 2, -l))$	$\langle \sigma^{mon}, \sigma_1^{-1}\sigma_2^{-1}\sigma_1 \rangle$	$\langle (\sigma^{mon})^2 \sigma_1^{-1}\sigma_2^{l^2-4}\sigma_1 \rangle$
$\mathcal{G}_6$	$C_{10}\ (\mathbb{P}^2),\ C_{11},\ C_{12}$	$\langle \sigma_2\sigma_1 \rangle$	$\langle \mathrm{id} \rangle$
$\mathcal{G}_7$	C_{13} (e.g. $(4, 4, 8)$)	$\langle \sigma_2\sigma_1^2 \rangle$	$\langle \mathrm{id} \rangle$
$\mathcal{G}_8$	C_{14} (e.g. $(3, 4, 6)$)	$\langle \sigma^{mon} \rangle$	$\langle \mathrm{id} \rangle$
$\mathcal{G}_9$	$C_{15},\ C_{16},\ C_{23},\ C_{24}$	$\langle \sigma^{mon} \rangle$	$\langle \mathrm{id} \rangle$
$\mathcal{G}_{10}$	C_{17} (e.g. $(-2, -2, 0)$)	$\langle \sigma^{mon}, \sigma_2 \rangle$	$\langle \sigma_2^2 \rangle$
$\mathcal{G}_{11}$	C_{18} (e.g. $(-3, -2, 0)$)	$\langle \sigma^{mon}, \sigma_2^2 \rangle$	$\langle \sigma_2^2 \rangle$
$\mathcal{G}_{12}$	C_{19} (e.g. $(-2, -1, 0)$)	$\langle \sigma^{mon}, \sigma_2^2, \sigma_2\sigma_1^3\sigma_2^{-1} \rangle$	$\langle \sigma_2^2, \sigma_2\sigma_1^3\sigma_2^{-1} \rangle$
$\mathcal{G}_{13}$	C_{20} (e.g. $(-2, -1, -1)$)	$\langle \sigma^{mon}, \sigma_2^3, \sigma_2\sigma_1^3\sigma_2^{-1} \rangle$	$\langle \sigma_2^3, \sigma_2\sigma_1^3\sigma_2^{-1} \rangle$
$\mathcal{G}_{14}$	$C_{21},\ C_{22}$	$\langle \sigma^{mon}, \sigma_2^3 \rangle$	$\langle \sigma_2^3 \rangle$

Proof: The reducible case $\mathcal{G}_1 \,\&\, C_1\,(A_1^3)$: Here $\underline{x} = (0, 0, 0)$ and $(\mathrm{Br}_3)_{\underline{x}/\{\pm 1\}^3} = \mathrm{Br}_3$.

$$\mathcal{B}^{dist} = \{(\varepsilon_1 e_{\sigma(1)}, \varepsilon_2 e_{\sigma(2)}, \varepsilon_3 e_{\sigma(3)}) \,|\, \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{\pm 1\}, \sigma \in S_3\},$$

and Br_3 acts by permutation of the entries of triples on $\mathcal{B}^{dist}$. Therefore $(\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3}$ is the kernel of the natural group homomorphism $\mathrm{Br}_3 \to S_3$, so it is the subgroup Br_3^{pure} of pure braids.

The case $\mathcal{G}_1 \,\&\, C_2\,(\mathcal{H}_{1,2})$: Here $\underline{x} = (2, 2, 2) \sim (-2, 2, -2)$ and $(\mathrm{Br}_3)_{\underline{x}/\{\pm 1\}^3} = \mathrm{Br}_3$. Recall the case $\mathcal{H}_{1,2}$ in the proof of Theorem 5.14, recall the $\mathbb{Z}$-basis $\widetilde{f}$ of $H_{\mathbb{Z}} = H_{\mathbb{Z},1} \oplus H_{\mathbb{Z},2}$, and recall

$$G_{\mathbb{Z}} = \{g \in \mathrm{Aut}(H_{\mathbb{Z}}, 1) \,|\, \det g = 1\} \times \mathrm{Aut}(H_{\mathbb{Z},2}) \cong SL_2(\mathbb{Z}) \times \{\pm 1\}.$$

We found in the proof of Theorem 5.14 (c)

$$Z(\delta_2\sigma_1) = (\widetilde{f} \mapsto \widetilde{f}\begin{pmatrix}1 & -1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}), \quad Z(\delta_3\sigma_2) = (\widetilde{f} \mapsto \widetilde{f}\begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}).$$

The group antihomomorphism $\overline{Z} : (\text{Br}_3)_{\underline{x}/\{\pm 1\}^3} = \text{Br}_3 \to G_{\mathbb{Z}}/\{\pm \text{id}\} \cong SL_2(\mathbb{Z})$ is surjective with $\overline{Z}(\sigma_1) \equiv A_1$ and $\overline{Z}(\sigma_2) \equiv A_2$. It almost coincides with the group homomorphism $\text{Br}_3 \to SL_2(\mathbb{Z})$ in Remark 4.15 (i). It has the same kernel $\langle(\sigma^{mon})^2\rangle$.

The reducible cases $\mathcal{G}_2$ & $C_3\,(A_2A_1), C_4\,(\mathbb{P}^1A_1), C_5$: Here $\underline{x} = (x_1, 0, 0)$ with $x_1 \leq -1$ and $(\text{Br}_3)_{\underline{x}/\{\pm 1\}^3} = \langle \sigma_1, \sigma_2^2\rangle$. The quotient group $(\text{Br}_3)_{\underline{x}/\{\pm 1\}^3}/(\text{Br}_3)_{\underline{e}/\{\pm 1\}^3}$ is by Theorem 3.28 (c) and Lemma 3.25 (e) isomorphic to the quotient group $G_{\mathbb{Z}}/Z((\{\pm 1\}^3)_{\underline{x}})$. Here $Z((\pm 1\}^3)_{\underline{x}}) = \langle(-1,-1,-1),(-1,-1,1)\rangle$ with

$$\begin{aligned} Z((-1,-1,-1)) &= -\text{id}, \\ Z((-1,-1,1)) &= (\underline{e} \mapsto (-e_1, -e_2, e_3)) = Q. \end{aligned}$$

Define

$$M^{root} := Z(\delta_2\sigma_1) = (\underline{e} \mapsto \underline{e}\begin{pmatrix}-x_1 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}),$$

and recall from Theorem 5.13 and Theorem 5.5

$$G_{\mathbb{Z}} = \{\pm(M^{root})^l \,|\, l \in \mathbb{Z}\} \times \{\text{id}, Q\}.$$

Therefore

$$(\text{Br}_3)_{\underline{x}/\{\pm 1\}^3}/(\text{Br}_3)_{\underline{e}/\{\pm 1\}^3} \cong G_{\mathbb{Z}}/Z((\{\pm 1\}^3)_{\underline{x}}) \cong \{(M^{root})^l \,|\, l \in \mathbb{Z}\}.$$

In the case $C_3\,(A_2A_1)$ $x_1 = -1$ and M^{root} has order three, so the quotient group $(\text{Br}_3)_{\underline{x}/\{\pm 1\}^3}/(\text{Br}_3)_{\underline{e}/\{\pm 1\}^3}$ is cyclic of order three with generator the class $[\sigma_1]$ of σ_1. In the cases C_4 and C_5 $x_1 \leq -2$ and M^{root} has infinite order, so the quotient group $(\text{Br}_3)_{\underline{x}/\{\pm 1\}^3}/(\text{Br}_3)_{\underline{e}/\{\pm 1\}^3}$ is cyclic of infinite order with generator the class $[\sigma_1]$ of σ_1.

The cases $C_4\,(\mathbb{P}^1A_1), C_5$: Theorem 7.1 (b) can be applied to the subbasis (e_2, e_3) with $x_3 = 0$. It shows $\sigma_2^2 \in (\text{Br}_3)_{\underline{e}/\{\pm 1\}^3}$. Therefore $(\text{Br}_3)_{\underline{e}/\{\pm 1\}^3}$ contains the normal closure of σ_2^2 in $(\text{Br}_3)_{\underline{x}/\{\pm 1\}^3} = \langle\sigma_1, \sigma_2^2\rangle$. This normal subgroup is obviously

$$\langle \sigma_1^l \sigma_2^2 \sigma_1^{-l} \,|\, l \in \mathbb{Z}\rangle.$$

It can also be written with two generators, namely it is

$$\langle \sigma_2^2, \sigma_1\sigma_2^2\sigma_1^{-1}\rangle = \langle \sigma_2^2, (\sigma^{mon})^{-1}\sigma_1^2\rangle.$$

The equality of left and right side follows from

$$\sigma_2^2\cdot\sigma_1\sigma_2^2\sigma_1^{-1}=\sigma_2^2\sigma_1\sigma_2^2\sigma_1\cdot\sigma_1^{-2}=\sigma^{mon}\sigma_1^{-2}.$$

The equality of this group with $\langle\sigma_1^l\sigma_2^2\sigma_1^{-l}\,|\,l\in\mathbb{Z}\rangle$ follows from the fact that σ^{mon} is in the center of Br_3. The quotient group $\langle\sigma_1,\sigma_2^2\rangle/\langle\sigma_1^l\sigma_2^2\sigma_1^{-l}\,|\,l\in\mathbb{Z}\rangle$ is cyclic of infinite order with generator the class $[\sigma_1]$ of σ_1. Therefore

$$\begin{aligned}(\mathrm{Br}_3)_{\underline{e}/\{\pm1\}^3} &= \langle\sigma_2^2,(\sigma^{mon})^{-1}\sigma_1^2\rangle=\langle\sigma_2^2,\sigma_1\sigma_2^2\sigma_1^{-1}\rangle\\ &= \langle\sigma_1^l\sigma_2^2\sigma_1^{-l}\,|\,l\in\mathbb{Z}\rangle\\ &= (\text{the normal closure of }\sigma_2^2\text{ in }\langle\sigma_1,\sigma_2^2\rangle).\end{aligned}$$

The case $C_3\,(A_2A_1)$: Theorem 7.1 (b) can be applied to the subbasis (e_1,e_2) with $x_1=-1$ and to the subbasis (e_2,e_3) with $x_3=0$. It shows σ_1^3 and $\sigma_2^2\in(\mathrm{Br}_3)_{\underline{e}/\{\pm1\}^3}$. Therefore $(\mathrm{Br}_3)_{\underline{e}/\{\pm1\}^3}$ contains the normal closure of σ_1^3 and σ_2^2 in $(Br_3)_{\underline{x}/\{\pm1\}^3}=\langle\sigma_1,\sigma_2^2\rangle$. The quotient group

$$\langle\sigma_1,\sigma_2^2\rangle/(\text{the normal closure of }\sigma_1^3\text{ and }\sigma_2^2\text{ in }\langle\sigma_1,\sigma_2^2\rangle)$$

is cyclic of order three with generator the class $[\sigma_1]$ of σ_1. Therefore

$$(\mathrm{Br}_3)_{\underline{e}/\{\pm1\}^3}=(\text{the normal closure of }\sigma_1^3\text{ and }\sigma_2^2\text{ in }\langle\sigma_1,\sigma_2^2\rangle).$$

It coincides with the subgroup generated by σ_1^3 and by the normal closure $\langle\sigma_2^2,(\sigma^{mon})^{-1}\sigma_1^2\rangle$ of σ_2^2 in $\langle\sigma_1,\sigma_2^2\rangle$. Therefore

$$(\mathrm{Br}_3)_{\underline{e}/\{\pm1\}^3} = \langle\sigma_2^2,(\sigma^{mon})^{-1}\sigma_1^2,\sigma_1^3\rangle=\langle\sigma_2^2,(\sigma^{mon})^{-1}\sigma_1^2,\sigma^{mon}\sigma_1\rangle.$$

The case $\mathcal{G}_3\,\&\,C_6\,(A_3)$: Here $\underline{x}=(-1,0,-1)$ and $(\mathrm{Br}_3)_{\underline{x}/\{\pm1\}^3}=\langle\sigma_1\sigma_2,\sigma_1^3\rangle$. By Theorem 5.14 (b) $G_{\mathbb{Z}}=\{\pm M^l\,|\,l\in\{0,1,2,3\}\}$, and M has order four. By Theorem 3.28 and Lemma 3.25 (f), the antihomomorphism

$$\overline{Z}:(\mathrm{Br}_3)_{\underline{x}/\{\pm1\}^3}/(\mathrm{Br}_3)_{\underline{e}/\{\pm1\}^3}\to G_{\mathbb{Z}}/\{\pm\,\mathrm{id}\}$$

is an antiisomorphism. Therefore the quotient group $(\mathrm{Br}_3)_{\underline{x}/\{\pm1\}^3}/(\mathrm{Br}_3)_{\underline{e}/\{\pm1\}^3}$ is cyclic of order four.

Theorem 7.1 (b) can be applied to the subbasis (e_1,e_2) with $x_1=-1$. It shows $\sigma_1^3\in(\mathrm{Br}_3)_{\underline{e}/\{\pm1\}^3}$. Observe also

$$\begin{aligned}\delta_1\sigma_1\sigma_2(\underline{e}) &= \delta_1\sigma_1(e_1,e_3+e_2,e_2)=\delta_1(e_1+e_2+e_3,e_1,e_2)\\ &= (-e_1-e_2-e_3,e_1,e_2)=-M^{-1}(\underline{e}),\\ \text{so }Z(\delta_1\sigma_1\sigma_2) &= -M^{-1}.\end{aligned}$$

M and $-M^{-1}$ have order four. Therefore $(\sigma_1\sigma_2)^4\in(\mathrm{Br}_3)_{\underline{e}/\{\pm1\}^3}$. Thus $(\mathrm{Br}_3)_{\underline{e}/\{\pm1\}^3}\supset\langle(\sigma_1\sigma_2)^4,\sigma_1^3\rangle$.

We will show first that $\langle(\sigma_1\sigma_2)^4,\sigma_1^3\rangle$ is a normal subgroup of $\langle\sigma_1\sigma_2,\sigma_1^3\rangle$ and then that the quotient group is cyclic of order four. This will imply $(\mathrm{Br}_3)_{\underline{e}/\{\pm1\}^3}=\langle(\sigma_1\sigma_2)^4,\sigma_1^3\rangle$.

Recall that $\sigma^{mon}=(\sigma_2\sigma_1)^3=(\sigma_1\sigma_2)^3$ generates the center of Br_3. Therefore

$$(\sigma_1\sigma_2)^l\sigma_1^3(\sigma_1\sigma_2)^{-l}=(\sigma_1\sigma_2)^{4l}\sigma_1^3(\sigma_1\sigma_2)^{-4l}\in\langle(\sigma_1\sigma_2)^4,\sigma_1^3\rangle \text{ for any } l\in\mathbb{Z}.$$

Thus $\langle(\sigma_1\sigma_2)^4,\sigma_1^3\rangle$ is a normal subgroup of $\langle\sigma_1\sigma_2,\sigma_1^3\rangle$. This also shows that the quotient group $\langle\sigma_1\sigma_1,\sigma_1^3\rangle/\langle(\sigma_1\sigma_2)^4,\sigma_1^3\rangle$ is cyclic of order four. Therefore $\langle(\sigma_1\sigma_2)^4,\sigma_1^3\rangle=(\mathrm{Br}_3)_{\underline{e}/\{\pm1\}^3}$.

The case $\mathcal{G}_4\,\&\,C_7\,(\widehat{A}_2)$: Here $\underline{x}=(-1,-1,-1)$ and $(\mathrm{Br}_3)_{\underline{x}/\{\pm1\}^3}=\langle\sigma_2\sigma_1,\sigma_1^3\rangle$. By Theorem 5.14 (b) $G_{\mathbb{Z}}=\{\pm(M^{root})^l\,|\,l\in\mathbb{Z}\}$, and M^{root} has infinite order. By Theorem 3.28 and Lemma 3.25 (f), the antihomomorphism

$$\overline{Z}:(\mathrm{Br}_3)_{\underline{x}/\{\pm1\}^3}/(\mathrm{Br}_3)_{\underline{e}/\{\pm1\}^3}\to G_{\mathbb{Z}}/\{\pm\mathrm{id}\}$$

is an antiisomorphism. By Theorem 3.26 (c) and Theorem 5.14 (b) $Z(\delta_3\sigma_2\sigma_1)=M^{root}$. Therefore the quotient group $(\mathrm{Br}_3)_{\underline{x}/\{\pm1\}^3}/(\mathrm{Br}_3)_{\underline{e}/\{\pm1\}^3}$ is cyclic of infinite order with generator the class $[\sigma_2\sigma_1]$ of $\sigma_2\sigma_1$.

Theorem 7.1 (b) can be applied to the subbasis (e_1,e_2) with $x_1=-1$. It shows $\sigma_1^3\in(\mathrm{Br}_3)_{\underline{e}/\{\pm1\}^3}$. Therefore $(\mathrm{Br}_3)_{\underline{e}/\{\pm1\}^3}$ contains the normal closure of σ_1^3 in $\langle\sigma_2\sigma_1,\sigma_1^3\rangle$. We will first determine this normal closure and then show that it equals $(\mathrm{Br}_3)_{\underline{e}/\{\pm1\}^3}$.

As $\sigma^{mon}=(\sigma_2\sigma_1)^3$ generates the center of Br_3,

$$(\sigma_2\sigma_1)^{\varepsilon+3l}\sigma_1^3(\sigma_2\sigma_1)^{-\varepsilon-3l}=(\sigma_2\sigma_1)^{\varepsilon}\sigma_1^3(\sigma_2\sigma_1)^{-\varepsilon}\quad\text{for }\varepsilon\in\{0;\pm1\},l\in\mathbb{Z}.$$

One sees

$$\begin{aligned}(\sigma_2\sigma_1)\sigma_1(\sigma_2\sigma_1)^{-1}&=\sigma_2\sigma_1\sigma_2^{-1},\quad\text{so}\\ (\sigma_2\sigma_1)\sigma_1^3(\sigma_2\sigma_1)^{-1}&=\sigma_2\sigma_1^3\sigma_2^{-1},\\ (\sigma_2\sigma_1)^{-1}\sigma_1(\sigma_2\sigma_1)&=\sigma_1^{-1}(\sigma_2^{-1}\sigma_1\sigma_2)\sigma_1\overset{(4.14)}{=}\sigma_1^{-1}(\sigma_1\sigma_2\sigma_1^{-1})\sigma_1=\sigma_2,\quad\text{so}\\ (\sigma_2\sigma_1)^{-1}\sigma_1^3(\sigma_2\sigma_1)&=\sigma_2^3.\end{aligned}$$

Therefore the normal closure of σ_1^3 in $\langle\sigma_2\sigma_1,\sigma_1^3\rangle$ is $\langle\sigma_1^3,\sigma_2^3,\sigma_2\sigma_1^3\sigma_2^{-1}\rangle$. The quotient group is an infinite cyclic group with generator the class $[\sigma_2\sigma_1]$ of $\sigma_2\sigma_1$. Therefore

$$\begin{aligned}(\mathrm{Br}_3)_{\underline{e}/\{\pm1\}^3}&=\langle\sigma_1^3,\sigma_2^3,\sigma_2\sigma_1^3\sigma_2^{-1}\rangle\\ &=(\text{the normal closure of }\sigma_1^3\text{ in }\langle\sigma_2\sigma_1,\sigma_1^3\rangle).\end{aligned}$$

The cases $\mathcal{G}_5\,\&\,C_8$: Here $\underline{x}=(-l,2,-l)$ with $l\geq3$ odd and $(\mathrm{Br}_3)_{\underline{x}/\{\pm1\}^3}=\langle\sigma^{mon},\sigma_1^{-1}\sigma_2^{-1}\sigma_1\rangle$.

By Theorem 5.14 (b) $G_{\mathbb{Z}} = \{\pm(M^{root})^l \,|\, l \in \mathbb{Z}\}$ with M^{root} as in Theorem 5.14 (b) with $(M^{root})^{l^2-4} = -M$. Because l is odd, the cyclic group $G_{\mathbb{Z}}/\{\pm \mathrm{id}\}$ with generator $[M^{root}]$ can also be written as

$$G_{\mathbb{Z}}/\{\pm \mathrm{id}\} = \langle [M], [(M^{root})^2] \rangle \quad \text{with } [M] = [M^{root}]^{l^2-4}.$$

By Theorem 3.28 and Lemma 3.25 (f), the antihomomorphism

$$\overline{Z} : (\mathrm{Br}_3)_{\underline{x}/\{\pm 1\}^3} \to G_{\mathbb{Z}}/\{\pm \mathrm{id}\}$$

is surjective with kernel $(\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3}$. By the proof of Theorem 5.14 (b)(iv) $[M] = \overline{Z}(\sigma^{mon})$ and $[(M^{root})^2] = \overline{Z}(\sigma_1^{-1}\sigma_2^{-1}\sigma_1)$. The single relation between $[M]$ and $[(M^{root})^2]$ is $[\mathrm{id}] = [M]^2([(M^{root})^2])^{4-l^2}$. Therefore the kernel $(Br_3)_{\underline{e}/\{\pm 1\}^3}$ of the group antihomomorphism $\overline{Z}$ above is generated by $(\sigma^{mon})^2(\sigma_1^{-1}\sigma_2^{-1}\sigma_1)^{4-l^2} = (\sigma^{mon})^2\sigma_1^{-1}\sigma_2^{l^2-4}\sigma_1$.

The cases $\mathcal{G}_6 \,\&\, C_9$: Here $\underline{x} = (-l, 2, -l)$ with $l \geq 4$ even and $(\mathrm{Br}_3)_{\underline{x}/\{\pm 1\}^3} = \langle \sigma^{mon}, \sigma_1^{-1}\sigma_2^{-1}\sigma_1 \rangle$.

By Theorem 3.28 and Lemma 3.25 (f), the antihomomorphism

$$\overline{Z} : (\mathrm{Br}_3)_{\underline{x}/\{\pm 1\}^3} \to G_{\mathbb{Z}}/\{\pm \mathrm{id}\}$$

is surjective with kernel $(\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3}$.

By Theorem 5.14 (a) and the proof of Theorem 5.14 (c)

$$\begin{aligned} G_{\mathbb{Z}} &= \langle -\mathrm{id}, \widetilde{M}, Q \rangle \\ \text{with} \quad \widetilde{M} &= Z(\delta_3\sigma_1^{-1}\sigma_2^{-1}\sigma_1),\ Q = Z(\sigma^{mon})Z(\delta_3\sigma_1^{-1}\sigma_2^{-1}\sigma_1)^{2-l^2/2}, \end{aligned}$$

$\widetilde{M}$ and Q commute, $\widetilde{M}$ has infinite order, Q has order two. Therefore the kernel $(\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3}$ of the group antihomomorphism $\overline{Z}$ above is generated by $(\sigma^{mon}(\sigma_1^{-1}\sigma_2^{-1}\sigma_1)^{2-l^2/2})^2 = (\sigma^{mon})^2\sigma_1^{-1}\sigma_2^{l^2-4}\sigma_1$.

The cases $\mathcal{G}_6 \,\&\, C_{10}\,(\mathbb{P}^2), C_{11}, C_{12}$: Here $(\mathrm{Br}_3)_{\underline{x}/\{\pm 1\}^3} = \langle \sigma_2\sigma_1 \rangle$. Here $Z(\sigma_2\sigma_1) = M^{root}$ is a third root of the monondromy. The monodromy M and M^{root} have infinite order. Therefore the kernel $(\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3}$ of the group antihomomorphism

$$\overline{Z} : (\mathrm{Br}_3)_{\underline{x}/\{\pm 1\}^3} = \langle \sigma_2\sigma_1 \rangle \to G_{\mathbb{Z}}/\{\pm \mathrm{id}\}$$

is $\langle \mathrm{id} \rangle$.

The cases $\mathcal{G}_7 \,\&\, C_{13}$ (e.g. $(4,4,8)$): Here $(\mathrm{Br}_3)_{\underline{x}/\{\pm 1\}^3} = \langle \sigma_2\sigma_1^2 \rangle$. Here $Z(\sigma_2\sigma_1) = M^{root}$ is a root of the monondromy. The monodromy M and M^{root} have infinite order. Therefore the kernel $(\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3}$ of the group antihomomorphism

$$\overline{Z} : (\mathrm{Br}_3)_{\underline{x}/\{\pm 1\}^3} = \langle \sigma_2\sigma_1^2 \rangle \to G_{\mathbb{Z}}/\{\pm \mathrm{id}\}$$

is $\langle \mathrm{id} \rangle$.

The cases $\mathcal{G}_8\,\&\,C_{14},\mathcal{G}_9\,\&\,C_{15},C_{16},C_{23},C_{24}$: Here $(\mathrm{Br}_3)_{\underline{x}/\{\pm 1\}^3} = \langle\sigma^{mon}\rangle$. The monodromy $M=Z(\sigma^{mon})$ has infinite order. Therefore the kernel $(\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3}$ of the group antihomomorphism

$$\overline{Z}:(\mathrm{Br}_3)_{\underline{x}/\{\pm 1\}^3}=\langle\sigma^{mon}\rangle\to G_{\mathbb{Z}}/\{\pm\mathrm{id}\}$$

is $\langle\mathrm{id}\rangle$.

The cases $\mathcal{G}_{10}\,\&\,C_{17}$ (e.g. $(-2,-2,0)$): Here $(\mathrm{Br}_3)_{\underline{x}/\{\pm 1\}^3} = \langle\sigma^{mon},\sigma_2\rangle$. Recall from Theorem 5.16 (c) that

$$G_{\mathbb{Z}}=\{\mathrm{id},Q\}\times\{\pm M^l\,|\,l\in\mathbb{Z}\}=\langle-\mathrm{id},Q,M\rangle,$$

Q and M commute, Q has order two, M has infinite order, $-Q=Z(\sigma_2)$ (see the proof of Theorem 5.17 (e)), $M=Z(\sigma^{mon})$. Therefore the kernel $(\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3}$ of the group antihomomorphism

$$\overline{Z}:(\mathrm{Br}_3)_{\underline{x}/\{\pm 1\}^3}=\langle\sigma^{mon},\sigma_2\rangle\to G_{\mathbb{Z}}/\{\pm\mathrm{id}\}$$

is $\langle\sigma_2^2\rangle$.

The cases $\mathcal{G}_{11}\,\&\,C_{18},\mathcal{G}_{12}\,\&\,C_{19},\mathcal{G}_{13}\,\&\,C_{20},\mathcal{G}_{14}\,\&\,C_{21},C_{22}$: By Theorem 4.16 in all cases $(\mathrm{Br}_3)_{\underline{x}/\{\pm 1\}^3}$ is generated by σ^{mon} and some other generators. We claim that the other generators are all in $(\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3}$. Application of Theorem 7.1 (b) to the subbasis (e_2,e_3) shows this for the following generators:
σ_2^2 in the cases $\mathcal{G}_{11}\&C_{18}$ and $G_{12}\&C_{19}$ because there $x_3=0$;
σ_2^3 in the cases $\mathcal{G}_{13}\&C_{20}$ and $G_{14}\&C_{21},C_{22}$ because there $x_3=-1$.

σ_2^{-1} maps $\underline{e}$ to the basis $(e_1,e_3,s_{e_3}^{(0)}(e_2))$. Therefore application of Theorem 7.1 (b) to the subbasis (e_1,e_3) with $x_2=-1$ in the cases $\mathcal{G}_{12}\&C_{19}$ and $\mathcal{G}_{13}\&C_{20}$ shows $\sigma_2\sigma_1^3\sigma_2^{-1}\in(\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3}$. The claim $\langle$other generators$\rangle\subset(\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3}$ is proved.

The monodromy $M=Z(\sigma^{mon})$ has infinite order. Therefore the kernel $(\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3}$ of the group antihomomorphism

$$\overline{Z}:(\mathrm{Br}_3)_{\underline{x}/\{\pm 1\}^3}=\langle\sigma^{mon},\text{other generators}\rangle\to G_{\mathbb{Z}}/\{\pm\mathrm{id}\}$$

is $\langle$other generators$\rangle$. □

Remarks 7.12. (i) In the cases $\mathcal{G}_6\,\&\,C_{10},C_{11},C_{12}$, $\mathcal{G}_7\,\&\,C_{13}$, $\mathcal{G}_8\,\&\,C_{14}$ and $\mathcal{G}_9\,\&\,C_{15},C_{16},C_{23},C_{24}$, the even monodromy group $\Gamma^{(0)}$ is a free Coxeter group with three generators by Theorem 6.11 (b) and (g). Example 3.23 (iv), which builds on Theorem 3.2, shows $(\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3}=\langle\mathrm{id}\rangle$.

Using this fact, one derives also $(\mathrm{Br}_3)_{\underline{x}/\{\pm 1\}^3}$ in the following way. In all cases $(H_{\mathbb{Z}},L,\underline{e})$ is irreducible. The group antihomomorphism

$$\overline{Z}:(\mathrm{Br}_3)_{\underline{x}/\{\pm 1\}^3}\to G_{\mathbb{Z}}/\{\pm\mathrm{id}\}$$

is injective because the kernel is $(\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3} = \langle \mathrm{id}\rangle$. By Theorem 3.28 $\overline{Z}$ is surjective in almost all cases. The proof of Theorem 3.28 provides in all cases preimages of generators of $\mathrm{Im}(\overline{Z}) \subset G_{\mathbb{Z}}/\{\pm \mathrm{id}\}$. These preimages generate $(\mathrm{Br}_3)_{\underline{x}/\{\pm 1\}^3}$.

So, the arguments here on one side and the Theorems 4.13, 4.16 and 7.11 on the other side offer two independent ways to derive the stabilizers $(\mathrm{Br}_3)_{\underline{x}/\{\pm 1\}^3}$ and $(\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3}$ in the considered cases.

(ii) But the arguments in (i) cannot easily be adapted to the other cases. In the cases $\mathcal{G}_{10}\,\&\,C_{17}$, $\mathcal{G}_{11}\,\&\,C_{18}$, $\mathcal{G}_{12}\,\&\,C_{19}$, $\mathcal{G}_{13}\,\&\,C_{20}$ and $\mathcal{G}_{14}\,\&\,C_{21}, C_{22}$, the even monodromy group $\Gamma^{(0)}$ is a non-free Coxeter group.

Theorem 3.2 (b) generalizes in Theorem 3.7 (b) to a statement on the size of $\mathcal{B}^{dist}$, but not to a statement on the stabilizer $(\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3}$.

The cases $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$ and $\mathcal{G}_5$ are with the exception of the reducible cases C_5 and the case C_{10} the cases with $r(\underline{x}) \in \{0, 1, 3, 4\}$. For them it looks possible, but difficult, to generalize the arguments in (i).

The conceptual derivation of the stabilizer groups for all cases with the Theorems 4.13, 4.16 and 7.11 is more elegant.

Remarks 7.13. (i) The table in Theorem 7.11 describes the stabilizer $(\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3}$ by generators, except for the case $\mathcal{G}_1\,\&\,C_1(A_1^3)$ where $(\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3} = \mathrm{Br}_3^{pure}$. In fact

$$\begin{aligned} \mathrm{Br}_3^{pure} &= \langle \sigma_1^2, \sigma_2^2, \sigma_2\sigma_1^2\sigma_2^{-1}, \sigma_2^{-1}\sigma_1^2\sigma_2\rangle \\ &= (\text{the normal closure of } \sigma_1^2 \text{ in } \mathrm{Br}_3), \end{aligned}$$

because $\sigma_2\sigma_1^2\sigma_2^{-1} = \sigma_1^{-1}\sigma_1^2\sigma_2$, $\sigma_2^{-1}\sigma_1^2\sigma_2 = \sigma_1\sigma_2^2\sigma_1^{-1}$ by (4.14).

(ii) In some cases the proof of Theorem 7.11 provides elements so that $(\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3}$ is the normal closure of these elements in $(\mathrm{Br}_3)_{\underline{x}/\{\pm 1\}^3}$:

	elements	$(\mathrm{Br}_3)_{\underline{x}/\{\pm 1\}^3}$
$\mathcal{G}_2\,\&\,C_3\,(A_2A_1)$	σ_1^3, σ_2^2	$\langle \sigma_1, \sigma_2^2\rangle$
$\mathcal{G}_2\,\&\,C_4\,(\mathbb{P}^1 A_1), C_5$	σ_2^2	$\langle \sigma_1, \sigma_2^2\rangle$
$\mathcal{G}_4\,\&\,C_7\,(\widehat{A}_2)$	σ_1^3	$\langle \sigma_2\sigma_1, \sigma_1^3\rangle$

(iii) In the case $\mathcal{G}_3\,\&\,C_6\,(A_3)$, the stabilizer $(\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3} = \langle (\sigma_1\sigma_2)^4, \sigma_1^3\rangle$ was determined already in [**Yu90**, Satz 7.3].

The pseudo-graph $\mathcal{G}(\underline{x})$ for $\underline{x} \in \bigcup_{i=1}^{24} C_i$ with vertex set $\mathcal{V} = \mathrm{Br}_3(\underline{x}/\{\pm 1\}^3)$ in Definition 4.9 (f), Lemma 4.10 and the Examples 4.11 had been very useful. All except two edges came from the generators $\varphi_1, \varphi_2, \varphi_3$ of the free Coxeter group G^{phi}, and two edges came from $\gamma(v_0)$ and $\gamma^{-1}(v_0)$. An a priori more natural choice of edges comes from the elementary braids σ_1 and σ_2. It is less useful, but also interesting.

Definition 7.14. Let $\mathcal{V}$ be a non-empty finite or countably infinite set on which Br_3 acts. The triple $\mathcal{G}_\sigma(\mathcal{V}) := (\mathcal{V}, \mathcal{E}_1, \mathcal{E}_2)$ with $\mathcal{E}_1 := \{(v, \sigma_1(v)) \,|\, v \in \mathcal{V}\}$ and $\mathcal{E}_2 := \{(v, \sigma_2(v)) \,|\, v \in \mathcal{V}\}$ is called *σ-pseudo-graph of $\mathcal{V}$*. Here $\mathcal{E}_1$ and $\mathcal{E}_2$ are two families of directed edges. A *loop* in $\mathcal{E}_i$ is an edge $(v, \sigma_i(v)) = (v, v)$.

Remarks 7.15. (i) In a picture of a σ-pseudo-graph, edges in $\mathcal{E}_1$ and in $\mathcal{E}_2$ are denoted as follows.

$\longrightarrow$ an edge in $\mathcal{E}_1$,

$\longrightarrow$ an edge in $\mathcal{E}_2$.

(ii) Consider a σ-pseudo-graph $\mathcal{G}_\sigma(\mathcal{V})$. Because $\sigma_1 : \mathcal{V} \to \mathcal{V}$ and $\sigma_2 : \mathcal{V} \to \mathcal{V}$ are bijections, each vertex $v \in \mathcal{V}$ is starting point of one edge in $\mathcal{E}_1$ and one edge in $\mathcal{E}_2$ and end point of one edge in $\mathcal{E}_1$ and one edge in $\mathcal{E}_2$. The σ-pseudo-graph is connected if and only if $\mathcal{V}$ is a single Br_3 orbit.

(iii) Let $(H_\mathbb{Z}, L, \underline{e})$ be a unimodular bilinear lattice with a triangular basis $\underline{e}$ with $L(\underline{e}^t, \underline{e})^t = S(\underline{x})$ for some $\underline{x} \in \mathbb{Z}^3$. Two σ-pseudo-graphs are associated to it, $\mathcal{G}_\sigma(\mathcal{B}^{dist}/\{\pm 1\}^3)$ and $\mathcal{G}_\sigma(\mathrm{Br}_3(\underline{x}/\{\pm 1\}^3))$. The natural map

$$\begin{array}{rcl} \mathcal{B}^{dist}/\{\pm 1\}^3 & \to & \mathrm{Br}_3(\underline{x}/\{\pm 1\}^3) \\ \underline{\widetilde{e}}/\{\pm 1\}^3 & \mapsto & \underline{\widetilde{x}}/\{\pm 1\}^3 \text{ with } L(\underline{\widetilde{e}}^t, \underline{\widetilde{e}})^t = S(\underline{\widetilde{x}}), \end{array}$$

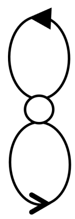
is Br_3 equivariant and surjective. It induces a *covering*

$$\mathcal{G}_\sigma(\mathcal{B}^{dist}/\{\pm 1\}^3) \quad \to \mathcal{G}_\sigma(\mathrm{Br}_3(\underline{x}/\{\pm 1\}^3))$$

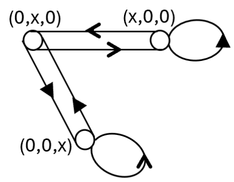
of σ-pseudo-graphs. This is even a *normal covering* with group of deck transformations $G_\mathbb{Z}^\mathcal{B}/Z((\{\pm 1\}^3)_{\underline{x}})$ where $G_\mathbb{Z}^\mathcal{B} = Z((\mathrm{Br}_3 \ltimes \{\pm 1\}^3)_{\underline{x}}) \subset G_\mathbb{Z}$ is as in Lemma 3.25 (e). Now we explain what this means and why it holds.

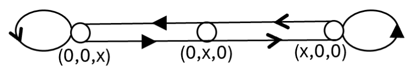
The group $G_\mathbb{Z}^\mathcal{B}$ acts transitively on the fiber over $\underline{x}$ of the map $\mathcal{B}^{dist} \to (\mathrm{Br}_3 \ltimes \{\pm 1\}^3)(\underline{x})$. By Lemma 3.22 (a) the action of this group $G_\mathbb{Z}^\mathcal{B}$ and the action of the group $\mathrm{Br}_3 \ltimes \{\pm 1\}^3$ on $\mathcal{B}^{dist}$ commute, so that $G_\mathbb{Z}^\mathcal{B}$ acts transitively on each fiber of the map $\mathcal{B}^{dist} \to (\mathrm{Br}_3 \ltimes \{\pm 1\}^3)(\underline{x})$. Therefore the group $G_\mathbb{Z}^\mathcal{B}/Z((\{\pm 1\}^3)_{\underline{x}})$ acts simply transitively on each fiber of the covering $\mathcal{G}_\sigma(\mathcal{B}^{dist}/\{\pm 1\}^3) \to \mathcal{G}_\sigma(\mathrm{Br}_3(\underline{x}/\{\pm 1\}^3))$ and is a group of automorphisms of the σ-pseudo-graph $\mathcal{G}_\sigma(\mathcal{B}^{dist}/\{\pm 1\}^3)$. The quotient by this group is the σ-pseudo-graph $\mathcal{G}_\sigma(\mathrm{Br}_3(\underline{x}/\{\pm 1\}^3))$. These statements are the meaning of the *normal covering* $\mathcal{G}_\sigma(\mathcal{B}^{dist}/\{\pm 1\}^3) \to \mathcal{G}_\sigma(\mathrm{Br}_3(\underline{x}/\{\pm 1\}^3))$.

(iv) In part (iii) the σ-pseudo-graph $\mathcal{G}_\sigma(\mathcal{B}^{dist}/\{\pm 1\}^3)$ contains no loops, and for any $v \in \mathcal{B}^{dist}/\{\pm 1\}^3$ $\sigma_1(v) \neq \sigma_2(v)$. The σ-pseudo-graph

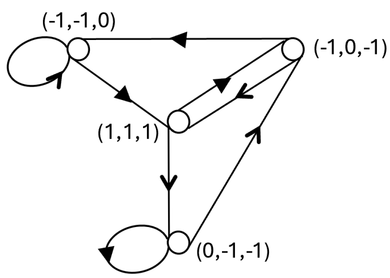


$\underline{x} \in C_1 \cup C_2 = \{(0,0,0),(2,2,2)\},\ A_1^3, \mathcal{H}_{1,2}$

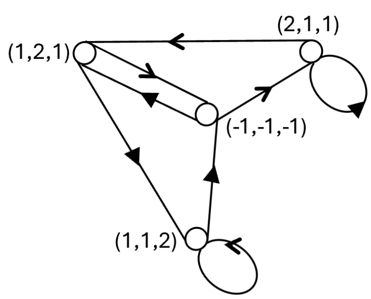




Two equivalent pictures for the cases
$\underline{x} = (x,0,0) \in C_3 \cup C_4 \cup C_5 = \{(\widetilde{x},0,0) \,|\, \widetilde{x} < 0\}$
($A_2A_1, \mathbb{P}^1A_1$, other reducible cases without A_1^3)



$\underline{x} = (-1,0,-1) \in C_6,\ A_3$



$\underline{x} = (-1,-1,-1) \in C_7,\ \widehat{A}_2$

FIGURE 7.1. Examples 7.16 (i): The σ-pseudo-graphs for the finite Br_3 orbits in $\mathbb{Z}^3/\{\pm 1\}^3$

$\mathcal{G}_\sigma(\mathrm{Br}_3(\underline{x}/\{\pm 1\}^3))$ contains loops in a few cases. It contains a vertex v with $\sigma_1(v) = \sigma_2(v)$ only in the cases $\underline{x} = (0,0,0)$ (A_1^3) and $\underline{x} = (2,2,2)$ ($\mathcal{H}_{1,2}$) where $\mathrm{Br}_3(\underline{x}/\{\pm 1\}^3)$ has only one vertex anyway.

EXAMPLES 7.16. (i) By Theorem 4.13 (a) $\mathbb{Z}^3/\{\pm 1\}^3$ consists of the Br_3 orbits $\mathrm{Br}_3(\underline{x}/\{\pm 1\}^3)$ for $\underline{x} \in \bigcup_{i=1}^{24} C_i$. Precisely for $\underline{x} \in \bigcup_{i=1}^{7} C_i$ such an orbit is finite. This led to the four pseudo-graphs $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$ in the Examples 4.11. The four corresponding σ-pseudo-graphs $\mathcal{G}_\sigma(\mathrm{Br}_3(\underline{x}/\{\pm 1\}^3))$ are listed in Figure 7.1. A vertex $\widetilde{\underline{x}}/\{\pm 1\}^3 \in \mathbb{Z}^3/\{\pm 1\}^3$ is denoted by a representative $\widetilde{\underline{x}} \in \mathbb{Z}^3_{>0} \cup \mathbb{Z}^3_{\leq 0}$. The vertices are positioned at the same places as in the pictures in the Examples 4.11 for $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$.

(ii) The case $\mathcal{H}_{1,2}$, $\underline{x} = (2,2,2)$: Here $\mathrm{Br}_3(\underline{x}/\{\pm 1\}^3) = \{\underline{x}/\{\pm 1\}^3\}$ has only one vertex, but the group

$$G_{\mathbb{Z}}^{\mathcal{B}}/Z((\{\pm 1\}^3)_{\underline{x}}) = G_{\mathbb{Z}}/\{\pm \mathrm{id}\} \cong SL_2(\mathbb{Z})$$

is big. There is a natural bijection $\mathcal{B}^{dist}/\{\pm 1\}^3 \to SL_2(\mathbb{Z})$, and the elementary braids σ_1 and σ_2 act by multiplication from the left with the matrices $A_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ on $SL_2(\mathbb{Z})$. This gives a clear description of the σ-pseudo-graph $\mathcal{G}_\sigma(\mathcal{B}^{dist}/\{\pm 1\}^3)$. We do not attempt a picture.

(iii) The reducible case A_1^3, $\underline{x} = (0,0,0)$: Also here $\mathrm{Br}_3(\underline{x}/\{\pm 1\}^3) = \{\underline{x}/\{\pm 1\}^3\}$ has only one vertex. The group

$$G_{\mathbb{Z}}^{\mathcal{B}}/Z((\{\pm 1\}^3)_{\underline{x}}) = G_{\mathbb{Z}}/\{\pm 1\}^3 \cong O_3(\mathbb{Z})/\{\pm 1\}^3 \cong S_3$$

has six elements. Therefore $\mathcal{B}^{dist}/\{\pm 1\}^3$ has six elements, and σ_1 and σ_2 act as involutions. The right hand side of the first line in Figure 7.3 gives the σ-pseudo-graph $\mathcal{G}_\sigma(\mathcal{B}^{dist}/\{\pm 1\}^3)$. Part (iv) offers a different description which applies also to A_1^3 if one sees it as $A_1^2 A_1$.

(iv) The reducible cases, $\underline{x} \in \bigcup_{i\in\{3,4,5\}} C_i$ (A_2A_1, $\mathbb{P}^1 A_1$, other reducible cases): Here $(H_{\mathbb{Z}}, L, \underline{e}) = (H_{\mathbb{Z},1}, L_1, (e_1, e_2)) \oplus (H_{\mathbb{Z},2}, L_2, e_3)$ with $H_{\mathbb{Z},1} = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ and $H_{\mathbb{Z},2} = \mathbb{Z}e_3$.

The group of deck transformations of the normal covering $\mathcal{G}_\sigma(\mathcal{B}^{dist}/\{\pm 1\}^3) \to \mathcal{G}_\sigma(\mathrm{Br}_3(\underline{x}/\{\pm 1\}^3))$ is

$$G_{\mathbb{Z}}^{\mathcal{B}}/Z((\{\pm 1\}^3)_{\underline{x}}) = G_{\mathbb{Z}}/\{\pm \mathrm{id}, \pm Q\} \cong \{(M^{root})^l \mid l \in \mathbb{Z}\}.$$

Here M^{root} has order 3 in the case A_2A_1 and infinite order in the other cases. Therefore the σ-pseudo-graph $\mathcal{G}_\sigma(\mathcal{B}^{dist}/\{\pm 1\}^3)$ can be obtained by a triple or infinite covering of the σ-pseudo-graph $\mathcal{G}_\sigma(\mathrm{Br}_3(\underline{x}/\{\pm 1\}^3)$.

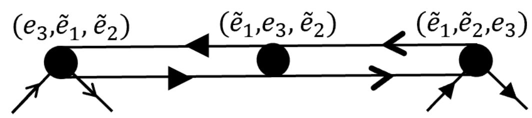


FIGURE 7.2. In the reducible cases (without A_1^3) one sheet of the covering $\mathcal{G}_\sigma(\mathcal{B}^{dist}/\{\pm 1\}^3) \to \mathcal{G}_\sigma(\mathrm{Br}_3(\underline{x}/\{\pm 1\}^3))$

More concretely, the type of the covering is determined by the Br_1 orbit of distinguished bases up to signs of $(H_{\mathbb{Z},1}, L_1, (e_1, e_2))$. One such distinguished basis modulo signs $(\widetilde{e}_1, \widetilde{e}_2)/\{\pm 1\}^2$ gives rise to one sheet in the covering $\mathcal{G}_\sigma(\mathcal{B}^{dist}/\{\pm 1\}^3) \to \mathcal{G}_\sigma(\mathrm{Br}_3(\underline{x}/\{\pm 1\}^3))$. Figure 7.2 shows the part of a σ-pseudo-graph which corresponds to one such sheet.

The six pictures in Figure 7.3 show on the left hand side analogous σ_1-pseudo-graphs for the distinguished bases modulo signs of the rank 2 cases A_1^2, A_2 and $\mathbb{P}^1$, and on the right hand side the σ-pseudo-graphs $\mathcal{G}_\sigma(\mathcal{B}^{dist}/\{\pm 1\}^3)$ for A_1^3, A_2A_1 and $\mathbb{P}^1A_1$ (respectively only a part of the σ-pseudo-graph in the case of $\mathbb{P}^1A_1$). The σ-pseudo-graph $\mathcal{G}_\sigma(\mathcal{B}^{dist}/\{\pm 1\}^3)$ for $\underline{x} = (x, 0, 0)$ with $x < -2$ looks the same as the one for $\mathbb{P}^1A_1$, though of course the distinguished bases are different.

(v) The case A_3, $\underline{x} = (-1, 0, -1)$: The σ-pseudo-graph $\mathcal{G}_\sigma(\mathcal{B}^{dist}/\{\pm 1\}^3)$ was first given in [**Yu90**, page 40, Figur 6]. We recall and explain it in our words. The group of deck transformations of the normal covering $\mathcal{G}_\sigma(\mathcal{B}^{dist}/\{\pm 1\}^3) \to \mathcal{G}_\sigma(\mathrm{Br}_3(\underline{x}/\{\pm 1\}^3))$ is

$$G_{\mathbb{Z}}^{\mathcal{B}})/Z((\{\pm 1\}^3)_{\underline{x}}) = G_{\mathbb{Z}}/\{\pm \mathrm{id}\} \cong \{M^l \mid l \in \{0, 1, 2, 3\}\}.$$

Here the monodromy M acts in the natural way,

$$M((\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3)/\{\pm 1\}^3) = (M(\widetilde{e}_1), M(\widetilde{e}_2), M(\widetilde{e}_3))/\{\pm 1\}^3),$$

on $\mathcal{B}^{dist}/\{\pm 1\}^3$ and has order four, $M^4 = \mathrm{id}$. Here M and its powers are

$$M(\underline{e}) = \underline{e}\begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}, M^2(\underline{e}) = \underline{e}\begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix},$$

$$M^3(\underline{e}) = \underline{e}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}.$$

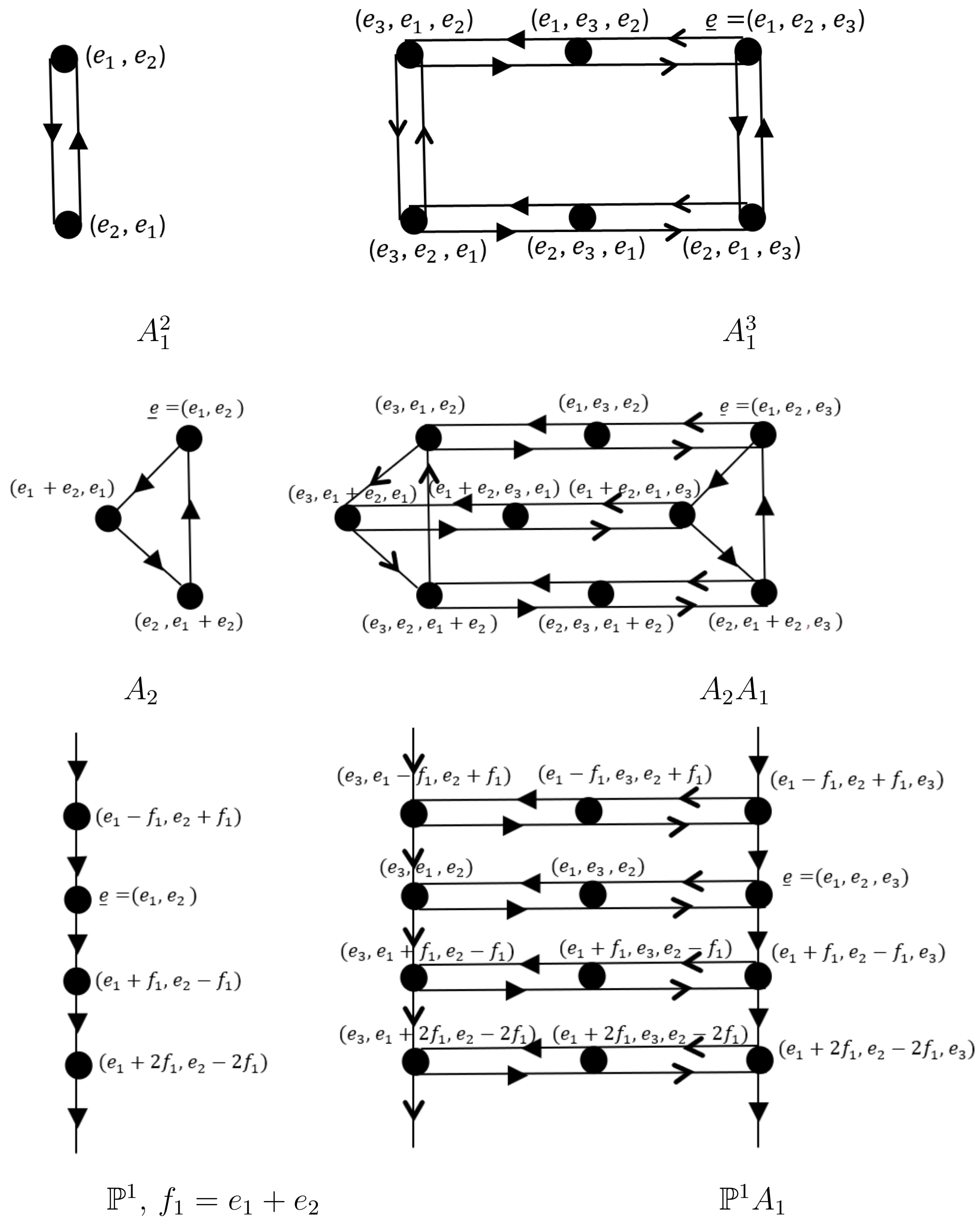


FIGURE 7.3. Examples 7.16 (iii): The σ-pseudo-graphs for distinguished bases modulo signs in the reducible cases

Because of the shape of $\mathcal{G}_\sigma(\mathrm{Br}_3(\underline{x}/\{\pm 1\}^3))$,

$$b^{1,0} := \underline{e}/\{\pm 1\}^3, \quad b^{2,0} := \sigma_1 b^{1,0}, \quad b^{3,0} := \sigma_1^2 b^{1,0}, \quad b^{4,0} := \sigma_2^{-1} b^{1,0}$$

form one sheet of the fourfold cyclic covering $\mathcal{G}_\sigma(\mathcal{B}^{dist}/\{\pm 1\}^3) \to \mathcal{G}_\sigma(\mathrm{Br}_3(\underline{x}/\{\pm 1\}^3))$. Define

$$b^{i,l} := M^l(b^{i,0}) \quad \text{for} \quad l \in \{1,2,3\}.$$

Then

$$\mathcal{B}^{dist}/\{\pm 1\}^3 = \{b^{i,l} \,|\, i \in \{1,2,3,4\}, l \in \{0,1,2,3\}\}$$

has sixteen elements. We claim for $l \in \{0,1,2,3\}$

$$\begin{array}{ll} \sigma_1 b^{1,l} = b^{2,l}, & \sigma_2 b^{1,l} = b^{3,l+3(\mathrm{mod}\,4)}, \\ \sigma_1 b^{2,l} = b^{3,l}, & \sigma_2 b^{2,l} = b^{2,l+2(\mathrm{mod}\,4)}, \\ \sigma_1 b^{3,l} = b^{1,l}, & \sigma_2 b^{3,l} = b^{4,l+1(\mathrm{mod}\,4)}, \\ \sigma_1 b^{4,l} = b^{4,l+2(\mathrm{mod}\,4)}, & \sigma_2 b^{4,l} = b^{1,l}. \end{array}$$

It is sufficient to prove the claim for $l = 0$. The equations $\sigma_1 b^{1,0} = b^{2,0}$, $\sigma_1 b^{2,0} = b^{3,0}$, $\sigma_2 b^{4,0} = b^{1,0}$ follow from the definitions of $b^{2,0}$, $b^{3,0}$, $b^{4,0}$. The inclusion $\sigma_1^3 \in (\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3})$ gives $\sigma_1 b^{3,0} = b^{1,0}$. It remains to show

$$\sigma_1 b^{4,0} = b^{4,2}, \sigma_2 b^{1,0} = b^{3,3},\ \sigma_2 b^{2,0} = b^{2,2},\ \sigma_2 b^{3,0} = b^{4,1}.$$

One sees

i	$b^{i,0}$	$\underline{\widetilde{e}}$	$\underline{\widetilde{x}}$ with $L(\underline{\widetilde{e}}^t, \underline{\widetilde{e}}) = S(\underline{\widetilde{x}})$
1	$\underline{e}/\{\pm\}^3$	$\underline{e}$	$(-1,0,-1)$
2	$\sigma_1(\underline{e})/\{\pm 1\}^3$	$\sigma_1(\underline{e}) = (e_1+e_2, e_1, e_3)$	$(1,-1,0)$
3	$\sigma_1^2(\underline{e})/\{\pm 1\}^3$	$\sigma_1^2(\underline{e}) = (-e_2, e_1+e_2, e_3)$	$(-1,1,-1)$
4	$\sigma_2^{-1}(\underline{e})/\{\pm 1\}^3$	$\sigma_2^{-1}(\underline{e}) = (e_1, e_3, e_2+e_3)$	$(0,-1,1)$

$$\begin{array}{rcl} \sigma_1 b^{4,0} & = & (e_3, e_1, e_2+e_3)/\{\pm 1\}^3 = M^2 b^{4,0} = b^{4,2}, \\ \sigma_2 b^{1,0} & = & (e_1, e_2+e_3, e_2)/\{\pm 1\}^3 = M^3 b^{3,0} = b^{3,3}, \\ \sigma_2 b^{2,0} & = & (e_1+e_2, e_3, e_1)/\{\pm 1\}^3 = M^2 b^{2,0} = b^{2,2}. \end{array}$$

$\sigma_2 b^{1,0} = b^{3,3}$, $\sigma_2 b^{4,0} = b^{1,0}$ and $\sigma_2^3 \in (\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3}$ show $\sigma_2 b^{3,3} = b^{4,0}$. This implies $\sigma_2 b^{3,0} = b^{4,1}$. The claim is proved. The σ-pseudo-graph $\mathcal{G}_\sigma(\mathcal{B}^{dist}/\{\pm 1\}^3)$ is given in Figure 7.4.

(vi) The case $\widehat{A}_2$, $\underline{x} = (-1,-1,-1)$: The group of deck transformations of the normal covering $\mathcal{G}_\sigma(\mathcal{B}^{dist}/\{\pm 1\}^3) \to \mathcal{G}_\sigma(\mathrm{Br}_3(\underline{x}/\{\pm 1\}^3))$ is

$$G_{\mathbb{Z}}^{\mathcal{B}}/Z((\{\pm 1\}^3)_{\underline{x}}) = G_{\mathbb{Z}}/\{\pm \mathrm{id}\} \cong \{(M^{root})^l \,|\, l \in \mathbb{Z}\}.$$

Here M^{root} acts in the natural way on $\mathcal{B}^{dist}/\{\pm 1\}^3$. M^{root} has infinite order and satisfies $(M^{root})^3 = -M$. Recall $f_1 = e_1 + e_2 + e_3$, $\mathbb{Z}f_1 =$

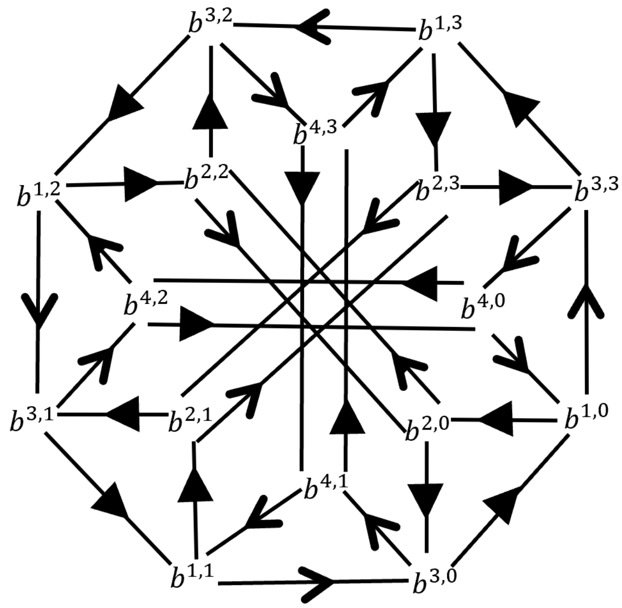


FIGURE 7.4. Example 7.16 (iv): The σ-pseudo-graph for distinguished bases modulo signs in the case A_3

$\operatorname{Rad} I^{(0)}$,

$$\begin{aligned} M^{root}(\underline{e}) &= \underline{e}\begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad M^{root}(f_1) = f_1, \\ (M^{root})^2(\underline{e}) &= \underline{e} + f_1(1,0,-1). \end{aligned}$$

Because of the shape of $\mathcal{G}_\sigma(\mathrm{Br}_3(\underline{x}/\{\pm 1\}^3))$,

$$b^{1,0} := \underline{e}/\{\pm 1\}^3, \quad b^{2,0} := \sigma_1 b^{1,0}, \quad b^{3,0} := \sigma_1^2 b^{1,0}, \quad b^{4,0} := \sigma_2 b^{1,0}$$

form one sheet of the infinite cyclic covering $\mathcal{G}_\sigma(\mathcal{B}^{dist}/\{\pm 1\}^3) \to \mathcal{G}_\sigma(\mathrm{Br}_3(\underline{x}/\{\pm 1\}^3))$. Define

$$b^{i,l} := (M^{root})^l(b^{i,0}) \quad \text{for} \quad i \in \{1,2,3,4\},\ l \in \mathbb{Z} - \{0\}.$$

Then

$$\mathcal{B}^{dist}/\{\pm 1\}^3 = \{b^{i,l} \mid i \in \{1,2,3,4\}, l \in \mathbb{Z}\}.$$

We claim for $l \in \mathbb{Z}$

$$\begin{array}{ll} \sigma_1 b^{1,l} = b^{2,l}, & \sigma_2 b^{1,l} = b^{4,l}, \\ \sigma_1 b^{2,l} = b^{3,l}, & \sigma_2 b^{2,l} = b^{1,l+1}, \\ \sigma_1 b^{3,l} = b^{1,l}, & \sigma_2 b^{3,l} = b^{3,l+2}, \\ \sigma_1 b^{4,l} = b^{4,l+2}, & \sigma_2 b^{4,l} = b^{2,l-1}. \end{array}$$

It is sufficient to prove the claim for $l = 0$. The equations $\sigma_1 b^{1,0} = b^{2,0}$, $\sigma_1 b^{2,0} = b^{3,0}$, $\sigma_2 b^{1,0} = b^{4,0}$ follow from the definitions of $b^{2,0}$, $b^{3,0}$, $b^{4,0}$. The inclusion $\sigma_1^3 \in (\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3}$ gives $\sigma_1 b^{3,0} = b^{1,0}$. It remains to show

$$\sigma_1 b^{4,0} = b^{4,2}, \sigma_2 b^{2,0} = b^{1,1},\ \sigma_2 b^{3,0} = b^{3,2},\ \sigma_2 b^{4,0} = b^{2,-1}.$$

One sees

i	$b^{i,0}$	$\underline{\widetilde{e}}$	$\underline{\widetilde{x}}$ with $L(\underline{\widetilde{e}}^t, \underline{\widetilde{e}}) = S(\underline{\widetilde{x}})$
1	$\underline{e}/\{\pm\}^3$	$\underline{e}$	$(-1,-1,-1)$
2	$\sigma_1(\underline{e})/\{\pm 1\}^3$	$\sigma_1(\underline{e}) = (e_1+e_2, e_1, e_3)$	$(1,-2,-1)$
3	$\sigma_1^2(\underline{e})/\{\pm 1\}^3$	$\sigma_1^2(\underline{e}) = (-e_2, e_1+e_2, e_3)$	$(-1,1,-2)$
4	$\sigma_2(\underline{e})/\{\pm 1\}^3$	$\sigma_2(\underline{e}) = (e_1, e_2+e_3, e_2)$	$(-2,-1,1)$

$$\begin{aligned} \sigma_1 b^{4,0} &= (2e_1+e_2+e_3, e_1, e_2)/\{\pm 1\}^3 \\ &= (e_1+f_1, e_2+e_3-f_1, e_2)/\{\pm 1\}^3 = (M^{root})^2 b^{4,0} = b^{4,2}, \\ \sigma_2 b^{2,0} &= (e_1+e_2, e_1+e_3, e_1)/\{\pm 1\}^3 = M^{root} b^{1,0} = b^{1,1}, \\ \sigma_2 b^{3,0} &= (-e_2, 2e_1+2e_2+e_3, e_1+e_2)/\{\pm 1\}^3 \\ &= (-e_2, e_1+e_2+f_1, -e_3+f_1)/\{\pm 1\}^3 = M^2 b^{3,0} = b^{3,2}. \end{aligned}$$

$\sigma_2 b^{2,0} = b^{1,1}$ implies $\sigma_2 b^{2,-1} = b^{1,0}$. This, $\sigma_2 b^{1,0} = b^{4,0}$ and $\sigma_2^3 \in (\mathrm{Br}_3)_{\underline{e}/\{\pm 1\}^3}$ show $\sigma_2 b^{4,0} = b^{2,-1}$. The claim is proved. A part of the σ-pseudo-graph $\mathcal{G}_\sigma(\mathcal{B}^{dist}/\{\pm 1\}^3)$ is given in Figure 7.5.

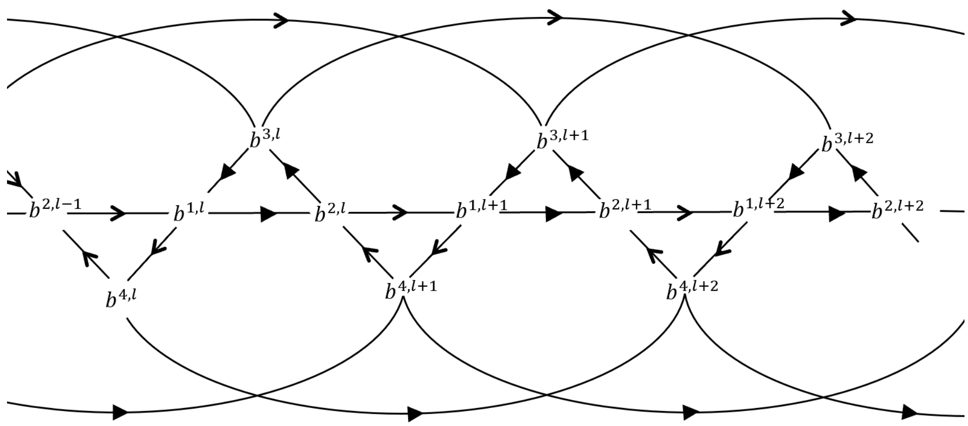


FIGURE 7.5. Example 7.16 (v): A part of the σ-pseudograph for distinguished bases modulo signs in the case $\widehat{A}_2$

APPENDIX A

Tools from hyperbolic geometry

The upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$ together with its natural metric (whose explicit form we will not need) is one model of the hyperbolic plane. In the study of the monodromy groups $\Gamma^{(0)}$ and $\Gamma^{(1)}$ for the cases with $n = 2$ or $n = 3$, we will often encounter subgroups of $\text{Isom}(\mathbb{H})$. The theorem of Poincaré-Maskit [**Po82**][**Ma71**] allows under some conditions to show for such a group that it is discrete, to find a fundamental domain and to find a presentation. Three special cases of this theorem, which will be sufficient for us, are formulated in Theorem A.2.

Before, we collect basic facts and set up some notations in the following Remarks and Notations A.1.

Subgroups of $\text{Isom}(\mathbb{H})$ arise here in two ways. Either they come from groups of real 2×2 matrices. This is covered by the Remarks and Notations A.1 (v). Or they come from the action of certain groups of real 3×3 matrices on $\mathbb{R}^3$ with an indefinite metric. This will treated in Theorem A.4.

Remarks and Notations A.1. (Some references for the following material are [**Fo51**][**Le66**][**Be83**])

(i) Let $n \in \mathbb{N}$. Recall the notions of the free group $G^{free,n}$ with n generators and of the free Coxeter group $G^{fCox,n}$ with n generators from Definition 3.1.

(ii) $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, $\widehat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$. The hyperbolic lines in $\mathbb{H}$ are the parts in $\mathbb{H}$ of circles and euclidean lines which meet $\mathbb{R}$ orthogonally. For any $z_1, z_2 \in \mathbb{H} \cup \widehat{\mathbb{R}}$ with $z_1 \neq z_2$, denote by $A(z_1, z_2)$ the part between z_1 and z_2 of the unique hyperbolic line whose closure in $\mathbb{H} \cup \widehat{\mathbb{R}}$ contains z_1 and z_2. Here $z_i \in A(z_1, z_2)$ if $z_i \in \mathbb{H}$, but $z_i \notin A(z_1, z_2)$ if $z_i \in \widehat{\mathbb{R}}$. Such sets are called *arcs*.

(iii) We simplify the definition of a polygon in [**Ma71**]. A *hyperbolic polygon* P is a contractible open subset $P \subset \mathbb{H}$ whose relative boundary in $\mathbb{H}$ consists of finitely many arcs $A_1 = A(z_{1,1}, z_{1,2}), ..., A_m = A(z_{m,1}, z_{m,2})$ (Maskit allows countably many arcs). The arcs and the points are numbered such that one runs through them in the order

$A_1, ..., A_m$ and $z_{1,1}, z_{1,2}, z_{2,1}, z_{2,2}, ..., z_{m,1}, z_{m,2}$ if one runs mathematically positive on the euclidean boundary of P in $\mathbb{H} \cup \widehat{\mathbb{R}}$. For A_i and A_{i+1} (with $A_{m+1} := A_1$) there are three possibilities:

(a) $z_{i,2} = z_{i+1,1} \in \mathbb{H}$; then this point is called a *vertex* of P;
(b) $z_{i,2} = z_{i+1,1} \in \widehat{\mathbb{R}}$;
(c) $z_{i,2} \in \widehat{\mathbb{R}}, z_{i+1,1} \in \widehat{\mathbb{R}}, z_{i,2} \neq z_{i+1,1}$; then the part of $\widehat{\mathbb{R}}$ between $z_{i,2}$ and $z_{i+1,1}$ (moving from smaller to larger values) is in the euclidean boundary of P between A_i and A_{i+1}.

In the second and third case $A_i \cap A_{i+1} = \emptyset$. A polygon has no vertices if and only if all arcs $A_1, ..., A_m$ are hyperbolic lines, and if and only if all points $z_{1,1}, ..., z_{m,2} \in \widehat{\mathbb{R}}$.

(iv) Denote

$$\begin{aligned} Gl_2^{(-1)}(\mathbb{R}) &:= \{A \in GL_2(\mathbb{R}) \mid \det A = -1\}, \\ Gl_2^{(\pm 1)}(\mathbb{R}) &:= \{A \in GL_2(\mathbb{R}) \mid \det A = \pm 1\} = SL_2(\mathbb{R}) \cup GL_2^{(-1)}(\mathbb{R}), \end{aligned}$$

and analogously $Gl_2^{(-1)}(\mathbb{Z})$, $Gl_2^{(\pm 1)}(\mathbb{Z})$. Recall

$$A^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A = \det A \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{for } A \in Gl_2^{(\pm 1)}(\mathbb{R}).$$

(v) The following map $\mu : Gl_2^{(\pm 1)}(\mathbb{R}) \to \text{Isom}(\mathbb{H})$ is a surjective group homomorphism with kernel $\ker \mu = \{\pm E_2\}$,

$$\begin{aligned} \mu(A) &= \Big(z \mapsto \frac{za+b}{cz+d}\Big) \qquad \text{if } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}), \\ \mu(A) &= \Big(z \mapsto \frac{\overline{z}a+b}{c\overline{z}+d}\Big) \qquad \text{if } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^{(-1)}(\mathbb{R}). \end{aligned}$$

$\mu(A)$ for $A \in SL_2(\mathbb{R})$ is orientation preserving and is called a *Möbius transformation*. $\mu(A)$ for $A \in GL_2^{(-1)}(\mathbb{R})$ is orientation reversing.

If $A \in SL_2(\mathbb{R}) - \{\pm E_2\}$, there are three possibilities:

(a) $|\operatorname{tr}(A)| < 2$; then A has a fixed point in $\mathbb{H}$ (and the complex conjugate number is a fixed point in $-\mathbb{H}$) and is called *elliptic*.

(b) $|\operatorname{tr}(A)| = 2$; then A has one fixed point in $\widehat{\mathbb{R}}$ and is called *parabolic*.
(c) $|\operatorname{tr}(A)| > 2$; then A has two fixed points in $\widehat{\mathbb{R}}$ and is called *hyperbolic*.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^{(-1)}(\mathbb{R})$ with $\mathrm{tr}(A) = 0$ then $\mu(A)$ is a reflection along the hyperbolic line

$$\{z \in \mathbb{H} \,|\, z = \mu(A)(z)\} = \{z \in \mathbb{H} \,|\, 0 = cz\overline{z} - 2a\,\mathrm{Re}(z) - b\}$$
$$\Big(= \{z \in \mathbb{H} \,|\, 0 = (z - \frac{a}{c})(\overline{z} - \frac{a}{c}) - \frac{1}{c^2}\} \quad \text{if } c \neq 0\Big).$$

The theorem of Poincaré-Maskit starts with a hyperbolic polygon P whose relative boundary in $\mathbb{H}$ consists of arcs $A_1, ..., A_m$, with an involution $\sigma \in S_m$ and with elements $g_1, ..., g_m \in \mathrm{Isom}(\mathbb{H})$ with $g_i(A_i) = A_{\sigma(i)}$ and $g_{\sigma(i)} = g_i^{-1}$. Under some additional conditions, it states that the group $G := \langle g_1, ..., g_m\rangle \subset \mathrm{Isom}(\mathbb{H})$ is discrete, that P is a fundamental domain (i.e. each orbit of G in $\mathbb{H}$ meets the relative closure of P in $\mathbb{H}$, no orbit of G in $\mathbb{H}$ meets P in more than one point), and it gives a complete set of relations with respect to $g_1, ..., g_m$ of G. Poincaré [**Po82**] had the case when $\mathbb{H}/G$ is compact, Maskit [**Ma71**] generalized it greatly. In [**Ma71**] the relative boundary of P in $\mathbb{H}$ may consist of countably many arcs. The following theorem singles out three special cases, which are sufficient for us. Remark A.3 (ii) illustrates them with pictures.

THEOREM A.2. [**Ma71**] *Let $P \subset \mathbb{H}$ be a hyperbolic polygon whose relative boundary in $\mathbb{H}$ consists of arcs $A_1, ..., A_m$ with $A_i = A(z_{i,1}z_{i,2})$ where one runs through these arcs and these points in the order $A_1, ..., A_m$ and $z_{1,1}, z_{1,2}, z_{2,1}, z_{2,2}, ..., z_{m,1}, z_{m,2}$ if one runs mathematically positive on the euclidean boundary of P in $\mathbb{H} \cup \widehat{\mathbb{R}}$.*

(a) Let $I \subset \{1, ..., m\}$ be the set of indices such that $z_{i,2}$ is a vertex, so $z_{i,2} = z_{i+1,1} \in \mathbb{H}$, with $A_{m+1} := A_1$ and $z_{m+1,i} := z_{1,i}$ (I may be empty). Suppose that at a vertex $z_{i,2}$ the arcs A_i and A_{i+1} meet at an angle $\frac{\pi}{n_i}$ for some number $n_i \in \mathbb{Z}_{\geq 2}$. For $i \in \{1, ..., m\}$ let $g_i \in \mathrm{Isom}(\mathbb{H})$ be the reflection along the hyperbolic line which contains A_i.

The group $G := \langle g_1, ..., g_m\rangle \subset \mathrm{Isom}(\mathbb{H})$ is discrete, P is a fundamental domain, and the set of relations

$$g_1^2 = ... = g_m^2 = \mathrm{id}, \quad (g_i g_{i+1})^{n_i} = \mathrm{id} \quad \text{for } i \in I,$$

form a complete set of relations. Especially, if $I = \emptyset$ then G is a free Coxeter group with generators $g_1, ..., g_m$.

(b) Let $P \subset \mathbb{H}$ have no vertices. Choose on each hyperbolic line A_i a point p_i, and let g_i be the elliptic element with fixed point p_i and rotation angle π.

The group $G := \langle g_1, ..., g_m\rangle \subset \mathrm{Isom}(\mathbb{H})$ is discrete, P is a fundamental domain, and the set of relations $g_1^2 = ... = g_m^2 = \mathrm{id}$ a complete set of relation, so G is a free Coxeter group with generators $g_1, ..., g_m$.

(c) Let $P \subset \mathbb{H}$ have no vertices. Suppose that m is even. Suppose $z_{2i-1,2} = z_{2i,1}$ for $i \in \{1, 2, ..., \frac{m}{2}\}$. Let g_i for $i \in \{1, 2, ..., \frac{m}{2}\}$ be the parabolic element with fixed point $z_{2i-1,2}$ which maps A_{2i-1} to A_{2i}.

The group $G := \langle g_1, ..., g_m \rangle \subset$ Isom($\mathbb{H}$) is discrete, P is a fundamental domain, and the group G is a free group with generators $g_1, ..., g_m$.

Remarks A.3. (i) The Cayley transformation $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, $\left(z \mapsto \frac{z-i}{z+i}\right)$ maps the upper half plane $\mathbb{H}$ to the unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. It leads to the unit disk model of the hyperbolic plane. The hyperbolic lines in this model are the parts in $\mathbb{D}$ of circles and euclidean lines which intersect $\partial\mathbb{D}$ orthogonally.

(ii) The following three pictures illustrate Theorem A.2 in the unit disk model instead of the upper half plane model.

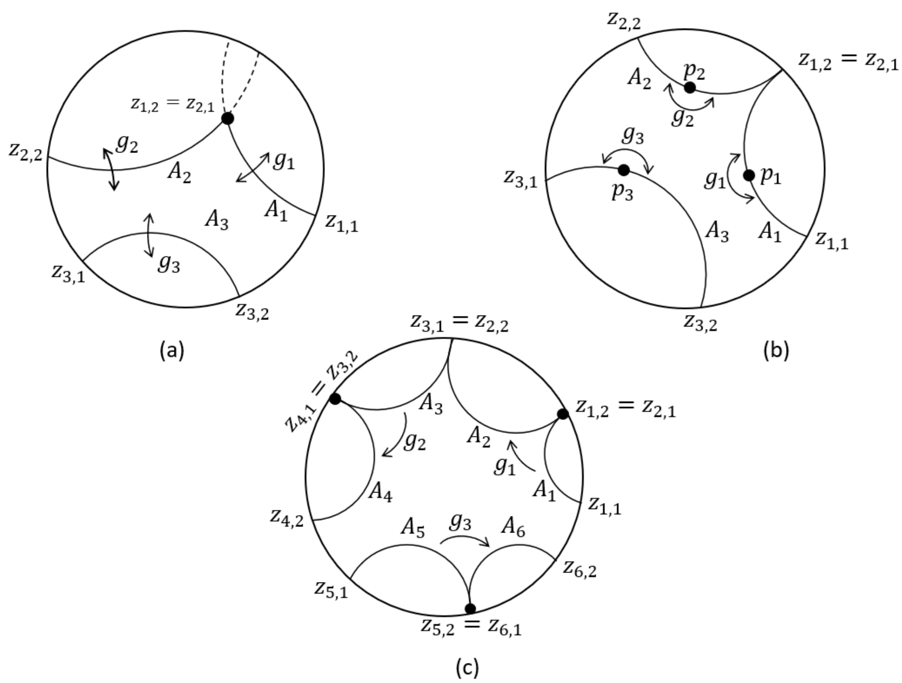


Figure A.1. Three pictures for Theorem A.2

The surjective group homomorphism $\mu : Gl_2^{(\pm 1)}(\mathbb{R}) \to$ Isom($\mathbb{H}$) in Remark A.1 (v) shows how to go from groups of real 2×2 matrices to subgroups of Isom($\mathbb{H}$). The next theorem shows how to go from groups of certain real 3×3 matrices to subgroups of Isom($\mathbb{H}$). It is classical. But as we need some details, we prefer to explain these details and not refer to some literature.

THEOREM A.4. *Let* $(H_\mathbb{R}, I^{[0]})$ *be a 3-dimensional real vector space with a symmetric bilinear form* $I^{[0]}$ *with signature* $(+--)$.

(a) (Elementary linear algebra) A vector $v \in H_\mathbb{R} - \{0\}$ *is called* positive *if* $I^{[0]}(v,v) > 0$, isotropic *if* $I^{[0]}(v,v) = 0$, negative *if* $I^{[0]}(v,v) < 0$. *The positive vectors form a (double) cone* $\mathcal{K} \subset H_\mathbb{R}$, *the isotropic vectors and the vector 0 form its boundary, the negative vectors form its complement. The orthogonal hyperplanes* $(\mathbb{R}\cdot v)^\perp$ *satisfy the following:*

(i) $(\mathbb{R}\cdot v)^\perp \cap \mathcal{K} \neq \emptyset$ *if* v *is negative.*
(ii) $(\mathbb{R}\cdot v)^\perp \cap \overline{\mathcal{K}} = \mathbb{R}\cdot v$ *if* v *is isotropic.*
(iii) $(\mathbb{R}\cdot v)^\perp \cap \overline{\mathcal{K}} = \{0\}$ *if* v *is positive.*

$\mathcal{K}/\mathbb{R}^*$ *denotes the lines in* $\mathcal{K}$, *i.e. the 1-dimensional subspaces.*

(b) (Basic properties of $\mathrm{Aut}(H_\mathbb{R}, I^{[0]})$*) Let* $\sigma : \mathrm{Aut}(H_\mathbb{R}, I^{[0]}) \to \{\pm 1\}$ *be the spinor norm map (see Remark 6.3 (iii)). The group* $\mathrm{Aut}(H_\mathbb{R}, I^{[0]})$ *is a real 3-dimensional Lie group with four components. The components are the fibers of the group homomorphism*

$$(\det, \sigma) : \mathrm{Aut}(H_\mathbb{R}, I^{[0]}) \to \{\pm 1\} \times \{\pm 1\}.$$

$-\mathrm{id} \in \mathrm{Aut}(H_\mathbb{R}, I^{[0]})$ *has value* $(\det,\sigma)(-\mathrm{id}) = (-1,1)$. *If* v *is positive then* $(\det,\sigma)(s_v^{(0)}) = (-1,1)$. *If* v *is negative then* $(\det,\sigma)(s_v^{(0)}) = (-1,-1)$. *An isometry* $g \in \mathrm{Aut}(H_\mathbb{R}, I^{[0]})$ *maps each of the two components of the cone* $\mathcal{K}$ *to itself if and only if* g *is in the two components which together form the kernel of* $\det\cdot\sigma$, *so if* $\det(g)\sigma(g) = 1$.

(c) Choose a basis $\underline{f} = (f_1, f_2, f_3)$ *of* $H_\mathbb{R}$ *with*

$$I^{[0]}(\underline{f}^t, \underline{f}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

(i) The map

$$\Theta : Gl_2^{(\pm 1)}(\mathbb{R}) \to \mathrm{Aut}(H_\mathbb{R}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\underline{f} \mapsto \underline{f} \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}\right),$$

is a group homomorphism with kernel $\ker\Theta = \{\pm E_2\}$ *and image* $\ker(\det\cdot\sigma : \mathrm{Aut}(H_\mathbb{R}, I^{[0]}) \to \{\pm 1\})$.

(ii) The map

$$\vartheta : \mathbb{H} \to (H_\mathbb{R} - \{0\})/\mathbb{R}^*, \quad z \mapsto \mathbb{R}^*(z\overline{z}f_1 + \mathrm{Re}(z)f_2 + f_3),$$

is a bijection $\vartheta : \mathbb{H} \to \mathcal{K}/\mathbb{R}^*$.

(iii) For $A \in Gl_2^{(\pm 1)}(\mathbb{R})$, *the automorphism* $\vartheta \circ \mu(A) \circ \vartheta^{-1} : \mathcal{K}/\mathbb{R}^* \to \mathcal{K}/\mathbb{R}^*$ *coincides with the action of* $\Theta(A)$ *on* $\mathcal{K}/\mathbb{R}^*$.

(iv) The natural maps

$$\begin{array}{ccccc} \mathrm{Aut}(H_{\mathbb{R}}, I^{[0]})/\{\pm \mathrm{id}\} & \longleftarrow & \ker(\det \cdot \sigma) & \longrightarrow & \mathrm{Isom}(\mathbb{H}) \\ \{\pm B\} & \longleftarrow & B\ ,\ \Theta(A) & \longmapsto & \mu(A) \end{array}$$

are group isomorphisms.

(v) For each hyperbolic line l there is a negative vector $v \in H_{\mathbb{R}}$ with $\vartheta(l) = ((\mathbb{R} \cdot v)^{\perp} \cap \mathcal{K})/\mathbb{R}^*$.

(vi) Let $\sigma_l \in \mathrm{Isom}(\mathbb{H})$ be the reflection along a hyperbolic line l, and let v be as in (v). The action of $s_v^{(0)}$ on $\mathcal{K}/\mathbb{R}^*$ coincides with $\vartheta \circ \sigma_l \circ \vartheta^{-1}$.

(vii) Let $\delta_p \in \mathrm{Isom}(\mathbb{H})$ be the elliptic element with fixed point $p \in \mathbb{H}$ and order 2 (so rotation angle π). Let $v \in H_{\mathbb{R}}$ be a positive vector with $\mathbb{R} \cdot v = \vartheta(p)$. The action of $s_v^{(0)}$ on $\mathcal{K}/\mathbb{R}^*$ coincides with $\vartheta \circ \delta_p \circ \vartheta^{-1}$.

Proof: (a) and (b) are elementary and classical, their proofs are skipped.

(c) (i) Start with a real 2-dimensional vector space $V_{\mathbb{R}}$ with basis $\underline{e} = (e_1, e_2)$ and a skew-symmetric bilinear form $I^{[1]}$ on $V_{\mathbb{R}}$ with matrix $I^{[1]}(\underline{e}^t, \underline{e}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The tensor product $V_{\mathbb{R}} \otimes V_{\mathbb{R}}$ comes equipped with an induced symmetric bilinear form $\widetilde{I}^{(0)}$ via (here I and J are finite index sets and $a_i, b_i, c_j, d_j \in V_{\mathbb{R}}$)

$$\widetilde{I}^{(0)}(\sum_{i\in I} a_i \otimes b_i, \sum_{j \in J} c_j \otimes d_j) = \sum_{i\in I} \sum_{j\in J} I^{[1]}(a_i, c_j) I^{[1]}(b_i, d_j).$$

An element $g \in \mathrm{Aut}(H_{\mathbb{R}})$ with $g\underline{e} = \underline{e}A$ and $A \in Gl_2^{(\pm 1)}(\mathbb{R})$ respects $I^{[1]}$ in the following weak sense:

$$\begin{array}{rcl} I^{[1]}(g(v_1), g(v_2)) & = & \det A \cdot I^{[1]}(v_1, v_2), \\ \text{because } A^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A & = & \det A \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{array}$$

It induces an element $\widetilde{\Theta}(g) \in \mathrm{Aut}(V_{\mathbb{R}} \otimes V_{\mathbb{R}}, \widetilde{I}^{(0)})$ via

$$\widetilde{\Theta}(g)(\sum_{i\in I} a_i \otimes b_i) = \sum_{i \in I} g(a_i) \otimes g(b_i).$$

The symmetric part $\widetilde{H}_{\mathbb{R}} \subset V_{\mathbb{R}} \otimes V_{\mathbb{R}}$ of the tensor product has the basis $\underline{\widetilde{f}} = (\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3) = (e_1 \otimes e_1, e_1 \otimes e_2 + e_2 \otimes e_1, e_2 \otimes e_2)$. One sees

$$\widetilde{I}^{(0)}(\underline{\widetilde{f}}^t, \underline{\widetilde{f}}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

From now on we identify $(\widetilde{H}_{\mathbb{R}}, \widetilde{I}^{(0)}|_{\widetilde{H}_{\mathbb{R}}}, \underline{\widetilde{f}})$ with $(H_{\mathbb{R}}, I^{[0]}, \underline{f})$.

For an element $g \in \mathrm{Aut}(V_{\mathbb{R}})$ with $g\underline{e} = \underline{e}A$ with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Gl_2^{(\pm 1)}(\mathbb{R})$, the automorphism $\widetilde{\Theta}(g)$ on $V_{\mathbb{R}} \otimes V_{\mathbb{R}}$ restricts to an automorphism of the symmetric part $H_{\mathbb{R}}$ with matrix

$$\widetilde{\Theta}(g)\underline{f} = \underline{f} \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}.$$

This fits to Θ. It shows especially $\Theta(A) \in \mathrm{Aut}(H_{\mathbb{R}}, I^{[0]})$.

The kernel of Θ is $\{\pm E_2\}$. The Lie groups $Gl_2^{(\pm 1)}(\mathbb{R})$ and $\mathrm{Aut}(H_{\mathbb{R}}, I^{[0]})$ are real 3-dimensional. The Lie group $Gl_2^{(\pm 1)}(\mathbb{R})$ has two components. $(\det, \sigma)(\Theta(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})) = (\det, \sigma)(s_{f_2}^{(0)}) = (-1, -1)$. Therefore the image of Θ consists of the two components of $\mathrm{Aut}(H_{\mathbb{R}}, I^{[0]})$ which together form the kernel of $\det \cdot \sigma$. This finishes the proof of part (i).

(ii) Define $\widetilde{\vartheta}(z) := z\overline{z}f_1 + \mathrm{Re}(z)f_2 + f_3$. It is a positive vector because

$$I^{[0]}(\widetilde{\vartheta}(z), \widetilde{\vartheta}(z)) = z\overline{z} \cdot 1 - 2(\mathrm{Re}(z))^2 + 1 \cdot z\overline{z} = 2(\Im(z))^2 > 0.$$

It is easy to see that ϑ is a bijection from $\mathbb{H}$ to $\mathcal{K}/\mathbb{R}^*$.

(iii) In fact, $\widetilde{\vartheta}(z)$ is the symmetric part of

$$(ze_1 + e_2) \otimes (\overline{z}e_1 + e_2) = (\underline{e}\begin{pmatrix} z \\ 1 \end{pmatrix}) \otimes (\underline{e}\begin{pmatrix} \overline{z} \\ 1 \end{pmatrix}) \in V_{\mathbb{C}} \otimes V_{\mathbb{C}}.$$

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Gl_2^{(\pm 1)}(\mathbb{R})$

$$\begin{aligned}
&\widetilde{\vartheta}(\mu(A)(z)) \\
= \ & \Big(\text{symmetric part of } (\underline{e}\begin{pmatrix} \mu(A)(z) \\ 1 \end{pmatrix}) \otimes (\underline{e}\begin{pmatrix} \mu(A)(\overline{z}) \\ 1 \end{pmatrix}\Big) \\
= \ & |cz+d|^{-2}\Big(\text{symmetric part of } (\underline{e}A\begin{pmatrix} z \\ 1 \end{pmatrix}) \otimes (\underline{e}A\begin{pmatrix} \overline{z} \\ 1 \end{pmatrix})\Big) \\
= \ & |cz+d|^{-2}\underline{f}\begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}\begin{pmatrix} z\overline{z} \\ \mathrm{Re}(z) \\ 1 \end{pmatrix} \\
= \ & |cz+d|^{-2}\Theta(A)(\underline{f})\begin{pmatrix} z\overline{z} \\ \mathrm{Re}(z) \\ 1 \end{pmatrix} \\
= \ & |cz+d|^{-2}\Theta(A)(\widetilde{\vartheta}(z)).
\end{aligned}$$

This shows part (iii). Part (iv) follows from part (iii).

(v) A hyperbolic line l is the fixed point set of a reflection $\mu(A)$ for a matrix $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in Gl_2^{(-1)}(\mathbb{R})$, so

$$l \;=\; \{z \in \mathbb{H} \,|\, z = \mu(A)(z)\} = \{z \in \mathcal{H} \,|\, 0 = cz\overline{z} - 2a\,\mathrm{Re}(z) - b\}.$$

Observe

$$cz\overline{z} - 2a\,\mathrm{Re}(z) - b = I^{[0]}(\underline{f}\begin{pmatrix} -b \\ a \\ c \end{pmatrix}, \underline{f}\begin{pmatrix} z\overline{z} \\ \mathrm{Re}(z) \\ 1 \end{pmatrix}).$$

Therefore

$$\vartheta(l) \;=\; \Big(\mathbb{R}(-bf_1 + af_2 + cf_3)^{\perp} \cap \mathcal{K}\Big)/\mathbb{R}^*.$$

(vi) and (vii) are clear now. □

Remarks A.5. (i) In the model $\mathcal{K}/\mathbb{R}^*$ of the hyperbolic plane, the isometries of $\mathbb{H}$ are transformed to linear isometries of $(H_{\mathbb{R}}, I^{[0]})$. The hyperbolic lines in $\mathbb{H}$ are transformed to linear hyperplanes in $H_{\mathbb{R}}$ (modulo $\mathbb{R}^*$) which intersect $\mathcal{K}$.

(ii) If one chooses an affine hyperplane in $H_{\mathbb{R}}$ which intersects one component of $\mathcal{K}$ in a disk, this disk gives a new disk model of the hyperbolic plane, which is not conformal to $\mathbb{H}$ and $\mathbb{D}$ (angles are not preserved), but where the hyperbolic lines in $\mathbb{H}$ are transformed to the segments in the new disk of euclidean lines in the affine hyperplane which intersect the disk. The following picture sketches three hyperbolic lines in the unit disk $\mathbb{D}$ and in the new disk which is part of the cone $\mathcal{K}$.

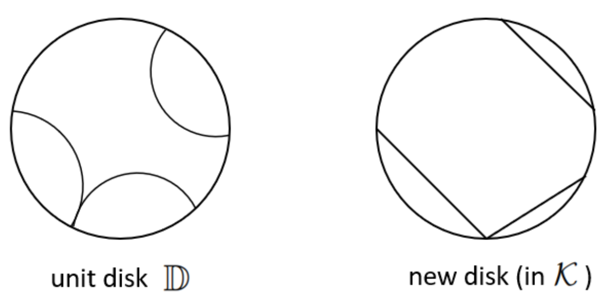


Figure A.2. Two disk models of the hyperbolic plane

APPENDIX B

Quadratic units via continued fractions

The purpose of this appendix is to prove Lemma B.1 with two statements on the units in certain rings of algebraic integers. The proof of Lemma B.1 will be given after the proof of Theorem B.6.

A convenient and very classical tool to prove this lemma is the theory of continued fractions as best approximations of irrational numbers, applied to the case of quadratic irrationals which are algebraic integers.

Theorem B.4 below cites standard results on the continued fractions of real irrationals. It is prepared by the Definitions B.2 and B.3

Lemma B.5 provides the less well known formulas (B.2) and (B.3) for the case of a quadratic irrational. Theorem B.6 describes the unit group $\mathbb{Z}[\alpha]^*$ where α is a quadratic irrational and an algebraic integer, in terms of the continued fractions of α.

LEMMA B.1. *(a) Let $x\in\mathbb{Z}_{\geq 3}$, and let $\kappa_{1/2}:=\frac{x}{2}\pm\frac{1}{2}\sqrt{x^2-4}$ be the zeros of the polynomial t^2-xt+1, so $\kappa_1+\kappa_2=x, \kappa_1\kappa_2=1, \kappa_1^2=x\kappa_1-1$. Then*

$$\mathbb{Z}[\kappa_1]^*=\begin{cases}\{\pm\kappa_1^l\,|\,l\in\mathbb{Z}\} & \text{if } x\in\mathbb{Z}_{\geq 4},\\ \{\pm(\kappa_1-1)^l\,|\,l\in\mathbb{Z}\} & \text{if } x=3.\end{cases}$$

κ_1 has norm 1. If $x=3$ then $(\kappa_1-1)^2=\kappa_1$, and κ_1-1 has norm -1.

(b) Let $x\in\mathbb{Z}_{\geq 2}$, and let $\lambda_{1/2}=x^2-1\pm x\sqrt{x^2-2}$ be the zeros of the polynomial $t^2-(2x^2-2)t+1$, so $\lambda_1+\lambda_2=2x^2-2, \lambda_1\lambda_2=1, \lambda_1^2=(2x^2-2)\lambda_1-1$. Then

$$\mathbb{Z}[\sqrt{x^2-2}]^*=\begin{cases}\{\pm\lambda_1^l\,|\,l\in\mathbb{Z}\} & \text{if } x\geq 3,\\ \{\pm(1+\sqrt{2})^l\,|\,l\in\mathbb{Z}\} & \text{if } x=2.\end{cases}$$

λ_1 has norm 1. If $x=2$ then $(1+\sqrt{2})^2=\lambda_1$, and $1+\sqrt{2}$ has norm -1.

Theorem B.4 is mainly taken from several theorems in [**Ai13**, 1.2 and 1.3], but with part (b) from [**Ca65**, I 2.]. It is preceded by two definitions. According to [**Bu00**, 5.9 Lagrange's Theorem], this part (b) is orginally due to Lagrange 1770. In fact, we will not use this part (b), but we find it enlightening.

Definition B.2. Let $\theta \in \mathbb{R} - \mathbb{Q}$ be an irrational number.

(a) Define recursively sequences $(a_n)_{n\geq 0}$, $(\theta_n)_{n\geq 0}$, $(p_n)_{n\geq -1}, (q_n)_{n\geq -1}$, $(r_n)_{n\geq 0}$ as follows:

$$\begin{array}{rcl} \theta_0 & := & \theta, \\ a_0 & := & \lfloor \theta_0 \rfloor \in \mathbb{Z}, \\ \theta_n & := & \dfrac{1}{\theta_{n-1} - a_{n-1}} \in \mathbb{R}_{>1} - \mathbb{Q} \quad \text{for } n \in \mathbb{N}, \\ a_n & := & \lfloor \theta_n \rfloor \in \mathbb{N} \quad \text{for } n \in \mathbb{N}, \\ (p_{-1}, p_0, q_{-1}, q_0) & := & (1, a_0, 0, 1), \\ p_n & := & a_n p_{n-1} + p_{n-2} \in \mathbb{Z} \quad \text{for } n \in \mathbb{N}, \\ q_n & := & a_n q_{n-1} + q_{n-2} \in \mathbb{N} \quad \text{for } n \in \mathbb{N}, \\ r_n & := & \dfrac{p_n}{q_n} \in \mathbb{Q} \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \end{array}$$

θ_n and a_n are defined for all $n \in \mathbb{N}$, because each θ_{n-1} is in $\mathbb{R} - \mathbb{Q}$, so $\theta_{n-1} - a_{n-1} \in (0, 1)$.

(b) Following [**Ca65**, Notation 2.] define

$$\|\theta\| := \min(\theta - \lfloor \theta \rfloor, \lceil \theta \rceil - \theta) \in (0, \frac{1}{2})$$

We are interested especially in the case when $\theta \in \mathbb{R} - \mathbb{Q}$ is a quadratic irrational. We recall some notations for this case.

Definition B.3. Let $\theta \in \mathbb{R} - \mathbb{Q}$ be a quadratic irrational, i.e. $\dim_{\mathbb{Q}} \mathbb{Q}[\theta] = 2$. The other root of the minimal polynomial of θ is called θ^{conj}, so $\theta + \theta^{conj} =: \widetilde{a_0} \in \mathbb{Q}$ and $-\theta\theta^{conj} =: d_0 \in \mathbb{Q}$. For any $\alpha = a + b\theta \in \mathbb{Q}[\theta]$ with $a, b \in \mathbb{Q}$ write $\alpha^{conj} := a + b\theta^{conj}$. It is the algebraic conjugate of α. The algebra homomorphism

$$\mathcal{N} : \mathbb{Q}[\theta] \to \mathbb{Q}, \quad \alpha \mapsto \alpha\alpha^{conj},$$

is the norm map. The number α is called *reduced* if $\alpha > 1$ and $\alpha^{conj} \in (-1, 0)$. Recall that α is an algebraic integer if and only if $\alpha + \alpha^{conj} \in \mathbb{Z}$ and $\mathcal{N}(\alpha) \in \mathbb{Z}$ and that in this case α is a unit in $\mathbb{Z}[\alpha]$ if and only if $\mathcal{N}(\alpha) \in \{\pm 1\}$.

Theorem B.4. *(Classical) In the situation of Definition B.2 the following holds.*

(a) [**Ai13**, 1.2] $a_0 \in \mathbb{Z}$, $a_n \in \mathbb{N}$ *for* $n \in \mathbb{N}$. *For* $n \in \mathbb{Z}_{\geq 0}$ *the rational number* r_n *is*

$$r_n = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \cfrac{1}{a_{n-1} + \frac{1}{a_n}}}} =: [a_0, a_1, ..., a_n].$$

It is called partial quotient *or* continued fraction *or* n-th convergent *of* θ*. These numbers approximate* θ*,*

$$r_0 < r_2 < r_4 < ... < \theta < ... < r_5 < r_3 < r_1,$$
$$|\theta - r_n| < \frac{1}{q_n^2}.$$

This allows to write $\theta = [a_0, a_1, ...]$ *as an infinite continued fraction. The numerator* p_n *and the denominator* q_n *of* r_n *are coprime,*

$$\begin{aligned} \gcd(p_n, q_n) &= 1, \quad \text{and more precisely} \\ p_n q_{n-1} - p_{n-1} q_n &= (-1)^{n-1} \text{ for } n \in \mathbb{Z}_{\geq 0}. \end{aligned}$$

The denominators grow strictly from $n = 1$ *on,*

$$1 = q_0 \leq q_1 < q_2 < q_3 < ....$$

(b) [**Ca65**, I 2.] *The partial quotients* r_n *are in the following precise sense the only best approximations of* θ*:*

$$\begin{aligned} |p_n - q_n\theta| &= \|q_n\theta\| \quad \text{for } n \in \mathbb{N}, \\ \|q_{n+1}\theta\| &< \|q_n\theta\| \quad \text{for } n \in \mathbb{N}, \\ \|q\theta\| &\geq \|q_n\theta\| \quad \text{for } n \in \mathbb{Z}_{\geq 0} \text{ and } q \in \mathbb{N} \text{ with } q < q_{n+1}, \\ |p_0 - q_0\theta| &= \|q_0\theta\| > \|q_1\theta\| \quad \text{if } q_1 > 1 (\iff a_1 > 1), \\ |p_0 - q_0\theta| &\in (\frac{1}{2}, 1) \text{ and } |p_0 - q_0\theta| > \|q_1\theta\| \quad \text{if } q_1 = 1 (\iff a_1 = 1). \end{aligned}$$

In any case

$$|p_{n+1} - q_{n+1}\theta| < |p_n - q_n\theta| \quad \text{for } n \in \mathbb{Z}_{\geq 0}.$$

(c) [**Ai13**, Theorem 1.19] *The partial quotients* r_n *are also in the following precise sense the only best approximations of* θ*: A rational number* $\frac{p}{q}$ *with* $p \in \mathbb{Z}, q \in \mathbb{N}$ *and* $\gcd(p, q) = 1$ *satisfies*

$$|\theta - \frac{p}{q}| < \frac{1}{2q^2} \implies (p, q) = (p_n, q_n) \quad \text{for a suitable} \quad n \in \mathbb{Z}_{\geq 0}.$$

(d) [**Ai13**, Theorem 1.17 and Proposition 1.18] *The continued fraction is* periodic*, i.e. there exist* $k_0 \in \mathbb{Z}_{\geq 0}$ *and* $k_1 \in \mathbb{N}$ *with*

$$a_{n+k_1} = a_n \text{ for } n \geq k_0,$$

if and only if θ *is a quadratic irrational, i.e.* $\dim_{\mathbb{Q}} \mathbb{Q}[\theta] = 2$*. Then one writes* $[a_0 a_1 ...] = [a_0 a_1 ... a_{k_0-1} \overline{a_{k_0} a_{k_0+1} ... a_{k_0+k_1-1}}]$*. Furthermore, then* k_0 *can be chosen as 0 if and only if* θ *is reduced. In this case the continued fraction* $[a_0 a_1 ...]$ *is called* purely periodic*.*

Lemma B.5 fixes useful additional observations for the case of a quadratic irrational θ. These observations are used in the proof of Theorem B.6. It considers an algebraic integer $\alpha \in \mathbb{R} - \mathbb{Q}$ which is a quadratic irrational. We are interested in the group $\mathbb{Z}[\alpha]^*$ of units in $\mathbb{Z}[\alpha]$. Theorem B.6 shows how to see a generator of this group (and a quarter of its elements) in the continued fractions of a certain reduced element θ in $\mathbb{Z}[\alpha]$. Theorem B.6 is not new. For example, [**Bu00**, Theorem 8.13] gives its main part. But the proof here is more elegant than what we found in the literature.

LEMMA B.5. *Let $\theta \in \mathbb{R} - \mathbb{Q}$ be a quadratic irrational which is reduced. Let $[\overline{a_0 a_1 ... a_{k-1}}]$ be its purely periodic continued fraction of some minimal length $k \in \mathbb{N}$. We consider the objects in Definition B.2 for this θ. Then*

$$\theta_{n+k} = \theta_n \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{B.1}$$

θ_m is reduced for $m \in \{0, 1, ..., k-1\}$, and its purely periodic continued fraction is $[\overline{a_m ... a_{k-1} a_0 ... a_{m-1}}]$. Write

$$\begin{aligned} \widetilde{a_0} &:= \theta + \theta^{conj} \in \mathbb{Q}_{>0}, \qquad d_0 := -\mathcal{N}(\theta) = -\theta\theta^{conj} \in \mathbb{Q}_{>0}, \\ \beta &:= p_{k-1} - q_{k-1}\theta \in \mathbb{Q}[\theta] - \mathbb{Q}. \end{aligned}$$

Then for $n \in \mathbb{Z}_{\geq -1}$

$$p_{n+k} - q_{n+k}\theta = \beta \cdot (p_n - q_n\theta) \tag{B.2}$$

and for $n \in \mathbb{Z}_{\geq 0}$

$$\theta_n = \frac{-p_{n-2}p_{n-1} + p_{n-2}q_{n-1}\widetilde{a_0} + q_{n-2}q_{n-1}d_0 + (-1)^n\theta}{\mathcal{N}(p_{n-1} - q_{n-1}\theta)}. \tag{B.3}$$

Proof: The natural generalization of the notation $[a_0, a_1, ..., a_m]$ to numbers $a_0 \in \mathbb{R}, a_1, ..., a_m \in \mathbb{R}_{>0}$ gives for $n \in \mathbb{Z}_{\geq 0}$

$$\theta = [a_0, a_1, ..., a_{n-1}, \theta_n] = \frac{\theta_n p_{n-1} + p_{n-2}}{\theta_n q_{n-1} + q_{n-2}},$$

see Proposition 1.9 in [**Ai13**]. One concludes that the continued fraction of θ_n is purely periodic, that $\theta_n = \theta_m$ if $n = kl + m$ with $l \in \mathbb{Z}_{\geq 0}$ and $m \in \{0, 1, ..., k-1\}$, and that its continued fraction is $[\overline{a_m ... a_{k-1} a_0 ... a_{m-1}}]$. Therefore θ_n is reduced. Recall $p_{n-1}q_{n-2} -$

$p_{n-2}q_{n-1} = (-1)^n$. Inverting the equation above gives

$$\begin{aligned}\theta_n &= \frac{\theta q_{n-2} - p_{n-2}}{\theta(-q_{n-1}) + p_{n-1}} \\ &= \frac{(-p_{n-2} + q_{n-2}\theta)(p_{n-1} - q_{n-1}\theta^{conj})}{\mathcal{N}(p_{n-1} - q_{n-1}\theta)} \\ &= \frac{-p_{n-2}p_{n-1} + p_{n-2}q_{n-1}\widetilde{a_0} + q_{n-2}q_{n-1}d_0 + (-1)^n\theta}{\mathcal{N}(p_{n-1} - q_{n-1}\theta)}.\end{aligned}$$

The formula $\theta = \theta_k = \frac{\theta q_{k-2} - p_{k-2}}{\theta(-q_{k-1}) + p_{k-1}}$ shows

$$(1, -\theta)\begin{pmatrix} p_{k-1} & p_{k-2} \\ q_{k-1} & q_{k-2}\end{pmatrix} = (p_{k-1} - q_{k-1}\theta)(1, -\theta) = \beta(1, -\theta).$$

The inductive definition of p_n and q_n shows

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1}\end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} a_n & 1 \\ 1 & 0\end{pmatrix}.$$

With the periodicity $a_n + k = a_n$ we obtain

$$\begin{aligned}(1, -\theta)\begin{pmatrix} p_{n+k} & p_{n-1+k} \\ q_{n+k} & q_{n-1+k}\end{pmatrix} &= (1, -\theta)\begin{pmatrix} p_{k-1} & p_{k-2} \\ q_{k-1} & q_{k-2}\end{pmatrix}\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1}\end{pmatrix} \\ &= \beta(1, -\theta)\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1}\end{pmatrix}.\end{aligned}$$

This gives formula (B.2). □

THEOREM B.6. *Let $\alpha \in \mathbb{R} - \mathbb{Q}$ be a quadratic irrational and an algebraic integer.*

(a) There are a unique sign $\varepsilon_\alpha \in \{\pm 1\}$ and a unique number $n_\alpha \in \mathbb{Z}$ such that $\theta := \varepsilon_\alpha \alpha + n_\alpha$ is reduced. Then $\mathbb{Z}[\alpha] = \mathbb{Z}[\theta]$, and any reduced element $\widetilde{\theta} \in \mathbb{Z}[\alpha]$ with $\mathbb{Z}[\alpha] = \mathbb{Z}[\widetilde{\theta}]$ satisfies $\widetilde{\theta} = \theta$. We consider the objects in Definition B.2 for this θ. We define

$$\begin{aligned}\widetilde{a_0} &:= \theta + \theta^{conj} \in \mathbb{N}, \qquad d_0 := -\mathcal{N}(\theta) = -\theta\theta^{conj} \in \mathbb{N}, \\ \beta &:= p_{k-1} - q_{k-1}\theta \in \mathbb{Z}[\theta] - \mathbb{Z}.\end{aligned}$$

as in Lemma B.5. Then $a_0 = \widetilde{a_0}$ and $d_0 \in \{1, 2, ..., a_0\}$.

(b) Then β is a unit and generates together with -1 the unit group $\mathbb{Z}[\alpha]^*$, the l-th power of β is $\beta^l = p_{lk-1} - q_{lk-1}\theta$ for $l \in \mathbb{Z}_{\geq 0}$, and

$$\begin{aligned}\{\pm\beta^l \,|\, l \in \mathbb{Z}\} &= \mathbb{Z}[\alpha]^*, \\ \{\beta^l \,|\, l \in \mathbb{Z}_{\geq 0}\} &= \mathbb{Z}[\alpha]^* \cap \{p_{n-1} - q_{n-1}\theta \,|\, n \in \mathbb{Z}_{\geq 0}\}.\end{aligned}$$

The element β is uniquely characterized by the following properties:

(i) -1 and β generate the unit group $\mathbb{Z}[\alpha]^*$,

(ii) $|\beta| < 1$,

(iii) $\beta = p - q\theta$ *with* $p \in \mathbb{Z}, q \in \mathbb{N}$ *(namely* $p = p_{k-1}, q = q_{k-1}$*).*

Proof: (a) Choose $\varepsilon_\alpha \in \{\pm 1\}$ such that $\varepsilon_\alpha(\alpha - \alpha^{conj}) > 0$. Then choose $n_\alpha \in \mathbb{Z}$ such that $\varepsilon_\alpha \alpha^{conj} + n_\alpha \in (-1, 0)$. Define $\theta := \varepsilon_\alpha \alpha + n_\alpha$. Then $\theta^{conj} = \varepsilon_\alpha \alpha^{conj} + n_\alpha \in (-1, 0)$. Also $\theta > \theta^{conj}$ and $\theta\theta^{conj} \in \mathbb{Z} - \{0\}$. This shows $\theta > 1$, so θ is reduced. Also $\mathbb{Z}[\alpha] = \mathbb{Z}[\theta]$ is clear.

Any reduced element $\widetilde{\theta} \in \mathbb{Z}[\alpha]$ with $\mathbb{Z}[\alpha] = \mathbb{Z}[\theta]$ has the shape $\widetilde{\theta} = \widetilde{\varepsilon_\alpha}\alpha + \widetilde{n_\alpha}$ with $\widetilde{\varepsilon_\alpha} \in \{\pm 1\}$ and $\widetilde{n_\alpha} \in \mathbb{Z}$. The sign $\widetilde{\varepsilon_\alpha}$ is because of $\widetilde{\theta} > 1 > 0 > \widetilde{\theta}^{conj}$ the unique sign with $\widetilde{\varepsilon_\alpha}(\alpha - \alpha^{conj}) > 0$, so $\widetilde{\varepsilon_\alpha} = \varepsilon_\alpha$. Now $\widetilde{n_\alpha}$ is the unique integer with $\varepsilon_\alpha \alpha^{conj} + n_\alpha \in (-1, 0)$, so $\widetilde{n_\alpha} = n_\alpha$. Therefore $\widetilde{\theta} = \theta$.

We have $a_0 = \lfloor \theta \rfloor = \theta + \theta^{conj} = \widetilde{a_0}$ and $d_0 = -\theta\theta^{conj} \leq a_0$, both because $\theta^{conj} \in (-1, 0)$.

(b) We apply Lemma B.5. It tells us which of the elements $p_n - q_n\theta$ for $n \in \mathbb{Z}_{\geq 0}$ are units, in the following way.

Consider $n \in \mathbb{Z}_{\geq 0}$ and write $n = lk + m$ with $l \in \mathbb{Z}_{\geq 0}$ and $m \in \{0, 1, ..., k-1\}$. Recall that θ_n is reduced, that $\theta_n = \theta_m$ because of formula (B.1) and that $\theta_m = \theta$ only for $m = 0$, because for $m \in \{1, ..., k-1\}$ the purely periodic continued fractions of θ and θ_m differ. Recall also from formula (B.2) that

$$p_{n-1} - q_{n-1}\theta = \beta^l (p_{m-1} - q_{m-1}\theta).$$

If for some $n \in \mathbb{Z}_{\geq 0}$ $\mathcal{N}(p_{n-1} - q_{n-1}\theta) \in \{\pm 1\}$, then by formula (B.3) θ_n satisfies $\mathbb{Z}[\alpha] = \mathbb{Z}[\theta] = \mathbb{Z}[\theta_n]$. The uniqueness of θ in part (a) implies that then $\theta_n = \theta$, so $m = 0$. Therefore for $n \in \mathbb{Z}_{\geq 0} - k\mathbb{Z}_{\geq 0}$, $\mathcal{N}(p_{n-1} - q_{n-1}\theta) \notin \{\pm 1\}$, so then $p_{n-1} - q_{n-1}\theta$ is not a unit.

On the other hand, if $n = kl$, so $m = 0$, then $\theta_n = \theta$, and formula (B.3) tells $\mathcal{N}(p_{n-1} - q_{n-1}\theta) = (-1)^n$, so $p_{n-1} - q_{n-1}\theta$ is a unit. In fact, formula (B.2) tells $p_{n-1} - q_{n-1}\theta = \beta^l$. We see

$$\{p_{n-1} - q_{n-1}\theta \mid n \in \mathbb{Z}_{\geq 0}\} \cap \mathbb{Z}[\theta]^* = \{\beta^l \mid l \in \mathbb{Z}_{\geq 0}\}. \tag{B.4}$$

It remains to see that -1 and β generate $\mathbb{Z}[\theta]^*$.

By Dirichlet's unit theorem [**BSh73**, Ch. 2 4.3 Theorem 5], the set $\mathbb{Z}[\alpha]^*$ is as a group isomorphic to $\{\pm 1\} \times \mathbb{Z}$. It has two generators $\pm\widetilde{\beta}$ with $|\pm\widetilde{\beta}| < 1$. They are the unique elements in $\mathbb{Z}[\alpha]^*$ with maximal absolute value < 1. One of them has the shape $p - q\theta$ with $q \in \mathbb{N}$. This is called $\widetilde{\beta}$. Then also $p \in \mathbb{N}$, because $|\widetilde{\beta}| = |p - q\theta| < 1$ and $q\theta > 1$.

1st case, $\theta \in (1, 2)$: Then $a_0 = d_0 = 1$, and $\theta = \frac{1+\sqrt{5}}{2}$ is the golden section with $\theta^2 = \theta + 1$. This case is well known. Here the continued fraction of θ is purely periodic with period $\overline{1}$ of length one, because

$a_0 = 1$ and

$$\theta_1 = (\theta_0 - a_0)^{-1} = \theta = \theta_0.$$

Here $\beta = 1 - \theta = -\theta^{-1} = \theta^{conj}$. It is well known that

$$\mathbb{Z}[\alpha]^* = \mathbb{Z}[\theta]^* = \{\pm\theta^l \,|\, l \in \mathbb{Z}\} = \{\pm\beta^l \,|\, l \in \mathbb{Z}\}$$

2nd case, $\theta > 2$: $\widetilde{\beta} = p - q\theta$ is a unit, so $\pm 1 = \mathcal{N}(p - q\theta)$. Also $|\widetilde{\beta}| < 1$ and $\theta > 2$ imply $p \geq 2q$. Therefore

$$\begin{aligned} \left|\frac{p}{q} - \theta\right| &= \frac{1}{q(p - q\theta^{conj})} = \frac{1}{q(p + q|\theta^{conj}|)} \\ &< \frac{1}{q(2q+0)} = \frac{1}{2q^2}. \end{aligned}$$

By Theorem B.4 (c) $n \in \mathbb{Z}_{\geq 0}$ with $(p, q) = (p_n, q_n)$ exists. By (B.4) $\widetilde{\beta}$ is a power of β, so $\widetilde{\beta} = \beta$. □

The parts (a) and (b) in the following proof of Lemma B.1 serve also as examples for Theorem B.6.

Proof of Lemma B.1: (a) Here

$$\begin{aligned} x \geq 3 \Rightarrow 2x > 5 \Rightarrow x^2 - 4 &> x^2 - 2x + 1 \\ \Rightarrow \sqrt{x^2 - 4} &\in (x - 1, x), \\ \Rightarrow \theta = \kappa_1 - 1 = \frac{x-2}{2} + \frac{1}{2}\sqrt{x^2 - 4} &\in (x - \frac{3}{2}, x - 1) \\ \text{and} \quad \theta^{conj} = \kappa_2 - 1 = \frac{x-2}{2} - \frac{1}{2}\sqrt{x^2 - 4} &\in (-1, -\frac{1}{2}). \end{aligned}$$

Observe

$$\theta + \theta^{conj} = x - 2, \quad \theta\theta^{conj} = -x + 2.$$

Therefore

$$\begin{aligned} \theta_0 &= \theta, \quad a_0 = \lfloor \theta_0 \rfloor = x - 2, \\ \theta_1 &= (\theta_0 - a_0)^{-1} = \frac{\theta^{conj} - (x-2)}{(\theta - (x-2))(\theta^{conj} - (x-2))} \\ &= \frac{-\theta}{-x+2} = \frac{\theta}{x-2}, \\ a_1 &= \lfloor \theta_1 \rfloor = 1, \\ \theta_2 &= (\theta_1 - a_1)^{-1} = \frac{x-2}{\theta - (x-2)} = \frac{(x-2)(\theta^{conj} - (x-2))}{-x+2} = \theta, \\ \theta &= [\overline{x-2, 1}]. \end{aligned}$$

The continued fraction of θ is purely periodic with period $\overline{x-2,1}$ of length 2 if $x \geq 4$ and purely periodic with period $\overline{1}$ if $x = 3$. The norm of $p - q\theta$ for $p, q \in \mathbb{Z}$ is

$$\mathcal{N}(p - q\theta) := (p - q\theta)(p - q\theta^{conj}) = p^2 - q(p+q)(x-2) \in \mathbb{Z}.$$

It is ± 1 if and only if $p - q\theta$ is a unit.

n	0	1	2	3
a_n	$x-2$	1	$x-2$	1
(p_n, q_n)	$(x-2, 1)$	$(x-1, 1)$	$(x^2-2x, x-1)$	(x^2-x-1, x)
$\mathcal{N}(p_n - q_n\theta)$	$-x+2$	1	$-x+2$	1

If $x = 3$ then $\beta = p_0 - q_0\theta = 1 - \theta$ in the notation of Lemma B.5, so $\mathbb{Z}[\theta]^*$ is generated by (-1) and β or $-\beta^{-1} = \theta = \kappa_1 - 1$, so

$$\mathbb{Z}[\theta]^* = \{\pm(1-\theta)^l \,|\, l \in \mathbb{Z}\} = \{\pm(\kappa_1 - 1)^l \,|\, l \in \mathbb{Z}\}.$$

This is also consistent with the 1st case in the proof of part (b) of Theorem B.6.

If $x \geq 4$ then $\beta = p_1 - q_1\theta = x - 1 - \theta$ in the notation of Corollary B.4, so $\mathbb{Z}[\theta]^*$ is generated by (-1) and β or $x - 1 - \theta^{conj} = \kappa_1$, so

$$\mathbb{Z}[\theta]^* = \{\pm(x-1-\theta)^l \,|\, l \in \mathbb{Z}\} = \{\pm\kappa_1^l \,|\, l \in \mathbb{Z}\}.$$

This proves part (a) of Lemma B.1.

(b) The case $x = 2$ is treated separately and first. This case is well known. Then $\theta = 1 + \sqrt{2} \in (2,3)$, $\theta^{conj} = 1 - \sqrt{2} \in (-1, 0)$, so $a_0 = 2$. The continued fraction of θ is purely periodic with period $\overline{2}$ of length one, because $a_0 = 2$ and

$$\theta_1 = (\theta_0 - a_0)^{-1} = (\sqrt{2} - 1)^{-1} = \theta_0.$$

The element

$$p_0 - q_0\theta = 2 - \theta = 1 - \sqrt{2} = \theta^{conj}$$

is a unit. This and Theorem B.6 (b) show

$$\mathbb{Z}[\alpha]^* = \mathbb{Z}[\theta]^* = \{\pm\theta^l \,|\, l \in \mathbb{Z}\} = \{\pm(1+\sqrt{2})^l \,|\, l \in \mathbb{Z}\}.$$

Now we treat the cases $x \geq 3$. Here

$$x \geq 3 \Rightarrow x^2 - 2 > x^2 - x + \frac{1}{4} \Rightarrow \sqrt{x^2-2} > x - \frac{1}{2},$$
$$\Rightarrow \theta = (x-1) + \sqrt{x^2-2} \in (2x - \frac{3}{2}, 2x-1)$$
$$\text{and} \quad \theta^{conj} = (x-1) - \sqrt{x^2-2} \in (-1, -\frac{1}{2}).$$

Observe

$$\theta + \theta^{conj} = 2x - 2, \quad \theta\theta^{conj} = -2x + 3.$$

Therefore

$$\begin{aligned}
\theta_0 &= \theta, \quad a_0 = \lfloor \theta_0 \rfloor = 2x-2,\\
\theta_1 &= (\theta_0 - a_0)^{-1} = ... = \frac{\theta}{2x-3} \in (1,2), \quad a_1 = 1,\\
\theta_2 &= (\theta_1 - a_1)^{-1} = ... = \frac{\theta - 1}{2} \in (x-2, x-1), \quad a_2 = x-2,\\
\theta_3 &= (\theta_2 - a_2)^{-1} = ... = \frac{\theta - 1}{2x-3} \in (1,2), \quad a_3 = 1,\\
\theta_4 &= (\theta_3 - a_3)^{-1} = ... = \theta = \theta_0,\\
\theta &= [\overline{2x-2, 1, x-2, 1}].
\end{aligned}$$

The continued fraction of θ is purely periodic with period $\overline{2x-2, 1, x-2, 1}$ of length four. The norm of $p - q\theta$ is

$$\mathcal{N}(p - q\theta) = (p - q\theta)(p - q\theta^{conj}) = p^2 + q^2 - q(p+q)(2x-2).$$

It is ± 1 if and only if $p - q\theta$ is a unit.

n	0	1	2	3
a_n	$2x-2$	1	$x-2$	1
(p_n, q_n)	$(2x-2, 1)$	$(2x-1, 1)$	$(2x^2-3x, x-1)$	$(2x^2-x-1, x)$
$\mathcal{N}(p_n - q_n\theta)$	$-2x+3$	2	$-2x+3$	1

We conclude with Theorem B.6 (and with the notation of Lemma B.5) that

$$\beta = p_3 - q_3\theta = (2x^2 - x - 1) - x\theta = (x^2-1) - x\sqrt{x^2-2} = \lambda_2$$

is together with (-1) a generator of $\mathbb{Z}[\sqrt{x^2-2}]^* = \mathbb{Z}[\theta]^*$. Therefore also λ_1 together with (-1) is a generator of $\mathbb{Z}[\sqrt{x^2-2}]^*$. This proves part (b) of Lemma B.1. $\square$

REMARK B.7. In the situation of Theorem B.6, Satz 9.5.2 in [**Ko97**] tells that the unit group $\mathbb{Z}[\theta]^*$ is generated by -1 and $q_{k-2}+q_{k-1}\theta$. This is consistent with Theorem B.6 because of the following. Here

$$\begin{aligned}
\theta &= \frac{\theta_k p_{k-1} + p_{k-2}}{\theta_k q_{k-1} + q_{k-2}} = \frac{\theta p_{k-1} + p_{k-2}}{\theta q_{k-1} + q_{k-2}},\\
\text{so} \quad 0 &= q_{k-1}\theta^2 - (p_{k-1} - q_{k-2})\theta - p_{k-2},\\
\text{but also} \quad 0 &= \theta^2 - \widetilde{a_0}\theta - d_0,\\
\text{so} \quad a_0 &= \widetilde{a_0} = \frac{p_{k-1} - q_{k-2}}{q_{k-1}}, \quad d_0 = \frac{p_{k-2}}{q_{k-1}},\\
p_{k-1} - q_{k-1}\theta^{conj} &= p_{k-1} - q_{k-1}(a_0 - \theta) = q_{k-2} + q_{k-1}\theta.
\end{aligned}$$

APPENDIX C

Powers of quadratic units

The following definition and lemma treat powers of units of norm 1 in the rings of integers of quadratic number fields. Though these powers appear explicitly only in Lemma C.2 (c). Lemma C.2 will be used in the proof of Theorem 5.18.

DEFINITION C.1. (a) Define the polynomials $b_l(a) \in \mathbb{Z}[a]$ for $l \in \mathbb{Z}_{\geq 0}$ by the following recursion.

$$b_0 := 0, \quad b_1 := 1, \quad b_l := ab_{l-1} - b_{l-2} \quad \text{for } l \in \mathbb{Z}_{\geq 2}. \tag{C.1}$$

(b) Define for $l \in \mathbb{Z}_{\geq 0}$ the polynomial $r_l \in \mathbb{Z}[a]$ and for $l \in \mathbb{N}$ the rational functions $q_{0,l}, q_{1,l}, q_{2,l} \in \mathbb{Q}(t)$,

$$\begin{aligned} r_0 &:= 0, \\ r_l &:= -ab_l + 2b_{l-1} + 2 \quad \text{for } l \in \mathbb{N}, \\ q_{0,l} &:= \frac{b_l - b_{l-1}}{b_l}, \\ q_{1,l} &:= \frac{b_l - b_{l-1} - 1}{r_l b_l}, \\ q_{2,l} &:= q_{0,l} - 2q_{1,l}. \end{aligned}$$

(c) A notation: For two polynomials $f_1, f_2 \in \mathbb{Z}[a]$, $(f_1, f_2)_{\mathbb{Z}[a]} := \mathbb{Z}[a]f_1 + \mathbb{Z}[a]f_2 \subset \mathbb{Z}[a]$ denotes the ideal generated by f_1 and f_2.

Remark: If $(f_1, f_2)_{\mathbb{Z}[a]} = \mathbb{Z}[a]$ then for any integer $c \in \mathbb{Z}$ $\gcd(f_1(c), f_2(c)) = 1$.

(d) For $a \in \mathbb{Z}_{\leq -3} \cup \mathbb{Z}_{\geq 3}$ define $\kappa_a := \frac{a}{2} + \frac{1}{2}\sqrt{a^2-4}$ and $\kappa_a^{conj} := \frac{a}{2} - \frac{1}{2}\sqrt{a^2-4}$ as the zeros of the polynomial $t^2 - at + 1$, so that $\kappa_a + \kappa_a^{conj} = a$, $\kappa_a \kappa_a^{conj} = 1$, $\kappa_a^2 = a\kappa_a - 1$. They are algebraic integers and units with norm 1.

The following table gives the first twelve of the polynomials $b_l(a)$. The software Maxima [**Maxima22**] claims that the factors in the products are irreducible polynomials as polynomials in $\mathbb{Q}[a]$. We will not use this claim.

$$
\begin{aligned}
b_0 &= 0 \\
b_1 &= 1 \\
b_2 &= a \\
b_3 &= (a-1)(a+1) \\
b_4 &= a(a^2-2) \\
b_5 &= (a^2-a-1)(a^2+a-1) \\
b_6 &= (a-1)a(a+1)(a^2-3) \\
b_7 &= (a^3-a^2-2a+1)(a^3+a^2-2a-1) \\
b_8 &= a(a^2-2)(a^4-4a^2+2) \\
b_9 &= (a-1)(a+1)(a^3-3a-1)(a^3-3a+1) \\
b_{10} &= a(a^2-a-1)(a^2+a-1)(a^4-5a^2+5) \\
b_{11} &= (a^5-a^4-4a^3+3a^2+3a-1)(a^5+a^4-4a^3-3a^2+3a+1)
\end{aligned}
$$

LEMMA C.2. *(a) For any* $l \in \mathbb{N}$

$$1 = b_l^2 - ab_lb_{l-1} + b_{l-1}^2 = b_l^2 - b_{l+1}b_{l-1}, \tag{C.2}$$

$$(b_{l-1}, b_l)_{\mathbb{Z}[a]} = (b_l - b_{l-1}, b_l)_{\mathbb{Z}[a]} = \mathbb{Z}[a], \tag{C.3}$$

$$r_l = \begin{cases} (2-a)(b_{(l+1)/2} + b_{(l-1)/2})^2 & \text{for } l \text{ odd,} \\ (2-a)(a+2)b_{l/2}^2 & \text{for } l \text{ even,} \\ & \text{(also } l = 0) \end{cases} \tag{C.4}$$

$$(r_{l-1}/(2-a), r_l/(2-a))_{\mathbb{Z}[a]} = \mathbb{Z}[a], \tag{C.5}$$

and

$$q_{2,l} = 1 - \frac{r_{l-1}/(2-a)}{r_l/(2-a)}. \tag{C.6}$$

(b) For $a \in \mathbb{Z}_{\le -3}$

$$
\begin{aligned}
b_l(a) \in (-1)^{l-1}\mathbb{N} \quad &\text{for} \quad l \ge 1, \\
b_1(a) = 1, \quad b_2(a) = a, \quad |b_2(a) + b_1(a)| = |a| - 1&, \\
|b_l(a)| > 2|b_{l-1}(a)| \ge |b_{l-1}(a)| + 1 \quad &\text{for} \quad l \ge 2, \\
|b_{l+1}(a) + b_l(a)| > |b_l(a) + b_{l-1}(a)| \quad &\text{for} \quad l \ge 1.
\end{aligned}
$$

For $a \in \mathbb{Z}_{\ge 3}$

$$
\begin{aligned}
b_l(a) > 0 \quad &\text{for} \quad l \ge 1, \\
b_1(a) = 1, \quad b_2(a) = a, \quad b_2(a) + b_1(a) = a + 1&, \\
b_l(a) > 2b_{l-1}(a) \quad &\text{for} \quad l \ge 1, \\
b_{l+1}(a) + b_l(a) > 2(b_l(a) + b_{l-1}(a)) \quad &\text{for} \quad l \ge 1.
\end{aligned}
$$

(c) Consider $a\in\mathbb{Z}_{\leq -3}\cup\mathbb{Z}_{\geq 3}$ and $l\in\mathbb{N}$. Then

$$\begin{aligned}
\kappa_a^l &= b_l(a)\kappa_a - b_{l-1}(a), && \text{(C.7)}\\
\kappa_a &= (1-q_{0,l}(a)) + (q_{0,l}(a) - r_l(a)q_{1,l}(a))\kappa_a^l, && \text{(C.8)}\\
\kappa_a^l &= \frac{2-r_l(a)}{2} + \frac{1}{2}\sqrt{r_l(a)(r_l(a)-4)}, && \text{(C.9)}
\end{aligned}$$

so κ_a^l is a zero of the polynomial $t^2-(2-r_l(a))t+1$.

Proof: (a) The recursive definition (C.1) of b_l shows immediately the equality of the middle and right term in (C.2), it shows

$$b_l^2 - ab_lb_{l-1} + b_{l-1}^2 = b_{l-1}^2 - ab_{l-1}b_{l-2} + b_{l-2}^2,$$

and it shows $b_1^2 - ab_1b_0 + b_0^2 = 1$. This proves (C.2). It implies (C.3).

The sequence $(r_l)_{l\in\mathbb{N}}$ satisfies the recursion

$$r_0 = 0,\quad r_1 = 2-a,\quad r_l = ar_{l-1} - r_{l-2} + 2(2-a)\quad \text{for } l\geq 2.$$

For $l=2$ one verifies this immediately. For $l\geq 3$ it follows inductively with (C.1),

$$\begin{aligned}
r_l &= -a(ab_{l-1}-b_{l-2}) + 2(ab_{l-2}-b_{l-3}) + 2\\
&= a(-ab_{l-1}+2b_{l-2}+2) - (-ab_{l-2}+2b_{l-3}+2) + 2(2-a)\\
&= ar_{l-1} - r_{l-2} + 2(2-a).
\end{aligned}$$

For $l=1$ and $l=0$ (C.4) is obvious. For odd $l=2k+1\geq 3$ as well as even $l=2k\geq 2$, one verifies (C.4) inductively with this recursion and with (C.2), for odd $l=2k+1\geq 3$:

$$\begin{aligned}
&(2-a)(b_{k+1}+b_k)^2 - ar_{2k} + r_{2k-1} - 2(2-a)\\
=\ &(2-a)[((ab_k-b_{k-1})+b_k)^2]\\
&-(2-a)[a(a+2)b_k^2 - (b_k+b_{k-1})^2 + 2]\\
=\ &(2-a)[2b_k^2 - 2ab_kb_{k-1} + 2b_{k-1}^2 - 2]\\
=\ &0\quad \text{(with (C.2))},
\end{aligned}$$

for even $l=2k\geq 2$:

$$\begin{aligned}
&(2-a)(a+2)b_k^2 - ar_{2k-1} + r_{2k-2} - 2(2-a)\\
=\ &(2-a)[(a+2)b_k^2]\\
&-(2-a)[a(b_k+b_{k-1})^2 - (a+2)b_{k-1}^2 + 2]\\
=\ &(2-a)[2b_k^2 - 2ab_kb_{k-1} + 2b_{k-1}^2 - 2]\\
=\ &0\quad \text{(with (C.2))}.
\end{aligned}$$

(C.5) claims for $k \geq 0$

$$((b_{k+1}+b_k)^2, (a+2)b_k^2)_{\mathbb{Z}[a]} = \mathbb{Z}[a]$$
$$\text{and} \quad ((a+2)b_{k+1}^2, (b_{k+1}+b_k)^2)_{\mathbb{Z}[a]} = \mathbb{Z}[a].$$

The following **claim** is basic: For $f_1, f_2, f_3 \in \mathbb{Z}[a]$

$$(f_1, f_3)_{\mathbb{Z}[a]} = (f_2, f_3)_{\mathbb{Z}[a]} = \mathbb{Z}[a] \quad \Rightarrow \quad (f_1 f_2, f_3)_{\mathbb{Z}[a]} = \mathbb{Z}[a].$$

To see this claim consider $1 = \alpha_1 f_1 + \alpha_2 f_3$, $1 = \beta_1 f_2 + \beta_2 f_3$. Then

$$\begin{aligned} 1 &= (\alpha_1 f_1 + \alpha_2 f_3)(\beta_1 f_2 + \beta_2 f_3) \\ &= \alpha_1\beta_1 f_1 f_2 + \alpha_2\beta_2 f_3^3 + \alpha_1\beta_2 f_1 f_3 + \alpha_2\beta_1 f_2 f_3. \end{aligned}$$

The claim and (C.3) show that for (C.5) it is sufficient to prove

$$(a+2, b_{k+1}+b_k)_{\mathbb{Z}[a]} = \mathbb{Z}[a].$$

This follows inductively in k with

$$b_{k+1}+b_k = (a+2)b_k - (b_k + b_{k-1}) \quad \text{and} \quad b_1 + b_0 = 1.$$

Finally, we calculate $q_{2,l}$:

$$\begin{aligned} q_{2,l} &= (r_l b_l)^{-1}(r_l(b_l - b_{l-1}) - 2(b_l - b_{l-1} - 1)) \\ &= 1 + (r_l b_l)^{-1}(-(-ab_l + 2b_{l-1} + 2)b_{l-1}) - 2b_l + 2b_{l-1} + 2) \\ &= 1 + (r_l b_l)^{-1}(b_l(ab_{l-1} - 2) - 2(b_{l-1}^2 - 1)) \\ &= \begin{cases} 1 + (r_l)^{-1}((ab_{l-1} - 2) - 2b_{l-2}) \quad (\text{with (C.2)}) & \text{for } l \geq 2, \\ 1 & \text{for } l = 1 \end{cases} \\ &= 1 - r_l^{-1} r_{l-1}. \end{aligned}$$

(b) All inequalities and signs follow inductively with (C.1).

(c) (C.7) is true for $l = 1$. It follows inductively in l with the following calculation, which uses $\kappa_a^2 = a\kappa_a - 1$.

$$\begin{aligned} \kappa_a^{l+1} &= \kappa_a(b_l(a)\kappa_a - b_{l-1}(a)) \\ &= b_l(a)(a\kappa_a - 1) - b_{l-1}(a)\kappa_a \\ &= (ab_l(a) - b_{l-1}(a))\kappa_a - b_l(a) \\ &= b_{l+1}(a)\kappa_a - b_l(a). \end{aligned}$$

The right hand side of (C.8) is

$$\frac{b_{l-1}(a)}{b_l(a)} + \frac{1}{b_l(a)}\kappa_a^l$$

which is κ_a by inverting (C.7). Writing $\kappa_a = \frac{a}{2} + \frac{1}{2}\sqrt{a^2-4}$ gives for κ_a^l

$$\kappa_a^l = \frac{ab_l(a) - 2b_{l-1}(a)}{2} + \frac{b_l(a)}{2}\sqrt{a^2-4}.$$

One verifies that this equals the right hand side of (C.9). □

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