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A note on Bertino and Fredricks–Nelsen copulas

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ABSTRACT

Bertino and Fredricks–Nelsen copulas are the most prominent examples of copulas which are constructed from a diagonal. For a given diagonal, it is known that the Bertino copula is dominated by the Fredricks–Nelsen copula. In this paper, we show that these copulas are either distinct or both equal to the upper Fréchet–Hoeffding bound. We also prove known and new representations of Spearman's rho, Gini's gamma and Kendall's tau for Bertino and Fredricks–Nelsen copulas in terms of their diagonal, and we show that, under any convex–linear measure of concordance or Kendall's tau, the range of the collection of all copulas having the same diagonal is bounded by the values of the respective Bertino and Fredricks–Nelsen copulas.

1. Introduction

Similar to (convex) Archimedean generators, diagonals $[0,1] \to [0,1]$ provide a useful tool for the construction of (bivariate) copulas. The most prominent copulas with a given diagonal are those proposed by Bertino [1] and Fredricks and Nelsen [5]. These copulas are of particular interest since, for every diagonal δ, the Bertino copula is the least element of the collection of all copulas with diagonal δ and the Fredricks–Nelsen copula is the greatest element of the collection of all *symmetric* copulas with diagonal δ.

In this paper, we first show that the Bertino and Fredricks–Nelsen copulas having the same diagonal are either distinct or equal to the upper Fréchet–Hoeffding bound (Theorem 1). We then turn to measures of concordance and prove representations of Spearman's rho (Theorem 3), Gini's gamma (Theorem 4) and Kendall's tau (Theorem 5) for Bertino and Fredricks–Nelsen copulas in terms of their diagonal, thus completing the literature in which some of these results are stated without proof; see Fredricks and Nelsen [5]. For measures of concordance which, like Spearman's rho and Gini's gamma, are convex–linear, and also for Kendall's tau, which fails to be convex–linear, we show that their range on the collection of all copulas (symmetric or not) having the same diagonal is bounded by the values of the respective Bertino and Fredricks–Nelsen copulas (Theorems 2 and 6).

As an application of Theorems 2 and 6, consider a random vector (X, Y) with continuous distribution function H, identical margins F and copula C. If the distribution of $\max\{X, Y\}$ is known, then the identity

$$P[\max\{X, Y\} \leq z] = H(z, z) = C(F(z), F(z)) = \delta(F(z))$$

implies that the diagonal δ of C is known as well, and evaluation of the Bertino and Fredricks–Nelsen copulas with diagonal δ under a convex–linear measure of concordance or Kendall's tau yields bounds for the degree of dependence between X and Y. Via the survival copula of C, the same argument applies when the distribution of $\min\{X, Y\}$ is known.

Let us recall some definitions:

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1. A function $C : [0,1]^2 \to \mathbb{R}$ is said to be a *copula* if it is 2–increasing in the sense that $C(u,v) - C(s,v) - C(u,t) + C(s,t) \geq 0$ holds for all $(s,t),(u,v) \in [0,1]^2$ such that $(s,t) \leq (u,v)$ and if it satisfies the boundary conditions $C(t,0) = 0 = C(0,t)$ and $C(t,1) = t = C(1,t)$ for all $t \in [0,1]$. Every copula C satisfies $|C(u,v) - C(s,t)| \leq |u-s| + |v-t|$ for all $(s,t),(u,v) \in [0,1]^2$, which shows that C is Lipschitz continuous and hence partially differentiable almost everywhere.
2. Each of the functions $\Pi, M, W : [0,1]^2 \to [0,1]$ given by $\Pi(u,v) := uv$, $M(u,v) := u \wedge v$ and $W(u,v) := (u+v-1)^+$ is a copula, and every copula C satisfies $W \leq C \leq M$.
3. A probability measure $Q : \mathcal{B}([0,1]^2) \to [0,1]$ is said to be a *copula measure* (or a *doubly stochastic measure*) if $Q[A \times [0,1]] = \lambda[A] = Q[[0,1] \times A]$ holds for every Borel set $A \in \mathcal{B}([0,1])$, where λ denotes the Lebesgue measure on $\mathcal{B}([0,1])$.

The correspondence theorem for copulas assert that there exists a bijection between copulas and copula measures and that a copula C and its copula measure Q^C satisfy $Q^C[[s,u] \times [t,v]] = C(u,v) - C(s,v) - C(u,t) + C(s,t)$ for all $(s,t),(u,v) \in [0,1]^2$ such that $(s,t) \leq (u,v)$.

2. Diagonals

A function $\delta : [0,1] \to \mathbb{R}$ is said to be a *diagonal* if it has the following properties:

(i) δ is increasing with $\delta(0) = 0$ and $\delta(1) = 1$.
(ii) $\delta(t) \leq t$ holds for all $t \in [0,1]$.
(iii) $|\delta(t) - \delta(s)| \leq 2\,|t-s|$ holds for all $s,t \in [0,1]$.

If δ is a diagonal, then $(2t-1)^+ \leq \delta(t)$ holds for all $t \in [0,1]$. For every copula C, the function $\delta_C : [0,1] \to [0,1]$ given by

$$\delta_C(t) := C(t,t)$$

is a diagonal and is said to be the *diagonal of* C.

Example 1.

1. **Product Copula:** The copula Π satisfies $\delta_\Pi(t) = t^2$ for all $t \in [0,1]$.
2. **Upper Fréchet–Hoeffding Bound:** A copula C satisfies $\delta_C(t) = t$ for all $t \in [0,1]$ if and only if $C = M$.
   Indeed: It is evident that $\delta_M(t) = t$ holds for all $t \in [0,1]$. Assume now that C is a copula such that $\delta_C(t) = t$ holds for all $t \in [0,1]$. Then we have
   $$M(u,v) = u \wedge v = \delta_C(u \wedge v) = C(u \wedge v, u \wedge v) \leq C(u,v) \leq M(u,v)$$
   for all $(u,v) \in [0,1]^2$, and hence $C = M$; see also Durante and Sempi [[2]; Example 2.6.4].
3. **Lower Fréchet–Hoeffding Bound:** The copula W satisfies $\delta_W(t) = (2t-1)^+$ for all $t \in [0,1]$.
4. **Fréchet Copulas:** For $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$, the function
   $$C_{\text{Fréchet};\alpha,\beta} := \alpha M + (1-\alpha-\beta)\Pi + \beta W$$
   is a copula with $\delta_{C_{\text{Fréchet};\alpha,\beta}} = \alpha\delta_M + (1-\alpha-\beta)\,\delta_\Pi + \beta\,\delta_W$.

The inequalities for diagonals imply that $\delta_W \leq \delta \leq \delta_M$ holds for every diagonal δ. Moreover, the diagonal δ_M is particular since it turns out that for every diagonal $\delta \neq \delta_M$ there exist distinct copulas with diagonal δ; see Theorem 1.

For a diagonal δ, we also consider the functions $\hat{\delta}, \tilde{\delta} : [0,1] \to \mathbb{R}$ given by

$$\hat{\delta}(t) := t - \delta(t) \qquad \text{and} \qquad \tilde{\delta}(t) := 2t - \delta(t)\,.$$

Then we have $0 \leq \hat{\delta}(t) \leq t \wedge (1-t)$ and $0 \leq \delta(t) \leq t \leq \tilde{\delta}(t) \leq (2t) \wedge 1$ for all $t \in [0,1]$, as well as $|\hat{\delta}(t) - \hat{\delta}(s)| \leq |t-s|$ and $|\tilde{\delta}(t) - \tilde{\delta}(s)| \leq 2\,|t-s|$ for all $s,t \in [0,1]$. Moreover, each of δ and $\tilde{\delta}$ is a distribution function on $[0,1]$ and hence has lower and upper quantile functions, and each of δ, $\hat{\delta}$ and $\tilde{\delta}$ is differentiable almost everywhere. In particular, we have $0 \leq \delta'(t) \leq 2$ and hence $-1 \leq \hat{\delta}'(t) \leq 1$ and $0 \leq \tilde{\delta}'(t) \leq 2$ for all $t \in [0,1]$ for which δ is differentiable at t.

3. Bertino and Fredricks–Nelsen copulas

For a diagonal δ, we consider the functions $C_{\mathbf{Ber},\delta}, C_{\mathbf{FN},\delta} : [0,1]^2 \to \mathbb{R}$ given by

$$C_{\mathbf{Ber},\delta}(u,v) := u \wedge v - \min\Big\{ r - \delta(r) \;\Big|\; u \wedge v \leq r \leq u \vee v \Big\}$$

and

$$C_{\mathbf{FN},\delta}(u,v) := u \wedge v \wedge \frac{\delta(u) + \delta(v)}{2}\,.$$

The function $C_{\mathbf{Ber},\delta}$ is said to be the *Bertino copula* with diagonal δ, and $C_{\mathbf{FN},\delta}$ is said to be the *Fredricks–Nelsen copula* with diagonal δ. The following results are well–known from the papers by Fredricks and Nelsen [5,6]; see also Durante and Sempi [[2]; Section 1.7] for a short proof of the fact that every Fredricks–Nelsen copula is indeed a copula.

Proposition 1. *For every diagonal δ, the function $C_{\mathbf{Ber},\delta}$ has the following properties:*

1. *$C_{\mathbf{Ber},\delta}$ is a symmetric copula with diagonal δ.*
2. *Every copula C with diagonal δ satisfies $C_{\mathbf{Ber},\delta} \le C$.*
3. *$C_{\mathbf{Ber},\delta} = M$ if and only if $\delta = \delta_M$.*
4. *$C_{\mathbf{Ber},\delta} = W$ if and only if $\delta = \delta_W$.*
5. *$C_{\mathbf{Ber},\delta} \le \Pi$ if and only if $\delta \le \delta_\Pi$.*

Moreover, if δ and η are diagonals such that $\delta \le \eta$, then $C_{\mathbf{Ber},\delta} \le C_{\mathbf{Ber},\eta}$.

Proposition 2. *For every diagonal δ, the function $C_{\mathbf{FN},\delta}$ has the following properties:*

1. *$C_{\mathbf{FN},\delta}$ is a symmetric copula with diagonal δ.*
2. *Every symmetric copula C with diagonal δ satisfies $C \le C_{\mathbf{FN},\delta}$.*
3. *$C_{\mathbf{FN},\delta} = M$ if and only if $\delta = \delta_M$.*
4. *$C_{\mathbf{FN},\delta_W} \neq W$.*
5. *$\Pi \le C_{\mathbf{FN},\delta}$ if and only if $\delta_\Pi \le \delta$.*

Moreover, if δ and η are diagonals such that $\delta \le \eta$, then $C_{\mathbf{FN},\delta} \le C_{\mathbf{FN},\eta}$.

The following example shows that the assumption of symmetry cannot be dropped in assertion (2) of Proposition 2; see also Nelsen et al. [[14]; Example 11] for another example:

Example 2. The function $D : [0,1]^2 \to [0,1]$ given by

$$D(u,v) = \begin{cases} u \wedge (v - \frac{3}{4}) & \text{if } (u,v) \in [0,\frac{1}{4}] \times [\frac{3}{4},1] \\ (u - \frac{1}{4}) \wedge v & \text{if } (u,v) \in [\frac{1}{4},1] \times [0,\frac{3}{4}] \\ W(u,v) & \text{else} \end{cases}$$

is an asymmetric copula and its diagonal satisfies

$$\delta_D(t) = \begin{cases} 0 & \text{if } t \in [0,\frac{1}{4}] \\ t - \frac{1}{4} & \text{if } t \in [\frac{1}{4},\frac{3}{4}] \\ 2t-1 & \text{if } t \in [\frac{3}{4},1] \end{cases}$$

We thus obtain $C_{\mathbf{FN},\delta_D}(\frac{1}{2},\frac{1}{4}) = 1/8 < 1/4 = D(\frac{1}{2},\frac{1}{4})$, and hence $D \nleq C_{\mathbf{FN},\delta_D}$.

With regard to assertion (2) of Proposition 2, we also note in passing that, for every diagonal δ, the function $\overline{C}_\delta : [0,1]^2 \to \mathbb{R}$ given by $\overline{C}_\delta(u,v) := \sup\{C(u,v) \mid C \text{ is a copula with } \delta_C = \delta\}$ is a quasi–copula. Since $\overline{C}_\delta$ is symmetric and satisfies $C_{\mathbf{FN},\delta} \le \overline{C}_\delta$, it follows that $\overline{C}_\delta$ is a copula if and only if $C_{\mathbf{FN},\delta} = \overline{C}_\delta$. Kokol Bukovšek et al. [8] present an explicit formula for $\overline{C}_\delta$.

By Propositions 1 and 2, we have $C_{\mathbf{Ber},\delta} \le C_{\mathbf{FN},\delta}$ for every diagonal δ, as well as $C_{\mathbf{Ber},\delta_M} = C_{\mathbf{FN},\delta_M}$. The following result completes the comparison of $C_{\mathbf{Ber},\delta}$ and $C_{\mathbf{FN},\delta}$:

Theorem 1. *For every diagonal δ, the following properties are equivalent:*

(a) $\delta = \delta_M$.
(b) $C_{\mathbf{Ber},\delta} = C_{\mathbf{FN},\delta}$.

Proof. By Propositions 1 and 2, we know that (a) implies (b). Assume now that $\delta \neq \delta_M$. Then there exists some $t \in (0,1)$ such that $\delta(t) < t$, and we put $\varepsilon := t - \delta(t)$. Since $\lim_{s\to 0}(s - \delta(s)) = 0 = \lim_{s\to 1}(s - \delta(s))$, continuity implies that each of the sets $A := \{s \in (0,t) \mid s - \delta(s) = \varepsilon/2\}$ and $B := \{s \in (t,1) \mid s - \delta(s) = \varepsilon/2\}$ is nonempty. Define now

$$u := \sup A \qquad \text{and} \qquad v := \inf B\,.$$

Then we have $u \le t \le v$ and continuity yields

$$u - \delta(u) = \varepsilon/2 = v - \delta(v)\,.$$

Since $t - \delta(t) = \varepsilon$, we conclude that

$$u < v \qquad \text{and} \qquad \min\{r - \delta(r) \mid u \le r \le v\} = \varepsilon/2\,.$$

We thus obtain

$$\begin{aligned} & C_{\mathbf{FN},\delta}(u,v) - C_{\mathbf{Ber},\delta}(u,v) \\ & = u \wedge v \wedge \frac{\delta(u)+\delta(v)}{2} - \Big(u \wedge v - \min\{r - \delta(r) \mid u \wedge v \le r \le u \vee v\}\Big) \\ & = u \wedge \frac{\delta(u)+\delta(v)}{2} - \Big(u - \min\{r - \delta(r) \mid u \le r \le v\}\Big) \\ & = 0 \wedge \left(\frac{\delta(u)+\delta(v)}{2} - u\right) + \frac{\varepsilon}{2}\,. \end{aligned}$$

In the case $(\delta(u)+\delta(v))/2 \geq u$ this yields $C_{\mathbf{FN},\delta}(u,v) - C_{\mathbf{Ber},\delta}(u,v) = \varepsilon/2 > 0$, and in the case $(\delta(u)+\delta(v))/2 < u$ we obtain

$$\begin{aligned} C_{\mathbf{FN},\delta}(u,v) - C_{\mathbf{Ber},\delta}(u,v) &= \frac{\delta(u)+\delta(v)}{2} - u + \frac{\varepsilon}{2} \\ &> \frac{\delta(u)+\delta(v)}{2} - \frac{u+v}{2} + \frac{\varepsilon}{2} \\ &= 0\,. \end{aligned}$$

This proves the assertion. □

For every diagonal δ, the collection of all copulas with diagonal δ is convex. Thus, for every $\delta \neq \delta_M$, there exist infinitely many copulas with diagonal δ.

4. Measures of concordance

Let $\mathcal{C}$ denote the collection of all copulas $[0,1]^2 \to [0,1]$. For every $C \in \mathcal{C}$, each of the functions $\pi(C)$ and $\nu_1(C)$ given by

$$(\pi(C))(u,v) := C(v,u) \qquad \text{and} \qquad (\nu_1(C))(u,v) := v - C(1-u,v)$$

is a copula. It follows that each of the mappings $\pi, \nu_1 : \mathcal{C} \to \mathcal{C}$ is an involution and that π and ν_1 generate a group Γ of transformations $\mathcal{C} \to \mathcal{C}$ which is a realization of the dihedral group with 8 elements; see Fuchs and Schmidt [7].

A functional $\kappa : \mathcal{C} \to \mathbb{R}$ is said to be a *measure of concordance* if it has the following properties:

(i) κ is continuous (with respect to uniform convergence on $\mathcal{C}$) and order preserving (with respect to the natural order relation on $\mathcal{C}$) with $\kappa[M] = 1$.
(ii) $\kappa[\pi(C)] = \kappa[C]$ holds for every copula C.
(iii) $\kappa[\nu_1(C)] = -\kappa[C]$ holds for every copula C.

Since $\nu_1(\Pi) = \Pi$ and $\nu_1(M) = W$, every measure of concordance κ satisfies $\kappa[\Pi] = 0$ and $\kappa[W] = -1$, and hence $\kappa(\mathcal{C}) \subseteq [-1,1]$. More generally, the identity $\kappa[C] = 0$ holds for every copula C which is invariant under the group Γ.

Invariant copulas also produce a wide class of measures of concordance since, for every invariant copula A, the functional $\kappa_A : \mathcal{C} \to \mathbb{R}$ given by

$$\kappa_A[C] := \frac{4\int_{[0,1]^2} C(u,v)\, dQ^A(u,v) - 1}{4\int_{[0,1]^2} M(u,v)\, dQ^A(u,v) - 1}$$

is a measure of concordance; see Edwards et al. [3]. The class of these *copula–induced* measures of concordance includes *Spearman's rho* $\kappa_\varrho := \kappa_\Pi$ and *Gini's gamma* $\kappa_\gamma := \kappa_{(M+W)/2}$.

Every copula–induced measure of concordance κ is *convex–linear* in the sense that

$$\kappa[aC + (1-a)D] = a\kappa[C] + (1-a)\kappa[D]$$

holds for all $C, D \in \mathcal{C}$ and all $a \in [0,1]$. Moreover, since each of π and ν_1, and hence every $\gamma \in \Gamma$, is convex–linear, the class of all convex–linear measures of concordance also includes the functionals $\kappa_{(s,t)} : \mathcal{C} \to \mathbb{R}$ with $(s,t) \in (0, \frac{1}{2}]^2$, given by

$$\kappa_{(s,t)}[C] := \frac{1}{4(s \wedge t)} \left(\sum_{\gamma \in \Gamma^{\pi,\tau}} (\gamma(C))(s,t) - \sum_{\gamma \in \Gamma \setminus \Gamma^{\pi,\tau}} (\gamma(C))(s,t) \right),$$

where $\Gamma^{\pi,\tau}$ denotes the subgroup of Γ generated by π and the transformation $\tau : \mathcal{C} \to \mathcal{C}$, which is defined by $(\tau(C))(u,v) := u + v - 1 + C(1-u, 1-v)$ and turns every copula C into its *survival copula*. These measures of concordance were introduced by Mesiar et al. [11]. Since $\kappa_{(1/2,1/2)}[C] = 4\,C(\frac{1}{2},\frac{1}{2}) - 1$, we see that $\kappa_{(1/2,1/2)}$ is *Blomqvist's beta*.

5. Convex–linear measures of concordance

Because of Proposition 1, the inequality

$$\kappa[C_{\mathbf{Ber},\delta}] \leq \kappa[C]$$

holds for every measure of concordance κ and every copula C with diagonal δ. Therefore, the first inequality of the following result is obvious, but the second is a bit surprising since Example 2 yields the existence of a copula D with diagonal δ and $D \not\leq C_{\mathbf{FN},\delta}$:

Theorem 2. *For every diagonal δ, the inequality*

$$\kappa[C_{\mathbf{Ber},\delta}] \leq \kappa[C] \leq \kappa[C_{\mathbf{FN},\delta}]$$

holds for every convex–linear measure of concordance κ and for every copula C with diagonal δ.

Proof. Consider a convex–linear measure of concordance κ and a copula C with diagonal δ. To prove the second inequality, we note that the copula $(C + \pi(C))/2$ is symmetric with diagonal δ. Therefore, Proposition 2 yields $(C + \pi(C))/2 \leq C_{\mathbf{FN},\delta}$. Since κ is convex–linear and $\kappa[\pi(C)] = \kappa[C]$, we obtain $\kappa[C] = \kappa[(C + \pi(C))/2] \leq \kappa[C_{\mathbf{FN},\delta}]$. □

6. Spearman's rho

Since Q^Π is the bivariate Lebesgue measure λ^2 on $\mathcal{B}([0,1]^2)$, Spearman's rho satisfies

$$\kappa_\rho[C] = 12 \int_{[0,1]^2} C(u,v)\, d\lambda^2(u,v) - 3\,.$$

We now study Spearman's rho for Bertino and Fredricks–Nelsen copulas.

For a diagonal δ, we consider the functions $L, U : [0,1] \to \mathbb{R}$ given by

$$\begin{aligned} L(t) &:= \inf\{s \in [0,t] \mid \hat{\delta}(t) \le \hat{\delta}(r) \text{ for all } r \in [s,t]\} \\ U(t) &:= \sup\{s \in [t,1] \mid \hat{\delta}(t) \le \hat{\delta}(r) \text{ for all } r \in [t,s]\}\,. \end{aligned}$$

Then we have $0 \le L(t) \le t \le U(t) \le 1$ for all $t \in [0,1]$. Moreover, we denote by $\delta^{\leftarrow}$ the lower quantile function of δ, given by $\delta^{\leftarrow}(t) := \inf\{u \in [0,1] \mid t \le \delta(u)\}$, and by $\tilde{\delta}^{\leftarrow}$ the lower quantile function of $\tilde{\delta}$. For all $t, u \in [0,1]$, we have $\delta^{\leftarrow}(t) \le u$ if and only if $t \le \delta(u)$.

The following result provides the values of Spearman's rho for Bertino and Fredricks–Nelsen copulas. The identity for Fredricks–Nelsen copulas is stated without proof in Fredricks and Nelsen [5] and we include a proof for the sake of completeness:

Theorem 3. *For every diagonal δ, Spearman's rho satisfies*

$$\kappa_\rho[C_{\mathbf{Ber},\delta}] = 1 + 12 \int_{[0,1]} t\,\hat{\delta}'(t) \Big((t - L(t))\, \chi_{[-1,0]}(\hat{\delta}'(t)) + (U(t) - t)\, \chi_{(0,1]}(\hat{\delta}'(t)) \Big) d\lambda(t)$$

and

$$\kappa_\rho[C_{\mathbf{FN},\delta}] = 12 \int_{[0,1]} \tilde{\delta}^{\leftarrow}(t)\, \delta^{\leftarrow}(t)\, d\lambda(t) - 3\,.$$

Proof. Let us first consider Bertino copulas. For every copula C, partial integration yields

$$\int_{[0,1]^2} C(u,v)\, d\lambda^2(u,v) = \frac{1}{2} - \int_{[0,1]^2} u\, (\partial_1 C)(u,v)\, d\lambda^2(u,v),$$

and hence

$$\kappa_\rho[C] = 3 - 12 \int_{[0,1]^2} u\, (\partial_1 C)(u,v)\, d\lambda^2(u,v)\,.$$

Moreover, Fernández–Sánchez and Trutschnig [[4]; Theorem 5.5] have shown that

$$(\partial_1 C_{\mathbf{Ber},\delta})(u,v) = \begin{cases} (1 + \hat{\delta}'(u))\chi_{[0,v]}(u) - \hat{\delta}'(u)\chi_{[0,v]}(L(u)) & \text{if } \hat{\delta}'(u) \le 0 \\ (1 - \hat{\delta}'(u))\chi_{[0,v]}(u) + \hat{\delta}'(u)\chi_{[0,v]}(U(u)) & \text{if } \hat{\delta}'(u) > 0 \end{cases}$$

We thus obtain

$$\begin{aligned} &\int_{[0,1]^2} u\, (\partial_1 C_{\mathbf{Ber},\delta})(u,v) d\lambda^2(u,v) \\ &= \int_{[0,1]^2} u\Big((1 + \hat{\delta}'(u))\chi_{[0,v]}(u) - \hat{\delta}'(u)\chi_{[0,v]}(L(u)) \Big) \chi_{[-1,0]}(\hat{\delta}'(u))\, d\lambda^2(u,v) \\ &\quad + \int_{[0,1]^2} u\Big((1 - \hat{\delta}'(u))\chi_{[0,v]}(u) + \hat{\delta}'(u)\chi_{[0,v]}(U(u)) \Big) \chi_{(0,1]}(\hat{\delta}'(u))\, d\lambda^2(u,v) \\ &= \int_{[0,1]^2} u\, \chi_{[0,v]}(u)\, d\lambda^2(u,v) \\ &\quad + \int_{[0,1]^2} u\, \hat{\delta}'(u) \Big(\chi_{[u,1]}(v) - \chi_{[L(u),1]}(v) \Big) \chi_{[-1,0)}(\hat{\delta}'(u))\, d\lambda^2(u,v) \\ &\quad + \int_{[0,1]^2} u\, \hat{\delta}'(u) \Big(\chi_{[U(u),1]}(v) - \chi_{[u,1]}(v) \Big) \chi_{(0,1]}(\hat{\delta}'(u))\, d\lambda^2(u,v) \\ &= \frac{1}{6} + \int_{[0,1]} u\, \hat{\delta}'(u)(L(u) - u)\, \chi_{[-1,0)}(\hat{\delta}'(u))\, d\lambda(u) + \int_{[0,1]} u\, \hat{\delta}'(u)(u - U(u))\, \chi_{(0,1]}(\hat{\delta}'(u))\, d\lambda(u) \\ &= \frac{1}{6} - \int_{[0,1]} u\, \hat{\delta}'(u) \Big((u - L(u))\, \chi_{[-1,0)}(\hat{\delta}'(u)) + (U(u) - u)\, \chi_{(0,1]}(\hat{\delta}'(u)) \Big) d\lambda(u), \end{aligned}$$

and hence

$$\begin{aligned} \kappa_\rho[C_{\mathbf{Ber},\delta}] &= 3 - 12 \int_{[0,1]^2} u\, (\partial_1 C_{\mathbf{Ber},\delta})(u,v)\, d\lambda^2(u,v) \\ &= 1 + 12 \int_{[0,1]} u\, \hat{\delta}'(u) \Big((u - L(u))\, \chi_{[-1,0)}(\hat{\delta}'(u)) + (U(u) - u)\, \chi_{(0,1]}(\hat{\delta}'(u)) \Big) d\lambda(u)\,. \end{aligned}$$

This proves the first identity.
Let us now consider Fredricks–Nelsen copulas. We have

$$\kappa_\varrho[C_{\mathbf{FN},\delta}] = 12\int_{[0,1]^2} C_{\mathbf{FN},\delta}(u,v)\, d\lambda^2(u,v) - 3 = 12\int_{[0,1]^2} u\wedge v\wedge \frac{\delta(u)+\delta(v)}{2}\, d\lambda^2(u,v) - 3\,.$$

Since every Fredricks–Nelsen copula is symmetric and since $\delta(u)\le\delta(v)\le\tilde{\delta}(v)$ holds for all $(u,v)\in[0,1]^2$ such that $u\le v$, we obtain

$$\begin{aligned}
&\int_{[0,1]^2} u\wedge v\wedge \frac{\delta(u)+\delta(v)}{2}\, d\lambda^2(u,v)\\
&= \int_{[0,1]\times[u,1]} (2u)\wedge\Big(\delta(u)+\delta(v)\Big)\, d\lambda^2(u,v)\\
&= \int_{[0,1]\times[u,1]} \tilde{\delta}(u)\wedge\delta(v)\, d\lambda^2(u,v) + \int_{[0,1]\times[u,1]} \delta(u)\, d\lambda^2(u,v)\\
&= \int_{[0,1]\times[u,1]} \tilde{\delta}(u)\wedge\delta(v)\, d\lambda^2(u,v) + \int_{[0,1]\times[u,1]} \tilde{\delta}(v)\wedge\delta(u)\, d\lambda^2(u,v)\\
&= \int_{[0,1]\times[u,1]} \tilde{\delta}(u)\wedge\delta(v)\, d\lambda^2(u,v) + \int_{[0,1]\times[0,v]} \tilde{\delta}(v)\wedge\delta(u)\, d\lambda^2(v,u)\\
&= \int_{[0,1]^2} \tilde{\delta}(u)\wedge\delta(v)\, d\lambda^2(u,v)\\
&= \int_{[0,1]^2}\int_{[0,1]} \chi_{[0,\tilde{\delta}(u)]}(t)\,\chi_{[0,\delta(v)]}(t)\, d\lambda(t)\, d\lambda^2(u,v)\\
&= \int_{[0,1]^2}\int_{[0,1]} \chi_{[\tilde{\delta}^{\leftarrow}(t),1]}(u)\,\chi_{[\delta^{\leftarrow}(t),1]}(v)\, d\lambda(t)\, d\lambda^2(u,v)\\
&= \int_{[0,1]} (1-\tilde{\delta}^{\leftarrow}(t))\,(1-\delta^{\leftarrow}(t))\, d\lambda(t)\\
&= \int_{[0,1]} \Big(1-\tilde{\delta}^{\leftarrow}(t)-\delta^{\leftarrow}(t)+\tilde{\delta}^{\leftarrow}(t)\,\delta^{\leftarrow}(t)\Big)\, d\lambda(t)\\
&= \int_{[0,1]} (1-\tilde{\delta}^{\leftarrow}(t))\, d\lambda(t) + \int_{[0,1]} (1-\delta^{\leftarrow}(t))\, d\lambda(t) - 1 + \int_{[0,1]} \tilde{\delta}^{\leftarrow}(t)\,\delta^{\leftarrow}(t)\, d\lambda(t)\\
&= \int_{[0,1]} \tilde{\delta}(t)\, d\lambda(t) + \int_{[0,1]} \delta(t)\, d\lambda(t) - 1 + \int_{[0,1]} \tilde{\delta}^{\leftarrow}(t)\,\delta^{\leftarrow}(t)\, d\lambda(t)\\
&= \int_{[0,1]} 2t\, d\lambda(t) - 1 + \int_{[0,1]} \tilde{\delta}^{\leftarrow}(t)\,\delta^{\leftarrow}(t)\, d\lambda(t)\\
&= \int_{[0,1]} \tilde{\delta}^{\leftarrow}(t)\,\delta^{\leftarrow}(t)\, d\lambda(t),
\end{aligned}$$

and hence

$$\begin{aligned}
\kappa_\varrho[C_{\mathbf{FN},\delta}] &= 12\int_{[0,1]^2} u\wedge v\wedge \frac{\delta(u)+\delta(v)}{2}\, d\lambda^2(u,v) - 3\\
&= 12\int_{[0,1]} \tilde{\delta}^{\leftarrow}(t)\,\delta^{\leftarrow}(t)\, d\lambda(t) - 3\,.
\end{aligned}$$

This proves the second identity. □

For certain diagonals, the first identity of Theorem 3 can be simplified as follows:

Corollary 1. *If δ is a diagonal for which $\hat{\delta}$ is symmetric and increasing on $[0,1/2]$, then*

$$\kappa_\rho[C_{\mathbf{Ber},\delta}] = 48\int_{[0,1/2]} (1-2t)\,\delta(t)\, d\lambda(t) - 1\,.$$

Proof. The assumption on δ yields $\hat{\delta}'(t)\le 0$ and $L(t)=1-t$ for all $t\in[1/2,1]$, $\hat{\delta}'(t)\ge 0$ and $U(t)=1-t$ for all $t\in[0,1/2]$, and $\hat{\delta}'(1-t)=-\hat{\delta}'(t)$ for all $t\in[0,1]$. We thus obtain

$$\begin{aligned}
&\int_{[0,1]} t\,\hat{\delta}'(t)\Big((t-L(t))\,\chi_{[-1,0)}(\hat{\delta}'(t)) + (U(t)-t)\,\chi_{(0,1]}(\hat{\delta}'(t))\Big)\, d\lambda(t)\\
&= \int_{[0,1]} t\,\hat{\delta}'(t)\Big((2t-1)\,\chi_{[1/2,1]}(t) + (1-2t)\,\chi_{[0,1/2]}(t)\Big)\, d\lambda(t)\\
&= \int_{[0,1]} t(2t-1)\,\hat{\delta}'(t)\,\chi_{[1/2,1]}(t)\, d\lambda(t) + \int_{[0,1]} t(1-2t)\,\hat{\delta}'(t)\,\chi_{[0,1/2]}(t)\, d\lambda(t)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{[0,1]} (1-t)(1-2t)\,\hat{\delta}'(1-t)\,\chi_{[0,1/2]}(t)\,d\lambda(t) + \int_{[0,1]} t(1-2t)\,\hat{\delta}'(t)\,\chi_{[0,1/2]}(t)\,d\lambda(t) \\
&= -\int_{[0,1]} (1-t)(1-2t)\,\hat{\delta}'(t)\,\chi_{[0,1/2]}(t)\,d\lambda(t) + \int_{[0,1]} t(1-2t)\,\hat{\delta}'(t)\,\chi_{[0,1/2]}(t)\,d\lambda(t) \\
&= -\int_{[0,1/2]} (1-2t)^2\,\hat{\delta}'(t)\,d\lambda(t),
\end{aligned}
$$

where the second last identity follows from substituting t with $1-t$. Now partial integration yields

$$
\begin{aligned}
\int_{[0,1/2]} (1-2t)^2\,\hat{\delta}'(t)\,d\lambda(t) &= \int_{[0,1/2]} 4\,(1-2t)\,\hat{\delta}(t)\,d\lambda(t) \\
&= 4\int_{[0,1/2]} (1-2t)(t-\delta(t))\,d\lambda(t) \\
&= 4\int_{[0,1/2]} (t-2t^2)\,d\lambda(t) - 4\int_{[0,1/2]} (1-2t)\,\delta(t)\,d\lambda(t) \\
&= \frac{1}{6} - 4\int_{[0,1/2]} (1-2t)\,\delta(t)\,d\lambda(t)\,.
\end{aligned}
$$

We thus obtain

$$
\begin{aligned}
\kappa_\varrho[C_{\mathbf{Ber},\delta}] &= 1 + 12\int_{[0,1]} t\,\hat{\delta}'(t)\Big((t-L(t))\,\chi_{[-1,0)}(\hat{\delta}'(t)) + (U(t)-t)\,\chi_{(0,1]}(\hat{\delta}'(t))\Big)\,d\lambda(t) \\
&= 1 + 12\left(-\frac{1}{6} + 4\int_{[0,1/2]} (1-2t)\,\delta(t)\,d\lambda(t)\right) \\
&= 48\int_{[0,1/2]} (1-2t)\,\delta(t)\,d\lambda(t) - 1,
\end{aligned}
$$

as was to be shown. □

By Corollary 1, Spearman's rho is convex–linear on the collection of all Bertino copulas having a diagonal δ for which $\hat{\delta}$ is symmetric and increasing on $[0,1/2]$. The collection of these diagonals includes δ_M, δ_Π and δ_W.

Example 3. (Fréchet Diagonals)

1. If $\delta = \alpha\,\delta_M + (1-\alpha-\beta)\,\delta_\Pi + \beta\,\delta_W$ holds for some $\alpha, \beta \in [0,1]$ such that $\alpha+\beta \le 1$, then
$$\kappa_\varrho[C_{\mathbf{Ber},\delta}] = \frac{3\alpha-\beta-1}{2}\,.$$
2. If $\delta = \alpha\,\delta_M + (1-\alpha)\,\delta_W$ holds for some $\alpha \in [0,1]$, then
$$2\alpha - 1 = \kappa_\varrho[C_{\mathbf{Ber},\delta}] \le \kappa_\varrho[C_{\mathbf{FN},\delta}] = 3 + 4\alpha + \frac{8\alpha-14}{(2-\alpha)^2}$$
and equality holds if and only if $\alpha = 1$.
3. $\kappa_\varrho[C_{\mathbf{FN},\delta_\Pi}] = 5 - (3/2)\pi \approx 0.2876$.

For a particular class of diagonals, Kokol Bukovšek and Stopar [[10]; Lemma 9] obtained another representation of $\kappa_\varrho[C_{\mathbf{FN},\delta}]$ in terms of δ.

7. Gini's gamma

Straightforward calculation shows that Gini's gamma satisfies

$$\kappa_\gamma[C] = 4\int_{[0,1]} \delta_C(t)\,d\lambda(t) + 4\int_{[0,1]} C(t,1-t)\,d\lambda(t) - 2\,.$$

The following result provides the values of Gini's gamma for Bertino and Fredricks–Nelsen copulas.

Theorem 4. *For every diagonal δ, Gini's gamma satisfies*

$$\kappa_\gamma[C_{\mathbf{Ber},\delta}] = 4\int_{[0,1]} \delta(t)\,d\lambda(t) - 8\int_{[0,1/2]} \min\{s-\delta(s) \mid t \le s \le 1-t\}\,d\lambda(t) - 1$$

and

$$\kappa_\gamma[C_{\mathbf{FN},\delta}] = 4\int_{[0,1/2]} \Big((\delta(1-t)-\tilde{\delta}(t)) + (\delta(1-t)-\tilde{\delta}(t)) \wedge 0\Big)\,d\lambda(t)\,.$$

Proof. Let us first consider Bertino copulas. Symmetry yields

$$\int_{[0,1]} C_{\mathbf{Ber},\delta}(t,1-t)\,d\lambda(t) = 2\int_{[0,1/2]} \Big(t - \min\{s-\delta(s) \mid t \le s \le (1-t)\Big)\,d\lambda(t)$$
$$= \frac{1}{4} - 2\int_{[0,1/2]} \min\{s-\delta(s) \mid t \le s \le (1-t)\Big)\,d\lambda(t),$$

and hence

$$\kappa_\gamma[C_{\mathbf{Ber},\delta}] = 4\int_{[0,1]} \delta(t)\,d\lambda(t) + 4\int_{[0,1]} C_{\mathbf{Ber},\delta}(t,1-t)\,d\lambda(t) - 2$$
$$= 4\int_{[0,1]} \delta(t)\,d\lambda(t) - 8\int_{[0,1/2]} \min\{s-\delta(s) \mid t \le s \le 1-t\}\,d\lambda(t) - 1\,.$$

Let us now consider Fredricks–Nelsen copulas. Symmetry yields

$$\int_{[0,1]} C_{\mathbf{FN},\delta}(t,1-t)\,d\lambda(t) = 2\int_{[0,1/2]} t \wedge \frac{\delta(t)+\delta(1-t)}{2}\,d\lambda(t)$$
$$= \int_{[0,1/2]} (2t) \wedge (\delta(t)+\delta(1-t))\,d\lambda(t)$$
$$= \int_{[0,1/2]} \Big(2t - (2t-\delta(t)-\delta(1-t))^+\Big)\,d\lambda(t)$$
$$= \frac{1}{4} - \int_{[0,1/2]} (\tilde{\delta}(t) - \delta(1-t))^+\,d\lambda(t),$$

and hence

$$\kappa_\gamma[C_{\mathbf{FN},\delta}] = 4\int_{[0,1]} \delta(t)\,d\lambda(t) + 4\int_{[0,1]} C_{\mathbf{FN},\delta}(t,1-t)\,d\lambda(t) - 2$$
$$= 4\int_{[0,1]} \delta(t)\,d\lambda(t) - 4\int_{[0,1/2]} (\tilde{\delta}(t) - \delta(1-t))^+\,d\lambda(t) - 1$$
$$= 4\int_{[0,1/2]} \Big(\delta(t)+\delta(1-t) - 2t - (\tilde{\delta}(t) - \delta(1-t))^+\Big)\,d\lambda(t)$$
$$= 4\int_{[0,1/2]} \Big((\delta(1-t) - \tilde{\delta}(t)) + (\delta(1-t) - \tilde{\delta}(t)) \wedge 0\Big)\,d\lambda(t)\,.$$

This completes the proof. □

As in the case of Spearman's rho the first identity of Theorem 4 can be simplified for certain diagonals:

Corollary 2. *If δ is a diagonal for which $\hat{\delta}$ is symmetric and increasing on* $[0,1/2]$*, then*

$$\kappa_\gamma[C_{\mathbf{Ber},\delta}] = 16\int_{[0,1/2]} \delta(t)\,d\lambda(t) - 1\,.$$

Proof. The assumption on δ yields

$$\kappa_\gamma[C_{\mathbf{Ber},\delta}] = 4\int_{[0,1]} \delta(t)\,d\lambda(t) - 8\int_{[0,1/2]} \min\{s-\delta(s) \mid t \le s \le 1-t\}\,d\lambda(t) - 1$$
$$= 4\int_{[0,1]} (\delta(t)-t)\,d\lambda(t) - 8\int_{[0,1/2]} (t-\delta(t))\,d\lambda(t) + 1$$
$$= 8\int_{[0,1/2]} (\delta(t)-t)\,d\lambda(t) + 8\int_{[0,1/2]} (\delta(t)-t)\,d\lambda(t) + 1$$
$$= 16\int_{[0,1/2]} \delta(t)\,d\lambda(t) - 1$$

as was to be shown. □

Corollary 2 shows that also Gini's gamma is convex–linear on the collection of all Bertino copulas having a diagonal δ for which $\hat{\delta}$ is symmetric and increasing on $[0,1/2]$.

Example 4. (Fréchet Diagonals)

1. If $\delta = \alpha\,\delta_M + (1-\alpha-\beta)\,\delta_\Pi + \beta\,\delta_W$ holds for some $\alpha,\beta \in [0,1]$ such that $\alpha+\beta \le 1$, then

$$\kappa_\gamma[C_{\mathbf{Ber},\delta}] = \frac{4\,\alpha - 2\,\beta - 1}{3}\,.$$

2. If $\delta = \alpha\,\delta_M + (1-\alpha)\,\delta_W$ holds for some $\alpha \in [0,1]$, then
$$2\,\alpha - 1 = \kappa_\gamma[C_{\mathbf{Ber},\delta}] \le \kappa_\gamma[C_{\mathbf{FN},\delta}] = 2\,\alpha - \frac{1}{2-\alpha}$$
and equality holds if and only if $\alpha = 1$.
3. $\kappa_\gamma[C_{\mathbf{FN},\delta_\Pi}] = 2 - (4/3)\sqrt{2} \approx 0.1144$.

8. Kendall's tau

Another important measure of concordance is *Kendall's tau* κ_τ, given by
$$\kappa_\tau[C] := 4 \int_{[0,1]^2} C(u,v)\, dQ^C(u,v) - 1\,.$$
Since
$$\kappa_\tau\left[\frac{\Pi + M}{2}\right] = \frac{5}{12} \neq \frac{1}{2} = \frac{\kappa_\tau[\Pi] + \kappa_\tau[M]}{2},$$
we see that Kendall's tau fails to be convex–linear. Nevertheless, we shall show in Theorem 6 that the inequality of Theorem 2 also holds for Kendall's tau. Before, we determine Kendall's tau for Bertino and Fredricks–Nelsen copulas.

For the computation of the values of Kendall's tau, it is sometimes convenient to use *Kendall's distribution function* $K_C : [0,1] \to [0,1]$ which is defined by
$$K_C(t) := Q^C[\{C \le t\}]$$
and satisfies
$$\int_{[0,1]^2} C(u,v)\, dQ^C(u,v) = \int_{[0,1]} (1 - K_C(t))\, d\lambda(t)\,.$$
The following result provides the values of Kendall's tau for Bertino and Fredricks–Nelsen copulas. The identity for Fredricks–Nelsen copulas is stated without proof in Fredricks and Nelsen [5] and Nelsen [[12]; Exercise 5.5] and we include a proof for the sake of completeness:

Theorem 5. *For every diagonal δ, Kendall's tau satisfies*
$$\kappa_\tau[C_{\mathbf{Ber},\delta}] = 8 \int_{[0,1]} \delta(t)\, d\lambda(t) - 3$$
and
$$\kappa_\tau[C_{\mathbf{FN},\delta}] = 4 \int_{[0,1]} \delta(t)\, d\lambda(t) - 1\,.$$
In particular, $2\,\kappa_\tau[C_{\mathbf{FN},\delta}] = 1 + \kappa_\tau[C_{\mathbf{Ber},\delta}] \ge 0$.

Proof. Let us first consider Bertino copulas. Nelsen et al. [13] have shown that
$$K_{C_{\mathbf{Ber},\delta}}(t) = 2\,\delta^{\to}(t) - t$$
with the upper quantile function $\delta^{\to}$ of δ, given by $\delta^{\to}(t) := \sup\{s \in [0,1] \mid \delta(s) \le t\}$. Since $s \le \delta^{\to}(t)$ if and only if $\delta(s) \le t$ holds for all $s,t \in [0,1]$, Fubini's theorem yields
$$\int_{[0,1]} \delta^{\to}(t)\, d\lambda(t) = \int_{[0,1]}\int_{[0,1]} \chi_{[0,\delta^{\to}(t)]}(s)\, d\lambda(s)\, d\lambda(t) = \int_{[0,1]}\int_{[0,1]} \chi_{[\delta(s),1]}(t)\, d\lambda(t)\, d\lambda(s) = \int_{[0,1]} (1-\delta(s))\, d\lambda(s)\,.$$
We thus obtain
$$\int_{[0,1]} K_{C_{\mathbf{Ber},\delta}}(t)\, d\lambda(t) = \int_{[0,1]} (2\,\delta^{\to}(t) - t)\, d\lambda(t) = \int_{[0,1]} (2 - 2\delta(t) - t)\, d\lambda(t) = \frac{3}{2} - 2\int_{[0,1]} \delta(t)\, d\lambda(t)\,,$$
and hence
$$\kappa_\tau[C_{\mathbf{Ber},\delta}] = 4\int_{[0,1]} (1 - K_{C_{\mathbf{Ber},\delta}}(t))\, d\lambda(t) - 1 = 8\int_{[0,1]} \delta(t)\, d\lambda(t) - 3\,.$$
Let us now consider Fredricks–Nelsen copulas. For $t \in [0,1]$, define
$$A_t := \{(u,v) \in (t,1]^2 \mid \delta(u) + \delta(v) > 2t\}$$
$$B_t := \{(u,v) \in (t,1]^2 \mid \delta(u) + \delta(v) \le 2t\}\,.$$
Then we have
$$C_{\mathbf{FN},\delta}(u,v) = \frac{\delta(u) + \delta(v)}{2}$$

for all $(u,v) \in B_t$, and it follows that $Q^{C_{\mathbf{FN},\delta}}[[p,u]\times[q,v]] = 0$ holds for every rectangle $[p,u]\times[q,v] \subseteq B_t$. This yields $Q^{C_{\mathbf{FN},\delta}}[B_t] = 0$. Therefore, we obtain

$$\begin{aligned} Q^{C_{\mathbf{FN},\delta}}[\{C_{\mathbf{FN},\delta} > t\}] &= Q^{C_{\mathbf{FN},\delta}}[A_t] \\ &= Q^{C_{\mathbf{FN},\delta}}[A_t] + Q^{C_{\mathbf{FN},\delta}}[B_t] \\ &= Q^{C_{\mathbf{FN},\delta}}[(t,1]^2] \\ &= 1 - 2t + \delta(t), \end{aligned}$$

hence

$$K_{C_{\mathbf{FN},\delta}}(t) = Q^{C_{\mathbf{FN},\delta}}[\{C_{\mathbf{FN},\delta} \le t\}] = 2t - \delta(t),$$

and thus

$$\kappa_\tau[C_{\mathbf{FN},\delta}] = 4\int_{[0,1]} (1 - K_{C_{\mathbf{FN},\delta}}(t))\, d\lambda(t) - 1 = 4\int_{[0,1]} \delta(t)\, d\lambda(t) - 1\,.$$

The final assertion is then immediate. □

By Theorem 5, Kendall's tau for Bertino and Fredricks–Nelsen copulas is convex–linear in δ.

Example 5. (Fréchet Diagonals) If $\delta = \alpha\,\delta_M + (1-\alpha-\beta)\,\delta_\Pi + \beta\,\delta_W$ holds for some $\alpha,\beta \in [0,1]$ such that $\alpha+\beta \le 1$, then

$$\frac{4\,\alpha - 2\,\beta - 1}{3} = \kappa_\tau[C_{\mathbf{Ber},\delta}] \le \kappa_\tau[C_{\mathbf{FN},\delta}] = \frac{2\,\alpha - \beta + 1}{3}$$

and equality holds if and only if $\alpha = 1$.

It is evident from Theorem 5 that the values $\kappa_\tau[C_{\mathbf{Ber},\delta}]$ and $\kappa_\tau[C_{\mathbf{FN},\delta}]$ are increasing in δ, and it follows that the differences $\kappa_\tau[C_{\mathbf{FN},\delta}] - \kappa_\tau[C_{\mathbf{Ber},\delta}] = 1 - \kappa_\tau[C_{\mathbf{FN},\delta}] \in [0,1]$ are decreasing in δ. The following results characterize the diagonals for which the bounds of $\kappa_\tau[C_{\mathbf{FN},\delta}] - \kappa_\tau[C_{\mathbf{Ber},\delta}]$ are attained:

Corollary 3. *For every diagonal δ, the following properties are equivalent:*

(a) $\delta = \delta_M$.
(b) $\kappa_\tau[C_{\mathbf{Ber},\delta}] = 1$.
(c) $\kappa_\tau[C_{\mathbf{FN},\delta}] = 1$.
(d) $\kappa_\tau[C_{\mathbf{FN},\delta}] - \kappa_\tau[C_{\mathbf{Ber},\delta}] = 0$.

Proof. If $\delta = \delta_M$, then Proposition 1 yields $C_{\mathbf{Ber},\delta} = M$, and hence $\kappa_\tau[C_{\mathbf{Ber},\delta}] = 1$. Assume now that $\kappa_\tau[C_{\mathbf{Ber},\delta}] = 1$. Then Theorem 5 yields $\int_{[0,1]} \delta(t)\, d\lambda(t) = 1/2$, and thus

$$\int_{[0,1]} (\delta_M(t) - \delta(t))\, d\lambda(t) = \int_{[0,1]} (t - \delta(t))\, d\lambda(t) = 0\,.$$

Since $\delta_M - \delta \ge 0$ and since diagonals are continuous, this implies $\delta_M(t) - \delta(t) = 0$ for all $t \in [0,1]$. The remaining implications are obvious. □

Corollary 4. *For every diagonal δ, the following properties are equivalent:*

(a) $\delta = \delta_W$.
(b) $\kappa_\tau[C_{\mathbf{Ber},\delta}] = -1$.
(c) $\kappa_\tau[C_{\mathbf{FN},\delta}] = 0$.
(d) $\kappa_\tau[C_{\mathbf{FN},\delta}] - \kappa_\tau[C_{\mathbf{Ber},\delta}] = 1$.

The proof of Corollary 4 is analogous to that of Corollary 3.

9. Spearman's footrule

It turns out that Kendall's tau for Bertino and Fredricks–Nelsen copulas is closely related to *Spearman's footrule* $\phi : \mathcal{C} \to \mathbb{R}$ given by

$$\phi[C] := 6\int_{[0,1]^2} C(u,v)\, dQ^M(u,v) - 2\,.$$

Since $\phi[W] = -1/2 \ne -1$, Spearman's footrule fails to be a measure of concordance. Nevertheless, it is of interest since Q^M is the push–forward measure λ_{T^M} of λ under the measurable mapping $T^M : [0,1] \to [0,1]^2$ given by $T^M(t) = (t,t)$, which yields

$$\phi[C] = 6\int_{[0,1]} \delta(t)\, d\lambda(t) - 2$$

for every copula C with diagonal δ and hence, in particular, for $C_{\mathbf{Ber},\delta}$ and $C_{\mathbf{FN},\delta}$.

For copulas with a given diagonal, the previous identity together with those of Theorem 5 reveals a close connection between the values of Spearman's footrule and those of Kendall's tau for Bertino and Fredricks–Nelsen copulas:

Corollary 5. *For every diagonal δ, the identities*

$$\phi[C] = \frac{\kappa_\tau[C_{\mathbf{Ber},\delta}] + \kappa_\tau[C_{\mathbf{FN},\delta}]}{2}$$

and

$$3\,\kappa_\tau[C_{\mathbf{Ber},\delta}] + 3 = 4\,\phi[C] + 2 = 6\,\kappa_\tau[C_{\mathbf{FN},\delta}]$$

hold for every copula C with diagonal δ.

For arbitrary copulas, Kokol Bukovšek and Stopar [9] obtained the following bounds for Kendall's tau in terms of Spearman's footrule:

Proposition 3. *The inequality*

$$4\,\phi[C] - 1 \le 3\,\kappa_\tau[C] \le 2\,\phi[C] + 1$$

holds for every copula C and the bounds are attained.

The final assertion of the previous result is also evident from Corollary 5 which yields $4\,\phi[C] - 1 = 3\,\kappa_\tau[C_{\mathbf{Ber},\delta}]$ and $2\,\phi[C] + 1 = 3\,\kappa_\tau[C_{\mathbf{FN},\delta}]$. For Kendall's tau, we thus obtain the following analogue of Theorem 2:

Theorem 6. *For every diagonal δ, the inequality*

$$\kappa_\tau[C_{\mathbf{Ber},\delta}] \le \kappa_\tau[C] \le \kappa_\tau[C_{\mathbf{FN},\delta}]$$

holds for every copula C with diagonal δ.

10. Fréchet copulas

For the diagonals δ_M, δ_Π, δ_W and $(\delta_M + \delta_W)/2$, the following table presents the values of Spearman's rho, Gini's gamma and Kendall's tau for Bertino and Fredricks–Nelsen copulas and the value of Spearman's footrule:

δ	$\kappa_\varrho[C_{\mathbf{Ber},\delta}]$	$\kappa_\gamma[C_{\mathbf{Ber},\delta}]$	$\kappa_\tau[C_{\mathbf{Ber},\delta}]$	$\kappa_\varrho[C_{\mathbf{FN},\delta}]$	$\kappa_\gamma[C_{\mathbf{FN},\delta}]$	$\kappa_\tau[C_{\mathbf{FN},\delta}]$	$\phi[C]$
δ_M	1	1	1	1	1	1	1
δ_Π	-1/2	-1/3	-1/3	5 - $(3/2)\pi$	$2 - (4/3)\sqrt{2}$	1/3	0
δ_W	-1	-1	-1	-1/2	-1/2	0	-1/2
$(\delta_M + \delta_W)/2$	0	0	0	5/9	1/3	1/2	1/4

CRediT authorship contribution statement

Sebastian Fuchs: Formal analysis; **Klaus D. Schmidt:** Formal analysis; **Yuping Wang:** Formal analysis.

Data availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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